EE2211 Tutorial 4

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Q1

Given $Xw = y$ where $X = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$, $y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(a) What kind of system is this? (even-, over- or under-determined?)

(b) Is X invertible? Why?

(c) Solve for w if it is solvable.

Q1(a) What kind of system is this?

Given $Xw = y$ where $X = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$, $y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

The number of equations and unknowns determines the type of system:

- Even-determined: Number of equations = number of unknowns.
- Over-determined: More equations than unknowns.
- Under-determined: Fewer equations than unknowns.

Here, we need to compare the number of equations (rows in X) and the number of unknowns (columns in X).

- X has 2 rows and 2 columns.
- Since the number of equations equals the number of unknowns, this is an even-determined system (or a square system).

Q1(b) Is X invertible? Why?

To determine if X is invertible, we need to check if its determinant is non-zero. The determinant of a 2×2 matrix $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by:

$$\det(X) = ad - bc$$

For $X = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$, we calculate:

$$\det(X) = (1 \times 4) - (1 \times 3) = 4 - 3 = 1$$

Since the determinant is non-zero, X is invertible.

Q1(c) Solve for w if it is solvable.

Since X is invertible, we can solve the system $Xw = y$ by multiplying both sides by X^{-1}:

$$w = X^{-1}y$$

To find X^{-1}, the inverse of a 2×2 matrix $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by:

$$X^{-1} = \frac{1}{\det(X)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

For $X = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$, we already found that $\det(X) = 1$. Thus:

$$X^{-1} = \frac{1}{1} \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$$

Now, multiply X^{-1} by y:

$$w = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Q1

```
import numpy as np
m_list = [[1, 1], [3, 4]]
X = np.array(m_list)

inv_X = np.linalg.inv(X) # np.linalg.inv(X) computes the inverse of the matrix X using NumPy's linear algebra module (np.linalg).

y = np.array([0, 1])
w = inv_X.dot(y)
print(w)
```

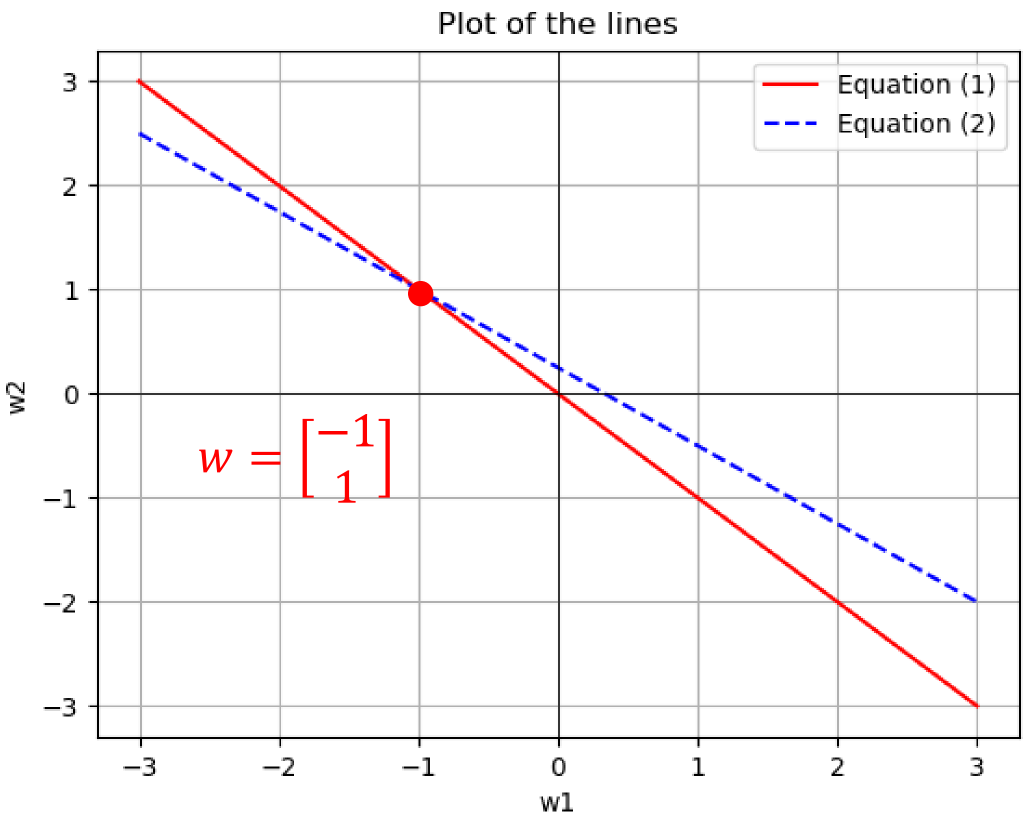
Converts m_list from a list (or a list of lists for a matrix) into a NumPy array, which are more efficient than Python lists for numerical operations.

```
[-1.  1.]
```

Q1

```python
import matplotlib.pyplot as plt

# Define the range for w1
w1 = np.linspace(-3, 3, 400)

# Define the equations of the lines
w2_1 = -w1  # From w1 + w2 = 0, w2 = -w1
w2_2 = (1 - 3*w1) / 4  # From 3w1 + 4w2 = 1, w2 = (1 - 3*w1) / 4

# Plot the lines
plt.plot(w1, w2_1, 'r-', label='Equation (1)')
plt.plot(w1, w2_2, 'b--', label='Equation (2)')

# Add labels and title
plt.xlabel('w1')
plt.ylabel('w2')
plt.title('Plot of the lines')

# Add a legend
plt.legend()

# Show the plot
plt.grid(True)
plt.axhline(0, color='black',linewidth=0.5)
plt.axvline(0, color='black',linewidth=0.5)
plt.show()
```

Q1



Plot of the lines
Equation (1)
Equation (2)
3
2
1
0
−1
−2
−3
−3
−2
−1
0
1
2
3
w1
w2
$w = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Q2

Given $Xw = y$ where $X = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, $y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(a) What kind of system is this? (even-, over- or under-determined?)

(b) Is X invertible? Why?

(c) Solve for w if it is solvable.

Q2(a) What kind of system is this? (even-, over-, or under-determined?)

The number of equations and unknowns determines the type of system:

- Even-determined: Number of equations = number of unknowns.
- Over-determined: More equations than unknowns.
- Under-determined: Fewer equations than unknowns.

Here, X is a 2×2 matrix, meaning there are 2 equations and 2 unknowns in the system. This is an even-determined system.

Q2(b) Is X invertible? Why?

To check if X is invertible, we need to find its determinant. For a 2x2 matrix, the determinant is given by:

$$\det(X) = \det\left(\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\right) = 1 \times 6 - 2 \times 3 = 6 - 6 = 0$$

Since the determinant of X is 0, the matrix is not invertible. A matrix is invertible only if its determinant is non-zero.

Q2(c) Solve for w if it is solvable.

The system is $Xw = y$, or:

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We can write the system as two linear equations:

$$w_1 + 2w_2 = 0$$

$$3w_1 + 6w_2 = 1$$

Now let's analyze the system:

- The first equation is $w_1 + 2w_2 = 0$, so $w_1 = -2w_2$.
- Substituting $w_1 = -2w_2$ into the second equation:

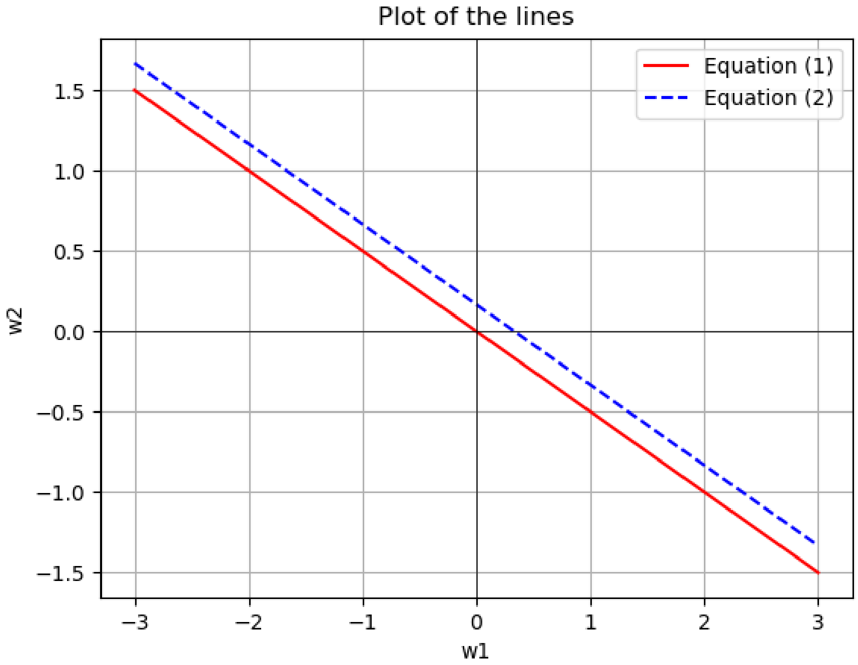
$$3(-2w_2) + 6w_2 = 1 \implies -6w_2 + 6w_2 = 1 \implies 0 = 1$$

This is a contradiction, meaning the system is inconsistent and has no solution. Therefore, it is not solvable.

Q2(c) Solve for *w* if it is solvable.

```
import numpy as np
import matplotlib.pyplot as plt

# Define the range for w1
w1 = np.linspace(-3, 3, 400)

# Define the equations of the lines
w2_1 = -w1 / 2  # From w1 + 2 * w2 = 0, w2 = -w1 /2
w2_2 = (1 - 3*w1) / 6  # From 3*w1 + 6 * w2 = 1, w2 = (1 - 3*w1) / 6

# Plot the lines
plt.plot(w1, w2_1, 'r-', label='Equation (1)')
plt.plot(w1, w2_2, 'b--', label='Equation (2)')
```

Q2(c) Solve for w if it is solvable.



There is no solution for w since the rows/columns of X are inter-dependent. The two lines shown in the plot are parallel and has no intersection.

Q3

Given $Xw = y$ where $X = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & -1 \end{bmatrix}$, $y = \begin{bmatrix} 0 \\ 0.1 \\ 1 \end{bmatrix}$.

(a) What kind of system is this? (even-, over- or under-determined?)

(b) Is X invertible? Why?

(c) Solve for w if it is solvable.

Q3(a) What kind of system is this? (even-, over-, or under-determined?)

The number of equations and unknowns determines the type of system:

- Even-determined: Number of equations = number of unknowns.
- Over-determined: More equations than unknowns.
- Under-determined: Fewer equations than unknowns.

Here, we need to compare the number of equations (rows in X) and the number of unknowns (columns in X).

- X is a 3×2 matrix (3 rows, 2 columns).
- There are 3 equations and 2 unknowns.

Since there are more equations than unknowns, the system is over-determined.

Q3 (b): Is X invertible? Why?

- For a matrix to be invertible, it must be a square matrix (i.e., the number of rows must equal the number of columns).
- Since X is a 3×2 matrix (not square), it is not invertible.
- But $X^T X = \begin{bmatrix} 6 & 9 \\ 9 & 21 \end{bmatrix}$ is invertible.

Q3 (c) Solve for w if it is solvable.

We want to solve:

$$Xw = y$$

where

$$X = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & -1 \end{bmatrix}, \qquad w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \qquad y = \begin{bmatrix} 0 \\ 0.1 \\ 1 \end{bmatrix}$$

Since this system is over-determined, it may not have an exact solution. The most common way to solve an over-determined system is to find a least squares solution. The least squares solution minimizes the error $\| Xw - y \|$, and it is given by:

$$w = (X^T X)^{-1} X^T y$$

Let's calculate it step by step

Step 1: Compute $X^T X$:

$$X^T X = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 9 & 21 \end{bmatrix}$$

Step 2: Compute $\left(X^T X\right)^{-1}$

$$(X^T X)^{-1} = \begin{bmatrix} 6 & 9 \\ 9 & 21 \end{bmatrix}^{-1} = \begin{bmatrix} 0.4667 & -0.2 \\ -0.2 & 0.1333 \end{bmatrix}$$

Step 3: Compute w

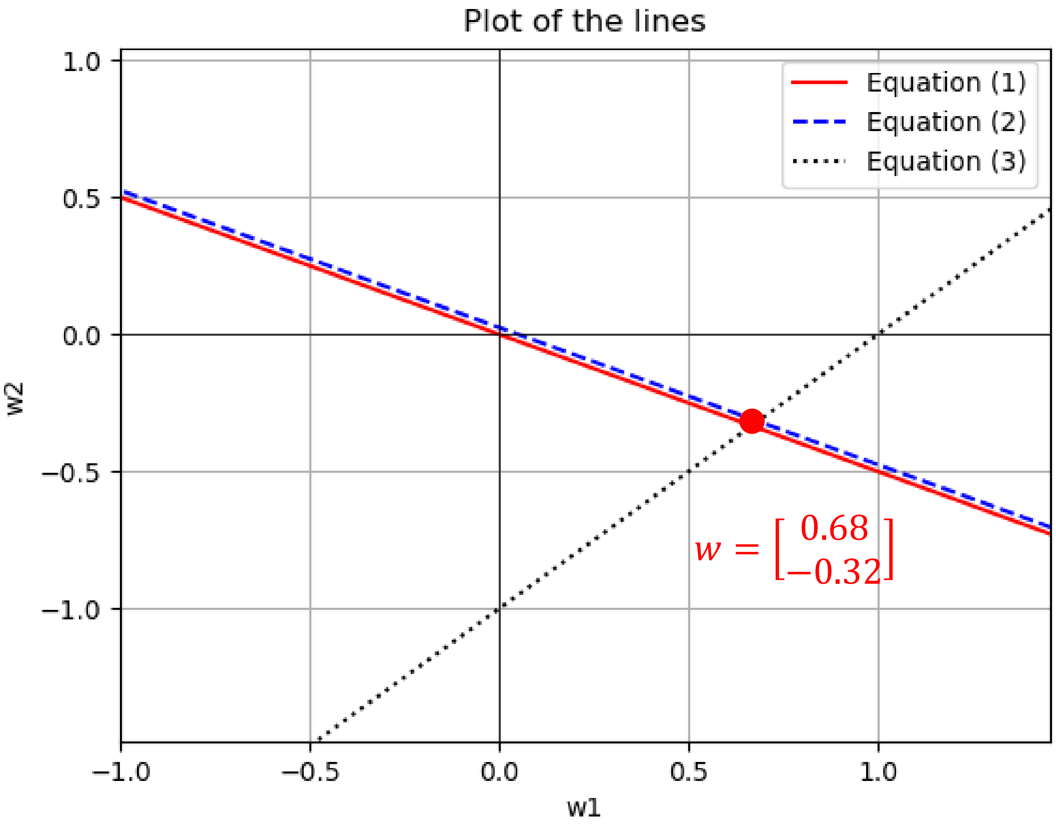
$$w = (X^T X)^{-1} X^T y = \begin{bmatrix} 0.4667 & -0.2 \\ -0.2 & 0.1333 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0.1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.68 \\ -0.32 \end{bmatrix}$$

Q3 (c) Solve for *w* if it is solvable.

```
import numpy as np
from numpy.linalg import inv

# Create a 3x2 matrix X:
X = np.array([[1, 2], [2, 4], [1, -1]])

# Create a vector y representing the target values
y = np.array([0, 0.1, 1])

# Compute the least squares solution for w
w = inv(X.T @ X) @ X.T @ y
print(w)
```

```
[ 0.68 -0.32]
```

Q3 (c) Solve for w if it is solvable.



$$w = \begin{bmatrix} 0.68 \\ -0.32 \end{bmatrix}$$

Question 4

Given $Xw = y$ where $X = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$, $y = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

(a) What kind of system is this? (even-, over- or under-determined?)

(b) Is X invertible? Why?

(c) Solve for w if it is solvable.

Q4 (a) What kind of system is this? (even-, over- or under-determined?)

The number of equations and unknowns determines the type of system:

- Even-determined: Number of equations = number of unknowns.
- Over-determined: More equations than unknowns.
- Under-determined: Fewer equations than unknowns.

To determine the type of system, we compare the number of equations (rows in X) and the number of unknowns (columns in X).

- X is a 3×4 matrix (3 rows, 4 columns).
- There are 3 equations and 4 unknowns.

In this case, there are 3 equations and 4 unknowns. So, the system is under-determined.

Q4 (b) Is X invertible? Why?

- For a matrix to be invertible, it must be a square matrix (i.e., the number of rows must equal the number of columns).
- Since X is a 3×4 matrix (not square), it is not invertible.
- But XX^T is invertible.

Q4 (c) Solve for w if it is solvable.

We want to solve:

$$Xw = y$$

where

$$X = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 0 & 0 \end{bmatrix}, w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, y = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Since this system is over-determined, it may have infinite solutions. We want to use the right inverse to find a constrained solution for w :

$$w = X^T(XX^T)^{-1}y$$

Let's calculate it step by step

Step 1: Compute XX^T:

$$XX^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Step 2: Compute $(XX^T)^{-1}$

$$(XX^T)^{-1} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 0.75 & 0.5 \\ -1 & 0.5 & 1 \end{bmatrix}$$

Step 3: Compute w

$$w = X^T(XX^T)^{-1}y = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 0.75 & 0.5 \\ -1 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$$

Q4

```python
import numpy as np
from numpy.linalg import inv

X = np.array([[1, 0, 1, 0], [1, -1, 1, -1], [1,1,0,0]])
y = np.array([[1], [0], [1]])

w1 = X.T @ inv(X @ X.T) @y
print('w1=')
print(w1)

# Another method
from numpy.linalg import pinv
w2 = pinv(X) @ y
print('w2=')
print(w2)
```

```
w1=
[[0.5]
 [0.5]
 [0.5]
 [0.5]]
w2=
[[0.5]
 [0.5]
 [0.5]
 [0.5]]
```

pinv: compute the (Moore-Penrose) pseudo-inverse of a matrix.

Q5

Given $\mathrm{w}^{\mathrm{T}}X = y^{\mathrm{T}}$ where $X = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, $y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(a) What kind of system is this? (even-, over- or under-determined?)

(b) Is X invertible? Why?

(c) Solve for w if it is solvable.

Q5 (a) What kind of system is this? (even-, over- or under-determined?)

A canonical formulation is given by

$$w^T X = y^T \;\rightarrow\; X^T w = y \;\rightarrow\; X_1 w = y$$

where

$$X^T = X_1 = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

Here, we compare the number of equations (rows) with the number of unknowns (columns).

- Number of equations: The number of rows in matrix X_1, which is 2.
- Number of unknowns: The number of columns in matrix X_1, which is 2.

Since the number of equations equals the number of unknowns, the system is even-determined.

Q5 (b) Is X invertible? Why?

A matrix is invertible if it is square (number of rows equals the number of columns) and its determinant is non-zero.

To check if X is invertible, calculate its determinant

$$\det(X) = \det\left(\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\right) = 1 \times 6 - 2 \times 3 = 0$$

Since the determinant of X is zero, the matrix is **not invertible**.

Q5 (c) Solve for w if it is solvable.

Similar to Question 2, the matrix is not invertible, and therefore no solution exists.

Q6

Given $\mathrm{w}^{\mathrm{T}}X = y^{\mathrm{T}}$ where $X = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & -1 \end{bmatrix}$, $y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(a) What kind of system is this? (even-, over- or under-determined?)

(b) Is X invertible? Why?

(c) Solve for w if it is solvable to obtain a constrained solution.

Q6 (a) What kind of system is this? (even-, over- or under-determined?)

A canonical formulation is given by taking transpose of both sides

$$w^T X = y^T \;\rightarrow\; X^T w = y$$

where

$$X^T = X_1 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & -1 \end{bmatrix}$$

To determine the type of system, we compare the number of equations (rows in X^T) and the number of unknowns (columns in X^T).

- X^T is a 2×3 matrix (2 rows, 3 columns).
- There are 2 equations and 3 unknowns.

Since there are more unknows than equations, the system is under-determined.

Q6 (b) Is X invertible? Why?

- For a matrix to be invertible, it must be square (i.e., the number of rows must equal the number of columns). X is a 3×2 matrix, so it is not square and therefore not invertible.
- But $X^T X$ is invertible.

$$\det\left(\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & -1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 6 & 9 \\ 9 & 21 \end{bmatrix}\right) = 6 \times 21 - 9 \times 9 = 45$$

Q6 (c) Solve for w if it is solvable.

A canonical formulation is given by

$$w^T X = y^T \;\rightarrow\; X^T w = y$$

where

$$X^T = X_1 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & -1 \end{bmatrix}$$

A constrained solution (exact) is given by

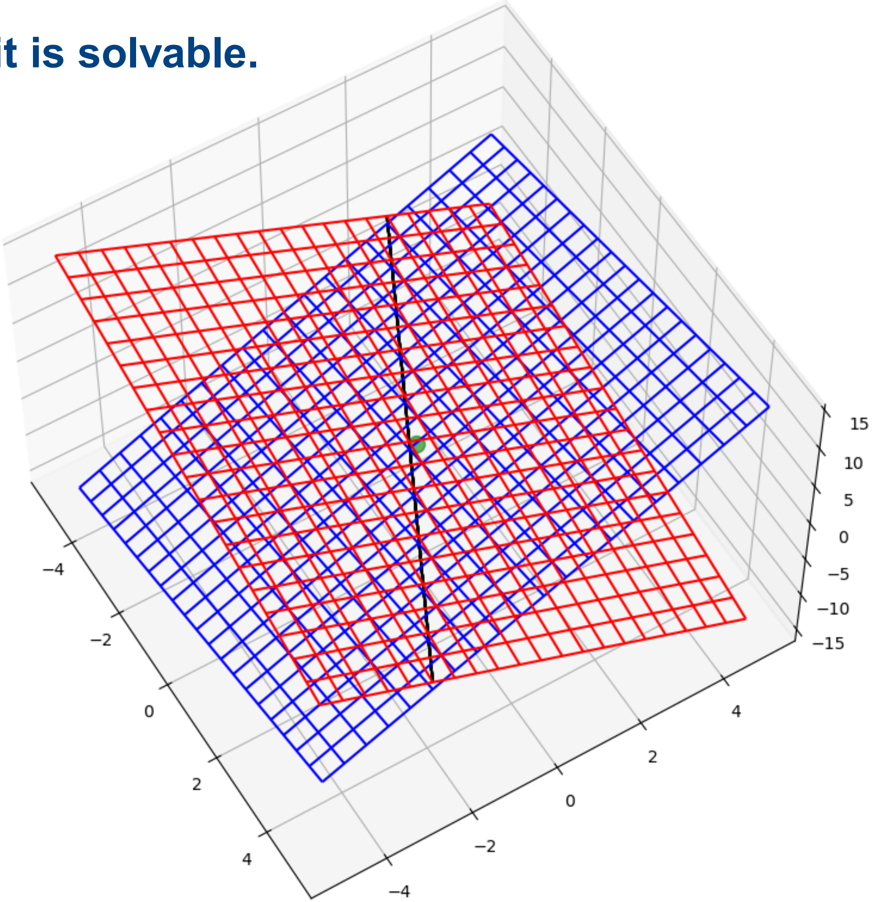
$$\hat{w} = X_1^T (X_1 X_1^T)^{-1} y = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & -1 \end{bmatrix}^T \left(\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & -1 \end{bmatrix}^T \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{bmatrix} 0.066667 \\ 0.133333 \\ -0.33333 \end{bmatrix}$$

Q6

```python
import numpy as np
from numpy.linalg import inv
X = np.array([[1, 2], [2, 4], [1,-1]])
X = X.T
print('X=')
print(X)

y = np.array([[0], [1]])

w1 = X.T @ inv(X @ X.T) @y
print('w1=')
print(w1)

# Another method
from numpy.linalg import pinv
w2 = pinv(X) @ y
print('w2=')
print(w2)
```

```
X=
[[ 1  2  1]
 [ 2  4 -1]]
w1=
[[ 0.06666667]
 [ 0.13333333]
 [-0.33333333]]
w2=
[[ 0.06666667]
 [ 0.13333333]
 [-0.33333333]]
```

Q6 (c) Solve for w if it is solvable.



Q7

This question is related to determination of types of system where an appropriate solution can be found subsequently. The following matrix has a left inverse.

$$X = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

a) True

b) False

Q7

Answer

Left inverse is given by $\left(\mathbf{X}^{T}\mathbf{X}\right)^{-1}\mathbf{X}^{T}$ where $\mathbf{X}^{T}\mathbf{X}$ should be invertible. In this case, $\mathbf{X}^{T}\mathbf{X}$ is not invertible so the matrix does not have a left inverse.

Q8

MCQ: Which of the following is/are true about matrix A below? There could be more than one answer.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

a) A is invertible

b) A is left invertible

c) A is right invertible

d) A has no determinant

e) None of the above

Q8

Answer

c and d.

THANK YOU