EE2012 Tutorial 6

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Instructor

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Location:

- E1A-04-01
- Level 6, #06-02, T-Lab Building

Summary of Chapter 6

➢ Sum of Independent Continuous RVs

- For independent continuous RVs X and Y:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\, dx \rightarrow \ \textbf{continuous convolution}$$

- Example: If $X \sim N(\mu_X, \sigma_X^2), Y \sim N(\mu_Y, \sigma_Y^2)$, then

$$Z = X + Y \sim N(\mu_X + \mu_Y,\ \sigma_X^2 + \sigma_Y^2)$$

- Generalization: For n i.i.d. Gaussians, the sum is also Gaussian, with mean scaled by n and variance scaled by n.

Summary of Chapter 6

- **Central Limit Theorem (CLT)**
  - For i.i.d. RVs X_i with mean μ, variance σ^2:

$$Z_n = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \quad \overset{d}{\rightarrow} \quad N(0, 1) \quad \text{as } n \rightarrow \infty$$

  - CLT holds regardless of the original distribution (discrete or continuous), as long as mean & variance are finite.
  - Applications: Approximation of sums (e.g., Binomial → normal for large n).
  - Continuity correction improves accuracy when approximating discrete RVs (e.g., Binomial, Poisson).

Summary of Chapter 6

➢ Joint Distribution of Continuous RVs

- **Joint PDF** $f_{X,Y}(x,y)$:
  - Must be nonnegative and integrate to 1.
  - Probability over a region A:

$$P\big((X,Y) \in A\big) = \iint_A f_{X,Y}(x,y)dxdy$$

- **Marginal PDFs**:

$$f_X(x) = \int f_{X,Y}(x,y)dy\,, \qquad f_Y(y) = \int f_{X,Y}(x,y)dy$$

Summary of Chapter 6

➢ Expectation, Covariance & Independence

- Expectation:

$$E[X+Y] = E[X] + E[Y] \quad \text{(always true)}$$

- Covariance:

$$\mathrm{Cov}(X,Y) = E[(X-E[X])(Y-E[Y])] = E[XY] - E[X]E[Y]$$

- Correlation coefficient:

$$\rho_{X,Y} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y}, \qquad -1 \le \rho \le 1$$

- Independence: $f_{X,Y}(x,y) = f_X(x) f_Y(y)$
- Independent ⇒ uncorrelated, but uncorrelated does **not** always ⇒ independent (except Gaussian case).

Summary of Chapter 6

➢ Bivariate Gaussian Distribution

- Joint PDF:

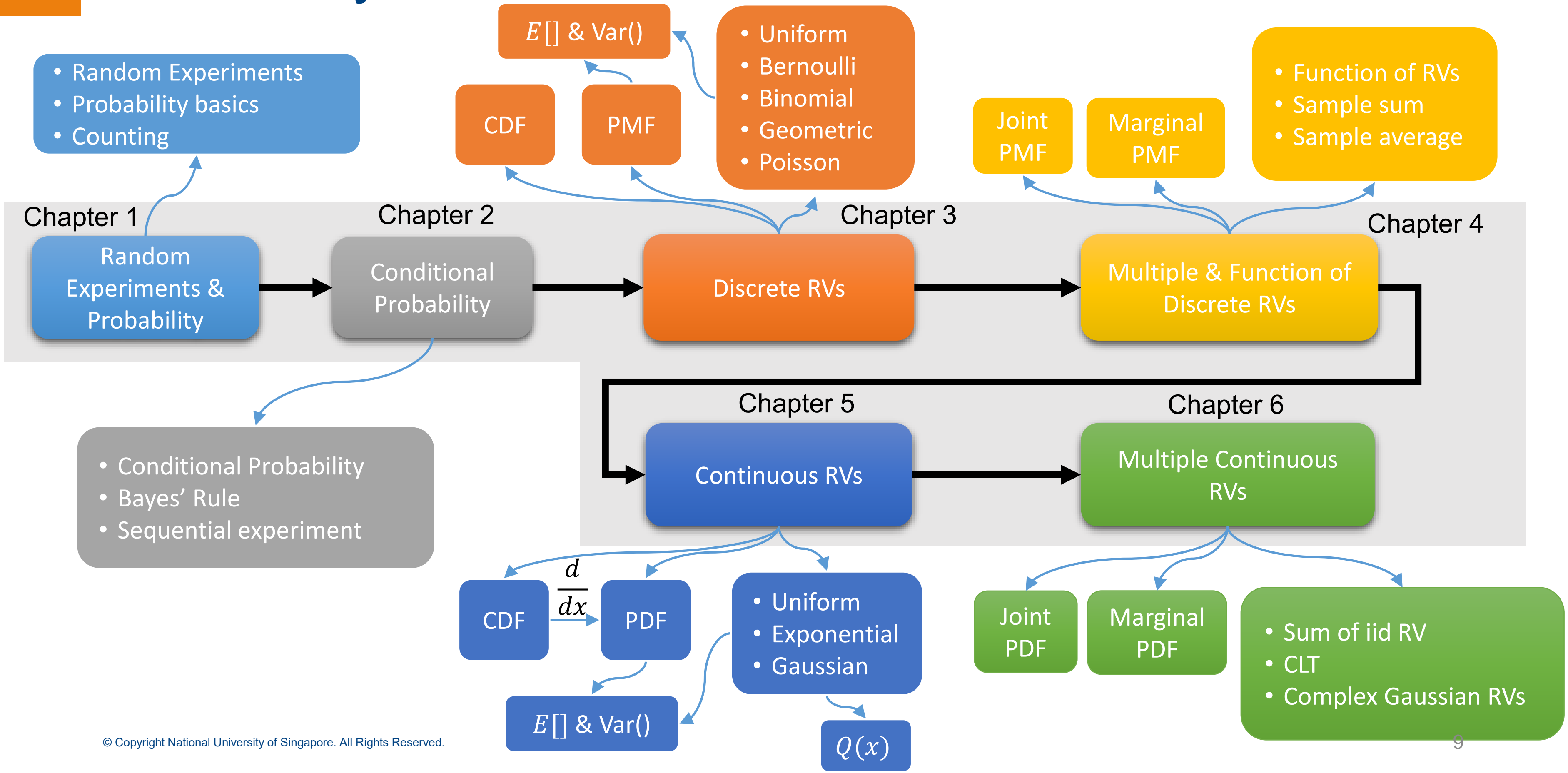
$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{(x-\mu_X)^2}{\sigma_X^2} - \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right)\right]$$

- Marginals are Gaussian.
- If correlation $\rho = 0$, then X and Y are independent.

Summary of Chapter 6

- **Complex Gaussian RVs**
  - Defined as $H = X + iY$ with X, Y Gaussian.
  - Expectation: $E[H] = \mu_X + i\mu_Y$.
  - Variance: $\sigma^2 = \sigma_X^2 + \sigma_Y^2$
  - **Circularly symmetric complex Gaussian RV:**
    - $X \sim N\left(0, \frac{\sigma^2}{2}\right), \ \ Y \sim N\left(0, \frac{\sigma^2}{2}\right)$ independent.
    - Distribution invariant under rotation in the complex plane.
    - Important in communications: models **AWGN** and wireless fading.
  - Magnitude $R = \sqrt{X^2 + Y^2} \ \rightarrow$ **Rayleigh distribution**.
  - Phase $\Theta = \tan^{-1}(Y/X) \rightarrow$ **Uniform distribution** on $[-\pi, \pi)$.

Summary of Chapter 1 to 6



Chapter 1

Random Experiments & Probability

- Random Experiments
- Probability basics
- Counting

Chapter 2

Conditional Probability

- Conditional Probability
- Bayes' Rule
- Sequential experiment

Chapter 3

Discrete RVs

CDF

PMF

$E[]$ & Var()

- Uniform
- Bernoulli
- Binomial
- Geometric
- Poisson

Chapter 4

Multiple & Function of Discrete RVs

Joint PMF

Marginal PMF

- Function of RVs
- Sample sum
- Sample average

Chapter 5

Continuous RVs

CDF $\xrightarrow{\frac{d}{dx}}$ PDF

$E[]$ & Var()

- Uniform
- Exponential
- Gaussian

$Q(x)$

Chapter 6

Multiple Continuous RVs

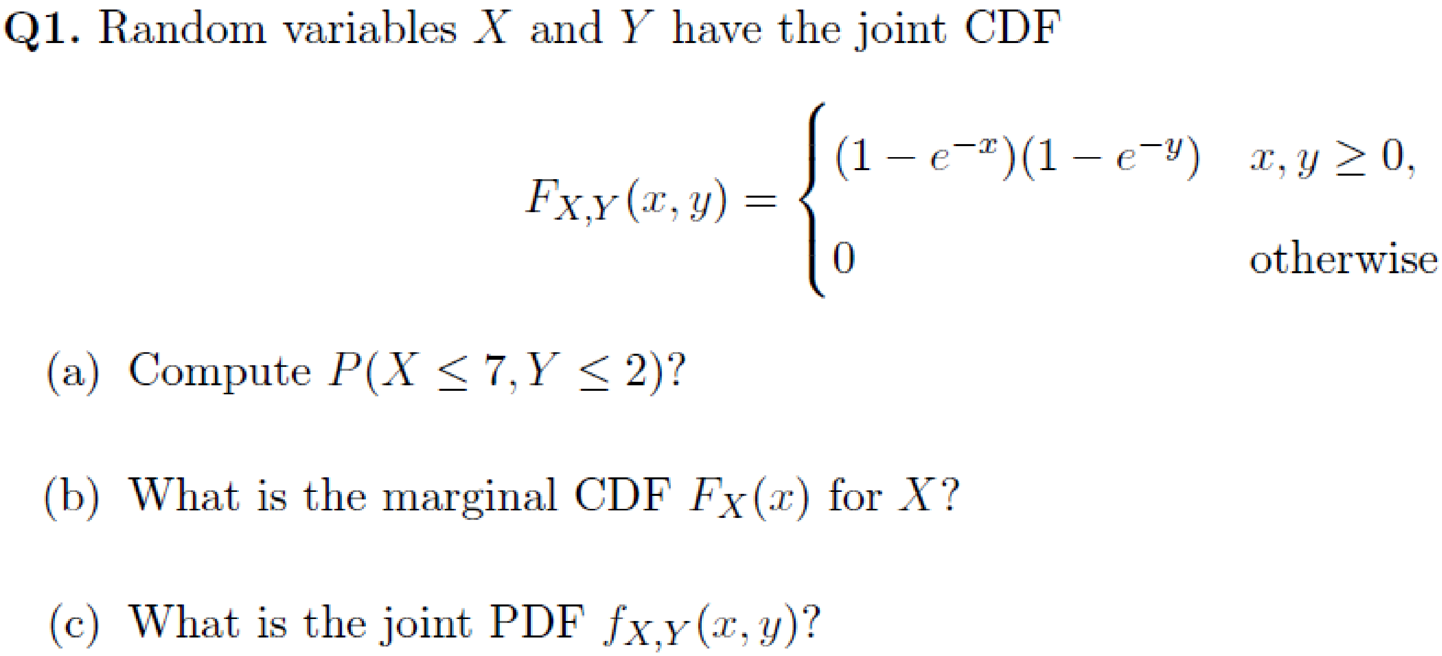
Joint PDF

Marginal PDF

- Sum of iid RV
- CLT
- Complex Gaussian RVs

Q1

(PDF, CDF)



Q1. Random variables X and Y have the joint CDF

$$F_{X,Y}(x,y) = \begin{cases} (1-e^{-x})(1-e^{-y}) & x, y \geq 0, \\ 0 & \text{otherwise} \end{cases}$$

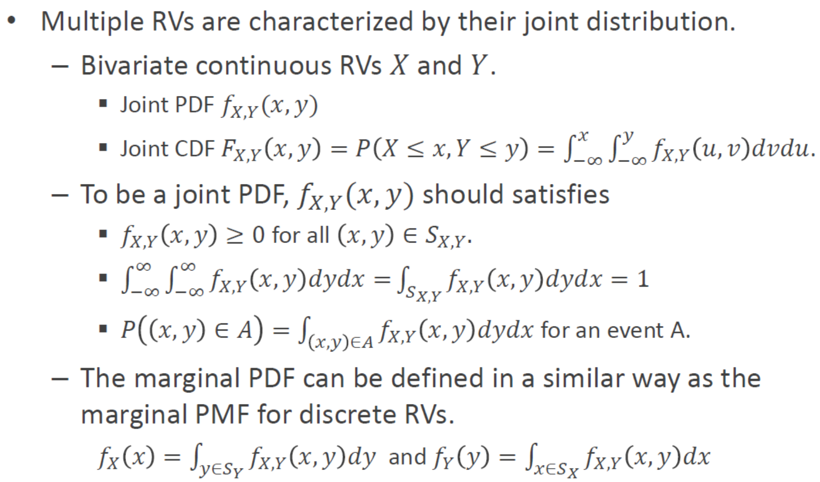
(a) Compute $P(X \leq 7, Y \leq 2)$?

(b) What is the marginal CDF $F_X(x)$ for X?

(c) What is the joint PDF $f_{X,Y}(x,y)$?

Q1(a) Compute $P(X \leq 7, Y \leq 2)$?

Page 12, Chapter6



- Multiple RVs are characterized by their joint distribution.
  - Bivariate continuous RVs X and Y.
    - Joint PDF $f_{X,Y}(x,y)$
    - Joint CDF $F_{X,Y}(x,y) = P(X \leq x, Y \leq y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u,v)dvdu$.
  - To be a joint PDF, $f_{X,Y}(x,y)$ should satisfies
    - $f_{X,Y}(x,y) \geq 0$ for all $(x,y) \in S_{X,Y}$.
    - $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)dydx = \int_{S_{X,Y}} f_{X,Y}(x,y)dydx = 1$
    - $P\big((x,y) \in A\big) = \int_{(x,y)\in A} f_{X,Y}(x,y)dydx$ for an event A.
  - The marginal PDF can be defined in a similar way as the marginal PMF for discrete RVs.

    $f_X(x) = \int_{y \in S_Y} f_{X,Y}(x,y)dy$ and $f_Y(y) = \int_{x \in S_X} f_{X,Y}(x,y)dx$

Joint CDF

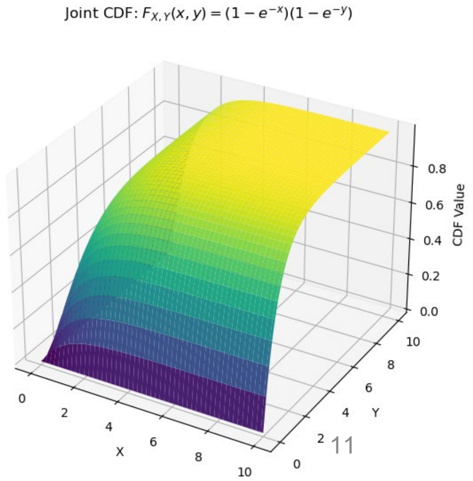
$$F_{X,Y}(x,y) = \begin{cases} (1-e^{-x})(1-e^{-y}) & x, y \geq 0, \\ 0 & \text{otherwise} \end{cases}$$

By definition, joint CDF

$$F_{X,Y}(x,y) = P(X \leq x, Y \leq y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u,v)dvdu.$$

Hence

$$P(X \leq 7, Y \leq 2) = F_{X,Y}(7,2) = (1-e^{-7})(1-e^{-2}) = 0.8639$$



Q1(b) What is the marginal CDF $F_X(x)$ for X?

Marginal CDF

Referred to joint CDF

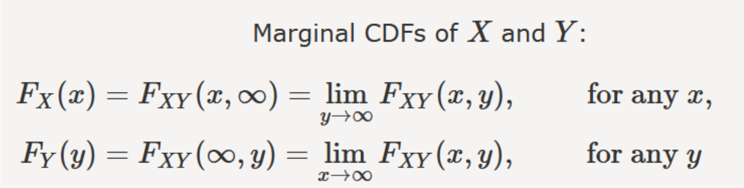
$$F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(u,v)\,dv\,du$$

Note that the marginal CDF

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{X,Y}(u,v)\,dv\,du = F_{X,Y}(x,\infty)$$

Marginal PDF

Joint PDF



Marginal CDFs of X and Y:

$$F_X(x) = F_{XY}(x,\infty) = \lim_{y\to\infty} F_{XY}(x,y), \qquad \text{for any } x,$$

$$F_Y(y) = F_{XY}(\infty,y) = \lim_{x\to\infty} F_{XY}(x,y), \qquad \text{for any } y$$

Hence,

$$F_X(x) = F_{X,Y}(x,\infty) \begin{cases} (1-e^{-x})(1-e^{-\infty}) = 1-e^{-x}, & x \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

The marginal CDF $F_Y(y)$ can be found in a similar way.

Q1(c) What is the joint PDF $f_{X,Y}(x,y)$?

Joint CDF and PDF

For continuous RVs X, Y, the relationship between their joint PDF and CDF are

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$$

The joint PDF can be derived as

$$f_{X,Y}(x,y) = \begin{cases} e^{-(x+y)}, & x, y \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $P(X \leq 7, Y \leq 2)$ and $F_X(x)$ can also be calculated from $f_{X,Y}(x,y)$ as below.

$$P(X \leq 7, Y \leq 2) = \int_0^7 \left(\int_0^2 f_{X,Y}(x,y)dy \right) dx.$$

$$F_X(x) = \int_0^\infty f_{X,Y}(x,y)dy.$$

Page 12, Chapter6

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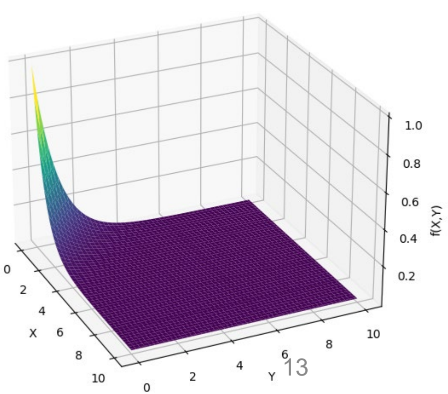
Joint PDF: $f_{X,Y}(x,y) = e^{-(x+y)}$



Q2

(Multiple continuous RVs)

Q2. Let $X_1, X_2, ..., X_n$ be random variables with the same expectation μ and with covariance function

$$\mathrm{Cov}[X_i, X_j] = \begin{cases} \sigma^2, & i = j \\ \rho\sigma^2, & |i - j| = 1 \\ 0, & \text{otherwise} \end{cases}$$

where $|\rho| < 1$, find the expectation and variance of $S_n = X_1 + X_2 + \cdots + X_n$.

Q2

The following two results are known to us

- $\mathrm{Var}[X_i] = \mathrm{Cov}[X_i, X_i] = \sigma^2$, for all i
- $E[X_i] = \mu$, for all i

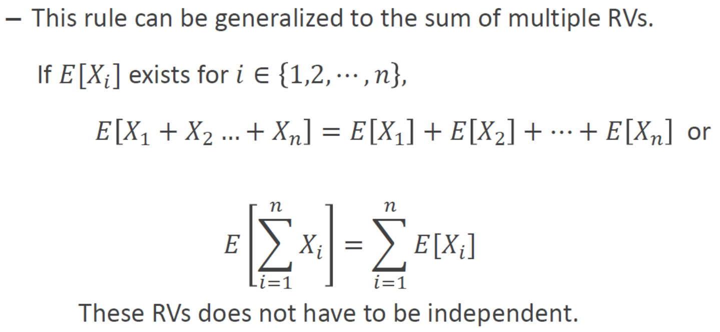
Therefore,

$$E(S_n) = E[X_1 + X_2 + \cdots + X_n] = E[X_1] + E[X_2] + \cdots + E[X_n] = n\mu$$

$$\mathrm{Var}[S_n] = \sum_{i=1}^{n} \mathrm{Var}[X_i] + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \mathrm{Cov}[X_i, X_j]$$

$$= n\sigma^2 + \sum_{i=1}^{n} \sum_{|i-j|=1}^{n} \mathrm{Cov}[X_i, X_j] = n\sigma^2 + 2(n-1)\rho\sigma^2$$

Page 32, Chapter 4



– This rule can be generalized to the sum of multiple RVs.

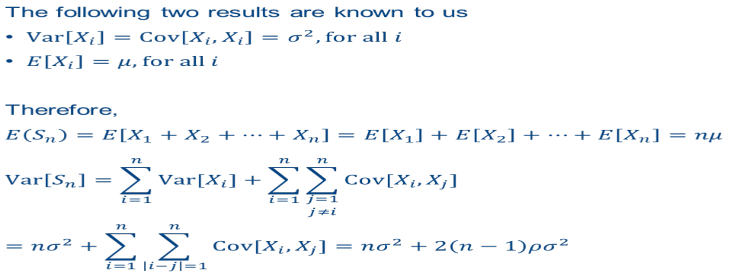
If $E[X_i]$ exists for $i \in \{1,2,\cdots,n\}$,

$$E[X_1 + X_2 \ldots + X_n] = E[X_1] + E[X_2] + \cdots + E[X_n] \text{ or}$$

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$

These RVs does not have to be independent.

Page 34, Chapter 4



The following two results are known to us

- $\mathrm{Var}[X_i] = \mathrm{Cov}[X_i, X_i] = \sigma^2$, for all i
- $E[X_i] = \mu$, for all i

Therefore,

$$E(S_n) = E[X_1 + X_2 + \cdots + X_n] = E[X_1] + E[X_2] + \cdots + E[X_n] = n\mu$$

$$\mathrm{Var}[S_n] = \sum_{i=1}^{n} \mathrm{Var}[X_i] + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \mathrm{Cov}[X_i, X_j]$$

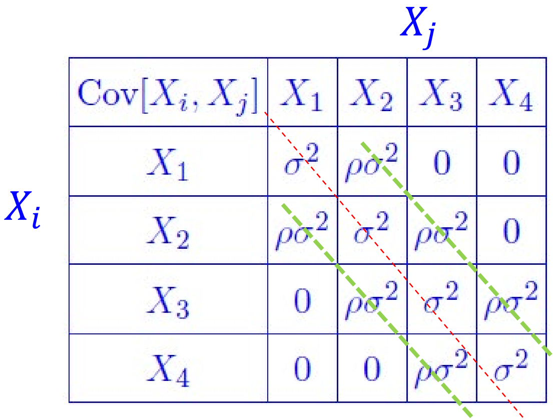
$$= n\sigma^2 + \sum_{i=1}^{n} \sum_{|i-j|=1}^{n} \mathrm{Cov}[X_i, X_j] = n\sigma^2 + 2(n-1)\rho\sigma^2$$

Q2

$$\mathrm{Var}[S_n] = n\sigma^2 + \sum_{i}^{n} \sum_{|i-j|=1}^{n} \mathrm{Cov}[X_i, X_j] = \boxed{n\sigma^2} + \boxed{2(n-1)\rho\sigma^2}$$

To see this derivation, we use the case of $n = 4$ as an example in the table below.

$$\mathrm{Cov}[X_i, X_j] = \begin{cases} \sigma^2, & i = j \\ \rho\sigma^2, & |i-j| = 1 \\ 0, & \text{otherwise} \end{cases}$$

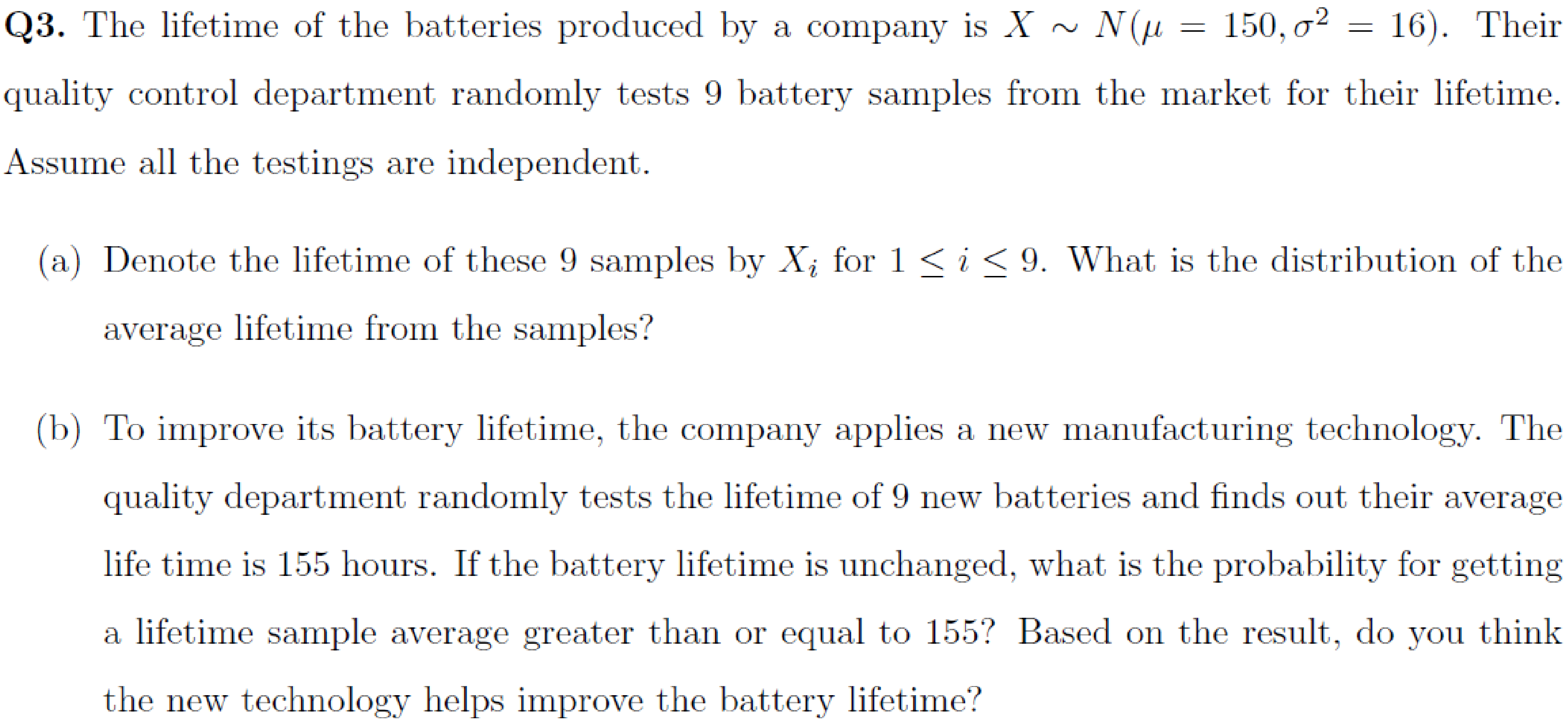


$\mathrm{Cov}[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	σ^2	$\rho\sigma^2$	0	0
X_2	$\rho\sigma^2$	σ^2	$\rho\sigma^2$	0
X_3	0	$\rho\sigma^2$	σ^2	$\rho\sigma^2$
X_4	0	0	$\rho\sigma^2$	σ^2

As you can see, in this table, each column, except for the first and the last column, has three nonzero elements, which have a sum of $\sigma^2 + 2\rho\sigma^2$. The first and the last column only has two nonzero elements with a sum of $\sigma^2 + \rho\sigma^2$. So $\mathrm{Var}[S_4] = 4\sigma^2 + 8\rho\sigma^2 - 2\rho\sigma^2 = 4\sigma^2 + 6\rho\sigma^2$.

Q3

(Expectation and Variance of Multiple continuous RVs)



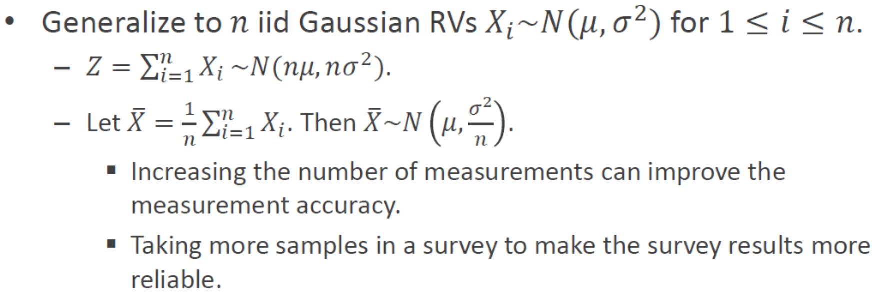
Q3. The lifetime of the batteries produced by a company is $X \sim N(\mu = 150, \sigma^2 = 16)$. Their quality control department randomly tests 9 battery samples from the market for their lifetime. Assume all the testings are independent.

(a) Denote the lifetime of these 9 samples by X_i for $1 \leq i \leq 9$. What is the distribution of the average lifetime from the samples?

(b) To improve its battery lifetime, the company applies a new manufacturing technology. The quality department randomly tests the lifetime of 9 new batteries and finds out their average life time is 155 hours. If the battery lifetime is unchanged, what is the probability for getting a lifetime sample average greater than or equal to 155? Based on the result, do you think the new technology helps improve the battery lifetime?

Q3(a)

Page 3 , Chapter 6



- Generalize to n iid Gaussian RVs $X_i \sim N(\mu, \sigma^2)$ for $1 \le i \le n$.
  - $Z = \sum_{i=1}^{n} X_i \sim N(n\mu, n\sigma^2)$.
  - Let $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Then $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$.
    - Increasing the number of measurements can improve the measurement accuracy.
    - Taking more samples in a survey to make the survey results more reliable.

Denote the sample average by $S = \frac{1}{9}\sum_{i=1}^{9} X_i$.

Since $X_i \sim N(150, 16)$, we have

$$\mathrm{E}[S] = \mathrm{E}\left[\frac{1}{9}\sum_{i=1}^{9} X_i\right] = \frac{1}{9}\sum_{i=1}^{9} \mathrm{E}[X_i] = 150.$$

Since X_i are also independent, we have

$$\mathrm{Var}[S] = \mathrm{Var}\left[\frac{1}{9}\sum_{i=1}^{9} X_i\right] = \left(\frac{1}{9}\right)^2 \sum_{i=1}^{9} \mathrm{Var}[X_i] = \frac{16 \times 9}{81} = \frac{16}{9}.$$

X_i's are independent Gaussian RVs, their average $S \sim N(150, 16/9)$. And it has a much smaller variance due to multiple sample average.

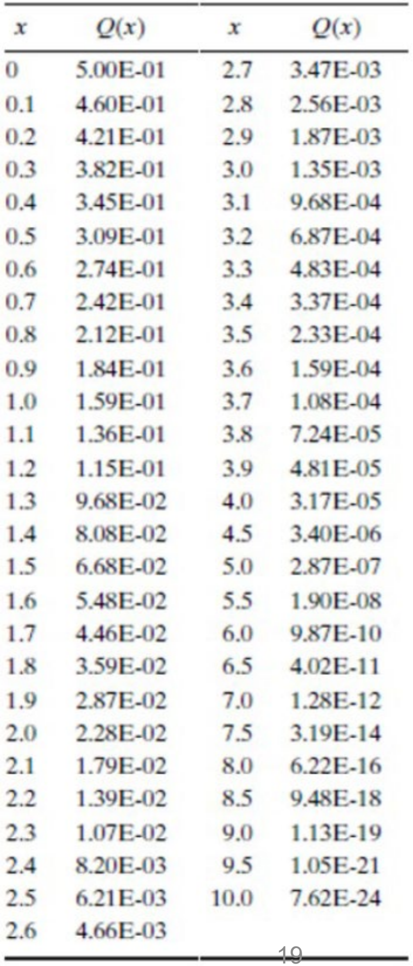
Q3(b)

If the battery lifetime is unchanged, the sample average lifetime is $S \sim N(150, 16/9)$ as derived in the last question. Then

$$P(S \geq 155) = P\left(\frac{S-150}{\sqrt{\frac{16}{9}}} \geq \frac{155-150}{\sqrt{\frac{16}{9}}}\right) = P(Z \geq 3.75)$$

$$= 0.8842 \times 10^{-4}.$$

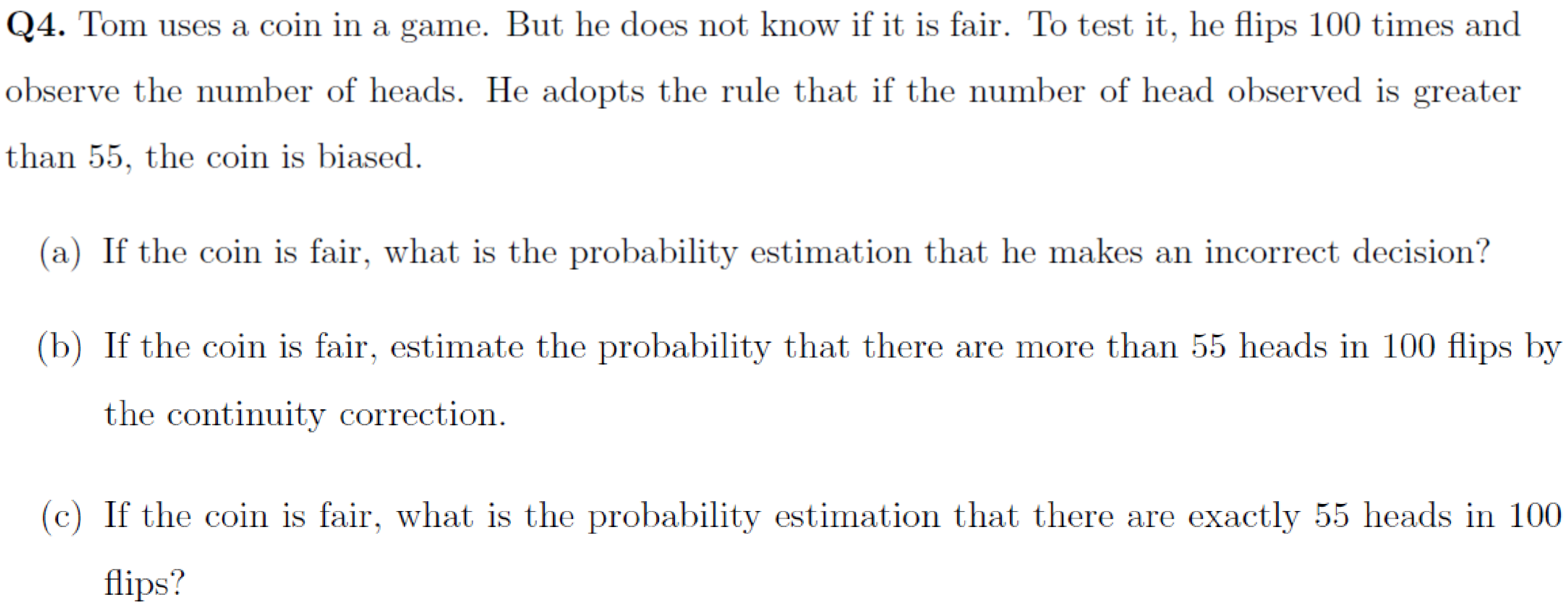
This is a small probability. It implies the event “a sample average of 155” unlikely happens if the battery lifetime is unchanged. This testing result suggests the new technology probably helps improve the battery lifetime.



x	$Q(x)$	x	$Q(x)$
0	5.00E-01	2.7	3.47E-03
0.1	4.60E-01	2.8	2.56E-03
0.2	4.21E-01	2.9	1.87E-03
0.3	3.82E-01	3.0	1.35E-03
0.4	3.45E-01	3.1	9.68E-04
0.5	3.09E-01	3.2	6.87E-04
0.6	2.74E-01	3.3	4.83E-04
0.7	2.42E-01	3.4	3.37E-04
0.8	2.12E-01	3.5	2.33E-04
0.9	1.84E-01	3.6	1.59E-04
1.0	1.59E-01	3.7	1.08E-04
1.1	1.36E-01	3.8	7.24E-05
1.2	1.15E-01	3.9	4.81E-05
1.3	9.68E-02	4.0	3.17E-05
1.4	8.08E-02	4.5	3.40E-06
1.5	6.68E-02	5.0	2.87E-07
1.6	5.48E-02	5.5	1.90E-08
1.7	4.46E-02	6.0	9.87E-10
1.8	3.59E-02	6.5	4.02E-11
1.9	2.87E-02	7.0	1.28E-12
2.0	2.28E-02	7.5	3.19E-14
2.1	1.79E-02	8.0	6.22E-16
2.2	1.39E-02	8.5	9.48E-18
2.3	1.07E-02	9.0	1.13E-19
2.4	8.20E-03	9.5	1.05E-21
2.5	6.21E-03	10.0	7.62E-24
2.6	4.66E-03		

Q4

(Central limit theorem)



Q4. Tom uses a coin in a game. But he does not know if it is fair. To test it, he flips 100 times and observe the number of heads. He adopts the rule that if the number of head observed is greater than 55, the coin is biased.

(a) If the coin is fair, what is the probability estimation that he makes an incorrect decision?

(b) If the coin is fair, estimate the probability that there are more than 55 heads in 100 flips by the continuity correction.

(c) If the coin is fair, what is the probability estimation that there are exactly 55 heads in 100 flips?

Q4

Understand the problem

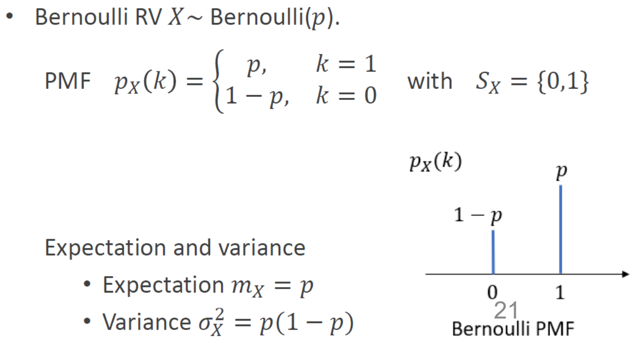
- Tom flips a coin 100 times.
- He considers the coin biased if he observes more than 55 heads.
- We assume the coin is fair, meaning the probability of getting a head on each flip is $p = 0.5$

Let $X_i \sim \text{Bernoulli}(0.5)$ for $1 \le i \le 100$ and the number of heads in $n = 100$ flips is

$$Y = \sum_{i=1}^{n} X_i$$

We have $E[X_i] = 0.5$ and the variance $\sigma^2 = 0.25$ for all X_i.

Page 24, Chapter 3



- Bernoulli RV $X \sim \text{Bernoulli}(p)$.

PMF $p_X(k) = \begin{cases} p, & k = 1 \\ 1-p, & k = 0 \end{cases}$ with $S_X = \{0,1\}$

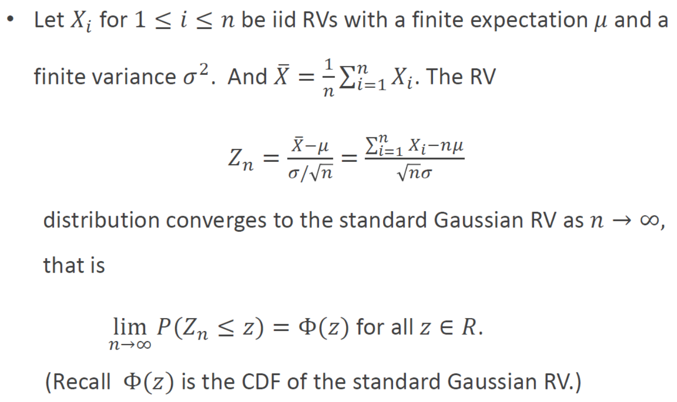
Expectation and variance

- Expectation $m_X = p$
- Variance $\sigma_X^2 = p(1-p)$

Bernoulli PMF

Q4(a)

Central limit theorem
Page 4, Chapter 6



- Let X_i for $1 \le i \le n$ be iid RVs with a finite expectation μ and a finite variance σ^2. And $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. The RV

$$Z_n = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\sigma}$$

distribution converges to the standard Gaussian RV as $n \to \infty$, that is

$$\lim_{n\to\infty} P(Z_n \le z) = \Phi(z) \text{ for all } z \in R.$$

(Recall $\Phi(z)$ is the CDF of the standard Gaussian RV.)

1. **Define Incorrect Decision**:
    - Tom makes an incorrect decision if he concludes the coin is biased when it is actually fair.
    - This happens if $Y > 55$.

2. **Approximate Using CLT**:

Since $Y \approx N(n\mu, n\sigma^2) = N(n\mu, (\sqrt{n}\sigma)^2) = N(50, 5^2)$, we can calculate $P(X > 55)$ by.

$$P_e = P(Y > 55) \approx P\left(\frac{Y-50}{5} > \frac{55-50}{5}\right) = P(Z > 1) = Q(1) = 0.1587$$

where $Z = \frac{Y-n\mu}{\sqrt{n}\sigma} = \frac{Y-50}{5}$ can be estimated as a standard Gaussian RV. Q(·) is the Q-function defined in our lecture

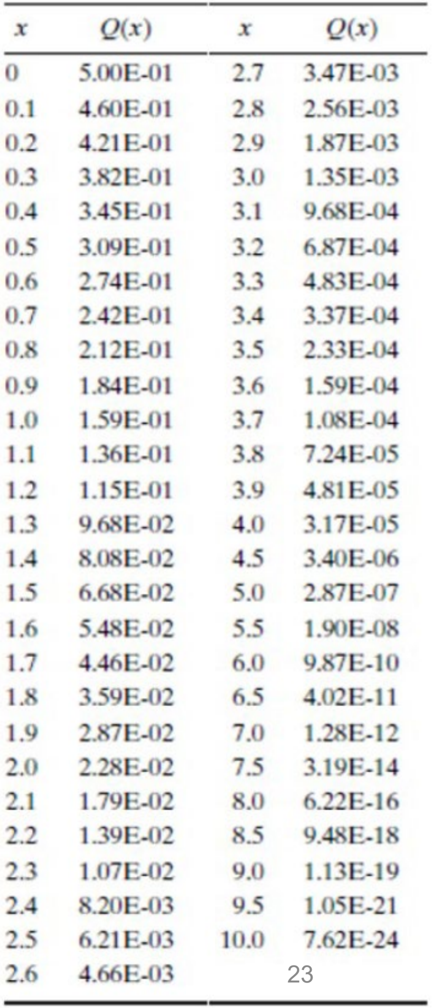
Q4(b)

Use the same definition of Y in (a). We need to calculate $P(Y > 55)$ by continuity correction because Y is discrete.

$$P(Y > 55) \approx P(Y > 55.5) = P\left(Z > \frac{55.5 - 50}{5}\right) = P(Z > 1.1)$$

$$= Q(1.1) = 0.1357.$$

The term "continuity correction" is used when approximating a discrete probability distribution with a continuous one



x	$Q(x)$	x	$Q(x)$
0	5.00E-01	2.7	3.47E-03
0.1	4.60E-01	2.8	2.56E-03
0.2	4.21E-01	2.9	1.87E-03
0.3	3.82E-01	3.0	1.35E-03
0.4	3.45E-01	3.1	9.68E-04
0.5	3.09E-01	3.2	6.87E-04
0.6	2.74E-01	3.3	4.83E-04
0.7	2.42E-01	3.4	3.37E-04
0.8	2.12E-01	3.5	2.33E-04
0.9	1.84E-01	3.6	1.59E-04
1.0	1.59E-01	3.7	1.08E-04
1.1	1.36E-01	3.8	7.24E-05
1.2	1.15E-01	3.9	4.81E-05
1.3	9.68E-02	4.0	3.17E-05
1.4	8.08E-02	4.5	3.40E-06
1.5	6.68E-02	5.0	2.87E-07
1.6	5.48E-02	5.5	1.90E-08
1.7	4.46E-02	6.0	9.87E-10
1.8	3.59E-02	6.5	4.02E-11
1.9	2.87E-02	7.0	1.28E-12
2.0	2.28E-02	7.5	3.19E-14
2.1	1.79E-02	8.0	6.22E-16
2.2	1.39E-02	8.5	9.48E-18
2.3	1.07E-02	9.0	1.13E-19
2.4	8.20E-03	9.5	1.05E-21
2.5	6.21E-03	10.0	7.62E-24
2.6	4.66E-03		

Q4(c)

Since Y is binomial, we can directly calculate the probability with some tedious work because of $\binom{100}{55}$.

$$P(Y = 55) = \binom{100}{55}\left(\frac{1}{2}\right)^{100} = 0.04847.$$

Since Y is discrete, by CLT and the continuity correction, we can estimate it as

$$P(Y = 55) \approx P(54.5 < Y < 55.5) = P(0.9 < Z < 1.1) = Q(0.9) - Q(1.1) = 0.04839.$$

They are very close and the continuity correction for binomial distribution is accurate.

Q5

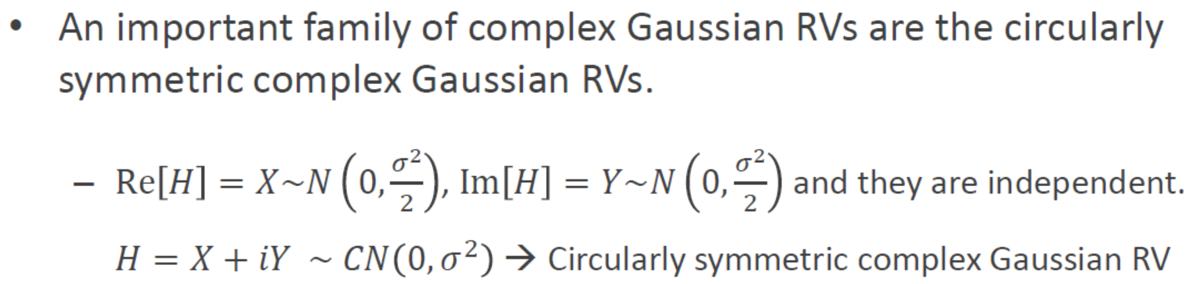
Q5. In a communication via a noisy channel, the signal detected by the receiver is modelled as $R = e^{-j\theta}M + H$. In this model, M is the transmitted message. The constant phase angle θ is due to the channel disturbance and it is perfectly estimated by the receiver. The channel additive noise H is a circularly symmetric complex Gaussian random variable with zero expectation and variance σ^2. Since θ is known to the receiver, the receiver can compensate the received signal as $R' = e^{j\theta}R = M + e^{j\theta}H$. Let $H' = e^{j\theta}H$.

(a) Find the expectation and variance of H'.

(b) Show the real and imaginary parts of H' are independent.

Q5(a)

Page 22, Chapter 6



- An important family of complex Gaussian RVs are the circularly symmetric complex Gaussian RVs.
  - $\mathrm{Re}[H] = X \sim N\left(0, \frac{\sigma^2}{2}\right)$, $\mathrm{Im}[H] = Y \sim N\left(0, \frac{\sigma^2}{2}\right)$ and they are independent.

    $H = X + iY \sim CN(0, \sigma^2)$ → Circularly symmetric complex Gaussian RV

Let $H = X + jY$ with both X and Y being real RVs. Since H is a circularly symmetric complex Gaussian RV, we have $X \sim N(0, \sigma^2/2)$,$Y \sim N(0, \sigma^2/2)$ and they are also independent.

By Euler formula , we have

$$H' = (\cos(\theta) + j\sin(\theta)) \times (X + jY) = [X\cos(\theta) - Y\sin(\theta)] + j[X\sin(\theta) + Y\cos(\theta)].$$

Q5(a)

Because both X and Y have zero expectation, we have

$$E[H'] = E[X\cos(\theta) - Y\sin(\theta)] + jE[X\sin(\theta) + Y\cos(\theta)]$$

$$= \cos(\theta)E[X] - \sin(\theta)E[Y] + j\{\sin(\theta)E[X] + \cos(\theta)E[Y]\}$$

$$= 0$$

$\cos(\theta)$ and $\sin(\theta)$ are constant values, since θ is a constant value.

Since $E[H'] = 0$,the variance of H' is

$$\mathrm{Var}[H'] = E[H' \times (H')^*]$$

$$= E[\left(X\cos(\theta) - Y\sin(\theta)\right)^2 + \left(X\sin(\theta) + Y\cos(\theta)\right)^2]$$

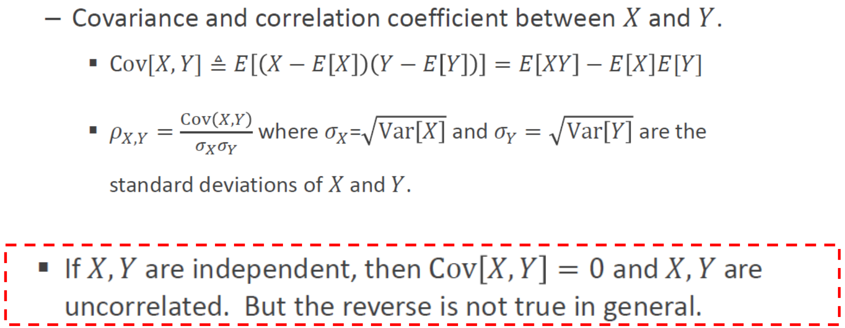
$$= E[X^2 + Y^2] = E[X^2] + E[Y^2] = \sigma^2/2 + \sigma^2/2 = \sigma^2.$$

- $\mathrm{Var}[H'] = \mathrm{E}[(H' - \mathrm{E}[H']) \times (H' - E[H'])^*]$
- Multiplying a complex number by its conjugate $(a + jb) \times (a - jb) = a^2 + b^2$

Q5(b)

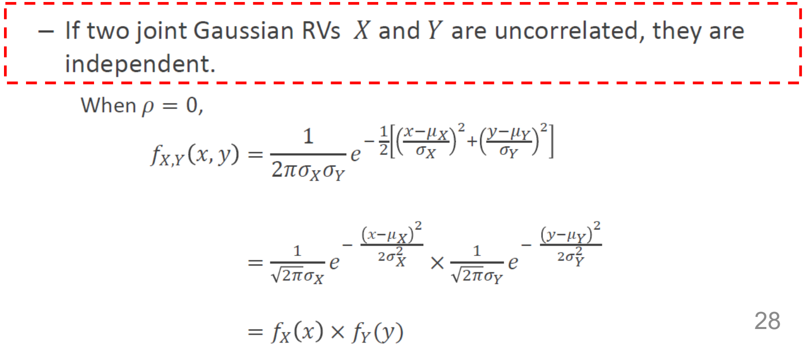
From (a), the real part of H' is $H_r{}' = X\cos(\theta) - Y\sin(\theta)$, and the imaginary part of H' is $H_i{}' = X\sin(\theta) + Y\cos(\theta)$. Since X and Y are independent Gaussian RVs and θ is a constant, both H_r' and H_i' are real Gaussian RVs. If they are uncorrelated, they will be independent.

Page 14, Chapter 6



- Covariance and correlation coefficient between X and Y.
  - $\mathrm{Cov}[X,Y] \triangleq E[(X-E[X])(Y-E[Y])] = E[XY] - E[X]E[Y]$
  - $\rho_{X,Y} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X\sigma_Y}$ where σ_X=$\sqrt{\mathrm{Var}[X]}$ and $\sigma_Y = \sqrt{\mathrm{Var}[Y]}$ are the standard deviations of X and Y.
- If X, Y are independent, then $\mathrm{Cov}[X,Y] = 0$ and X, Y are uncorrelated. But the reverse is not true in general.

Page 20, Chapter 6



- If two joint Gaussian RVs X and Y are uncorrelated, they are independent.

When $\rho = 0$,

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y} e^{-\frac{1}{2}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_X} e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}} \times \frac{1}{\sqrt{2\pi}\sigma_Y} e^{-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}}$$

$$= f_X(x) \times f_Y(y)$$

Q5(b)

From (a), we also have $E[H_r'] = E[H_i'] = 0$. The covariance of H_r' and H_i' is calculated as

$$
\begin{aligned}
\mathrm{Cov}[{H_r}', H_i'] &= E[(H_r' - 0)(H_i' - 0)] \\
&= E[(X\cos(\theta) - Y\sin(\theta))(X\sin(\theta) + Y\cos(\theta))] \\
&= E[X^2\cos(\theta)\sin(\theta) - Y^2\cos(\theta)\sin(\theta)] \\
&= \cos(\theta)\sin(\theta) \times \{E[X^2] - E[Y^2]\} \\
&= \cos(\theta)\sin(\theta) \times \{\sigma^2/2 - \sigma^2/2\} = 0
\end{aligned}
$$

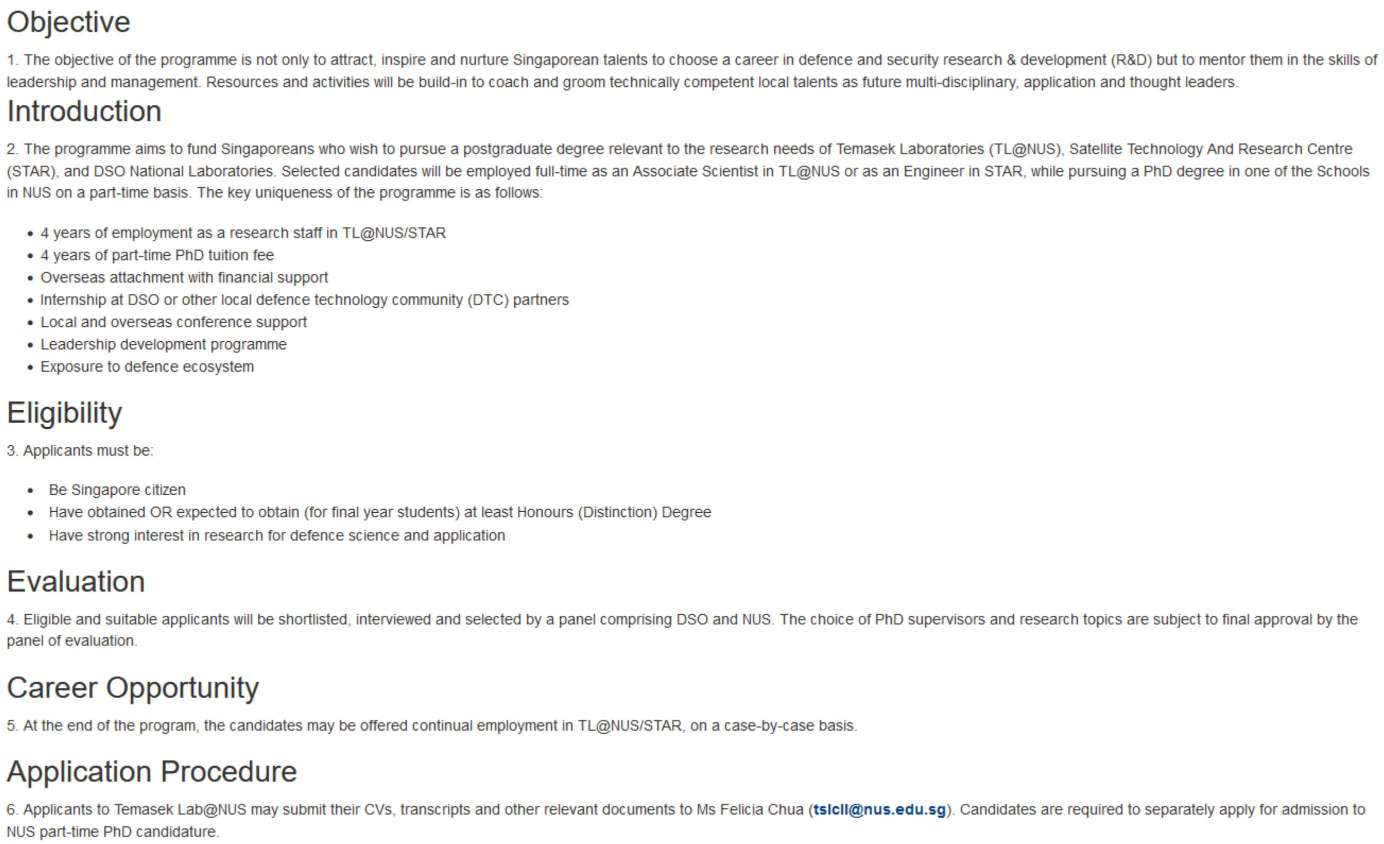
Page 17, Chapter 5

- $\mathrm{Var}[X] = E[X^2] - (E[X])^2$.

So $H_r'$and $H_i'$are uncorrelated. Since they are both Gaussian, they are independent. Combining the results in (a) and (b), we have $H' \sim CN(0, \sigma^2)$ .It has the same distribution as H. This is exactly the property of the cyclic symmetric Gaussian RVs.

https://temasek-labs.nus.edu.sg/ndgp/

NUS-DSO Graduate Programme



Objective

1. The objective of the programme is not only to attract, inspire and nurture Singaporean talents to choose a career in defence and security research & development (R&D) but to mentor them in the skills of leadership and management. Resources and activities will be build-in to coach and groom technically competent local talents as future multi-disciplinary, application and thought leaders.

Introduction

2. The programme aims to fund Singaporeans who wish to pursue a postgraduate degree relevant to the research needs of Temasek Laboratories (TL@NUS), Satellite Technology And Research Centre (STAR), and DSO National Laboratories. Selected candidates will be employed full-time as an Associate Scientist in TL@NUS or as an Engineer in STAR, while pursuing a PhD degree in one of the Schools in NUS on a part-time basis. The key uniqueness of the programme is as follows:

- 4 years of employment as a research staff in TL@NUS/STAR
- 4 years of part-time PhD tuition fee
- Overseas attachment with financial support
- Internship at DSO or other local defence technology community (DTC) partners
- Local and overseas conference support
- Leadership development programme
- Exposure to defence ecosystem

Eligibility

3. Applicants must be:

- Be Singapore citizen
- Have obtained OR expected to obtain (for final year students) at least Honours (Distinction) Degree
- Have strong interest in research for defence science and application

Evaluation

4. Eligible and suitable applicants will be shortlisted, interviewed and selected by a panel comprising DSO and NUS. The choice of PhD supervisors and research topics are subject to final approval by the panel of evaluation.

Career Opportunity

5. At the end of the program, the candidates may be offered continual employment in TL@NUS/STAR, on a case-by-case basis.

Application Procedure

6. Applicants to Temasek Lab@NUS may submit their CVs, transcripts and other relevant documents to Ms Felicia Chua (**tslcll@nus.edu.sg**). Candidates are required to separately apply for admission to NUS part-time PhD candidature.

THANK YOU