EE2012 Tutorial 2

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Instructor

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Location:

- E1A-04-01
- Level 6, #06-02, T-Lab Building

Summary of Chapter 2

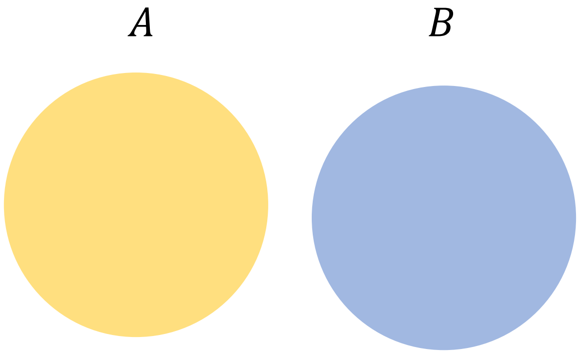
Conditional Probability

- Definition: $P(A|B) = \frac{P(A \cap B)}{P(B)}$
- Probability of A given B has occurred
- Properties:
  - Non-negativity: $P(A|B) \geq 0$
  - Normalization: $P(S|B) = 1$
  - Additivity for disjoint events

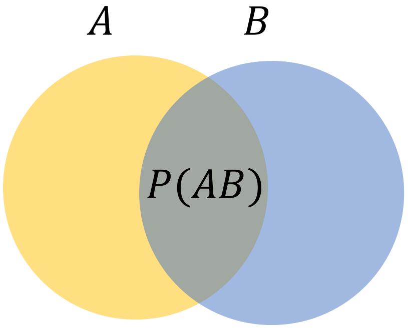
Summary of Chapter 2

Independence and Disjoint

- Independent: $P(A|B) = P(A) \leftrightarrow P(A \cap B) = P(A)P(B)$
- Disjoint: $A \cap B = \emptyset \rightarrow P(A \cap B) = 0$

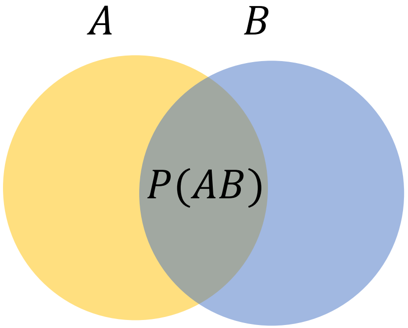


Disjoint



$P(AB) = P(A)P(B)$

Independent



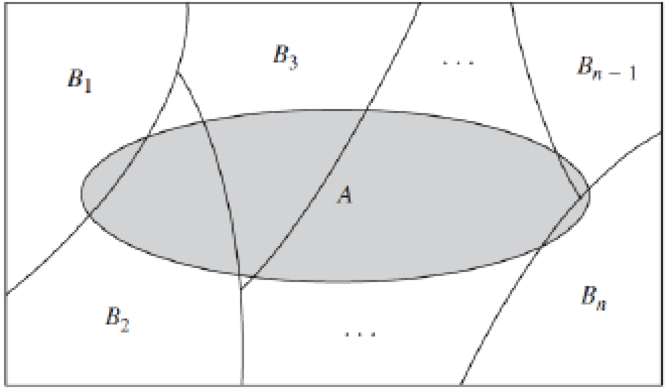
$P(AB) \neq P(A)P(B)$

Dependent

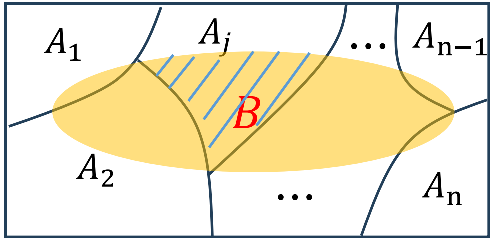
Summary of Chapter 2

Total Probability Theorem

- Partition $\{B_1, B_2, \dots, B_n\}$ of sample space S
- $P(A) = \sum_{i=1}^{n} P(A|B_i)\, P(B_i)$
- An event A can be represented as union of disjoint sets.



Summary of Chapter 2



Bayes' Rule

Formula:

$$P(A_j|B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(B|A_j)P(A_j)}{\sum_{j=1}^{n} P(B|A_j)P(A_j)}$$

- Numerator: $\text{likelihood} \times \text{prior}$
- Denominator: total probability of A
- Application: posterior probability from evidence

Sequential Experiments

Sequential Experiments

Random experiments conducted in steps

- Independent case: product of probabilities

$$P(A_1, A_2, \dots, A_n) = P(A_1)P(A_2)\cdots P(A_n) = \prod_{k=1}^{n} P(A_k)$$

- Dependent case:

$$P(A_1 A_2 A_3) = P(A_3|A_1 A_2)P(A_2|A_1)P(A_1)$$

Q1

A fair dice is rolled twice and all the possible outcomes are equally likely. Let X and Y be the results of the two rolls, respectively.

(a) Let $A = \{\max(X, Y) = m\}$ and $B = \{\min(X, Y) = 2\}$. Find $P(A|B)$ for $m = 1, 2, 5$.

(b) Find the probability $P(X < Y \mid Y \geq 2)$.

$A = \{\max(X, Y) = m\}$ and $B = \{\min(X, Y) = 2\}$

Q1 (a) Find $P(A|B)$ for $m = 1, 2, 5$.

By conditional probability, we have $P(A|B) = P(AB)/P(B)$. We need to find $P(AB)$ and $P(B)$.

Step 1: Find $P(B)$

- Denote an outcome by (X, Y). There are $6 \times 6 = 36$ outcomes, illustrated in Figure 1 below.
- For event B, the minimum can be X and/or Y. The outcomes in event
  $B = \{(6,2),(5,2),(4,2),(3,2),(2,2),(2,3),(2,4),(2,5),(2,6)\}$, $|B| = 9$
- The outcomes in event are marked in the Figure 1.
- Since all outcomes have the same probability $\frac{1}{36}$, we have $P(B) = \frac{9}{36} = \frac{1}{4}$.

- **Step 2: Find $P(A \cap B)$ for $m = 1, 2, 5$**
  - When $m = 1$, $A \cap B = \emptyset$. $P(A|B) = P(AB)/P(B) = 0$.
  - When $m = 2$, $A \cap B = \{(2,2)\}$. $P(A|B) = P(AB)/P(B) = \frac{\frac{1}{36}}{\frac{1}{4}} = \frac{1}{9}$.
  - When $m = 5$, $A \cap B = \{(5,2),(2,5)\}$. $P(A|B) = \frac{P(AB)}{P(B)} = \frac{\frac{2}{36}}{\frac{1}{4}} = \frac{2}{9}$.

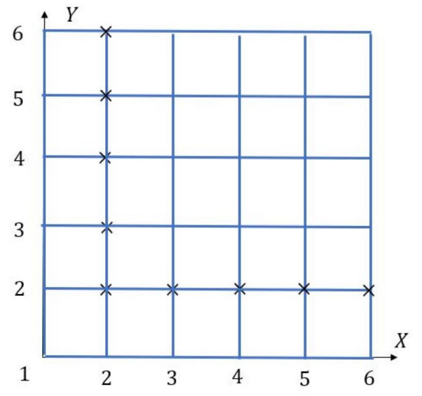
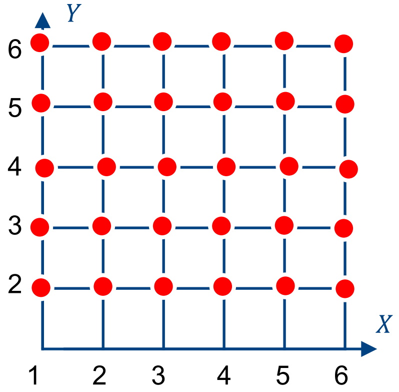


FIG. 1. Outcomes of two dices.

Q1 (b) Find the probability $P(X < Y | Y \geq 2)$



Step 1: Find $P(Y \geq 2)$

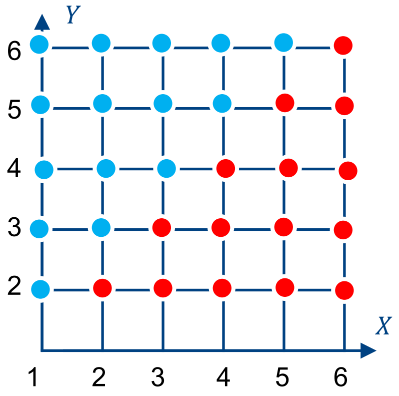
- For $Y \geq 2$, the outcomes for Y are $2, 3, 4, 5,$ or 6.
- This includes all possible outcomes except those where $Y = 1$.
- There are 30 outcomes where $Y \geq 2$
- Therefore, $P(Y \geq 2) = \frac{30}{36} = \frac{5}{6}$

Step2: Find $P(X < Y \cap Y \geq 2)$

In the left-top triangle, there are 15 outcomes in the event $\{X < Y\} \cap \{Y \geq 2\}$.

- $(1,6), (2,6), (3,6), (4,6), (5,6)$
- $(1,5), (2,5), (3,5), (4,5)$
- $(1,4), (2,4), (3,4)$
- $(1,3), (2,3)$
- $(1,2)$

Therefore, $P(X < Y \cap Y \geq 2) = \frac{15}{36} = \frac{5}{12}$



Step3: Compute $P(X < Y \mid Y \geq 2)$

$$P(X < Y \mid Y \geq 2) = \frac{P(X < Y \cap Y \geq 2)}{P(Y \geq 2)} = \frac{\frac{5}{12}}{\frac{5}{6}} = \frac{1}{2}$$

Q2

Bayes' Rule

Suppose you have two coins, one biased and one fair, but you don't know which coin is which.

Coin A is biased, i.e., it comes up heads with a probability 3/4, while Coin B is fair, i.e., it comes up heads with a probability 1/2. Suppose you pick a coin at random and flip it. Let C_X denote the event that coin X is picked and let H and T be the possible outcomes of the flip.

(a) Given that the outcome of the flip is a head, what is $P(C_A|H)$? That is, what is the conditional probability of picking coin A given the outcome is a head?

(b) What is $P(C_A|T)$?

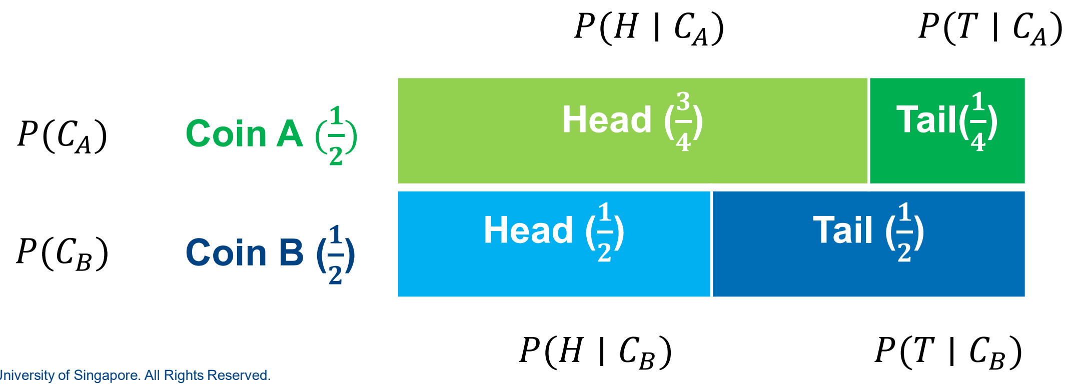
Q2

Understand the problem

You have two coins:

- Coin A (biased): The probability of getting heads (H) is $P(H \mid C_A) = \frac{3}{4}$.
- Coin B (fair): The probability of getting heads is $P(H \mid C_B) = \frac{1}{2}$.

Suppose you pick a coin at random and flip it

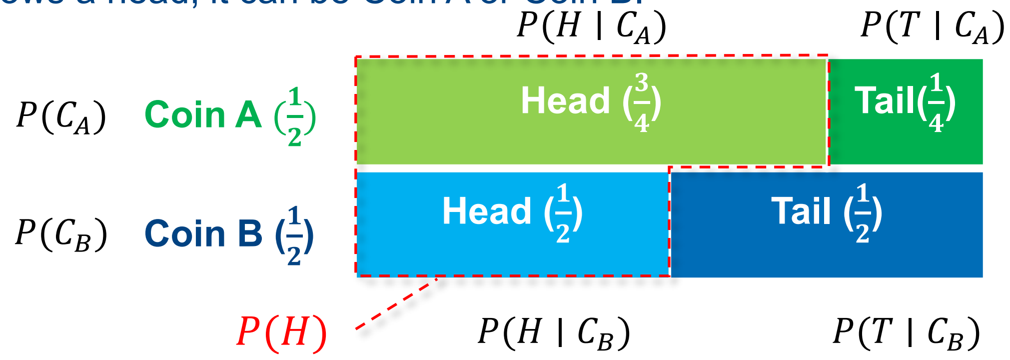


Q2 (a) Finding $P(C_A \mid H)$

The probability $P(C_A|H)$ is the chance that Coin A is picked after we know the result of the flipping is a head. Note that when the picked coin shows a head, it can be Coin A or Coin B.

To calculate $P(C_A|H)$ using Bayes' rule

$$P(C_A \mid H) = \frac{P(H \mid C_A) \cdot P(C_A)}{P(H)}$$



Step 1: Find $P(H)$

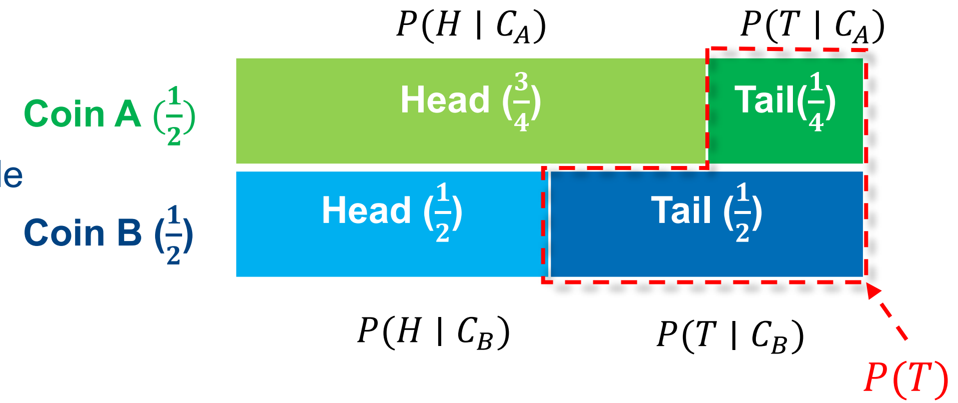
The total probability of getting heads ($P(H)$) is a weighted sum of the probabilities of getting heads from each coin. Since you pick either coin with equal probability, $P(C_A) = P(C_B) = \frac{1}{2}$.

$$P(H) = P(H \mid C_A) \cdot P(C_A) + P(H \mid C_B) \cdot P(C_B) = \frac{3}{4} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{5}{8}$$

Step 2: Apply Bayes' Theorem to find $P(C_A \mid H)$

$$P(C_A \mid H) = \frac{P(H \mid C_A) \cdot P(C_A)}{P(H)} = \frac{\frac{3}{4} \times \frac{1}{2}}{\frac{5}{8}} = \frac{3}{5}$$

Q2 (b) Finding $P(C_A \mid T)$



Similarly, to calculate $P(C_A|T)$ by using Bayes' rule

$$P(C_A \mid T) = \frac{P(T \mid C_A) \cdot P(C_A)}{P(T)}$$

Step 1: Find $P(T)$

The total probability of getting tails $P(T)$ is the complement of $P(H)$, so:

$$P(T) = 1 - P(H) = 1 - 5/8 = 3/8$$

Step 2: Apply Bayes' Theorem to find $P(C_A \mid T)$

Now we substitute the values into Bayes' Theorem:

$$P(C_A \mid T) = \frac{P(T \mid C_A) \cdot P(C_A)}{P(T)} = \frac{\frac{1}{4} \times \frac{1}{2}}{\frac{3}{8}} = \frac{1}{3}$$

Q3

Conditional Probability

A machine produces photo-detectors in pairs. Tests show that the first photo-detector is acceptable with probability 3/5. When the first photo-detector is acceptable, the second photodetector is acceptable with probability 4/5. If the first photodetector is defective, the second photodetector is acceptable with probability 2/5.

(a) What is the probability that exactly one photodetector of a pair is acceptable? That is, only one photo-detector is acceptable.

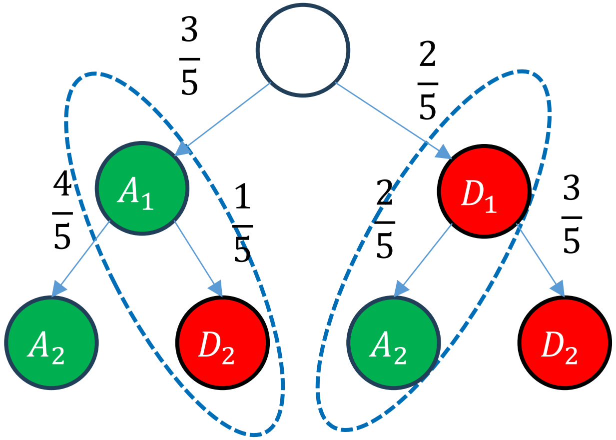
(b) What is the probability that both photo-detectors in a pair are defective?

Q3 (a) What is the probability that exactly one photodetector of a pair is acceptable? That is, only one photo-detector is acceptable.

The whole process can be described by a tree diagram of three layers

- From this tree diagram, we can see that the two desired outcomes can be represented by the two paths A_1D_2 and D_1A_2.
- Since these two outcomes are mutually exclusive, the probability of only one acceptable is

$$P((A_1D_2) \cup (D_1A_2)) = P(A_1D_2) + P(D_1A_2)$$



To compute the probability of path $P((A_1D_2)$ and $P(D_1A_2)$

$$P(A_1D_2) = P(A_1) \times P(D_2|A_1) = 3/5 \times 1/5 = 3/25$$

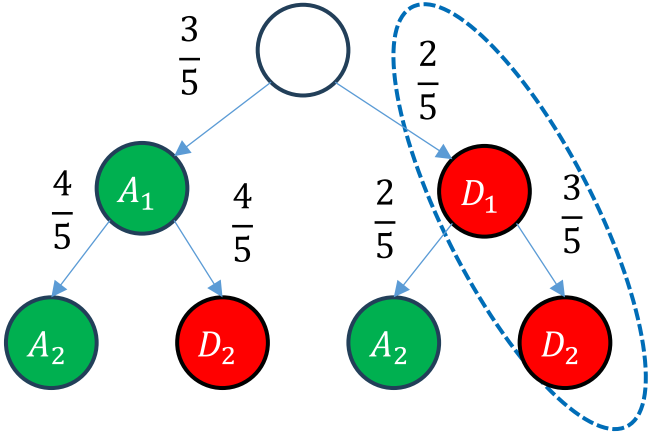
$$P(D_1A_2) = P(D_1) \times P(A_2|D_1) = 2/5 \times 2/5 = 4/25$$

Therefore

$$P((A_1D_2) \cup (D_1A_2)) = 3/25 + 4/25 = 7/25$$

Q3 (b) What is the probability that both photo-detectors in a pair are defective?

Applying the same method as Q3(a)



- We can observe from the tree diagram that it corresponds to the path D_1D_2.
- The probability is the product of the probabilities associated with the segments along this path

$$P(D_1D_2) = 2/5 \times 3/5 = 6/25$$

Q4

Bayes' Rule

People may be infected by a virus with a chance of 10^{-4}. To detect the infection, a testing kit is used. If one is infected, it is detected as positive by the testing kit with probability 99%. The testing kit responses positive with probability 10% when people, who do not infect, are tested by the kit. Those tested as positive will be treated in the hospital.

(a) What is the probability of one being hospitalized based on the testing result?

(b) Do you think this testing kit is useful? Why?

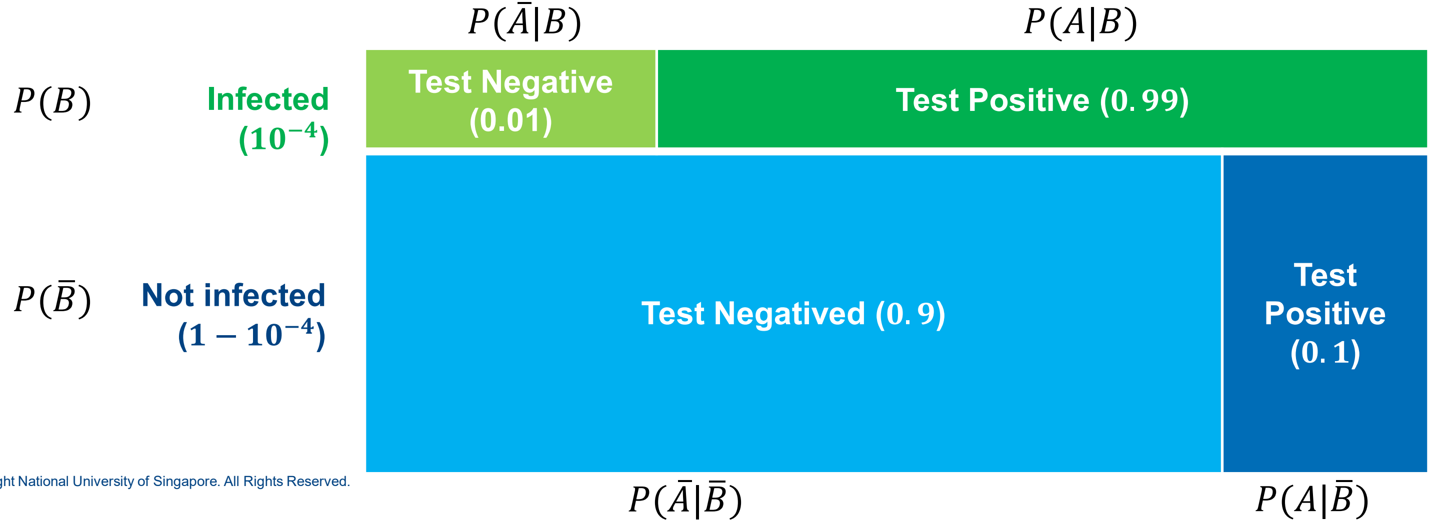
Q4(a) What is the probability of one being hospitalized based on the testing result?

Step 1: Define the events

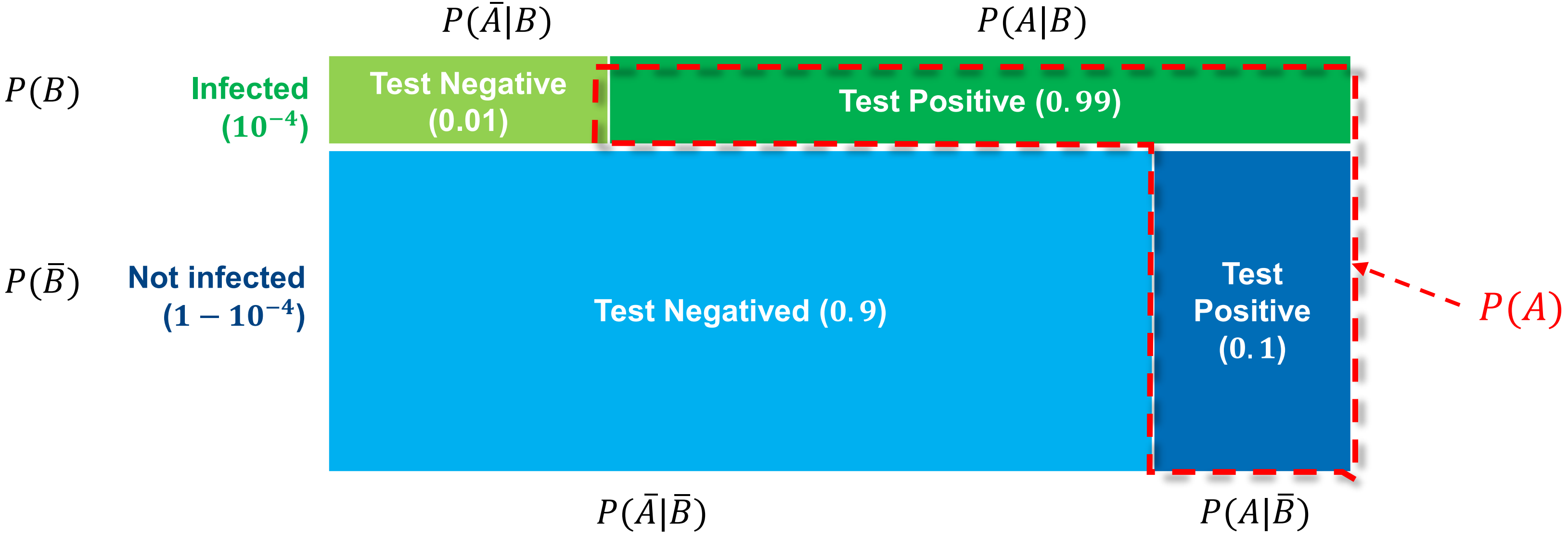
- Let A be "one is hospitalized", which is the same as "one's testing is positive".
- Let B be "one is infected".

Step 2: Define the probabilities

- $P(B) = 10^{-4}$: prior probability that a person is infected.
- $P(A \mid B) = 0.99$: probability that the test is positive given the person is infected (sensitivity).
- $P(A \mid \bar{B}) = 0.10$: probability that the test is positive given the person is not infected (false positive rate).
- $P(\bar{B}) = 1 - P(B) = 1 - 10^{-4} = 0.9999$: probability that a person is not infected.



Q4 (a) What is the probability of one being hospitalized based on the testing result?



Step 3: Compute $P(A)$: The total probability of testing positive

$P(A)$ is the sum of two mutually exclusive cases:

1. The person is infected and tests positive.
2. The person is not infected and still tests positive (false positive)

$$P(A) = \boldsymbol{P(AB)} + \boldsymbol{P(A\bar{B})} = P(A|B)P(B) + P(A|\bar{B})P(\bar{B})$$
$$= 0.99 \times 10^{-4} + 0.1 \times (1 - 10^{-4}) = \mathbf{0.000099} + \mathbf{0.099999} \approx 0.10009$$

Q4(a)

Probability that a person is infected given the person's test is positive

Step 4: Apply Bayes' Theorem to find $P(B \mid A)$

Now we substitute the values into Bayes' Theorem:

$$P(B \mid A) = \frac{P(A \mid B) \cdot P(B)}{P(A)} = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|\bar{B})P(\bar{B})} = \frac{0.99 \times 10^{-4}}{0.99 \times 10^{-4} + 0.1 \times (1 - 10^{-4})} \approx 0.00098912$$

If a person is tested positive the 2nd time

Optional

$P(B) = 0.00098912$ (The current Prior is previous Posterior probability: tested positive the 1st time)

$P(\bar{B}) = 1 - 0.00098912$

$P(A|B)$: Sensitivity = 0.99

$P(A|\bar{B})$: False positive rate = 0.1

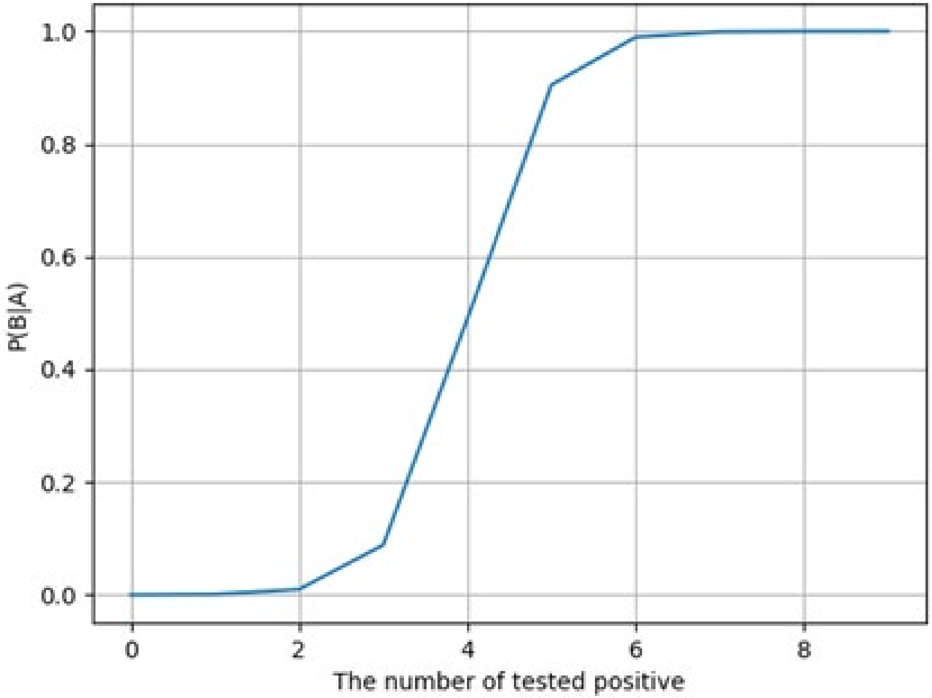
$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|\bar{B})P(\bar{B})} = \frac{0.99 \times 0.00098912}{0.99 \times 0.00098912 + 0.1 \times (1 - 0.00098912)} = 0.0097$$

If a person is tested positive the 5th time

$$P(B|A) = 0.9048$$

Q4 (a)

The plot illustrates the change of $P(B|A)$ with respect to the number of tested positive.



Demo code can be found in the repository.

Q4 (b) Do you think this testing kit is useful? Why?

- Let $\bar{A}$ be the complement event of A. Since $P(A|B) = 0.99$ and $P(\bar{A}|B) = 0.01$, this testing kit is good at identifying the infected cases.
- But if we check the intermediate steps above when calculating $P(A)$, we can see that the second term $P(A|\bar{B})P(\bar{B})$ contributes much more than the first term $P(A|B)P(B)$. More precisely, the second term actually contributes more than 99.9% to $P(A)$. However, the second term is the probability for "false positive" cases that people are not infected but tested positive.
- That is more than 99.9% of the people hospitalized actually do not need to be hospitalized. In this sense, this testing kit is not very successful.

Q5

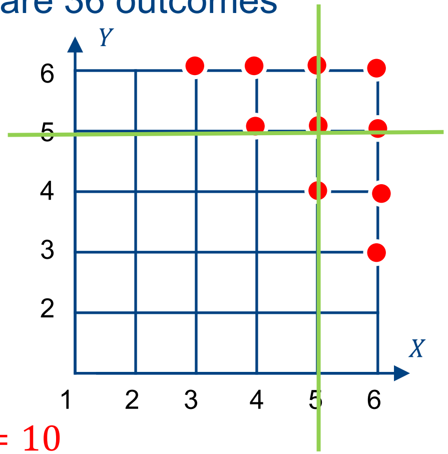
A fair dice is rolled twice and the sum of the numbers is observed.

(a) What is the probability that the sum is at least 9 if it is known that a 5 was rolled?

(b) What is the probability that a 5 was rolled given that the sum is at least 9?

Q5 (a) What is the probability that the sum is at least 9 if it is known that a 5 was rolled?

It is rolled twice and the outcomes can be denoted by a 2-tuple (X_1, X_2). There are 36 outcomes and each occurs with the same probability 1/36 because of fair dice.



Step 1: Define the events

- Let A = "the sum is at least 9"
- Let B = "a 5 is rolled".
- We want to compute $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{|A\cap B|}{|B|}$

Step 2: Counting

- $A = \{(3,6),(4,5),(5,4),(6,3),(4,6),(5,5),(6,4),(5,6),(6,5),(6,6)\}, |A| = 10$
- $B = \{(1,5),(2,5),(3,5),(4,5),(5,5),(6,5),(5,1),(5,2),(5,3),(5,4),(5,6)\}, |B| = 11.$
- $A \cap B = \{(4,5),(5,5),(6,5),(5,4),(5,6)\}, |A \cap B| = 5.$

Step 3: Compute Conditional Probability $P(A|B)$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{|A \cap B|}{|B|} = \frac{5}{11}$$

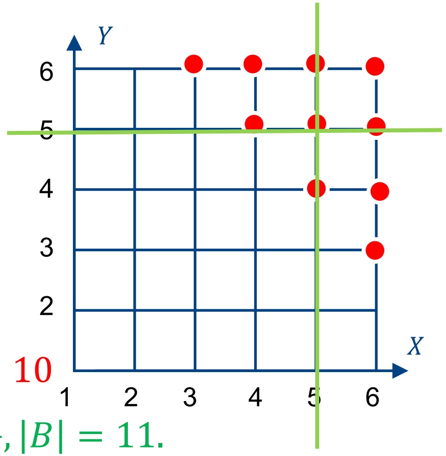
because all the elementary outcomes have the same probability.

Q5 (b) What is the probability that a 5 was rolled given that the sum is at least 9?

This question asks for $P(B|A)$. It seems like a Bayes probability problem. But it turns out we don't need to find out the likelihood $P(A|B)$ and a prior probability $P(B)$.

Step 1: Define Events

- Let A = "the sum is at least 9"
- Let B = "a 5 is rolled".
- We want to compute $P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{[A \cap B]}{|A|}$



Step 2: Counting

- $A = \{(3,6),(4,5),(5,4),(6,3),(4,6),(5,5),(6,4),(5,6),(6,5),(6,6)\}, |A| = 10$
- $B = \{(1,5),(2,5),(3,5),(4,5),(5,5),(6,5),(5,1),(5,2),(5,3),(5,4),(5,6)\}, |B| = 11.$
- $A \cap B = \{(4,5),(5,5),(6,5),(5,4),(5,6)\}, |A \cap B| = 5.$

Step 3: Compute the Conditional Probability $P(B|A)$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{|A \cap B|}{|A|} = \frac{5}{10} = 0.5$$

Q6

Sequential Experiment & Bayes' Rule

Symbols 0 and 1 are transmitted in a noisy channel with probability p and $1 - p$, respectively. Due to the channel noise, they are received correctly with probability u and v, respectively. Assume the transmissions for any two symbols are independent.

(a) What is the probability that the k^{th} symbol is received correctly?

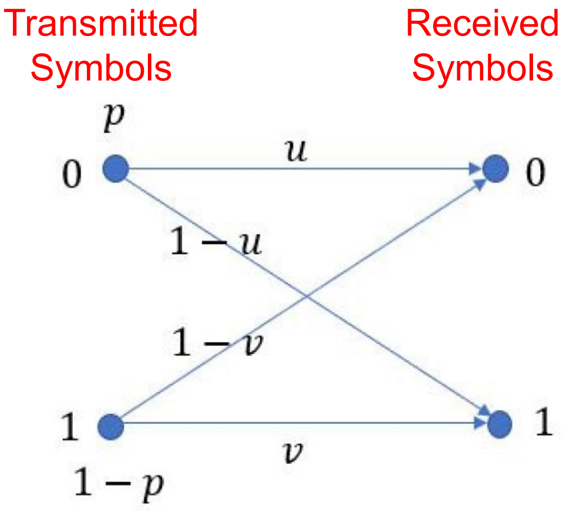
(b) What is the probability that the sequence 1101 is received correctly?

(c) To reduce the error probability, each symbol is transmitted three times and the receiver applies a majority vote to decode it. What is the probability that 0 is correctly decoded? If 101 is received, what is the probability this symbol is 0?

Q6 (a) What is the probability that the k^{th} symbol is received correctly?

Understand the information:

- **Symbol 0**
  - transmitted with probability p,
  - received correctly with probability u
  - received incorrectly with probability $1 - u$.
- **Symbol 1**
  - transmitted with probability $1 - p$,
  - received correctly with probability v and
  - received incorrectly with probability $1 - v$.



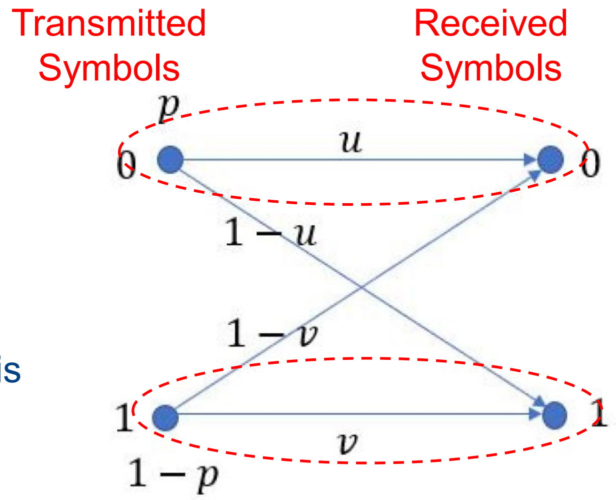
Noisy channel model.

Q6 (a) What is the probability that the k^{th} symbol is received correctly?

We define

S^k be the k^{th} transmitted symbol.

C^k be the event the k^{th} symbol is received correctly



Noisy channel model.

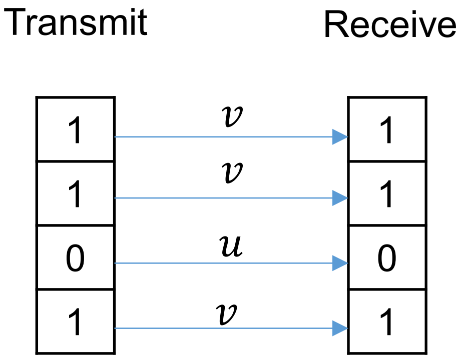
Thus, the total probability that the k^{th} symbol is received correctly is

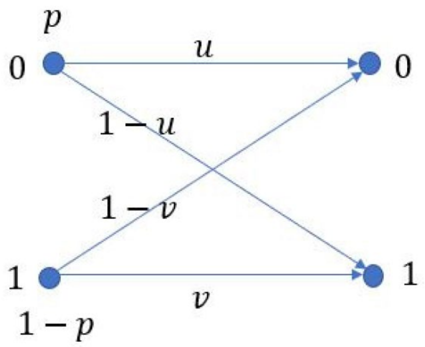
$$P(C_k) = \sum_{S_k \in \{0,1\}} P(C_k, S_k) = \sum_{s_k \in \{0,1\}} P(C_k|S_k)P(S_k)$$

$$= P(C_k|S_k = 0)P(S_k = 0) + P(C_k|S_k = 1)P(S_k = 1) = up + v(1 - p).$$

Q6 (b) What is the probability that the sequence 1101 is received correctly?

There are one zero and three 1 in the sequence 1101.





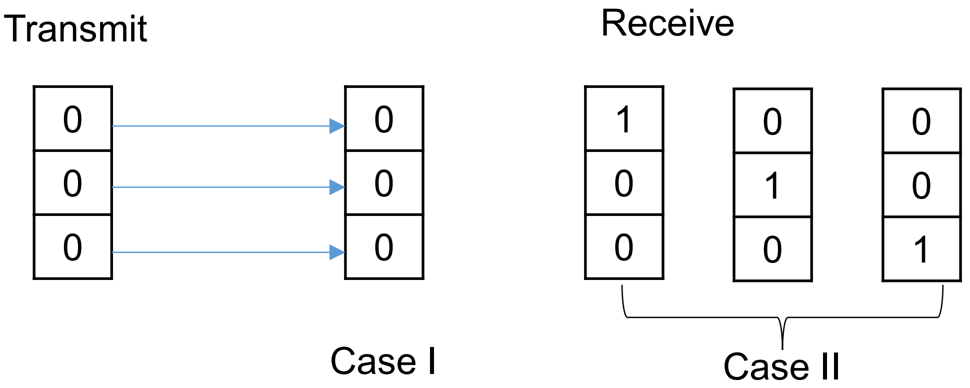
Noisy channel model.

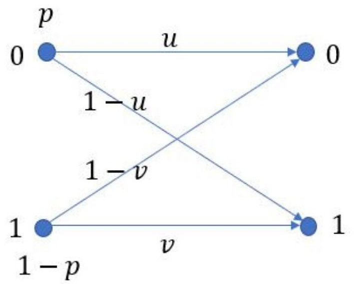
The probability they are received correctly is

$$v \times v \times u \times v = uv^3$$

Q6(c) To reduce the error probability, each symbol is transmitted three times and the receiver applies a majority vote to decode it. What is the probability that 0 is correctly decoded?

Symbol 0 is transmitted as 000 by the coding method. To decode the received sequence to symbol 0 with a majority vote, the received sequence should have at least two 0's out of 3 received symbols.





Noisy channel model.

Case I: $P(\text{three 0's received}|\text{000 transmitted}) = u^3$

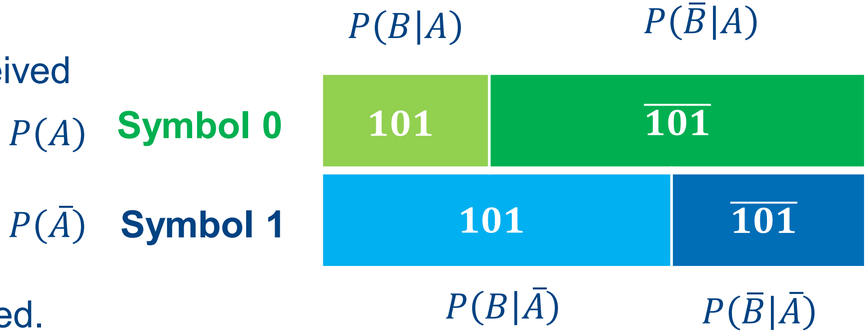
Case II: $P(\text{two 0's received}|\text{000 transmitted}) = \binom{3}{2}u^2(1-u) = 3u^2(1-u)$

So the probability 0 is correctly decoded:

$$P(\text{at least two 0's received}|\text{000 transmitted}) = P(\text{two 0's received}|\text{000 transmitted}) + P(\text{three 0's received}|\text{000 transmitted}) = 3u^2(1-u) + u^3$$

Q6 (c) To reduce the error probability, each symbol is transmitted three times and the receiver applies a majority vote to decode it. If 101 is received, what is the probability this symbol is 0?

- Even A = "symbol 0 is transmitted"
- Even B = "101 received".



The probability that symbol 0 is transmitted when 101 is received

Bayes' theorem. $P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}$

Step 1: Define probability

- $P(A) = p$, the prior probability that 0 was transmitted.
- $P(\bar{A}) = 1 - p$, the prior probability that 1 was transmitted.
- $P(B|A) = u(1-u)^2$, the probability that the sequence 101 was received given 0 was transmitted
- $P(B|\bar{A}) = v^2(1-v)$, the probability that the sequence 101 was received given 1 was transmitted.

Step 2: Compute $P(B)$

$$P(B) = P(B,A) + P(B,\bar{A}) = P(B|A)P(A) + P(B|\bar{A})P(\bar{A})$$
$$= u(1-u)^2 p + v^2(1-v)(1-p)$$

Step 3: Compute $P(A|B)$

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(B|A)P(A)}{P(B)} = \frac{u(1-u)^2 p}{u(1-u)^2 p + v^2(1-v)(1-p)}$$

Q7

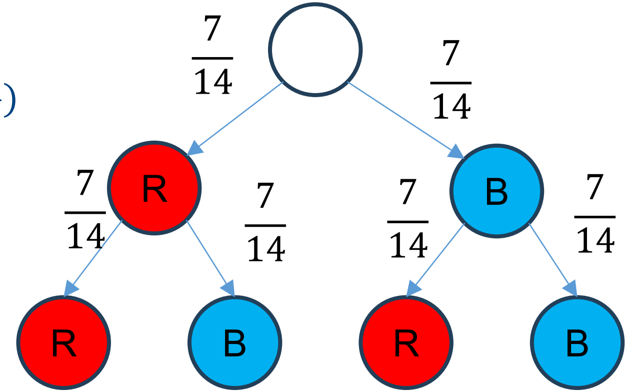
An urn contains 7 red balls and 7 blue balls. The experiment asks that you draw from the urn two times and that you need to get a red ball and a blue ball. You can choose to either draw with replacement or without replacement. Which strategy would give you the higher chance of success?

Q7 Method I

We need to compare the probabilities of success for these two strategies. The desired event of getting a red ball and a blue ball includes the outcomes $\{r_1 b_2, b_1 r_2\}$, where r_i and b_j represent the red and blue ball, and subscript i and j indicates the pick order.

Case I: If you draw with replacement,

The probability that you get a red ball and a blue ball is



$$P(\text{“one red and one blue”}) = P(\{r_1 b_2, b_1 r_2\}) = P(\{r_1 b_2\}) + P(\{b_1 r_2\})$$
$$= P(\{r_1\}) \times P(\{b_2\}) + P(\{b_1\}) \times P(\{r_2\}).$$

where

$$P(\{r_1\}) = P(\{r_2\}) = P(\{b_1\}) = P(\{b_2\}) = 7/14 = 1/2$$

Then

$$P(\text{“one red and one blue”}) = 1/2 \times 1/2 + 1/2 \times 1/2 = 1/2$$

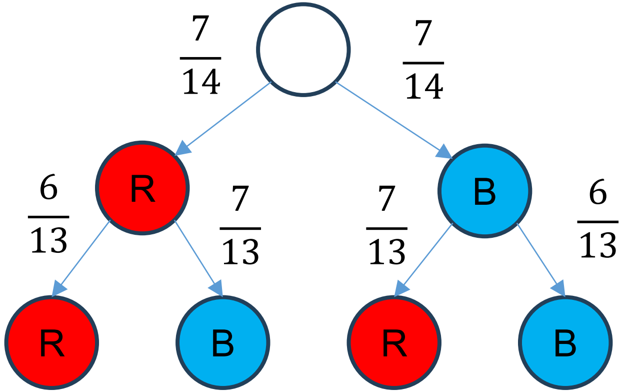
Q7 Method I

Case II: If you draw without replacement, the desired event is also $\{r_1b_2, b_1r_2\}$. Its probability is

$$P(\text{“one red and one blue”}) = P(\{r_1b_2, b_1r_2\}) = P(\{r_1b_2\}) + P(\{b_1r_2\}).$$

But due to no replacement, the picking results are not independent anymore,

and we need to calculate it based on the conditional probability.



$$P(\{r_1b_2\}) = P(\{b_2\}|\{r_1\}) \times P(\{r_1\}) = \frac{7}{13} \times \frac{7}{14}$$

and

$$P(\{b_1r_2\}) = P(\{r_2\}|\{b_1\}) \times P(\{b_1\}) = \frac{7}{13} \times \frac{7}{14}.$$

So

$$P(\text{“one red and one blue”}) = \frac{7}{13} \times \frac{7}{14} + \frac{7}{13} \times \frac{7}{14} = \frac{49}{91} \approx 0.5385$$

Based on the probabilities of success calculation, you should draw without replacement.

Q7 Method II

There is another way to find the probability of success without replacement. Any two out of these 14 balls can be picked with the same probability, which form a uniform sample space. The probability of the desired event can be calculated by the ratio of the desired event size and the sample space size.

Case I: If you draw with replacement

Step 1: Total outcomes:

- Since there are 14 balls (7 red, 7 blue) and you are drawing twice with replacement, each draw has 14 possible outcomes.
- So, the total number of possible outcomes is $14 \times 14 = 196$.

Step 2: Favorable outcomes (red and blue):

- The number of ways to get red first and blue second: $7 \times 7 = 49$ (7 choices for red, 7 choices for blue).
- The number of ways to get blue first and red second: $7 \times 7 = 49$.

Step 3: Probability of success:

- Probability of drawing one red and one blue ball

$$P = \frac{49 + 49}{196} = \frac{1}{2}$$

Q7 Method II

Case II: If you draw without replacement

Step 1: Total outcomes

- The total number of possible outcomes is $14 \times 13 = 182$ (because the second draw has only 13 balls to choose from).

Step 2: Favorable outcomes (red and blue):

- For the red first, then blue: There are 7 choices for the red ball (out of 14), and once a red is chosen, 7 choices remain for the blue (out of 13 remaining balls).

$$\text{favorable outcomes} = 7 \times 7 = 49.$$

- For the blue first, then red: There are 7 choices for the blue ball (out of 14), and once a blue is chosen, 7 choices remain for the red (out of 13 remaining balls). So, favorable outcomes

$$\text{favorable outcomes} = 7 \times 7 = 49.$$

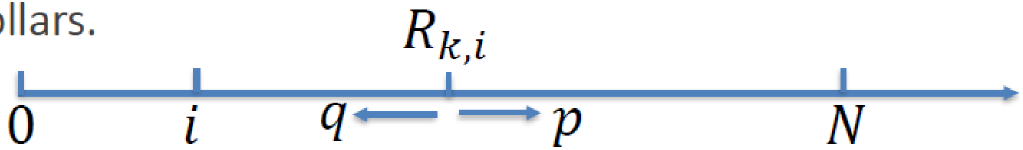
Step 3: Probability of success:

$$P = \frac{49 + 49}{182} = \frac{98}{182} \approx 0.5358$$

Example – Markov Chain

Example – Markov Chain

[Gambler's ruin] A player starts with i dollar. In each game, he will win 1 dollar with probability p and lose 1 dollar with probability q. He will stop playing when either he loses all his money or have total N dollars ($N > i$). Find the probability he stops playing with N dollars.



- In each game, the player's money varies by 1 . The money the player has is $i + \Delta_1 + \Delta_2 + \cdots + \Delta_k$ after the k^{th} game $\Delta_j \in \{1, -1\}$ for k $\geq j \geq 1$.
- If player starts with amount i, denote his money after k^{th} game as $R_{k,i}$ which is the outcome of the random sequential experiment. Current step outcome depends on only the previous step outcome. – Markov chain
- The player stops playing only when $R_{k,i} = 0$ (lose) or N (win).

Example – Markov Chain

Let $P_i = P(R_i)$ be the probability that the player stops playing with winning when he has initial money i. It is obvious $P_0 = 0$ and $P_N = 1$.

Then, using the Law of Total Probability, we have

$$P(R_i) = P(R_i|W)P(W) + P(R_i|\bar{W})P(\bar{W})$$

where W denotes the event that the player wins the first bet. Then clearly $P(W) = p$ and $P(\bar{W}) = 1 - p = q$. We also have

- $P(R_i|W) = P(R_{i+1})$ is the probability that a player experiences gambler's ruin having started with $i + 1$ amount of money;
- $P(R_i|\bar{W}) = P(R_{i-1})$ is the probability that a player experiences gambler's ruin having started with $i - 1$ amount of money.

Denoting $P(R_{i+1}) = P_{i+1}$ and $P(R_{i-1}) = P_{i-1}$, we get

$$P_i = pP_{i+1} + qP_{i-1}$$

Example – Markov Chain

- Since $p + q = 1$, above equation can be $q(P_i - P_{i-1}) = p(P_{i+1} - P_i)$.
  - If $r = \frac{q}{p} \neq 1$, $P_{i+1} - P_i = r(P_i - P_{i-1}) = r^i(P_1 - P_0) = r^i P_1$ and

$$
\left.
\begin{aligned}
P_{i+1} - \cancel{P_i} &= r^i P_1 \\
+ \quad \cancel{P_i} - \cancel{P_{i-1}} &= r^{i-1} P_1 \\
&\vdots \\
\cancel{P_2} - P_1 &= r P_1
\end{aligned}
\right\}
\quad
\begin{aligned}
P_{i+1} - P_1 &= P_1 \frac{r(1 - r^i)}{1 - r} = P_1 \frac{r - r^{i+1}}{1 - r} \\
P_{i+1} &= P_1 \frac{1 - r^{i+1}}{1 - r}
\end{aligned}
$$

The Geometric Series Formula is given as

$$S = a + ar + ar^2 + ar^3 + \cdots + ar^{i-1}$$

The sum then becomes

$$S = \frac{a(1 - r^i)}{1 - r};\ |r| < 1$$

$$P_N = P_1 \frac{1 - r^N}{1 - r} = 1 \rightarrow \quad P_1 = \frac{1 - r}{1 - r^N} \rightarrow \quad P_i = P_1 \frac{1 - r^i}{1 - r} = \frac{1 - r}{1 - r^N} \times \frac{1 - r^i}{1 - r} = \frac{1 - r^i}{1 - r^N}$$

Example – Markov Chain

- If $r = \frac{q}{p} = 1$, $P_{i+1} - P_1 = iP_1$ and $P_{i+1} = (i+1)P_1$.

  $P_N = NP_1 = 1 \rightarrow \; P_1 = \frac{1}{N} \rightarrow \; P_i = \frac{i}{N}$.

- For a greedy player want to stop at a larger N, the probability for him to win becomes smaller whatever r is.

- The winning probability also depends on the initial money i the player has. In a two-player setting, the chance is in favor of the player with more money. Who is the player in casino with the most money?

Read more about Gambler's ruin problem: https://towardsdatascience.com/the-gamblers-ruin-problem-9c97a7747171/

THANK YOU