MA 17: HOW TO SOLVE IT
HANDOUT 7: DISCUSSIONS OF PSET 6 AND SOME FURTHER STRATEGIES

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Selected problems from PSet 6:

Problem 1. Compute $\sum_{k=0}^{\infty} \frac{2^k}{5^{2^k}+1}$.

Problem 4. Prove that $\int_0^1 x^{-x} dx = \sum_{n=1}^{\infty} n^{-n}$.

Problem 5. A rectangle in $\mathbb{R}^2$ is called *great* if either its width or its height is an integer. Prove that if a rectangle X can be dissected into great rectangles, then the rectangle X is itself great.

Problem 7. Show that

$$\frac{1}{(n-1)!} \int_n^{\infty} w(t) e^{-t}\, dt < \frac{1}{(e-1)^n},$$

where t is real, n is a positive integer, and $w(t) = (t-1)(t-2)\cdots(t-n+1)$.

For the next problems, try to use symmetry.

Problem 1. Choose n points at random (uniformly and independently) on the circumference of a circle. Find the probability p_n that all the points lie on a semicircle. (For instance, $p_1 = p_2 = 1$.)

Problem 2. Passengers $P_1, \ldots, P_n$ enter a plane with n seats. Each passenger has a different assigned seat. The first passenger sits in the wrong seat. Thereafter, each passenger either sits in their own seat if it is unoccupied, or otherwise chooses a random unoccupied seat. What is the probability that the last passenger sits in his or her own seat?

Problem 3. There are n parking spaces $1, 2, \ldots, n$ (in that order) on a one-way street. Cars $C_1, \ldots, C_n$ enter the street in that order and try to park. Each car C_i has a preferred space a_i. A car will drive to its preferred space and try to park there. If the space is already occupied, the car will park in the next available space. If the car must leave the street without parking, then the process fails. If $\alpha = (a_1, \ldots, a_n)$ is a sequence of preferences that allows every car to park, then we call α a *parking function*. For instance, there are 16 parking functions of length 3, given by (abbreviating $(1,1,1)$ as 111, etc.)

$$111, 112, 121, 211, 113, 131, 311, 122, 212, 221, 123, 132, 213, 231, 312, 321.$$

Show that the number of parking functions of length n is equal to $(n+1)^{n-1}$.

Some Putnam problems of recent years:

Problem 4. (Putnam 2024, A2)
For which real polynomials p is there a real polynomial q such that

$$p(p(x)) - x = (p(x) - x)^2 q(x)$$

for all real x?

Problem 5. (Putnam 2024, A5)
Consider a circle Ω with radius 9 and center at the origin $(0,0)$, and a disc Δ with radius 1 and center at $(r,0)$, where $0 \leq r \leq 8$. Two points P and Q are chosen independently and uniformly at random on Ω. Which value(s) of r minimize the probability that the chord $\overline{PQ}$ intersects Δ?

Problem 6. (Putnam 2024, B2)
Two convex quadrilaterals are called *partners* if they have three vertices in common and they can be labeled $ABCD$ and $ABCE$ so that E is the reflection of D across the perpendicular bisector of the diagonal $\overline{AC}$. Is there an infinite sequence of convex quadrilaterals such that each quadrilateral is a partner of its successor and no two elements of the sequence are congruent?

Problem 7. (Putnam 2024, B3)
Let r_n be the nth smallest positive solution to $\tan x = x$, where the argument of tangent is in radians. Prove that for $n \geq 1$,

$$0 < r_{n+1} - r_n - \pi < \frac{1}{(n^2+n)\pi}.$$

Problem 8. (Putnam 2023, A1)
For a positive integer n, let $f_n(x) = \cos(x)\cos(2x)\cos(3x)\cdots\cos(nx)$. Find the smallest n such that $|f_n''(0)| > 2023$.

Problem 9. (Putnam 2023, A2)
Let n be an even positive integer. Let p be a monic, real polynomial of degree $2n$; that is to say, $p(x) = x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_1 x + a_0$ for some real coefficients $a_0, \ldots, a_{2n-1}$. Suppose that $p(1/k) = k^2$ for all integers k such that $1 \leq |k| \leq n$. Find all other real numbers x for which $p(1/x) = x^2$.

Problem 10. (Putnam 2023, B1)
Consider an m-by-n grid of unit squares, indexed by (i,j) with $1 \leq i \leq m$ and $1 \leq j \leq n$. There are $(m-1)(n-1)$ coins, which are initially placed in the squares (i,j) with $1 \leq i \leq m-1$ and $1 \leq j \leq n-1$. If a coin occupies the square (i,j) with $i \leq m-1$ and $j \leq n-1$ and the squares $(i+1,j), (i,j+1)$, and $(i+1,j+1)$ are unoccupied, then a legal move is to slide the coin from (i,j) to $(i+1,j+1)$. How many distinct configurations of coins can be reached starting from the initial configuration by a (possibly empty) sequence of legal moves?

Problem 11. (Putnam 2023, A3)
Determine the smallest positive real number r such that there exist differentiable functions $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ satisfying

(a) $f(0) > 0$,
(b) $g(0) = 0$,
(c) $|f'(x)| \leq |g(x)|$ for all x,
(d) $|g'(x)| \leq |f(x)|$ for all x, and
(e) $f(r) = 0$.