BROWNIAN BRIDGE LIMIT OF PATH MEASURES IN THE UPPER TAIL OF KPZ MODELS

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ABSTRACT. For models in the KPZ universality class, such as the zero temperature model of planar last passage-percolation (LPP) and the positive temperature model of directed polymers, its upper tail behavior has been a topic of recent interest, with particular focus on the associated path measures (i.e., geodesics or polymers). For Exponential LPP, diffusive fluctuation had been established in [BG23]. In the directed landscape, the continuum limit of LPP, the limiting Gaussianity at one point, as well as of related finite-dimensional distributions of the KPZ fixed point, were established, using exact formulas in [Liu22, LW22]. It was further conjectured in these works that the limit of the corresponding geodesic should be a Brownian bridge. We prove it in both zero and positive temperatures; for the latter, neither the one-point limit nor the scale of fluctuations was previously known. Instead of relying on formulas (which are still missing in the positive temperature literature), our arguments are geometric and probabilistic, using the results on the shape of the weight and free energy profiles under the upper tail from [GH22] as a starting point. Another key ingredient involves novel coalescence estimates, developed using the recently discovered shift-invariance [BGW22] in these models. Finally, our proof also yields insight into the structure of the polymer measure under the upper tail conditioning, establishing a quenched localization exponent around a random backbone.

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1. Introduction

Planar last passage percolation (LPP) models are paradigm examples of models believed to exhibit the features of the Kardar-Parisi-Zhang (KPZ) universality class. In such models, one studies the weight and geometry of the maximum weight directed path (called henceforth a geodesic) between two far away points in a 2D i.i.d. random field, such as $\mathbb{Z}^2$ with i.i.d. random variables at each vertex, or a homogenous Poission point process in $\mathbb{R}^2$. A handful of such models, when the underlying noise is given by i.i.d Exponentials, a Poisson point process (connected to the problem of the longest increasing subsequence in a random permutation) or even Brownian motions, admit exact-solvability. Pioneered by the seminal work of Baik, Deift and Johansson [BDK99], for these models exact formulae stemming from algebraic combinatorics and representation theory have been employed to rigorously deduce the predicted KPZ behaviors: the weight of a geodesic between points with Euclidean separation n fluctuates by $n^{1/3}$ while the geodesic itself deviates from the straight line joining the endpoints by $n^{2/3}$ (leading to the well known $1 : 2 : 3$ scaling of KPZ). Moreover, the geodesic weight scaled by $n^{1/3}$ is shown to converge to the GUE Tracy-Widom distribution. More recently, in a breakthrough work [DOV22] the directed landscape was constructed as a putative scaling limit of a large class of planar random geometry models and further in [DV21b] it was shown that all the known exactly solvable LPP models indeed converge to it under the KPZ scaling.

For the geometry of geodesics, other than being an interesting object to study on its own, in practice it has been a useful and robust tool. Historically, such geometric techniques were first developed for general LPP or the related first passage percolation (FPP) models (where one considers the minimum weight path rather than maximum directed path connecting two points). For these models much less has been rigorously established due to the absence of exact formulae and most of the progress has crucially relied on developing an understandings of the geodesics. For summaries of the results and techniques in this direction, see [New95] and the survey [ADH17]. More recently, techniques via this geometric perspective, combined with algebraic inputs, have led to the resolution of many problems for exactly-solvable LPP and related models. Some important techniques were put forth, for example, in the solution of the slow bond problem of TASEP [BSS14, SSZ21], the study of the correlation structure in KPZ [BG21, BGZ21, CGH21], the study of local statistics around LPP geodesics [MSZ21], and the ruling out bi-infinite geodesics in LPP [BHS22, BBS20].

The pre-limiting geodesics in exactly-solvable LPP were shown in [DOV22] and [DV21b] to converge to their scaling limits, which may be termed as geodesics in the directed landscape. For these processes various properties have been established, such as being $2/3^{-}$-Hölder regular and possessing a finite $3/2$-variation in [DSV22]. Associated local time processes were constructed and shown to be $1/3^{-}$ Hölder regular in [GZ22]. Finally, an explicit, albeit somewhat complicated, expression for the one point distribution of the geodesic was established in [Liu22] relying on exact formulae.

Atypical behavior. A natural and particularly interesting problem which has witnessed significant interest isto analyze the behavior of a geodesic given that its weight is atypical. Consistent with the disparity between the upper and lower tail behaviors of the Tracy-Widom distribution, it turns out that the upper tail probability, i.e., the probability of the geodesic weight being atypically large, is significantly larger than the lower tail probability, i.e., the probability of an atypically low geodesic weight. This stems from the fact that having an atypically large weight geodesic can be viewed as a somewhat local event requiring the existence of simply one large weight path whereas an atypically small weight geodesic requires *all* path weights to be small. Such a discrepancy between the upper and lower tails has been noted in the rigorous literature even beyond exactly-solvable models: in [Kes86] Kesten showed that in FPP, the upper and lower tail (which correspond to lower and upper tail for LPP) large deviation rates are of order n^2 and n, respectively. More precisely, for two points with Euclidean distance n, the logarithm of the probability for the geodesic weight to be δn larger (resp. smaller) than its expectation is of order n^2 (resp. n). Further, [Kes86] also proves the existence of a large deviation rate function in the lower tail; while the corresponding result for the upper tail was left open and resolved much later in [BGS21]. For various exact-solvable LPPs, exact expressions for the rate functions have been obtained using a range of methods, across the papers [LS77, Sep98a, Sep98b, DZ99, Joh00].

In correspondence to the difference in tail probabilities, the behaviors of geodesics are also quite different conditional on these tail events. Under lower tail large deviation of LPP (i.e., the rate n^2 tail) the geodesics are delocalized in the sense of having order n transversal fluctuation (thus transversal fluctuation exponent equals 1), as has been proven in [BGS19] for LPP models for a large class of weight distributions, including and going beyond exactly-solvable models. Under upper tail large deviation (i.e., the rate n tail), for exponential LPP it was shown in [BG23] that the geodesics are more localized: the transversal fluctuation exponent changes from its typical behavior of $2/3$ to $1/2$. The disparity in the exponents arises due to the heuristic reasoning that the upper tail event entails local changes effected by picking a random directed path connecting the endpoints and making its weight large whereas the lower tail event forces every path to have a small weight thereby making even paths with large transversal fluctuations competitive.

The above reasoning in particular suggests that LPP geodesics under upper tail large deviation should scale to a Brownian bridge. An evidence in this direction was provided in [LW22], where it was shown that in the directed landscape, if the weight between two points is conditioned to be $> L$, for a process that one expects to be close to the geodesic, as $L \to \infty$, the multi-point distribution converge to a joint Gaussian with correlation structure matching that of a Brownian bridge leading them to conjecture the Brownian bridge limit for the geodesic. Such a statement was previously posed as a question in the prior work [BG23] in the context of Exponential LPP.

Polymer models. Models of LPP can be viewed as the zero-temperature version of the more general directed polymer models, where one still has a 2D i.i.d. random field, but now considers a Gibbs measure on the space of all directed paths between two points parametrized by temperature. More precisely, the probability density of a path γ is proportional to $\exp(\beta H(\gamma))$, where $\beta > 0$ is the inverse temperature and $H(\gamma)$ is the weight of the path in the random field. We henceforth refer to the random path under this measure as the polymer (between the two points). At least formally, when $\beta \to \infty$, the polymer degenerates into the corresponding geodesic, which can therefore be viewed the zero-temperature polymer. Certain integrable features persist even for positive temperature models, such as the log-gamma polymer and the O'Connell-Yor polymer [OY01, Sep12]. Exploiting such special properties, KPZ behavior has also been established, at least to some degree, for these examples as well. In particular, for the KPZ equation from the original paper of Kardar, Parisi, and Zhang [KPZ86], the Cole-Hopf solution turns out to be the free energy of the Continuum Directed Random Polymer (CDRP) model [AKQ14], and the KPZ scaling convergence to the directed landscape has been established in a series of recent works [QS23, DM21, Wu21, Wu23].

The tail probabilities of these positive temperature models have also been studied. In particular, there have been extensive works on tails of the KPZ equation: its one-point upper tail probability is similar to that of the Tracy-Widom distribution, as established in [KKX17, CG20a, DT21, GH22]; the lower tail probability is much more involved with a cross-over behavior, as has been shown in [CG20b, CG20a, CC22, Tsa22]. Besides the KPZ equation, estimates on both tails have also been obtained for the O'Cennell-Yor polymer in [LS22].

We now move on to the main results of this paper which prove the full Brownian bridge conjecture for both the directed landscape (zero temperature) and the CDRP (positive temperature). Further, in the latter case, our results also establish a quenched localization phenomenon, where the polymer localizes around a random backbone, the law of the latter being a Brownian bridge.

1.1. **Main results.** The decision behind working with the directed landscape is guided by the various symmetries and scaling invariance properties that are absent in the pre-limit, which help make our arguments more transparent. However, we emphasize that we rely on probabilistic and geometric arguments rather than exact formulae, in contrast to [LW22]. Our proofs are robust and can be adapted to any pre-limiting exact-solvable LPP or polymer models, using (through the RSK correspondence) the respective line ensembles given by different types of random walks or Brownian motions.

We now proceed to formally define some of the key objects to help set up the framework towards stating our main results. The directed landscape $\mathcal{L}$, constructed in [DOV22], is a continuous random function from the parameter space

$$\mathbb{R}^4_\uparrow = \{u = (p; q) = (x, s; y, t) \in \mathbb{R}^4 : s < t\}$$

to $\mathbb{R}$. It satisfies the 'reverse triangle inequality':

$$\mathcal{L}(x, r; z, t) = \max_y \mathcal{L}(x, r; y, s) + \mathcal{L}(y, s; z, t)$$

for any $r < s < t$, making it a 'directed metric'.

We next describe the 'directed geometry' induced by $\mathcal{L}$ and record some facts about it, following [DOV22, Section 12]. For any $s < t$, and x, y, a path from (x, s) to (y, t) is a continuous function $\pi : [s, t] \to \mathbb{R}$ with $\pi(s) = x$ and $\pi(t) = y$; and its length is given by

$$\|\pi\|_{\mathcal{L}} = \inf_{k\in\mathbb{N}} \inf_{s=t_0<t_1<\ldots<t_k=t} \sum_{i=1}^{k} \mathcal{L}(\pi(t_{i-1}), t_{i-1}; \pi(t_i), t_i).$$

A path π is a geodesic if $\|\pi\|_{\mathcal{L}}$ is maximal among all paths with the same start and endpoints. Equivalently, a geodesic between (x, s) and (y, t) is any path π with $\|\pi\|_{\mathcal{L}} = \mathcal{L}(x, s; y, t)$. Almost surely, geodesics exist between every pair $(x, s), (y, t)$ with $s < t$, and we use $\pi_{(x,s;y,t)}$ to denote any such a geodesic. Moreover, there is almost surely a unique geodesic between any fixed pair $(x, s), (y, t)$. Let $\pi_0 : [0, 1] \to \mathbb{R}$ denote the geodesic from $(0, 0)$ to $(0, 1)$.

We now arrive at our first main result concerning its limit, under the upper tail event.

Theorem 1.1. *As $L \to \infty$, $2L^{1/4}\pi_0$ conditioned on $\mathcal{L}(0, 0; 0, 1) > L$ converges to a standard Brownian bridge, weakly in the topology of uniform convergence.*

For our next result we switch to the positive temperature model of CDRP. Let $\tilde{\mathcal{Z}}$ denote the $\beta = 1$ random field constructed in [AKQ14, Theorem 3.1], which is also a continuous random function from $\mathbb{R}^4_\uparrow$ to $\mathbb{R}$. The function $\tilde{\mathcal{Z}}(0, 0; \cdot, \cdot)$ solves the multiplicative stochastic heat equation (which is connected to the KPZ equation via a Cole-Hopf transformation) with the Dirac delta initial condition.

For the convenience of notations in our proofs, we denote $\mathcal{Z}(x, s; y, t) = 2\tilde{\mathcal{Z}}(2x, 2s; 2y, 2t)$. A key property of $\mathcal{Z}$ (inherited from $\tilde{\mathcal{Z}}$), allowing one to define the CDRP, is the following semi-group property: almost surely, for any $x, y \in \mathbb{R}$ and $s < r < t$, we have

$$\mathcal{Z}(x, s; y, t) = \int \mathcal{Z}(x, s; z, r)\mathcal{Z}(z, r; y, t)dz. \tag{1.1}$$

See e.g. [AJRAS22, Theorem 2.6(v)]. As in [AKQ14], conditional on $\mathcal{Z}$ we can define the random polymer from (x, s) to (y, t) (for any $s < t$), as the continuous random function $\Gamma : [s, t] \to \mathbb{R}$, such that for any $k \in \mathbb{N}$ and $s = s_0 < s_1 < \cdots < s_k < s_{k+1} = t$, and $x = x_0, x_1, \ldots, x_k, x_{k+1} = y$, the probability density for $\Gamma(s_1) = x_1, \ldots, \Gamma(s_k) = x_k$ is proportional to

$$\prod_{i=0}^{k} \mathcal{Z}(x_i, s_i; x_{i+1}, s_{i+1}). \tag{1.2}$$

We will use $\mathcal{P}$ to denote the (quenched) measure of the polymer.

Letting Γ_0 denote the random polymer from $(0,0)$ to $(0,1)$, the counterpart of Theorem 1.1 in the setting of CDRP is the following.

Theorem 1.2. *As $L \to \infty$, $2L^{1/4}\Gamma_0$ conditioned on $\log \mathcal{Z}(0,0;0,1) > L$ converges to a standard Brownian bridge, weakly in the topology of uniform convergence.*

Further, as our proof will show, in this case, a further exponent arises. This is because, under the upper tail event, the quenched polymer fluctuates at a scale $L^{-1/2}$ which is much smaller than $L^{-1/4}$, and hence, this leads to the following structural result. Under the upper tail event , there is a random backbone which when scaled by $L^{-1/4}$ converges to a Brownian bridge, while the quenched polymer fluctuates at a scale $L^{-1/2}$ around the backbone. A further discussion on this will appear shortly when we review some of the key ideas in our proofs. We postpone a further discussion on this to later in the article, once the relevant arguments have already been presented (see Remark 8.3).

Our arguments involve several ingredients which we now provide an overview of.

1.2. **Ideas of proofs.** We start by outlining our proof for Theorem 1.1, and then move on to the extra ingredients involved in proving Theorem 1.2.

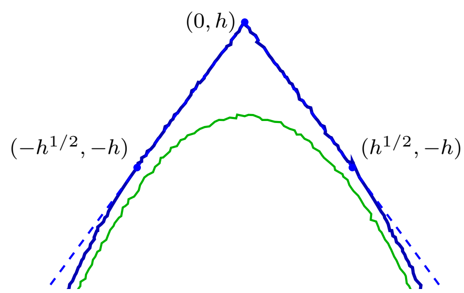

FIGURE 1. An illustration of the Airy$_2$ process conditional on that it equals h at 0, and the second line of the Airy line ensemble.

The arguments consist of two components: multi-point joint Gaussian limit, as well as tightness of the paths.

1.2.1. *One-point Gaussianity.* For Gaussianity, to illustrate the ideas let us first consider the one point distribution, say $\pi_0(1/2)$. It precisely is the argmax of $x \mapsto \mathcal{L}(0,0;x,1/2) + \mathcal{L}(x,1/2;0,1)$, whereas the conditioning of $\mathcal{L}(0,0;0,1) > L$ is precisely $\max_x \mathcal{L}(0,0;x,1/2) + \mathcal{L}(x,1/2;0,1) > L$. For these two processes $\mathcal{L}(0,0;\cdot,1/2)$ and $\mathcal{L}(\cdot,1/2;0,1)$, (without conditioning on the upper tail) they are independent, and up to a rescaling each has the distribution of the so-called (parabolic) Airy$_2$ process, which is a stationary process minus x^2 (see [QR14] for a survey about it). This process is also the top line of the Airy line ensemble, a family of continuous processes constructed by Corwin and Hammond [CH14] with the so-called 'Brownian Gibbs property'. Roughly speaking, this implies that in any given interval, given the second line of the Airy line ensemble, and the Airy$_2$ process at the boundary of this interval, inside the interval it is the Brownian bridge conditioned to stay above the second line (which can be thought of as the negative parabola $-x^2$ for our purpose).

From this one can deduce a tent picture under upper tail. Namely, conditional on that the Airy$_2$ process at 0 equals h for some large h, it behaves like two independent Brownian bridges in $[-h^{1/2},0]$ and $[0,h^{1/2}]$ respectively, and is approximately $-h$ at $-h^{1/2}$ or $h^{1/2}$ (see Figure 1). By stationarity, there is a similar picture when the Airy$_2$ process is conditioned to be large at any other point. In [GH22], such a tent picture in the positive temperature setting of the KPZ line ensemble has been used to obtain upper tail estimates for the KPZ equation. For our purpose, we use this to deduce the probability of $\max_x \mathcal{L}(0,0;x,1/2)+\mathcal{L}(x,1/2;0,1) \le h_1+h_2$, conditional on that $\mathcal{L}(0,0;x_*,1/2)=h_1$ and $\mathcal{L}(x_*,1/2;0,1)=h_2$, for given $h_1+h_2>L$ and x_*. We then wish to integrate over such h_1 and h_2 to obtain the probability density of $\pi_0(1/2)=x_*$. For this, one needs an accurate (up to a factor of $1+o(1)$) estimate of the probability densities of $\mathcal{L}(0,0;x_*,1/2)=h_1$ and $\mathcal{L}(x_*,1/2;0,1)=h_2$, for large h_1 and h_2. However, existing estimates such as those in [GH22] are not sufficient; and to get the desired estimate seems a technically demanding task. We on the other hand get around of it by instead estimating the ratio of two, namely, the probability density of $\mathcal{L}(0,0;x_*,1/2)=h_1$ over that of $\mathcal{L}(0,0;x_*,1/2)=h_1'$ for some h_1' close to h_1; and do the same for $\mathcal{L}(x_*,1/2;0,1)$. It turns out that for our purpose, it suffices to estimate such a ratio up to a factor of $1+o(1)$. This ratio estimate can be achieved via resampling the top line of the Airy line ensemble using the Brownian Gibbs property, and is implemented in Section 4.

1.2.2. *Multi-point: coalescence of geodesics.* We then consider two-point joint distribution of π_0, such as that of $\pi_0(1/3)$ and $\pi_0(2/3)$, which are the argmax of $x,y\mapsto \mathcal{L}(0,0;x,1/3)+\mathcal{L}(x,1/3;y,2/3)+\mathcal{L}(y,2/3;0,1)$. Even under the typical regime, while any one-point distribution of π_0 could be described as the argmax of the sum of two independent Airy$_2$ process, such a two-point joint distribution could hardly be explicitly computed or even described. The main reason is that there is no exact formula for the two-variable process $\mathcal{L}(\cdot,1/3;\cdot,2/3)$. However, in the upper tail large deviation regime, we manage to decouple the argmax problem using a strong coalescence resulted from the upper tail conditioning.

A key observation we make is that, assuming $\mathcal{L}(0,0;0,1)>L$ for a large L, all the geodesics from $(x,1/3)$ to $(y,2/3)$ for any $|x|,|y|$ of order smaller than $L^{1/2}$ would tend to coalescence (see Figure 2). This can be described as the following quadrangle equality (see also (2.3) below): with high probability there is

$$\mathcal{L}(x,1/3;y,2/3)=\mathcal{L}(x_*,1/3;y,2/3)+\mathcal{L}(x,1/3;y_*,2/3)-\mathcal{L}(x_*,1/3;y_*,2/3),$$

for all $|x|,|y|$ and $|x_*|,|y_*|$ that are of order smaller than $L^{1/2}$. Such a quantitative description of coalescence can also be generalized to the positive temperature setting of CDRP, to be discussed shortly. Therefore, to have $\pi_0(1/3)=x_*$ and $\pi_0(2/3)=y_*$, roughly one just need to ensure that

$$x_*=\operatorname*{argmax}_x \mathcal{L}(0,0;x,1/3)+\mathcal{L}(x,1/3;y_*,2/3),\quad y_*=\operatorname*{argmax}_y \mathcal{L}(x_*,1/3;y,2/3)+\mathcal{L}(y,2/3;0,1).$$

We further have that these two events are nearly independent, and this can also be deduced from the above mentioned coalescence. We can then estimate the probability densities of these events respectively, using the same arguments for the one-point distribution above.

We note that to rigorously justify the coalesence and the independence of the events, we will actually resort to some less intuitive tools, such as the multi-point passage times (which in the setting of the directed landscape is studied in [DZ21]) and a shift-invariance symmetry of LPP from [BGW22]. The details are presented in Section 3.

Such analysis could also be done for multi-point joint distributions. It remains to carry out all these computations to reach the desired multi-point joint-Gaussian, which turns out to require much technical efforts. We achieve that in the last two sections of this paper.

1.2.3. *Tightness.* To prove tightness of π_0 as continuous random functions under upper tail, using the Kolmogorov-Chentsov criterion for tightness (e.g. [Kal22, Theorem 23.7]), the task is essentially to estimate the transversal fluctuation at two points. More precisely, we will prove an upper tail for $|\pi_0(s)-\pi_0(t)|(t-s)^{-1/2}L^{1/4}$ conditional on $\mathcal{L}(0,0;0,1)>L$, that is uniform in L and $0<s<t<1$. This can be achieved by considering the probability that there exist some x,y with $|x-y|$ of order at least $(t-s)^{1/2}L^{-1/4}$, and $\mathcal{L}(0,0;x,s)+\mathcal{L}(x,s;y,t)+\mathcal{L}(y,t;0,1)>L$; and show that it is much

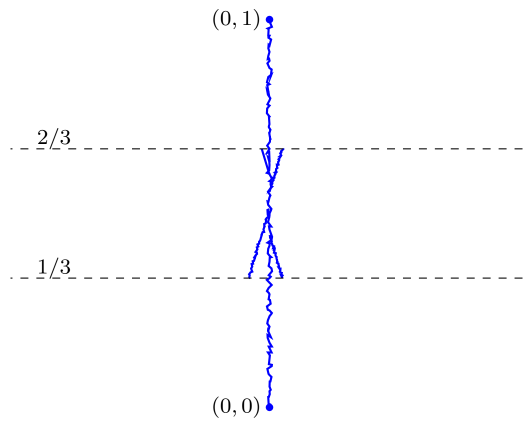

FIGURE 2. An illustration of the coalescence phenomenon in time [1/3.2/3], under upper tail.

smaller than the probability of $\mathcal{L}(0,0;0,1) > L$. Using the shear invariance property of $\mathcal{L}(\cdot, s; \cdot, t)$ (to be introduced in Section 2.1 below), the former event can be reduced to that $\mathcal{L}(0,0;0,1) - L$ is at least of order $(t-s)^{-1}$. Then by applying the ratio estimate from Section 4 we get the desired transversal fluctuation bound. The details of these will be given in Section 5 below.

Organization of the remaining text. In Section 2 we give the formal setup and quote existing tools and estimates that we will use. Section 3 establishes coalescence under the upper tail, and Section 4 proves the estimate on the ratio of upper tail probabilities. The next four sections prove the tightness; in particular, the zero temperature tightness is done in Section 5, while the positive temperature tightness is much more involved and is proved in Section 8, along with the localization around a random backbone. The last four sections are devoted to proving the finite-point Gaussian limit.

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2. Preliminaries

Notations. Throughout this paper, we will use $C, c > 0$ to denote large and small constants, whose values may and often change from line to line. We will also use the Bachmann-Landau notations: for $A > 0$, $O(A)$ denotes a number B such that $|B| < CA$, and $\Omega(A)$ denotes a number B such that $B > cA$. For any $x, y \in \mathbb{R} \cup \{-\infty, \infty\}$, $x \leq y$, we denote $[\![x, y]\!] = [x, y] \cap \mathbb{Z}$.

For any $m \in \mathbb{N}$, we introduce the following simplexes,

$$\Lambda_m = \{(x_1, \cdots, x_m) : x_1 \leq \cdots \leq x_m\},$$

and for any $a < b$,

$$\Lambda_m([a, b]) = \{(x_1, \cdots, x_m) : a \leq x_1 \leq \cdots \leq x_m \leq b\}.$$

We use $\mathring{\Lambda}_m$ and $\mathring{\Lambda}_m([a, b])$ to denote the interiors of Λ_m and $\Lambda_m([a, b])$, respectively. For any $\mathbf{x} = (x_1, \cdots, x_m) \in \Lambda_m$, let $\Delta(\mathbf{x}) = \prod_{i<j}(x_i - x_j)$.

For any μ and σ, we let $\mathcal{N}(\mu, \sigma^2)$ denote the normal distribution with mean μ and variance σ^2.

For any topology space X, we use $\mathcal{C}(X,\mathbb{R})$ to denote the space of real continuous functions on X with the uniform topology.

We next move on to some properties of the directed landscapes and the associated geodesics therein which will appear in our arguments repeatedly.

2.1. **The directed landscape and geodesics.** The directed landscape $\mathcal{L}$, which is a random continuous function on $\mathbb{R}^4_\uparrow$, is shift, shear, reflection, and $1:2:3$ scaling invariant. More precisely, (as given by [DOV22, Lemma 10.2]) $\mathcal{L}$ has the same distribution as

- (Shift and shear) $(x,s;y,t)\mapsto \mathcal{L}(x+\nu s+\alpha, s+\eta; y+\nu t+\alpha, t+\eta)+2\nu(y-x)+\nu^2(t-s)$, for any $\nu,\alpha,\eta\in\mathbb{R}$;
- (Reflection) $(x,s;y,t)\mapsto\mathcal{L}(-x,s;-y,t)$, and $(x,s;y,t)\mapsto\mathcal{L}(y,-t;x,-s)$;
- (Scaling) $(x,s;y,t)\mapsto w\mathcal{L}(w^{-2}x,w^{-3}s;w^{-2}y,w^{-3}t)$, for any $w>0$.

We next describe the multi-point passage times, studied in [DZ21]. For any $k\in\mathbb{N}$, $\mathbf{x}=(x_1,\cdots,x_k)\in\Lambda_k$, $\mathbf{y}=(y_1,\cdots,y_k)\in\Lambda_k$, and $s<t$, we define

$$\mathcal{L}(\mathbf{x},s;\mathbf{y},t)=\sup_{\pi_1,\ldots,\pi_k}\sum_{i=1}^k \|\pi_i\|_{\mathcal{L}},$$

where the supremum is over all k-tuples of paths $\pi=(\pi_1,\ldots,\pi_k)$ where each π_i is a path from (x_i,s) to (y_i,t), satisfying the disjointness condition $\pi_i(r)<\pi_j(r)$ for all $i<j$ and $r\in(s,t)$. It is shown (in [DZ21, Theorem 1.7]) that, almost surely, for every set of endpoints the supremum is achieved by some paths satisfying the disjointness condition. Therefore the following statement holds.

Lemma 2.1 ([DZ21, Corollary 1.11])**.** *Almost surely the following holds. For any $k\in\mathbb{N}$, $\mathbf{x}=(x_1,\cdots,x_k),\mathbf{y}=(y_1,\cdots,y_k)\in\Lambda_k$, and $s<t$, we have*

$$\mathcal{L}(\mathbf{x},s;\mathbf{y},t)=\sum_{i=1}^k\mathcal{L}(x_i,s;y_i,t),$$

if and only if there exist geodesics $\pi_1,\ldots,\pi_k$, where π_i is from (x_i,s) to (y_i,t), satisfying $\pi_i(r)<\pi_{i+1}(r)$ for each $1\le i<k$ and $r\in(s,t)$.

For any $s<t$ and x,y, we also denote

$$\mathcal{L}_k(x,s;y,t)=\mathcal{L}(x\mathbf{1}_k,s;y\mathbf{1}_k,t),$$

where $\mathbf{1}_k\in\mathbb{R}^k$ is the vector with each coordinate equal 1.

A key property of the directed landscape is the following inequality due to planarity.

Lemma 2.2 ([DZ21, Lemma 5.7])**.** *The following holds almost surely. Take any $k\in\mathbb{N}$ and $\mathbf{x},\mathbf{y},\mathbf{x}',\mathbf{y}'\in\Lambda^k$. Define $\mathbf{x}^\ell,\mathbf{y}^\ell,\mathbf{x}^r,\mathbf{y}^r\in\Lambda^k$ by setting $x_i^\ell=x_i\wedge x_i'$, $y_i^\ell=y_i\wedge y_i'$, and $x_i^r=x_i\vee x_i'$, $y_i^r=y_i\vee y_i'$, for each $1\le i\le k$. Take any $s<t$. Then we have*

$$\mathcal{L}(\mathbf{x}^\ell,s;\mathbf{y}^\ell,t)+\mathcal{L}(\mathbf{x}^r,s;\mathbf{y}^r,t)\ge\mathcal{L}(\mathbf{x},s;\mathbf{y},t)+\mathcal{L}(\mathbf{x}',s;\mathbf{y}',t).$$

In the case of $k=1$, this is the quadrangle inequality: for any $s<t$, $x_1<x_2$, $y_1<y_2$, we have

$$\mathcal{L}(x_1,s;y_1,t)+\mathcal{L}(x_2,s;y_2,t)\ge\mathcal{L}(x_1,s;y_2,t)+\mathcal{L}(x_2,s;y_1,t). \tag{2.1}$$

See e.g. [DOV22, Lemma 9.1]. Besides, the strict inequality is known to be equivalent to the disjointness of geodesics.

Lemma 2.3 ([GZ22, Lemma 3.15])**.** *For any fixed $s < t$, $x_1 < x_2$, $y_1 < y_2$, almost surely the inequality*

$$\mathcal{L}(x_1, s; y_1, t) + \mathcal{L}(x_2, s; y_2, t) > \mathcal{L}(x_1, s; y_2, t) + \mathcal{L}(x_2, s; y_1, t)$$

is equivalent to that $\pi_{(x_1,s;y_1,t)}$ and $\pi_{(x_2,s;y_2,t)}$ are disjoint; i.e., $\pi_{(x_1,s;y_1,t)}(r) < \pi_{(x_2,s;y_2,t)}(r)$, $\forall r \in [s, t]$.

Another degeneration of Lemma 2.2 is the following inequality which will be used later. Take any x and $y_1 \leq y_2 \leq y_3$, and $s < t$, we have almost surely,

$$\mathcal{L}((x, x), s; (y_1, y_3), t) + \mathcal{L}(x, s; y_2, t) \geq \mathcal{L}((x, x), s; (y_1, y_2), t) + \mathcal{L}(x, s; y_3, t). \tag{2.2}$$

Proof of (2.2). We take $z < x \wedge y_1$ and apply Lemma 2.2 with $k = 2$, $\mathbf{x} = (x, x)$, $\mathbf{x}' = (z, x)$, $\mathbf{y} = (y_1, y_2)$, $\mathbf{y}' = (z, y_3)$, to get

$$\mathcal{L}((x, x), s; (y_1, y_3), t) + \mathcal{L}((z, x), s; (z, y_2), t) \geq \mathcal{L}((x, x), s; (y_1, y_2), t) + \mathcal{L}((z, x), s; (z, y_3), t). \tag{2.3}$$

We note that by Lemma 2.1, and the fact that any geodesic is almost surely continuous (therefore bounded), we have

$$\lim_{z\to-\infty} \mathbb{P}\left(\mathcal{L}(x, s; y_2, t) = \mathcal{L}((z, x), s; (z, y_2), t) - \mathcal{L}(z, s; z, t)\right) = 1,$$
$$\lim_{z\to-\infty} \mathbb{P}\left(\mathcal{L}(x, s; y_3, t) = \mathcal{L}((z, x), s; (z, y_3), t) - \mathcal{L}(z, s; z, t)\right) = 1.$$

Plugging these into (2.3) we get (2.2). □

2.2. **Continuum-directed random polymer.** We define the multi-line continuum partition functions through the chaos expansion, as done in [OW16]. For CDRP, we take the inverse temperature $\beta = 1$ throughout this paper for simplicity of notations, while our arguments go through verbatim for any fixed β.

2.2.1. *Partition function.* Let W be a cylindrical Brownian motion on $L^2(\mathbb{R})$, and $\dot{W}$ be the space-time white noise associated with W. Denote

$$p_t(x) = \frac{1}{\sqrt{2\pi t}} \exp(-x^2/2t).$$

For any $s < t$, $x, y \in \mathbb{R}$, and $k \in \mathbb{Z}_+$, we let

$$\tilde{\mathcal{Z}}_k(x, s; y, t) = p_{t-s}(y-x)^k \left(1 + \sum_{m=1}^{\infty} \int_{\Lambda_m([s,t])} \int_{\mathbb{R}^m} R((x_1, t_1), \dots, (x_m, t_m)) W(dt_1, dx_1) \cdots W(dt_m, dx_m)\right),$$

where R denotes the m point correlation function for a collection of k non-intersecting Brownian bridges which all start at x at time s and end at y at time t. We write $\tilde{\mathcal{Z}} = \tilde{\mathcal{Z}}_1$, which, historically, was introduced as the solution to the multiplicative stochastic heat equation with Dirac delta initial condition, via the Feynman-Kac representation. Specifically, $\tilde{u}(x, t) = \tilde{\mathcal{Z}}(0, 0; x, t)$ satisfies

$$\partial_t \tilde{u} = \frac{1}{2} \partial_x^2 \tilde{u} + \tilde{u} \dot{W},$$

with $\tilde{u}(\cdot, 0)$ being the delta mass at 0.

In what sense is $\tilde{\mathcal{Z}}_k(x, s; y, t)$ defined? In [OW16] this is defined for any fixed k and x, s, y, t, by proving the $L^2(W)$ convergence of the chaos expansion. In [Nic21], it is shown that

$$(y, k) \mapsto \log(\tilde{\mathcal{Z}}_k(0, 0; y, t)/\tilde{\mathcal{Z}}_{k-1}(0, 0; y, t))$$

is a (scaled) KPZ$_t$ line ensemble, as defined in [CH16, Theorem 2.15]; therefore $\tilde{\mathcal{Z}}_k(x, s; y, t)$ can be thought of as a continuous function of y, for any fixed k and x, s, t (see [Nic21, Corollary 1.9, 1.11]). In [LW20], it is further shown that $(y, t) \mapsto \tilde{\mathcal{Z}}_k(x, s; y, t)$ can be defined as a continuous function, for any fixed k, x, s.

Moreover, for $\tilde{\mathcal{Z}}$, it can be defined as a four-parameter random continuous function. It is also shift, shear, and reflection invariant (in distribution). (See [AKQ14, Theorem 3.1] and [AJRAS22, Proposition 2.3].)

Scaling. Under certain limiting transitions (either $t \to \infty$ or $\beta \to \infty$) and appropriate scaling, the logarithm of $\tilde{\mathcal{Z}}$, which can be understood as a solution to the KPZ equation, converges to the directed landscape [QS23, Wu23]. While we do not actually use this convergence in this paper, in light of the scaling involved in this limit transition, and for the purposes of being consistent with the directed landscape setting and reducing notations (which will be clear shortly), we denote $\mathcal{Z}_k(x, s; y, t) = 2^k \tilde{\mathcal{Z}}_k(2x, 2s; 2y, 2t)$ and $\mathcal{Z} = \mathcal{Z}_1$. Now $u(x, t) = \mathcal{Z}(0, 0; x, t)$ satisfies

$$\partial_t u = \frac{1}{4}\partial_x^2 u + u\dot{W},$$

with $u(\cdot, 0)$ being the delta mass at 0. The shift, shear, and reflection invariance of $\mathcal{Z}$ states that $\mathcal{Z}$ has the same distribution as

- (Shift and shear) $(x, s; y, t) \mapsto \mathcal{Z}(x+\nu s+\alpha, s+\eta; y+\nu t+\alpha, t+\eta) \exp(\nu^2(t-s)+2\nu(y-x))$, for any $\nu, \alpha, \eta \in \mathbb{R}$;
- (Reflection) $(x, s; y, t) \mapsto \mathcal{Z}(-x, s; -y, t)$, and $(x, s; y, t) \mapsto \mathcal{Z}(y, -t; x, -s)$.

As already mentioned in the introduction, for any $s < t$ and x, y, in [AKQ14] a measure (denoted by $\mathbf{\mathcal{P}}$) on $\mathcal{C}([s, t], \mathbb{R})$ is defined and gives the random polymer from (x, s) to (y, t), with finite-dimensional distribution given by (1.2).

2.2.2. *Multi-point partition function with distinct endpoints.* For any $\mathbf{x} = (x_1, \cdots, x_k), \mathbf{y} = (y_1, \cdots, y_k) \in \mathring{\Lambda}_k$, and $s < t$, we define

$$\mathcal{M}(\mathbf{x}, s; \mathbf{y}, t) = \det[\mathcal{Z}(x_i, s; y_j, t)]_{i,j=1}^k \Delta(\mathbf{x})^{-1}\Delta(\mathbf{y})^{-1}.$$

Then from the continuity of $\mathcal{Z} = \mathcal{Z}_1$, we have that $\mathcal{M}(\mathbf{x}, s; \mathbf{y}, t)$ is almost surely continuous in all the variables.

Positivity and implications. Using the Karlin-McGregor theorem, it is straightforward to deduce that $\mathcal{M}$ is non-negative, as shown in [OW16, Proposition 5.5]. The simultaneous strict inequality is proved in [LW20, Theorem 1.4], and also in [AJRAS22, Theorem 2.17] with a different method.

Lemma 2.4. *Almost surely, for any $s < t$, $k \in \mathbb{N}$, and $\mathbf{x}, \mathbf{y} \in \mathring{\Lambda}_k$, there is $\mathcal{M}(\mathbf{x}, s; \mathbf{y}, t) > 0$.*

The case of Lemma 2.4 where $k = 2$ can be viewed as an analog to Equation (2.1); namely, for any $s < t$, $x_1 < x_2$, $y_1 < y_2$, we have

$$\mathcal{Z}(x_1, s; y_1, t)\mathcal{Z}(x_2, s; y_2, t) > \mathcal{Z}(x_1, s; y_2, t)\mathcal{Z}(x_2, s; y_1, t). \tag{2.4}$$

Another useful statement that can be deduced from Lemma 2.4 is the following monotonicity.

Lemma 2.5. *Almost surely the following is true. For any $s < t$, $x_1 < x_2 < x_3$ and $y_1 < y_2 < y_3$, we have*

$$\begin{aligned} &(y_3 - y_1)\mathcal{M}((x_1, x_3), s; (y_1, y_3), t)\mathcal{Z}(x_2, s; y_2, t) \\ >&(y_2 - y_1)\mathcal{M}((x_1, x_3), s; (y_1, y_2), t)\mathcal{Z}(x_2, s; y_3, t) \\ &+ (y_3 - y_2)\mathcal{M}((x_1, x_3), s; (y_2, y_3), t)\mathcal{Z}(x_2, s; y_1, t). \end{aligned}$$

Proof. This is equivalent to $\mathcal{M}((x_1, x_2, x_3), s; (y_1, y_2, y_3), t) > 0$, which holds by Lemma 2.4. $\square$

Composition. There is also a composition law of $\mathcal{M}$, which can be obtained from (1.1) and the Cauchy-Binet formula: almost surely, for any $k \in \mathbb{N}$, $\mathbf{x}, \mathbf{y} \in \mathring{\Lambda}_k$ and $s < r < t$, we have

$$\mathcal{M}(\mathbf{x}, s; \mathbf{y}, t) = \int_{\mathring{\Lambda}_k} \mathcal{M}(\mathbf{x}, s; \mathbf{z}, r)\mathcal{M}(\mathbf{z}, r; \mathbf{y}, t)\Delta(\mathbf{z})^2 d\mathbf{z}. \tag{2.5}$$

Continuous extension. The function $\mathcal{M}$ is connected to the multi-layer partition function, through the following extension of $\mathcal{M}$ to the boundary of $\Lambda_k \times \Lambda_k$.

Lemma 2.6 ([OW16, Lemma 6.1])**.** *For any $s < t$ and $k \in \mathbb{N}$, the function $\mathbf{x}, \mathbf{y} \mapsto \mathcal{M}(\mathbf{x}, s; \mathbf{y}, t)$ extends continuously in $L^2(W)$ to $\Lambda_k \times \Lambda_k$, and the extension satisfies*

$$2^{-k(k-1)/2}(t-s)^{k(k-1)/2} \prod_{i=1}^{k-1} i! \mathcal{M}(x\mathbf{1}, s; y\mathbf{1}, t) = \mathcal{Z}_k(x, s; y, t),$$

where $\mathbf{1}$ is the vector in $\mathbb{R}^k$ where each entry equals 1.

2.3. **Line ensembles and Gibbs properties.** As already indicated, a tool widely used in the study of the directed landscape is the Airy line ensemble constructed in [CH14]. Analogously, a tool widely used in the study of the KPZ equation and the free energy of the CDRP is the related KPZ_t line ensemble from [CH16], and its Gibbs property. To be concise, instead of giving the complete line ensemble setups, we quote some useful results from these connections.

For any $t > 0$, $x \in \mathbb{R}$, we denote

$$\mathfrak{h}_{t,1}^{\beta=1}(x) = \log \mathcal{Z}(0,0;x,t) + t/12, \quad \mathfrak{h}_{t,2}^{\beta=1}(x) = \log(\mathcal{Z}_2(0,0;x,t)/\mathcal{Z}(0,0;x,t)) + t/12,$$

and

$$\mathfrak{h}_{t,1}^{\beta=\infty}(x) = \mathcal{L}(0,0;x,t), \quad \mathfrak{h}_{t,2}^{\beta=\infty}(x) = \mathcal{L}_2(0,0;x,t) - \mathcal{L}(0,0;x,t).$$

We note that $\mathcal{H}(x,t) = \mathfrak{h}_{t,1}^{\beta=1}(x)$ solves the KPZ equation

$$\partial_t \mathcal{H} = \tfrac{1}{4}\partial_x^2 \mathcal{H} + \tfrac{1}{4}(\partial_x \mathcal{H})^2 + \dot{W}.$$

(Recall that $\dot{W}$ is the space-time white noise.)

Thanks to the scaling in defining $\mathcal{Z}$, $\mathfrak{h}_{t,1}^{\beta=1}$ has the same parabolic decay of $-x^2/t$ as $\mathfrak{h}_{t,1}^{\beta=\infty}$. More precisely, $\mathfrak{h}_{t,1}^{\beta}(x) + x^2/t$ for both $\beta = 1$ and ∞ are stationary (which can be deduced from the shear invariance of $\mathcal{Z}$ and $\mathcal{L}$). We also denote

$$\hat{\mathfrak{h}}_{t,1}^{\beta}(x) := t^{-1/3}\mathfrak{h}_{t,1}^{\beta}(t^{2/3}x), \quad \hat{\mathfrak{h}}_{t,2}^{\beta}(x) := t^{-1/3}\mathfrak{h}_{t,2}^{\beta}(t^{2/3}x).$$

As will be clear from later texts, using $\hat{\mathfrak{h}}_{t,1}^{\beta}$ and $\hat{\mathfrak{h}}_{t,2}^{\beta}$ instead of $\mathfrak{h}_{t,1}^{\beta}$ and $\mathfrak{h}_{t,2}^{\beta}$ could reduce some notations, since $\hat{\mathfrak{h}}_{t,1}^{\beta}$ and $\hat{\mathfrak{h}}_{t,2}^{\beta}$ have the parabolic decay of $-x^2$.

Another consequence of the scaling in defining $\mathcal{Z}$ is that $\mathfrak{h}_{t,1}^{\beta=1}$ is locally absolutely continuous with respect to a rate 2 Brownian bridge, which is also true for $\mathfrak{h}_{t,1}^{\beta=\infty}$. To be more precise, we quote the following Gibbs properties of $\mathfrak{h}_{t,1}^{\beta}$ given $\mathfrak{h}_{t,2}^{\beta}$. For any $a < b$, denote by $\mathcal{F}_{\mathrm{ext}}([a,b])$ the σ-algebra generated by $\mathfrak{h}_{t,1}^{\beta}$ in $\mathbb{R} \setminus (a,b)$, and $\mathfrak{h}_{t,2}^{\beta}$.

Lemma 2.7. *Take any $a < b$ and $t > 0$. Conditional on $\mathcal{F}_{\mathrm{ext}}([a,b])$, for (1) law of $\mathfrak{h}_{t,1}^{\beta}$ in $[a,b]$, (2) the rate 2 Brownian bridge connecting $\mathfrak{h}_{t,1}^{\beta}(a)$ and $\mathfrak{h}_{t,1}^{\beta}(b)$, the former is absolutely continuous with respect to the latter, with Radon-Nykodim derivative (for a path B) proportional to $W(B, \mathfrak{h}_{t,2}^{\beta})$, where*

$$W(f,g) = \begin{cases} \exp\Big(-2\int_a^b \exp(f(x)-g(x))dx\Big) & \text{when } \beta = 1, \\ \mathbb{1}[g(x) \le f(x), \forall x \in [a,b]] & \text{when } \beta = \infty. \end{cases} \tag{2.6}$$

For the case where $\beta = \infty$ (the Airy line ensemble setting), this was established in [CH14]. For $\beta = 1$ (the KPZ$_t$ line ensemble setting), such a Gibbs property was first introduced in [CH16]; and the connection between the KPZ$_t$ line ensemble and CDRP was formally established in [Nic21]. The form of the Gibbs property presented here is from [GH22, Proposition 2.6, Theorem 2.8].

A useful consequence of the Gibbs property is the monotonicity recorded below.

Lemma 2.8 (Monotonicity in boundary data). *Fix $k_1 \le k_2 \in \mathbb{Z}$, $a < b$, two pairs of vectors $w^{(i)}, z^{(i)} \in \mathbb{R}^k$ and two pairs of measurable functions $(f^{(i)}, g^{(i)})$ for $i \in \{1,2\}$ such that $w_j^{(1)} \le w_j^{(2)}$ and $z_j^{(1)} \le z_j^{(2)}$ for all $j = 1, \dots, k$ and $f^{(i)} : (a,b) \to \mathbb{R} \cup \{\infty\}$, $g^{(i)} : (a,b) \to \mathbb{R} \cup \{-\infty\}$ and for all $s \in (a,b)$, $f^{(1)}(s) \le f^{(2)}(s)$ and $g^{(1)}(s) \le g^{(2)}(s)$.*

For $i \in \{1,2\}$, let $\mathcal{Q}^{(i)} = \{\mathcal{Q}_j^{(i)}\}_{j=1}^k$ be a $\{1, \dots, k\} \times [a,b]$-indexed line ensemble such that $\mathcal{Q}^{(i)}(a) = w^{(i)}$ and $\mathcal{Q}^{(i)}(b) = z^{(i)}$, with Radon-Nykodim derivative over a family of Brownian bridges given by $\prod_{j=0}^k W(\mathcal{Q}_j^{(i)}, \mathcal{Q}_{j+1}^{(i)})$ for $\beta = 1$, where (for simplicity of notations) $\mathcal{Q}_0^{(i)} = f^{(i)}$ and $\mathcal{Q}_{k+1}^{(i)} = g^{(i)}$. Then there exists a coupling of the laws of $\{\mathcal{Q}_j^{(1)}\}$ and $\{\mathcal{Q}_j^{(2)}\}$ such that almost surely $\mathcal{Q}_j^{(1)}(s) \le \mathcal{Q}_j^{(2)}(s)$ for all $j \in \{k_1, \dots, k_2\}$ and all $s \in (a,b)$.

The same is true in the $\beta = \infty$ (zero-temperature) case if additionally $w_j^{(i)} > w_{j+1}^{(i)}$ and $z_j^{(i)} > z_{j+1}^{(i)}$ for $j = 1, \dots, k-1$, and $f^{(i)}(a) > w_1^{(i)}$, $f^{(i)}(b) > z_1^{(i)}$, $g^{(i)}(a) < w_k$, $g^{(i)}(b) < z_k$, for $i = 1, 2$.

The positive temperature ($\beta = 1$) statements are Lemmas 2.6 and 2.7 of [CH16]. The zero temperature ($\beta = \infty$) statements are Lemmas 2.6 and 2.7 of [CH14]. See also [DM21] and [Dim22] for more detailed proofs of the respective cases.

The following two correlation inequalities can be deduced using the monotonicity property. They are contained in [GH22, Theorem 2.8], and their proofs can be found in [GH22, Appendix A].

Lemma 2.9. *For any $t > 0$, $a < b$, $\beta = 1$ or ∞, and any pair of increasing events A and B in the space of all real continuous functions on $[a,b]$,*

$$\mathbb{P}(\mathfrak{h}_{t,1}^{\beta}|_{[a,b]} \in A,\, \mathfrak{h}_{t,1}^{\beta}|_{[a,b]} \in B) \ge \mathbb{P}(\mathfrak{h}_{t,1}^{\beta}|_{[a,b]} \in A) \cdot \mathbb{P}(\mathfrak{h}_{t,1}^{\beta}|_{[a,b]} \in B).$$

This is the FKG inequality for $\mathfrak{h}_{t,1}^{\beta}|_{[a,b]}$; and for discrete models such as the exponential LPP, such a result follows from the classical FKG inequality.

For any $a < b$, an event A in the space of all real continuous functions on $[a,b]$ is called 'increasing', if for any $f \in A$ and $f \le g$ point-wisely, there is also $g \in A$.

Lemma 2.10. *For any $t > 0$, $a < b$, $\beta = 1$ or ∞, and any increasing event A in the space of all real continuous functions on $[a,b]$,*

$$\mathbb{P}\left(\mathfrak{h}_{t,2}^{\beta}|_{[a,b]} \in A \mid \mathfrak{h}_{t,1}^{\beta}|_{[a,b]}\right) \le \mathbb{P}\left(\mathfrak{h}_{t,1}^{\beta}|_{[a,b]} \in A\right).$$

This is also called the BK inequality for (reweighted) Brownian bridge ensemble, due to the connection between line ensembles and disjoint paths through the RSK correspondence (see also the footnote in [GH22, Page 14]).

2.4. **Existing estimates.** In this subsection, we list some existing estimates of the line ensembles, which hold in both zero and positive temperature settings. For the convenience of notations, we state them in terms of the scaled version $\hat{\mathfrak{h}}_{t,1}^{\beta}$ and $\hat{\mathfrak{h}}_{t,2}^{\beta}$.

All of the results below hold for all $t \ge t_0$, where t_0 is an arbitrary positive number, and all the constants (including all $C, c > 0$) may depend on t_0. We will omit this point in the statements of these results for conciseness.

We next state the one-point upper- and lower-tails for $\hat{\mathfrak{h}}_{t,1}^{\beta}$.

Theorem 2.11 ([GH22, Theorem 1 and Proposition 9.5]). *There exists $L_0 > 0$ such that for any $L > L_0$, we have*

$$\exp\Big(-\tfrac{4}{3}L^{3/2} - CL^{3/4}\Big) < \mathbb{P}\left(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) \in (L, L + \mathrm{d}L)\right) / \,\mathrm{d}L < \exp\Big(-\tfrac{4}{3}L^{3/2} + CL^{3/4}\Big).$$

Using Theorem 2.11 and stochastic domination, one can deduce an upper-tail bound for one-point distribution of $\hat{\mathfrak{h}}^{\beta}_{t,2}$, which will be useful later.

Lemma 2.12. *There exists $L_0 > 0$, such that for any $L > L_0$, we have*

$$\mathbb{P}\left(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L, \hat{\mathfrak{h}}^{\beta}_{t,1}(0) + \hat{\mathfrak{h}}^{\beta}_{t,2}(0) > 2L\right) < \exp\Big(-\tfrac{8}{3}L^{3/2} + CL^{3/4}\Big).$$

Proof. We can write the left-hand side as

$$\int_L^{\infty} \mathbb{P}\left(\hat{\mathfrak{h}}^{\beta}_{t,2}(0) > 2L - \vartheta \mid \hat{\mathfrak{h}}^{\beta}_{t,1}(0) \in (\vartheta, \vartheta + \mathrm{d}\vartheta)\right) \mathbb{P}\left(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) \in (\vartheta, \vartheta + \mathrm{d}\vartheta)\right).$$

By Lemma 2.10, this is upper bounded by

$$\int_L^{\infty} \mathbb{P}\left(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > 2L - \vartheta\right) \mathbb{P}\left(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) \in (\vartheta, \vartheta + \mathrm{d}\vartheta)\right).$$

Then the conclusion follows by using Theorem 2.11. □

We have the following estimate on the one-point lower tail of $\hat{\mathfrak{h}}^{\beta}_{t,1}$.

Theorem 2.13. *There exists $L_0 > 0$, such that for any $L > L_0$, we have*

$$\mathbb{P}\left(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) < -L\right) < \exp(-cL^{5/2}).$$

For $\beta = 1$ this can be deduced from [CG20b, Theorem 1]. For $\beta = \infty$, $\hat{\mathfrak{h}}^{\beta=\infty}_{t,1}(0)$ has GUE Tracy-Widom distribution, whose lower tail is more classical (see e.g., [TW94] or [CG20b, Proposition 5.1])

The process $\hat{\mathfrak{h}}^{\beta}_{t,1}$ is also 1/2-Hölder.

Proposition 2.14. *For any $0 < d \le 1$ and $M > 0$, we have*

$$\mathbb{P}\Big(\sup_{x\in[0,d]} |\hat{\mathfrak{h}}^{\beta}_{t,1}(x) - \hat{\mathfrak{h}}^{\beta}_{t,1}(0)| > Md^{1/2}\Big) < C\exp(-cM^2).$$

For $\beta = \infty$ this is proved in [DV21a, Lemma 6.1] and [Dau23, Lemma 3.4], and their arguments carry over to the $\beta = 1$ setting. We omit the details here.

Finally, we quote the following tent behavior of $\hat{\mathfrak{h}}^{\beta}_{t,1}$, under the one-point upper-tail event.

Theorem 2.15 ([GH22, Theorem 9]). *There exists $L_0 > 0$, such that for any $t > 0$ and $L > t^{1/3}L_0$, we have*

$$\mathbb{P}\left(\sup_{x\in[-L^{1/2},L^{1/2}]} \left|\hat{\mathfrak{h}}^{\beta}_{t,1}(x) - L + 2L^{1/2}|x|\right| > ML^{1/4} \;\middle|\; \hat{\mathfrak{h}}^{\beta}_{t,1}(0) \in (L, L + \mathrm{d}L)\right) < \exp(-cM^2),$$

for any $0 < M < cL^{3/4}$. The same is true under the conditioning $\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L$.

We note that the $\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L$ conditioning version is not contained in [GH22, Theorem 9]; and we will give a proof of it in Appendix B.

We next give a more precise comparison of the tent with Brownian bridges.

Lemma 2.16. *Take any $L > 0$. Consider the following processes, each defined on $[0, L^{1/2}/2]$,*

$$x \mapsto \hat{\mathfrak{h}}^{\beta}_{t,1}(x) - L, \quad x \mapsto \hat{\mathfrak{h}}^{\beta}_{t,1}(-x) - L,$$

conditional on $\hat{\mathfrak{h}}^{\beta}_{t,1}(0) \in (L, L + \mathrm{d}L)$. Also, consider the processes (each defined on $[0, L^{1/2}/2]$ as well)

$$x \mapsto B_1(x) + 2xL^{-1/2}(\hat{\mathfrak{h}}^{\beta}_{t,1}(L^{1/2}/2) - L), \quad x \mapsto B_2(x) + 2xL^{-1/2}(\hat{\mathfrak{h}}^{\beta}_{t,1}(-L^{1/2}/2) - L,$$

also conditional on $\hat{\mathfrak{h}}^{\beta}_{t,1}(0) \in (L, L + \mathrm{d}L)$, where B_1, B_2 are rate 2 Brownian bridges in $[0, L^{1/2}/2]$, independent of each other and independent of $\hat{\mathfrak{h}}^{\beta}_{t,1}$. They can be coupled so that under an event with probability $> 1 - C\exp(-cL^{3/2})$ for both, their Radon-Nikodym derivative is $1 + O(\exp(-cL))$.

The proof of this as well as the following lemma will be given in Appendix B.

Lemma 2.17. *For any $I \subseteq [-\frac{1}{2}L^{1/2}, \frac{1}{2}L^{1/2}]$ and $\sigma_I = \sup_{x\in I} |x|^{1/2}$, and $0 < M < L^{3/4}$,*

$$\mathbb{P}\left(\sup_{x\in I} |\hat{\mathfrak{h}}^{\beta}_{t,1}(x) - (L - 2L^{1/2}|x|)| > M\sigma_I \;\Big|\; \hat{\mathfrak{h}}^{\beta}_{t,1}(0) \in (L, L + \mathrm{d}L)\right) < C\exp(-cM^2),$$

$$\mathbb{P}\left(\sup_{x\in I} |\hat{\mathfrak{h}}^{\beta}_{t,1}(x) - (L - 2L^{1/2}|x|)| > M\sigma_I \;\Big|\; \hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L\right) < C\exp\left(-c(M^2 \wedge M\sigma_I L^{1/2})\right).$$

Using Theorem 2.15 and Lemma 2.17, one can quickly deduce the following using a union bound.

Corollary 2.18. *There exists $L_0 > 0$ such that for any $L > L_0$ and $0 < M < L^{3/4}$, and any $a > 0$, we have*

$$\mathbb{P}\left(\sup_{|x|\le L^{1/2}} |\hat{\mathfrak{h}}^{\beta}_{t,1}(x) - (L - 2L^{1/2}|x|)|(|\log(|x|a)| + 1)^{-1}|x|^{-1/2} > M \;\Big|\; \hat{\mathfrak{h}}^{\beta}_{t,1}(0) \in (L, L + \mathrm{d}L)\right)$$
$$< C\exp(-cM^2).$$

2.4.1. *Note on the lower bound of t.* As mentioned at the beginning of this subsection, all these estimates are stated with a uniform lower bound on t in the positive temperature setting. (In the zero temperature setting all these estimates for smaller t can also be deduced, using scaling invariance of the directed landscape.) Our proofs, as written in later sections using these results, will actually lead to the convergence of Γ_0 in a weaker topology.

To achieve the convergence in the stronger topology as stated in Theorem 1.2, analogs of the estimates in this subsection for smaller t would be needed. One issue is that the small t versions of Theorem 2.11 and Theorem 2.15 are not currently available in [GH22]. More precisely, the arguments in [GH22] take as input a priori tail estimates from [CG20a], which hold for $t > t_0$ and $L > L_0$ with L_0 possibly depending on t_0. However, analogous estimates are also available for arbitrary $t > 0$ from [DG23], namely Theorems 1.4 (upper bound on upper tail) and 1.7 (upper bound on lower tail) there. Using these inputs the estimates quoted in this subsection can be upgraded to cover small $t > 0$, leading to the convergence in Theorem 1.2. We will give more explanations at the end of Section 8.

2.5. **Gaussian and Brownian bridge estimates.** Here we recall two well-known bounds for Gaussian random variables and Brownian bridges. We start with a standard bound on the tail of centered Gaussian random variables.

Lemma 2.19. *For $\sigma > 0$ and $x > 0$,*

$$\frac{1}{\sqrt{2\pi}} \cdot \frac{\sigma}{x}\left(1 - \frac{\sigma^2}{x^2}\right)\exp\left(-\frac{x^2}{2\sigma^2}\right) \le \mathbb{P}\left(\mathcal{N}(0, \sigma^2) \ge x\right) \le \frac{1}{\sqrt{2\pi}} \cdot \frac{\sigma}{x}\exp\left(-\frac{x^2}{2\sigma^2}\right).$$

Proof. We set $\sigma = 1$ without loss of generality. Now we write $\mathbb{P}\left(\mathcal{N}(0,\sigma^2) \geq x\right)$ as an integral of the normal density and obtain the claimed bounds by doing integration by parts one for the upper bound and again for the lower bound. $\square$

The following is a tail bound on the supremum of a rate one Brownian bridge B over a subinterval of its domain of definition I. Note that the scale of fluctuation adapts to the interval and the bound holds for all $M > 0$, which will be useful for our applications. The result is stated and proved as [GH22, Lemma 3.6], and makes use of the well-known fact that $\sup_I B$ has Rayleigh distribution.

Lemma 2.20 ([GH22, Lemma 3.6])**.** *Let $I, J \subseteq \mathbb{R}$ be intervals with $J \subseteq I$, and $B : I \to \mathbb{R}$ be a Brownian bridge with $B(\inf I) = B(\sup I) = 0$. Let $\sigma_J^2 = \max_{x\in J} \mathrm{Var}(B(x))$. Then, for all $M > 0$,*

$$\mathbb{P}\left(\sup_{x\in J} B(x) \geq M\sigma_J\right) \leq 3\exp\left(-\tfrac{1}{8}M^2\right).$$

3. Coalescence and Brownian bridge comparison under upper tail

This section develops the key coalescence estimates that our analysis relies on. It will be convenient to denote (within this section)

$$\mathcal{S}_t^{\beta=1}(x,y) = t^{-1/3}(\log \mathcal{Z}(t^{2/3}x, 0; t^{2/3}y, t) + t/12),$$

for any $t > 0$. One may interpret this as a 'rescaled KPZ sheet'. The zero temperature analog, that is the Airy sheet, will be denoted by

$$\mathcal{S}_t^{\beta=\infty}(x,y) = t^{-1/3}\mathcal{L}(t^{2/3}x, 0; t^{2/3}y, 1).$$

Note that actually the law of $\mathcal{S}_t^{\beta=\infty}$ is the same for any $t > 0$.

From these definitions, we see that for any $t > 0$ and $\beta = 1, \infty$, we have $\mathcal{S}_t^{\beta}(0,x) = \hat{\mathfrak{h}}_{t,1}^{\beta}(x) = t^{-1/3}\mathfrak{h}_{t,1}^{\beta}(t^{2/3}x)$. By the shear, shift, and reflection invariance properties of $\mathcal{L}$ and $\mathcal{Z}$ (introduced in Section 2.1 and Section 2.2.1), we have that $\mathcal{S}_t^{\beta}$ has the same law as

$$(x,y) \mapsto \mathcal{S}_t^{\beta}(x+a, y+b) + (x+a-y-b)^2 - (x-y)^2, \quad (x,y) \mapsto \mathcal{S}_t^{\beta}(-x,-y), \quad (x,y) \mapsto \mathcal{S}_t^{\beta}(y,x).$$

We next deduce two uniform bounds of $\mathcal{S}_t^{\beta}$. They hold for $\beta = 1, \infty$ and any $t > t_0$, where t_0 is an arbitrary positive number, and the constants $C, c > 0$ may depend on t_0. The first is a Hölder estimate.

Lemma 3.1. *For any $0 < d \leq 1$ and $M > 0$, we have*

$$\mathbb{P}\left(\sup_{x,y\in[0,d]} |\mathcal{S}_t^{\beta}(x,y) - \mathcal{S}_t^{\beta}(0,0)| > Md^{1/2}\right) < C\exp(-cM^2).$$

Proof. Using (2.1) or (2.4), for any $x, y \in [0,d]$, we have

$$-\mathcal{S}_t^{\beta}(0,0) + \mathcal{S}_t^{\beta}(0,y) + \mathcal{S}_t^{\beta}(x,0) \leq \mathcal{S}_t^{\beta}(x,y) \leq -\mathcal{S}_t^{\beta}(d,0) + \mathcal{S}_t^{\beta}(d,y) + \mathcal{S}_t^{\beta}(x,0).$$

By Proposition 2.14, and symmetries of $\mathcal{S}_t^{\beta}$, we have

$$\mathbb{P}\left(\sup_{x\in[0,d]} |\mathcal{S}_t^{\beta}(0,x) - \mathcal{S}_t^{\beta}(0,0)| > Md^{1/2}\right), \mathbb{P}\left(\sup_{x\in[0,1]} |\mathcal{S}_t^{\beta}(x,0) - \mathcal{S}_t^{\beta}(0,0)| > Md^{1/2}\right) < C\exp(-cM^2),$$

and

$$\mathbb{P}\left(\sup_{x\in[0,d]} |\mathcal{S}_t^{\beta}(d,x) - \mathcal{S}_t^{\beta}(d,0)| > Md^{1/2}\right), \mathbb{P}\left(|\mathcal{S}_t^{\beta}(d,0) - \mathcal{S}_t^{\beta}(0,0)| > Md^{1/2}\right) < C\exp(-cM^2),$$

therefore the conclusion holds. $\square$

Lemma 3.2. *There exists a random variable $H > 0$, such that $\mathbb{P}(H > M) < C\exp(-cM^{3/2})$, and*

$$|\mathcal{S}_t^\beta(x,y) + (x-y)^2| < H + \log(|x| + |y| + 2), \quad \forall x, y \in \mathbb{R}.$$

Proof. It suffices to show that, for any $M > 0$,

$$\mathbb{P}\left(\sup_{x,y\in[0,1]} |\mathcal{S}_t^\beta(x,y) + (x-y)^2| > M\right) < C\exp(-cM^{3/2}). \tag{3.1}$$

Then the conclusion follows by splitting $\mathbb{R}$ into intervals of length 1, using the shear and shift invariance properties of $\mathcal{S}_t^\beta$, and taking a union bound.

As for (3.1), we just apply Lemma 3.1 for $d = 1$, and use that $\mathbb{P}(|\mathcal{S}_t^\beta(0,0)| > M) < C\exp(-cM^{3/2})$, which can be obtained from Theorems 2.11 and 2.13. $\square$

FIGURE 3. An illustration of the shift-invariance: the joint distributions of the three passage times/partition functions in the left and right panels are the same.

The following remarkable shift-invariance property (illustrated in Figure 3) proved in [BGW22] will also be a key input. It follows from [BGW22, Theorems 7.8, 7.10] immediately.

Lemma 3.3. *Take any $m \in \mathbb{N}$, and $x_1 \le \cdots \le x_m$, $y_1 \ge \cdots \ge y_m$, and $x_1' \le \cdots \le x_m'$, $y_1' \ge \cdots \ge y_m'$, such that $x_i - y_i = x_i' - y_i'$ for any $i \in [\![1, m]\!]$. Then $\{\mathcal{S}_t^\beta(x_i, y_i)\}_{i=1}^m$ and $\{\mathcal{S}_t^\beta(x_i', y_i')\}_{i=1}^m$ are equal in distribution.*

We next give the behavior of $\mathcal{S}_t^\beta$ under upper tail events. To be concise, for the rest of this section we fix $t > 0$. All the constants (including all $C, c > 0$) are allowed to depend on t. And β is taken to equal either 1 or ∞.

3.1. **Coalescence and independent tents with Brownian bridges.** We now give our main coalescence estimate, in the form of stating that the quadrangle inequalities from (2.1) and (2.4) are sharp under the upper tail. Its proof will be given in Section 3.2.

Proposition 3.4. *Take any $L > 0$ and $L^+ > L + \exp(-0.001L^{3/2})$, and denote $H = 10^{-6}L^{1/2}$. When $\beta = \infty$, we have that*

$$\mathbb{P}\left(\mathcal{S}_t^\beta(x,y) = \mathcal{S}_t^\beta(x,0) + \mathcal{S}_t^\beta(0,y) - \mathcal{S}_t^\beta(0,0),\ \forall |x|, |y| \le H \mid L < \mathcal{S}_t^\beta(0,0) < L^+\right) > 1 - C\exp(-cL^{3/2}).$$

When $\beta = 1$, the same bound holds when the event is replaced by

$$|\mathcal{S}_t^\beta(x,y) - (\mathcal{S}_t^\beta(x,0) + \mathcal{S}_t^\beta(0,y) - \mathcal{S}_t^\beta(0,0))| < C\exp(-cL), \quad |x|, |y| \le H.$$

The connection between this estimate and coalescence is that, at zero temperature ($\beta = \infty$), the equality is equivalent to the coalescence of a family of geodesics, according to Lemma 2.3. At positive temperature, almost surely the quadrangle inequality is strict (see (2.4)), so the equality is replaced by an upper bound of $C\exp(-cL)$, which is roughly the probability for the corresponding polymers to be disjoint given the field $\mathcal{Z}$.

As we have seen from e.g. Theorem 2.15, there are tent behaviors under the upper tail event. The following proposition states that conditional on the upper tail event, the two tents seen from both positive and negative directions are roughly independent, and are close to Brownian bridges. It is a two-sided version of Lemma 2.16.

Proposition 3.5. *Take any $L > 0$ and $L^+ > L + \exp(-0.001L^{3/2})$. (The choice of L^+ is to ensure a lower bound of $\mathbb{P}(L < \mathcal{S}_t^\beta(0,0) < L^+)$ to be used in the proof.) Let $H = 10^{-6}L^{1/2}$. Consider the following processes, each defined on $[0,H]$,*

$$x \mapsto \mathcal{S}_t^\beta(0,x) - \mathcal{S}_t^\beta(0,0), \quad x \mapsto \mathcal{S}_t^\beta(0,-x) - \mathcal{S}_t^\beta(0,0),$$
$$x \mapsto \mathcal{S}_t^\beta(x,0) - \mathcal{S}_t^\beta(0,0), \quad x \mapsto \mathcal{S}_t^\beta(-x,0) - \mathcal{S}_t^\beta(0,0),$$

conditional on $L < \mathcal{S}_t^\beta(0,0) < L^+$. Further, also consider the processes (each defined on $[0,H]$ as well)

$$x \mapsto B_1(x) + x(\mathcal{S}_t^\beta(0,H) - \mathcal{S}_t^\beta(0,0))/H, \quad x \mapsto B_2(x) + x(\mathcal{S}_t^\beta(0,-H) - \mathcal{S}_t^\beta(0,0))/H,$$
$$x \mapsto B_3(x) + x(\mathcal{S}_t^\beta(H,0) - \mathcal{S}_t^\beta(0,0))/H, \quad x \mapsto B_4(x) + x(\mathcal{S}_t^\beta(-H,0) - \mathcal{S}_t^\beta(0,0))/H,$$

also conditional on $L < \mathcal{S}_t^\beta(0,0) < L^+$, where B_1, B_2, B_3, B_4 are four rate 2 Brownian bridges in $[0,H]$, independent of each other and independent of $\mathcal{S}_t^\beta$.

There is another measure on $\mathcal{C}([0,H],\mathbb{R})^4$, such that (1) its Radon-Nikodym derivative over the first set of processes is $1 + O(\exp(-cL))$, and (2) it can be coupled with the second set of processes such that with probability $> 1 - C\exp(-cL^{3/2})$, the L^∞ distance between them is $< C\exp(-cL)$.

The proof of Proposition 3.5 involves reducing it to the following statement, using the shift-invariance of Lemma 3.3 and Proposition 3.4.

Lemma 3.6. *Take any $L > 0$ and denote $H = 10^{-6}L^{1/2}$. Consider the following processes, each defined on $[0,H]$,*

$$x \mapsto \mathcal{S}_t^\beta(0,x) - \mathcal{S}_t^\beta(0,0), \quad x \mapsto \mathcal{S}_t^\beta(0,-x) - \mathcal{S}_t^\beta(0,0),$$
$$x \mapsto \mathcal{S}_t^\beta(0,-x-H) - \mathcal{S}_t^\beta(0,-H), \quad x \mapsto \mathcal{S}_t^\beta(0,x+H) - \mathcal{S}_t^\beta(0,H),$$

conditional on $\mathcal{S}_t^\beta(0,0) \in (L, L + \mathrm{d}L)$. Also, consider the processes (each defined on $[0,H]$ as well)

$$x \mapsto B_1(x) + x(\mathcal{S}_t^\beta(0,H) - \mathcal{S}_t^\beta(0,0))/H, \quad x \mapsto B_2(x) + x(\mathcal{S}_t^\beta(0,-H) - \mathcal{S}_t^\beta(0,0))/H,$$
$$x \mapsto B_3(x) + x(\mathcal{S}_t^\beta(0,-2H) - \mathcal{S}_t^\beta(0,-H))/H, \quad x \mapsto B_4(x) + x(\mathcal{S}_t^\beta(0,2H) - \mathcal{S}_t^\beta(0,H))/H,$$

also conditional on $\mathcal{S}_t^\beta(0,0) \in (L, L + \mathrm{d}L)$, where B_1, B_2, B_3, B_4 are four rate 2 Brownian bridges in $[0,H]$, independent of each other and independent of $\mathcal{S}_t^\beta$. They can be coupled so that under an event with probability $> 1 - C\exp(-cL^{3/2})$ for both, their Radon-Nikodym derivative is $1 + O(\exp(-cL))$.

This lemma directly follows from Lemma 2.16, and we omit the details. We now prove Proposition 3.5 assuming Proposition 3.4 and Lemma 3.6.

Proof of Proposition 3.5. By shift-invariance (Lemma 3.3), in Lemma 3.6 we can replace $\mathcal{S}_t^\beta(0,x+H)$ by $\mathcal{S}_t^\beta(-x,H)$ and $\mathcal{S}_t^\beta(0,-x-H)$ by $\mathcal{S}_t^\beta(x,-H)$ (see Figure 4). Note that Lemma 3.3 is stated in terms of finitely many points, but we can do the replacement for each $x \in [0,H]$ simultaneously since $\mathcal{S}_t^\beta$ is continuous. Therefore we get the following statement. Consider the processes

$$x \mapsto \mathcal{S}_t^\beta(0,x) - \mathcal{S}_t^\beta(0,0), \quad x \mapsto \mathcal{S}_t^\beta(0,-x) - \mathcal{S}_t^\beta(0,0),$$
$$x \mapsto \mathcal{S}_t^\beta(x,-H) - \mathcal{S}_t^\beta(0,-H), \quad x \mapsto \mathcal{S}_t^\beta(-x,H) - \mathcal{S}_t^\beta(0,H), \tag{3.2}$$

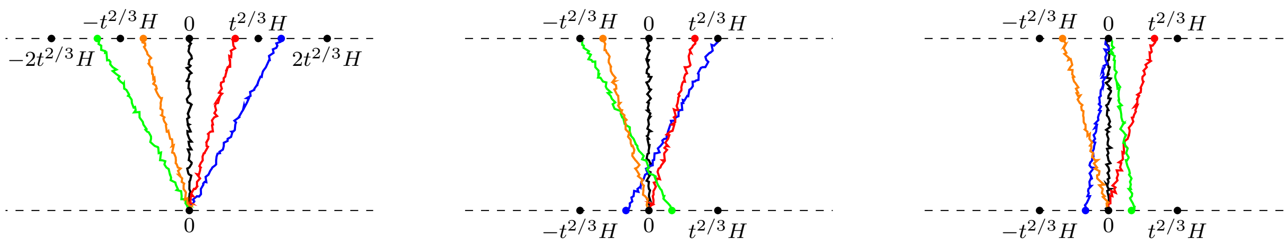

FIGURE 4. An illustration of transforming the weights of varying one side (left panel) into varying both sides (middle panel) using shift-invariance, then to two tents (right panel) by coalescence.

on $[0, H]$, conditional on $L < \mathcal{S}_t^\beta(0,0) < L^+$; and the processes (also on $[0, H]$)

$$x \mapsto B_1(x) + x(\mathcal{S}_t^\beta(0,H) - \mathcal{S}_t^\beta(0,0))/H, \quad x \mapsto B_2(x) + x(\mathcal{S}_t^\beta(0,-H) - \mathcal{S}_t^\beta(0,0))/H,$$

$$x \mapsto B_3(x) + x(\mathcal{S}_t^\beta(H,-H) - \mathcal{S}_t^\beta(0,-H))/H, \quad x \mapsto B_4(x) + x(\mathcal{S}_t^\beta(-H,H) - \mathcal{S}_t^\beta(0,H))/H, \tag{3.3}$$

conditional on $L < \mathcal{S}_t^\beta(0,0) < L^+$, where B_1, B_2, B_3, B_4 are four rate 2 Brownian bridges in $[0, H]$, independent of each other and independent of $\mathcal{S}_t^\beta$. They can be coupled so that under an event with probability $> 1 - C\exp(-cL^{3/2})$ for both, their Radon-Nikodym derivative is $1 + O(\exp(-cL))$.

By Proposition 3.4, with probability $> 1 - C\exp(-cL^{3/2})$, we have

$$|(\mathcal{S}_t^\beta(x,-H) - \mathcal{S}_t^\beta(0,-H)) - (\mathcal{S}_t^\beta(x,0) - \mathcal{S}_t^\beta(0,0))| < C\exp(-cL),$$

$$|(\mathcal{S}_t^\beta(-x,H) - \mathcal{S}_t^\beta(0,H)) - (\mathcal{S}_t^\beta(-x,0) - \mathcal{S}_t^\beta(0,0))| < C\exp(-cL),$$

for any $x \in [0, H]$; in particular,

$$|(\mathcal{S}_t^\beta(H,-H) - \mathcal{S}_t^\beta(0,-H)) - (\mathcal{S}_t^\beta(H,0) - \mathcal{S}_t^\beta(0,0))| < C\exp(-cL),$$

$$|(\mathcal{S}_t^\beta(-H,H) - \mathcal{S}_t^\beta(0,H)) - (\mathcal{S}_t^\beta(-H,0) - \mathcal{S}_t^\beta(0,0))| < C\exp(-cL).$$

By plugging these estimates into (3.2) and (3.3) respectively, we get the conclusion. □

It remains to prove Proposition 3.4, which we accomplish in the next subsection.

3.2. **Coalescence of polymers.** We note that at zero temperature, there have been many studies on the coalescence of geodesics, in the directed landscape as well as the exactly solvable LPP models (in e.g. [Pim16, Zha20, BSS19, SS20, BF22]). However, as far as we know there have not been many previous studies on polymer coalescence in the positive temperature setting.

Using the quadrangle inequalities (2.1) and (2.4), Proposition 3.4 can be reduced to the following lemma.

Lemma 3.7. *Take any $L > 0$ and $L^+ > L + \exp(-0.001L^{3/2})$, and denote $H = 10^{-6}L^{1/2}$. Then when $\beta = \infty$, we have*

$$\mathbb{P}\left(\mathcal{S}_t^\beta(-H,-H) + \mathcal{S}_t^\beta(H,H) > \mathcal{S}_t^\beta(-H,H) + \mathcal{S}_t^\beta(H,-H) \mid L < \mathcal{S}_t^\beta(0,0) < L^+\right) < C\exp(-cL^{3/2}).$$

And when $\beta = 1$, the same estimate holds with the event replaced by

$$\mathcal{S}_t^\beta(-H,-H) + \mathcal{S}_t^\beta(H,H) - \mathcal{S}_t^\beta(-H,H) - \mathcal{S}_t^\beta(H,-H) > C\exp(-cL).$$

In the $\beta = \infty$ setting, in light of Lemma 2.3, the event whose probability we wish to bound is equivalent to that (in the directed landscape) the geodesic from $(-t^{2/3}H, 0)$ to $(-t^{2/3}H, t)$ and the geodesic from $(t^{2/3}H, 0)$ to $(t^{2/3}H, t)$ are disjoint. Such disjointness is unlikely to happen under the upper large since then geodesics tend to merge into the geodesic from $(0, 0)$ to $(0, t)$, shortly away from the endpoints. When $\beta = 1$, the event is instead interpreted as that the multi-point partition function from $-t^{2/3}H, t^{2/3}H$ at time 0 to $-t^{2/3}H, t^{2/3}H$ at time t is comparable to the product of the two individual ones. This can be then understood as a positive temperature form of 'disjointness'.

For multi-point passage times and multi-point partition functions of size 2, we can only estimate them when each side of the endpoints is at a single point, using Lemma 2.12. In the proof of Lemma 3.7 we will need to upper bound those from $-t^{2/3}H, t^{2/3}H$ at time 0 to $-t^{2/3}H, t^{2/3}H$ at time t. Our strategy is to do surgeries around time 0 and time 1 (see Figure 5): we instead upper bound those from $0, 0$ at time $-Kt$ to $0, 0$ at time $(1 + K)t$. Here $K > 0$ is a small constant. Then we use the composition laws and lower bound various passage times and partition functions between time $-Kt$ and 0, and time t and $(1 + K)t$. We note that such surgery arguments have appeared in the directed landscape and LPP models, see e.g. [Ham20].

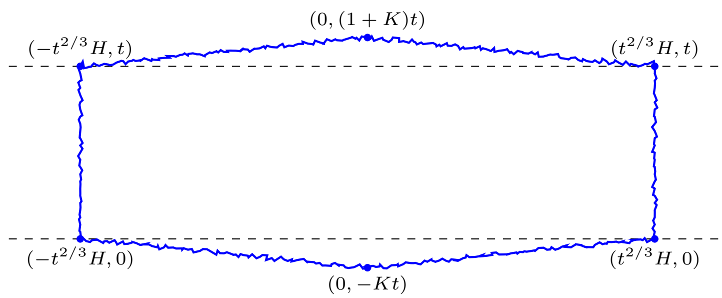

FIGURE 5. An illustration of the surgery, bringing the endpoints of multi-point passage times and multi-point partition functions together.

We now give the details. For correctness, we take $K = 10^{-6}$ in the rest of this section. First, we note that by taking C large and c small, it suffices to prove Lemma 3.7 for large enough L. By Lemma 3.3, $\mathcal{S}_t^\beta(-H, H) + \mathcal{S}_t^\beta(H, -H)$ and $\mathcal{S}_t^\beta(0, 0)$ have the same joint distribution as $\mathcal{S}_t^\beta(0, 2H) + \mathcal{S}_t^\beta(0, -2H)$ and $\mathcal{S}_t^\beta(0, 0)$. Then by Theorem 2.15 we have

$$\mathbb{P}\left(\mathcal{S}_t^\beta(-H, H) + \mathcal{S}_t^\beta(H, -H) < 2L - 8L^{1/2}H - 0.001L \mid L < \mathcal{S}_t^\beta(0, 0) < L^+\right) < \exp(-cL^{3/2}). \tag{3.4}$$

Also by Theorem 2.11, for large L we have

$$\mathbb{P}\left(L < \mathcal{S}_t^\beta(0, 0) < L^+\right) > c\exp(-(4/3 + 0.005)L^{3/2}). \tag{3.5}$$

3.2.1. *Directed landscape ($\beta = \infty$) setting.* In this case, we can assume $t = 1$ since the law of $\mathcal{S}_t^{\beta=\infty}$ is independent of t. Recall that $\mathcal{S}_1^{\beta=\infty} = \mathcal{L}(\cdot, 0; \cdot, 1)$. For the event in Lemma 3.7 whose probability we wish to bound, by Lemma 2.1 and Lemma 2.3, it implies that

$$\mathcal{L}((-H, H), 0; (-H, H), 1) = \mathcal{L}(-H, 0; -H, 1) + \mathcal{L}(H, 0; H, 1) > \mathcal{L}(-H, 0; H, 1) + \mathcal{L}(H, 0; -H, 1).$$

Therefore, to prove Lemma 3.7, it suffices to prove the following estimate.

Lemma 3.8. *For $L > 0$ large enough and $H = 10^{-6}L^{1/2}$, we have*

$$\mathbb{P}\big[\mathcal{L}((-H, H), 0; (-H, H), 1) > 1.99L\big] < C\exp\big(-(4/3 + 0.01)L^{3/2}\big).$$

Using this lemma and (3.5), we can bound the probability of the same event conditional on $L < \mathcal{S}_t^\beta(0,0) < L^+$, by $C\exp(-0.005L^{3/2})$. Then by (3.4), and note that $2L - 8L^{1/2}H - 0.001L > 1.99L$, we get Lemma 3.7.

Proof of Lemma 3.8. By using (2.2) a couple of times, we have

$$\mathcal{L}((0,0),-K;(-H,H),0) \geq \mathcal{L}((0,0),-K;(0,0),0)+\mathcal{L}(0,-K;-H,0)+\mathcal{L}(0,-K;H,0)-2\mathcal{L}(0,-K;0,0).$$

Denote $w := \mathbb{P}(\mathcal{L}((0,0),-K;(0,0),0) > 0)$. There exists $c_* > 0$, such that

$$\mathbb{P}\Big(\mathcal{L}(0,-K;-H,0)+H^2/K < -c_*\Big),\ \mathbb{P}\Big(\mathcal{L}(0,-K;H,0)+H^2/K < -c_*\Big),\ \mathbb{P}\Big(\mathcal{L}(0,-K;0,0) > c_*\Big) < w/4.$$

We note that w, c_* are independent of L. Therefore

$$\mathbb{P}\Big(\mathcal{L}((0,0),-K;(-H,H),0) > -2H^2/K - 4c_*\Big) > w/4.$$

Similarly,

$$\mathbb{P}\Big(\mathcal{L}((-H,H),1;(0,0),1+K) > -2H^2/K - 4c_*\Big) > w/4.$$

Note that

$$\begin{aligned}&\mathcal{L}_2(0,-K;0,1+K)\\ &\qquad \geq \mathcal{L}((0,0),-K;(-H,H),0) + \mathcal{L}((-H,H),0;(-H,H),1) + \mathcal{L}((-H,H),1;(0,0),1+K).\end{aligned}$$

So we have

$$\mathbb{P}\left(\mathcal{L}_2(0,-K;0,1+K) > 1.99L - 4H^2/K - 8c_*\right) \geq \mathbb{P}\Big(\mathcal{L}((-H,H),0;(-H,H),1) > 1.99L\Big)w^2/16.$$

Using Lemma 2.12 and the fact that $\mathcal{L}_2(0,-K;0,1+K) \leq 2\mathcal{L}(0,-K;0,1+K)$ almost surely, we can bound the left-hand side by $\exp\big(-(4/3+0.01)L^{3/2}\big)$. Thus the conclusion follows. $\square$

3.2.2. *Positive temperature ($\beta = 1$) setting.* Recall from Section 2.2.2 the notation $\mathcal{M}$ for multi-point partition functions. We can then write

$$\begin{aligned}&\exp\left(t^{1/3}(\mathcal{S}_t^\beta(-H,-H) + \mathcal{S}_t^\beta(H,H) - \mathcal{S}_t^\beta(-H,H) - \mathcal{S}_t^\beta(H,-H))\right) - 1\\ &\qquad = \frac{2t^{4/3}H^2\mathcal{M}((-t^{2/3}H,t^{2/3}H),0;(-t^{2/3}H,t^{2/3}H),t)}{\exp\left(t^{1/3}(\mathcal{S}_t^\beta(-H,H) + \mathcal{S}_t^\beta(H,-H)) - t/6\right)}.\end{aligned} \tag{3.6}$$

Therefore, to prove Lemma 3.7, the main task is to prove the following estimate.

Lemma 3.9. *For $L > 0$ large enough and $H = 10^{-6}L^{1/2}$, we have*

$$\mathbb{P}\Big(\mathcal{M}\big((-t^{2/3}H,t^{2/3}H),0;(-t^{2/3}H,t^{2/3}H),t)\big) > \exp\big(t^{1/3}\cdot 1.99L\big)\Big) < C\exp\big(-(4/3+0.01)L^{3/2}\big).$$

Using this lemma and (3.5), we can bound the probability of the same event conditional on $L < \mathcal{S}_t^\beta(0,0) < L^+$, by $C\exp(-0.005L^{3/2})$. Then by (3.4), conditional on $L < \mathcal{S}_t^\beta(0,0) < L^+$, with probability $> 1 - C\exp(-cL^{3/2})$ the expression (3.6) is upper bounded by

$$\frac{2t^{4/3}H^2\exp\left(t^{1/3}\cdot 1.99L\right)}{\exp\left(t^{1/3}\cdot(2L - 8L^{1/2}H - 0.001L) - t/6\right)} < C\exp(-cL).$$

Thus we get Lemma 3.7.

The rest of this section is devoted to proving Lemma 3.9. We let $\mathcal{E}$ denote the event whose probability we wish to bound, in the statement of Lemma 3.9. By Lemma 2.12, we have

$$\mathbb{P}\Big(\mathcal{Z}(0,-Kt;0,(1+K)t) > \exp\left(t^{1/3}\cdot 0.98L\right),\ \mathcal{Z}_2(0,-Kt;0,(1+K)t) > \exp\left(t^{1/3}\cdot 1.96L\right)\Big)$$

$$< C \exp\big(-(8/3-0.1)L^{3/2}\big).$$

By the continuity of $\mathcal{M}$ (Lemma 2.6), we have almost surely

$$2^{-1}(1+2K)t\mathcal{M}((-\delta,\delta),-Kt;(-\delta,\delta),(1+K)t) \to \mathcal{Z}_2(0,-Kt;0,(1+K)t),$$

as $\delta \to 0$ from the right. Then there exists small enough $\delta_L > 0$, such that $\mathbb{P}(\mathcal{E}^+) < C\exp\big(-(8/3-0.1)L^{3/2}\big)$, with $\mathcal{E}^+$ being the event where

$$\mathcal{Z}(0,-Kt;0,(1+K)t) > \exp\big(t^{1/3}\cdot 0.98L\big),$$
$$\mathcal{M}\big((-\delta_L,\delta_L),-Kt;(-\delta_L,\delta_L),(1+K)t\big) > \exp\big(t^{1/3}\cdot 1.98L\big).$$

We will show that conditional on $\mathcal{E}$, with positive probability (independent of L), $\mathcal{E}^+$ holds. For this, we define the following events.

By Lemma 2.6, there exists a small number $w > 0$, such that

$$\mathbb{P}\left(\mathcal{M}((-x,x),-Kt;(-w,w),0) > w\right) > w$$

for any $|x| \le w$. We let $\mathcal{E}_1$ be the event where

$$\mathcal{M}\big((-\delta_L,\delta_L),-Kt;(-w,w),0\big) > w, \quad \mathcal{M}\big((-w,w),t;(-\delta_L,\delta_L,(1+K)t\big) > w.$$

Then $\mathbb{P}(\mathcal{E}_1) > w^2$.

We let $\mathcal{E}_2$ be the event where for any $x,y \in \mathbb{R}$,

$$\left|(Kt)^{-1/3}(\log \mathcal{Z}((Kt)^{2/3}x,-Kt;(Kt)^{2/3}y,0)+Kt/12)+(x-y)^2\right| < 10^{-3}L + \log(|x-y|+2),$$

and

$$\left|(Kt)^{-1/3}(\log \mathcal{Z}((Kt)^{2/3}x,t;(Kt)^{2/3}y,(1+K)t)+Kt/12)+(x-y)^2\right| < 10^{-3}L + \log(|x-y|+2).$$

Then by Lemma 3.2, we have $\mathbb{P}(\mathcal{E}_2) > 1 - C\exp(-cL^{3/2})$.

By Lemma 3.1 and the shift invariance of $\mathcal{Z}$, we have $\mathbb{P}(\mathcal{E}^*) > 1 - C\exp(-cL^2)$, for $\mathcal{E}^*$ denoting the event where

$$\sup_{x,y\in[-H-1,-H]} \left|\mathcal{S}_t^\beta(x,y) - \mathcal{S}_t^\beta(-H,-H)\right|, \quad \sup_{x,y\in[H,H+1]} \left|\mathcal{S}_t^\beta(x,y) - \mathcal{S}_t^\beta(H,H)\right| < 10^{-10}L.$$

Lemma 3.10. *We have $\mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}^* \cap \mathcal{E} \subset \mathcal{E}^+$.*

Proof. In this proof we always take

$$x,x' \in [-t^{2/3}(H+1),-t^{2/3}H], \quad y,y' \in [t^{2/3}H, t^{2/3}(H+1)].$$

We will use $\mathcal{E}_1 \cap \mathcal{E}_2$ to lower bound $\mathcal{M}((-\delta_L,\delta_L),-Kt;(x,y),0)$ and $\mathcal{M}((x',y'),t;(-\delta_L,\delta_L),(1+K)t)$, and $\mathcal{Z}(0,-Kt;x,0)$, $\mathcal{Z}(x',t;0,(1+K)t)$; then we will lower bound $\mathcal{M}((x,y),0;(x',y'),t)$ and $\mathcal{Z}(x,0;x',t)$ for these x,x',y,y', using $\mathcal{E}\cap\mathcal{E}^*$. Then using the composition laws (1.1) and (2.5), we obtain $\mathcal{E}^+$.

Step 1. Using Lemma 2.5, we have that

$$(y-x)\mathcal{M}((-\delta_L,\delta_L),-Kt;(x,y),0) > 2w\mathcal{M}((-\delta_L,\delta_L),-Kt;(-w,w),0) \times \frac{\mathcal{Z}(-\delta_L,-Kt;x,0)\mathcal{Z}(\delta_L,-Kt;y,0)}{\mathcal{Z}(-\delta_L,-Kt;-w,0)\mathcal{Z}(\delta_L,-Kt;w,0)}.$$

Under $\mathcal{E}_1$, the first factor in the right-hand side is $> 2w^2$; and under $\mathcal{E}_2$, the second factor in the right hand side is $> \exp(-t^{1/3}10^{-3}L)$, when L is large enough. Therefore, we have

$$\mathcal{M}((-\delta_L,\delta_L),-Kt;(x,y),0) > c\exp(-t^{1/3}10^{-3}L). \tag{3.7}$$

By symmetry, the same lower bound holds for $\mathcal{M}((x',y'),t;(-\delta_L,\delta_L),(1+K)t)$.

The event $\mathcal{E}_2$ also implies that

$$\mathcal{Z}(0,-Kt;x,0),\mathcal{Z}(x',2t;0,(2+K)t)>\exp(-t^{1/3}10^{-3}L). \tag{3.8}$$

Step 2. Using Lemma 2.5, we have that

$$2\mathcal{M}((x,y),0;(x',y'),t)>\mathcal{M}((-t^{2/3}H,t^{2/3}H),0;(-t^{2/3}H,t^{2/3}H),t)$$
$$\times\frac{\mathcal{Z}(x',0;x,t)\mathcal{Z}(y',-0;y,t)}{\mathcal{Z}(-t^{2/3}H,0;-t^{2/3}H,t)\mathcal{Z}(t^{2/3}H,-0;t^{2/3}H,t)}.$$

By $\mathcal{E}$, the first factor in the right-hand side is $>\exp(t^{1/3}\cdot 1.99L)$; and by $\mathcal{E}^*$, the second factor in the right-hand side is $>\exp(-t^{1/3}10^{-9}L)$. Therefore we have

$$\mathcal{M}((x,y),0;(x',y'),t)>c\exp(t^{1/3}\cdot(1.99-10^{-9})L). \tag{3.9}$$

From $\mathcal{E}$, we also have that

$$\mathcal{Z}(-t^{2/3}H,0;-t^{2/3}H,t)\vee\mathcal{Z}(t^{2/3}H,0;t^{2/3}H,t)>\exp(t^{1/3}\cdot 0.99L).$$

Without loss of generality, we assume that

$$\mathcal{Z}(-t^{2/3}H,0;-t^{2/3}H,t)>\exp(t^{1/3}\cdot 0.99L).$$

Then by $\mathcal{E}^*$, we have

$$\mathcal{Z}(x,0;x',t)>\exp(t^{1/3}\cdot(0.99-10^{-9})L). \tag{3.10}$$

Step 3. By the composition law (2.5), and (3.7), (3.9), we have

$$\mathcal{M}((-\delta_L,\delta_L),-Kt;(-\delta_L,\delta_L),(1+K)t)>c\exp(t^{1/3}\cdot(1.99-2\cdot 10^{-3}-10^{-9})L).$$

By the composition law (1.1), and (3.8), (3.10), we have

$$\mathcal{Z}(0,-Kt;0,(1+K)t)>c\exp(t^{1/3}\cdot(0.99-2\cdot 10^{-3}-10^{-9})L).$$

These imply the event $\mathcal{E}^+$. □

Proof of Lemma 3.9. By Lemma 3.10, and the fact that $\mathcal{E}_1,\mathcal{E}_2$ are independent of $\mathcal{E}^*,\mathcal{E}$, we have $\mathbb{P}(\mathcal{E}_1\cap\mathcal{E}_2)\mathbb{P}(\mathcal{E}^*\cap\mathcal{E})\le\mathbb{P}(\mathcal{E}^+)$. Then by the bounds on $\mathbb{P}(\mathcal{E}_1)$, $\mathbb{P}(\mathcal{E}_2)$, $\mathbb{P}(\mathcal{E}^*)$, $\mathbb{P}(\mathcal{E})$, and $\mathbb{P}(\mathcal{E}^+)$, we have that

$$\mathbb{P}(\mathcal{E})\le C\exp\big(-(8/3-0.1)L^{3/2}\big)(w^2-C\exp(-cL^{3/2}))^{-1}-C\exp(-cL^2),$$

and this is bounded by $C\exp\big(-(4/3+0.01)L^{3/2}\big)$ for L large enough. □

4. Tail comparison estimates

To reduce notations, in the rest of this paper we denote

$$\mathcal{L}^{\beta=1}(x,s;y,t)=\log\mathcal{Z}(x,s;y,t)+(t-s)/12,\quad \mathcal{L}^{\beta=\infty}(x,s;y,t)=\mathcal{L}(x,s;y,t).$$

We take $\beta=1$ or ∞ systematically unless otherwise noted. From the above definition we have $\mathfrak{h}^{\beta}_{t,1}(x)=\mathcal{L}^{\beta}(0,0;x,t)$.

In this section, t is taken to be any $t>t_0$, where t_0 is an arbitrary positive number. All the constants may depend on t_0, but are uniform in t.

As indicated in Section 1.2, the following tail ratio estimate would be central in much of our analysis.

Theorem 4.1. *For any $L \geq 2$, and $0 < \delta < L^{1/4}$,*

$$\frac{\mathbb{P}\left(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L + \delta\right)}{\mathbb{P}(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L)} = \exp(-2\delta L^{1/2} + O(\delta L^{-1/4}\log(L) + L^{-3/2})). \tag{4.1}$$

For $\delta \geq L^{1/4}$, the same ratio equals $\exp(-\Omega(\delta L^{1/2}))$.

The main term $\exp(-2\delta L^{1/2})$ comes from the following: from the tent behavior, one considers a Brownian bridge in $[-L^{-1/2}, L^{1/2}]$ that equals $-L$ at the two ends; then the ratio is roughly the probability that it is $> L + \delta$ at 0 versus the probability that it is $> L$ at 0.

The general strategy will be to use the Gibbs property (Lemma 2.7) to resample $\hat{\mathfrak{h}}^{\beta}_{t,1}$ in an interval $[-L^{1/2} + M, L^{1/2} - M]$, with some M large but much smaller than $L^{1/2}$. The choice of M is due to that, conditioned on the upper tail large deviation event, $\hat{\mathfrak{h}}^{\beta}_{t,1}$ in the interval is not much affected by the second line $\hat{\mathfrak{h}}^{\beta}_{t,2}$ by the tent behavior, and can be analyzed as a Brownian bridge. For this, we need the following estimate of $\hat{\mathfrak{h}}^{\beta}_{t,1}$ at the end point $-L^{1/2} + M$ (and also for $L^{1/2} - M$ by symmetry).

Lemma 4.2. *For any $0 < M < L^{1/2}$ and $L \geq 2$,*

$$\mathbb{P}\left(\hat{\mathfrak{h}}^{\beta}_{t,1}(-L^{1/2} + M) \leq -(L^{1/2} - M)^2 + \tfrac{1}{2}M^2 \mid \hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L\right) < C\exp(-cM^{5/2}).$$

By the tent behavior, we expect the expectation of $\hat{\mathfrak{h}}^{\beta}_{t,1}$ to be around $-L + 2ML^{1/2}$, with a Gaussian fluctuation of order $M^{1/2}$. This estimate bounds the probability that it is order M^2 smaller than its mean. The reason the upper bound is $\exp(-cM^{5/2})$ rather than $\exp(-c(M^2)^2/M) = \exp(-cM^3)$ is due to interactions with $\hat{\mathfrak{h}}^{\beta}_{t,2}$.

Proof of Lemma 4.2. Let $\mathcal{F} = \mathcal{F}_{\mathrm{ext}}([-t^{2/3}L^{1/2}, 0])$ (recall the definition of $\mathcal{F}_{\mathrm{ext}}$ from Section 2.3) and $L_M = (L^{1/2} - M)^2$. Then we can write the probability of the complement of the event in the LHS in the statement of the lemma as

$$\frac{\mathbb{E}\left[\mathbb{P}_{\mathcal{F}}\left(\hat{\mathfrak{h}}^{\beta}_{t,1}(-L_M^{1/2}) > -L_M + \tfrac{1}{2}M^2\right)\mathbb{1}_{\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L}\right]}{\mathbb{P}(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L)}. \tag{4.2}$$

Let us focus on the conditional probability in the numerator. Let $\tilde{B}$ be a Brownian bridge from $(-L^{1/2}, \hat{\mathfrak{h}}^{\beta}_{t,1}(-L^{1/2}))$ to $(0, \hat{\mathfrak{h}}^{\beta}_{t,1}(0))$, such that $x \mapsto t^{1/3}\tilde{B}(t^{-2/3}x)$ in $[-t^{2/3}L^{1/2}, 0]$ interacts with $\mathfrak{h}^{\beta}_{t,2}$ by the Radon Nikodym derivative reweighting (2.6). Then the Brownian Gibbs property says that the conditional probability in the previous display equals

$$\mathbb{P}_{\mathcal{F}}\left(\tilde{B}(-L_M^{1/2}) > -L_M + \tfrac{1}{2}M^2\right). \tag{4.3}$$

Next let B be a Brownian bridge from $(-L^{1/2}, -L - M)$ to $(0, L)$ (with no lower boundary conditioning). Then, on the $\mathcal{F}$-measurable event

$$A(L, M) = \left\{\hat{\mathfrak{h}}^{\beta}_{t,1}(-L^{1/2}) > -L - M,\ \hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L\right\}, \tag{4.4}$$

it holds that $\tilde{B}$ dominates B by monotonicity (Lemma 2.8), so (4.3) is lower bounded by

$$\mathbb{P}_{\mathcal{F}}\left(B(-L_M^{1/2}) > -L_M + \tfrac{1}{2}M^2\right)\mathbb{1}_{A(L,M)}. \tag{4.5}$$

Now $B(-L_M^{1/2})$ is distributed as a normal random variable with mean

$$-L - M + M \cdot \frac{L - (-L - M)}{L^{1/2}} = -L - M + 2L^{1/2}M + M^2L^{-1/2}$$

and variance $\sigma^2 = \frac{2M(L^{1/2}-M)}{L^{1/2}} \le 2M$. Since $-L_M = -(L^{1/2}-M)^2 = -L + 2L^{1/2}M - M^2$, we see that (4.5) equals, on $A(L, M)$,

$$\mathbb{P}\left(\mathcal{N}(0,\sigma^2) > -M^2(\tfrac{1}{2} + L^{-1/2}) + M\right) \ge 1 - \exp\left(-cM^3\right)$$

using standard tail bounds for the normal distribution (Lemma 2.19). Putting this back in (4.2) and recalling the definition (4.4) of $A(L, M)$, we see that the LHS in the lemma is upper bounded by

$$1 - (1-\exp(-cM^3))\mathbb{P}\left(\hat{\mathfrak{h}}^{\beta}_{t,1}(-L^{1/2}) > -L - M \;\middle|\; \hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L\right).$$

By the FKG inequality (Lemma 2.9) and Theorem 2.13, we have

$$\mathbb{P}\left(\hat{\mathfrak{h}}^{\beta}_{t,1}(-L^{1/2}) > -L - M \;\middle|\; \hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L\right) > 1 - \exp(-cM^{5/2})).$$

This completes the proof. □

Proof of Theorem 4.1. We can assume that L is large since otherwise the conclusion follows from Theorem 2.11. For some constant C_0 sufficiently large, the case of $\delta > C_0 L^{1/4}$ also follows from Theorem 2.11. In the remainder of the proof, we prove (4.1) for $\delta < C_0 L^{1/4}$.

Denote $M = \log(L)$. It suffices to show that

$$\frac{\mathbb{P}\left(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L + \delta\right)}{\mathbb{P}(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L)} \cdot \exp(2\delta L^{1/2}) < (1 + CL^{-3/2})\exp(C\delta M L^{-1/4}) \qquad \text{and}$$
$$\frac{\mathbb{P}\left(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L + \delta\right)}{\mathbb{P}(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L)} \cdot \exp(2\delta L^{1/2}) > (1 - CL^{-3/2})\exp(-C\delta M L^{-1/4}).$$

We let $L_M = (L^{1/2} - M)^2$, and $\mathcal{F} = \mathcal{F}_{\mathrm{ext}}([-t^{2/3}L_M^{1/2}, t^{2/3}L_M^{1/2}])$. We start by considering the ratio of conditional probabilities

$$\frac{\mathbb{P}_{\mathcal{F}}\left(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L + \delta\right)}{\mathbb{P}_{\mathcal{F}}\left(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L\right)}.$$

We adopt the notation $\mathcal{B}$ for the law of a Brownian bridge B from $(-t^{2/3}L_M^{1/2}, \mathfrak{h}^{\beta}_{t,1}(-t^{2/3}L_M^{1/2}))$ to $(t^{2/3}L_M^{1/2}, \mathfrak{h}^{\beta}_{t,1}(t^{2/3}L_M^{1/2}))$, as well as the associated expectation. With this notation, by the Brownian Gibbs property, the previous display equals

$$\frac{\mathcal{B}\big(\mathbb{1}_{B(0)>t^{1/3}(L+\delta)}W(B,\mathfrak{h}^{\beta}_{t,2})\big)}{\mathcal{B}\big(\mathbb{1}_{B(0)>t^{1/3}L}W(B,\mathfrak{h}^{\beta}_{t,2})\big)} = \frac{\mathcal{B}\big(B(0) > t^{1/3}(L+\delta)\big)}{\mathcal{B}\big(B(0) > t^{1/3}L\big)} \cdot \frac{\mathcal{B}\big(W(B,\mathfrak{h}^{\beta}_{t,2}) \mid B(0) > t^{1/3}(L+\delta)\big)}{\mathcal{B}\big(W(B,\mathfrak{h}^{\beta}_{t,2}) \mid B(0) > t^{1/3}L\big)},$$

where $W(B, \mathfrak{h}^{\beta}_{t,2})$ is the weight factor from (2.6). Now the second ratio of terms in the previous display is lower bounded by 1 using stochastic monotonicity properties of Brownian bridges and that W is increasing in B. To upper bound the second ratio, we note that, since $W(B, \mathfrak{h}^{\beta}_{t,2}) \le 1$, it suffices to lower bound the denominator $\mathcal{B}\big(W(B,\mathfrak{h}^{\beta}_{t,2}) \mid B(0) > t^{1/3}L\big)$. To do this we consider the $\mathcal{F}$-measurable event $\mathrm{BdyCtrl} = \mathrm{BdyCtrl}(L, M)$ defined by

$$\left\{\hat{\mathfrak{h}}^{\beta}_{t,1}(\pm L_M^{1/2}) \ge -L_M + \tfrac{1}{2}M^2\right\} \cap \bigcap_{i=0}^{M^{-1}L_M^{1/2}} \left\{ \sup_{|x| \in [L_M^{1/2} - iM, L_M^{1/2} - (i+1)M]} \hat{\mathfrak{h}}^{\beta}_{t,2}(x) + x^2 \le (i+1)M \right\}.$$

By Lemma 4.2, the BK inequality (Lemma 2.10), and the upper bound on $\sup \hat{\mathfrak{h}}^{\beta}_{t,1}$ (following from Theorem 2.11 and Proposition 2.14), we have

$$\mathbb{P}\left(\mathrm{BdyCtrl}^c \mid \hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L+\delta\right) \leq C\exp(-cM^{5/2}) + \sum_{i=0}^{M^{-1}L_M^{1/2}} C\exp\left(-ci^{3/2}M^{3/2}\right) < C\exp(-cM^{3/2}). \tag{4.6}$$

Let ℓ^- be the line joining $(-L_M^{1/2}, -L_M + \frac{1}{4}M^2)$ and $(0, L-\frac{1}{4}M^2)$, ℓ^+ be the line joining $(0, L-\frac{1}{4}M^2)$ and $(L_M^{1/2}, -L_M + \frac{1}{4}M^2)$, and ℓ be their concatenation. Consider the high corridor event

$$\mathrm{HighCorr} = \left\{t^{-1/3}B(t^{2/3}x) \geq \ell(x) - L_M^{1/2} + |x| \text{ for all } x \in [-L_M^{1/2}, L_M^{1/2}]\right\};$$

it is easy to obtain by standard Brownian bridge estimates that, on BdyCtrl, $\mathcal{B}[\mathrm{HighCorr} \mid B(0) > t^{1/3}L] > 1 - C\exp(-cM^2)$.

Now on HighCorr and BdyCtrl, we have $t^{-1/3}B(t^{2/3}x) \geq \hat{\mathfrak{h}}^{\beta}_{t,2}(x) + \frac{1}{4}M^2 - M$; and it is easy to check with the formula (2.6) that

$$W(B, \mathfrak{h}^{\beta}_{t,2}) > 1 - Ct^{2/3}L_M^{1/2}\exp(-ct^{1/3}M^2) > 1 - C\exp(-cM^2).$$

Thus, these bounds on $W(B, \mathfrak{h}^{\beta}_{t,2})$ and $\mathcal{B}(\mathrm{HighCorr} \mid B(0) > t^{1/3}L)$ yield that, on BdyCtrl,

$$\mathcal{B}\big(W(B, \mathfrak{h}^{\beta}_{t,2}) \mid B(0) > t^{1/3}L\big) > 1 - C\exp(-cM^2).$$

So we see that, on $\mathrm{BdyCtrl}(L, M)$,

$$\frac{\mathbb{P}_{\mathcal{F}}\left(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L+\delta\right)}{\mathbb{P}_{\mathcal{F}}\left(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L\right)} = (1 + O(e^{-cM^2})) \cdot \frac{\mathcal{B}\big(B(0) > t^{1/3}(L+\delta)\big)}{\mathcal{B}\big(B(0) > t^{1/3}L\big)}. \tag{4.7}$$

Let $\mu = \frac{1}{2}(\hat{\mathfrak{h}}^{\beta}_{t,1}(-L_M^{1/2}) + \hat{\mathfrak{h}}^{\beta}_{t,1}(L_M^{1/2}))$, and observe that $B(0)$ under $\mathcal{B}$ is distributed as a normal random variable with mean $t^{1/3}\mu$ and variance $t^{2/3}L_M^{1/2}$. Consider the $\mathcal{F}$-measurable event

$$\mathrm{MeanCtrl}(L) = \left\{\mu \in [-L + 2L^{1/2}M - ML^{1/4}, -L + 2L^{1/2}M + ML^{1/4}]\right\}.$$

We know from Theorem 2.15 that

$$\mathbb{P}(\mathrm{MeanCtrl}(L)^c \mid \hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L+\delta) < \exp(-cM^2). \tag{4.8}$$

Using standard bounds on the tail of the normal distribution (Lemma 2.19), on $\mathrm{MeanCtrl}(L)$ (and on $\mathrm{BdyCtrl}(L, M)$ as we are assuming throughout), the RHS of (4.7) equals

$$(1 + O(e^{-cM^2}))(1 + O(L^{-3/2})) \cdot \frac{L-\mu}{L+\delta-\mu} \cdot \exp\left(-\frac{1}{2L_M^{1/2}}\Big[(L+\delta-\mu)^2 - (L-\mu)^2\Big]\right)$$

and the last factor can be further written as

$$\begin{aligned}\exp\left(-\frac{2\delta(L-\mu)+\delta^2}{2L_M^{1/2}}\right) &= \exp\left(-\frac{2\delta(L + L - 2L^{1/2}M + O(ML^{1/4})) + \delta^2}{2L_M^{1/2}}\right) \\ &= \exp\left(-2\delta L^{1/2} + O(\delta M L^{-1/4})\right).\end{aligned}$$

Overall, we have at this point established that, on $\mathrm{BdyCtrl}(L, M) \cap \mathrm{MeanCtrl}(L)$,

$$\frac{\mathbb{P}_{\mathcal{F}}\left(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L+\delta\right)}{\mathbb{P}_{\mathcal{F}}\left(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L\right)} = (1+O(e^{-cM^2}))(1+O(L^{-3/2}))\cdot\frac{L-\mu}{L+\delta-\mu}\exp\left(-2\delta L^{1/2} + O(\delta M L^{-1/4})\right). \tag{4.9}$$

We next convert this into upper and lower bounds on $\mathbb{P}(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L+\delta)$, respectively.

Upper bound. We see from (4.9) that

$$\begin{aligned}\mathbb{P}_{\mathcal{F}}\left(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L+\delta\right) &= \mathbb{P}_{\mathcal{F}}\left(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L+\delta\right)\left(\mathbb{1}_{\mathrm{BdyCtrl}(L,M)\cap\mathrm{MeanCtrl}(L)} + \mathbb{1}_{(\mathrm{BdyCtrl}(L,M)\cap\mathrm{MeanCtrl}(L))^c}\right)\\ &\le (1+Ce^{-cM^2})(1+CL^{-3/2})\exp\left(-2\delta L^{1/2} + C\delta M L^{-1/4}\right)\mathbb{P}_{\mathcal{F}}\left(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L\right)\\ &\qquad + \mathbb{P}_{\mathcal{F}}\left(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L+\delta\right)\mathbb{1}_{(\mathrm{BdyCtrl}(L,M)\cap\mathrm{MeanCtrl}(L))^c},\end{aligned}$$

so that, by taking expectations,

$$\begin{aligned}\mathbb{P}\left(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L+\delta\right) &\le (1+Ce^{-cM^2})(1+CL^{-3/2})\exp\left(-2\delta L^{1/2} + C\delta M L^{-1/4}\right)\mathbb{P}\left(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L\right)\\ &\qquad + \mathbb{P}\left(\left\{\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L+\delta\right\}\cap\left(\mathrm{BdyCtrl}(L,M)\cap\mathrm{MeanCtrl}(L)\right)^c\right).\end{aligned}$$

We focus on the last term. It equals

$$\mathbb{P}\left(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L+\delta\right)\cdot\mathbb{P}\left(\left(\mathrm{BdyCtrl}(L,M)\cap\mathrm{MeanCtrl}(L)\right)^c \,\middle|\, \hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L+\delta\right).$$

If we can show that the second factor is small, we will be done. By a union bound, it is at most

$$\mathbb{P}(\mathrm{BdyCtrl}(L,M)^c \mid \hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L+\delta) + \mathbb{P}(\mathrm{MeanCtrl}(L)^c \mid \hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L+\delta),$$

which, from (4.6) and (4.8), is upper bounded by $C\exp(-cM^{3/2}) + \exp(-cM^2)$. This completes the proof of the upper bound on $\mathbb{P}(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L+\delta)$.

Lower bound. We observe that, on MeanCtrl, $\frac{L-\mu}{L+\delta-\mu} = \frac{2L+O(ML^{1/2})}{2L+O(ML^{1/2})+\delta} > 1 - C\delta L^{-1}$. Then we have, using (4.9),

$$\begin{aligned}&\mathbb{P}_{\mathcal{F}}(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L+\delta)\\ &\ge \mathbb{P}_{\mathcal{F}}(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L+\delta)\cdot\mathbb{1}_{\mathrm{BdyCtrl}(L,M)\cap\mathrm{MeanCtrl}(L)}\\ &\ge (1-Ce^{-cM^2})(1-CL^{-3/2})\exp\left(-2\delta L^{1/2} - C\delta M L^{-1/4}\right)\mathbb{P}_{\mathcal{F}}(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L)\cdot\mathbb{1}_{\mathrm{BdyCtrl}(L,M)\cap\mathrm{MeanCtrl}(L)},\end{aligned}$$

absorbing the factor of $1 - C\delta L^{-1}$ into $\exp(-C\delta M L^{-1/4})$. Taking expectations yields

$$\begin{aligned}\mathbb{P}\left(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L+\delta\right) &\ge (1-Ce^{-cM^2})(1-CL^{-3/2})\exp\left(-2\delta L^{1/2} - C\delta M L^{-1/4}\right)\\ &\qquad\times\mathbb{P}\left(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L, \mathrm{BdyCtrl}(L,M), \mathrm{MeanCtrl}(L)\right)\\ &\ge (1-Ce^{-cM^2})(1-CL^{-3/2})\exp\left(-2\delta L^{1/2} - C\delta M L^{-1/4}\right)\\ &\qquad\times\mathbb{P}\left(\mathrm{BdyCtrl}(L,M), \mathrm{MeanCtrl}(L) \mid \hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L\right)\cdot\mathbb{P}\left(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L\right).\end{aligned}$$

As we saw above, the latter conditional probability is lower bounded by $1 - C\exp(-cM^{3/2})$, thus completing the proof of the lower bound. ☐

We next establish the following tightness of the relevant path measures.

As in Theorems 1.1 and 1.2, let π_0 be the geodesic from $(0,0)$ to $(0,1)$, in the directed landscape $\mathcal{L}^{\beta=\infty}$; and Γ_0 be sampled from the annealed polymer measure from $(0,0)$ to $(0,1)$, under $\mathcal{L}^{\beta=1}$.

Proposition 5.1 (Tightness)**.** *As random elements in $\mathcal{C}([0,1],\mathbb{R})$, $L^{1/4}\pi_0$ or $L^{1/4}\Gamma_0$ conditional on $\mathcal{L}^{\beta}(0,0;0,1) > L$ for all $L \geq 2$ are tight.*

As mentioned in Section 1.2, to prove this requires tail bounds on two-point deviations, which relies on shear invariance of the directed landscape and the CDRP free energy field. The proof in zero temperature is much simpler than in positive temperature as in the former shear invariance alone suffices to give tightness on the $L^{-1/4}$ scale. In positive temperature, the analogous argument only yields tightness on the $O(1)$ scale, and additional arguments are needed to obtain the correct scale. The reason is that in zero temperature, given the environment, the path location at a given height is determined; while in positive temperature there is no a priori concentration of the polymer location.

In this section, we give the zero temperature proof and some rough bounds for the transversal fluctuation of polymers. The positive temperature part of Proposition 5.1 will be proved in Section 8. We note that, as pointed out in Section 2.4.1, we will write our proofs using the estimates there, which leads to tightness in a weaker topology of uniform convergence in $[t_0, 1-t_0]$, for arbitrarily small $t_0 > 0$. Explanations on upgrading will be given at the end of Section 8.

5.1. **Geodesic tightness.** We start with a one-point estimate and later give the two-point estimate.

Lemma 5.2. *For all $K > 0$, $L \geq 2$, and $s \in (0,1)$,*

$$\mathbb{P}\left(|\pi_0(s)| > K(s(1-s))^{1/2}L^{-1/4} \;\middle|\; \mathcal{L}(0,0;0,1) > L\right) < C\exp\left(-cK^2\right).$$

In fact, for K up to $L^{1/2}$ we will obtain a tail of $\exp(-2K^2)$ (via the first case of the comparison statement Theorem 4.1). Note that it exactly corresponds to our ultimate goal, namely that $L^{-1/4}\pi_0(s)$ converges to $\frac{1}{2}N(0,s(1-s))$, which on scale $(s(1-s))^{1/2}$ would have a tail at depth x that satisfies the approximate asymptotics of $\exp(-2x^2)$.

Proof of Lemma 5.2. Observe that, for any K, $\mathbb{P}(|\pi_0(s)| > K \mid \mathcal{L}(0,0;0,1) > L)$ is upper bounded by

$$\begin{aligned}
&\mathbb{P}\left(\sup_{|x|>K}\Big(\mathcal{L}(0,0;x,s) + \mathcal{L}(x,s;0,1)\Big) > L \;\middle|\; \mathcal{L}(0,0;0,1) > L\right) \\
&\qquad\qquad \leq 2\cdot\mathbb{P}\left(\sup_{x>K}\Big(\mathcal{L}(0,0;x,s) + \mathcal{L}(x,s;0,1)\Big) > L \;\middle|\; \mathcal{L}(0,0;0,1) > L\right) \\
&\qquad\qquad \leq 2\cdot\frac{\mathbb{P}\left(\sup_{x>K}\Big(\mathcal{L}(0,0;x,s) + \mathcal{L}(x,s;0,1)\Big) > L\right)}{\mathbb{P}(\mathcal{L}(0,0;0,1) > L)},
\end{aligned} \tag{5.1}$$

the factor of 2 coming from a union bound and using the distributional symmetry of $\mathcal{L}(0,0;x,s)$ and $\mathcal{L}(x,s;0,1)$ under $x \mapsto -x$ (as well as the independence of the two processes) to remove the absolute value in the condition under the supremum. Then using the stationarity (and independence) properties of $\mathcal{L}$ and that $x > K$ for the inequality in the second line,

$$\begin{aligned}
\mathcal{L}(0,0;x,s) + \mathcal{L}(x,s;0,1) &\stackrel{d}{=} \mathcal{L}(0,0;x-K,s) + \mathcal{L}(x-K,s;0,1) + (s(1-s))^{-1}\left[(x-K)^2 - x^2\right] \\
&\leq \mathcal{L}(0,0;x-K,s) + \mathcal{L}(x-K,s;0,1) - (s(1-s))^{-1}K^2
\end{aligned}$$

as a process in x. Thus we see that the RHS of (5.1) is upper bounded by

$$\frac{\mathbb{P}\left(\sup_{x>0}\Big(\mathcal{L}(0,0;x,s)+\mathcal{L}(x,s;0,1)\Big) > L + (s(1-s))^{-1}K^2\right)}{\mathbb{P}(\mathcal{L}(0,0;0,1) > L)}$$

Now using that $\mathcal{L}(0,0;0,1) = \sup_{x\in\mathbb{R}}\Big(\mathcal{L}(0,0;x,s)+\mathcal{L}(x,s;0,1)\Big)$, it follows that the previous display is upper bounded by

$$\frac{\mathbb{P}\left(\mathcal{L}(0,0;0,1) > L + (s(1-s))^{-1}K^2\right)}{\mathbb{P}(\mathcal{L}(0,0;0,1) > L)}.$$

Applying Theorem 4.1 now bounds the previous display by

$$C\exp\left(-cK^2(s(1-s))^{-1}L^{1/2}\right).$$

Replacing K by $K(s(1-s))^{1/2}L^{-1/4}$ completes the proof. □

We next give a two-point estimate. Combining it with the Kolmogorov-Censtov criterion for tightness (see e.g. [Kal22, Theorem 23.7]), the $\beta = \infty$ case of Proposition 5.1 follows.

Proposition 5.3. *For all $K > 0$, $L \geq 2$, and $0 < s < t < 1$,*

$$\mathbb{P}\Big(|\pi_0(s) - \pi_0(t)| > K(t-s)^{1/2}L^{-1/4} \mid \mathcal{L}(0,0;0,1) > L\Big) \leq C\exp(-cK^2).$$

Proof. We give the proof under the assumption that $t - s \in (0, \frac{1}{2})$. The case where $t - s \in [\frac{1}{2}, 1]$ follows from Lemma 5.2 easily.

Similar to the previous proof, the LHS of the display in the lemma is upper bounded by

$$\begin{aligned}
&\mathbb{P}\left(\sup_{|x-y|>K}\Big(\mathcal{L}(0,0;x,s)+\mathcal{L}(x,s;y,t)+\mathcal{L}(y,t;0,1)\Big) > L \;\Big|\; \mathcal{L}(0,0;0,1) > L\right)\\
&\qquad \leq 2\cdot\mathbb{P}\left(\sup_{x-y>K}\Big(\mathcal{L}(0,0;x,s)+\mathcal{L}(x,s;y,t)+\mathcal{L}(y,t;0,1)\Big) > L \;\Big|\; \mathcal{L}(0,0;0,1) > L\right)\\
&\qquad \leq 2\cdot\frac{\mathbb{P}\left(\sup_{x-y>K}\Big(\mathcal{L}(0,0;x,s)+\mathcal{L}(x,s;y,t)+\mathcal{L}(y,t;0,1)\Big) > L\right)}{\mathbb{P}\left(\mathcal{L}(0,0;0,1) > L\right)}.
\end{aligned} \tag{5.2}$$

Now, using the stationarity (and independence) properties of $\mathcal{L}$,

$$\begin{aligned}
&\mathcal{L}(0,0;x,s)+\mathcal{L}(x,s;y,t)+\mathcal{L}(y,t;0,1)\\
&\stackrel{d}{=} \mathcal{L}(-K,0;x-K,s)+\mathcal{L}(x-K,s;y,t)+\mathcal{L}(y,t;0,1)+(t-s)^{-1}\left[(x-y-K)^2-(x-y)^2\right]
\end{aligned}$$

as a process in (x,y). Since $x - y > K$, we have that

$$(t-s)^{-1}\left[(x-y-K)^2-(x-y)^2\right] < -(t-s)^{-1}K^2.$$

Thus we see that the RHS of (5.2) is upper bounded by

$$2\cdot\frac{\mathbb{P}\left(\sup_{x-y>K}\Big(\mathcal{L}(-K,0;x-K,s)+\mathcal{L}(x-K,s;y,t)+\mathcal{L}(y,t;0,1)\Big) > L + (t-s)^{-1}K^2\right)}{\mathbb{P}\left(\mathcal{L}(0,0;0,1) > L\right)}$$

Now using that $\mathcal{L}(-K,0;0,1) = \sup_{x,y\in\mathbb{R}}\Big(\mathcal{L}(-K,0;x,s)+\mathcal{L}(x,s;y,t)+\mathcal{L}(y,t;0,1)\Big)$, and that $\mathcal{L}(-K,0;0,1) \stackrel{d}{=} \mathcal{L}(0,0;0,1) - K^2$, it follows that the previous display is upper bounded by

$$\frac{\mathbb{P}\left(\mathcal{L}(-K,0;0,1) > L + (t-s)^{-1}K^2\right)}{\mathbb{P}(\mathcal{L}(0,0;0,1) > L)} = \frac{\mathbb{P}\left(\mathcal{L}(0,0;0,1) > L + ((t-s)^{-1}+1)K^2\right)}{\mathbb{P}(\mathcal{L}(0,0;0,1) > L)}.$$

By Theorem 4.1, this is upper bounded by

$$C\exp\left(-c(t-s)^{-1}K^2L^{1/2}\right).$$

Replacing K by $K(t-s)^{1/2}L^{-1/4}$ completes the proof. $\square$

5.2. **Polymer transversal estimates.** We now adapt the zero temperature arguments above to the positive temperature setting. Although the bounds in this subsection are not sufficient to derive the $\beta=1$ case of Proposition 5.1, they will be used in the proof to be given in Section 8.

We start with the positive temperature analog of Lemma 5.2.

Lemma 5.4. *For all $K>0$, $s\in(0,1)$, and $L\geq 2$,*

$$\mathbb{P}\left(\boldsymbol{\mathcal{P}}\left[|\Gamma_0(s)|\geq K(s(1-s))^{1/2}\right]>\exp(-\tfrac{1}{2}K^2)\;\middle|\;\mathcal{L}^\beta(0,0;0,1)>L\right)<C\exp(-cK^2L^{1/2}).$$

Remark 5.5. Unlike the zero-temperature case, here the concentration is on an $O(1)$ scale. Some more work is needed to obtain an $L^{-1/4}$ scale concentration, and we will turn to that shortly.

Proof of Lemma 5.4. By the convolution formula, for any $K>0$ and $\varepsilon>0$,

$$\begin{aligned}
&\mathbb{P}\left(\boldsymbol{\mathcal{P}}\left[|\Gamma_0(s)|\geq K\right]>\varepsilon\mid\mathcal{L}^\beta(0,0;0,1)>L\right)\\
&\qquad=\mathbb{P}\left(\int_{|x|\geq K}\mathcal{Z}(0,0;x,s)\mathcal{Z}(x,s;0,1)\,\mathrm{d}x>\varepsilon\mathcal{Z}(0,0;0,1)\;\middle|\;\mathcal{L}^\beta(0,0;0,1)>L\right)\\
&\qquad\leq\frac{\mathbb{P}\left(\int_{|x|\geq K}\mathcal{Z}(0,0;x,s)\mathcal{Z}(x,s;0,1)\,\mathrm{d}x>\varepsilon e^{L-1/12}\right)}{\mathbb{P}\left(\mathcal{L}^\beta(0,0;0,1)>L\right)}\\
&\qquad\leq\frac{2\cdot\mathbb{P}\left(\int_{x\geq K}\mathcal{Z}(0,0;x,s)\mathcal{Z}(x,s;0,1)\,\mathrm{d}x>\frac{1}{2}\varepsilon e^{L-1/12}\right)}{\mathbb{P}\left(\mathcal{L}^\beta(0,0;0,1)>L\right)},
\end{aligned}\tag{5.3}$$

where the last inequality uses the reflection symmetry of $\mathcal{Z}(0,0;\cdot,s)$ and $\mathcal{Z}(\cdot,s;0,1)$ and their independence. By the shear invariance and independence of the same two processes, we have that the following distributional equality holds as processes in x for every fixed s:

$$\begin{aligned}
\mathcal{Z}(0,0;x,s)\mathcal{Z}(x,s;0,1)&\stackrel{d}{=}\mathcal{Z}(0,0;x-K,s)\mathcal{Z}(x-K,s;0,1)e^{(s(1-s))^{-1}[(x-K)^2-x^2]}\\
&\leq\mathcal{Z}(0,0;x-K,s)\mathcal{Z}(x-K,s;0,1)e^{-(s(1-s))^{-1}K^2},
\end{aligned}$$

where the inequality is due to that $x\geq K$. Substituting this into (5.3) yields that

$$\begin{aligned}
&\mathbb{P}\left(\boldsymbol{\mathcal{P}}\left[|\Gamma_0(s)|\geq K\right]>\varepsilon\mid\mathcal{L}^\beta(0,0;0,1)>L\right)\\
&\qquad\leq\frac{2\cdot\mathbb{P}\left(\int_{x\geq 0}\mathcal{Z}(0,0;x,s)\mathcal{Z}(x,s;0,1)\,\mathrm{d}x>\frac{1}{2}\varepsilon e^{L-1/12+(s(1-s))^{-1}K^2}\right)}{\mathbb{P}\left(\mathcal{L}^\beta(0,0;0,1)>L\right)}\\
&\qquad\leq\frac{2\cdot\mathbb{P}\left(\int_{\mathbb{R}}\mathcal{Z}(0,0;x,s)\mathcal{Z}(x,s;0,1)\,\mathrm{d}x>\frac{1}{2}\varepsilon e^{L-1/12+(s(1-s))^{-1}K^2}\right)}{\mathbb{P}\left(\mathcal{L}^\beta(0,0;0,1)>L\right)}\\
&\qquad=\frac{2\cdot\mathbb{P}\left(\mathcal{L}^\beta(0,0;0,1)>L+(s(1-s))^{-1}K^2+\log(\varepsilon/2)\right)}{\mathbb{P}\left(\mathcal{L}^\beta(0,0;0,1)>L\right)}
\end{aligned}$$

We set $\varepsilon=\exp\left(-(s(1-s))^{-1}K^2/2\right)$ and invoke Theorem 4.1 to obtain that the previous display is upper bounded by

$$C\exp\left(-cL^{1/2}(s(1-s))^{-1}K^2\right).$$

Replacing K by $K(s(1-s))^{1/2}$ completes the proof. $\square$

We next derive a two-point estimate.

Lemma 5.6. *For all $K > 0$, $0 < s < t < 1$, and $L \geq 2$,*

$$\mathbb{P}\left(\boldsymbol{\mathcal{P}}\left[|\Gamma_0(s) - \Gamma_0(t)| > K(t-s)^{1/2}\right] > \exp(-\tfrac{1}{2}K^2) \;\middle|\; \mathcal{L}^\beta(0,0;0,1) > L\right) < C\exp(-cK^2L^{1/2}).$$

Proof. We give the proof under the assumption that $t - s \in (0, \frac{1}{2})$, since case where $t - s \in [\frac{1}{2}, 1]$ follows from Lemma 5.4 easily. Observe that, for any $\varepsilon > 0$,

$$\begin{aligned}
&\mathbb{P}\left(\boldsymbol{\mathcal{P}}\left(|\Gamma_0(s) - \Gamma_0(t)| > K\right) \geq \varepsilon \;\middle|\; \mathcal{L}^\beta(0,0;0,1) > L\right) \\
&\quad = \mathbb{P}\left(\int_{|x-y|\geq K} \mathcal{Z}(0,0;x,s)\mathcal{Z}(x,s;y,t)\mathcal{Z}(y,t;0,1)\,\mathrm{d}x\,\mathrm{d}y > \varepsilon\mathcal{Z}(0,0;0,1) \;\middle|\; \mathcal{L}^\beta(0,0;0,1) > L\right) \\
&\quad \leq \frac{\mathbb{P}\left(\int_{|x-y|\geq K} \mathcal{Z}(0,0;x,s)\mathcal{Z}(x,s;y,t)\mathcal{Z}(y,t;0,1)\,\mathrm{d}x\,\mathrm{d}y > \varepsilon e^{L-1/12}\right)}{\mathbb{P}\left(\mathcal{L}^\beta(0,0;0,1) > L\right)} \\
&\quad \leq \frac{2\cdot\mathbb{P}\left(\int_{x-y\geq K} \mathcal{Z}(0,0;x,s)\mathcal{Z}(x,s;y,t)\mathcal{Z}(y,t;0,1)\,\mathrm{d}x\,\mathrm{d}y > \frac{1}{2}\varepsilon e^{L-1/12}\right)}{\mathbb{P}\left(\mathcal{L}^\beta(0,0;0,1) > L\right)},
\end{aligned} \tag{5.4}$$

where the factor of 2 comes from removing the absolute value in the condition under the supremum by a union bound and using that $\mathcal{L}^\beta(x,s;y,t) \stackrel{d}{=} \mathcal{L}^\beta(-x,s;-y,t)$, $\mathcal{L}^\beta(0,0;x,s) \stackrel{d}{=} \mathcal{L}^\beta(0,0;-x,s)$, and $\mathcal{L}^\beta(y,t;0,1) \stackrel{d}{=} \mathcal{L}^\beta(-y,t;0,1)$ each as processes in the relevant spatial variables, as well as the independence of all three processes on the LHS of the equalities.

Now, using the stationarity (and independence) properties of $\mathcal{L}^\beta$,

$$\begin{aligned}
\mathcal{L}^\beta(0,0;x,s) + \mathcal{L}^\beta(x,s;y,t) + \mathcal{L}^\beta(y,t;0,1) \stackrel{d}{=} \mathcal{L}^\beta(-K,0;x-K,s) &+ \mathcal{L}^\beta(x-K,s;y,t) + \mathcal{L}^\beta(y,t;0,1) \\
&+ (t-s)^{-1}\left[(x-y-K)^2 - (x-y)^2\right]
\end{aligned}$$

as a process in (x,y). Now since $x - y > K$, we see that

$$(t-s)^{-1}\left[(x-y-K)^2 - (x-y)^2\right] < -(t-s)^{-1}K^2.$$

Thus we see that the RHS of (5.4) is upper bounded by

$$2\cdot\frac{\mathbb{P}\left(\int_{x-y>K} \mathcal{Z}(-K,0;x-K,s)\mathcal{Z}(x-K,s;y,t)\mathcal{Z}(y,t;0,1)\,\mathrm{d}x\,\mathrm{d}y > \frac{1}{2}\varepsilon e^{L-1/12+(t-s)^{-1}(K)^2}\right)}{\mathbb{P}\left(\mathcal{L}^\beta(0,0;0,1) > L\right)} \tag{5.5}$$

Now using that

$$\mathcal{L}^\beta(-K,0;0,1) - 1/12 = \log\int_{x,y\in\mathbb{R}} \mathcal{Z}(-K,0;x-K,s)\mathcal{Z}(x-K,s;y,t)\mathcal{Z}(y,t;0,1)\,\mathrm{d}x\,\mathrm{d}y,$$

and that $\mathcal{L}^\beta(-K,0;0,1) \stackrel{d}{=} \mathcal{L}^\beta(0,0;0,1) - K^2$, it follows that (5.5) is upper bounded by

$$2\cdot\frac{\mathbb{P}\left(\mathcal{L}^\beta(0,0;0,1) > L + ((t-s)^{-1}+1)K^2 + \log(\varepsilon/2)\right)}{\mathbb{P}(\mathcal{L}^\beta(0,0;0,1) > L)}.$$

Set $\varepsilon = \exp(-(t-s)^{-1}K^2/2)$. Applying Theorem 4.1 gives that the previous display is upper bounded by

$$C\exp\left(-c(t-s)^{-1}K^2L^{1/2}\right).$$

Replacing K by $K(t-s)^{1/2}$ completes the proof. $\square$

In this section, we record a number of estimates on the sum of passage times or free energies. Consider $(s_1, \cdots, s_k) \in \mathring{\Lambda}_k([0,\infty))$ and $\vec{y} \in \mathbb{R}^k$ for $k \in \mathbb{N}$. For the convenience of notations we denote $s_0 = y_0 = 0$, and adopt the shorthands $\mathcal{L}^\beta = \mathcal{L}^\beta(0,0;0,s_k)$ and $\mathcal{L}_i^\beta = \mathcal{L}^\beta(s_{i-1}, y_{i-1}; s_i, y_i)$ for each $1 \le i \le k$. Estimates in this section provide control on tail probabilities for the sum $\sum_i \mathcal{L}_i^\beta$ (such as Lemma 6.1, Lemma 6.3) or on the deviation of each $\mathcal{L}_i^\beta$ from $(s_i - s_{i-1})L$ conditional on $\sum_i \mathcal{L}_i^\beta > L$ (Lemma 6.4).

We assume that $\min_{1\le i\le k} s_i - s_{i-1} > t_0$ for some $t_0 > 0$. All the constants (within this section) can depend on k and t_0.

Our first statement bounds the upper tail of $\sum_i \mathcal{L}_i^\beta$, in terms of the upper tail of $\mathcal{L}^\beta$.

Lemma 6.1. *For any M and large enough L, and any $\vec{z} \in \mathbb{R}^k$, we have*

$$\mathbb{P}\left(\sup_{\vec{y}:\|\vec{y}-\vec{z}\|_\infty \le L^{-2}} \sum_{i=1}^k \mathcal{L}_i^\beta > M,\ \mathcal{L}^\beta(0,0;z_k,s_k) < M - C\log L \right) < C\exp(-cL^2).$$

Proof. This is immediate in the case of $\beta = \infty$ by subadditivity, i.e., $\mathcal{L}^\beta \ge \sum_{i=1}^k \mathcal{L}_i^\beta$. We next turn to $\beta = 1$.

We note from the unconditional fluctuation bound (Lemma 3.1) that, for some intervals $I_1, \ldots, I_k$, each with length L^{-2}, and each $z_i \in I_i$,

$$\inf_{\vec{y}\in\prod_{i=1}^k I_i} \sum_{i=1}^k \mathcal{L}_i^\beta > \sup_{\vec{y}:\|\vec{y}-\vec{z}\|_\infty \le cL^{-2}} \sum_{i=1}^k \mathcal{L}_i^\beta - 1,$$

with probability at least $1-C\exp(-cL^2)$. Under this event, and assuming that $(\sup_{\vec{y}:\|\vec{y}-\vec{z}\|_\infty\le L^{-2}} \sum_{i=1}^k \mathcal{L}_i^\beta > M$, we have (with $y_k = z_k$)

$$\begin{aligned}\mathcal{L}^\beta(0,0;z_k,s_k) &\ge \log \int_{\prod_{i=1}^{k-1} I_i} \exp\left(\sum_{i=1}^k \mathcal{L}_i^\beta\right) \prod_{y=1}^{k-1} \mathrm{d}y_i \\ &> M - 1 + (k-1)\log(L^{-2}).\end{aligned}$$

This completes the proof. $\square$

By taking $L = M$ and using Theorem 2.11, we get the following.

Corollary 6.2. *For any large enough L, and any $\vec{z} \in \mathbb{R}^k$, we have*

$$\mathbb{P}\left(\sup_{\vec{y}:\|\vec{y}-\vec{z}\|_\infty \le L^{-2}} \sum_{i=1}^k \mathcal{L}_i^\beta > L \right) < \exp\left(-\frac{4}{3} s_k^{-1/2} L^{3/2} + CL^{3/4} \right).$$

The previous results provide control on the upper tail of the sum $\sum_{i=1}^k \mathcal{L}_i^\beta$, and the following lemma provides control on the lower tail of the same, conditional on the upper tail of $\mathcal{L}^\beta$.

Lemma 6.3. *For any large enough L,*

$$\mathbb{P}\left(\inf_{\vec{y}:\|\vec{y}\|_\infty \le L^{-1/4}\log L} \sum_{i=1}^k \mathcal{L}_i^\beta < L - L^{5/8}\log L \;\middle|\; \mathcal{L}^\beta > L \right) < C\exp(-c(\log L)^2).$$

Proof. We first claim that, conditional on $\mathcal{L}^\beta > L$, it holds with probability at least $1-C\exp(-c(\log L)^2)$ that $\sup_{\vec{y}:\|\vec{y}\|_\infty \le L^{-1/4}\log L} \sum_{i=1}^k \mathcal{L}_i^\beta > L - C\log L$. Indeed, suppose that this inequality does not hold. In the $\beta=1$ case, we know from Lemma 5.4 that with conditional probability at least $1-C\exp(-c(\log L)^2)$,

$$\exp(L) < \left(1-e^{-cL^{-1/2}(\log L)^2}\right)^{-1} \int_{[-L^{-1/4}\log L, L^{-1/4}\log L]^{k-1}} \exp\left(\sum_{i=1}^k \mathcal{L}_i^\beta\right) \prod_{i=1}^{k-1} \mathrm{d}y_i,$$

where $y_k = 0$ in the integral. If $\sup_{\vec{y}:\|\vec{y}\|_\infty \le L^{-1/4}\log L} \sum_{i=1}^k \mathcal{L}_i^\beta < L - C\log L$, the RHS is upper bounded by

$$CL^{1/2}(\log L)^{-2}(2L^{-1/4}\log L)^k \exp\left(L - C\log L\right) \ll \exp(L),$$

which is a contradiction.

In the $\beta=\infty$ case, by Lemma 5.2, with conditional probability at least $1-C\exp(-c(\log L)^2)$ it holds that $\mathcal{L}^\beta = \sup_{\vec{y}:\|\vec{y}\|_\infty \le L^{-1/4}\log L} \sum_{i=1}^k \mathcal{L}_i^\beta$, which implies our claim since we have conditioned on $\mathcal{L}^\beta > L$.

Next, we know from Lemma 3.1 that, with (unconditional) probability at least $1-C\exp(-cL^{3/2}\log L)$,

$$\sup_{\|\vec{y}\|_\infty \le L^{-1/4}\log L} \sum_{i=1}^k \left|\mathcal{L}_i^\beta - \mathcal{L}^\beta(0, s_{i-1}; 0, s_i)\right| \le \tfrac{1}{2} L^{3/4}(\log L)^{1/2} \cdot (L^{-1/4}\log L)^{1/2} = \tfrac{1}{2} L^{5/8}\log L.$$

By Theorem 2.11 we know $\mathbb{P}(\mathcal{L}^\beta > L) > \exp(-CL^{3/2})$, the previous bound also holds conditionally on $\mathcal{L}^\beta > L$ with probability at least $1-C\exp(-cL^{3/2}\log L)$. This completes the proof. $\square$

The following statement asserts that, conditional on the sum of independent free energies being large, the individual terms are with high probability proportionate to the total, up to a certain scale of fluctuation.

Lemma 6.4. *Fix each $y_i = 0$. For any L large enough, $K > CL^{3/8}$ (so that $KL^{1/4} > CL^{5/8}$), and each $j = 1, \ldots, k$,*

$$\mathbb{P}\left(\mathcal{L}_j^\beta < (s_j - s_{j-1})L - KL^{1/4} \;\middle|\; \sum_{i=1}^k \mathcal{L}_i^\beta > s_k L\right) < \exp(-cK^2).$$

The above bound is optimal except for the fact that we require $K > CL^{3/8}$, while it should hold for $K > C$; the loss is due to the non-optimal error term in our tail bound Theorem 2.11.

Proof of Lemma 6.4. By the independence of these $\mathcal{L}_j^\beta$, without loss of generality, we prove the estimate for $j = k$.

Let $Y = \sum_{i=1}^{k-1} \mathcal{L}_i^\beta$, and $X = \mathcal{L}_k^\beta - (s_k - s_{k-1})L$. Then the probability in the LHS equals

$$\begin{aligned}
&\sum_{\ell=KL^{1/4}}^\infty \mathbb{P}\left(X \in -\ell + [-1,0] \;\middle|\; \sum_{i=1}^k \mathcal{L}_i^\beta > s_k L\right) \\
&= \frac{\sum_{\ell=KL^{1/4}}^\infty \mathbb{P}\left(\sum_{i=1}^k \mathcal{L}_i^\beta > s_k L \;\middle|\; X \in -\ell + [-1,0]\right) \cdot \mathbb{P}\left(X \in -\ell + [-1,0]\right)}{\mathbb{P}\left(\sum_{i=1}^k \mathcal{L}_i^\beta > s_k L\right)} \\
&\le \frac{\sum_{\ell=KL^{1/4}}^\infty \mathbb{P}\left(Y > s_{k-1}L + \ell \;\middle|\; X \in -\ell + [-1,0]\right) \cdot \mathbb{P}\left(X \in -\ell + [-1,0]\right)}{\mathbb{P}\left(\sum_{i=1}^k \mathcal{L}_i^\beta > s_k L\right)}.
\end{aligned} \tag{6.1}$$

We can lower bound the denominator by $\prod_{i=1}^{k}\mathbb{P}(\mathcal{L}_i^{\beta}>(s_i-s_{i-1})L)$, which is $>\exp\left(-\frac{4}{3}s_kL^{3/2}-CL^{3/4}\right)$ by Theorem 2.11. For the numerator, note that Y is independent of X, so the summand indexed by ℓ in (6.1) can be upper bounded, using Corollary 6.2 and Theorem 2.11 by

$$\begin{aligned}&\exp\left(-\tfrac{4}{3}s_{k-1}^{-1/2}\left(s_{k-1}L+\ell\right)^{3/2}-\tfrac{4}{3}(s_k-s_{k-1})^{-1/2}\left((s_k-s_{k-1})L-\ell\right)^{3/2}+CL^{3/4}\right)\\&\quad\leq\exp\left(-\tfrac{4}{3}s_kL^{3/2}-c\ell^2L^{-1/2}+CL^{3/4}\right).\end{aligned}$$

Substituting this into (6.1) and using that $K>CL^{3/8}$ for a large enough constant C (so that $(KL^{1/4})^2L^{-1/2}>C^2L^{3/4}$) yields the claim. $\square$

7. Concentration of polymers

In this section, we mainly work with polymers, i.e., set $\beta=1$. We prove the following fact: the polymer measure at a given height s is a Dirac mass spread out over an $L^{-1/2}$ interval around a random location. More specifically, we define

$$\pi(s)=\operatorname*{argmax}_{x\in\mathbb{R}}\mathcal{L}^{\beta}(0,0;x,s)+\mathcal{L}^{\beta}(x,s;0,1),$$

for any $s\in(0,1)$. Note that this is a function purely of the environment. Take an arbitrarily small number $t_0>0$. All the constants in this section can depend on t_0. The main result of this section asserts that under the polymer measure $\boldsymbol{\mathcal{P}}$, the polymer path in $[t_0,1-t_0]$ stays within $O(L^{-1/2}\log L)$ of π with high probability.

Proposition 7.1. *There exists $M_0>0$ such that for any $L\geq 2$, $M>M_0$ and $s\in[t_0,1-t_0]$,*

$$\mathbb{P}\left(\boldsymbol{\mathcal{P}}\left(|\Gamma_0(s)-\pi(s)|>ML^{-1/2}\log L\right)>L^{-2M}\;\middle|\;\mathcal{L}^{\beta}(0,0;0,1)>L\right)<C\exp(-c(\log(L))^2).$$

In the rest of this section, we shall always take $s\in[t_0,1-t_0]$. For the convenience of notations, we adopt the notation $s_1=s$ and $s_2=1-s$, $\mathcal{L}_1^{\beta}(x)=\mathcal{L}^{\beta}(0,0;x,s)$ and $\mathcal{L}_2^{\beta}(x)=\mathcal{L}^{\beta}(x,s;0,1)$, and $\mathcal{L}^{\beta}=\mathcal{L}^{\beta}(0,0;0,1)$.

7.1. Global to two-segments conditioning.

In the upcoming proof of Proposition 7.1, it will be useful to go from conditioning on $\mathcal{L}^{\beta}>L$ to conditioning on $\mathcal{L}_i^{\beta}(x)\in(h_i,h_i+\mathrm{d}h_i)$ for each $i=1,2$, with x,h_1,h_2 belonging to a set of nice values. The following is the general statement that allows us to do this.

Let $\mathrm{Val}\subseteq\mathbb{R}^2$ be defined by

$$\mathrm{Val}=\left\{(h_1,h_2)\in\mathbb{R}^2:L-(\log L)^2<h_1+h_2<L+(\log L)^2,h_i>s_iL-2\log(L)L^{5/8}\text{ for }i=1,2\right\}.$$

Lemma 7.2 (Global to two-segments conditioning)**.** *For any $L>0$, $0<K\leq\log L$, and any event A,*

$$\begin{aligned}\mathbb{P}\left(A\mid\mathcal{L}^{\beta}>L\right)<e^{CL^{1/2}\log L}\max_{x\in L^{-2}\mathbb{Z},|x|\leq KL^{-1/4}}\sup_{(h_1,h_2)\in\mathrm{Val}}\mathbb{P}\left(A\;\middle|\;\mathcal{L}_i^{\beta}(x)\in(h_i,h_i+\mathrm{d}h_i),i=1,2\right)\\+C\exp(-cK^2).\end{aligned}$$

Proof. We assume that $K>1$ since otherwise the conclusion follows obviously.

We observe from Lemma 5.4 that, with probability at least $1-C\exp(-cK^2)$, conditionally on $\mathcal{L}^{\beta}>L$,

$$\int_{[-KL^{-1/4},KL^{-1/4}]}\exp\left((\mathcal{L}_1^{\beta}+\mathcal{L}_2^{\beta})(x)\right)\mathrm{d}x\geq\left(1-\exp\left(-\tfrac{1}{2}K^2L^{-1/2}\right)\right)\exp(L)$$

$$\geq \exp\left(L+\log\left(\tfrac{1}{2}K^2L^{-1/2}\right)\right),$$

which implies that

$$\mathbb{P}\left(\max_{[-KL^{-1/4},KL^{-1/4}]}\mathcal{L}_1^\beta+\mathcal{L}_2^\beta\geq L-\log L \;\middle|\; \mathcal{L}^\beta>L\right)>1-C\exp(-cK^2).$$

This with Proposition 2.14 implies that

$$\mathbb{P}\left(\max_{x\in L^{-2}\mathbb{Z},|x|\leq KL^{-1/4}}(\mathcal{L}_1^\beta+\mathcal{L}_2^\beta)(x)\leq L-(\log L)^2 \;\middle|\; \mathcal{L}^\beta>L\right)<C\exp(-cL^2(\log L)^4)+C\exp(-cK^2).$$

Also, for each $|x|\leq KL^{-1/4}$, by Theorem 4.1 and Lemma 6.1, we have

$$\mathbb{P}\left((\mathcal{L}_1^\beta+\mathcal{L}_2^\beta)(x)\geq L+(\log L)^2\mid\mathcal{L}^\beta>L\right)<C\exp(-c(\log L)^2L^{1/2});$$

and by Lemma 6.3 and Lemma 6.4, we have

$$\mathbb{P}\left(\mathcal{L}_i^\beta(x)\leq s_iL-2\log(L)L^{5/8}\mid\mathcal{L}^\beta>L\right)<C\exp(-c(\log L)^2),$$

for each $i=1,2$. Combining the above three estimates gives that

$$\mathbb{P}(\mathrm{ValueCtrl}^c\mid\mathcal{L}^\beta>L)<C\exp(-cK^2),$$

where

$$\mathrm{ValueCtrl}_x=\left\{L-(\log L)^2<(\mathcal{L}_1^\beta+\mathcal{L}_2^\beta)(x)<L+(\log L)^2,\mathcal{L}_i^\beta(x)>s_iL-2\log(L)L^{5/8}\text{ for }i=1,2\right\},$$

and $\mathrm{ValueCtrl}=\cup_{x\in L^{-2}\mathbb{Z},|x|\leq KL^{-1/4}/2}\mathrm{ValueCtrl}_x$. Now we have

$$\mathbb{P}(A\mid\mathcal{L}^\beta>L)<\mathbb{P}(A\cap\mathrm{ValueCtrl}\mid\mathcal{L}^\beta>L)+C\exp(-cK^2).$$

Note that the first term in the RHS is bounded by

$$e^{CL^{1/2}\log L}\max_{x\in L^{-2}\mathbb{Z},|x|\leq KL^{-1/4}}\sup_{(h_1,h_2)\in\mathrm{Val}}\mathbb{P}\left(A\;\middle|\;\mathcal{L}_i^\beta(x)\in(h_i,h_i+\,\mathrm{d}h_i),i=1,2\right)\frac{\mathbb{P}(\mathrm{ValueCtrl})}{\mathbb{P}(\mathcal{L}^\beta>L)}.$$

By Lemma 6.1 and Theorem 4.1, we have

$$\mathbb{P}(\mathrm{ValueCtrl})\leq\sum_{x\in L^{-2}\mathbb{Z},|x|\leq KL^{-1/4}}\mathbb{P}(\mathrm{ValueCtrl}_x)\leq\mathbb{P}((\mathcal{L}_1^\beta+\mathcal{L}_2^\beta)(x)>L-(\log L)^2)$$
$$<C\exp(C(\log L)^2L^{1/2})\mathbb{P}(\mathcal{L}^\beta>L).$$

Combining the last three displays leads to the conclusion. □

7.2. **Random location and concentration.** We next use Lemma 7.2 to control the location of $\pi(s)$, as well as prove Proposition 7.1.

The next result asserts that $\pi(s)$ is of order $L^{-1/4}$.

Proposition 7.3. *For any L large enough, $s\in[t_0,1-t_0]$, and $0<K\leq\log(L)$,*

$$\mathbb{P}\left(|\pi(s)|>KL^{-1/4}\mid\mathcal{L}^\beta>L\right)<C\exp(-cK^2).$$

The general idea to prove Proposition 7.3 is to (1) upper bound $\max_{(-\log(L)L^{-1/4},\log(L)L^{-1/4})^c}\mathcal{L}_1^\beta+\mathcal{L}_2^\beta$ conditional on $\mathcal{L}_1^\beta(x)$ and $\mathcal{L}_2^\beta(x)$ for some $x=O(KL^{-1/4})$, using Corollary 2.18; and (2) connect the conditioning $\mathcal{L}^\beta>L$ and the conditioning on the values of $\mathcal{L}_1^\beta(x)$ and $\mathcal{L}_2^\beta(x)$, using Lemma 7.2.

Proof of Proposition 7.3. We assume that $K > 1$ since otherwise the conclusion follows obviously. As in the proof of Lemma 7.2, we have

$$\mathbb{P}\left(\max \mathcal{L}_1^\beta + \mathcal{L}_2^\beta \geq L - \log L \;\middle|\; \mathcal{L}^\beta > L\right) > 1 - C\exp(-c(\log L)^2). \tag{7.1}$$

We also have

$$\mathbb{P}\left(\max_{(-(t_0L)^{1/2}/2,(t_0L)^{1/2}/2)^c} \mathcal{L}_1^\beta + \mathcal{L}_2^\beta \geq (1 - t_0/10)L\right) < C\exp(-cL^{3/2})\mathbb{P}(\mathcal{L}^\beta > L), \tag{7.2}$$

by upper bounding the LHS using Corollary 6.2 plus shear invariance and a union bound, and lower bounding $\mathbb{P}(\mathcal{L}^\beta > L)$ using Theorem 2.11. Combining (7.1) and (7.2) implies that

$$\begin{aligned} &\mathbb{P}\left(|\pi(s)| > KL^{-1/4} \mid \mathcal{L}^\beta > L\right) \\ &< \mathbb{P}\left(\max_{[-(t_0L)^{1/2}/2,-KL^{-1/4}]\cup[KL^{-1/4},(t_0L)^{1/2}/2]} \mathcal{L}_1^\beta + \mathcal{L}_2^\beta \geq L - \log L \;\middle|\; \mathcal{L}^\beta > L\right) + C\exp(-c(\log L)^2). \end{aligned} \tag{7.3}$$

By Corollary 2.18 with $a = KL^{-1/4}$, for any $|x| \leq KL^{-1/4}/2$, and $(h_1, h_2) \in \mathrm{Val}$, we have

$$\begin{aligned} &\mathbb{P}\left(\max_{[-(t_0L)^{1/2}/2,-KL^{-1/4}]\cup[KL^{-1/4},(t_0L)^{1/2}/2]} \mathcal{L}_1^\beta + \mathcal{L}_2^\beta \geq L - \log L \;\middle|\; \mathcal{L}_i^\beta(x) \in (h_i, h_i + \mathrm{d}h_i), i = 1,2\right) \\ &\qquad < C\exp(-cKL^{3/4}) \end{aligned}$$

Thus by Lemma 7.2, the first term in the RHS of (7.3) can be bounded by $C\exp(-cK^2)$, and the conclusion follows. $\square$

We next finish proving the polymer concentration result.

Proof of Proposition 7.1. Consider the event $\mathrm{Tent}(M, L)$ defined by

$$\mathrm{Tent}(M, L) := \left\{\max_{|x-\pi(s)|\geq ML^{-1/2}\log L} \mathcal{L}_1^\beta(x) + \mathcal{L}_2^\beta(x) - (L - 2L^{1/2}|x - \pi(s)|) < 0\right\} \tag{7.4}$$

To understand this definition, recall that we expect each of $\mathcal{L}_1^\beta$ and $\mathcal{L}_2^\beta$ to essentially adopt tent shapes under the conditioning $\mathcal{L}^\beta > L$, where the tents each have slope approximately $\pm 2L^{1/2}$. Thus the sum $\mathcal{L}_1^\beta + \mathcal{L}_2^\beta$ can be expected to be a line of slope $\pm 4L^{1/2}$ up to random fluctuations; in the definition of the event, we have reduced the slope magnitude by 1/2 for the benefit of ignoring the random fluctuation.

Now, on $\mathrm{Tent}(M, L)$,

$$\begin{aligned} \int_{|x-\pi(s)|\geq ML^{-1/2}\log L} \exp\left((\mathcal{L}_1^\beta + \mathcal{L}_2^\beta)(x)\right)\,\mathrm{d}x &\leq \int_{|x-\pi(s)|\geq ML^{-1/2}\log L} \exp\left(L - 2L^{1/2}|x - \pi(s)|\right)\,\mathrm{d}x \\ &= 2\int_{ML^{-1/2}\log L}^{\infty} \exp\left(L - 2L^{1/2}x\right)\,\mathrm{d}x = L^{-1/2}\exp\left(L - 2M\log L\right), \end{aligned}$$

which implies that, on $\mathrm{Tent}(M, L)$ and $\mathcal{L}^\beta > L$,

$$\boldsymbol{\mathcal{P}}\left(|\Gamma_0(s) - \pi(s)| > ML^{-1/2}\log L\right) < L^{-2M}.$$

Thus our task is now to upper bound $\mathbb{P}(\mathrm{Tent}(M, L)^c \mid \mathcal{L}^\beta > L)$. We note that by Proposition 7.3,

$$\mathbb{P}\left(|\pi(s)| > \log(L)L^{-1/4} \mid \mathcal{L}^\beta > L\right) < C\exp(-c(\log L)^2). \tag{7.5}$$

Then it remains to upper bound $\mathbb{P}(\mathrm{Tent}(M, L)^c \cap \{|\pi(s)| \leq \log(L)L^{-1/4}\} \mid \mathcal{L}^\beta > L)$.

To apply Lemma 7.2, we take $x \in L^{-2}\mathbb{Z}$, $|x| \le \log(L)L^{-1/4}$ and $(h_1, h_2) \in \mathrm{Val}$, and consider

$$\mathbb{P}\left(\mathrm{Tent}(M, L)^c \cap \{|\pi(s)| \le \log(L)L^{-1/4}\} \,\middle|\, \mathcal{L}_i^\beta(x) \in (h_i, h_i + \mathrm{d}h_i), i = 1, 2\right). \tag{7.6}$$

Assuming that for each $i = 1, 2$, $\mathcal{L}_i^\beta(x) \in (h_i, h_i + \mathrm{d}h_i)$, and

$$\max_{ML^{-1/2}\log(L)/5 \le |y| \le (s_i h_i)^{1/2}} \mathcal{L}_i^\beta(x + y) \le h_i - \frac{3}{2}|y|(h_i/s_i)^{1/2},$$

and $|\pi(s)| \le \log(L)L^{-1/4}$, we must have $|\pi(s) - x| < ML^{-1/2}\log(L)/5$, and Tent holds. Therefore by Corollary 2.18 (with $a = ML^{-1/2}\log L$), we have that (7.6) is bounded by $C\exp(-cML^{1/2}\log L) + C\exp(-cL^{3/2})$. Then by Lemma 7.2, and taking M_0 large enough, we have $\mathbb{P}(\mathrm{Tent}(M, L)^c \mid \mathcal{L}^\beta > L) < C\exp(-cL^{1/2}\log L) + C\exp(-c(\log L)^2)$. Thus the conclusion follows. □

8. Tightness for polymers

We prove the $\beta = 1$ case of Proposition 5.1 in this section. Throughout this section, we fix an arbitrarily small number $t_0 > 0$, and all the constants can depend on it. We also denote $\mathcal{L}^\beta = \mathcal{L}^\beta(0, 0; 0, 1)$ for simplicity of notations.

The main task is to prove the following two points estimate, which refines Lemma 5.6.

Proposition 8.1. *For all $L \ge 2$, $t_0 \le s < t \le 1 - t_0$ and $K > 0$,*

$$\mathbb{P}\left(\boldsymbol{\mathcal{P}}\left(|\Gamma_0(s) - \Gamma_0(t)| > K(t-s)^{1/10}L^{-1/4}\right) > L^{-K} \,\middle|\, \mathcal{L}^\beta > L\right) < C\exp(-c(K \wedge \log L)^2).$$

Compared to the zero temperature setting (i.e., Proposition 5.3), we weaken our demand to a Hölder $\frac{1}{10}-$ bound instead of $\frac{1}{2}-$, due to technical reasons which will be clear from its proof.

Proposition 8.1 immediately implies the following.

Corollary 8.2. *For all $L \ge 2$, $K > 0$ and $t_0 \le s < t \le 1 - t_0$,*

$$\mathbb{E}\left[\boldsymbol{\mathcal{P}}\left(|\Gamma_0(s) - \Gamma_0(t)| > K(t-s)^{1/10}L^{-1/4}\right) \,\middle|\, \mathcal{L}^\beta > L\right] < C\exp(-c(K \wedge \log L)^2).$$

Remark 8.3. As indicated in the introduction, from these tightness results and the polymer concentration of Proposition 7.1, one can define a random backbone $\tilde{\pi}$, by e.g., taking $\tilde{\pi}(s) = \pi(s)$ for each $s \in [0, 1] \cap L^{-10}\mathbb{Z}$, and linearly interpolate between them. Then with probability $> 1 - C\exp(-c(\log L)^2)$, Γ_0 is within distance $L^{-1/2}(\log L)^2$ from $\tilde{\pi}$ at each $s \in [0, 1] \cap L^{-10}\mathbb{Z}$, by Proposition 7.1; and between any two points, Γ_0 can deviate at most L^{-1}, by Corollary 8.2. Therefore (with the same probability) Γ_0 is within distance $L^{-1/2}(\log L)^2$ from the backbone $\tilde{\pi}$ throughout $[0, 1]$.

Proof of the $\beta = 1$ case of Proposition 5.1 (away from the ends). With Corollary 8.2, via a union bound over all $s = 2^{-i}j$, $t = s^{-i}(j + 1)$, $t_0 \le s < t \le 1 - t_0$, with $i, j \in \mathbb{Z}$, we have the following. For any $K > 0$, with probability $> 1 - C\exp(-c(K \wedge \log L)^2)$ conditional on $\mathcal{L}^\beta > L$,

$$\sup_{t_0 \le s < t \le 1 - t_0} |\Gamma_0(s) - \Gamma_0(t)|(t-s)^{-1/11} \le KL^{-1/4}.$$

This completes the proof. □

To prove Proposition 8.1, the key idea is to upgrade Lemma 5.6 using the concentration result Proposition 7.1. It allows us to essentially say that (with high probability under $\mathbb{P}$) if the law of $\Gamma_0(s)$ under $\boldsymbol{\mathcal{P}}$ assigns a probability greater than ε (for a carefully chosen small ε) to the event of large transversal fluctuation, it must assign a close to 1 probability to the event of having at least say half of the same transversal fluctuation. The latter event's probability is bounded by Lemma 5.6.

Note that in this argument it is actually not important that the interval around which most of the mass is spread is centered around $\pi(s)$, only that the interval is small.

We start by first upgrading the one-point estimate, away from the endpoints.

Proposition 8.4. *For all $K > 0$, $L \geq 2$, and $s \in [t_0, 1-t_0]$,*

$$\mathbb{P}\left(\boldsymbol{\mathcal{P}}\left(|\Gamma_0(s)| \geq KL^{-1/4}\right) > L^{-K} \;\middle|\; \mathcal{L}^\beta > L\right) < C\exp(-c(K \wedge \log L)^2).$$

Proof. We assume that K is large enough since otherwise we just choose C large to make the estimate hold. We also assume that L is large enough since otherwise the conclusion follows from Lemma 5.4. Let $X(s,K) = \boldsymbol{\mathcal{P}}(|\Gamma_0(s)| \geq KL^{-1/4})$. By Proposition 7.1, with probability $\geq 1 - C\exp(-c(\log L)^2)$ conditional on $\mathcal{L}^\beta > L$ we have

$$\begin{aligned} X(s,K) &\leq \boldsymbol{\mathcal{P}}\left(|\Gamma_0(s) - \pi(s)| \geq KL^{-1/2}\log L\right) + \mathbb{1}_{|\pi(s)| \geq \frac{1}{2}KL^{-1/4}} \\ &\leq L^{-2K} + \mathbb{1}_{|\pi(s)| \geq \frac{1}{2}KL^{-1/4}}, \end{aligned}$$

and

$$\begin{aligned} X(s, \tfrac{1}{4}K) &\geq \boldsymbol{\mathcal{P}}\left(|\Gamma_0(s) - \pi(s)| \leq KL^{-1/2}\log L\right)\mathbb{1}_{|\pi(s)| \geq \frac{1}{2}KL^{-1/4}} \\ &\geq (1 - L^{-2K})\mathbb{1}_{|\pi(s)| \geq \frac{1}{2}KL^{-1/4}}. \end{aligned}$$

Here we used that $KL^{-1/2}\log L < \frac{1}{4}KL^{-1/4}$. These two bounds show that

$$X(s,K) > L^{-2K} \implies |\pi(s)| \geq \tfrac{1}{2}KL^{-1/4} \implies X(s, \tfrac{1}{4}K) \geq 1 - L^{-2K},$$

and thus

$$\mathbb{P}\left(X(s,K) > L^{-2K} \;\middle|\; \mathcal{L}^\beta > L\right) < \mathbb{P}\left(X(s, \tfrac{1}{4}K) \geq 1 - L^{-2K} \;\middle|\; \mathcal{L}^\beta > L\right) + C\exp(-c(\log L)^2).$$

By Lemma 5.4, $\mathbb{P}\left(X(s, \frac{1}{4}K) > \exp(-K^2L^{-1/2}/32) \;\middle|\; \mathcal{L}^\beta > L\right) < C\exp(-cK^2)$. This completes the proof. ∎

Next, we turn to the two-point estimates. The proof strategies are analogous to what we just saw for the one-point: we combine a cruder estimate coming from shear invariance with the information that the polymer measure is localized on a smaller scale. We initially get the following two-point estimate, which is under the additional constraint that the two points are not too close. To get Proposition 8.1 from it, it turns out that the cruder estimate Lemma 5.6 is sufficient, since we just prove an Hölder $\frac{1}{10}-$ bound instead of $\frac{1}{2}-$.

Proposition 8.5. *For all $K > 0$, $L \geq 2$, and $t_0 \leq s < t \leq 1-t_0$, $t - s \geq L^{-5/8}$,*

$$\mathbb{P}\left(\boldsymbol{\mathcal{P}}\left(|\Gamma_0(s) - \Gamma_0(t)| > K(t-s)^{1/3}L^{-1/4}\right) > L^{-K} \;\middle|\; \mathcal{L}^\beta > L\right) < C\exp(-c(K \wedge \log L)^2).$$

We note that the exponents $1/3$ and $5/8$ can be replaced by any other numbers $< 1/2$ and $> 1/2$ respectively, as long as their product is $< 1/4$. We chose these numbers merely for correctness.

Proof of Proposition 8.5. This proof is very similar to that of Proposition 8.4. Again, we can assume that L, K are large enough. Let $X(s,t,K) = \boldsymbol{\mathcal{P}}\left(|\Gamma_0(s) - \Gamma_0(t)| > K(t-s)^{1/3}L^{-1/4}\right)$. By Proposition 7.1, with probability $\geq 1 - C\exp(-c(\log L)^2)$ conditional on $\mathcal{L}^\beta > L$,

$$\begin{aligned} X(s,t,K) &\leq \boldsymbol{\mathcal{P}}\left(\max_{r \in \{s,t\}} |\Gamma_0(r) - \pi(r)| \geq KL^{-1/2}\log L\right) + \mathbb{1}_{|\pi(s)-\pi(t)| \geq \frac{1}{2}K(t-s)^{1/3}L^{-1/4}} \\ &\leq 2L^{-2K} + \mathbb{1}_{|\pi(s)-\pi(t)| \geq \frac{1}{2}K(t-s)^{1/3}L^{-1/4}} \end{aligned}$$

and

$$X(s,t,\tfrac{1}{4}K) \geq \boldsymbol{\mathcal{P}}\left(\max_{r\in\{s,t\}} |\Gamma_0(r)-\pi(r)| \leq KL^{-1/2}\log L\right) \mathbb{1}_{|\pi(s)-\pi(t)|\geq \frac{1}{2}(t-s)^{1/3}KL^{-1/4}}$$
$$\geq (1-2L^{-2K})\mathbb{1}_{|\pi(s)-\pi(t)|\geq \frac{1}{2}(t-s)^{1/3}KL^{-1/4}}.$$

Here we used that $KL^{-1/2}\log L < \frac{1}{4}K(t-s)^{1/3}L^{-1/4}$. These imply that

$$X(s,t,K) > 2L^{-2K} \implies |\pi(s)-\pi(t)| \geq \tfrac{1}{2}K(t-s)^{1/3}L^{-1/4} \implies X(s,t,\tfrac{1}{4}K) \geq 1-2L^{-2K}.$$

Besides, by Lemma 5.6 we have $\mathbb{P}\left(X(s,t,\frac{1}{4}K) > \exp(-K^2L^{-1/2}/32) \;\middle|\; \mathcal{L}^\beta > L\right) < C\exp(-cK^2)$. These complete the proof. □

Proof of Proposition 8.1. We assume that K is large enough by taking C large (if necessary), and assume that L is large enough by applying Lemma 5.6 otherwise.

For the case where $t-s \geq L^{-5/8}$, we are done by Proposition 8.5. For the case where $t-s < L^{-5/8}$. By Lemma 5.6, we can bound the conditional probability in the statement of Proposition 8.1 by

$$C\exp(-cK^2(t-s)^{-4/5}L^{1/2}) < C\exp(-cK^2),$$

since $\exp(-\frac{1}{2}K^2(t-s)^{-4/5}) < L^{-K}$. Therefore the conclusion follows. □

Now that we have proven the tightness of $L^{1/4}\Gamma_0$ away from the ends, we next explain how to upgrade that to tightness in $\mathcal{C}([0,1],\mathbb{R})$.

From the arguments in this section, it can be seen that the main task is to upgrade Proposition 7.1 to hold for any $s \in [L^{-5/8}, 1-L^{-5/8}]$, since then one can similarly replace t_0 by $L^{-5/8}$ in Section 5.2. One can prove Proposition 8.1 for any $0 < s < t < 1$, by treating the case of $s < L^{-5/8}$ or $t > 1-L^{-5/8}$ using Lemma 5.6.

The proof of Proposition 7.1 relies on the tail and tent estimates in Section 2.4, and it suffices to upgrade these estimates. More precisely, we need to upgrade Theorem 2.11 and Theorem 2.15 from [GH22] to the following estimates.

Theorem 2.11*. Fix $\varepsilon > 0$. There exists $L_0 > 0$ such that, for any $t > 0$ and $L > (t^{-1/3+\varepsilon} \vee 1)L_0$,

$$\exp\Big(-\tfrac{4}{3}L^{3/2} - CL^{3/4}\Big) < \mathbb{P}\left(\hat{\mathfrak{h}}^\beta_{t,1}(0) \in (L, L+\mathrm{d}L)\right)/\,\mathrm{d}L < \exp\Big(-\tfrac{4}{3}L^{3/2} + CL^{3/4}\Big).$$

Theorem 2.15*. There exists $L_0 > 0$, such that for any $t > 0$ and $L > (t^{-1/3+\varepsilon} \vee 1)L_0$, we have

$$\mathbb{P}\left(\sup_{x\in[-L^{1/2},L^{1/2}]} \left|\hat{\mathfrak{h}}^\beta_{t,1}(x) - L + 2L^{1/2}|x|\right| > ML^{1/4} \;\middle|\; \hat{\mathfrak{h}}^\beta_{t,1}(0) \in (L, L+\mathrm{d}L)\right) < \exp(-cM^2),$$

for any $0 < M < cL^{3/4}$. The same is true under the conditioning $\hat{\mathfrak{h}}^\beta_{t,1}(0) > L$.

The lower tail bound Theorem 2.13 also needs to be expanded to small t. That directly follows from [DG23, Theorem 1.7], as stated in (8.1) below. We note that these tail and tent bounds now hold for t starting from order $L^{-3+10\varepsilon}$, which would suffice to deduce Proposition 7.1 in $[L^{-5/8}, 1-L^{-5/8}]$.

We next explain how to adapt the proofs in [GH22] to get Theorem 2.11* and Theorem 2.15*, using inputs from [DG23]. Written with our scaling, [DG23, Theorem 1.4] asserts that, for any $\varepsilon > 0$, there exist t_0, c, and s_0, all depending on ε, such that, for $0 < t < t_0$ and $M > M_0 t^{-1/12}$,

$$\mathbb{P}\left(\hat{\mathfrak{h}}^\beta_{t,1}(0) > M + t^{-1/3}\log t^{-1}\right) \leq \exp\left(-c\frac{M^2t^{1/6}}{\sqrt{1+Mt^{1/3-\varepsilon/2}}}\right).$$

If we additionally assume that $M > M_0 t^{-1/3-\varepsilon}$ and $t < 1$, so that $M^{1/2} \geq t^{-1/6-\varepsilon/2} \geq t^{-\varepsilon/2}$, the previous display implies that

$$\mathbb{P}\left(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > 2M\right) \leq \mathbb{P}\left(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > M + t^{-1/3}\log t^{-1}\right) \leq \exp\left(-cM^{3/2}t^{\varepsilon/2}\right) \leq \exp(-cM).$$

For control on the lower tail, [DG23, Theorem 1.7] and a similar calculation shows that, if $M > t^{-1/6}$,

$$\mathbb{P}\left(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) < -M\right) \leq \exp(-cM^2 t^{1/6}) \leq \exp(-cM); \tag{8.1}$$

and one can also instead assume $M > t^{-1/12-\varepsilon}$ and obtain a weaker bound of the form $\exp(-cM^{\alpha})$ for some $\alpha = \alpha(\varepsilon)$.

To apply the arguments of [GH22] that yield Theorems 2.11 and 2.15, we also need an a priori lower bound on $\mathbb{P}(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > M)$. For this it is easy to check that the proof of [GH22, Lemma 5.4] applies verbatim if we assume $M > t^{-1/6}$ (or $M > t^{-1/12-\varepsilon}$, as above) and use (8.1) in place of lower tail tightness (over $t \geq t_0$) of $\hat{\mathfrak{h}}^{\beta}_{t,1}(0)$. This will then yield, for $t > 0$ and $M > t^{-1/6}$,

$$\mathbb{P}\left(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > M\right) \geq \exp(-5M^{3/2}).$$

With these estimates in place, we satisfy Assumptions (i)–(iv) from [GH22, Section 2.2] (more precisely, the above verifies Assumption (iv) on tail bounds, as the other assumptions are qualitative and hold for any $t > 0$). In particular, [GH22, Theorems 4.1, 5.1, and 5.3] will apply as they hold only assuming these assumptions, modulo controlling the partition function arising from the Gibbs property of $\hat{\mathfrak{h}}^{\beta}_{t}$, and we explain this point next.

In [GH22], the estimates lower bounding the partition function are captured in Lemma 3.11, Corollary 3.12, Proposition 3.14, and Corollary 3.15. The latter two are already stated for $t > 0$. Lemma 3.11 says that the partition function Z_t of a single curve with respect to a lower boundary curve p on an interval $[z_1, z_2]$ is lower bounded by $\exp(-2t^{2/3}e^{-t^{1/6}})\int_{z_1}^{z_2}\exp(-t^{1/3}g(u))\,\mathrm{d}u)\cdot\mathbb{P}(B(u) > p(u) + g(u) + t^{-1/6}, \forall u \in [z_1, z_2])$ for any non-negative function g, and a quick inspection of the proof shows that a minor modification yields a lower bound of $\exp(-2t^{2/3})\int_{z_1}^{z_2}\exp(-t^{1/3}g(u))\,\mathrm{d}u)\cdot\mathbb{P}(B(u) > p(u) + g(u), \forall u \in [z_1, z_2])$, which is better for small t. Carrying this change to Corollary 3.12 and then making use of these estimates in the proofs of [GH22, Theorems 4.1, and 5.1] will then yield Theorems 2.11* and 2.15*.

The remaining sections will be devoted to proving finite dimensional convergence.

9. Estimates on free energies under conditionings

As indicated in Section 1.2, our strategy of deducing finite dimensional limit heavily relies on realizing the conditioning on $\mathcal{L}^{\beta}(0,0;0,1)$ as conditioning on the existence of peaks at certain heights, of certain values, at certain locations. As such, we will often need estimates on the probability of the existence of such peaks given the global conditioning $\mathcal{L}^{\beta}(0,0;0,1)$, or on the probability of the latter conditioned on the former. One such estimate (Lemma 6.3) has appeared before, and in this section we provide some more refined ones.

Let us introduce the setup, and some notations needed to state these estimates. We will work under a setting similar to that in Section 6. Namely, we consider $(s_1, \cdots, s_{k-1}) \in \mathring{\Lambda}_{k-1}([0,1])$ and $\vec{x} \in \mathbb{R}^{k-1}$ for $k \in \mathbb{N}$, and denote $s_0 = x_0 = x_k = 0$ and $s_k = 1$. All the constants within this section can depend on k and $(s_1, \cdots, s_{k-1})$. For the convenience of notations we adopt the shorthands $\mathcal{L}^{\beta} = \mathcal{L}^{\beta}(0,0;0,1)$ and $\mathcal{L}^{\beta}_i = \mathcal{L}^{\beta}(s_{i-1}, x_{i-1}; s_i, x_i)$ for each $1 \leq i \leq k$. We also write $\vec{\mathcal{L}}^{\beta}$ for the vector $\{\mathcal{L}^{\beta}_i\}_{i=1}^{k}$.

Below we use $x \approx y$ to denote that $x \in y + [0, e^{-L}]$; and for any vector $\vec{h} \in \mathbb{R}^k$, $\vec{\mathcal{L}}^\beta \approx \vec{h}$ is the event where $\mathcal{L}_i^\beta \approx h_i$ for each i. A main reason for introducing this notation is that we will need to invoke coalescence or Brownian comparison statements (Proposition 3.4 and Proposition 3.5) which do not allow conditioning on exact values.

Take $\vec{a} = (a_0, \dots, a_k) \in [-L^{5/16} \log L, L^{5/16} \log L]^{k-1}$. For each $i = 1, \dots, k-1$, we write

$$\pi^*(s_i) = \operatorname*{argmax}_x \mathcal{L}^\beta(a_{i-1}, s_{i-1}; x, s_i) + \mathcal{L}^\beta(x, s_i; a_{i+1}, s_{i+1}).$$

We introduce useful notations generalizing the maximum to positive temperature: for $f : \mathbb{R}^{k=1} \to \mathbb{R}$ and a set $I \subseteq \mathbb{R}^{k-1}$,

$$\max_{\vec{x} \in I}{}^{(\beta)} f = \begin{cases} \log \displaystyle\int_I \exp\big(f(x_1, \dots, x_{k-1})\big) \, \mathrm{d}x_1 \cdots \mathrm{d}x_{k-1} & \beta = 1 \\ \displaystyle\max_{\vec{x} \in I} f(x_1 \cdots \mathrm{d}x_{k-1}) & \beta = \infty. \end{cases} \tag{9.1}$$

We further define the restricted free energy $\mathcal{L}^\beta[\vec{y}, R]$ for $R > 0$ and $\vec{y} \in \mathbb{R}^{k-1}$ by

$$\mathcal{L}^\beta[\vec{y}, R] = \max_{\|\vec{x} - \vec{y}\|_\infty \le R}{}^{(\beta)} \sum_{i=1}^k \mathcal{L}_i^\beta.$$

Let $r_{\beta=1} = 1$ and $r_{\beta=\infty} = L^{-1/2}$. These are the fluctuation scales of the total free energy conditional on the peak heights and locations. Let $w_{\beta=1} = L^{-1/2}$ and $w_{\beta=\infty} = L^{-1}$. These are the window sizes around each $\pi^*(s_i)$ that would affect $\mathcal{L}^\beta$, under the upper tail.

The next two statements record the above-mentioned complementary estimates and will be the goal of this section.

Proposition 9.1. *Take any large enough L, M, and $\vec{h} \in \mathbb{R}^k$, $\vec{a} \in [-L^{5/16} \log L, L^{5/16} \log L]^{k+1}$, $\vec{x} \in [-L^{5/16} \log L, L^{5/16} \log L]^{k-1}$. Denote $H = \sum_{i=1}^k h_i$ and assume that $H > L/2$ and each $|h_i - (s_i - s_{i-1})H| < L^{8/9}$. Then we have*

$$\mathbb{P}\Bigg(\mathcal{L}^\beta > H - (k-1)\beta^{-1} \log(2H^{1/2}) + Mr_\beta, \\ \max_{i=1,\dots,k-1} |\pi^*(s_i) - x_i| \le w_\beta \;\Bigg|\; \vec{\mathcal{L}}^\beta \approx \vec{h}\Bigg) < C \exp\left(-cM^2 L^{1/2} r_\beta\right) + C \exp(-cL^{3/2}), \tag{9.2}$$

$$\mathbb{P}\Bigg(\mathcal{L}^\beta[\vec{x}, L^{-1/2} (\log L)^2] > H - (k-1)\beta^{-1} \log(2H^{1/2}) + Mr_\beta, \\ \max_{i=1,\dots,k-1} |\pi^*(s_i) - x_i| \le w_\beta \;\Bigg|\; \vec{\mathcal{L}}^\beta \approx \vec{h}\Bigg) > c \exp\left(-CM^2 L^{1/2} r_\beta\right) - C \exp(-cL^{3/2}). \tag{9.3}$$

Moreover, we also have

$$\mathbb{P}\left(\mathcal{L}^\beta > H - (k-1)\beta^{-1} \log(2H^{1/2}) + M \;\middle|\; \vec{\mathcal{L}}^\beta \approx \vec{h}\right) < C \exp\left(-cML^{1/2}\right). \tag{9.4}$$

The term $(k-1) \log(2H^{1/2})$ is meant to be present in the case $\beta = 1$ and absent in the case $\beta = \infty$, and multiplying the term by β^{-1} is a convenient notational tool to this effect (though in fact if one were to work out the arguments in the case of general β, the term would be $(k-1)\beta^{-1} \log(2\beta H^{1/2})$). The source of the log term for $\beta < \infty$ comes from the fact that $\int_{-\infty}^\infty \exp(-4H^{1/2}|x|) \, \mathrm{d}x = (2H^{1/2})^{-1}$, which itself is a result that the dominant contribution to the integral is from an interval of scale $L^{-1/2}$ around zero. Since on $\vec{\mathcal{L}}^\beta \approx \vec{h}$ the terms in the exponential in the convolution formula for

$\exp(\mathcal{L}^\beta)$ are essentially sums of two Brownian bridges with slope $-2H^{1/2}$ each, heuristically taking logarithms will yield that $\mathcal{L}^\beta$ loses $(k-1)\log(2H^{1/2})$ compared to the peak height of H.

We next give an estimate on the probability of $\sum_{i=1}^k \mathcal{L}_i^\beta$ being much smaller than L, conditional on $\mathcal{L}^\beta > L$ and π^*. It can be viewed as a refinement of Lemma 6.3.

Proposition 9.2. *For all large enough L, M with $M < L^{0.01}$, and $\vec{x} \in [-L^{5/16}\log L, L^{5/16}\log L]^{k-1}$, $\vec{x} \in [-L^{5/16}\log L, L^{5/16}\log L]^{k-1}$, we have*

$$\mathbb{P}\left(E_{k,M,w,L} \;\middle|\; \mathcal{L}^\beta > L, \max_{i=1,\dots,k} |\pi^*(s_i) - x_i| \le w_\beta\right) \le \exp\left(-cM^2 L^{1/2} r_\beta\right),$$

where

$$E_{k,M,w,L} = \left\{\sum_{i=1}^k \mathcal{L}_i^\beta \in L + (k-1)\beta^{-1}\log(2L^{1/2}) + [-L^{8/9}, -Mr_\beta]\right\} \cap \bigcap_{i=1}^k \left\{\mathcal{L}_i^\beta > (s_i - s_{i-1})L - k^{-1}L^{8/9}\right\}. \quad (9.5)$$

Proposition 9.2 is proved by invoking Bayes' theorem and Proposition 9.1, and we give its proof now; we will return to proving Proposition 9.1 later. For any vector $\vec{h} \in \mathbb{R}^k$, denote

$$A_{\vec{h}} = \bigcap_{i=1}^k \left\{\mathcal{L}_i^\beta \in h_i + [0, r_\beta]\right\}.$$

Proof of Proposition 9.2. We assume without loss of generality that $M \in \mathbb{N}$. Define Val_M by

$$\mathrm{Val}_M = \left\{\vec{h} \in (r_\beta\mathbb{Z})^k : \begin{array}{c} \sum_{i=1}^k h_i \in L + (k-1)\beta^{-1}\log(2L^{1/2}) + [-L^{8/9} - r_\beta, -Mr_\beta] \\ h_i \in (s_i - s_{i-1})L + [-L^{8/9}, L^{8/9}], \quad i = 1, \dots, k \end{array}\right\}.$$

By doing a disjoint decomposition and applying Bayes' theorem,

$$\begin{aligned} &\mathbb{P}\left(E_{k,M,w,L} \;\middle|\; \mathcal{L}^\beta > L, \max_{i=1,\dots,k} |\pi^*(s_i) - x_i| \le w_\beta\right) \\ &\le \frac{\sum_{\vec{h}\in\mathrm{Val}_M} \mathbb{P}\left(\mathcal{L}^\beta > L, \max_{i=1,\dots,k} |\pi^*(s_i) - x_i| \le w_\beta \;\middle|\; A_{\vec{h}}\right) \cdot \mathbb{P}\left(A_{\vec{h}}\right)}{\mathbb{P}\left(\mathcal{L}^\beta > L, \max_{i=1,\dots,k} |\pi^*(s_i) - x_i| \le w_\beta\right)}. \end{aligned}$$

For each $\vec{h} \in \mathrm{Val}_M$, we wish to bound the ratio

$$\frac{\mathbb{P}\left(\mathcal{L}^\beta > L, \max_{i=1,\dots,k} |\pi^*(s_i) - x_i| \le w_\beta \;\middle|\; A_{\vec{h}}\right) \cdot \mathbb{P}(A_{\vec{h}})}{\mathbb{P}\left(\mathcal{L}^\beta > L, \max_{i=1,\dots,k} |\pi^*(s_i) - x_i| \le w_\beta \;\middle|\; A_{\vec{h}'}\right) \cdot \mathbb{P}(A_{\vec{h}'})},$$

where $\vec{h}' \in (r_\beta\mathbb{Z})^k$ is defined as follows: for each $i = 2, \dots, k$, $h'_i = h_i$, while $\sum_{i=1}^k h'_i \in L + (k-1)\beta^{-1}\log(2L^{1/2}) + [0, 1]$, $h'_1 - h_1 \in \lfloor h'_1 - h_1 \rfloor + [0, r_\beta)$.

By Theorem 4.1 we have

$$\frac{\mathbb{P}(A_{\vec{h}})}{\mathbb{P}(A_{\vec{h}'})} < C\exp(C(h'_1 - h_1)L^{1/2}),$$

and by Proposition 9.1 we have

$$\frac{\mathbb{P}\left(\mathcal{L}^\beta > L, \max_{i=1,\dots,k} |\pi^*(s_i) - x_i| \le w_\beta \;\middle|\; A_{\vec{h}}\right)}{\mathbb{P}\left(\mathcal{L}^\beta > L, \max_{i=1,\dots,k} |\pi^*(s_i) - x_i| \le w_\beta \;\middle|\; A_{\vec{h}'}\right)} < C\exp(-c(h'_1 - h_1)^2 L^{1/2} r_\beta^{-1}) + C\exp(-cL^{3/2}).$$

By combining these two estimates, we can now bound the ratio as desired. Then by summing over all $\vec{h} \in \mathrm{Val}_M$ the conclusion follows. $\square$

We next give the proof of Proposition 9.1.

9.1. $k = 2$ **setting.** The basic step is to prove the following $k = 2$ version: from this, we can obtain the general k-version by invoking coalescence (Proposition 3.4) to break down the case of general k to a collection of $k = 2$ cases.

The notations within this subsection are slightly different, and we setup now. We take large enough $L > 0$, h_1 and $h_2 = \Theta(L)$, $0 < s_1, s_2 < 1$, $|x_*| \le 2L^{5/16} \log L$. All the constants in this section can depend on $s_1 \wedge s_2$. We write $\vec{\mathcal{L}}^\beta = (\mathcal{L}^\beta(0, 0; x_*, s_1), \mathcal{L}^\beta(x_*, s_1; 0, s_1 + s_2))$. For any $R > 0$ we denote

$$\mathcal{L}^\beta[x_*, R] = \max_{|x - x_*| \le R}{}^{(\beta)}\, \mathcal{L}^\beta(0, 0; x, s_1) + \mathcal{L}^\beta(x, s_1; 0, s_1 + s_2).$$

And we write $\mathcal{L}^\beta[R] = \mathcal{L}^\beta[0, R]$. We also denote $\lambda = (s_1^{-1/2} h_1^{1/2} + s_2^{-1/2} h_2^{1/2})$. We take some x_-, x_+ with $|x_-|, |x_+| \le 2L^{5/16} \log L$ and denote

$$\pi^*(s_1) = \operatorname{argmax} \mathcal{L}^\beta(x_-, 0; \cdot, s_1) + \mathcal{L}^\beta(\cdot, s_1; x_+, s_1 + s_2).$$

Lemma 9.3. *Denote $W = 10^{-6}(h_1 s_1)^{1/2} \wedge (h_2 s_2)^{1/2}$. In the case of $\beta = 1$, for any $M > L^{1/16} \log L$ we have*

$$\mathbb{P}\Big(\mathcal{L}^\beta[W] > h_1 + h_2 - \beta^{-1} \log(\lambda) + ML^{-1/4}, |\pi^*(s_1) - x_*| \le w_\beta \;\Big|\; \vec{\mathcal{L}}^\beta \approx \vec{h}\Big) \\ < C \exp(-cM^2) + C \exp(-cL^{3/2}), \quad (9.6)$$

and for any $M > 0$,

$$\mathbb{P}\Big(\mathcal{L}^\beta[x_*, L^{-1/2}(\log L)^2] > h_1 + h_2 - \beta^{-1} \log(\lambda) + ML^{-1/4}, |\pi^*(s_1) - x_*| \le w_\beta \;\Big|\; \vec{\mathcal{L}}^\beta \approx \vec{h}\Big) \\ > c \exp(-CM^2) - C \exp(-cL^{3/2}). \quad (9.7)$$

In the case of $\beta = \infty$, the first two bounds hold for any $M > 0$, after (1) replacing $ML^{-1/4}$ with $ML^{-1/2}$ in both; (2) in the lower bound, replacing $\mathcal{L}^\beta[x_*, L^{-1/2}(\log L)^2]$ with $\mathcal{L}^\beta[x_*, L^{-1}]$.

Moreover, for both $\beta = 1$ and $\beta = \infty$, and M large enough, we have

$$\mathbb{P}\Big(\mathcal{L}^\beta[W] > h_1 + h_2 - \beta^{-1} \log(\lambda) + M \;\Big|\; \vec{\mathcal{L}}^\beta \approx \vec{h}\Big) < C \exp(-cML^{1/2}). \quad (9.8)$$

Proof. The general idea of this proof is to invoke Lemma 2.16, then do computations of Brownian motions.

We can assume that M is large enough since otherwise the estimates obviously hold. For the convenience of notations, we denote $S(x) = \mathcal{L}^\beta(0, 0; x, s_1) + \mathcal{L}^\beta(x, s_1; 0, s_1 + s_2)$, and write $B(x) = S(x_* + x) - S(x_*) + 2\lambda|x|$. In light of Lemma 2.16, we shall think of $B(x)$ as a Brownian motion in the interval $[-CL^{-1/2} \log L, CL^{-1/2} \log L]$.

Positive temperature. We first consider the case of $\beta = 1$ (where $w_{\beta=1} = L^{-1/2}$), and turn to the zero temperature ($\beta = \infty$) case later.

Upper bound. By the convolution formula,

$$\mathcal{L}^\beta[W] = \log \int_{-W}^{W} \exp(S(x))\, \mathrm{d}x. \quad (9.9)$$

Since $h_1, h_2 = \Theta(L)$ and $|x_*|, |x_-|, |x_+| \le 2L^{5/16} \log L$, by Corollary 2.18, conditional on $\vec{\mathcal{L}}^\beta \approx \vec{h}$, with probability at least $1 - C\exp(-cM^2)$,

$$S(x) < h_1 + h_2 - 2\lambda|x - x_*| + C|x - x_*|L^{5/16} \log L + ML^{-1/4},$$

for any $|x - x_*| \le L^{-1/2}$, and

$$S(x) < h_1 + h_2 - 2\lambda|x - x_*| + C|x - x_*|L^{5/16} \log L + M|x - x_*|^{1/2}(|\log(|x - x_*|LM^{-2})| + 1),$$

for any x with $|x - x_*| \ge L^{-1/2}$, $|x| \le W$.

To get (9.8), we note that using the above two estimates, (9.9) implies that $\mathcal{L}^\beta[W] < h_1 + h_2 + \log(\lambda^{-1}) + CM^2L^{-1/2}$ when $M > L^{1/4}$. Relabelling M completes the proof.

To get (9.6), we assume the above two estimates as well as $|\pi^*(s_1) - x_*| \le L^{-1/2}$, which (by Proposition 3.4) further implies that $S(x) < h_1 + h_2 + ML^{-1/4} + C\exp(-cL)$ for all $|x| \le W$, outside an event with probability $< C\exp(-cL^{3/2})$. These together with (9.9) imply that $\mathcal{L}^\beta[W] < h_1 + h_2 + \log(\lambda^{-1}) + CML^{-1/4} + CL^{-3/16}\log L$. Using that $M > L^{1/16}\log L$ the estimate (9.6) follows.

Lower bound. As the comparison in Lemma 2.16 is for some interval with length of order $L^{1/2}$, we need to do a truncation for $\pi^*(s_1)$. Namely, we have

$$\mathbb{P}(|\pi^*(s_1)| < W \mid \vec{\mathcal{L}}^\beta \approx \vec{h}) < C\exp(-cL^{3/2}). \tag{9.10}$$

Indeed, from Theorem 2.11 and Proposition 2.14, and Proposition 3.4, we can deduce that

$$\frac{\mathbb{P}\left(\sup_{|x|>W} \mathcal{L}^\beta(x_-, 0; x, s_1) \ge \mathcal{L}^\beta(x_-, 0; x_*, s_1), \mathcal{L}_1^\beta \approx h_1\right)}{\mathbb{P}(\mathcal{L}_1^\beta \approx h_1)} < C\exp(-cL^{3/2}),$$

$$\frac{\mathbb{P}\left(\sup_{|x|>W} \mathcal{L}^\beta(x, s_1; x_+, s_1 + s_2) \ge \mathcal{L}^\beta(x_*, s_1; x_+, s_1 + s_2), \mathcal{L}_2^\beta \approx h_2\right)}{\mathbb{P}(\mathcal{L}_2^\beta \approx h_2)} < C\exp(-cL^{3/2}).$$

These together imply (9.10).

Now by definition

$$\begin{aligned}
\mathcal{L}^\beta[x_*, L^{-1/2}(\log L)^2] &= \log \int_{-L^{-1/2}(\log L)^2}^{L^{-1/2}(\log L)^2} \exp(S(x_* + x))\,\mathrm{d}x \\
&= S(x_*) + \log \int_{-L^{-1/2}(\log L)^2}^{L^{-1/2}(\log L)^2} \exp\left(-2\lambda|x| + B(x)\right)\,\mathrm{d}x.
\end{aligned} \tag{9.11}$$

Define events A_M and E as

$$\begin{aligned}
A_M = &\left\{\min_{|x|\in[L^{-1/2}/4, L^{-1/2}/2]} B(x) \ge ML^{-1/4}\right\} \cap \left\{\min_{|x|\le L^{-1/2}/4} B(x) \ge -L^{-1/4}\right\} \\
&\cap \left\{B(x) \ge -|x|^{1/2}\log(|x|L^{1/2}) \text{ for all } |x| \in [L^{-1/2}/2, L^{-1/2}\log L]\right\}, \\
E = &\left\{\max_{|x|\ge L^{-1/2}, |x+x_*|\le W} B(x) - 2\lambda|x| < \max_{|x|\le L^{-1/2}} B(x) - 2\lambda|x| - C\exp(-cL)\right\}.
\end{aligned}$$

If B were replaced by a two-sided Brownian motion with bounded drift, the probability of $A_M \cap E$ would be $> c\exp(-CM^2)$, by standard Brownian motion computations. Then by Lemma 2.16,

$$\begin{aligned}
\mathbb{P}(A_M \cap E \mid \vec{\mathcal{L}}^\beta \approx \vec{h}) &> c\exp(-CM^2)(1 - C\exp(-cL)) - C\exp(-cL^{3/2}) \\
&> c\exp(-CM^2) - C\exp(-cL^{3/2}).
\end{aligned}$$

Assuming $A_M \cap E$, we can lower bound (9.11) by $h_1 + h_2 + \log(\lambda^{-1}) + cML^{-1/4} - CL^{-1/4}$; and $|\pi^*(s_1)| \le L^{-1/2}$ outside an event with probability $< C\exp(-cL^{3/2})$, by Proposition 3.4. Noting that M is taken large enough, together with (9.10), the lower bound (9.7) follows.

These complete the proof of the lemma in the $\beta = 1$ case. We now turn to the $\beta = \infty$ case.

Zero temperature: We now need to bound the maximum of S instead of its integral. For the upper bound, since $h_1, h_2 = \Theta(L)$ and $|x_*|, |x_-|, |x_+| \le 2L^{5/16}\log L$, by Corollary 2.18, conditional on $\vec{\mathcal{L}}^\beta \approx \vec{h}$, with probability at least $1 - C\exp(-cM^2)$,

$$S(x) < h_1 + h_2 - 2\lambda|x - x_*| + C|x - x_*|L^{5/16}\log L + ML^{-1/2},$$

for any $|x - x_*| \le L^{-1}$, and

$$S(x) < h_1 + h_2 - 2\lambda|x - x_*| + C|x - x_*|L^{5/16}\log L + M|x - x_*|^{1/2}(|\log(|x - x_*|LM^{-2})| + 1),$$

for any x with $|x - x_*| \ge L^{-1}$, $|x| \le W$. Then $S(x)$ in $[-W, W]$ is at most $h_1 + h_2 + CM^2L^{-1/2}$, and we get (9.6), by relabeling M.

Under the additional assumption that $|\pi^*(s_1) - x_*| \le L^{-1}$ (note that $w_{\beta=\infty} = L^{-1}$), by Proposition 3.4, we have that $S(x)$ in $[-W, W]$ is at most $h_1 + h_2 + ML^{-1/2}$ outside another event of probability $< C\exp(-cL^{3/2})$.

For the lower bound, we note that (9.10) still holds with the same proof. Then by Proposition 3.4, we just need to consider the probability of

$$\max_{|x| \le L^{-1}} B(x) \ge ML^{-1/2}, \qquad \left|\operatorname*{argmax}_{|x + x_*| \le W} B(x) - 2\lambda|x|\right| \le L^{-1}.$$

Again by Lemma 2.16, conditional on $\vec{\mathcal{L}}^\beta \approx \vec{h}$, the above happens with probability $> c\exp(-CM^2) - C\exp(-cL^{3/2})$. This completes the proof. □

Below we return to the notations defined before this subsection.

9.2. **Proof of Proposition 9.1.** Note that (with $y_0 = y_k = 0$)

$$\mathcal{L}^\beta = \max_{y_1, \cdots, y_{k-1}}{}^{(\beta)} \sum_{i=1}^{k} \mathcal{L}\left(y_{i-1}, s_{i-1}; y_i, s_i\right).$$

We want to simplify the RHS using coalescence (Proposition 3.4). Since we only know that coalescence holds up to a distance of order $L^{1/2}$, we have to first argue that we can restrict the $\max^{(\beta)}$ to be over an interval centered at 0 of size much smaller than $L^{1/2}$, with high probability conditionally on $\vec{\mathcal{L}}^\beta \approx \vec{h}$.

For notational convenience let, for $R > 0$ and $i = 1, \ldots, k - 1$, and $x \in \mathbb{R}$,

$$\mathcal{L}_i^\beta[x, R] = \max_{|y - x| \le R}{}^{(\beta)} \mathcal{L}^\beta(x_{i-1}, s_{i-1}; y, s_i) + \mathcal{L}^\beta(y, s_i; x_{i+1}, s_{i+1}).$$

Restricting the interval and coalescence: Let $\delta = 10^{-6}\min_{i=1,\ldots,k}(s_i - s_{i-1})$. By Lemma 5.4, conditionally on $\vec{\mathcal{L}}^\beta \approx \vec{h}$, it holds with probability at least $1 - C\exp(-cL^{3/2})$ that

$$\mathcal{L}^\beta \le \mathcal{L}^\beta[\vec{0}, \delta L^{1/2}] + C\exp(-cL),$$

where $\vec{0} \in \mathbb{R}^{k-1}$ is the vector with every entry being 0. By the convolution formula, this with Proposition 3.4 implies that $\mathbb{P}(\mathrm{Coal} \mid \vec{\mathcal{L}}^\beta \approx \vec{h}) > 1 - C\exp(-cL^{3/2})$, where the coalescence event

Coal is defined by

$$\mathrm{Coal} = \left\{ \mathcal{L}^\beta \in \sum_{i=1}^{k-1} \mathcal{L}_i^\beta[0, \delta L^{1/2}] - \sum_{i=2}^{k-1} \mathcal{L}_i^\beta + [-C\exp(-cL), C\exp(-cL)] \right\}.$$

Upper bound: Under $\vec{\mathcal{L}}^\beta \approx \vec{h}$, the sum $\sum_{i=2}^{k-1} \mathcal{L}_i^\beta \in \sum_{i=2}^{k-1} h_i + [0, (k-2)e^{-L}]$. Therefore, assuming $\vec{\mathcal{L}}^\beta \approx \vec{h}$ and Coal,

$$\begin{aligned} &\left\{ \mathcal{L}^\beta > H - (k-1)\beta^{-1}\log(2H^{1/2}) + Mr_\beta \right\} \\ &\qquad \subseteq \left\{ \sum_{i=1}^{k-1} \mathcal{L}_i^\beta[\delta L^{1/2}] \geq \sum_{i=1}^{k-1} (h_i + h_{i+1}) - (k-1)\beta^{-1}\log(2H^{1/2}) + Mr_\beta - C\exp(-cL) \right\}. \end{aligned}$$

Let P_i be the probability of

$$\mathcal{L}_i^\beta[\delta L^{1/2}] \geq h_i + h_{i+1} - \beta^{-1}\log(2H^{1/2}) + k^{-1}M,$$

conditional on $\vec{\mathcal{L}}^\beta \approx \vec{h}$. Let P_i' be the probability of

$$\mathcal{L}_i^\beta[\delta L^{1/2}] \geq h_i + h_{i+1} - \beta^{-1}\log(2H^{1/2}) + k^{-1}Mr_\beta,$$

and that $|\pi^*(s_i) - x_i| \leq w_\beta$, conditional on $\vec{\mathcal{L}}^\beta \approx \vec{h}$. By a union bound, it suffices to upper bound $\sum_{i=1}^{k-1} P_i$ and $\sum_{i=1}^{k-1} P_i'$.

For (9.4), we can then invoke (9.8) in Lemma 9.3 (using shear and translation invariance properties) to obtain an upper bound on each P_i. Similarly, for (9.2) we invoke (9.6) in Lemma 9.3 to obtain an upper bound on P_i'. We note that, since $|h_i - (s_i - s_{i-1})H| < L^{8/9}$), it follows that $\log((s_i - s_{i-1})^{-1/2}h_i^{1/2} + (s_{i+1} - s_i)^{-1/2}h_{i+1}^{1/2}) = \log(2H^{1/2}) + O(L^{-1/9})$, while $\beta^{-1}L^{-1/9} < r_\beta$. This completes the proof.

Lower bound: By the convolution formula and Proposition 3.4, conditional on $\vec{\mathcal{L}}^\beta \approx \vec{h}$, with probability at least $1 - C\exp(-cL^{3/2})$,

$$\begin{aligned} \mathcal{L}^\beta[\vec{x}, L^{-1/2}(\log L)^2] &\geq \sum_{i=1}^{k-1} \mathcal{L}_i^\beta[x_i, L^{-1/2}(\log L)^2] - \sum_{i=2}^{k-1} \mathcal{L}_i^\beta - C\exp(-cL) \\ &\geq \sum_{i=1}^{k-1} \mathcal{L}_i^\beta[x_i, L^{-1/2}(\log L)^2] - \sum_{i=2}^{k-1} h_i - C\exp(-cL). \end{aligned} \tag{9.12}$$

Take $2 \leq i \leq k-1$. Consider the processes $\mathcal{L}^\beta(x_{i-1}, s_{i-1}; \cdot, s_i)$ and $\mathcal{L}^\beta(\cdot, s_{i-1}; x_i, s_i)$ conditional on $\mathcal{L}_i^\beta \approx h_i$, as a measure on $\mathcal{C}([0,1], \mathbb{R})^2$. It can be replaced by the product measure of its marginals, on an event with probability $> 1 - C\exp(-cL^{3/2})$, with Radon-Nikodym derivative bounded between $1 - C\exp(-cL)$ and $1 + C\exp(-cL)$. By Lemma 9.3 (plus shear and translation invariance properties), we can lower bound the probability of

$$\mathcal{L}_i^\beta[x_i, L^{-1/2}(\log L)^2] \geq h_i + h_{i+1} - \beta^{-1}\log(2H^{1/2}) + Mr_\beta, |\pi^*(s_i) - x_i| \leq w_\beta, \tag{9.13}$$

conditional on $\vec{\mathcal{L}}^\beta \approx \vec{h}$, by $c\exp(-CM^2L^{1/2}r_\beta) - C\exp(-cL^{3/2})$. (Here we also use that $\log((s_i - s_{i-1})^{-1/2}h_i^{1/2} + (s_{i+1} - s_i)^{-1/2}h_{i+1}^{1/2}) = \log(2H^{1/2}) + O(L^{-1/9})$ and $\beta^{-1}L^{-1/9} < r_\beta$.) By the above coupling to independence, the probability of (9.13) for each i, conditional on $\vec{\mathcal{L}}^\beta \approx \vec{h}$, is also $> c\exp(-CM^2L^{1/2}r_\beta) - C\exp(-cL^{3/2})$. Plugging these into (9.12) finishes the proof of (9.3). $\square$

10. Global and segment maximizers

In Proposition 7.1 we have shown that (conditional on the upper tail) the polymer Γ_0 at any $s \in (0,1)$ is close to the maximize $\pi(s)$, whose definition we recall is

$$\pi(s) = \operatorname*{argmax}_{x\in\mathbb{R}} \mathcal{L}^\beta(0,0;x,s) + \mathcal{L}^\beta(x,s;0,1).$$

In zero temperature, we also use π to denote the geodesic π_0, which is defined through the same expression.

As already alluded, in our computations to obtain finite dimensional Gaussianity, it is more convenient to use another optimizer

$$\pi^*(s) = \operatorname*{argmax}_{x\in\mathbb{R}} \mathcal{L}^\beta(0,s_-;x,s) + \mathcal{L}^\beta(x,s;0,s_+).$$

Here $s_- \in [0,s)$ and $s_+ \in (s,1]$. In the rest of this section, all the constants can depend on s_-, s, s_+.

The main goal of this section is the following closeness between $\pi(s)$ and $\pi^*(s)$. As in previous sections, we write $\mathcal{L}^\beta = \mathcal{L}^\beta(0,0;0,1)$.

Proposition 10.1. *For any $L \geq 2$, we have*

$$\mathbb{P}\left(|\pi(s) - \pi^*(s)| > L^{-1/2}(\log L)^2 \;\middle|\; \mathcal{L}^\beta > L\right) < C\exp(-c(\log L)^2).$$

The case where $s_- = 0$ and $s_+ = 1$ is obvious, since then $\pi(s) = \pi^*(s)$. Below we write the proof for the case where $s_- > 0$ and $s_+ < 1$. The proof for the other two cases should be similar and we omit them.

For the rest of this section, we shall take the shorthand $\mathcal{L}_i^\beta = \mathcal{L}^\beta(x_{i-1},s_{i-1};x_i,s_i)$ for some $\vec{x} = (x_1,x_2,x_3) \in \mathbb{R}^3$, and $x_0 = x_4 = 0$, $s_0 = 0$, $s_1 = s_-$, $s_2 = s$, $s_3 = s_+$, $s_4 = 1$. We also write $\vec{\mathcal{L}}^\beta$ for the vector $\{\mathcal{L}_i^\beta\}_{i=1}^4$. For any vector $\vec{h} \in \mathbb{R}^k$, $\vec{\mathcal{L}}^\beta \approx \vec{h}$ denotes the event where $\mathcal{L}_i^\beta \in h_i + [0, e^{-L}]$ for each i.

Our general idea is to replace the conditioning $\mathcal{L}^\beta > L$ by $\vec{\mathcal{L}}^\beta \approx \vec{h}$, for some reasonable $\vec{x} \in \mathbb{R}^3$ and $\vec{h}$. Then we bound $|\pi(s) - x_2|$ conditional on $\vec{\mathcal{L}}^\beta \approx \vec{h}$, using coalescence and the tent picture.

We first restrict π^* to an interval of length $< L^{1/2}$.

Lemma 10.2. *For each $i = 1,2,3$ and L large,*

$$\mathbb{P}\left(|\pi^*(s_i)| > L^{5/16}\log L \;\middle|\; \mathcal{L}^\beta > L\right) < C\exp(-c(\log L)^2).$$

Proof. We write the proof for $i = 2$; the other two cases should follow similarly.

Let $\mathcal{E}$ denote the event where $|\pi^*(s)| > L^{5/16}\log L$. By Lemma 6.3 we have $\mathbb{P}(\mathcal{E}_1 \mid \mathcal{L}^\beta > L) > 1 - C\exp(-c(\log L)^2)$, where $\mathcal{E}_1$ is the event

$$\inf_{|x_1|,|x_2|,|x_3| \leq L^{-1/4}\log L} \sum_{i=1}^4 \mathcal{L}_i^\beta \geq L - L^{5/8}\log L.$$

We have that $\mathcal{E} \cap \mathcal{E}_1 \subset \mathcal{E}_2$, where $\mathcal{E}_2$ is the event

$$\sup_{|x_2| > L^{5/16}\log L, x_1 = x_3 = 0} \sum_{i=1}^4 \mathcal{L}_i^\beta \geq L - L^{5/8}\log L.$$

By the shear invariance property and Corollary 6.2, and the continuity estimate Proposition 2.14, we have $\mathbb{P}(\mathcal{E}_2) < C\exp\left(-\frac{4}{3}L^{3/2} - cL^{9/8}(\log L)^2\right)$. By combining this with the tail estimate Theorem 2.11 we get that $\mathbb{P}(\mathcal{E}_2 \mid \mathcal{L}^\beta > L) < C\exp(-cL^{9/8}(\log L)^2)$. As we can upper bound $\mathbb{P}(\mathcal{E} \mid \mathcal{L}^\beta > L)$ by $\mathbb{P}(\mathcal{E}_1^c \mid \mathcal{L}^\beta > L) + \mathbb{P}(\mathcal{E}_2 \mid \mathcal{L}^\beta > L)$, the conclusion follows. $\square$

In replacing the conditioning $\mathcal{L}^\beta > L$ into $\vec{\mathcal{L}}^\beta \approx \vec{h}$, we will need to bound the ratio $\mathbb{P}(\vec{\mathcal{L}}^\beta \approx \vec{h})\mathbb{P}(\mathcal{L}^\beta > L)^{-1}$, for which we need that $\sum_{i=1}^4 h_i > L - \log L$. For this we will invoke Proposition 9.2, as well as a rough lower bound of $\mathcal{L}_i^\beta$ conditional on the upper tail $\mathcal{L}^\beta > L$.

Lemma 10.3. *For any $\vec{x} \in [-L^{5/16}\log L, L^{5/16}\log L]^3$, and $i \in \{1,2,3,4\}$, we have*

$$\mathbb{P}\left(\mathcal{L}_i^\beta < (s_i - s_{i-1})L - L^{27/32}\log L) \;\middle|\; \mathcal{L}^\beta > L\right) < C\exp(-c(\log L)^2).$$

Proof. We prove this for $i = 2$, and the other cases would follow verbatim.

We use $\mathcal{L}_{i,0}^\beta$ to denote $\mathcal{L}_i^\beta$ with each of x_1, x_2, x_3 replaced by 0. Let $\mathcal{E}$ denote the event $cL_i^\beta < (s_i - s_{i-1})L - L^{27/32}\log L$, and $\mathcal{E}_0$ denote the event

$$\mathcal{L}_{i,0}^\beta > (s_i - s_{i-1})L - L^{7/8},$$

for each $i = 1,2,3,4$. By Lemma 6.3, we have

$$\mathbb{P}\left(\sum_{i=1}^4 \mathcal{L}_{i,0}^\beta < L - L^{5/8}\log L \;\middle|\; \mathcal{L}^\beta > L\right) < C\exp(-c(\log L)^2). \tag{10.1}$$

By Lemma 6.4, we have

$$\mathbb{P}\left(\mathcal{E}_0^c, \sum_{i=1}^4 \mathcal{L}_{i,0}^\beta \geq L - L^{5/8}\log L\right) < C\exp(-cL^{5/4})\mathbb{P}\left(\sum_{i=1}^4 \mathcal{L}_{i,0}^\beta \geq L - L^{5/8}\log L\right).$$

By Theorem 2.11 and Corollary 6.2, we can bound the ratio of the last factor over $\mathbb{P}(\mathcal{L}^\beta > L)$ by $C\exp(CL^{9/8}\log L)$. Therefore we have

$$\mathbb{P}\left(\mathcal{E}_0^c, \sum_{i=1}^4 \mathcal{L}_{i,0}^\beta \geq L - L^{5/8}\log L \;\middle|\; \mathcal{L}^\beta > L\right) < C\exp(-cL^{5/4}).$$

Combining this with (10.1) implies that $\mathbb{P}(\mathcal{E}_0^c \mid \mathcal{L}^\beta > L) > 1 - C\exp(-c(\log L)^2)$.

We next upper bound $\mathbb{P}(\mathcal{E} \mid \mathcal{E}_0)$. In the case where $x_1x_2 < 0$, by the shift-invariance (Lemma 3.3)

$$\mathbb{P}(\mathcal{E} \mid \mathcal{E}_0) = \mathbb{P}\left(\mathcal{L}^\beta(0, s_-; x_2 - x_1, s) < (s_- - s_{i-1})L - L^{27/32}\log L \;\middle|\; \mathcal{E}_0\right).$$

By Lemma 2.17 and using that $|x_1|, |x_2| \leq L^{5/16}\log L$, this is bounded by $C\exp(-cL^{11/8}\log L)$.

In the case where $x_1x_2 \geq 0$, by Lemma 2.17 and using that $|x_1|, |x_2| \leq L^{5/16}\log L$, we have

$$\mathbb{P}\left(\mathcal{L}^\beta(0, s_-; x_2, s) < (s_- - s_{i-1})L - L^{27/32}(\log L)/2 \;\middle|\; \mathcal{E}_0\right) < C\exp(-cL^{11/8}\log L),$$

$$\mathbb{P}\left(\mathcal{L}^\beta(x_1, s_-; 0, s) < (s_- - s_{i-1})L - L^{27/32}(\log L)/2 \;\middle|\; \mathcal{E}_0\right) < C\exp(-cL^{11/8}\log L).$$

Then by the quadrangle inequalities (2.1) and (2.4), we still have $\mathbb{P}(\mathcal{E} \mid \mathcal{E}_0) < C\exp(-cL^{11/8}\log L)$. Then since (by Theorem 2.11) $\mathbb{P}(\mathcal{E}_0)\mathbb{P}(\mathcal{L}^\beta > L)^{-1} < C\exp(CL^{11/8})$, we have

$$\begin{aligned}
\mathbb{P}(\mathcal{E} \mid \mathcal{L}^\beta > L) &< C\exp(-c(\log L)^2) + \mathbb{P}(\mathcal{E} \cap \mathcal{E}_0 \mid \mathcal{L}^\beta > L) \\
&\leq C\exp(-c(\log L)^2) + \mathbb{P}(\mathcal{E} \cap \mathcal{E}_0)\mathbb{P}(\mathcal{L}^\beta > L)^{-1} \\
&\leq C\exp(-c(\log L)^2) + \mathbb{P}(\mathcal{E} \mid \mathcal{E}_0)\mathbb{P}(\mathcal{E}_0)\mathbb{P}(\mathcal{L}^\beta > L)^{-1} \\
&< C\exp(-c(\log L)^2).
\end{aligned}$$

Thus the conclusion follows. $\square$

We next prove that, given the tail $\vec{\mathcal{L}}^\beta \approx \vec{h}$, $\pi(s)$ should be close to x_2. Define

$$\mathrm{Val}^4 = \Bigg\{ \vec{h} \in \mathbb{R}^4 : h_i \geq (s_i - s_{i-1})L - L^{27/32}\log L,\ \forall i = 1,2,3,4; \\ L - \log L \leq \sum_{i=1}^4 h_i < L + (\log L)^2 \Bigg\}.$$

Lemma 10.4. *For any $\vec{x} \in [-L^{5/16}\log L, L^{5/16}\log L]^3$, and $\vec{h} \in \mathrm{Val}^4$,*

$$\mathbb{P}\left(|\pi(s) - x_2| > L^{-1/2}(\log L)^2/2 \,\middle|\, \vec{\mathcal{L}}^\beta \approx \vec{h} \right) < C\exp(-cL^{1/2}(\log L)^2).$$

Proof. By Theorem 2.11 we have

$$\mathbb{P}(\vec{\mathcal{L}}^\beta \approx \vec{h}) > c\exp\left(-\frac{4}{3}L^{3/2} - CL^{9/8}(\log L)^2 \right). \tag{10.2}$$

Therefore, by Lemma 6.1 we have

$$\mathbb{P}\left(\mathcal{L}^\beta(0,0;x_2,s) + \mathcal{L}^\beta(x_2,s;0,1) < L - C\log L \,\middle|\, \vec{\mathcal{L}}^\beta \approx \vec{h} \right) < C\exp(-cL^2). \tag{10.3}$$

We next restrict the intervals. We define $\mathcal{L}^\beta_{out}$ as follows. For $\beta = \infty$, we let

$$\mathcal{L}^\beta_{out} = \sup_{\vec{x} \in \mathbb{R}^3 \setminus [-L^{3/8}, L^{3/8}]^3} \sum_{i=1}^4 \mathcal{L}^\beta_i.$$

For $\beta = 1$, we let

$$\mathcal{L}^\beta_{out} = \left(\sup_{|x_2| > L^{3/8}} \mathcal{L}^\beta(x_2,s;0,1) + \mathcal{L}^\beta(0,0;x_2,s) \right) \vee \\ \left(\sup_{|x_2| \leq L^{3/8}} (\mathcal{L}^\beta_{out,-}(x_2) + \mathcal{L}^\beta(x_2,s;0,1)) \vee (\mathcal{L}^\beta(0,0;x_2,s) + \mathcal{L}^\beta_{out,+}(x_2)) \right),$$

where

$$\mathcal{L}^\beta_{out,-}(x_2) = \log \int_{|x_1| \geq L^{3/8}} \exp\left(\mathcal{L}^\beta_1 + \mathcal{L}^\beta_2 \right) \mathrm{d}x_1,$$
$$\mathcal{L}^\beta_{out,+}(x_2) = \log \int_{|x_3| \geq L^{3/8}} \exp\left(\mathcal{L}^\beta_3 + \mathcal{L}^\beta_4 \right) \mathrm{d}x_3.$$

Using Corollary 6.2 and shear invariance, and the continuity estimate Proposition 2.14, we can upper bound $\mathbb{P}(\mathcal{L}^\beta_{out} > L - (\log L)^{3/2})$ by $C\exp\left(-\frac{4}{3}L^{3/2} - cL^{5/4}(\log L)^2\right)$. Then by (10.2) we have

$$\mathbb{P}\left(\mathcal{L}^\beta_{out} > L - (\log L)^{3/2} \mid \vec{\mathcal{L}}^\beta \approx \vec{h} \right) < C\exp(-cL^{5/4}(\log L)^2). \tag{10.4}$$

We next consider the restricted part.

$$\mathcal{L}^\beta_{res}(x_2) = \max_{x_1, x_3 \in [-L^{3/8}, L^{3/8}]} \sum_{i=1}^4 \mathcal{L}^\beta_i,$$

for $\beta = \infty$, and

$$\mathcal{L}^\beta_{res}(x_2) = \log \int_{|x_1| \leq L^{3/8}} \exp\left(\mathcal{L}^\beta_1 + \mathcal{L}^\beta_2 \right) \mathrm{d}x_1 + \log \int_{|x_3| \leq L^{3/8}} \exp\left(\mathcal{L}^\beta_3 + \mathcal{L}^\beta_4 \right) \mathrm{d}x_3,$$

for $\beta = 1$. Using Proposition 3.4 and Corollary 2.18, one can upper bound $\mathcal{L}^{\beta}_{res}$ assuming a coalescence event, as in the proof of Proposition 9.1. Omitting the details, we can deduce that

$$\mathbb{P}\left(\sup_{x\in I}\mathcal{L}^{\beta}_{res}(x) > L - (\log L)^{3/2} \;\middle|\; \vec{\mathcal{L}}^{\beta} \approx \vec{h}\right) < C\exp(-cL^{1/2}(\log L)^2),$$

where $I = [-L^{3/8}, L^{3/8}] \setminus [x_2 - L^{-1/2}(\log L)^2/2, x_2 + L^{-1/2}(\log L)^2/2]$. This with (10.4) implies that

$$\mathbb{P}\left(\sup_{|x-x_2|>L^{-1/2}(\log L)^2/2}\mathcal{L}^{\beta}(0,0;x_2,s) + \mathcal{L}^{\beta}(x_2,s;0,1) > L - (\log L)^{3/2} \;\middle|\; \vec{\mathcal{L}}^{\beta} \approx \vec{h}\right)$$
$$< C\exp(-cL^{1/2}(\log L)^2).$$

This and (10.3) imply the conclusion. □

We now finish proving Proposition 10.1, using all the above ingredients as well as Proposition 9.2, which lower bounds the sum $\sum_{i=1}^{4}\mathcal{L}^{\beta}_{i}$ by $L - \log L$, given that $\mathcal{L}^{\beta} > L$.

Proof of Proposition 10.1. We can assume that L is large enough, since otherwise the conclusion follows obviously.

Denote $w_{\beta=1} = L^{-1/2}$ and $w_{\beta=\infty} = L^{-1}$ as in the previous seciton. By (10.2) and Lemma 10.3, we can bound the LHS in the display by

$$\sum_{\vec{x}\in([-L^{5/16}\log L, L^{5/16}\log L]\cap w_\beta\mathbb{Z})^3}\mathbb{P}\Big(\max_{i=1,2,3}|\pi^*(s_i) - x_i| \le w_\beta, |\pi(s) - x_2| > L^{-1/2}(\log L)^2/2$$
$$\mathcal{L}^{\beta}_i \ge (s_i - s_{i-1})L - L^{27/32}\log L, \forall i = 1,2,3,4 \;\Big|\; \mathcal{L}^{\beta} > L\Big) + C\exp(-c(\log L)^2). \quad (10.5)$$

By Lemma 6.1 and Theorem 4.1, we have

$$\mathbb{P}\left(\sum_{i=1}^{4}\mathcal{L}^{\beta}_i \ge L + (\log L)^2 \;\middle|\; \mathcal{L}^{\beta} > L\right) < C\exp(-cL^{1/2}(\log L)^2).$$

This together with Proposition 9.2 implies that, for any $\vec{x} \in ([-L^{5/16}\log L, L^{5/16}\log L] \cap w_\beta\mathbb{Z})^3$,

$$\mathbb{P}\Big(\max_{i=1,2,3}|\pi^*(s_i) - x_i| \le w_\beta, \vec{\mathcal{L}}^{\beta} \notin \mathrm{Val}^4,$$
$$\mathcal{L}^{\beta}_i \ge (s_i - s_{i-1})L - L^{27/32}\log L, \forall i = 1,2,3,4 \;\Big|\; \mathcal{L}^{\beta} > L\Big) < C\exp(-c(\log L)^2).$$

Then each summand in (10.5) can be bounded by

$$\sum_{\vec{h}}\mathbb{P}\Big(\max_{i=1,2,3}|\pi^*(s_i) - x_i| \le w_\beta, |\pi(s) - x_2| > L^{-1/2}(\log L)^2/2, \vec{\mathcal{L}}^{\beta} \approx \vec{h}, \;\Big|\; \mathcal{L}^{\beta} > L\Big)$$
$$+ C\exp(-c(\log L)^2),$$

where the sum is over $< \exp(CL)$ many $\vec{h} \in \mathrm{Val}^4$, such that any element in Val^4 is $\approx \vec{h}$ for one of them. We note that the summand for each $\vec{h}$ is bounded by

$$\mathbb{P}\left(|\pi(s) - x_2| > L^{-1/2}(\log L)^2/2 \;\middle|\; \vec{\mathcal{L}}^{\beta} \approx \vec{h}\right)\mathbb{P}(\vec{\mathcal{L}}^{\beta} \approx \vec{h})\mathbb{P}(\mathcal{L}^{\beta} > L)^{-1}.$$

By Lemma 10.4, the first factor is bounded by $C\exp(-cL^{1/2}(\log L)^2)$. Then the sum over $\vec{h}$ is bounded by

$$C\exp(-cL^{1/2}(\log L)^2)\mathbb{P}(\vec{\mathcal{L}}^{\beta} \in \mathrm{Val}^4)\mathbb{P}(\mathcal{L}^{\beta} > L)^{-1}.$$

By Lemma 6.1 and Theorem 4.1, we can bound this by $C\exp(CL^{1/2}\log L)$, since $\vec{\mathcal{L}}^\beta \in \mathrm{Val}^4$ implies that $\sum_{i=1}^4 \mathcal{L}_i^\beta \geq L - \log L$. Now that each summand in (10.5) is bounded by $C\exp(-c(\log L)^2)$, plugging this estimate back leads to the conclusion. □

11. Finite dimensional Brownian bridge limit

In this section, we prove finite dimensional convergence of π_0 and Γ_0 to Brownian bridge (under upper tails). As in previous sections we take the setup of $(s_1, \cdots, s_{k-1}) \in \mathring{\Lambda}_{k-1}([0,1])$ for $k \in \mathbb{N}$, and denote $s_0 = 0$ and $s_k = 1$; and all the constants within this section can depend on k and $(s_1, \cdots, s_{k-1})$. Also recall that we define

$$\pi^*(s_i) = \operatorname*{argmax}_x \mathcal{L}^\beta(0, s_{i-1}; x, s_i) + \mathcal{L}^\beta(x, s_i; 0, s_{i+1}),$$

for each $i = 1, \ldots, k-1$. We adopt the shorthand $\mathcal{L}^\beta = \mathcal{L}^\beta(0,0;0,1)$.

We shall prove that as $L \to \infty$, $L^{1/4}\{\pi^*(s_i)\}_{i=1}^{k-1}$ conditional on $\mathcal{L}^\beta > L$ converges to a joint Gaussian, matching that for a Brownian bridge. We note that even in zero temperature, $\pi^*(s_i)$ does not a priori coincide with $\pi_0(s_i) = \pi(s_i)$, which is instead given as the maximizer of processes from height 0 to s_i and s_i to 1 (instead of s_{i-1} to s_i and s_i to s_{i+1} here), i.e., $\pi(s_i) = \operatorname{argmax}_z \mathcal{L}^\beta(0,0;z,s_i) + \mathcal{L}^\beta(z,s_i;0,1)$. However, we have shown that, conditional on the upper tail, with high probability $|\pi^*(s_i) - \pi(s_i)|$ is of order smaller than $L^{-1/4}$ (Proposition 10.1); and in positive temperature the polymer measure concentrates at height s_i in a window of order smaller than $L^{-1/4}$ around $\pi(s_i)$ (Proposition 7.1). Therefore, the Gaussian limit of $\{\pi^*(s_i)\}_{i=1}^{k-1}$ as follows suffices for us to deduce our main results.

We denote $w_{\beta=1} = L^{-1/2}$ and $w_{\beta=\infty} = L^{-1}$ as before. Fix any compact set $\mathcal{K} \subseteq \mathbb{R}^{k-1}$, and all the constants below can depend on $\mathcal{K}$.

Theorem 11.1. *As $L \to \infty$, uniformly over $\vec{x}, \vec{y} \in \mathcal{K}$ (with $x_0 = x_k = y_0 = y_k = 0$ for the convenience of notations),*

$$\frac{\mathbb{P}\left(\max_{i=1,\ldots,k-1} |\pi^*(s_i) - x_i L^{-1/4}| \leq w_\beta \mid \mathcal{L}^\beta > L\right)}{\mathbb{P}\left(\max_{i=1,\ldots,k-1} |\pi^*(s_i) - y_i L^{-1/4}| \leq w_\beta \mid \mathcal{L}^\beta > L\right)} \to \exp\left(-2\left[\sum_{i=1}^k \frac{(x_i - x_{i-1})^2 - (y_i - y_{i-1})^2}{s_i - s_{i-1}}\right]\right).$$

We note that the RHS is the ratio of the joint density of $(\frac{1}{2}B(s_1), \ldots, \frac{1}{2}B(s_{k-1}))$ evaluated at $\vec{x}$ and $\vec{y}$, where B is a standard Brownian bridge on $[0,1]$.

Note that this is a comparison of probabilities, while we wish to show weak convergence. The following lemma allows the transition. Its proof is fairly straightforward real analysis and we relegate it to Appendix A.

Lemma 11.2. *Let $d \geq 1$ and suppose $\{\vec{X}_\varepsilon\}_{\varepsilon>0}$ is a family of $\mathbb{R}^d$-valued random vectors such that as $\varepsilon \to 0$, $\vec{X}_\varepsilon \to \vec{X}$ in distribution for some random vector $\vec{X}$. Suppose also that there is a continuous strictly positive integrable function $f : \mathbb{R}^d \to (0,\infty)$ such that, for every compact set $\mathcal{K} \subseteq \mathbb{R}^d$, uniformly over $\vec{x}, \vec{y} \in \mathcal{K}$ as $\varepsilon \to 0$,*

$$\frac{\mathbb{P}\left(\vec{X}_\varepsilon \in \vec{x} + [-\varepsilon,\varepsilon)^d\right)}{\mathbb{P}\left(\vec{X}_\varepsilon \in \vec{y} + [-\varepsilon,\varepsilon)^d\right)} \cdot \frac{f(\vec{y})}{f(\vec{x})} \to 1. \tag{11.1}$$

Then $\vec{X}$ is absolutely continuous with respect to the Lebesgue measure of $\mathbb{R}^d$, and has density given by $f(\vec{x}) / \int_{\mathbb{R}^d} f(\vec{z})\,\mathrm{d}\vec{z}$.

With these results in hand, we may prove our main theorems.

Proofs of Theorems 1.1 and 1.2. The tightness of $\{L^{1/4}\pi_0\}_{L\geq 2}$ and $\{L^{1/4}\Gamma_0\}_{L\geq 2}$ conditional on $\mathcal{L}^\beta > L$ in the space $\mathcal{C}([0,1],\mathbb{R})$ is given by Proposition 5.1, and it remains to establish finite dimensional convergence.

For B being a standard Brownian bridge on $[0,1]$, Theorem 11.1 and Lemma 11.2 imply that $\{2L^{1/4}\pi^*(s_i)\}_{i=1}^{k-1} \to \{B(s_i)\}_{i=1}^{k-1}$ in distribution. Proposition 10.1 along with the Borel-Cantelli lemma guarantees that $L^{1/4}\pi(s_i) - L^{1/4}\pi^*(s_i) \to 0$ almost surely for every i. This implies that, for $\beta = \infty$, $\{2L^{1/4}\pi_0(s_i)\}_{i=1}^{k-1} \to \{B(s_i)\}_{i=1}^{k-1}$ in distribution, completing the proof of Theorem 1.1.

For $\beta = 1$, taking expectations in Proposition 7.1 further yields that $\mathbb{P}(\max_{i=1,\dots,k-1} |\Gamma_0(s_i) - \pi^*(s_i)| \leq ML^{-1/2}\log L \mid \mathcal{L}^\beta > L) > 1 - L^{-cM} - C\exp(-c(\log L)^2)$, for M being a large enough constant. Thus $L^{1/4}\Gamma_0(s_i) - L^{1/4}\pi^*(s_i) \to 0$ almost surely for every i. We then obtain that $\{2L^{1/4}\Gamma_0(s_i)\}_{i=1}^{k-1} \to \{B(s_i)\}_{i=1}^{k-1}$ in distribution, completing the proof of Theorem 1.2. $\square$

The rest of this section and the next section are devoted to proving Theorem 11.1.

11.1. **Finite dimensional convergence.** We introduce some useful shorthand to make the notation simpler: $\vec{\mathcal{L}}^{\beta,\vec{x}}$ is a vector whose i^{th} component is given by

$$\mathcal{L}_i^{\beta,\vec{x}} = \mathcal{L}^\beta(x_{i-1}L^{-1/4}, s_{i-1}; x_iL^{-1/4}, s_i)$$

Below we also use $o(1)$ to denote any quantity that $\to 0$ as $L \to \infty$.

Proof of Theorem 11.1. We start by noting that, by Lemma 6.3,

$$\mathbb{P}\left(\sum_{i=1}^k \mathcal{L}_i^{\beta,\vec{x}} < L - L^{5/8}\log L \;\middle|\; \mathcal{L}^\beta > L\right) < C\exp(-c(\log L)^2);$$

and by Lemma 6.1, plus Theorem 2.11 and Theorem 4.1, for some $C_0 > 0$

$$\mathbb{P}\left(\sum_{i=1}^k \mathcal{L}_i^{\beta,\vec{x}} > L + C_0\log L \;\middle|\; \mathcal{L}^\beta > L\right) < C\exp(-cL^{1/2}\log L).$$

So we can upper bound $\mathbb{P}(\max_{i=1,\dots,k-1} |\pi^*(s_i) - x_iL^{-1/4}| \leq w_\beta, \mathcal{L}^\beta > L)$ by

$$\mathbb{P}\left(\mathcal{E}_{\text{sum}}, \max_{i=1,\dots,k-1} |\pi^*(s_i) - x_iL^{-1/4}| < w_\beta, \mathcal{L}^\beta > L\right) + C\exp(-c(\log L)^2)\cdot\mathbb{P}(\mathcal{L}^\beta > L),$$

where $\mathcal{E}_{\text{sum}}$ is the event

$$L - L^{5/8}\log L \leq \sum_{i=1}^k \mathcal{L}_i^{\beta,\vec{x}} \leq L + C_0\log L.$$

Take M_* to be a large constant, and let $\mathcal{E}_{\text{prop}}(M_*)$ be defined by

$$\mathcal{E}_{\text{prop}}(M_*) = \bigcap_{i=1}^k \left\{\mathcal{L}_i^{\beta,\vec{x}} \geq (s_i - s_{i-1})L - M_*L^{7/8}\right\}.$$

Using Lemma 6.4 (proportionality statement), and Corollary 6.2 and Theorem 2.11 (tail bounds for the sum $\sum_{i=1}^k \mathcal{L}_i^{\beta,\vec{x}}$ and $\mathcal{L}^\beta$), we have

$$\mathbb{P}\left(\mathcal{E}_{\text{prop}}^c, \sum_{i=1}^k \mathcal{L}_i^{\beta,\vec{x}} \geq L - L^{5/8}\log L\right)$$

$$
\begin{aligned}
&\leq \exp\left(-cM_*^2 L^{5/4}\right) \cdot \mathbb{P}\left(\sum_{i=1}^{k} \mathcal{L}_i^{\beta,\vec{x}} \geq L - L^{5/8}\log L\right) \\
&\leq \exp\left(-cM_*^2 L^{5/4} - \frac{4}{3}(L - L^{5/8}\log L)^{3/2} + CL^{3/4}\right) \\
&\leq \exp\left(-cM_*^2 L^{5/4} + CL^{9/8}\log L\right) \cdot \mathbb{P}\left(\mathcal{L}^\beta > L\right).
\end{aligned}
$$

As M_* is large enough, this is upper bounded by $\exp(-cM_*^2 L^{5/4}) \cdot \mathbb{P}(\mathcal{L}^\beta > L)$.

Thus far, we have overall shown that

$$
\begin{aligned}
&\mathbb{P}\left(\max_{i=1,\ldots,k-1} |\pi^*(s_i) - x_i L^{-1/4}| < w_\beta, \mathcal{L}^\beta > L\right) \\
&\leq \mathbb{P}\left(\mathcal{E}_{\mathrm{prop}}(M_*), \mathcal{E}_{\mathrm{sum}}, \max_{i=1,\ldots,k-1} |\pi^*(s_i) - x_i L^{-1/4}| < w_\beta, \mathcal{L}^\beta > L\right) + C\exp(-c(\log L)^2) \cdot \mathbb{P}(\mathcal{L}^\beta > L).
\end{aligned}
$$

Set (as before) $r_{\beta=1} = 1$ and $r_{\beta=\infty} = L^{-1/2}$. Using Proposition 9.2, we conclude that

$$
\begin{aligned}
&\mathbb{P}\left(\max_{i=1,\ldots,k-1} |\pi^*(s_i) - x_i L^{-1/4}| < w_\beta, \mathcal{L}^\beta > L\right) \\
&\leq \left(1 - e^{-cM_*^2 L^{1/2} r_\beta}\right)^{-1} \mathbb{P}\begin{pmatrix} \mathcal{E}_{\mathrm{prop}}(M_*), \mathcal{E}_{\mathrm{sum}}, \max_{i=1,\ldots,k-1} |\pi^*(s_i) - x_i L^{-1/4}| < w_\beta, \mathcal{L}^\beta > \\ L \\ \sum_{i=1}^k \mathcal{L}_i^{\beta,\vec{x}} > L + (k-1)\beta^{-1}\log(2L^{1/2}) - M_* r_\beta \end{pmatrix} \\
&\qquad + C\exp(-c(\log L)^2) \cdot \mathbb{P}\left(\mathcal{L}^\beta > L\right).
\end{aligned} \tag{11.2}
$$

We next do a restriction. Recall the notation $\max^{(\beta)}_{\vec{x}\in I} f$ for $f : \mathbb{R}^{k-1} \to \mathbb{R}$ and a set $I \subseteq \mathbb{R}^{k-1}$ from (9.1). Define

$$
\mathcal{L}^\beta[1] = \max_{\|\vec{z}\|_\infty \leq 1}{}^{(\beta)} \sum_{i=1}^{k} \mathcal{L}^\beta(s_{i-1}, z_{i-1}; s_i, z_i), \tag{11.3}
$$

where $z_0 = z_k = 0$. Then, by the (quenched) one-point transversal fluctuation estimate (Proposition 8.4 for $\beta = 1$ and Lemma 5.2 for $\beta = \infty$), (11.2) is bounded by

$$
\begin{aligned}
&\left(1 - e^{-cM_*^2 L^{1/2} r_\beta}\right)^{-1} \mathbb{P}\begin{pmatrix} \mathcal{E}_{\mathrm{prop}}(M_*), \mathcal{E}_{\mathrm{sum}}, \max_{i=1,\ldots,k-1} |\pi^*(s_i) - x_i L^{-1/4}| < w_\beta, \\ \mathcal{L}^\beta[1] > (1 - \beta^{-1} e^{-cL^{1/4}})L, \\ \sum_{i=1}^k \mathcal{L}_i^{\beta,\vec{x}} > L + (k-1)\beta^{-1}\log(2L^{1/2}) - M_* r_\beta \end{pmatrix} \\
&\qquad + C\exp(-c(\log L)^2) \cdot \mathbb{P}\left(\mathcal{L}^\beta > L\right).
\end{aligned} \tag{11.4}
$$

To analyze the first term in the previous display, we define

$$
E_{\mathrm{val}} = \left\{ (h_1, \ldots, h_k) \in (e^{-L}\mathbb{Z})^k : \begin{matrix} L + (k-1)\beta^{-1}\log(2L^{1/2}) - (M_*+1) r_\beta \leq \sum_{i=1}^k h_i \leq L + C_0 \log L, \\ h_i \in (s_i - s_{i-1})L + [-2M_* L^{7/8}, kM_* L^{7/8}] \text{ for } i = 1, \ldots, k \end{matrix} \right\}. \tag{11.5}
$$

Here we take the fine mesh $(e^{-L}\mathbb{Z})^k$ instead of $\mathbb{R}^k$ because (as mentioned before) we will need to apply Proposition 3.4 and Proposition 3.5. We also take the notation $\vec{\mathcal{L}}^{\beta,\vec{x}} \approx \vec{h}$ for the event that each $\mathcal{L}_i^{\beta,\vec{x}} \in h_i + [0, e^{-L}]$.

Now we see that the probability in the first term of (11.4) is bounded by

$$
\sum_{\vec{h}\in E_{\mathrm{val}}} \mathbb{P}\Big(\max_{i=1,\ldots,k-1} |\pi^*(s_i) - x_i L^{-1/4}| < w_\beta, \mathcal{L}^\beta[1] > (1 - \beta^{-1} e^{-cL^{1/4}})L \;\Big|\; \vec{\mathcal{L}}^{\beta,\vec{x}} \approx \vec{h}\Big) \cdot \mathbb{P}\Big(\vec{\mathcal{L}}^{\beta,\vec{x}} \approx \vec{h}\Big). \tag{11.6}
$$

Our goal is to relate each of these terms to the corresponding one with $\vec{x}$ replaced by $\vec{y}$, which we state precisely in the next two lemmas to be proved later. The first factor is essentially unchanged:

Lemma 11.3. *For $\vec{h} = (h_1, \ldots, h_k) \in E_{\mathrm{val}}$,*

$$\begin{aligned}\mathbb{P}\Big(\max_{i=1,\ldots,k-1} |\pi^*(s_i) - x_i L^{-1/4}| < w_\beta, \mathcal{L}^\beta[1] > (1-\beta^{-1}e^{-cL^{1/4}})L \;\Big|\; \vec{\mathcal{L}}^{\beta,\vec{x}} \approx \vec{h}\Big)\\ \leq (1+o(1))\mathbb{P}\Big(\max_{i=1,\ldots,k-1} |\pi^*(s_i) - y_i L^{-1/4}| < w_\beta, \mathcal{L}^\beta[1] > L \;\Big|\; \mathcal{L}^{\beta,\vec{y}} \approx \vec{h}\Big).\end{aligned}$$

The proof of this lemma is somewhat involved and will be given in Section 12. For the second factor in (11.6), we have the following statement, which is the source of the Brownian bridge density in our result. Its proof is a straightforward consequence of Theorem 4.1, and we give it after completing the proof of Theorem 11.1.

Lemma 11.4. *For $(h_1, \ldots, h_k) \in E_{\mathrm{val}}$,*

$$\mathbb{P}\left(\vec{\mathcal{L}}^{\beta,\vec{x}} \approx \vec{h}\right) = (1+o(1))\exp\left(-2\sum_{i=1}^{k} \frac{(x_i - x_{i-1})^2 - (y_i - y_{i-1})^2}{s_i - s_{i-1}}\right)\cdot \mathbb{P}\left(\mathcal{L}^{\beta,\vec{y}} \approx \vec{h}\right).$$

Inputting the information from the previous two lemmas into (11.6) and (11.4) yields that

$$\begin{aligned}&\mathbb{P}\left(\max_{i=1,\ldots,k-1} |\pi^*(s_i) - x_i L^{-1/4}| \leq w_\beta, \mathcal{L}^\beta > L\right)\\ &\quad < C\exp(-c(\log L)^2)\cdot \mathbb{P}\left(\mathcal{L}^\beta > L\right)\\ &\qquad + (1+o(1))\left(1+\exp(-cM_*^2 L^{1/2} r_\beta)\right)\exp\left(-2\sum_{i=1}^{k} \frac{(x_i - x_{i-1})^2 - (y_i - y_{i-1})^2)}{s_i - s_{i-1}}\right)\\ &\qquad\quad \times \sum_{\vec{h}\in E_{\mathrm{val}}} \mathbb{P}\Big(\max_{i=1,\ldots,k-1} |\pi^*(s_i) - y_i L^{-1/4}| < w_\beta, \mathcal{L}^\beta[1] > L \;\Big|\; \mathcal{L}^{\beta,\vec{y}} \approx \vec{h}\Big)\cdot \mathbb{P}(\mathcal{L}^{\beta,\vec{y}} \approx \vec{h}).\end{aligned}$$

Therefore we have

$$\begin{aligned}&\mathbb{P}\left(\max_{i=1,\ldots,k-1} |\pi^*(s_i) - x_i L^{-1/4}| \leq w_\beta \mid \mathcal{L}^\beta > L\right)\\ &<(1+o(1))\left(1+e^{-cM_*^2 L^{1/2} r_\beta}\right)\exp\left(-2\sum_{i=1}^{k} \frac{(x_i - x_{i-1})^2 - (y_i - y_{i-1})^2)}{s_i - s_{i-1}}\right)\\ &\quad \times \mathbb{P}\left(\max_{i=1,\ldots,k-1} |\pi^*(s_i) - y_i L^{-1/4}| \leq w_\beta \mid \mathcal{L}^\beta > L\right) + C\exp(-c(\log L)^2).\end{aligned} \tag{11.7}$$

We need to lower bound $\mathbb{P}(\max_{i=1,\ldots,k-1} |\pi^*(s_i) - y_i L^{-1/4}| \leq w_\beta \mid \mathcal{L}^\beta > L)$ in order to ensure that the error term $C\exp(-c(\log L)^2)$ is not dominating. For this we sum over $\vec{x} \in [-KL^{-1/4}, KL^{-1/4}]^k \cap (w_\beta Z)^k$, where K is a large constant, such that

$$\sum_{\vec{x}\in[-KL^{-1/4},KL^{-1/4}]^k\cap(w_\beta Z)^k} \mathbb{P}\left(\max_{i=1,\ldots,k-1} |\pi^*(s_i) - x_i L^{-1/4}| \leq w_\beta \;\Big|\; \mathcal{L}^\beta > L\right) \geq \frac{1}{2};$$

that this is possible follows from combining Proposition 7.1 and Proposition 7.3 (for $\beta = 1$) or Lemma 5.2 (for $\beta = \infty$). Then from (11.7) we have that

$$\left(\tfrac{1}{2} - C\exp(-c(\log L)^2)\right) \leq C\cdot \mathbb{P}\left(\max_{i=1,\ldots,k-1} |\pi^*(s_i) - y_i L^{-1/4}| \leq w_\beta \;\Big|\; \mathcal{L}^\beta > L\right)$$

$$\times \sum_{\vec{x}\in[-KL^{-1/4},KL^{-1/4}]^k\cap(w_\beta Z)^k} \exp\left(-2\sum_{i=1}^k \frac{(x_i-x_{i-1})^2-(y_i-y_{i-1})^2)}{s_i-s_{i-1}}\right).$$

This yields that $\mathbb{P}(\max_{i=1,\dots,k-1}|\pi^*(s_i)-y_iL^{-1/4}|\le w_\beta \mid \mathcal{L}^\beta > L)$ is lower bounded by a polynomial in L^{-1}. Thus we conclude that

$$\begin{aligned}
&\frac{\mathbb{P}\left(\max_{i=1,\dots,k-1}|\pi^*(s_i)-x_iL^{-1/4}|\le w_\beta \mid \mathcal{L}^\beta > L\right)}{\mathbb{P}\left(\max_{i=1,\dots,k-1}|\pi^*(s_i)-y_iL^{-1/4}|\le w_\beta \mid \mathcal{L}^\beta > L\right)}\\
&\qquad < (1+o(1))\left(1+e^{-cM_*^2L^{1/2}r_\beta}\right)\exp\left(-2\sum_{i=1}^k \frac{(x_i-x_{i-1})^2-(y_i-y_{i-1})^2)}{s_i-s_{i-1}}\right)\\
&\qquad + C\exp(-c(\log L)^2).
\end{aligned}$$

We can then get a lower bound of the same ratio by swapping $\vec{x}$ and $\vec{y}$. Taking $L\to\infty$ followed by $M_*\to\infty$ completes the proof. $\square$

Proof of Lemma 11.4. It suffices to prove the case $\vec{y}=\vec{0}$ and apply the resulting statement twice. By independence, $\mathbb{P}\left(\vec{\mathcal{L}}^{\beta,\vec{x}}\approx\vec{h}\right)=\prod_{i=1}^k\mathbb{P}\left(\mathcal{L}_i^{\beta,\vec{x}}\in h_i+[0,e^{-L}]\right)$. So we have to show that

$$\mathbb{P}\left(\mathcal{L}_i^{\beta,\vec{x}}\in h_i+[0,e^{-L}]\right)=(1+o(1))\exp\left(-\frac{2x_i^2}{s_i-s_{i-1}}\right)\cdot\mathbb{P}\left(\mathcal{L}_i^{\beta,\vec{0}}\in h_i+[0,e^{-L}]\right).$$

Now, by shear invariance and using Theorem 4.1 in the third line,

$$\begin{aligned}
&\mathbb{P}\left(\mathcal{L}_i^{\beta,\vec{x}}\in h_i+[0,e^{-L}]\right)\\
&\quad=\mathbb{P}\left(\mathcal{L}_i^{\beta,\vec{x}}\ge h_i\right)-\mathbb{P}\left(\mathcal{L}_i^{\beta,\vec{x}}\ge h_i+e^{-L}\right)\\
&\quad=\mathbb{P}\left(\mathcal{L}_i^{\beta,\vec{0}}\ge h_i+\frac{(x_i-x_{i-1})^2L^{-1/2}}{s_i-s_{i-1}}\right)-\mathbb{P}\left(\mathcal{L}_i^{\beta,\vec{0}}\ge h_i+e^{-L}+\frac{(x_i-x_{i-1})^2L^{-1/2}}{s_i-s_{i-1}}\right)\\
&\quad=(1+o(1))\exp\left(-2(s_i-s_{i-1})^{-3/2}h_i^{1/2}(x_i-x_{i-1})^2L^{-1/2}\right)\cdot\mathbb{P}\left(\mathcal{L}_i^{\beta,\vec{0}}\ge h_i\right)\\
&\qquad-(1+o(1))\exp\left(-2(s_i-s_{i-1})^{-3/2}(h_i+e^{-L})^{1/2}(x_i-x_{i-1})^2L^{-1/2}\right)\cdot\mathbb{P}\left(\mathcal{L}_i^{\beta,\vec{0}}\ge h_i+e^{-L}\right).
\end{aligned}$$

Since $(h_i+e^{-L})^{1/2}=h_i^{1/2}+O(e^{-L})$, and $h_i^{1/2}=(s_i-s_{i-1})^{1/2}L^{1/2}+o(L^{1/2})$ due to that $h_i=(s_i-s_{i-1})L+o(L)$, the previous display equals

$$(1+o(1))\exp\left(-\frac{2(x_i-x_{i-1})^2}{s_i-s_{i-1}}\right)\left[\mathbb{P}\left(\mathcal{L}_i^{\beta,\vec{0}}\ge h_i\right)-\mathbb{P}\left(\mathcal{L}_i^{\beta,\vec{0}}\ge h_i+e^{-L}\right)\right],$$

which is what we wanted to show. $\square$

12. Joint comparison of maximizer location and free energy across peaks

In this section, we give the proof of Lemma 11.3. We use the setup there: in particular, C_0, M_* are large constants, L is taken to be large (depending on C_0, M_*), $\vec{h}\in E_{\mathrm{val}}$ for E_{val} defined in (11.5), $\vec{x},\vec{y}\in\mathcal{K}$ for a compact set $\mathcal{K}\subseteq\mathbb{R}^{k-1}$, and $(s_1,\cdots,s_{k-1})\in\mathring{\Lambda}_{k-1}([0,1])$ for $k\in\mathbb{N}$, with $s_0=0$ and $s_k=1$. All the constants within this section can depend on C_0, M_*, $\mathcal{K}$, k, and $(s_1,\cdots,s_{k-1})$.

The proof strategy is to do a resampling on a small interval I (with size of order $L^{-1/2}\log L$) in a way that a certain conditional probability of the event in question is a function of the endpoint values at the boundary of I. The proof then comes down to showing that the density of these endpoint values under the conditioning $\mathcal{L}^{\vec{x}} \approx \vec{h}$ (which recall is shorthand for $\mathcal{L}_i^{\beta,\vec{x}} \in h_i + [0, e^{-L}]$ for $i = 1, \ldots, k$) is $1 + o(1)$ of the density of the same under the conditioning $\mathcal{L}^{\vec{y}} \approx \vec{h}$.

However, note that neither of the events $\max_{i=1,\ldots,k-1} |\pi^*(s_i) - x_i L^{-1/4}| < w_\beta$ and $\mathcal{L}^\beta[1] > (1 - \beta^{-1}\exp(-cL^{1/4}))L$ are functions of the profile on an interval of size of order $L^{-1/2}\log L$. Thus we will need to first argue that we can consider different events which do have this localized property. For the first event, we simply consider the maximizer on I as a proxy, instead of on $\mathbb{R}$; clearly, if the first event holds, then it also holds that the restricted maximizer is w_β-close to $x_i L^{-1/4}$. We will then show (Lemma 12.7) that with high probability the restricted maximizer and true maximizer coincide. In the $\beta = \infty$ case, on this event we also have that $\mathcal{L}^\beta[1]$ is a function of the profile on an interval of size of order $L^{-1/2}\log L$.

Modifying the second event, $\mathcal{L}^\beta[1] > (1 - \exp(-cL^{1/4}))L$, to be a local function of the profile in the $\beta = 1$ case is more difficult. A naive argument (invoking Proposition 7.1 and Proposition 10.1) shows that the free energy when restricted to an interval of size $ML^{-1/2}\log L$ (for a large number M) would capture a $(1 - L^{-M})$ fraction of the total free energy. But at the end of the comparison we need to be able to return to the event of being larger than $\mathcal{L}^\beta[1] > L$, and so we need to argue that, conditional on $\mathcal{L}^{\vec{x}} \approx \vec{h}$ and $\mathcal{L}^\beta[1] > (1 - \exp(-cL^{1/4}))L$ (as well as the location of the restricted maximizer), with high probability $\mathcal{L}^\beta$ will actually be larger than $L + L^{-E_*}$ for some E_*, i.e., the free energy overshoots by a polynomial amount. Then we know that restricting to a $ML^{-1/2}\log L$ window still results in the free energy being larger than L (by picking M large enough), and we can do our localized comparison argument with this event.

In the following Section 12.1 we give the argument for saying that, conditional on $\mathcal{L}^{\vec{x}} \approx \vec{h}$ and $\mathcal{L}^\beta[1] > (1 - \exp(-cL^{1/4}))L$, it holds with high probability that $\mathcal{L}^\beta[1] > L + L^{-E_*}$ (Lemma 12.1). In Section 12.2 we argue that the free energy (in a form suitable for analysis in the upcoming proof of Lemma 11.3) from the smaller interval of size $ML^{-1/2}\log L$ is also $L + L^{-E_*}$ (Lemma 12.5). Then in Section 12.3 we give the proof of Lemma 11.3.

Denote $\mathcal{E}_{\mathrm{cond}}$ to be the event where $\max_{i=1,\ldots,k-1} |\pi^*(s_i)| < w_\beta, \mathcal{L}^\beta[1] > (1 - \beta^{-1}e^{-cL^{1/4}})L, \vec{\mathcal{L}}^{\beta,\vec{x}} \approx \vec{h}$. The following will be frequently used, to relate the conditioning on $\mathcal{E}_{\mathrm{cond}}$ and $\vec{\mathcal{L}}^{\beta,\vec{x}} \approx \vec{h}$.

$$\mathbb{P}\left(\max_{i=1,\ldots,k-1} |\pi^*(s_i)| < w_\beta, \mathcal{L}^\beta[1] > (1 - \beta^{-1}e^{-cL^{1/4}})L \;\middle|\; \vec{\mathcal{L}}^{\beta,\vec{x}} \approx \vec{h}\right) > c\exp(-CL^{1/2}). \tag{12.1}$$

It follows from the lower bound in Proposition 9.1 and $\vec{h} \in E_{\mathrm{val}}$.

12.1. **Conditional overshoot at positive temperature.** We work in the case of $\beta = 1$ only in this subsection, in which case we recall that $w_\beta = CL^{-1/2}$.

Lemma 12.1. *There exists a constant $\rho > 0$ such that*

$$\mathbb{P}\Big(\mathcal{L}^\beta[1] < L + \rho L^{-3} \;\Big|\; \mathcal{E}_{\mathrm{cond}}\Big) < CL^{-1}.$$

This lemma is proved by a resampling argument, and we explain the setup next. For a function $f : \mathbb{R} \to \mathbb{R}$ and an interval $[a, b]$, we define the *bridge of f on $[a, b]$*, denoted $f^{[a,b]}$, by

$$f^{[a,b]}(x) = f(x) - \frac{x-a}{b-a}f(b) - \frac{b-x}{b-a}f(a);$$

in words, it is the function obtained by affinely shifting f to equal 0 at a and b.

For any $x \in \mathbb{R}$, we define

$$\mathcal{L}_+^\beta[1](x) = \max_{|z_2|,\ldots,|z_{k-1}|\le 1}{}^{(\beta)} \sum_{i=2}^{k} \mathcal{L}^\beta(s_{i-1}, z_{i-1}; s_i, z_i),$$

with $z_1 = x$, $z_k = 0$.

Let $\mathfrak{h}^{\beta}_{s_1,1}$ and $\mathfrak{h}^{\beta}_{s_1,2}$ be the top two lines in the line ensemble associated with $\mathcal{L}^{\beta}(0,0;\cdot,s_1)$; in particular $\mathfrak{h}^{\beta}_{s_1,1} = \mathcal{L}^{\beta}(0,0;\cdot,s_1)$. Let $\mathcal{F}$ be the σ-algebra generated by

- $\mathfrak{h}^{\beta}_{s_1,1}(z)$ for $z \in (-\infty, x_1L^{-1/4}] \cup [x_1L^{-1/4} + w_\beta, \infty)$,
- $(\mathfrak{h}^{\beta}_{s_1,1})^{x_1L^{-1/4}+[0,\frac{1}{2}w_\beta]}$, $(\mathfrak{h}^{\beta}_{s_1,1})^{x_1L^{-1/4}+[\frac{1}{2}w_\beta,w_\beta]}$,
- $\mathfrak{h}^{\beta}_{s_1,2}$ and $\mathcal{L}^{\beta}_{+}[1]$.

In particular, conditional on $\mathcal{F}$, the remaining randomness (to determine $\mathfrak{h}^{\beta}_{s_1,1}$) is the value of $U := \mathcal{L}^{\beta}(0,0;x_1L^{-1/4} + \frac{1}{2}w_\beta, s_1) = \mathfrak{h}^{\beta}_{s_1,1}(x_1L^{-1/4} + \frac{1}{2}w_\beta)$, by linear interpolation. Therefore, for $z \in [x_1L^{-1/4}, x_1L^{-1/4} + \frac{1}{2}w_\beta]$ and $u \in \mathbb{R}$, we denote

$$\mathfrak{h}^{\beta,u}_{s_1,1}(z) := \frac{z - x_1L^{-1/4}}{\frac{1}{2}w_\beta}\big(u - \mathfrak{h}^{\beta}_{s_1,1}(x_1L^{-1/4})\big) + \mathfrak{h}^{\beta}_{s_1,1}(x_1L^{-1/4}) + (\mathfrak{h}^{\beta}_{s_1,1})^{x_1L^{-1/4}+[0,\frac{1}{2}w_\beta]}(z) \qquad (12.2)$$

and a similar expression for $z \in [x_1L^{-1/4} + \frac{1}{2}w_\beta, x_1L^{-1/4} + w_\beta]$. Further, given $\mathcal{F}$ and U, we also determine the value of $\mathcal{L}^{\beta}[1]$ via the formula for $\mathfrak{h}^{\beta}_{s_1,1}$ as well as the convolution formula, since $\mathcal{L}^{\beta}_{+}[1]$ is $\mathcal{F}$-measurable. We also observe that the dependence of $\mathcal{L}^{\beta}[1]$ on U is increasing since the convolution formula has an increasing dependence on $\mathfrak{h}^{\beta}_{s_1,1}$.

Now we apply the Brownian Gibbs property. It implies that the distribution of U is a normal random variable of $\mathcal{F}$-measurable mean μ and variance σ^2 given by

$$\begin{aligned} \mu &= \tfrac{1}{2}\left(\mathfrak{h}^{\beta}_{s_1,1}(x_1L^{-1/4} + w_\beta) + \mathfrak{h}^{\beta}_{s_1,1}(x_1L^{-1/4})\right) \\ \sigma^2 &= 2 \cdot \frac{\frac{1}{2}w_\beta \cdot \frac{1}{2}w_\beta}{w_\beta} = \tfrac{1}{2}w_\beta, \end{aligned} \qquad (12.3)$$

tilted by the Radon-Nikodym derivative $W^{\mathrm{pt}}(U)/Z^{\mathrm{pt}}$, where W^{pt} and Z^{pt} are given by

$$W^{\mathrm{pt}}(u) = W(\mathfrak{h}^{\beta,u}_{s_1,1}, \mathfrak{h}^{\beta}_{s_1,2}), \quad Z^{\mathrm{pt}} = \mathbb{E}_{\mathcal{F}}\left[W^{\mathrm{pt}}(U)\right].$$

where $W(\mathfrak{h}^{\beta,u}_{s_1,1}, \mathfrak{h}^{\beta}_{s_1,2})$ is from (2.6) for the interval $[x_1L^{-1/4}, x_1L^{-1/4} + w_\beta]$.

Now, if we require that $\mathcal{L}^{\beta}[1] > (1 - \beta^{-1}e^{-cL^{1/4}})L$, this is equivalent to U being larger than some $\mathcal{F}$-determined value, due to the increasing dependence of $\mathcal{L}^{\beta}$ on U already noted. Further, if we also require that $|\pi^*(s_1)| \le w_\beta$, it is not hard to see that this also is equivalent to a lower bound on U (which may be $-\infty$).

These in turn imply that, conditional on $\mathcal{E}_{\mathrm{cond}}$, and $\vec{\mathcal{L}}^{\beta,\vec{x}} \approx \vec{h}$, the distribution of U is given by a Gaussian random variable of mean μ and variance σ^2 as given in (12.3) tilted by $W^{\mathrm{pt}}(U)/Z^{\mathrm{pt}}$, and is further conditioned to be larger than an $\mathcal{F}$-measurable random variable, which we denote by Cor. Now, since W^{pt} is an increasing function, this Gaussian random variable stochastically dominates the Gaussian random variable X with the same mean and variance which is conditioned only to be larger than Cor.

If we know that Cor is not too high (say less than order L) and μ is not too low (say $-\mu$ less than order L), then the Gaussian random variable will overshoot by an amount at least polynomial in L^{-1}. This upper bound on Cor and lower bound on μ are recorded in the next two lemmas and will be proved shortly.

Lemma 12.2. *There exists K_0 such that for $K > K_0$,*

$$\mathbb{P}\left(\mathrm{Cor} > h_1 + K \;\middle|\; \mathcal{E}_{\mathrm{cond}}\right) \le \exp(-cKL^{1/2}).$$

Lemma 12.3. *There exists K_0 such that,*

$$\mathbb{P}\left(\mu < -K_0 L \;\middle|\; \mathcal{E}_{\mathrm{cond}}\right) \le \exp(-cL^{3/2}).$$

However, knowing that the Gaussian overshoots by a polynomial-in-L^{-1} amount does not immediately imply that the free energy overshoots L by a comparable amount. For this we need to additionally know that the contribution to the free energy from the interval that gets perturbed by resampling U is non-trivial (in fact, it contributes a positive proportion of the free energy, as we will show). This is recorded in the next lemma.

Lemma 12.4. *There exists $\rho > 0$ such that, conditional on $\mathcal{E}_{\mathrm{cond}}$, with probability at least $1 - \exp(-cL^{1/2})$,*

$$\int_{x_1 L^{-1/4}+\frac{1}{4}w_\beta}^{x_1 L^{-1/4}+\frac{3}{4}w_\beta} \exp\left(\mathfrak{h}^{\beta}_{s_1,1}(x) + \mathcal{L}^{\beta}_{+}[1](x)\right)\,\mathrm{d}x \ge \rho \mathcal{L}^{\beta}[1].$$

With the previous three lemmas in hand, we now give the proof of Lemma 12.1. Afterward, we will give the proofs of those lemmas.

Proof of Lemma 12.1. For every $u \in \mathbb{R}$, we define $\mathcal{L}^{\beta,u}[1]$ the same way as $\mathcal{L}^{\beta}[1]$ through (11.3), except for replacing $\mathcal{L}^{\beta}(0,0;s_1,\cdot) = \mathfrak{h}^{\beta}_{s_1,1}$ by $\mathfrak{h}^{\beta,u}_{s_1,1}$. We observe that, for any $\delta > 0$,

$$\begin{aligned}\exp\left(\mathcal{L}^{\beta,U-\delta}[1]\right) &= \int_{[-1,1]\setminus[x_1 L^{-1/4}+\frac{1}{4}w_\beta,\, x_1 L^{-1/4}+\frac{3}{4}w_\beta]} \exp\left(\mathfrak{h}^{\beta,U-\delta}_{1}(x) + \mathcal{L}^{\beta}_{+}[1](x)\right)\,\mathrm{d}x \\ &\quad + \int_{x_1 L^{-1/4}+\frac{1}{4}w_\beta}^{x_1 L^{-1/4}+\frac{3}{4}w_\beta} \exp\left(\mathfrak{h}^{\beta,U-\delta}_{1}(x) + \mathcal{L}^{\beta}_{+}[1](x)\right)\,\mathrm{d}x.\end{aligned}$$

Notice that $\mathfrak{h}^{\beta,U-\delta}_{s_1,1}(x) \le \mathfrak{h}^{\beta}_{s_1,1}(x) - \frac{1}{2}\delta$ on $x_1 L^{-1/4} + [\frac{1}{4}w_\beta, \frac{3}{4}w_\beta]$ (by the formula (12.2)). So the second term in the previous display is upper bounded by

$$\int_{x_1 L^{-1/4}+\frac{1}{4}w_\beta}^{x_1 L^{-1/4}+\frac{3}{4}w_\beta} \exp\left(\mathfrak{h}^{\beta}_{s_1,1}(x) + \mathcal{L}^{\beta}_{+}[1](x) - \tfrac{1}{2}\delta\right)\,\mathrm{d}x.$$

Then by Lemma 12.4, there exists $\rho > 0$ such that, with conditional probability at least $1 - \exp(-cL^{1/2})$, $\exp(\mathcal{L}^{\beta,U-\delta}[1])$ is upper bounded by

$$\left[1-\rho+e^{-\frac{1}{2}\delta}\rho\right]\int_{[-1,1]} \exp\left(\mathfrak{h}^{\beta}_{s_1,1}(x) + \mathcal{L}^{\beta}_{+}[1](x)\right)\,\mathrm{d}x = \left[1-\rho+e^{-\frac{1}{2}\delta}\rho\right]\exp(\mathcal{L}^{\beta}[1]).$$

Since $e^{-x} \le 1 - x/2$ for $x \in [0,1]$, we see that the square bracket factor is at most $1 - \rho\delta/4$. In summary, with conditional probability at least $1 - \exp(-cL^{1/2})$, for any $\delta > 0$ sufficiently small,

$$\exp\left(\mathcal{L}^{\beta,U-\delta}[1]\right) \le (1-\rho\delta/4)\exp\left(\mathcal{L}^{\beta}[1]\right). \tag{12.4}$$

Recall from the discussion preceding Lemma 12.2 that, conditionally on $\mathcal{F}$, U stochastically dominates a random variable X which is Gaussian with mean μ and variance σ^2 as given in (12.3) and conditioned to stay above Cor (i.e., X is not tilted by $W^{\mathrm{pt}}(U)/Z^{\mathrm{pt}}$). Now, since (by Lemma 12.2 and Lemma 12.3) with conditional probability at least $1-\exp(-cL^{1/2})$ it holds that $\mathrm{Cor} \le h_1 + K_0$ (so that in particular $\mathrm{Cor} = O(L)$) and $\mu \ge -K_0 L$, it follows that, with conditional probability at least $1 - L^{-1}$ that

$$X \ge \mathrm{Cor} + L^{-3},$$

using standard estimates for the normal tail bound (Lemma 2.19).

In particular, since U stochastically dominates X, taking $\delta = L^{-3}$ implies that $U - \delta \geq \mathrm{Cor}$. Since Cor is such that for any $u \geq \mathrm{Cor}$ it holds that $\mathcal{L}^{\beta,u}[1] \geq (1-\beta^{-1}e^{-cL^{1/4}})L$, we obtain from combining these observations with (12.4) that

$$\exp(\mathcal{L}^{\beta}[1]) \geq (1 + \rho L^{-3}/5)\exp(L).$$

Taking logarithms and relabeling ρ completes the proof. $\square$

We next prove the three lemmas that have been assumed.

Proof of Lemma 12.2. We observe that by the definition of Cor, it holds almost surely that $\mathrm{Cor} \leq \mathfrak{h}^{\beta}_{s_1,1}(x_1L^{-1/4} + \frac{1}{2}w_\beta)$. So we obtain

$$\mathbb{P}\left(\mathrm{Cor} > h_1 + K \;\middle|\; \vec{\mathcal{L}}^{\beta,\vec{x}} \approx \vec{h}\right) \leq \mathbb{P}\left(\mathfrak{h}^{\beta}_{s_1,1}(x_1L^{-1/4} + \tfrac{1}{2}w_\beta) > h_1 + K \;\middle|\; \vec{\mathcal{L}}^{\beta,\vec{x}} \approx \vec{h}\right).$$

By Theorem 2.15 this is upper bounded by $\exp(-cKL^{1/2})$. Combining this with (12.1) and taking K large enough, we obtain the claimed bound. $\square$

Proof of Lemma 12.3. Since $\mu = \frac{1}{2}(\mathfrak{h}^{\beta}_{s_1,1}(x_1L^{-1/4} + w_\beta) + \mathfrak{h}^{\beta}_{s_1,1}(x_1L^{-1/4}))$, it holds by Theorem 2.15 and the independence of $\mathcal{L}^{\beta,\vec{x}}_i$ across i that $\mathbb{P}\left(\mu < -K_0L \mid \vec{\mathcal{L}}^{\beta,\vec{x}} \approx \vec{h}\right) < \exp(-cL^{3/2})$. Combining this with (12.1) completes the proof. $\square$

Proof of Lemma 12.4. Take C_* to be a large constant. By using Proposition 9.1 for a mesh of L^2 many $x \in [-1,1]$, taking a union bound, and applying the unconditional local fluctuations estimates Proposition 2.14, we have

$$\mathbb{P}\left(\max_{|x|\leq 1}\mathcal{L}^{\beta}(x, s_1; 0, 1) > \sum_{i=2}^{k} h_i - (k-2)\log(2L^{1/2}) + \frac{1}{2}C_* \;\middle|\; \vec{\mathcal{L}}^{\beta,\vec{x}} \approx \vec{h}\right) < \exp(-cC_*L^{1/2}). \quad (12.5)$$

Further by Corollary 2.18,

$$\mathbb{P}\left(\mathfrak{h}^{\beta}_{s_1,1}(x) \leq h_1 - C_* - L^{1/2}|x - x_1L^{-1/4}|/2,\ \forall x : |x - x_1L^{-1/4}| \geq 10C_*w_\beta, |x| \leq 1 \;\middle|\; \vec{\mathcal{L}}^{\beta,\vec{x}} \approx \vec{h}\right)$$
$$> 1 - C\exp(-cC_*L^{1/2}).$$

We can replace the conditioning $\vec{\mathcal{L}}^{\beta,\vec{x}} \approx \vec{h}$ in the above two estimates by $\mathcal{E}_{\mathrm{cond}}$, using (12.1). Then from these, we have that with probability $> 1 - C\exp(-cL^{1/2})$ conditioning on $\mathcal{E}_{\mathrm{cond}}$,

$$\int_{|x-x_1L^{-1/4}|\geq 10C_*w_\beta, |x|\leq 1} \exp\left(\mathfrak{h}^{\beta}_{s_1,1}(x) + \mathcal{L}^{\beta}(x, s_1; 0, 1)\right) \mathrm{d}x$$
$$\leq \exp\left(\sum_{i=1}^{k} h_i - (k-2)\log(2L^{1/2}) - \log(L^{1/2}) - \frac{1}{2}C_*\right) \leq \exp(L - \tfrac{1}{4}C_*) \leq \exp(-\tfrac{1}{4}C_*)\mathcal{L}^{\beta}[1],$$

using that $\vec{h} \in E_{\mathrm{val}}$ and C_* is large for the second inequality, and that we have conditioned on $\mathcal{L}^{\beta}[1] > L$ for the last inequality. This implies that

$$\mathcal{L}^{\beta}[1] \leq (1 - \exp(-\tfrac{1}{4}C_*))^{-1} \int_{|x-x_1L^{-1/4}|\leq 10C_*w_\beta} \exp\left(\mathfrak{h}^{\beta}_{s_1,1}(x) + \mathcal{L}^{\beta}_{+}[1](x)\right) \mathrm{d}x. \quad (12.6)$$

It also follows from Corollary 2.18 that, with probability at least $1 - C\exp(-cC_*L^{1/2})$ conditioning on $\vec{\mathcal{L}}^{\beta,\vec{x}} \approx \vec{h}$,

$$h_1 - 20C_* \leq \inf_{|x-x_1L^{-1/4}|\leq 10C_*w_\beta} \mathfrak{h}^{\beta}_{s_1,1}(x) \leq \sup_{|x-x_1L^{-1/4}|\leq 10C_*w_\beta} \mathfrak{h}^{\beta}_{s_1,1}(x) \leq h_1 + C_*.$$

And by Corollary 2.18 and Proposition 3.4, with probability at least $1-C\exp(-cC_*L^{1/2})$ conditioning on $\vec{\mathcal{L}}^{\beta,\vec{x}} \approx \vec{h}$, there is

$$\mathcal{L}^\beta(x, s_{i-1}; y, s_i) > h_i - 30C_*,$$

for any $i = 2, \ldots, k$, and $|x - x_{i-1}L^{-1/4}|, |y - x_iL^{-1/4}| \le 10C_*w_\beta$; thus

$$\inf_{|x-x_1L^{-1/4}|\le 10C_*w_\beta} \mathcal{L}^\beta_+[1](x) \ge \sum_{i=2}^k h_i - (k-2)\log(2L^{1/2}) - 30kC_*.$$

Combining these two estimates with (12.5), we get that, with probability at least $1-C\exp(-cC_*L^{1/2})$ conditioning on $\vec{\mathcal{L}}^{\beta,\vec{x}} \approx \vec{h}$,

$$\frac{\int_{x_1L^{-1/4}+\frac{1}{4}w_\beta}^{x_1L^{-1/4}+\frac{3}{4}w_\beta} \exp\left(\mathfrak{h}^\beta_{s_1,1}(x) + \mathcal{L}^\beta(x, s_1; 0, 1)\right) \mathrm{d}x}{\int_{x_1L^{-1/4}-10C_*w_\beta}^{x_1L^{-1/4}+10C_*w_\beta} \exp\left(\mathfrak{h}^\beta_{s_1,1}(x) + \mathcal{L}^\beta(x, s_1; 0, 1)\right) \mathrm{d}x} \ge (40C_*)^{-1}\exp(-40kC_*).$$

We can further replace the conditioning by $\mathcal{E}_{\mathrm{cond}}$, using (12.1). Then together with (12.6) the proof completes. □

12.2. **Free energy of restricted interval.** In this subsection, we prove the following statement.

We introduce some notation to describe the free energy profiles in disjoint temporal strips defined by $[s_{i-1}, s_i]$. We define

$$\mathfrak{h}_1^{\mathrm{top},i,\vec{x}} = \mathcal{L}^\beta(x_{i-1}L^{-1/4}, s_{i-1}; x_iL^{-1/4}+\cdot, s_i), \quad \mathfrak{h}_1^{\mathrm{bot},i,\vec{x}} = \mathcal{L}^\beta(x_{i-1}L^{-1/4}+\cdot, s_{i-1}; x_iL^{-1/4}, s_i), \tag{12.7}$$

for each $i = 1, \ldots, k$. For any vector $\vec{z} \in \mathbb{R}^{k-1}$, we always write $z_0 = z_k = 0$ for the convenience of notations. We denote

$$\mathfrak{h}^{\mathrm{sum}}(\vec{z}) = \sum_{i=1}^{k-1} \mathfrak{h}_1^{\mathrm{top},i,\vec{x}}(z_i) + \sum_{i=2}^{k} \mathfrak{h}_1^{\mathrm{bot},i,\vec{x}}(z_{i-1}), \tag{12.8}$$

We also write $I_{M,L} = [-ML^{-1/2}\log L, ML^{-1/2}\log L]$, and $I_{\vec{x}} = L^{-1/4}\vec{x} + I_{M,L}^{k-1} \subset \mathbb{R}^{k-1}$. Recall the $\max^{(\beta)}$ notation from (9.1).

Lemma 12.5. *When $\beta = 1$, there exists $\rho > 0$ such that, conditional on $\mathcal{E}_{\mathrm{cond}}$, it holds with probability at least $1 - CL^{-1}$ that*

$$\max^{(\beta)}_{\vec{z}\in I_{M,L}^{k-1}} \mathfrak{h}^{\mathrm{sum}}(\vec{z}) \ge L + \rho L^{-3} + \sum_{i=2}^{k-1} \mathcal{L}_i^{\beta,\vec{x}}.$$

When $\beta = \infty$, conditional on $\mathcal{E}_{\mathrm{cond}}$, it holds with probability at least $1 - \exp(-cL^{1/2})$ that

$$\max_{\vec{z}\in I_{M,L}^{k-1}} \mathfrak{h}^{\mathrm{sum}}(\vec{z}) \ge L + \sum_{i=2}^{k-1} \mathcal{L}_i^{\beta,\vec{x}}.$$

To prove this in the $\beta = 1$ case, the main thing we need is that the contribution to the free energy $\mathcal{L}^\beta[1]$ from outside $I_{\vec{x}}$ is small, which we isolate in the following statement.

Lemma 12.6. *When $\beta = 1$, conditional on $\mathcal{E}_{\mathrm{cond}}$, with probability at least $1 - \exp(-cML^{1/2})$,*

$$\max^{(\beta)}_{\vec{z}\in[-1,1]^{k-1}\setminus I_{\vec{x}}} \sum_{i=1}^k \mathcal{L}^\beta(z_{i-1}, s_{i-1}; z_i, s_i) \le L - \frac{1}{2}M\log L.$$

To handle the $\beta = \infty$ case, we will need the following statement, which is that the maximizer restricted to $I_{\vec{x}}$ is with high probability the same as the global maximize. We note that it also holds when $\beta = 1$.

We define the restricted (in terms of the interval over which the maximization is performed) version of $\pi^*(s_i)$: for $i = 1, \ldots, k-1$ we let

$$\pi^{*,\mathrm{res},\vec{x}}(s_i) = \operatorname*{argmax}_{|z| \le ML^{-1/2}\log L} \Big(\mathcal{L}^\beta(0, s_{i-1}; x_i L^{-1/4} + z, s_i) + \mathcal{L}^\beta(x_i L^{-1/4} + z, s_i; 0, s_{i+1}) \Big), \tag{12.9}$$

for M being a large number.

Lemma 12.7. *With probability* $1 - \exp(-cML^{1/2}\log L)$ *conditional on* $\vec{\mathcal{L}}^{\beta,\vec{x}} \approx \vec{h}$*, for each* $i = 1, \ldots, k$*,*

$$\pi^*(s_i) = x_i L^{-1/4} + \pi^{*,\mathrm{res},\vec{x}}(s_i).$$

We give the proof of Lemma 12.5 before turning to proving Lemmas 12.6 and 12.7.

Proof of Lemma 12.5. In this proof, by 'conditional probability', we refer to the conditioning $\mathcal{E}_{\mathrm{cond}}$.

Case of $\beta = \infty$: We let $\mathcal{E}_{\mathrm{eq}}$ denote the event where $\pi^*(s_i) = x_i L^{-1/4} + \pi^{*,\mathrm{res},\vec{x}}(s_i)$ for each $i = 1, \ldots, k-1$. We first note that by Lemma 12.7 and (12.1), for all large enough M, $\mathcal{E}_{\mathrm{eq}}$ holds with conditional probability we $> 1 - \exp(-cML^{1/2})$. Now on $\mathcal{E}_{\mathrm{eq}}$ it follows by definition that $\mathcal{L}^\beta[1] = \max_{\vec{z} \in I_{\vec{x}}} \sum_{i=1}^k \mathcal{L}^\beta(z_{i-1}, s_{i-1}; z_i, s_i)$. Further by coalescence, which holds with conditional probability $> 1 - C\exp(-cL^{3/2})$ by Proposition 3.4 and (12.1), it follows that

$$\max_{\vec{z} \in I_{M,L}^k} \mathfrak{h}^{\mathrm{sum}}(\vec{z}) = \max_{\vec{z} \in I_{\vec{x}}} \sum_{i=1}^k \mathcal{L}^\beta(z_{i-1}, s_{i-1}; z_i, s_i) + \sum_{i=2}^{k-1} \mathcal{L}_i^{\beta,\vec{x}} \ge L + \sum_{i=2}^{k-1} \mathcal{L}_i^{\beta,\vec{x}},$$

completing the proof.

Case of $\beta = 1$: By Lemma 12.1 there exists $\rho > 0$ such that, with conditional probability at least $1 - CL^{-1}$, we have $\mathcal{L}^\beta[1] \ge L + \rho L^{-3}$. By Lemma 12.6, with conditional probability at least $1 - \exp(-cML^{1/2})$,

$$\exp(\mathcal{L}^\beta[1]) - \exp\left(\max_{\vec{z} \in I_{\vec{x}}}{}^{(\beta)} \sum_{i=1}^k \mathcal{L}^\beta(z_{i-1}, s_{i-1}; z_i, s_i) \right) \le L^{-\frac{1}{2}M} \exp(L).$$

The previous two inequalities, along with $\exp(x) \ge 1 + x$, imply that

$$\begin{aligned}
\exp\left(\max_{\vec{z} \in I_{\vec{x}}}{}^{(\beta)} \sum_{i=1}^k \mathcal{L}^\beta(z_{i-1}, s_{i-1}; z_i, s_i) \right) &\ge \exp(L + \rho L^{-3}) - L^{-\frac{1}{2}M} \exp(L) \\
&\ge \exp(L) \left[1 + \rho L^{-3} - L^{-\frac{1}{2}M} \right] \\
&\ge \exp(L) \left[1 + \tfrac{1}{2}\rho L^{-3} \right].
\end{aligned}$$

By coalescence (Proposition 3.4), with conditional probability at least $1 - C\exp(-cL^{3/2})$,

$$\begin{aligned}
\max_{\vec{z} \in I_{M,L}^k}{}^{(\beta)} \mathfrak{h}^{\mathrm{sum}}(\vec{z}) &\ge \max_{\vec{z} \in I_{\vec{x}}}{}^{(\beta)} \sum_{i=1}^k \mathcal{L}^\beta(z_{i-1}, s_{i-1}; z_i, s_i) + \sum_{i=2}^{k-1} \mathcal{L}_i^{\beta,\vec{x}} - kCe^{-cL} \\
&> L + \tfrac{1}{4}\rho L^{-3} - kCe^{-cL} + \sum_{i=2}^{k-1} \mathcal{L}_i^{\beta,\vec{x}},
\end{aligned}$$

Since $kCe^{-cL} \ll \rho L^{-1}$, the proof is complete by relabelling ρ. $\square$

The main step in proving Lemma 12.6 is to handle the case of $k = 2$, which is given in the following statement; then we can obtain the statement for general k by making use of coalescence (Proposition 3.4).

Lemma 12.8. *Conditional on $\mathcal{E}_{\mathrm{cond}}$, it holds with probability at least $1-\exp(-cML^{1/2})$ that, for each $i = 1, \dots, k-1$,*

$$\int_{(-x_iL^{-1/4}+[-1,1])\setminus I_{M,L}} \exp\big(\mathfrak{h}_1^{\mathrm{top},i,\vec{x}}(z) + \mathfrak{h}_1^{\mathrm{bot},i+1,\vec{x}}(z)\big)\,\mathrm{d}z \le L^{-M}\exp\left(h_i + h_{i+1}\right).$$

and

$$\int_{-x_iL^{-1/4}+[-1,1]} \exp\big(\mathfrak{h}_1^{\mathrm{top},i,\vec{x}}(z) + \mathfrak{h}_1^{\mathrm{bot},i+1,\vec{x}}(z)\big)\,\mathrm{d}z \le \exp\Big(h_i + h_{i+1} - \log(2L^{1/2}) + M\Big).$$

Proof. The second inequality follows from (9.6) in Lemma 9.3 and (12.1).

For the first inequality, by (12.1), it suffices to show that the displayed inequality holds with probability at least $1-\exp(-cML^{1/2})$ given only that $\vec{\mathcal{L}}^{\beta,\vec{x}} \approx \vec{h}$. Indeed, conditional on $\vec{\mathcal{L}}^{\beta,\vec{x}} \approx \vec{h}$, from Corollary 2.18, it holds with probability at least $1-C\exp(-cML^{1/2}\log L)$ that, for all $z \notin I_{M,L}$, $z + x_iL^{-1/4} \in [-1,1]$,

$$\mathfrak{h}_1^{\mathrm{top},i,\vec{x}}(z) \le h_i - \tfrac{1}{2}M\log L - \tfrac{1}{4}L^{1/2}|z|,$$
$$\mathfrak{h}_1^{\mathrm{bot},i+1,\vec{x}}(z) \le h_{i+1} - \tfrac{1}{2}M\log L - \tfrac{1}{4}L^{1/2}|z|.$$

By plugging these into the integral, the conclusion follows. ☐

Proof of Lemma 12.6. By coalescence Proposition 3.4 and (12.1), with probability at least $1-C\exp(-cL^{3/2})$ conditional on $\mathcal{E}_{\mathrm{cond}}$, the LHS is at most

$$\max^{(\beta)}_{\vec{z}+L^{-1/4}\vec{x}\in[-1,1]^{k-1}\setminus I_{\vec{x}}} \sum_{i=1}^{k-1} \mathfrak{h}_1^{\mathrm{top},i,\vec{x}}(z_i) + \mathfrak{h}_1^{\mathrm{bot},i+1,\vec{x}}(z_i) + Ce^{-cL} - \sum_{i=2}^{k-1} h_i.$$

We can upper bound this by replacing $[-1,1]^{k-1}\setminus I_{\vec{x}}$ with the $k-1$ dimension product where all but the i^{th} one is $[-1,1]$ and the i^{th} one is $[-1,1]\setminus(x_iL^{-1/4}+I_{M,L})$, and summing over i. Using Lemma 12.8, we get an upper bound of

$$\sum_{i=1}^{k} h_i + Ce^{-cL} - M\log L - (k-2)\log(2L^{1/2}) + M.$$

Using that $\vec{h} \in E_{\mathrm{val}}$, by taking M large enough the conclusion follows. ☐

Proof of Lemma 12.7. We need to show that with high probability, conditional on $\vec{\mathcal{L}}^{\beta,\vec{x}} \approx \vec{h}$, it holds that $\max_{i=1,\dots,k-1}|\pi^*(s_i) - x_iL^{-1/4}| \le ML^{-1/2}\log L$ so that $\pi^*(s_i) = x_iL^{-1/4} + \pi^{*,\mathrm{res},\vec{x}}(s_i)$.

Let J_i denote the interval $\{z : |z + x_iL^{-1/4}| \le 10^{-6}(s_i - s_{i-1})^{1/2} \wedge (s_{i+1}-s_i)^{1/2}\}$. By Proposition 3.4, conditional on $\vec{\mathcal{L}}^{\beta,\vec{x}} \approx \vec{h}$, with probability $> 1-C\exp(-cL^{3/2})$ the event $\pi^*(s_i) \ne x_iL^{-1/4} + \pi^{*,\mathrm{res},\vec{x}}(s_i)$ implies that

$$\sup_{z\in J_i\setminus I_{M,L}} \mathfrak{h}_1^{\mathrm{top},i,\vec{x}}(z) + \mathfrak{h}_1^{\mathrm{bot},i+1,\vec{x}}(z) > h_i + h_{i+1} - C\exp(-cL), \tag{12.10}$$

or

$$\sup_{z\notin J_i} \mathcal{L}^\beta(0, s_{i-1}; x_iL^{-1/4}+z, s_i) + \mathcal{L}^\beta(x_iL^{-1/4}+z, s_i; 0, s_{i+1}) \ge \mathcal{L}^\beta(0, s_{i-1}; x_iL^{-1/4}, s_i) + \mathcal{L}^\beta(x_iL^{-1/4}, s_i; 0, s_{i+1}). \tag{12.11}$$

We have that (12.10) further implies that either

$$\sup_{z\in J_i\setminus I_{M,L}} \mathfrak{h}_1^{\mathrm{top},i,\vec{x}}(z) > h_{i+1} - C\exp(-cL),$$

or

$$\sup_{z\in J_i\setminus I_{M,L}} \mathfrak{h}_1^{\mathrm{bot},i+1,\vec{x}}(z) > h_i - C\exp(-cL),$$

happens; and by Corollary 2.18, each happens with conditional on $\vec{\mathcal{L}}^{\beta,\vec{x}} \approx \vec{h}$, probability $< \exp(-cML^{1/2}\log L)$.

As for (12.11), by Corollary 2.18, with probability $> 1 - C\exp(-cL^{3/4})$ conditional on $\vec{\mathcal{L}}^{\beta,\vec{x}} \approx \vec{h}$, we have that the RHS is $> h_i + h_{i+1} - L^{1/4}\log L$. For the LHS, for any fixed $z \notin J_i$ we have

$$\mathbb{P}\left(\mathcal{L}^\beta(0, s_{i-1}; x_iL^{-1/4} + z, s_i) + \mathcal{L}^\beta(x_iL^{-1/4} + z, s_i; 0, s_{i+1}) > h_i + h_{i+1} - L^{1/4}\log L\right)$$
$$< C\exp\left(-\tfrac{4}{3}(s_{i+1} - s_{i-1})L^{3/2} - cz^2L^{1/2}\right)$$

by Theorem 2.11, and that $\vec{h} \in E_{\mathrm{val}}$. Then by a union bound over all $z \notin J_i$, $z \in L^{-2}L$, and using the continuity estimate Proposition 2.14, we get that the LHS is $\leq h_i + h_{i+1} - L^{1/4}\log L$ with probability $> 1 - C\exp\left(-\frac{4}{3}(s_{i+1} - s_{i-1})L^{3/2} - cL^{3/2}\right)$. Then since $\mathbb{P}\left(\mathcal{L}_i^{\beta,\vec{x}} \in h_i + [0, e^{-L}]\right) > c\exp(-\frac{4}{3}(s_i - s_{i-1})L^{3/2} - CL^{3/4})$ and $\mathbb{P}\left(\mathcal{L}_{i+1}^{\beta,\vec{x}} \in h_{i+1} + [0, e^{-L}]\right) > c\exp(-\frac{4}{3}(s_{i+1} - s_i)L^{3/2} - CL^{3/4})$, by Theorem 2.11, we conclude that (12.11) happens with probability $< C\exp(-cL^{3/4})$ conditional on $\vec{\mathcal{L}}^{\beta,\vec{x}} \approx \vec{h}$. This completes the proof. □

12.3. **The density comparison.** We now finish the proof of Lemma 11.3.

Recall the setup at the beginning of this section. Also recall the definition of $\pi^{*,\mathrm{res},\vec{x}}(s_i)$ from (12.9) as a version of $\pi^*(s_i)$ defined for $i = 1, \ldots, k-1$ but with a restricted interval over which the maximization is performed, and the definition of $\mathfrak{h}_1^{\mathrm{top},i,\vec{x}}$ and $\mathfrak{h}_1^{\mathrm{bot},i,\vec{x}}$ from (12.7), and $\mathfrak{h}^{\mathrm{sum}}$ from (12.8).

As before, we let $I_{M,L} = [-ML^{-1/2}\log L, ML^{-1/2}\log L]$, and $w_{\beta=1} = L^{-1/2}$, $w_{\beta=\infty} = L^{-1}$.

Proof of Lemma 11.3. For $\iota = 0$ or 1, we have that

$$\mathbb{P}\Big(\max_{i=1,\ldots,k-1} |\pi^*(s_i) - x_iL^{-1/4}| < w_\beta, \mathcal{L}^\beta[1] > (1 - \iota\beta^{-1}e^{-cL^{1/4}})L \;\Big|\; \vec{\mathcal{L}}^{\beta,\vec{x}} \approx \vec{h}\Big) = (1 + O(L^{-1}))$$
$$\times\,\mathbb{P}\left(\max_{i=1,\ldots,k-1} |\pi^*(s_i) - x_iL^{-1/4}| < w_\beta,\ \max_{\vec{z}\in I_{M,L}^{k-1}}{}^{(\beta)}\, \mathfrak{h}^{\mathrm{sum}}(\vec{z}) \geq L + \beta^{-1}\rho L^{-3} + \sum_{i=2}^{k-1} \mathcal{L}_i^{\beta,\vec{x}} \;\Big|\; \vec{\mathcal{L}}^{\beta,\vec{x}} \approx \vec{h}\right). \tag{12.12}$$

Indeed, by Lemma 12.5 we have that the LHS is upper bounded by the RHS; while by Proposition 3.4, we have that

$$\mathcal{L}^\beta[1] + \sum_{i=2}^{k-1} \mathcal{L}_i^{\beta,\vec{x}} \geq \max_{\vec{z}\in I_{M,L}^{k-1}}{}^{(\beta)}\, \mathfrak{h}^{\mathrm{sum}}(\vec{z}),$$

with probability $> 1 - C\exp(-cL^{3/2})$, conditional on $\mathcal{L}^{\beta,\vec{x}} \approx \vec{h}$. Then since the LHS is lower bounded by $c\exp(-CL^{1/2})$ ($\beta = 1$) or c ($\beta = \infty$) by Proposition 9.1 and $\vec{h} \in E_{\mathrm{val}}$, the LHS is lower bounded by the RHS.

We now let $\mathcal{E}_\pm$ denote the event where for each $i = 1, \ldots, k-1$,

$$\sup_{|z|\leq w_\beta} \mathfrak{h}_1^{\mathrm{top},i,\vec{x}}(z) + \mathfrak{h}_1^{\mathrm{bot},i+1,\vec{x}}(z) > \sup_{|z|\geq w_\beta, z\in I_{M,L}} \mathfrak{h}_1^{\mathrm{top},i,\vec{x}}(z) + \mathfrak{h}_1^{\mathrm{bot},i+1,\vec{x}}(z) \mp C\exp(-cL),$$

and

$$\max_{\vec{z}\in I_{M,L}^{k-1}}^{(\beta)} \mathfrak{h}^{\mathrm{sum}}(\vec{z}) \geq L + \beta^{-1}\rho L^{-3} + \sum_{i=2}^{k-1} h_i \mp C\exp(-cL).$$

By Lemma 12.7 and Proposition 9.1, the probability in the RHS of (12.12) is in

$$\left[\mathbb{P}(\mathcal{E}_- \mid \vec{\mathcal{L}}^{\beta,\vec{x}} \approx \vec{h}) - \exp(-cML^{1/2}),\ \mathbb{P}(\mathcal{E}_+ \mid \vec{\mathcal{L}}^{\beta,\vec{x}} \approx \vec{h}) + \exp(-cML^{1/2})\right]. \tag{12.13}$$

Now we wish to make use of Proposition 3.5 to replace the processes $\mathfrak{h}_1^{\mathrm{top},i,\vec{x}}$ and $\mathfrak{h}_1^{\mathrm{bot},i,\vec{x}}$ by a collection of processes given by independent Brownian bridges, up to an exponentially small L^∞ error and on an event with probability $1-\exp(-cL)$.

Denote

$$G = L + \beta^{-1}\rho L^{-3} - \sum_{i=1}^{k} h_i$$

Since $\vec{h} \in E_{\mathrm{val}}$ from (11.5), we have $G < -(k-1)\beta^{-1}\log(2L^{1/2}) + Cr_\beta$ (recall that $r_{\beta=1}=1$ and $r_{\beta=\infty} = L^{-1/2}$).

Reduce to Brownian bridges. We denote $\theta = 10^{-7}\min_{i=1,\ldots,k}(s_i - s_{i-1})L^{1/2}$. We let Θ denote the collection of all

$$\vec{b} = (\vec{b}^{\mathrm{bot,L}}, \vec{b}^{\mathrm{bot,R}}, \vec{b}^{\mathrm{top,L}}, \vec{b}^{\mathrm{top,R}}) = ((b_i^{\mathrm{bot,L}})_{i=2}^k, (b_i^{\mathrm{bot,R}})_{i=2}^k, (b_i^{\mathrm{top,L}})_{i=1}^{k-1}, (b_i^{\mathrm{top,R}})_{i=1}^{k-1}).$$

For any $\vec{b}\in\Theta$, we denote by $B^{\dagger,i,\vec{b}}$ a rate 2 Brownian bridge from $(-\theta, -2\theta L^{1/2} + b^{\dagger_i,\mathrm{L}})$, to $(0,0)$, then to $(\theta, -2\theta L^{1/2} + b_i^{\dagger,\mathrm{R}})$, for $\dagger = \mathrm{top}$, $i=1,\ldots,k-1$, and $\dagger=\mathrm{bot}$, $i=2,\ldots,k$. All these $B^{\dagger,i}$ are independent of each other.

We then denote by $\mathcal{E}_{\vec{b}}$ the event where for each $i=1,\ldots,k-1$,

$$\sup_{|z|\leq w_\beta} B^{\mathrm{top},i,\vec{b}}(z) + B^{\mathrm{bot},i+1,\vec{b}}(z) > \sup_{|z|\geq w_\beta, z\in I_{M,L}} B^{\mathrm{top},i,\vec{b}}(z) + B^{\mathrm{bot},i+1,\vec{b}}(z),$$

and

$$\max_{\vec{z}\in I_{M,L}^{k-1}}^{(\beta)} \sum_{i=1}^{k-1} B^{\mathrm{top},i,\vec{b}}(z_i) + \sum_{i=2}^{k} B^{\mathrm{bot},i,\vec{b}}(z_{i-1}) \geq G.$$

We also denote by $\mathcal{E}_{\vec{b},\pm}$ the event where for each $i=1,\ldots,k-1$,

$$\sup_{|z|\leq w_\beta} B^{\mathrm{top},i,\vec{b}}(z) + B^{\mathrm{bot},i+1,\vec{b}}(z) > \sup_{|z|\geq w_\beta, z\in I_{M,L}} B^{\mathrm{top},i,\vec{b}}(z) + B^{\mathrm{bot},i+1,\vec{b}}(z) \mp C\exp(-cL),$$

and

$$\max_{\vec{z}\in I_{M,L}^{k-1}}^{(\beta)} \sum_{i=1}^{k-1} B^{\mathrm{top},i,\vec{b}}(z_i) + \sum_{i=2}^{k} B^{\mathrm{bot},i,\vec{b}}(z_{i-1}) \geq G \mp C\exp(-cL).$$

It is straightforward to check that

$$|\mathbb{P}(\mathcal{E}_{\vec{b}}) - \mathbb{P}(\mathcal{E}_{\vec{b},\pm})| < C\exp(-cL). \tag{12.14}$$

For each $a>0$ we let Θ_a denote the collection of all $\vec{b}\in\Theta$, where each coordinate is $\leq a$. We write $D = \log L$.

Since that $\vec{x}\in\mathcal{K}$ which is a compact set, applying Proposition 3.5, and using Lemma 2.17 to bound each $\mathfrak{h}_1^{\mathrm{top},i}$ and $\mathfrak{h}_1^{\mathrm{bog},i}$ at $\pm\theta$, we have that

$$\mathbb{P}(\mathcal{E}_- \mid \vec{\mathcal{L}}^{\beta,\vec{x}} \approx \vec{h}) \geq (1 - C\exp(-cD^2)) \min_{\vec{b}\in\Theta_{DL^{1/4}}} \mathbb{P}(\mathcal{E}_{\vec{b},-}) - C\exp(-cL),$$

and

$$\mathbb{P}(\mathcal{E}_+ \mid \vec{\mathcal{L}}^{\beta,\vec{x}} \approx \vec{h}) \le \max_{\vec{b}\in\Theta_{DL^{1/4}}} \mathbb{P}(\mathcal{E}_{\vec{b},+}) + \sum_{i=2}^{\lfloor L^{1/4}\rfloor} C\exp(-ci^2D^2) \max_{\vec{b}\in\Theta_{iDL^{1/4}}} \mathbb{P}(\mathcal{E}_{\vec{b},+}) + C\exp(-cL) + C\exp(-cD^2L^{1/2}).$$

By (12.14), we can replace each $\mathbb{P}(\mathcal{E}_{\vec{b},\pm})$ by $\mathbb{P}(\mathcal{E}_{\vec{b}})$. We also note that these bounds are independent of $\vec{x}$.

We next state a comparison lemma for different $\vec{b}$.

Lemma 12.9. *For any large enough $K > 0$, and $\vec{b}, \vec{g} \in \Theta$ such that $\|\vec{b} - \vec{g}\|_\infty < KL^{1/4}$, we have*

$$\mathbb{P}(\mathcal{E}_{\vec{b}}) = (1 + O(K^2L^{-1/4}\log L))\mathbb{P}(\mathcal{E}_{\vec{g}}) + O(\exp(-cK^2M^{-1}L^{1/2}\log L)).$$

By repeatedly using this lemma, we have (for each $i = 1, \ldots, \lfloor L^{1/4}\rfloor$)

$$\max_{\vec{b}\in\Theta_{iDL^{1/4}}} \mathbb{P}(\mathcal{E}_{\vec{b}}) < \exp(CiD^2L^{-1/4}\log L)\left(\min_{\vec{b}\in\Theta_{DL^{1/4}}} \mathbb{P}(\mathcal{E}_{\vec{b}}) + C\exp(-cD^2M^{-1}L^{1/2}\log L)\right).$$

Therefore we have

$$\mathbb{P}(\mathcal{E}_+ \mid \vec{\mathcal{L}}^{\beta,\vec{x}} \approx \vec{h}) < (1 + C\exp(-cD^2)) \max_{\vec{b}\in\Theta_{DL^{1/4}}} \mathbb{P}(\mathcal{E}_{\vec{b}}) + C\exp(-cD^2M^{-1}L^{1/2}\log L),$$

and

$$\mathbb{P}(\mathcal{E}_- \mid \vec{\mathcal{L}}^{\beta,\vec{x}} \approx \vec{h}) < (1 - C\exp(-cD^2)) \max_{\vec{b}\in\Theta_{DL^{1/4}}} \mathbb{P}(\mathcal{E}_{\vec{b}}) - C\exp(-cD^2M^{-1}L^{1/2}\log L).$$

Since that (as we have seen above) (12.12) is lower bounded by $c\exp(-CL^{1/2})$ ($\beta = 1$) or c ($\beta = \infty$) by Proposition 9.1, and the RHS of (12.12) is in the interval (12.13), we conclude that (12.12) equals $(1 + o(1)) \max_{\vec{b}\in\Theta_{DL^{1/4}}} \mathbb{P}(\mathcal{E}_{\vec{b}})$, which is independent of $\vec{x}$. ☐

It remains to prove the comparison lemma on Brownian bridges.

Proof of Lemma 12.9. We note that the events $\mathcal{E}_{\vec{b}}$ and $\mathcal{E}_{\vec{g}}$ are measurable with respect to the processes $B^{\dagger,i,\vec{b}}$ in the interval $I_{M,L}$. Therefore, conditional on that each $B^{\dagger,i,\vec{b}}$ and $B^{\dagger,i,\vec{g}}$ are the same at $\pm ML^{-1/2}\log L$, the conditional probabilities of $\mathcal{E}_{\vec{b}}$ and $\mathcal{E}_{\vec{g}}$ would be the same.

We let BdyCtrl be the event where for each $\dagger = \text{top}$, $i = 1, \ldots, k-1$, and $\dagger = \text{bot}$, $i = 2, \ldots, k$, we have

$$|B^{\dagger,i,\vec{b}}(ML^{-1/2}\log L) - b_i^{\dagger,\mathrm{R}}\theta^{-1}ML^{-1/2}\log L + 2M\log L| < K\log L,$$
$$|B^{\dagger,i,\vec{b}}(-ML^{-1/2}\log L) - b_i^{\dagger,\mathrm{L}}\theta^{-1}ML^{-1/2}\log L + 2M\log L| < K\log L,$$
$$|B^{\dagger,i,\vec{g}}(ML^{-1/2}\log L) - b_i^{\dagger,\mathrm{R}}\theta^{-1}ML^{-1/2}\log L + 2M\log L| < K\log L,$$
$$|B^{\dagger,i,\vec{g}}(-ML^{-1/2}\log L) - b_i^{\dagger,\mathrm{L}}\theta^{-1}ML^{-1/2}\log L + 2M\log L| < K\log L.$$

It follows from Gaussian tail bounds (Lemma 2.19) (and noting that $\|\vec{b} - \vec{g}\| \le KL^{1/4}$) that

$$\mathbb{P}(\mathrm{BdyCtrl}^c) \le \exp\left(-cK^2M^{-1}L^{1/2}\log L\right).$$

It then suffices to control the ratio of the probability densities of $B^{\dagger,i,\vec{b}}(\pm ML^{-1/2}\log L)$ and $B^{\dagger,i,\vec{g}}(\pm ML^{-1/2}\log L)$, in the interval $-ML^{1/2}\log L + [-K\log L, K\log L]$. For this, note that $B^{\dagger,i,\vec{b}}(\pm ML^{-1/2}\log L)$ and $B^{\dagger,i,\vec{g}}(\pm ML^{-1/2}\log L)$ are Guassian random variables with the same variance of order $ML^{-1/2}\log L$, and mean differ by $\le K\theta^{-1}ML^{-1/4}\log L$. Therefore the density ratio is

$$\exp(O((K\log L)(K\theta^{-1}ML^{-1/4}\log L)(ML^{-1/2}\log L)^{-1})) = 1 + O(K^2L^{-1/4}\log L).$$

Therefore the conclusion follows. $\square$

Appendix A. Weak convergence lemma

Proof of Lemma 11.2. Fix $M > 0$. We then have that

$$\max_{\|\vec{x}\|_\infty,\|\vec{y}\|_\infty\le M,\|\vec{x}-\vec{y}\|_\infty\le\epsilon}\left|\frac{f(\vec{x})}{f(\vec{y})}-1\right| \to 1. \tag{A.1}$$

This follows from the continuity and strict positivity of f combined with the compactness of $[-M,M]^d$. Now we take any $\delta > 0$, and $\vec{x},\vec{y} \in [-M+\delta, M-\delta]^d$. For any ε such that $\delta\varepsilon^{-1}\in\mathbb{N}$, by splitting $\vec{x}+[-\delta,\delta)^d$ into $(\delta\varepsilon^{-1})^d$ many translations of $[-\varepsilon,\varepsilon)^d$, applying (11.1) across them, and using (A.1), we have

$$\frac{\mathbb{P}\left(\vec{X}_\varepsilon \in \vec{x}+[-\delta,\delta)^d\right)}{\mathbb{P}\left(\vec{X}_\varepsilon \in \vec{y}+[-\delta,\delta)^d\right)} \to \frac{\int_{\vec{x}+[-\delta,\delta)^d} f(\vec{z})\,\mathrm{d}\vec{z}}{\int_{\vec{y}+[-\delta,\delta)^d} f(\vec{z})\,\mathrm{d}\vec{z}},$$

as $\epsilon\to 0$. Then by the continuity of f, and that $\vec{X}_\epsilon \to \vec{X}$ in distribution, we conclude that

$$\frac{\mathbb{P}\left(\vec{X} \in \vec{x}+[-\delta,\delta)^d\right)}{\mathbb{P}\left(\vec{X} \in \vec{y}+[-\delta,\delta)^d\right)} = \frac{\int_{\vec{x}+[-\delta,\delta)^d} f(\vec{z})\,\mathrm{d}\vec{z}}{\int_{\vec{y}+[-\delta,\delta)^d} f(\vec{z})\,\mathrm{d}\vec{z}}.$$

We note that by sending $M\to\infty$, this holds for arbitrary $\vec{x},\vec{y}\in\mathbb{R}^d$. Then by the integrability of f, we get

$$\mathbb{P}\left(\vec{X} \in \vec{x}+[-\delta,\delta)^d\right) = \frac{\int_{\vec{x}+[-\delta,\delta)^d} f(\vec{z})\,\mathrm{d}\vec{z}}{\int_{\mathbb{R}^d} f(\vec{z})\,\mathrm{d}\vec{z}}.$$

Then the conclusion follows by sending $\delta\to 0$. $\square$

Appendix B. Tent Brownian comparison and estimates

In this appendix, we provide the proofs of Theorem 2.15, Lemma 2.16, and Lemma 2.17.

Proof of Theorem 2.15. The case where the conditioning is on $\hat{\mathfrak{h}}^\beta_{t,1}(0)\in(L, L+\mathrm{d}L)$ is exactly [GH22, Theorem 9]. We now prove the other case using the known case. We wish to apply the tail comparison of Theorem 4.1; however, the proof of Theorem 4.1 we give uses Theorem 2.15, and we next explain how circular arguments are avoided.

First, recall the notation $L_M^{1/2} = (L-M)^{1/2}$ from the proof of Theorem 4.1. That proof used Theorem 2.15 in the special case of

$$\mathbb{P}\left(\max_{x\in\{-L_M^{1/2},L_M^{1/2}\}}\left|\hat{\mathfrak{h}}^\beta_{t,1}(x) - L + 2L^{1/2}|x|\right| > ML^{1/4} \;\middle|\; \hat{\mathfrak{h}}^\beta_{t,1}(0) > L\right) < \exp(-cM^2),$$

for $M = \log L$, i.e., the process is considered only at the two points $\pm L_M^{1/2}$ and not on the entire interval $[-L^{1/2}, L^{1/2}]$; this was done in the proof of Theorem 4.1 right before (4.8). This application of Theorem 2.15 with this value of M is the only one in the proof of Theorem 4.1. We will show how to prove the above display, and then we will be allowed to use Theorem 4.1 to prove the full version of Theorem 2.15 under the conditioning that $\hat{\mathfrak{h}}^\beta_{t,1}(0) > L$.

Call the event in the previous display $A_{M,L}$. Take a large $C_1 > 0$, then the previous display is bounded by

$$\mathbb{P}\left(A_{M,L}, \hat{\mathfrak{h}}^{\beta}_{t,1}(0) < L + C_1 L^{1/4} \mid \hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L\right) + \mathbb{P}\left(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L + C_1 L^{1/4} \mid \hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L\right). \tag{B.1}$$

The second term is upper bounded, using a trivial upper bound and then Theorem 2.11, by

$$\frac{\mathbb{P}\left(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L + C_1 L^{1/4}\right)}{\mathbb{P}\left(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L\right)} \leq \frac{\exp\left(-\frac{4}{3}(L + C_1 L^{1/4})^{3/2} + CL^{3/4}\right)}{\exp\left(-\frac{4}{3}L^{3/2} - CL^{3/4}\right)} \leq \exp\left(-cL^{3/4}\right), \tag{B.2}$$

as C_1 is large. The first term of (B.1) is upper bounded by

$$\mathbb{P}\left(A_{M,L} \;\middle|\; \hat{\mathfrak{h}}^{\beta}_{t,1}(0) \in (L, L + C_1 L^{1/4})\right) \leq \sup_{L' \in (L, L + C_1 L^{1/4})} \mathbb{P}\left(A_{M,L} \;\middle|\; \hat{\mathfrak{h}}^{\beta}_{t,1}(0) \in (L', L' + \mathrm{d}L')\right).$$

Now applying the known case of Theorem 2.15 where we condition on the value of $\hat{\mathfrak{h}}^{\beta}_{t,1}(0)$ yields that the final term is upper bounded by $\exp(-cM^2)$ when $0 < M < cL^{3/4}$. Thus the bound on (B.1) is $\exp(-cM^2) + \exp(-cL^{3/4}) \leq \exp(-cM^2)$ when $M < L^{3/8}$, as is certainly the case for $M = \log L$.

This establishes the needed bound on (B.1) and so we may now make use of Theorem 4.1, as noted above. We now perform the same analysis as above but with

$$\tilde{A}_{M,L} = \left\{ \sup_{|x| \leq L^{1/2}} \left| \hat{\mathfrak{h}}^{\beta}_{t,1}(x) - L + 2L^{1/2}|x| \right| > ML^{1/4} \right\}$$

replacing $A_{M,L}$ and with $M^2 L^{-1/2}$ replacing $C_1 L^{1/4}$; the only difference is that the bound in (B.2) now follows from Theorem 4.1 instead of Theorem 2.11 (and is now $\exp(-cM^2)$ instead of $\exp(-cL^{3/4})$). The remaining analysis is the same and yields an overall bound on the LHS of (B.1) of $\exp(-cM^2)$ for $0 < M < cL^{3/4}$. ☐

Proof of Lemma 2.16. Let $\mathcal{E}_0$ denote the event where $\hat{\mathfrak{h}}^{\beta}_{t,1}(0) \in (L, L + \mathrm{d}L)$. Denote by $\mathcal{E}_1$ the event

$$\hat{\mathfrak{h}}^{\beta}_{t,2}(x) < -x^2 + 0.1L, \quad \forall x \in [-L^{1/2}/2, L^{1/2}/2],$$

and by $\mathcal{E}_2$ the event where

$$\hat{\mathfrak{h}}^{\beta}_{t,1}(-L^{1/2}/2), \hat{\mathfrak{h}}^{\beta}_{t,1}(L^{1/2}/2) > -0.1L.$$

By Lemma 2.10 and Lemma 3.2, we have

$$\mathbb{P}(\mathcal{E}_1 \mid \mathcal{E}_0) > \mathbb{P}\left(\hat{\mathfrak{h}}^{\beta}_{t,1}(x) < -x^2 + 0.1L, \ \forall x \in [-\tfrac{1}{2}L^{1/2}, \tfrac{1}{2}L^{1/2}]\right) > 1 - C\exp(-cL^{3/2}).$$

By Theorem 2.15 we have $\mathbb{P}(\mathcal{E}_2 \mid \mathcal{E}_0) > 1 - C\exp(-cL^{3/2})$.

We now assume the event $\mathcal{E}_0 \cap \mathcal{E}_1 \cap \mathcal{E}_2$, and consider the Radon-Nikodym derivative between the two sets of processes. Let $\mathcal{F} = \mathcal{F}_{\mathrm{ext}}([-t^{2/3}L^{1/2}/2, t^{2/3}L^{1/2}/2])$ be the sigma-algebra generated by $\hat{\mathfrak{h}}^{\beta}_{t,1}$ on $(-\infty, -L^{1/2}/2] \cup \{0\} \cup [L^{1/2}/2, \infty)$ and $\hat{\mathfrak{h}}^{\beta}_{t,2}$. Then according to the Gibbs property (Lemma 2.7), it suffices to bound

$$Z^{-1} W(B, \mathfrak{h}^{\beta}_{t,2}), \tag{B.3}$$

where

- $\hat{B} : [-L^{1/2}/2, L^{1/2}/2] \to \mathbb{R}$ is a rate 2 Brownian bridge, conditional on $\hat{B}(-L^{1/2}/2) = \hat{\mathfrak{h}}^{\beta}_{t,1}(-L^{1/2}/2)$, $\hat{B}(0) = L$, $\hat{B}(L^{1/2}/2) = \hat{\mathfrak{h}}^{\beta}_{t,1}(L^{1/2}/2)$; and $B : [-t^{2/3}L^{1/2}/2, t^{2/3}L^{1/2}/2] \to \mathbb{R}$ satisfies that $\hat{B}(x) = t^{-1/3}B(t^{2/3}x)$;
- $W(B, \mathfrak{h}^{\beta}_{t,2}) < 1$ is the weight defined through (2.6), in the interval $[-t^{2/3}L^{1/2}/2, t^{2/3}L^{1/2}/2]$;
- $Z = \mathbb{E}[W(B, \mathfrak{h}^{\beta}_{t,2}) \mid \mathcal{F}]$ is a renormalization constant.

Assuming $\mathcal{E}_0 \cap \mathcal{E}_1 \cap \mathcal{E}_2$, we have $\mathbb{P}(\hat{B}(x) \geq 0.2L, \forall x \in [-L^{1/2}/2, L^{1/2}/2] \mid \mathcal{F}) > 1 - C\exp(-cL^2)$; and whenever $\inf_{x\in[-L^{1/2}/2,L^{1/2}/2]} \hat{B}(x) \geq 0.2L$ and also under $\mathcal{E}_1$, we have

$$\mathfrak{h}^{\beta}_{t,2}(t^{2/3}x) - B(t^{2/3}x) > t^{1/3} \cdot 0.1L, \quad \forall x \in [-L^{1/2}/2, L^{1/2}/2].$$

Therefore, from (2.6) we have

$$W(B, \mathfrak{h}^{\beta}_{t,2}) > 1 - 2t^{2/3}L^{1/2}\exp(-t^{1/3} \cdot 0.1L) > 1 - C\exp(-cL).$$

Thus we have $Z > 1 - C\exp(-cL)$, and (B.3) is $1 + O(\exp(-cL))$ under $\mathcal{E}_0 \cap \mathcal{E}_1 \cap \mathcal{E}_2$. This combined with the estimates on $\mathbb{P}(\mathcal{E}_1 \mid \mathcal{E}_0)$ and $\mathbb{P}(\mathcal{E}_2 \mid \mathcal{E}_0)$ leads to the conclusion. $\square$

We next prove Lemma 2.17.

Proof of Lemma 2.17. For the first bound, by Theorem 2.15 we have that

$$\mathbb{P}\left(|\hat{\mathfrak{h}}^{\beta}_{t,1}(\pm L^{1/2}/2)| > ML^{1/4}/5 \;\middle|\; \hat{\mathfrak{h}}^{\beta}_{t,1}(0) \in (L, L + \mathrm{d}L)\right) < \exp(-cM^2).$$

Then the bound follows from Lemma 2.16 and standard Brownian bridge estimates.

For the second bound, we use a strategy similar to the proof of the $\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L$ case of Theorem 2.15. Namely, the LHS is bounded by

$$\sup_{L' \in (L, L+M\sigma_I/2)} \mathbb{P}\left(\sup_{x\in I}\left|\hat{\mathfrak{h}}^{\beta}_{t,1}(x) - (L - 2L^{1/2}|x|)\right| > M\sigma_I \;\middle|\; \hat{\mathfrak{h}}^{\beta}_{t,1}(0) \in (L', L' + \mathrm{d}L')\right)$$
$$+ \mathbb{P}\left(\hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L + M\sigma_I/2 \;\middle|\; \hat{\mathfrak{h}}^{\beta}_{t,1}(0) > L\right).$$

Then by the first bound and Theorem 4.1, the second bound follows. $\square$

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