MA 17: HOW TO SOLVE IT
HANDOUT 3: DISCUSSIONS AFTER MOCK 1

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Base numbers. (left from last time)
For any integer $g > 1$, every positive integer can be uniquely written as

$$a_n g^n + a_{n-1} g^{n-1} + \cdots + a_1 g + a_0,$$

where each $a_i \in \{0, 1, \ldots, g-1\}$ and $a_n \neq 0$.
(For example, $g = 2$ gives the binary representation, and $g = 10$ gives the decimal representation.)

Problem 8. (Putnam 2020, B1)
For a positive integer n, define $d(n)$ to be the sum of the digits of n when written in binary (for example, $d(13) = 1 + 1 + 0 + 1 = 3$). Let

$$S = \sum_{k=1}^{2020} (-1)^{d(k)} k^3.$$

Determine S modulo 2020.

*Consider an **invariant** under a given transformation. It may be a number, an algebraic structure, or a certain property.*
*In some cases, one may instead consider a **monovariant**, which is a quantity that, although not constant under a specific transformation, keeps increasing (or decreasing).*

Problem 1.
There is a heap of 1001 stones on a table. You are allowed to perform the following operation: you choose one of the heaps containing more than one stone, throw away a stone from the heap, then divide it into two smaller (not necessarily equal) heaps. Is it possible to reach a situation in which all the heaps on the table contain exactly 3 stones by performing the operation finitely many times?

Problem 2. (Russian Mathematical Olympiad, 1995)
Four congruent right triangles are given. One can cut one of them along the altitude and repeat the operation several times with the newly obtained triangles. Prove that no matter how we perform the cuts, we can always find among the triangles two that are congruent.

Problem 3.
A real number is written in each square of an $n \times n$ chessboard. We can perform the operation of changing all signs of the numbers in a row or a column. Prove that by performing this operation a finite number of times we can produce a new table for which the sum of each row or column is non-negative.

Another problem on the sum of numbers.
Problem 4. (Putnam 1993, A4)
Let $x_1, x_2, \ldots, x_{19}$ be positive integers each of which is less than or equal to 93. Let $y_1, y_2, \ldots, y_{93}$

be positive integers each of which is less than or equal to 19. Prove that there exists a (nonempty) sum of some x_i's equal to a sum of some y_j's.

Another problem on recurrence.
Problem 5. (Putnam 2006, B6)
Let k be an integer greater than 1. Suppose $a_0 > 0$, and define

$$a_{n+1} = a_n + \frac{1}{\sqrt[k]{a_n}}$$

for $n > 0$. Evaluate

$$\lim_{n\to\infty} \frac{a_n^{k+1}}{n^k}.$$

Reminder PSet 3 to be released tomorrow, and due by the end of Oct 29 (on Canvas). No problem session this week.