MA 17: HOW TO SOLVE IT
PROBLEM SET 2

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Due: 11:59PM of October 29. Late submissions will not be accepted.

Instructions:

- *Please submit solutions to at most three problems, written carefully. You are encouraged to work on more, but the TA will provide feedback on only three. I recommend selecting the three for which you most want feedback.*
- *Collaboration on homework before you have solved at least three problems independently is discouraged, though not strictly forbidden. You should make a serious effort to work through each problem independently before turning to classmates, the internet, or tools like ChatGPT.*
- *As long as your submission clearly shows attempts on at least three problems, you will earn 8 points toward the final grade. The correctness of your solutions will not affect your score, although feedback will be provided.*
- *Submission should be made through Canvas.*

Problem 1.
Let $x_0, x_1, x_2, \ldots$ be the sequence such that $x_0 = 1$ and for $n \geq 0$,

$$x_{n+1} = \ln(e^{x_n} - x_n)$$

(as usual, the function ln is the natural logarithm). Show that the infinite series

$$x_0 + x_1 + x_2 + \cdots$$

converges and find its sum.
Be sure to write your proof carefully.

Problem 2.
Let $Q_0(x) = 1$, $Q_1(x) = x$, and

$$Q_n(x) = \frac{(Q_{n-1}(x))^2 - 1}{Q_{n-2}(x)}$$

for all $n \geq 2$. Show that, whenever n is a positive integer, $Q_n(x)$ is equal to a polynomial with integer coefficients.

Hint: Consider the roots.

Problem 3.
Let a_0, b_0, c_0 be real numbers. Define the sequences $(a_n)_n$, $(b_n)_n$, $(c_n)_n$ recursively by

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \frac{b_n + c_n}{2}, \quad c_{n+1} = \frac{c_n + a_n}{2}, \quad \text{for } n \geq 0.$$

Prove that the sequences are convergent and find their limits.

Problem 4.
Let $\times$ represent the cross product in $\mathbb{R}^3$. For what positive integers n does there exist a set $S \subset \mathbb{R}^3$ with exactly n elements such that

$$S = \{\mathbf{v} \times \mathbf{w} : \mathbf{v}, \mathbf{w} \in S\}?$$

Cross product: Let $\mathbf{a} = (a_1, a_2, a_3)$ *and* $\mathbf{b} = (b_1, b_2, b_3)$ *be vectors in* $\mathbb{R}^3$*. Their cross product is given by:*

$$\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1).$$

Problem 5.
Given a real number a, we define a sequence by $x_0 = 1$, $x_1 = x_2 = a$, and $x_{n+1} = 2x_n x_{n-1} - x_{n-2}$ for $n \geq 2$. Prove that if $x_n = 0$ for some n, then the sequence is periodic.

Hint: Show that if $x_n = 0$ for some n, necessarily $|a| < 1$; then write $a = \cos(\theta)$.

Problem 6.
There are n markers, each with one side white and the other side black, aligned in a row with their white sides up. At each step, if possible, we choose a marker with the white side up (but not one of the outermost markers), remove it, and reverse the two neighboring markers. Prove that one can reach a configuration with only two markers left if and only if $n - 1$ is not divisible by 3.

Hint: Find the appropriate invariant.

Reminder No problem session this week. Next Class: Oct 28, 7PM-9PM.