EE2012 Tutorial 1

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About Me

Undergraduate: Control Engineering at Beihang University

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Research Interests

- Flight Control Systems and Robust Control
- Vision-aided Control
- Vision-aided Inertial Navigation
- Autonomous Unmanned Aerial Vehicles

Summary of Chapter 1

- **Models**
  - Deterministic: predictable outcome
  - Probabilistic: random experiments → random outcomes
- **Random Experiments**
  - Sample space $S = \{\zeta_1, \zeta_2, \dots\}$
  - Different outcomes are disjoint
  - Results are random every time
- **Random Events**
  - Subsets of sample space
  - Event operations: union, intersection, complement

Summary of Chapter 1

➢ Probability Basics

- Defined via relative frequency
- Axioms: Assume events $A \subseteq S$ and $B \subseteq S$.
  1) [**Nonnegativity**] $P(A) \geq 0$ for every event A.
  2) [**Normalization**] $P(S) = 1$.
  3) [**Additivity**] If $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$.
     It can be extended to more events:
     If $A_1, A_2, \ldots$ are disjoint, then $P(\cup_{k=1}^{\infty} A_k) = \Sigma_{k=1}^{\infty} P(A_k)$.
- Properties: Let A and B be two events.
  1) $P(A^c) = 1 - P(A)$.
  2) $P(A) \leq 1$.
  3) $P(\emptyset) = 0$.
  4) If $A \subset B$, then $P(A) \leq P(B)$.
  5) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

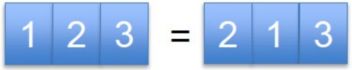
Summary of Chapter 1

$$P(A) = \frac{|A|}{|S|} = \frac{\text{\# of favorable outcomes}}{\text{\# of total outcomes}}$$

➢Counting Principles.

$$n \times n \times n = n^3$$

	With ordering	Without ordering
With replacement	n^k	$\binom{n+k-1}{k}$
Without replacement	$\frac{n!}{(n-k)!}$	$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$



$$n \times (n-1) \times (n-2) = \frac{n(n-1)(n-2)\cdots 1}{(n-3)(n-4)\cdots 1} = \frac{n!}{(n-3)!}$$

permutation

combination

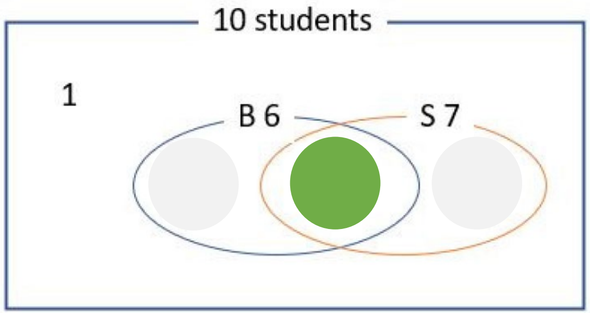
Q1

In a group of 10 students, 6 students play basketball, 7 students play soccer ball and 1 student plays neither.

(a) Find the probability that a randomly selected student plays both of the sports.

(b) If two students AB are selected from the group in order, what is the probability that A is a basketball player and B is a soccer player?

Q1 (a) Probability that a randomly selected student plays both of the sports



Step 1: Determine the number of students who play both basketball and soccer:

Let the number of students who play both sports be x.

Total students =(playing only basketball)+(playing only soccer)+(playing both sports)+(playing neither)

$$10 = (6 - x) + (7 - x) + x + 1$$

$$x = 4$$

$$P(A) = \frac{\text{\# of favorable outcomes}}{\text{\# of total outcomes}}$$

Step 2: Calculate the probability

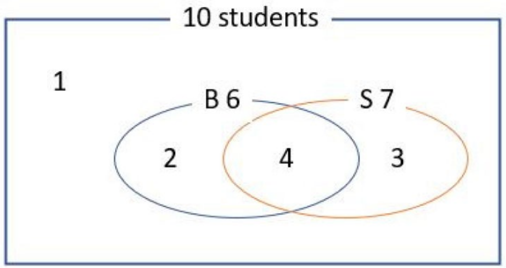
The probability that a randomly selected student plays both sports is the ratio of students who play both sports to the total number of students:

$$\text{Probability} = \frac{\text{\# of students playing both sports}}{\text{Total \# of students}} = \frac{4}{10} = 0.4$$

Q1 (b) Probability that A is a basketball player and B is a soccer player

This part involves two steps:

1. Selecting a basketball player (student A).
2. Selecting a soccer player (student B) from the remaining students.



Since any 2 students can be selected with equal probability, it is a permutation for 2 out of 10.

$$P(n,r) = \frac{n!}{(n-r)\,!} = \frac{10!}{(10-2)!} = 10 \times 9 = 90$$

Since there are 4 students playing both sports, we need to consider two cases.

Case I: If student A only plays basketball, there are 7 choices for B being a soccer ball player.

$$N_1 = 2 \times 7 = 14$$

Basketball only × soccer

Case II: if student A plays both, there are 6 choices for B being a soccer ball player.

$$N_2 = 4 \times 6 = 24$$

(Both Basketball and soccer) × (soccer-1)

So the total number of choices for desired AB is $N = N_1 + N_2 = 38$. The probability is $\frac{N}{P(10,2)} = \frac{38}{90}$

Q2

A sequential experiment

A six-sided dice is rolled 6 times independently. Find out the probability of the sequence "1 2 3 4 5 6 " in these 6 rolls for the following two settings.

(a) The dice is fair.

(b) The dice is loaded in a way that each even number is twice as likely as each odd number. All even numbers are equally likely, as are all odd numbers.

Q2 (a) The Dice is Fair

When the dice is fair, all the numbers (1 to 6) come out with **equal probability**. So the desired event probability is the ratio between the desired event size and the sample space size.

Method I

Only in the sequence “1 2 3 4 5 6 ” one outcome is in the desired event.

$$P(A) = \frac{\text{\# of favorable outcomes}}{\text{\# of total outcomes}}$$

- There are a total of $6^6 = 46656$ outcomes in the sample space.
- So the probability for the desired event is $\frac{1}{46656}$ if the dice is fair.

Method II

For a fair six-sided dice, each face (1, 2, 3, 4, 5, 6) has an equal probability of $\frac{1}{6}$

Since the rolls are **independent**, the probability of getting the sequence "1 2 3 4 5 6" is the product of the probabilities of each roll:

$$P(\text{"}1,2,3,4,5,6\text{"}) = \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \left(\frac{1}{6}\right)^6$$

Events A and B are independent
$\Leftrightarrow P(AB) = P(A)P(B)$

Q2 (b) The dice is loaded in a way that each even number is twice as likely as each odd number. All even numbers are equally likely, as are all odd numbers.

For the loaded die, we know:

- Each even number (2, 4, 6) is twice as likely as each odd number (1, 3, 5).
- All even numbers are equally likely, and all odd numbers are equally likely.

Step 1: Find Probabilities to Each Face

- Let the probability of rolling an odd number (1, 3, or 5) be p.
- Since each even number is twice as likely as each odd number, the probability of rolling an even number (2, 4, or 6) will be $2p$.

The sum of all probabilities must equal 1:

$P(S)$=1

$$p + 2p + p + 2p + p + 2p = 1 \quad \Rightarrow \quad p = \frac{1}{9}$$

Step 2: Calculate the probability of the sequence "1 2 3 4 5 6"

$$P('1,2,3,4,5,6') = P(1) \times P(2) \times P(3) \times P(4) \times P(5) \times P(6)$$

$$= p \times 2p \times p \times 2p \times p \times 2p = 8p^6$$

Q3

Assume a fair six-sided dice is rolled.

(a) If the dice is rolled 10 times, find the probability that six shows at least once.

(b) Find the probability that one shows twice, three shows twice, and six shows once if the dice is rolled 5 times.

(c) Let event A = "the number is less than 3". If the dice is rolled an infinite number of times, what is the probability that A occurs k times at the n^{th} roll but not earlier?

Q3 (a) If the dice is rolled 10 times, find the probability that six shows at least once.

Step 1: Calculate the probability of not rolling a six in one roll.

Since the dice is fair, $P(\text{'no 6 in one roll'}) = \left(\frac{5}{6}\right)$

Step 2: Calculate the probability of not rolling a six in 10 rolls.

$$P(\text{'no 6 in 10 rolls'}) = \left(\frac{5}{6}\right)^{10}$$

Step 3: Calculate the probability of rolling at least one six.

This probability is the complement of the probability of not rolling a six in 10 rolls:

$$P(\text{'at least one 6 in 10 rolls'}) = 1 - P(\text{'no 6 in 10 rolls'})$$
$$= 1 - \left(\frac{5}{6}\right)^{10}$$

Q3(b) Probability that “1” shows twice, “3” shows twice, and “6” shows once in 5 rolls

Step 1: Total number of possible outcomes.

Since the die is rolled 5 times, and each roll has 6 possible outcomes, the total number of possible outcomes is:

$$\text{Total outcomes} = 6^5$$

Step 2: Number of favourable outcomes.

- Choose 2 positions out of 5 for “1": $\binom{5}{2} = 10$
- Choose 2 positions out of the remaining 3 for “3": $\binom{3}{2} = 3$
- The last remaining position will automatically be for “6".

Number of favourable outcomes $= \binom{5}{2} \times \binom{3}{2} \times 1 = 30$

Step 3: Calculate the probability of event

$$P = \frac{\text{\# of favorable outcomes}}{\text{Total \# of outcomes}} = \frac{30}{6^5}$$

Alternative method using the product of the probabilities

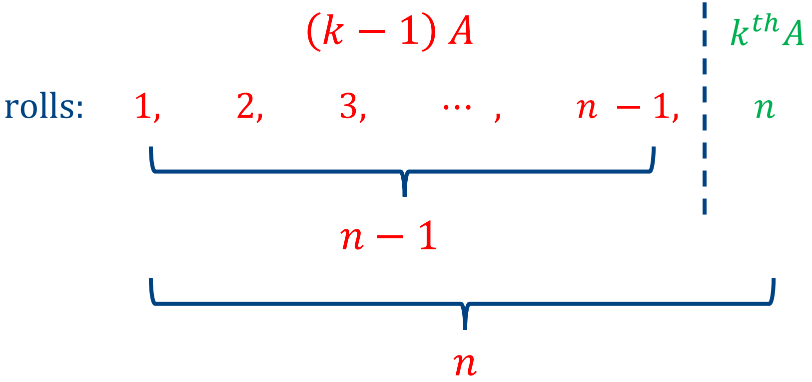
“1”	$\binom{5}{2}$	$\frac{1}{6}$	$\frac{1}{6}$
“3”	$\binom{3}{2}$	$\frac{1}{6}$	$\frac{1}{6}$
“6”	1	$\frac{1}{6}$	

Q3 (c) Probability that A={number is less than 3},occurs k times at the n^{th} roll but not earlier

Given A={number is less than 3}, the probability of event A occurring on any single roll is

$$p_A = P(\text{number is 1 or 2}) = \frac{2}{6}$$

- The k^{th} occurrence of A must be the outcome of the n^{th} roll.
- It also requires A to occur exactly $k - 1$ times in the first $n - 1$ rolls.



$(k-1)\ A$ | $k^{th} A$

rolls: $\underbrace{1, \quad 2, \quad 3, \quad \cdots, \quad n-1,}_{n-1}$ | n

$\underbrace{1, 2, 3, \cdots, n-1, n}_{n}$

Q3 (c)

	$(k-1)\ A$	$k^{th} A$
rolls:	$\underbrace{1, \quad 2, \quad 3, \quad \cdots, \quad n-1,}_{n-1}$	n
Possible cases	$\binom{n-1}{k-1}$	1
Probability of each case	$p = p_A^{k-1} p_{A^c}^{(n-1)-(k-1)}$ $= p_A^{k-1} p_{A^c}^{(n-k)}$ $= p_A^{k-1}(1-p_A)^{n-k}$	p_A

Probability of the interested event is $\binom{n-1}{k-1} \times p \times 1 \times p_A$

Q3 (c)

Summary

$$p_A = P(\text{number is } 1 \text{ or } 2) = \frac{2}{6}$$

Part 1: Probability of A occurring $k-1$ times in the first $n-1$ rolls.

- There are $\binom{n-1}{k-1}$ possible cases for A occurring $k - 1$ times in $n - 1$ rolls.
- Each of them has the probability $p = p_A^{k-1}(1-p_A)^{n-k}$

Part 2: Probability of the k^{th} success occurring at the n^{th} roll.

A must occur on the n^{th} roll, which has probability p_A

Finally, the probability of the interested event is $\binom{n-1}{k-1} \times p \times p_A$

Q4

Binomial expansion

Denote the number of “choosing k out of n” combinations by $\binom{n}{k}$. Find the values of the equations below if n is a constant positive integer.

(a) $1+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n-1}+\binom{n}{n}$

(b) $\binom{n}{1}-\binom{n}{2}+\binom{n}{3}+\cdots+(-1)^{n+1}\binom{n}{n}$

Q4 (a) $1 + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n}$

Denote the value of the above equation as A

Since $\binom{n}{0} = 1$, the equation can be rewritten as $A = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k}$

Recall the expansion of $(a+b)^n$ is

$$\underbrace{(a+b) \times (a+b) \times \cdots \times (a+b)}_{\text{Multiplication of } n \text{ terms}} = \sum_{k=0}^{n} \binom{n}{k} a^k \times b^{n-k}$$

Binomial expansion

Let $a = 1$ and $b = 1$. We have

$$(a+b)^n = 2^n = \sum_{k=0}^{n} \binom{n}{k} 1^k \times 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k} = A$$

$$\Rightarrow A = 2^n$$

Q4 (b) $\binom{n}{1} - \binom{n}{2} + \binom{n}{3} + \cdots + (-1)^{n+1}\binom{n}{n}$

Denote the value of the above equation as $A = \binom{n}{1} - \binom{n}{2} + \binom{n}{3} + \cdots + (-1)^{n+1}\binom{n}{n}$

Consider the summation $\binom{n}{0} - A = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^{n}\binom{n}{n} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}$

Recall the expansion of $(a+b)^n$ is

$$(a+b)\times(a+b)\times\cdots\times(a+b) = \sum_{k=0}^{n}\binom{n}{k}a^k \times b^{n-k}$$

with $a = -1$ and $b = 1$. We have

$$(-1+1)^n = 0 = \sum_{k=0}^{n}\binom{n}{k}(-1)^k \times 1^{n-k} = \sum_{k=0}^{n}(-1)^k\binom{n}{k} = \binom{n}{0} - A$$

$$\Rightarrow 1 - A = 0 \;\; \Rightarrow A = 1$$

Q5

Counting

Each of the five players competing in a quiz is asked 10 quick-fire questions. The score of each player depends on the number of correct answers, and each player gets all scores with equal probability. What is the probability that

(a) all players get a different score?

(b) exactly 2 of them get the same score?

(c) exactly 3 of them get the same score?

Q5 (a) Probability of all Players Getting a Different Score

Step 1: Calculate the Total Possible Outcomes

- Each player has **11 possible scores** (0 through 10).
- Since there are 5 players, the total number of possible outcomes for the players' scores is:

$$\text{Total possible outcomes} = 11^5$$

Step 2: Calculate the Number of Favourable Outcomes

- All 5 players get different scores. This is a permutation problem, where the 5 players must each take a unique score out of the 11 available.
- The number of favourable outcomes is the number of ways to choose 5 distinct scores out of 11 and arrange them among the 5 players:

$$\text{Favorable outcomes} = P(11,5) = \frac{11!}{(11-5)!} = \frac{11!}{6!}$$

Step 3: Calculate the Probability

$$P(\text{event}) = \frac{P(11,5)}{11^5}$$

Q5 (b) Probability that Exactly 2 of the Players Get the Same Score

Step 1: Calculate the Total Possible Outcomes

Total possible outcomes=11^5

Step 2: Calculate the Number of Favourable Outcomes

Exactly 2 players get the same score, and the other 3 to have different scores from each other and the pair.

Step 2a: Choose the score that will be duplicated:

Ways to choose the score for the pair=11

Step 2b: Choose the 2 players out of 5 who will get this score:

Ways to choose 2 players from $\binom{5}{2} = 10$

Step 2c: Assign different scores to the remaining 3 players from the remaining 10 scores:

Ways to assign 3 different scores= $P(10,3) = \frac{10!}{7!} = 720$

Step 2d: The number of favourable outcomes is:

Favorable outcomes= $11 \times 10 \times 720 = 79200$

Step 3: Calculate the Probability

$$P(\text{exactly 2 same scores}) = \frac{79200}{11^5}$$

Q5 (c) Probability that Exactly 3 of the Players Get the Same Score

Step 1: Calculate the Total Possible Outcomes

Total possible outcomes=11^5

Step 2: Calculate the Number of Favorable Outcomes

Exactly 3 players get the same score, and the other 2 to have different scores from each other and the pair.

Step 2a: Choose the score that will be duplicated:

Ways to choose the score for the pair=11

Step 2b: Choose the 3 players out of 5 who will get this score:

Ways to choose 3 players from $\binom{5}{3} = 10$

Step 2c: Assign different scores to the remaining 2 players from the remaining 10 scores:

Ways to assign 2 different scores= $P(10,2) = \frac{10!}{8!} = 90$

Step 2d: The number of favorable outcomes is:

Favorable outcomes= $11 \times 10 \times 90 = 9900$

Step 3: Calculate the Probability

$$P(\text{exactly 3 same scores}) = \frac{9900}{11^5}$$

Q6

Counting

One rolls two six-sided fair dice once.

(a) Find the probability of two numbers being the same.

(b) Find the probability of their sum being 5 and 7, respectively.

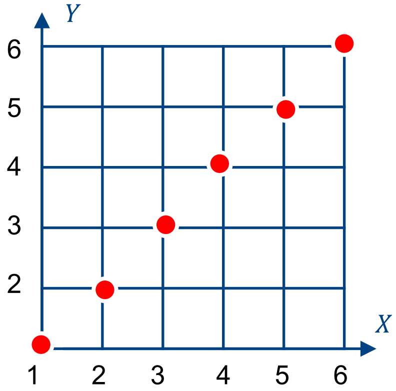
(c) Find the probability sum 5 happens before sum 7

Q6 (a) Probability of Rolling Two of the Same Number

Step 1: Determine the Total Number of Possible Outcomes

Since each die has 6 faces, the total number of possible outcomes when rolling two dice is:

Total outcomes$= 6 \times 6 = 36$



Step 2: Determine the Number of Favorable Outcomes

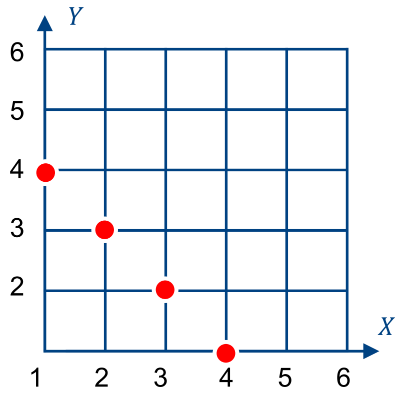
- For the two numbers to be the same, the outcomes are (1,1), (2,2), (3,3), (4,4), (5,5), and (6,6).
- There are 6 favorable outcomes.

Step 3: Calculate the Probability

The probability of rolling two of the same number is the ratio of the number of favorable outcomes to the total number of outcomes:

$$P(\text{two of same number}) = \frac{6}{36}$$

Q6 (b) Probability of Their Sum Being 5



Step 1: List All Possible Outcomes that Sum to 5

- The pairs of numbers that sum to 5 are

  (1,4), (2,3), (3,2), (4,1)

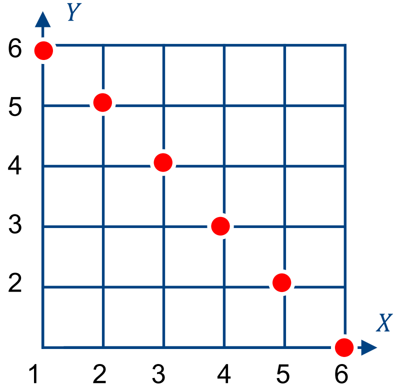
  There are 4 such outcomes.

Step 2: Calculate the Probability

The probability of the sum being 5 is the ratio of the number of favorable outcomes to the total number of outcomes:

$$P(\text{sum is } 5) = \frac{4}{36} = \frac{1}{9}$$

Q6 (b) Probability of Their Sum Being 7



Step 1: List All Possible Outcomes that Sum to 7

- The pairs of numbers that sum to 7 are (1,6), (2,5), (3,4), (4,3), (5,2), and (6,1).
- There are 6 such outcomes.

Step 2: Calculate the Probability

The probability of the sum being 7 is the ratio of the number of favorable outcomes to the total number of outcomes:

$$P(\text{sum is } 7) = \frac{6}{36} = \frac{1}{6}$$

Q6 (c)) Find the probability sum 5 happens before sum 7

Step 1: **List All Possible Outcomes for Sum 5 and Sum 7**

- Outcomes for sum 5: (1,4), (2,3), (3,2), (4,1) — 4 outcomes.
- Outcomes for sum 7: (1,6), (2,5), (3,4), (4,3), (5,2), (6,1) — 6 outcomes.

Step 2: Understanding the Process

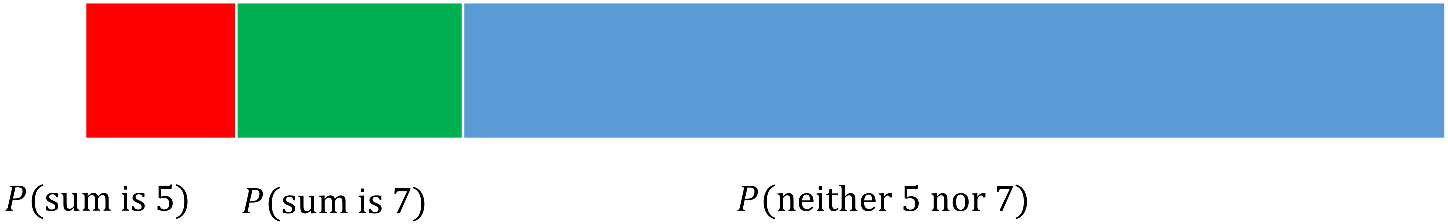
- The question is asking for the probability that a sum of 5 occurs before a sum of 7 in repeated rolls of the dice.
- We consider this as a sequential probability problem where we continue rolling until we get either a sum of 5 or 7.

Q6 (c) Find the probability sum 5 happens before sum 7

Step 3: Determine the Required Probability

- The probability of getting a sum of 5 on a single roll is: $P(\text{sum is } 5) = \frac{4}{36} = \frac{1}{9}$
- The probability of getting a sum of 7 on a single roll is: $P(\text{sum is } 7) = \frac{6}{36} = \frac{1}{6}$
- The probability that neither sum 5 nor sum 7 occurs and we have to roll again is:

$$P(\text{neither 5 nor 7}) = 1 - P(\text{sum is } 5) - P(\text{sum is } 7) = 1 - \frac{1}{9} - \frac{1}{6} = \frac{13}{18}$$



Q6 (c) Find the probability sum 5 happens before sum 7

Step 4: Compute the probability (Method I)

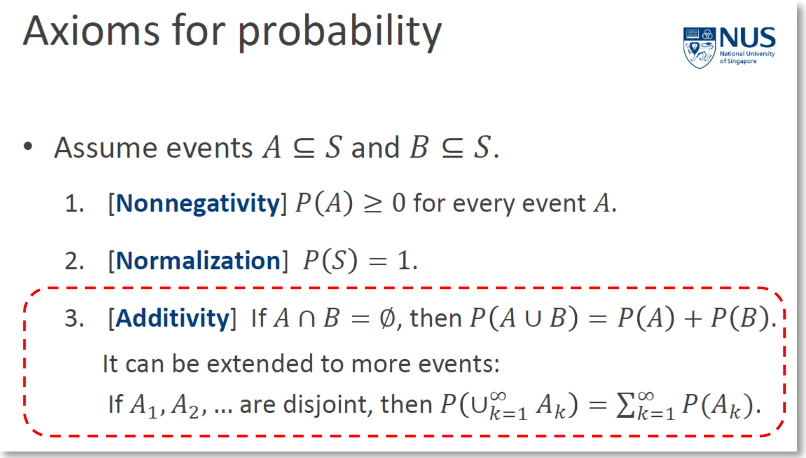
Let E_n = “no sum 5 and sum 7 in the first $n - 1$ rolls and sum 5 occurring at the n^{th} roll”.

$$P(E_n) = \left(P(\text{neither 5 nor 7})\right)^{n-1} \times \frac{1}{9} = \left(\frac{13}{18}\right)^{n-1} \times \frac{1}{9}$$

Let E = “sum 5 happens before sum 7”.

Since $\{E_n | n \in Z, n \geq 1\}$ are disjoint and $E = \cup_{n=1}^{\infty} E_n$,

$$P(E) = \sum_{n=1}^{\infty} P(E_n) = \frac{1}{9}\sum_{n=1}^{\infty}\left(\frac{13}{18}\right)^{n-1} = \frac{1}{9} \times \frac{1}{1 - 13/18} = \frac{2}{5}$$



Axioms for probability

- Assume events $A \subseteq S$ and $B \subseteq S$.
  1. [**Nonnegativity**] $P(A) \geq 0$ for every event A.
  2. [**Normalization**] $P(S) = 1$.
  3. [**Additivity**] If $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$.

     It can be extended to more events:

     If $A_1, A_2, \ldots$ are disjoint, then $P(\cup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} P(A_k)$.

The Infinite Geometric Series Formula is given as

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$$

As n approaches infinity, the sum then becomes

$$S_\infty = \frac{a}{1-r}; \boxed{|r| < 1}$$

Q6 (c) Find the probability sum 5 happens before sum 7

Step 4: Compute the probability (Method II)

Consider the Probabilities of the Event:

- Define the probability of sum 5 occurring before sum 7 as P_5,
- Define the probability of sum 7 occurring before sum 5 as P_7.
- Clearly, these two probabilities must sum to 1, i.e., $P_5 + P_7 = 1$

Each time a dice roll occurs, it can result in one of three possible scenarios:

- The sum is 5, which results in an immediate "win" for sum 5.
- The sum is 7, which results in an immediate "win" for sum 7.
- A sum other than 5 or 7 is rolled, and the process continues, and the situation is exactly the same as before.

So, the probability of P_5 can be written as:

$$P_5 = \frac{1}{9} + \frac{13}{18}P_5$$

Solve for P_5

$$P_5\left(1 - \frac{13}{18}\right) = \frac{1}{9} \quad \Longrightarrow \quad P_5 = \frac{1}{9} \times \frac{18}{5} = \frac{2}{5}$$

Q6 (c) Find the probability sum 5 happens before sum 7

Step 4: Compute the probability (Method III)

Reconsider the Process

- We continue rolling until we get either a sum of 5 or 7, and then stop
- The probability that sum 5 happens before sum 7 actually is the probability of sum 5 occurs this stopping point.

The probability that 5 occurs before 7 is:

$$P_5 = \frac{P(\text{sum is } 5)}{P(\text{sum is } 5) + P(\text{sum is } 7)} = \frac{\frac{1}{9}}{\frac{1}{9} + \frac{1}{6}} = \frac{2}{5}$$

Q7

You are given a biased coin that comes up heads with probability 1/4 on any flip.

(a) What is the probability that you observe two tails in two flips?

(b) What is the probability that you observe all heads in five flips?

(c) What is the probability that you observe at least one tails in five flips?

(d) What is the probability that you observe at least two tails in five flips?

(e) If the coin is flipped until the first head is observed, find the probability that 5 flips are needed.

Q7 (a) Probability of Observing Two Tails in Two Flips

Understand the Scenario

- We are given a biased coin where the probability of getting heads (H) on any flip is $P(\mathrm{H}) = \frac{1}{4}$.
- Therefore, the probability of getting tails (T) is $P(\mathrm{T}) = 1 - P(\mathrm{H}) = \frac{3}{4}$.

The probability of getting tails on both flips is the product of the probabilities of getting tails on each individual flip, since the flips are independent:

$$P(\text{Two Tails}) = P(\mathrm{T}) \times P(\mathrm{T}) = \frac{3}{4} \times \frac{3}{4} = \frac{9}{16}$$

Q7 (b) What is the probability that you observe all heads in five flips?

The probability of getting heads on all five flips is the product of the probabilities of getting heads on each individual flip:

$$P(\text{All heads in five flips}) = P(\text{H}) \times P(\text{H}) \times P(\text{H}) \times P(\text{H}) \times P(\text{H}) = \left(\frac{1}{4}\right)^5$$

Q7 (c) Probability of Observing at Least One Tails in Five Flips

Method I

The keyword in this question is "at least". It means the desired event is the union of outcomes with any nonzero tails in 5 flips. They are listed together with their probabilities below.

Probability of 1 tail: $\binom{5}{1} \times \frac{3}{4} \times \left(\frac{1}{4}\right)^4$

Probability of 2 tails: $\binom{5}{2} \times \left(\frac{3}{4}\right)^2 \times \left(\frac{1}{4}\right)^3$

Probability of 3 tails: $\binom{5}{3} \times \left(\frac{3}{4}\right)^3 \times \left(\frac{1}{4}\right)^2$

Probability of 4 tails: $\binom{5}{4} \times \left(\frac{3}{4}\right)^4 \times \left(\frac{1}{4}\right)$

Probability of 5 tails: $\binom{5}{5} \times \left(\frac{3}{4}\right)^5$

Since these outcomes are disjoint (mutually exclusive), the desired event probability is the sum of individual outcome probabilities. But this computation is tedious.

Q7 (c) Probability of Observing at Least One Tails in Five Flips

Method II

Step 1: Understand the Complement

- Instead of calculating the probability of getting at least one tail directly, it's easier to calculate the probability of the complementary event (i.e., no tails at all, which means all heads) and then subtract it from 1.
- The probability of getting no tails (i.e., all heads) in five flips is:

$$P(\text{All Heads}) = \left(\frac{1}{4}\right)^{5}$$

Step 2: Calculate the Probability of At Least One Tails

$$P(\text{At least one Tail}) = 1 - P(\text{All Heads}) = 1 - \frac{1}{1024} = \frac{1023}{1024}$$

Q7(d) Probability of Observing at Least Two Tails in Five Flips

Step 1: Understand the Complement

To find the probability of getting at least two tails, consider the complementary events:

- Getting 0 tails (all heads)
- Getting exactly 1 tail

Step 2: Calculate the Probability of Getting Exactly 0 Tail

$$P(\text{All heads}) = \left(\frac{1}{4}\right)^5$$

Step 3: Calculate the Probability of Getting Exactly 1 Tail

There are 5 ways to get exactly 1 tail (T) in 5 flips. Each of these involves getting T in one position and H in the remaining four.

$$P(\text{One Tails}) = \binom{5}{1} \times \frac{3}{4} \times \left(\frac{1}{4}\right)^4 = 5 \times \frac{3}{4} \times \frac{1}{256} = \frac{15}{1024}$$

Step 4: Calculate the Probability of At Least Two Tails

$$P(\text{At least two Tails}) = 1 - (P(\text{All Heads}) + P(\text{One Tail})) = 1 - (\frac{1}{1024} + \frac{15}{1024}) = \frac{1008}{1024}$$

Q7(e) Probability of That 5 Flips Are Needed to Observe the First Head

Step 1: Understand the Problem

- We need to find the probability that the first head is observed on the 5th flip.
- This means the first 4 flips are tails and the 5^{th} flip is heads.

Step 2: Calculate the Probability

The probability of getting tails on the first 4 flips and heads on the 5th flip is:

$$P(\text{four flips are tails and the fifth flips is head}) = \left(\frac{3}{4}\right)^4 \times \frac{1}{4} = \frac{81}{1024}$$

Q8

Suppose in an ECE graduation party there are 23 students. What is the probability that at least two students have the same birthday? That is if you randomly ask any two students. Assume that there are 365 days in a year and that all days are equally likely to be the birthday of a given student.

Q8

Problem Restatement

- There are 23 students.
- We want to calculate the probability that **at least** two students share the same birthday.
- Assume there are 365 days in a year, and each day is equally likely to be a birthday.

Step 1: Understand the Complementary Probability

$$P(\text{at least 2 students share the same birthday}) = 1 - P(\text{no students share the same birthday})$$

Step 2: Calculate the Probability that No Students Share the Same Birthday

Probability of first student's birthday: $\frac{365}{365}$

Probability of second student's birthday: $\frac{364}{365}$

Probability of third student's birthday: $\frac{363}{365}$

⋮

Probability of n^{th} student's birthday: $\frac{365-n+1}{365}$

$$P(\text{no students share the same birthday}) = \frac{365!}{(365-n)!365^n} \approx 0.4927.$$

Step 3: Calculate the Probability that At Least Two Students Share the Same Birthday

$$P(\text{At least one shared birthday}) = 1 - P(\text{No shared birthday})$$

THANK YOU