Concentration.

Sunday, March 30, 2025 5:56 PM

We have seen Chebyshev's inequality:

$$\mathbb{P}[\,|X-\mathbb{E}X|>\beta\,] \le \frac{\operatorname{Var}X}{\beta^2}$$

- Today: more concentration inequalities.
  - Chernoff - Cramér bound $\Rightarrow$ large-deviation
  - Bernstein inequality / Hoeffding inequality (sum of independent)
  - Azuma - Hoeffding inequality / bounded difference estimates (Martingale)

Applications:
- norm bound of random matrices
- first-passage percolation
- chromatic number of graph.

- Chernoff - Cramér: $\mathbb{P}[X\ge\beta] \le e^{-s\beta}\,\underline{\mathbb{E}[e^{sX}]}$, $\forall s$

  moment generating function (MGF) $= \sum_{k=0}^{\infty} \frac{s^k \mathbb{E}[X^k]}{k!}$

  proof: Markov inequality.

- Useful for sum of independent:

  MGF factors: $\mathbb{E}[e^{s\sum_{i=1}^m X_i}] = \prod_{i=1}^m \mathbb{E}[e^{sX_i}]$

- Example: Large deviation

  $\Lambda(s) = \log \mathbb{E}_\mu[e^{sX}]$ cumulant-generating function $\left[\begin{array}{l}\Lambda \text{ convex} \\ \Lambda_* \text{ convex}\end{array}\right]$

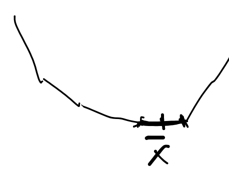
  Legendre transform: $\Lambda_*(x) = \sup_s\, sx-\Lambda(s)$ $\;\to\; \ge 0$ since $\Lambda(0)=0$ $\quad(\bar{X}=\mathbb{E}X)$

  $\Lambda_*(\bar{x})=0$, since $\Lambda(s) = \log \mathbb{E}_\mu[e^{sX}] \ge \mathbb{E}_\mu[\log e^{sX}] = s\bar{x}$

  $A_m = \frac{1}{m}(X_1+\cdots+X_m)$, iid $\sim\mu$

  (Cramér) $-\frac{1}{m}\log(\mathbb{P}[A_m\in I]) \to \inf_{x\in I} \Lambda_*(x)$ as $m\to\infty$

<u>proof</u>. Upper bound: $\mathbb{P}[A_m\in I] \le \exp(-m\inf_{x\in I}\Lambda_*(x))$



If $\bar{x}\in I$, obvious

Assume $\inf I = x_- > \bar{x}$

$$\mathbb{P}[A_m \ge x_-] \le \exp(-smx_- + m\Lambda(s)),\ \forall s$$
$$\le \exp(-m\Lambda_*(x_-))$$

Lower bound: $\mathbb{P}[A_m\in I] \gtrsim \exp(-m\Lambda_*(x))$ for any $x\in I$.

Change of measure: $\frac{\tilde{\mu}}{\mu}(x) = \exp(sx-\Lambda(s))$

$$\mathbb{P}[A_m\in(x-\varepsilon,x+\varepsilon)] \approx \mathbb{E}\left[\exp(\underbrace{-m\tilde{A}_m s+m\Lambda(s)})\,\mathbb{1}[\tilde{A}_m\in(x-\varepsilon,x+\varepsilon)]\right]$$
$$\approx \exp(-m(sx-\Lambda(s)))\,\mathbb{P}[\tilde{A}_m\in(x-\varepsilon,x+\varepsilon)]$$

choose s such that $\mathbb{P}[\tilde{A}_m\in(x-\varepsilon,x+\varepsilon)]\to 1$ $\quad(\mathbb{E}_{\tilde{\mu}}X = x)$

- Some concentration inequalities

  Take $X_1,\dots,X_n$ independent. with $\mu_i=\mathbb{E}X_i$, $\sigma_i^2=\operatorname{Var}(X_i)$ and $|X_i-\mu_i|\le c_i$ $\quad c=\max c_i$

  For $T=X_1+\cdots+X_n$. $\Rightarrow \mathbb{P}[T-\mathbb{E}T>\beta] < \begin{cases}\exp(-\beta^2/4\sum_{i=1}^n\sigma_i^2) & 0\le\beta\le\frac{\sum\sigma_i^2}{c} \\ \exp(-\beta/4c) & \beta>\frac{\sum\sigma_i^2}{c}\end{cases}$ $\Big/\ \exp\left(-\frac{\beta^2}{2\sum_{i=1}^n\sigma_i^2+2c\beta/3}\right)$

  (Bernstein)

  $$\mathbb{P}[T-\mathbb{E}T>\beta] < \exp\left(-2\beta^2/\sum_{i=1}^n c_i^2\right)$$

  (Hoeffding) $\qquad$ (Both for sub-Gaussian as well)

<u>proof</u> Using Chernoff-Cramér; need: compute MGF

(Bernstein) $\mathbb{E}\exp(s(X_i-\mu_i))$

$$= \sum_{k=0}^{\infty}\frac{s^k\mathbb{E}(X_i-\mu_i)^k}{k!} \le 1+\sum_{k=2}^{\infty}\frac{s^k}{k!}\sigma_i^2\cdot c^{k-2} \le 1+\frac{s^2\sigma_i^2}{2}+\frac{s^3\sigma_i^2}{6}\sum_{k=3}^{\infty}(sc)^{k-2} = 1+\frac{s^2\sigma_i^2}{2}\left(1+\frac{1}{3}\frac{sc}{1-sc}\right) \le 1+s^2\sigma_i^2 \le \exp(s^2\sigma_i^2)$$

$(\mathbb{E}(X_i-\mu_i)^k \le \mathbb{E}(X_i-\mu_i)^2\cdot c^{k-2})$ $\qquad$ when $s\le\frac{1}{2c}$

(Hoeffding) $\mathbb{E}\exp(s(X_i-\mu_i)) \le \frac{1}{2}(e^{c_is}+e^{-c_is}) \le e^{2c_i^2s^2}$

- Operator norm for random matrices

$X=(X_{ij})$ $m\times n$ random matrix,

$$\|X\|_2 = \sup_{\substack{\|u\|_2=\|v\|_2=1\\ u\in\mathbb{R}^m,\ v\in\mathbb{R}^n}} u^TXv$$

Suppose X_{ij} iid rademacher $(\mathbb{P}[X_{ij}=1]=\mathbb{P}[X_{ij}=-1]=\frac{1}{2})$

<u>Claim</u>. $\mathbb{P}[\|X\|_2\ge C(\sqrt{m}+\sqrt{n}+t)] < e^{-t^2}$

($C>0$ is some constant)

<u>proof</u>. ① Bound u^TXv for fixed u,v

$u^TXv = \sum_{i=1}^m\sum_{j=1}^n u_iv_jX_{ij}$

$$\mathbb{P}[u^TXv \ge C(\sqrt{m}+\sqrt{n}+t)] \underset{\text{(Hoeffding)}}{<} \exp\left(-\frac{2C^2(\sqrt{m}+\sqrt{n}+t)^2}{\sum_{i=1}^m\sum_{j=1}^n u_i^2v_j^2}\right) = \exp(-2C^2(\sqrt{m}+\sqrt{n}+t)^2) \le \exp(-2C^2(m+n+t^2))$$

② For S^{m-1}

(i.e. $\{u\in\mathbb{R}^m: \|u\|_2=1\}$) $\qquad$ ($\frac{1}{4}$-net)

$\exists\ P_m\subseteq S^{m-1}$, $|P_m|=12^m$, $\bigcup_{x\in P_m} B_{\frac{1}{4}}(x) \supseteq S^{m-1}$

$\Rightarrow \mathbb{P}[\max_{\substack{u\in P_m\\ v\in P_n}} u^TXv \ge C(\sqrt{m}+\sqrt{n}+t)] < e^{-t^2}$ (union bound)

③ $\forall\, u\in P_m$, $v\in P_n$, $\tilde{u}\in S^{m-1}$, $\tilde{v}\in S^{n-1}$, $\|u-\tilde{u}\|_2\le\frac{1}{4}$, $\|v-\tilde{v}\|_2\le\frac{1}{4}$,

$$|u^TXv-\tilde{u}^TX\tilde{v}| \le |(u^T-\tilde{u}^T)Xv| + |\tilde{u}^TX(v-\tilde{v})|$$
$$\le \frac{1}{4}\|X\|_2+\frac{1}{4}\|X\|_2 = \frac{1}{2}\|X\|_2$$

$\Rightarrow$ $\tilde{u}^TX\tilde{v} \le \frac{1}{2}\|X\|_2 + C(\sqrt{m}+\sqrt{n}+t)$, with prob $>1-e^{-t^2}$

$\Rightarrow$ $\|X\|_2 \le 2C(\sqrt{m}+\sqrt{n}+t)$, with prob $>1-e^{-t^2}$

Martingale concentration

Recall: Martingale $Z_1,Z_2,\dots,Z_n,\dots$

s.t. $\mathbb{E}|Z_n|<\infty$

$\mathbb{E}[Z_{n+1}\mid Z_1,\dots,Z_n]=Z_n$ $\qquad$ (A_i, B_i can dependend on $X_1,\dots,X_{i-1}$) (but c_i is constant)

Azuma / Azuma-Hoeffding ineq: $A_i\le Z_i-Z_{i-1}\le B_i$, $\quad B_i-A_i\le c_i$

$$\mathbb{P}[\max_{0\le i\le n} Z_i-Z_0\ge\beta] \le \exp\left(-\frac{2\beta^2}{\sum_{i=1}^n c_i^2}\right)$$

<u>proof</u> $\mathbb{P}[\max_{0\le i\le n} Z_i-Z_0\ge\beta]$

$\le \mathbb{P}[\max_{0\le i\le n}\exp(s(Z_i-Z_0))\ge e^{s\beta}]$

$\le \mathbb{E}[\exp(s(Z_\tau-Z_0))]\,e^{-s\beta}$

$\tau := \min\{t: Z_t-Z_0\ge\beta\}$ or n $\qquad$ (stopping time)

$\le \mathbb{E}[\exp(s(Z_n-Z_0))]\,e^{-s\beta} = \mathbb{E}\left[\exp\left(s\sum_{i=1}^n(Z_i-Z_{i-1})\right)\right]e^{-s\beta}$

Then Chernoff-Cramér.

- Bounded difference estimates.

  $X_1, X_2,\dots,X_n$ independent

  each $X_i\in\Omega_i$

  $f:\Omega_1\times\cdots\times\Omega_n\to\mathbb{R}$

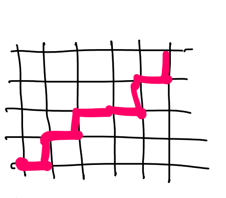
  $$D_i = \max_{\substack{x_1\in\Omega_1,\dots,x_n\in\Omega_n\\ \tilde{x}_i\in\Omega_i}} |f(x_1,\dots,x_i,\dots,x_n)-f(x_1,\dots,\tilde{x}_i,\dots,x_n)|$$

  Let $Z_i=\mathbb{E}[f(X_1,\dots,X_n)\mid X_1,\dots,X_i]$ $\qquad$ (McDiarmid's ineq)

  $\Rightarrow A_i\le Z_i-Z_{i-1}\le B_i$, $\quad B_i-A_i\le D_i$

  $\Rightarrow \mathbb{P}[f(X_1,\dots,X_n)-\mathbb{E}f(X_1,\dots,X_n)\ge\beta] \le \exp\left(-\frac{2\beta^2}{D_1^2+\cdots+D_n^2}\right)$

- Example: $\frac{\text{First-}}{\text{Last}}$ Passage Percolation



$w_e\sim\text{Unif}[0,1]$ $\forall e\in E$ (edge weights)

$T_n = \min_\gamma \sum_\gamma w_e$

γ: up-right path from $(1,1)$ to (n,n)

scaling limit of T_n as $n\to\infty$?

$T_{n+m}\le T_n+T_m$ $\qquad$ (sub-additivity)

$\Rightarrow \mathbb{E}T_{n+m}\le\mathbb{E}T_n+\mathbb{E}T_m$, $\quad \frac{1}{n}\mathbb{E}T_n$ converges as $n\to\infty$

(Exercise: $\lim_{n\to\infty}\frac{1}{n}\mathbb{E}T_n>0$, by large deviation)

By bounded difference estimates,

(X_i: w_e for e between $x+y=i-1$ and $x+y=i$)

$\mathbb{P}[T_n-\mathbb{E}T_n\ge\beta] \le \exp\left(-\frac{\beta^2}{4n}\right)$ $\qquad$ $(D_1=D_2=\cdots=D_{2n}=1)$

(In particular, $\operatorname{var}(T_n)$ is $O(n)$)

(KPZ universality conjecture: $\operatorname{var}(T_n)$ is of order $O(n^{\frac{2}{3}})$)

(state of art: $\underset{\text{(Newman-Piza)}}{c\log(n)} < \operatorname{Var}(T_n) < \underset{\text{(Benjamini-Kalai-Schramm)}}{\frac{cn}{\log(n)}}$)

Example. • Pattern matching

$X_1,X_2,\dots,X_n$, iid, each uniform from $\{1,\dots,s\}$

For $a=(a_1,\dots,a_k)\in\{1,\dots,s\}^k$

N_n: # i such that $(X_i,\dots,X_{i+k-1})=(a_1,\dots,a_k)$

$\mathbb{E}N_n=(n-k+1)s^{-k}$

$$\mathbb{P}[|N_n-(n-k+1)s^{-k}|\ge bk\sqrt{n}] \le 2\exp\left(-\frac{2b^2k^2n}{k^2n}\right) \le 2e^{-2b^2}$$

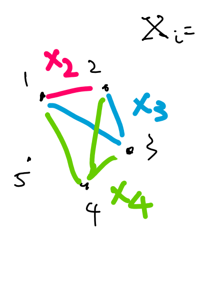
$\downarrow$ $D_i\le k$

- Chromatic number

  Take Erdős-Rényi graph $G(n,p)$

  χ: minimum number of colors to properly color $G(n,p)$

  $X_i := \{\mathbb{1}[(i,j)\in G(n,p)]: 1\le j\le i\}$



Claim: change X_i alters χ by ≤ 1

$(D_i\le 1)$

$$\Rightarrow \mathbb{P}[|\chi-\mathbb{E}\chi|\ge b\sqrt{n-1}] \le 2\exp\left(-\frac{2b^2(n-1)}{n-1}\right) = 2e^{-2b^2}$$