# MA 17: HOW TO SOLVE IT
# HANDOUT 5: DISCUSSIONS AFTER MOCK 2

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*Elements in the theory of matrix: trace, determinant, eigenvalues, eigenvectors, characteristic equation/polynomial.*

**The Cayley–Hamilton Theorem.** Any $n \times n$ matrix $A$ satisfies its characteristic equation, which means that if $P_A(\lambda) = \det(\lambda I_n - A)$, then $P_A(A) = O_n$.

**The Perron–Frobenius Theorem.** Any square matrix with positive entries has a unique eigenvector with positive entries (up to multiplication by a positive scalar), and the corresponding eigenvalue has multiplicity one and is strictly greater than the absolute value of any other eigenvalue.

*Some more problems:*

**Problem 1.** (Putnam 1986, B6)
Suppose that $A, B, C, D$ are $n \times n$ matrices (with entries in some field), such that $AB^\mathsf{T}$ and $CD^\mathsf{T}$ are symmetric, and $AD^\mathsf{T} - BC^\mathsf{T} = I$. Prove that $A^\mathsf{T}D - C^\mathsf{T}B = I$.

*Hint: find a more "matricial" interpretation of the condition $AD^\mathsf{T} - BC^\mathsf{T} = I$.*

**Problem 2.** (Putnam 1995, A5)
Let $x_1, x_2, \ldots, x_n$ be differentiable (real-valued) functions of a single variable $f$ which satisfy

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
\frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
&\vdots \qquad\qquad\qquad\qquad \vdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n
\end{aligned}$$

for some constants $a_{ij} > 0$. Suppose that for all $i$, $x_i(t) \to 0$ as $t \to \infty$. Are the functions $x_1, x_2, \ldots, x_n$ necessarily linearly dependent?

**Problem 3.** Consider bipartite graphs where the vertices on each side are labeled $\{1, 2, \ldots, n\}$. Find the number of such graphs for which there are an odd number of perfect matchings.

**Problem 4.** (Putnam 1999, B5)
For an integer $n \geq 3$, let $\theta = 2\pi/n$. Evaluate the determinant of the $n \times n$ matrix $I + A$, where $I$ is the $n \times n$ identity matrix and $A = (a_{jk})$ has entries $a_{jk} = \cos(j\theta + k\theta)$ for all $j, k$.

*Hint: find the eigenvectors, and use complex numbers.*

*Another problems on complex numbers:*

**Problem 5.** (Putnam 1991, B2)
Suppose $f$ and $g$ are non-constant, differentiable, real-valued functions defined on $(-\infty, \infty)$. Furthermore, suppose that for each pair of real numbers $x$ and $y$,

$$f(x+y) = f(x)f(y) - g(x)g(y),$$
$$g(x+y) = f(x)g(y) + g(x)f(y).$$

If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all $x$.

**Reminder** PSet 5 to be released tomorrow, and due by the end of Nov 24 (on Canvas). The fourth problem session (to discuss PSet 4) will be on Nov 13, 7-8 PM, at the same location.