MA 17: HOW TO SOLVE IT
PROBLEM SET 4

Instructor Lingfu Zhang (lingfuz@caltech) **Office** Linde Hall 358
TA Minghao Pan (mpan2@caltech)

Due: 11:59PM of November 5. Late submissions will not be accepted.

Instructions:

- *Please submit solutions to at most three problems, written carefully. You are encouraged to work on more, but the TA will provide feedback on only three. I recommend selecting the three for which you most want feedback.*
- *Collaboration on homework before you have solved at least three problems independently is discouraged, though not strictly forbidden. You should make a serious effort to work through each problem independently before turning to classmates, the internet, or tools like ChatGPT.*
- *As long as your submission clearly shows attempts on at least three problems, you will earn 8 points toward the final grade. The correctness of your solutions will not affect your score, although feedback will be provided.*
- *Submission should be made through Canvas.*

Problem 1.
Let $p_n(k)$ be the number of permutations of $\{1, 2, \ldots, n\}$ which have exactly k fixed points. Prove that

$$\sum_{k=0}^{n} k p_n(k) = n!$$

Problem 2.
Two real numbers x and y are chosen at random in the interval (0,1) with respect to the uniform distribution. What is the probability that the closest integer to x/y is even? Give the answer in the form $r + s\pi$, where r and s are rational numbers.

Hint: you may use the Taylor series expansion of the arctangent function.

Problem 3.
If a needle of length 1 is dropped at random on a surface ruled with parallel lines at distance 2 apart, what is the probability that the needle will cross one of the lines?

Problem 4.
Two airplanes are supposed to park at the same gate of a concourse. The arrival times of the airplanes are independent and randomly distributed throughout the 24 hours of the day. What is the probability that both can park at the gate, provided that the first to arrive will stay for a period of two hours, while the second can wait behind it for a period of one hour?

Problem 5.
Let C be the unit circle $x^2 + y^2 = 1$. A point p is chosen randomly on the circumference of C and

another point q is chosen randomly from the interior of C (these points are chosen independently and uniformly over their domains). Let R be the rectangle with sides parallel to the x- and y-axes with diagonal pq. What is the probability that no point of R lies outside of C?

Problem 6.
Let n be a positive integer. Define a sequence $(a_i)_{i\geq 1}$ as follows: set $a_1 = n$, and for each $i \geq 1$, given a_i, choose an integer X_i uniformly at random from $\{1, 2, \ldots, a_i\}$. If $X_i = a_i$, the process stops; otherwise, set $a_{i+1} = a_i - X_i$.

Let I_n denote the (random) index i at which the process stops. Prove that

$$\frac{\mathbb{E}[I_n]}{\log n} \to 1 \quad \text{as } n \to \infty.$$

Problem 7.
Let k be a positive integer. Suppose that the integers $1, 2, 3, \ldots, 3k+1$ are written down in random order. What is the probability that at no time during this process, the sum of the integers that have been written up to that time is a positive integer divisible by 3? Your answer should be in closed form, but may include factorials.

Problem 8.
Suppose X is a random variable that takes on only nonnegative integer values, with $\mathbb{E}[X] = 1$, $\mathbb{E}\left[X^2\right] = 2$, and $\mathbb{E}\left[X^3\right] = 5$. (Here $\mathbb{E}[Y]$ denotes the expectation of the random variable Y.) Determine the smallest possible value of the probability of the event $X = 0$.

Problem 9.
The points $1, 2, \ldots, 1000$ are paired up at random to form 500 intervals $[i, j]$. What is the probability that among these intervals is one which intersects all the others?

Reminder Problem session: Oct 30, 7PM to 8PM (to discuss PSet 3). Next Class: Nov 4, 7PM-10PM. (Mock Putnam 2; you will be free to leave early, and there will be snacks.)