Implementation of Trend Following Strategy

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HJB Equations

- Define value functions, for $i = 0, 1$

$$W_i(p,t) = \sup_{\Lambda_i} J_i(p,t,\Lambda_i)$$

- It is easy to see

$$W_0(p,t) = \sup_{\tau_1} E_t \left\{ \int_t^{\tau_1} \rho ds - log(1+\theta) + W_1(p_{\tau_1}, \tau_1) \right\}$$

$$W_1(p,t) = \sup_{\overline{v}_1} E_t \left\{ \int_t^{\overline{v}_1} \widetilde{f}(p_s) ds + log(1-\alpha) + W_0(p_{\overline{v}_1}, \overline{v}_1) \right\}$$

where

$$\widetilde{f}(p) = (\mu_1 - \mu_2)p + \mu_2 - \frac{\sigma^2}{2}$$

HJB Equations

- Let $Z(p,t) = W_1(p,t) - W_0(p,t)$. Then Z is the unique solution to the double obstacle problem

$$max\left\{min\left\{-\mathcal{A}Z - f(p), Z - log(1-\alpha)\right\}, Z - log(1+\theta)\right\} = 0$$

where

$$\mathcal{A} = \frac{\partial}{\partial t} + \frac{1}{2}\left(\frac{(\mu_1 - \mu_2)p(1-p)}{\sigma}\right)^2 \frac{\partial}{\partial p^2} + \left[-(\lambda_1 + \lambda_2)p + \lambda_2\right]\frac{\partial}{\partial p}$$

$$f(p) = \widetilde{f}(p) - \rho$$

in $p \in (0,1)$, with boundary condition:

$$Z(p,T) = log(1-\alpha)$$
$$Z(0,t) = log(1-\alpha)$$
$$Z(1,t) = log(1+\theta)$$

Implicit Finite Difference Grid Method

- Truncate the domain $p \in [0,1]$, $t \in [0,T]$ and use the finite difference grid:

$$\left\{\left(p_n^i, t_n\right) : p_n^i = i\Delta p, t_n = n\Delta t, i = 0,1,\ldots,I, n = 0,1,\ldots,N\right\}$$

and discretize the PDE at $\left(p_n^i, t_n\right)$.

Implicit Finite Difference Grid Method

- Let $\epsilon = \frac{1}{2}\left(\frac{(\mu_1-\mu_2)p(1-p)}{\sigma}\right)^2$ and $b = -\left(\lambda_1 + \lambda_2\right)p + \lambda_2$.
  As $p \to 0$ or $p \to 1$, $|b| >> |\epsilon|$, an **upwind treatment** should be applied:

$$\left.\frac{\partial z}{\partial p}\right|_{(p_n^j, t_n)} = \begin{cases} \frac{z_n^{i+1} - z_n^i}{\Delta p} + O(\Delta p), & \text{if} \quad b \geq 0 \\ \frac{z_n^i - z_n^{i-1}}{\Delta p} + O(\Delta p), & \text{if} \quad b < 0 \end{cases}$$

- For the other partial derivatives, use the following approximation:

$$\left.\frac{\partial z}{\partial t}\right|_{(p_n^i, t_n)} = \frac{z_{n+1}^i - z_n^i}{\Delta t} + O(\Delta t)$$

$$\left.\frac{\partial^2 z}{\partial p^2}\right|_{(p_n^i, t_n)} = \frac{z_n^{i-1} - 2z_n^i + z_n^{i+1}}{(\Delta p)^2} + O\left((\Delta p)^2\right)$$

Implicit Finite Difference Grid Method

- After considering **upwind treatment**, dropping truncation error,collating and re-arranging terms gives the PDE :

if $\quad b \geq 0$

$$-\frac{\Delta t\epsilon}{(\Delta p)^2} Z_n^{i-1} + \left(1 + \frac{2t\epsilon}{(\Delta p)^2} + \frac{\Delta tb}{\Delta p}\right) Z_n^i + \left(-\frac{t\epsilon}{(\Delta p)^2} - \frac{\Delta tb}{\Delta p}\right) Z_n^{i+1} = \Delta tf\left(i\Delta p\right) + Z_{n+}^i$$

if $\quad b<0$

$$\left(-\frac{\Delta t\epsilon}{(\Delta p)^2} + \frac{\Delta tb}{\Delta p}\right) Z_n^{i-1} + \left(1 + \frac{2t\epsilon}{(\Delta p)^2} - \frac{\Delta tb}{\Delta p}\right) Z_n^i - \frac{\Delta t\epsilon}{(\Delta p)^2} Z_n^{i+1} = \Delta tf\left(i\Delta p\right) + Z_{n+1}^i$$

with $i = 0, 1, \cdots, I$

Implicit Finite Difference Grid Method

- We can express the "square" system of $I-1$ equations in the matrix-vector form:

$$\begin{bmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{I-2} & b_{I-2} & c_{I-2} \\ & & & a_{I-1} & b_{I-1} \end{bmatrix} \begin{bmatrix} Z_n^1 \\ Z_n^2 \\ \vdots \\ Z_n^{I-2} \\ Z_n^{I-1} \end{bmatrix} = \begin{bmatrix} Z_{n+1}^1 \\ Z_{n+1}^2 \\ \vdots \\ Z_{n+1}^{I-2} \\ Z_{n+1}^{I-1} \end{bmatrix} + \begin{bmatrix} -a_1 Z_n^0 \\ \Delta t f\left(\Delta p\right) \\ \vdots \\ \Delta t f\left(\left(I-1\right)\Delta p\right) \\ -c_{I-1} Z_n^I + \Delta t f\left(I \Delta p\right) \end{bmatrix}$$

  can be showed as $\mathbf{A Z_n = Z_{n+1} + F_n}$

- Solving every Z_n with **Projected SOR Method**.

Projected SOR

- **Input** A, b, V, n, ϵ, ω, $upper$, $lower$
- **Output** V_n
- While not converged, for i=1,2, ..., n

$$x_{i,gs}^{k+1} = \frac{1}{a_{ii}} \left[-\sum_{j=1}^{i-1} a_{ij} x_j^{k+1} - \sum_{j=i+1}^{n} a_{ij} x_j^k + b_i \right]$$

$$x_i^{k+1} = min\left(max\left((1-\omega)x_i^k + \omega x_{i,gs}^{k+1}, lower\right), upper\right)$$

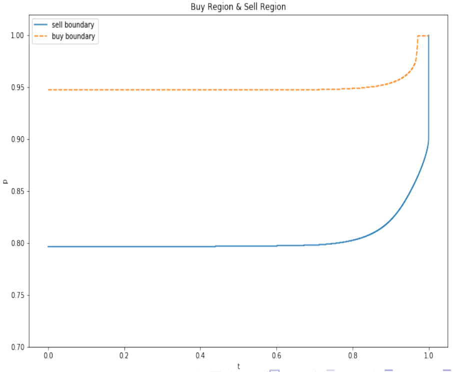
If $\|x^{k+1} - x^k\| < \epsilon$, then set Converged=true, and $V_n = x$.

- Solve every $V_n(n = 0, 1, \ldots, N)$, then get every node value of Z_Grid.

Test Example - SP500

λ_1	λ_2	μ_1	μ_2	σ	ρ	α	θ
0.360	2.530	0.180	-0.770	0.184	0.068	0.001	0.001

p_b^*	p_s^*
0.9475	0.7965



Define Bull & Bear Market

Definition

Define an up trend to be rally at least **25%** and a down trend decline at least **25%**.

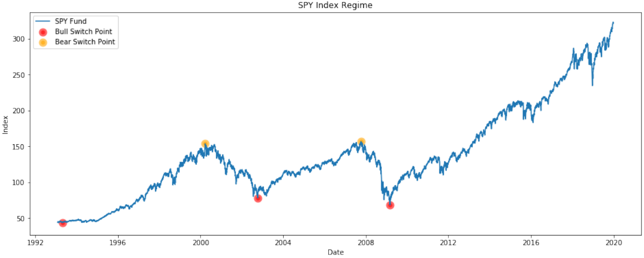
Up Trend	Down Trend
25%	25%

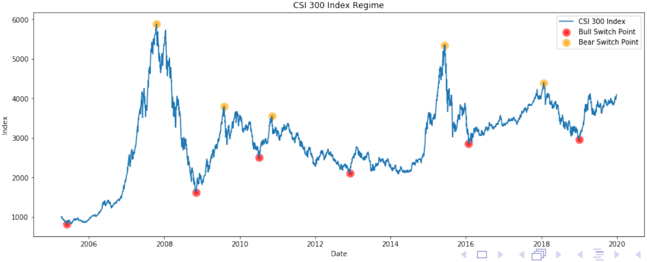
Number of bull and bear

For the period of our data, we counted the bull and bear market.

	No. Bull	No. Bear
SPY	2	2
CSI 300	5	5

Bull & Bear Market - SPY & CSI 300





Estimate λ

- $\lambda_1(\lambda_2)$ stands for the switching intensity from bull to bear(from bear to bull).
- $\lambda_1(\lambda_2)$ can be showed as average of reciprocal of the duration during the bull(bear) market:

$$\hat{\lambda}_1 = \frac{1}{N_1}\sum_{i=1}^{N_1}\frac{1}{T_i^{bull}}, \quad \hat{\lambda}_2 = \frac{1}{N_2}\sum_{i=1}^{N_2}\frac{1}{T_i^{bear}}$$

where, $T_i^{bull}(T_i^{bear})$ means the duration of the i-th bull(bear) market period, and $N_1(N_2)$ means total number of bull(bear) market.

Estimate μ, σ

- $\mu_1(\mu_2)$ and $\sigma_1(\sigma_2)$ stand for the average annualized return and volatility during bull(bear) market.
- According to Geometric Brownian Motion:

$$dlnS_t = (\mu - \frac{\sigma^2}{2})dt + \sigma dW_t$$

- Then we solve:

$$\Sigma(dlnS_t)^2 = \sigma^2 T$$

$$E(dlnS_t) = E\left[(\mu - \frac{\sigma^2}{2})dt + \sigma dW_t\right]$$

$$\Sigma dlnS_t = (\mu - \frac{\sigma^2}{2})T$$

$$\hat{\sigma} = \sqrt{\frac{1}{T}\Sigma(dlnS_t)^2}, \quad \hat{\mu} = \frac{\Sigma(dlnS_t)}{T} + \frac{\hat{\sigma}^2}{2}$$

Estimate p_t

- To estimate p_t, the conditional probability in a bull market, consider the below stochastic differential equation:

$$dp_t = g(p_t)dt + \frac{(\mu_1 - \mu_2)p_t(1-p_t)}{\sigma^2} dlogS_t$$

where,

$$g(p_t) = -(\lambda_1 + \lambda_2)p_t + \lambda_2 - \frac{(\mu_1 - \mu_2)p_t(1-p_t)((\mu_1 - \mu_2)p_t + \mu_2 - \frac{\sigma^2}{2})}{\sigma^2}$$

- Discrete the SDE for $t = 0, 1, \cdots, N$ with $dt = 1/252$, and ensure the discrete approximation p_t stays in $[0, 1]$:

$$p_{t+1} = min\left(max\left(p_t + g(p_t)dt + \frac{(\mu_1 - \mu_2)p_t(1-p_t)}{\sigma^2} log(\frac{S_{t+1}}{S_t}), 0\right), 1\right)$$

Initialization of Parameters: p_0

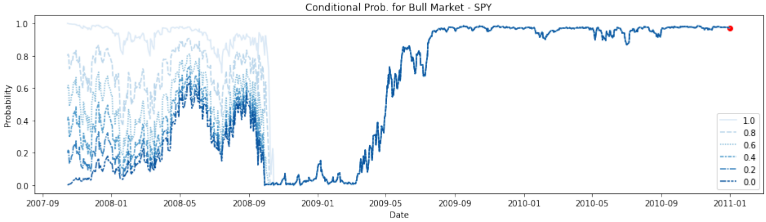
Start Date

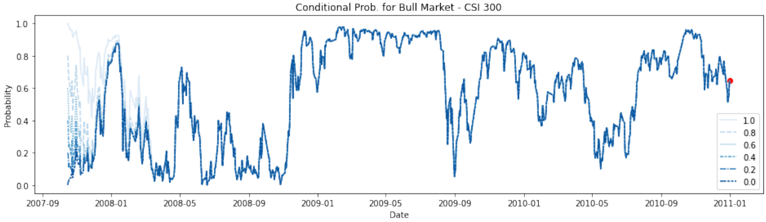
- Choose a starting point for a bear market with a significant decline as the start date(2007-10-16).
- After the estimation process of p_t, the p_0 of the initial date(2011-01-01) can be determined.
- Assume $p = 0$ at the starting point, calculate p_0:

SPY	CSI 300
0.961	0.797

- Actually, p value at the start point would not affect the value of initialized p_0.

Initialization of p_0 - SPY & CSI 300





Assumption

Basic Assumption

- The investment period is from **2011-01-01** to **2019-12-31**.
- Any trading action will take place at the **close** of the market.
- Use SPY and CSI 300 daily **closing price** for the tests.
- Assume $\rho^A = 2.86\%$ and $\rho^C = 3.89\%$.
- Assume $\alpha = 0.1\%$ and $\theta = 0.1\%$.
- Use σ^{bull} to calculate p_b^*, and σ^{bear} to calculate p_s^*.

SPY - Estimated Parameters

Estimation

- Since SPY experienced a bull market from 2011 to 2020, there is no need to conduct a rolling-base estimation [a].
- Use the data from 2001-01-01 to 2010-12-31 to estimate the parameters.

[a] The available data will always be before 2011.

SPY - Estimated Parameters (Con't)

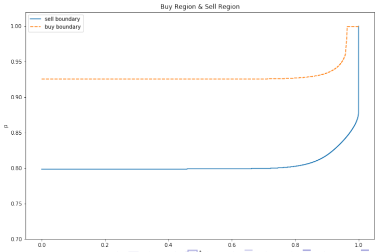
- Parameters:

λ_1	λ_2	μ_1	μ_2	σ	ρ	α	θ
0.200	0.706	0.143	-0.510	0.219	0.035	0.001	0.001

Table: Estimated Parameters (SPY)

- Buy & Sell Boundary:

p_b^*	p_s^*
0.9255	0.7985



Trend Following Strategy - SPY

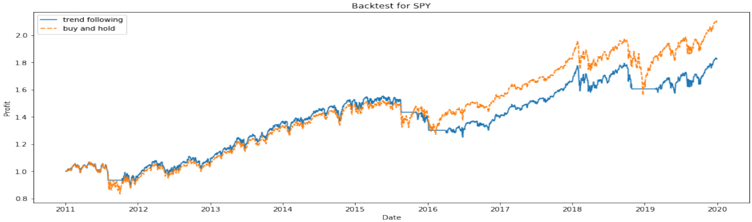
End Date	TF	Sharp Ratio	BH	10y Bonds	No. Trade
2020-01-01	83.92%	0.173	110.15%	22.49%	9
2020-03-19	53.74%	0.084	47.22%	23.19%	10

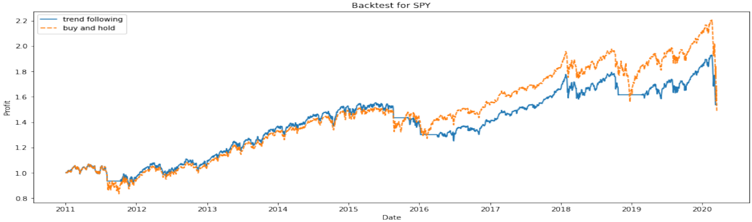
Table: TF Strategy Return (SPY)

Remark

- TF Strategy doesn't beat the SPY from 2011 to 2019.
- However, TF Strategy outperforms the SPY from 2011-01-01 to 2020-03-19.

Strategy Return on SPY





CSI 300 - Estimated Parameters

Rolling-Base Estimation

- Determine the parameters by beginning with the statistical estimate of the **10-year data** [a].
- Update the parameters and the corresponding thresholds at the beginning of a new year [b].
- Use **exponential average method** to update the parameters:

$$update = (1 - 2/N)old + (2/N)new$$

[a] If available data is less than 10 years, we use the available data to estimate the parameters

[b] using the new data that become available if a new up or down trend is completed.

CSI 300 - Estimated Parameters (Con't)

	λ_1	λ_2	μ_1	μ_2	σ_1	σ_2	σ
2011	1.829	5.241	0.959	-1.070	0.306	0.409	0.333
2012	2.096	5.241	0.967	-1.070	0.300	0.409	0.318
2013	2.096	5.241	0.967	-1.070	0.300	0.409	0.305
2014	2.096	4.051	0.967	-0.612	0.300	0.316	0.297
2015	2.096	4.051	0.967	-0.612	0.300	0.316	0.288
2016	1.785	4.165	0.797	-0.706	0.281	0.339	0.306
2017	2.123	4.165	0.788	-0.706	0.292	0.339	0.306
2018	2.291	5.948	0.658	-0.632	0.271	0.290	0.284
2019	2.070	5.948	0.504	-0.632	0.226	0.290	0.247

Table: Estimated Parameters (CSI 300)

CSI 300 - Estimated Parameters (Con't)

	p_b^*	p_s^*	p_b^u	p_s^u
2011	0.726	0.501	0.726	0.501
2012	0.727	0.499	0.727	0.500
2013	0.727	0.499	0.727	0.499
2014	0.608	0.366	0.637	0.399
2015	0.608	0.366	0.608	0.366
2016	0.696	0.469	0.674	0.443
2017	0.694	0.468	0.694	0.468
2018	0.708	0.483	0.705	0.479
2019	0.748	0.531	0.738	0.519

Table: Buy & Sell Region (CSI 300)

TF Strategy on CSI300

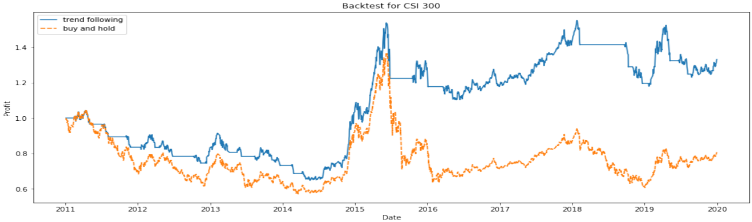
Estimation	TF	Sharp Ratio	BH	10y Bonds	No. Trade
Rolling	42.2%	0.030	-19.2%	36.96%	33
Fixing	8.47%	-0.151	-19.2%	36.96%	43

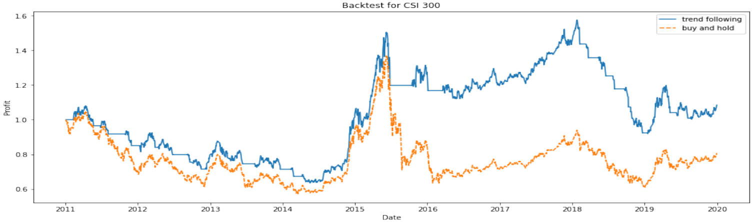
Table: TF Strategy Return (CSI 300)

Remark

- TF Strategy outperforms BH Strategy.
- Compared to 10-year bond's return and Sharp Ratio, the TF strategy doesn't perform that well.

TF Strategy on CSI300

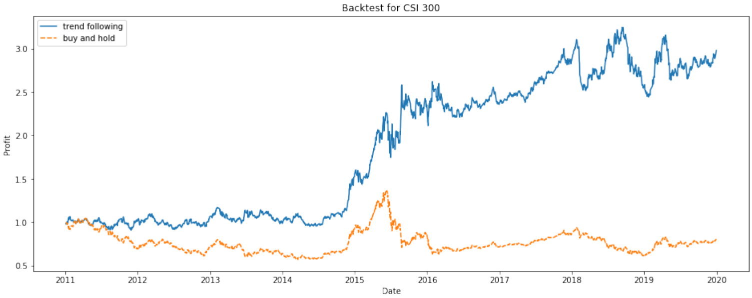




TF Strategy on CSI300 (Short)

Long & Short

TF (short)	SR (short)	TF (flat)	SR (flat)	BH	10y Bonds
197.8%	0.410	42.2%	0.018	-19.2%	25.856%



Sensitivity Analysis

- Do sensitivity analysis on these **six parameters**: $[\lambda_1, \lambda_2, \mu_1, \mu_2, \sigma_1, \sigma_2]$ respectively. Control the other five parameters stable when one is changed.
- Use estimated parameters in 2011 for CSI 300 and estimated parameters showed before for SPY, as the basic value, and the **degree of change** are: $[-15\%, -10\%, -5\%, 5\%, 10\%, 15\%]$
- Observe the effect of parameter changes on the conditional possibility p_b^* and p_s^*.
- Observe the effect of parameter changes on the strategy.

Sensitivity of λ on SPY

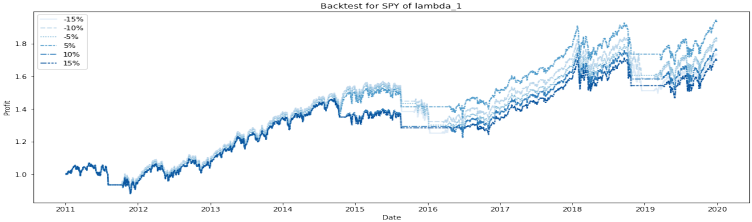
degree	-15%	-10%	-5%	5%	10%	15%
p_b^*	0.9265	0.927	0.927	0.9275	0.9275	0.9275
p_s^*	0.8015	0.802	0.802	0.8025	0.803	0.803
$p_b^*\_per$	-0.054%	0%	0%	0.054%	0.054%	0.054%
$p_s^*\_per$	-0.125%	-0.062%	-0.062%	0%	0.062%	0.062%

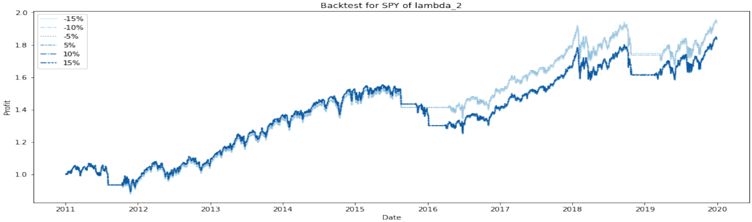
Table: Sensitivity of λ_1

degree	-15%	-10%	-5%	5%	10%	15%
p_b^*	0.9275	0.927	0.927	0.927	0.927	0.927
p_s^*	0.8025	0.8025	0.8025	0.802	0.802	0.802
$p_b^*\_per$	0.054%	0%	0%	0%	0%	0%
$p_s^*\_per$	0%	0%	0%	-0.062%	-0.062%	-0.062%

Table: Sensitivity of λ_2

Sensitivity Backtest of λ on SPY





Sensitivity of μ on SPY

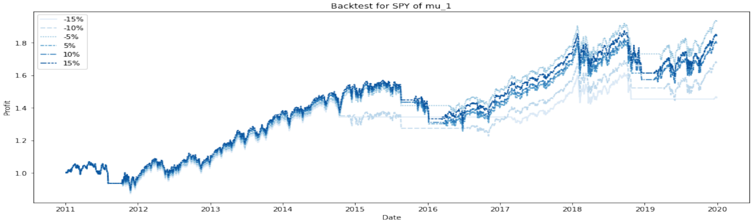
degree	-15%	-10%	-5%	5%	10%	15%
p_b^*	0.9485	0.9415	0.934	0.9205	0.9135	0.907
p_s^*	0.843	0.829	0.8155	0.79	0.7785	0.767
$p_b^*\_per$	2.319%	1.564%	0.755%	-0.701%	-1.456%	-2.157%
$p_s^*\_per$	5.047%	3.302%	1.620%	-1.558%	-2.991%	-4.424%

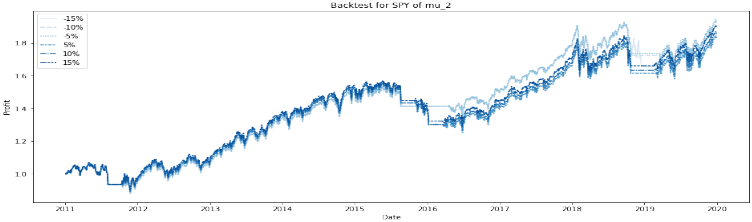
Table: Sensitivity of μ_1

degree	-15%	-10%	-5%	5%	10%	15%
p_b^*	0.914	0.919	0.923	0.9305	0.934	0.937
p_s^*	0.7865	0.7925	0.7975	0.8065	0.811	0.8145
$p_b^*\_per$	-1.402%	-0.863%	-0.431%	0.378%	0.755%	1.079%
$p_s^*\_per$	-1.994%	-1.246%	-0.623%	0.498%	1.059%	1.495%

Table: Sensitivity of μ_2

Sensitivity Backtest of μ on SPY





Sensitivity of σ on SPY

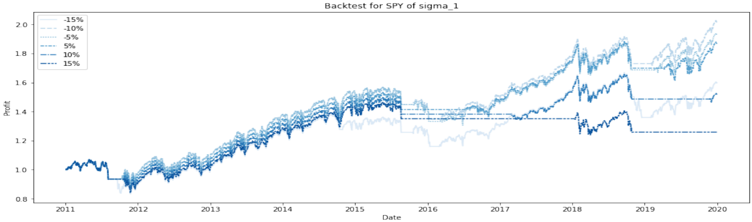
degree	-15%	-10%	-5%	5%	10%	15%
p_b^*	0.924	0.9245	0.9255	0.929	0.931	0.933
p_s^*	0.8025	0.8025	0.8025	0.8025	0.8025	0.8025
$p_b^*\_per$	-0.324%	-0.270%	-0.162%	0.216%	0.431%	0.647%
$p_s^*\_per$	0%	0%	0%	0%	0%	0%

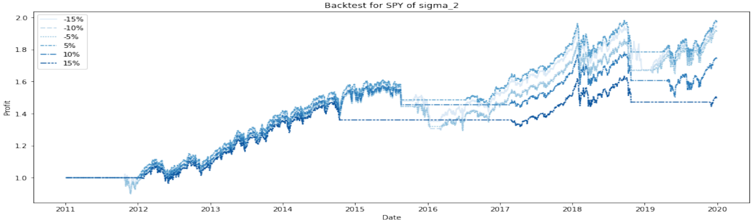
Table: Sensitivity of σ_1

degree	-15%	-10%	-5%	5%	10%	15%
p_b^*	0.927	0.927	0.927	0.927	0.927	0.927
p_s^*	0.779	0.787	0.7945	0.8105	0.8185	0.8265
$p_b^*\_per$	0%	0%	0%	0%	0%	0%
$p_s^*\_per$	-2.928%	-1.931%	-0.997%	0.997%	1.994%	2.991%

Table: Sensitivity of σ_2

Sensitivity Backtest of σ on SPY





Sensitivity of λ on CSI 300

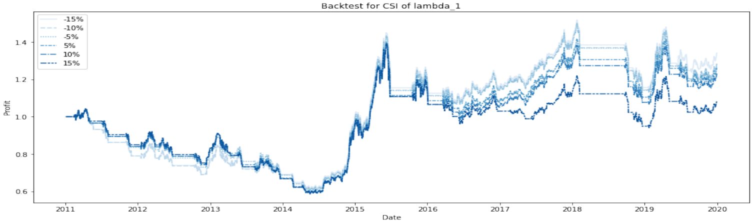
degree	-15%	-10%	-5%	5%	10%	15%
p_b^*	0.725	0.7255	0.7255	0.726	0.726	0.726
p_s^*	0.5005	0.5005	0.501	0.501	0.5015	0.5015
p_b^*_per	-0.069%	0%	0%	0.069%	0.069%	0.069%
p_s^*_per	-0.100%	-0.100%	0%	0%	0.100%	0.100%

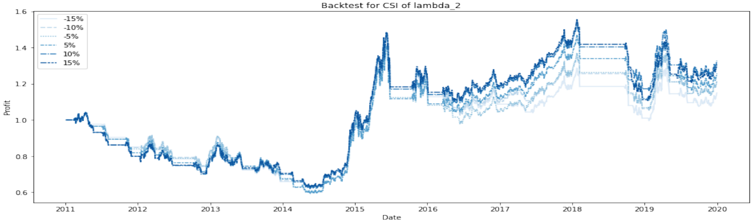
Table: Sensitivity of λ_1

degree	-15%	-10%	-5%	5%	10%	15%
p_b^*	0.7265	0.726	0.726	0.7255	0.725	0.725
p_s^*	0.502	0.502	0.5015	0.5005	0.5005	0.5
p_b^*_per	0.138%	0.069%	0.069%	0%	-0.069%	-0.069%
p_s^*_per	0.200%	0.200%	0.100%	-0.100%	-0.100%	-0.200%

Table: Sensitivity of λ_2

Sensitivity Backtest of λ on CSI 300





Sensitivity of μ on CSI 300

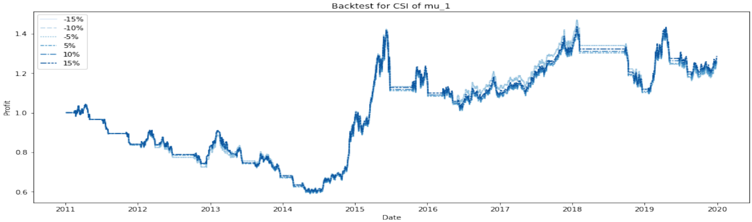
degree	-15%	-10%	-5%	5%	10%	15%
p_b^*	0.751	0.742	0.7335	0.718	0.7115	0.705
p_s^*	0.5445	0.5295	0.515	0.488	0.4755	0.464
$p_b^*\_per$	3.515%	2.274%	1.103%	-1.034%	-1.930%	-2.826%
$p_s^*\_per$	8.683%	5.689%	2.794%	-2.595%	-5.090%	-7.385%

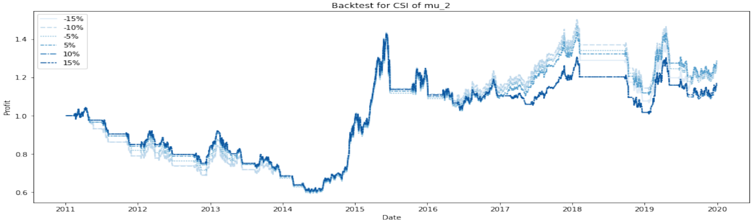
Table: Sensitivity of μ_1

degree	-15%	-10%	-5%	5%	10%	15%
p_b^*	0.692	0.704	0.715	0.7355	0.7445	0.753
p_s^*	0.467	0.479	0.4905	0.511	0.5205	0.5295
$p_b^*\_per$	-4.618%	-2.963%	-1.447%	1.378%	2.619%	3.790%
$p_s^*\_per$	-6.786%	-4.391%	-2.096%	1.996%	3.892%	5.689%

Table: Sensitivity of μ_2

Sensitivity Backtest of μ on CSI 300





Sensitivity of σ on CSI 300

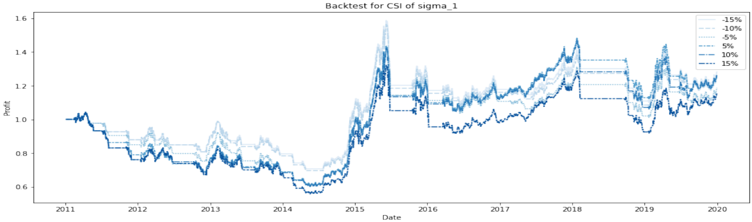
degree	-15%	-10%	-5%	5%	10%	15%
p_b^*	0.748	0.7395	0.732	0.72	0.715	0.7105
p_s^*	0.501	0.501	0.501	0.501	0.501	0.501
$p_b^*\_per$	3.101%	1.930%	0.896%	-0.758%	-1.447%	-2.068%
$p_s^*\_per$	0%	0%	0%	0%	0%	0%

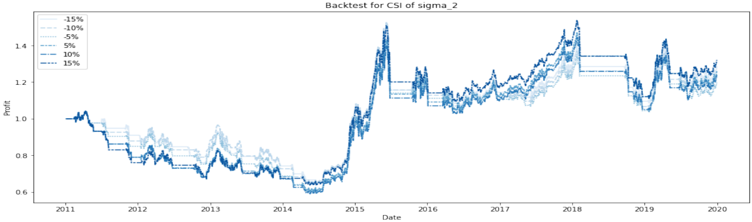
Table: Sensitivity of σ_1

degree	-15%	-10%	-5%	5%	10%	15%
p_b^*	0.7255	0.7255	0.7255	0.7255	0.7255	0.7255
p_s^*	0.4895	0.493	0.497	0.5055	0.5105	0.5155
$p_b^*\_per$	0%	0%	0%	0%	0%	0%
$p_s^*\_per$	-2.295%	-1.597%	-0.798%	0.898%	1.896%	2.894%

Table: Sensitivity of σ_2

Sensitivity Backtest of σ on CSI 300





Conclusion

Limitation

- **Data Limitation:** CSI 300 was released in 2005, there is no enough data for us to do parameters estimation.
- **Parameter Estimation:** The frequency of estimation is not high enough, maybe rolling parameters based on month would be better.
- **Reaction Speed to Market:** Compared with market, every switching between bull and bear is not quick enough in our strategy.

Conclusion

Contribution

- We implement a trend following strategy. The strategy helps to identify bull and bear market and then the investor can **avoid the plunge**, which make our strategy performed better than buy-and-hold.
- The **comparison** of backtest on **SPY and CSI 300** helps to realize that trend following strategy has different effects in mature and immature markets. In mature market, which is more sensitive with parameters, maybe the frequency of parameters estimation need to be higher.

Conclusion

Thank You for Watching!