

**MA 17: HOW TO SOLVE IT**  
**HANDOUT 5: DISCUSSIONS AFTER MOCK 2**

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*Elements in the theory of matrix: trace, determinant, eigenvalues, eigenvectors, characteristic equation/polynomial.*

**The Cayley–Hamilton Theorem.** Any  $n \times n$  matrix  $A$  satisfies its characteristic equation, which means that if  $P_A(\lambda) = \det(\lambda I_n - A)$ , then  $P_A(A) = O_n$ .

**The Perron–Frobenius Theorem.** Any square matrix with positive entries has a unique eigenvector with positive entries (up to multiplication by a positive scalar), and the corresponding eigenvalue has multiplicity one and is strictly greater than the absolute value of any other eigenvalue.

*Some more problems:*

**Problem 1.** (Putnam 1986, B6)

Suppose that  $A, B, C, D$  are  $n \times n$  matrices (with entries in some field), such that  $AB^T$  and  $CD^T$  are symmetric, and  $AD^T - BC^T = I$ . Prove that  $A^T D - C^T B = I$ .

*Hint: find a more “matricial” interpretation of the condition  $AD^T - BC^T = I$ .*

**Problem 2.** (Putnam 1995, A5)

Let  $x_1, x_2, \dots, x_n$  be differentiable (real-valued) functions of a single variable  $t$  which satisfy

$$\begin{aligned} \frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ \frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\vdots \\ \frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{aligned}$$

for some constants  $a_{ij} > 0$ . Suppose that for all  $i$ ,  $x_i(t) \rightarrow 0$  as  $t \rightarrow \infty$ . Are the functions  $x_1, x_2, \dots, x_n$  necessarily linearly dependent?

**Problem 3.** Consider bipartite graphs where the vertices on each side are labeled  $\{1, 2, \dots, n\}$ . Find the number of such graphs for which there are an odd number of perfect matchings.

**Problem 4.** (Putnam 1999, B5)

For an integer  $n \geq 3$ , let  $\theta = 2\pi/n$ . Evaluate the determinant of the  $n \times n$  matrix  $I + A$ , where  $I$  is the  $n \times n$  identity matrix and  $A = (a_{jk})$  has entries  $a_{jk} = \cos(j\theta + k\theta)$  for all  $j, k$ .

*Hint: find the eigenvectors, and use complex numbers.*

*Another problems on complex numbers:*

**Problem 5.** (Putnam 1991, B2)

Suppose  $f$  and  $g$  are non-constant, differentiable, real-valued functions defined on  $(-\infty, \infty)$ . Furthermore, suppose that for each pair of real numbers  $x$  and  $y$ ,

$$f(x+y) = f(x)f(y) - g(x)g(y),$$

$$g(x+y) = f(x)g(y) + g(x)f(y).$$

If  $f'(0) = 0$ , prove that  $(f(x))^2 + (g(x))^2 = 1$  for all  $x$ .

**Reminder** PSet 5 to be released tomorrow, and due by the end of Nov 19 (on Canvas). No problem session this week. The fourth problem session (to discuss PSet 4) will be on Nov 13, 7-8 PM, at the same location.