MA 17: HOW TO SOLVE IT
MOCK PUTNAM 2

B1. Prove that at least one integer in any set of ten consecutive integers is relatively prime to the others in the set.

B2. Let M be a finite set of points in the plane. Prove that it is possible to color the points of M red or blue in such a way that, on every line parallel to the coordinate axes, the number of red points and the number of blue points differ by at most 1.

B3. At a career fair, $2n$ students and $2n$ employers stand on two concentric circles, each consisting of $2n$ people. Each circle may contain a mix of students and employers.

The interaction rules are as follows:

- Each person in the outer circle faces a person in the inner circle directly across from them.
- If a facing pair consists of one student and one employer, they conduct a quick interview.
- Every 10 minutes, each person in the outer circle moves one position clockwise, while the inner circle remains fixed.

Prove that at some time during this process, at least n interviews take place simultaneously.

B4. Let a, b, c, d, e, f be independent random variables uniformly distributed over $\{1, 2, \ldots, 100\}$. Define

$$M = a + 2b + 4c + 8d + 16e + 32f.$$

What is the expected value of the remainder when M is divided by 64?

B5. Consider n locked boxes. Each box has a unique key, and each key is randomly placed in one of the boxes (it is possible that a box contains more than one key, or no key at all). For $1 \leq k < n$, we randomly choose k boxes and break them open. What is the probability that the remaining $n - k$ boxes can be opened using keys?

B6. Let A be an $n \times n$ matrix of real numbers for some $n \geq 1$. For each positive integer k, let $A^{[k]}$ be the matrix obtained by raising each entry to the kth power. Show that if $A^k = A^{[k]}$ for $k = 1, 2, \ldots, n+1$, then $A^k = A^{[k]}$ for all $k \geq 1$.