Interlacing adjacent levels of β-Jacobi corners processes

Vadim Gorin [*] Lingfu Zhang [†]

Abstract

We study the asymptotic of the global fluctuations for the difference between two adjacent levels in the β–Jacobi corners process (multilevel and general β extension of the classical Jacobi ensemble of random matrices). The limit is identified with the derivative of the $2d$ Gaussian Free Field. Our main tool is integral forms for the (Macdonald-type) difference operators originating from the shuffle algebra.

Contents

1 Introduction **2**

2 Background and setup **4**
- 2.1 β-Jacobi corners process . . . 4
- 2.2 Signed measures and their diagrams . . . 5
- 2.3 Pullback of Gaussian Free Field . . . 6

3 Main results **8**

4 Dimension reduction **13**

5 Discrete joint moments **15**
- 5.1 Macdonald processes and asymptotic relations . . . 16
- 5.2 Differential operator . . . 17
- 5.3 Joint higher order moments . . . 22

6 Law of large numbers: proofs of Theorems 3.1, 3.2, 3.3, 3.5 **28**
- 6.1 First moment of adjacent rows . . . 29
- 6.2 Convergence of diagrams . . . 29
- 6.3 Convergence of discrete signed measures . . . 32
- 6.4 Asymptote of roots of Jacobi polynomials . . . 33

7 Central limit theorem and gaussianity of fluctuation: proofs of Theorems 3.6 and 3.8 **34**
- 7.1 Computation of covariance . . . 35
- 7.2 Gaussian type asymptote of joint moments . . . 39
- 7.3 Gaussianity of discrete levels: proof of Theorem 3.6 . . . 44
- 7.4 Gaussianity of integral over levels: proof of Theorem 3.8 . . . 45

[*]Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA, USA and Institute for Information Transmission Problems of Russian Academy of Sciences, Moscow, Russia. e-mail: vadicgor@gmail.com

[†]Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA, USA, e-mail: lfzhang@mit.edu

8 Connecting the limit field with Gaussian Free Field: proofs of Theorems 3.11 and 3.16 **46**
8.1 Identification of 1-dimensional integral 47
8.2 Identification of 2-dimensional integral 49

1 Introduction

A prototypical setup in the random matrix theory is to take an $N \times N$ Hermitian matrix X and to study its eigenvalues $x_1 < \cdots < x_N$ as $N \to \infty$. One observable of interest is the linear statistics

$$\mathfrak{L}_f := \sum_{i=1}^{N} f(x_i), \tag{1}$$

for suitable (usually smooth) functions f. In many cases $\frac{1}{N}\mathfrak{L}_f$ converges to a constant as $N \to \infty$, while $\mathfrak{L}_f - \mathbb{E}\mathfrak{L}_f$ is asymptotically Gaussian, see e.g. the textbooks [AGZ10], [PS11], [For10]. Such limit results are usually referred to as the Law of Large Numbers and the Central Limit theorem for the *global fluctuations* of X.

The asymptotic covariance for $\mathfrak{L}_f$ is best understood through the *corners processes* — a 2d extension obtained by looking at the joint distribution of the eigenvalues of all principal corners of X. In more details, let $x_1^k < x_2^k < \cdots < x_k^k$ be eigenvalues of the $k \times k$ top–left corner of X, $k = 1, 2, \dots, N$. The global fluctuations of the array $\{x_i^j\}_{1 \le i \le j \le N}$ as $N \to \infty$ can be then described by a pullback of the two–dimensional Gaussian Free Field, as was proven in [Bor14], [BG15], [DP14], [JP14], [GP14] for numerous ensembles of random matrix theory: Wigner, Wishart, β–Jacobi, adjacency matrices of random graphs.

The corners processes also pave a way for a sequential construction of $\mathfrak{L}_f$. Define

$$\mathfrak{L}'_f(k) = \left(\sum_{i=1}^{k} f(x_i^k) \right) - \left(\sum_{i=1}^{k-1} f(x_i^{k-1}) \right), \tag{2}$$

then clearly $\mathfrak{L}_f = \sum_{k=1}^{N} \mathfrak{L}'_f(k)$. The aforementioned Central Limit Theorems for the corners processes imply that for any $0 < \alpha_1 < \cdots < \alpha_m < 1$, and smooth functions $f_1, \dots, f_m$ the m–dimensional vector

$$\sum_{k=1}^{\lfloor N\alpha_i \rfloor} (\mathfrak{L}'_{f_i}(k) - \mathbb{E}\mathfrak{L}'_{f_i}(k)), \quad i = 1, \cdots, m, \tag{3}$$

is asymptotically Gaussian and its covariance can be identified with the joint covariance of certain integrals of the Gaussian Free Field.

One way to analyze the sums (3) is through the asymptotic of its individual terms, $\mathfrak{L}'_f(k) - \mathbb{E}\mathfrak{L}'_f(k)$. As far as the authors know, such analysis in the setting of the Central Limit Theorem escaped the attention up until recently and it is the main topic of the present article. Let us however note that in the Law of Large Numbers context, $\mathfrak{L}'_f(k)$ was previously considered for Wigner matrices by Kerov [Ker93], [Ker94], [Ker98] and Bufetov [Buf13] leading to interesting connections with orthogonal polynomials and random partitions.

The stochastic system that we work with is the β–Jacobi corners process, first introduced in [BG15]. This is a random array of particles split into levels, and such that the distribution of particles at level N can be identified with the classical β–Jacobi ensemble of random matrices.

The special values of the parameter $\beta = 1, 2, 4$ arise when considering real, complex or quaternion matrices and for such values of the parameter the Jacobi corners process can be identified with eigenvalues of the MANOVA ensemble $A^*A(A^*A + B^*B)^{-1}$ with rectangular Gaussian matrices A and B that vary with N, see [BG15, Section 1.5] and [Sun16] for the details. More generally, extrapolating from $\beta = 1, 2, 4$ cases, the definition also makes sense for any value of $\beta > 0$, see Section 2.1 for the details.

We study $\mathfrak{L}'_f(k)$ in two separate asymptotic regimes: for individual $k = \lfloor yN \rfloor$ as $N \to \infty$, and in the integrated form by averaging $\mathfrak{L}'_f(\lfloor yN \rfloor)$ with a smooth weight function on y. These two regimes have very different behaviors. For the first one, $\mathfrak{L}'_f(\lfloor yN \rfloor)$ converges as $N \to \infty$ to a constant (depending on f, see Theorem 3.3), while $\mathfrak{L}'_f(\lfloor yN \rfloor) - \mathbb{E}\mathfrak{L}'_f(\lfloor yN \rfloor)$ decays as $N^{-1/2}$ and becomes asymptotically Gaussian upon rescaling. Somewhat surprisingly, for $y \neq y'$ the random variables $N^{1/2}\left(\mathfrak{L}'_f(\lfloor yN \rfloor) - \mathbb{E}\mathfrak{L}'_f(\lfloor yN \rfloor)\right)$ and $N^{1/2}\left(\mathfrak{L}'_f(\lfloor y'N \rfloor) - \mathbb{E}\mathfrak{L}'_f(\lfloor y'N \rfloor)\right)$ are asymptotically independent, see Theorem 3.6 for the exact statement and details. The scaling is also different for the second limit regime, as the weighted averages of the form

$$\int_0^1 g(y)\left(\mathfrak{L}'_f(\lfloor yN \rfloor) - \mathbb{E}\mathfrak{L}'_f(\lfloor yN \rfloor)\right) dy \tag{4}$$

decay as N^{-1} and become Gaussian upon rescaling, see Theorem 3.8 for the exact statement and details.

The results in both scalings are best understood if we recall the main theorem of [BG15]: $\mathfrak{L}_f(\lfloor yN \rfloor) - \mathbb{E}\mathfrak{L}_f(\lfloor yN \rfloor)$ is asymptotically Gaussian (jointly in several y's and f's), and the limit can be identified with the integral of a generalized Gaussian field, which in turn is a pullback of the Gaussian Free Field. Then our results yield that $\mathfrak{L}'_f(\lfloor yN \rfloor) - \mathbb{E}\mathfrak{L}'_f(\lfloor yN \rfloor)$ converges to the y–derivative of this generalized Gaussian field. Rigorously speaking, the field is not differentiable in y direction, and this is what leads to the appearance of two scalings; we make this connection more precise in Theorems 3.11, 3.16.

From this perspective, our main results strengthen the convergence of $\mathfrak{L}_f(\lfloor yN \rfloor) - \mathbb{E}\mathfrak{L}_f(\lfloor yN \rfloor)$ to the pullback of the Gaussian Free Field up to the convergence of the derivatives in y–direction. We emphasize that there is no a priori reason why such an upgrade for the CLT should hold. Indeed, in a parallel work [ES16] Erdős and Schröder show that this is not the case for general Wigner matrices; in that article the limit might even fail to be Gaussian.

On a more technical side, our result can be linked to an observation that the Gaussian convergence of $\mathfrak{L}_f(\lfloor yN \rfloor) - \mathbb{E}\mathfrak{L}_f(\lfloor yN \rfloor)$ is faster than it might have been: the cumulants (of order 3 and greater) decay much faster than just $o(1)$, see Proposition 7.6 for the details. In several $2d$ stochastic systems, which have no direct connection with our setup, but also lead to the asymptotic appearance of the Gaussian Free Field, somewhat similar *strong Central Limit Theorems* were observed e.g. in [CS14], [BBNY16].

For the proofs, we adopt parts of the methodology of [BG15] and exploit the fact that the β–Jacobi process is a particular case of a Heckman–Opdam process, which, in turn, is a limit of Macdonald processes of [BC14], [BCGS16]. A critical new ingredient is the use of a family of difference operators arising from the work of Negut [Neg13], [Neg14] (see also [FHH$^+$09]) on Macdonald operators and shuffle algebra. These operators were first introduced by Borodin and the first author in the appendix to [FD16] (without detailed proofs), and here we further develop them to the full power.

Independently of the present article, the Central Limit Theorem for $\mathfrak{L}'_f(k)$ in the context of Wigner matrices was also considered recently by Erdős–Schröder [ES16] and by Sodin [Sod16].

Since these authors consider only the asymptotic for single k, the link to the Gaussian Free Field is less visible there, although we believe that it should be also present (at least in the case when the Wigner matrices have Gaussian entries, i.e. for GOE, GUE, GSE). Despite the connections, our setup is quite different from [ES16], [Sod16]. In particular, these papers rely on matrix models and independence of matrix elements; we do not know how to extend such an approach to our settings of β–Jacobi corners process. And in the opposite direction: Although there is a simple and well-known limit from $\beta = 1, 2, 4$ Jacobi corners process to GOE/GUE/GSE, but it is non-trivial to perform such a limit transition in the exact moment formulas that we use for the asymptotic analysis.

Acknowledgments

The authors would like to thank Alexei Borodin and Alexey Bufetov for helpful discussions. V.G. was partially supported by the NSF grant DMS-1407562 and by the Sloan Research Fellowship. L.Z. was partially supported by Undergraduate Research Opportunity Program (UROP) in Massachusetts Institute of Technology.

2 Background and setup

2.1 β-Jacobi corners process

Definition 2.1. The *K-particle Jacobi ensemble* is a probability distribution on K-tuples of real numbers $0 \le x_1 < \cdots < x_K \le 1$ with density (with respect to Lebesgue measure) proportional to

$$\prod_{1 \le i < j \le K} (x_i - x_j)^\beta \prod_{i=1}^{K} x_i^p (1 - x_i)^q, \tag{5}$$

where $\beta > 0, p, q > -1$ are real parameters.

This is the distribution of eigenvalues of the MANOVA ensemble of random matrices. Specifically, consider two infinite matrices X_{ij} and Y_{ij}, $i, j = 1, 2, \cdots$ where entries are i.i.d. real, complex, or quaternion Gaussian, corresponding to $\beta = 1, 2, 4$, respectively. For integers $A \ge M > 0$ and $N > 0$, let X^{AM} be the $A \times M$ top–left corner of X, and Y^{NM} the $N \times M$ top–left corner of Y. For the $M \times M$ matrix,

$$\mathcal{M}^{ANM} = (X^{AM})^* X^{AM} \left((X^{AM})^* X^{AM} + (Y^{NM})^* Y^{NM} \right)^{-1}, \tag{6}$$

almost surely $K = \min(N, M)$ of its M eigenvalues are different from 0 and 1; they are distributed as K-particle Jacobi ensemble, for $\beta = 1, 2, 4$, and $p = \frac{\beta}{2}(A - M + 1) - 1$, $q = \frac{\beta}{2}(|M - N| + 1) - 1$, see e.g. [For10, Section 3.6].

Following [BG15], we further introduce the β–Jacobi corners process by coupling a sequence of Jacobi ensembles. Let χ^M be the set of infinite families of sequences $x^1, x^2, \cdots$, where for each $N \ge 1$, x^N is an increasing sequence with length $\min(N, M)$:

$$0 \le x_1^N < \cdots < x_{\min(N,M)}^N \le 1 \tag{7}$$

and for each $N > 1$, x^N and x^{N-1} interlace:

$$x_1^N < x_1^{N-1} < x_2^N < \cdots. \tag{8}$$

Definition 2.2. The β–Jacobi corners process is a random element of χ^M with distribution $\mathbb{P}^{\alpha,M,\theta}$, such that the sequence x^N, $N = 1, 2, \dots$ is a Markov chain with marginal distribution of a single x^N of density (with respect to Lebesgue measure) proportional to

$$\prod_{1\le i<j\le \min(N,M)} (x_i^N - x_j^N)^{2\theta} \prod_{i=1}^{\min(N,M)} (x_i^N)^{\theta\alpha-1}(1-x_i^N)^{\theta(|M-N|+1)-1}, \tag{9}$$

and conditional distribution of x^{N-1} given x^N of the density given for $N \le M$ by

$$\frac{\Gamma(N\theta)}{\Gamma(\theta)^N}\prod_{i=1}^{N}(x_i^N)^{(N-1)\theta}\prod_{1\le i<j<N}(x_j^{N-1}-x_i^{N-1})\prod_{1\le i<j\le N}(x_j^N-x_i^N)^{1-2\theta} \\ \times\prod_{i=1}^{N-1}\prod_{j=1}^{N}|x_j^N-x_i^{N-1}|^{\theta-1}\prod_{i=1}^{N-1}\frac{1}{(x_i^{N-1})^{N\theta}}, \tag{10}$$

and for $N > M$ by

$$\frac{\Gamma(N\theta)}{\Gamma(\theta)^M\Gamma(N\theta-M\theta)}\prod_{1\le i<j\le M}(x_i^{N-1}-x_j^{N-1})(x_i^N-x_j^N)^{1-2\theta} \\ \times\prod_{j=1}^{M}(x_i^N)^{(N-1)\theta}(1-x_i^N)^{\theta(M-N-1)+1}\prod_{i,j=1}^{M}|x_j^N-x_i^{N-1}|^{\theta-1}\prod_{i=1}^{M}\frac{(1-x_i^{N-1})^{\theta(N-M)-1}}{(x_i^{N-1})^{N\theta}}. \tag{11}$$

The proof that the distribution $\mathbb{P}^{\alpha,M,\theta}$ is well-defined (i.e., that the formulas (9), (10), and (11) agree with each other) can be found in [BG15, Proposition 2.7]. It is based on integral identities due to Dixon [Dix05] and Anderson [And91].

Sun proved in [Sun16, Section 4] that the joint distribution of (different from $0, 1$) eigenvalues in $\mathcal{M}^{AnM}$, $n = 1, \cdots, N$ is the same as the first N rows of β-Jacobi corners process with $\alpha = A-M+1$, and $\theta = \frac{\beta}{2}$, for $\beta = 1, 2$ (correspond to real and complex entries, respectively).

2.2 Signed measures and their diagrams

Our main object of study is a pair of interlacing sequences x^{N-1}, x^N from β–Jacobi corners process. We assign to such pair two closely related objects: a *signed measure* and a *diagram*.

Definition 2.3. Given an interlacing sequence $x_1 \le y_1 \le \cdots \le y_{n-1} \le x_n$, the corresponding *signed interlacing measure* $\mu^{\{x_i\},\{y_i\}}$ is an atomic signed measure on $\mathbb{R}$ of total mass 1 given by

$$\mu^{\{x_i\},\{y_i\}}(A) = \sum_{i=1}^{n}\mathbb{1}_{x_i\in A} - \sum_{i=1}^{n-1}\mathbb{1}_{y_i\in A}, \quad \forall A\subset\mathbb{R}. \tag{12}$$

An alternative way to describe interlacing sequences (due to Kerov [Ker93], see also [Buf13]) relies on the notion of a diagram.

Definition 2.4. A *diagram* $w : \mathbb{R} \to \mathbb{R}$ is a function satisfying:

1. Lipschitz condition: $|w(u_1) - w(u_2)| \le |u_1 - u_2|$, $\forall u_1, u_2 \in \mathbb{R}$.

2. There is $u_0 \in \mathbb{R}$, the *center* of w, such that $w(u) = |u - u_0|$ for $|u|$ large enough.

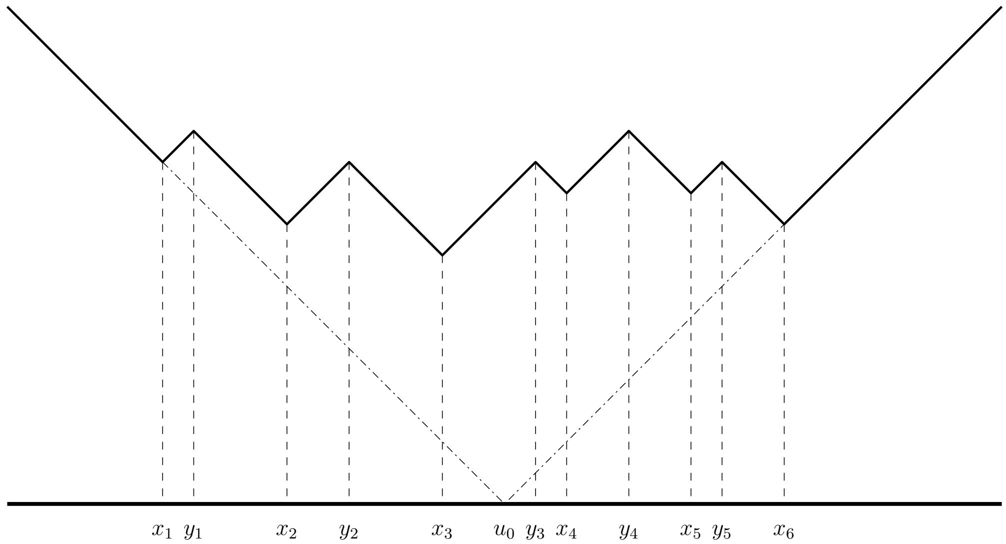


Figure 1: The diagram of an interlacing sequence

Any diagram w that is piecewise linear and satisfies $\frac{d}{du}w = \pm 1$ (except for finitely many points) is called *rectangular*.

We draw a connection between interlacing sequences and diagrams, see Figure 1 for an example.

Definition 2.5. For any interlacing sequence $x_1 \le y_1 \le \cdots \le y_{n-1} \le x_n$, define its diagram $w : \mathbb{R} \to \mathbb{R}$ as follows:

1. For $u \le x_1$ or $u > x_n$, let $w(u) = |u - u_0|$, where $u_0 = \sum_{i=1}^{n} x_i - \sum_{i=1}^{n-1} y_i$.
2. For $i = 1, \cdots, n$, let $w(x_i) = \sum_{1 \le j < i}(y_j - x_j) + \sum_{i < j \le n}(x_j - y_{j-1})$.
3. For $i = 1, \cdots, n-1$, let $w(y_i) = \sum_{1 \le j < i}(y_j - x_j) - x_i + x_{i+1} + \sum_{i+1 < j \le n}(x_j - y_{j-1})$.
4. In all the intervals $[x_i, y_i]$ and $[y_i, x_{i+1}]$, w is linear.

It's easy to verify that the defined w is a rectangular diagram; to be more precise, it satisfies the following conditions:

1. $\frac{d}{du}w(u) = 1$, for any $u \in \left(\bigcup_{i=1}^{n-1}(x_i, y_i)\right) \bigcup (x_n, \infty)$.
2. $\frac{d}{du}w(u) = -1$, for any $u \in \left(\bigcup_{i=1}^{n-1}(y_i, x_{i+1})\right) \bigcup (-\infty, x_1)$.

Remark 2.6. For an interlacing sequence $x_1 < y_1 < \cdots < y_{n-1} < x_n$, and its diagram w, the second derivative $\frac{d^2}{du^2}w$ can be identified with $2\mu^{\{x_i\},\{y_i\}}$.

2.3 Pullback of Gaussian Free Field

In this section we briefly define a pullback of the Gaussian Free Field, and review the results of [BG15] about the appearance of GFF in the asymptotic of β-Jacobi corners process.

Detailed surveys of the 2–dimensional Gaussian Free Field are given in [She07], [Dub09, Section 4], [Wer14], and here we will omit some details. Informally, the Gaussian Free Field with Dirichlet boundary conditions in the upper half plane $\mathbb{H}$ is defined as a mean 0 (generalized) Gaussian random field $\mathcal{G}$ on $\mathbb{H}$, whose covariance (for any $z, w \in \mathbb{H}$) is

$$\mathbb{E}(\mathcal{G}(z)\mathcal{G}(w)) = -\frac{1}{2\pi} \ln \left| \frac{z-w}{z-\bar{w}} \right|. \tag{13}$$

Since (13) has a singularity at the diagonal $z = w$, the value of the GFF at a point is not defined, however, GFF can be well-defined as an element of a certain functional space. In particular, the integrals of $\mathcal{G}(z)$ against sufficiently smooth measures are bona fide Gaussian random variables.

The next step is to define a correspondence which maps to the upper half–plane the space where particles of β–Jacobi corners process live.

Definition 2.7. Let $D \subset [0,1] \times \mathbb{R}_{>0}$ be defined by the following inequality

$$\left| x - \frac{\hat{M}\hat{N} + (\hat{M}+\hat{\alpha})(\hat{N}+\hat{\alpha})}{(\hat{N}+\hat{\alpha}+\hat{M})^2} \right| \le \frac{2\sqrt{\hat{M}\hat{N}(\hat{M}+\hat{\alpha})(\hat{N}+\hat{\alpha})}}{(\hat{N}+\hat{\alpha}+\hat{M})^2}. \tag{14}$$

Let $\Omega : D \bigcup \{\infty\} \to \mathbb{H} \bigcup \{\infty\}$ be such that the horizontal section of D at height $\hat{N}$ is mapped to the half-plane part of the circle, centered at

$$\frac{\hat{N}(\hat{\alpha}+\hat{M})}{\hat{N}-\hat{M}} \tag{15}$$

with radius

$$\frac{\sqrt{\hat{M}\hat{N}(\hat{M}+\hat{\alpha})(\hat{N}+\hat{\alpha})}}{\left|\hat{N}-\hat{M}\right|} \tag{16}$$

(when $\hat{N} = \hat{M}$ the circle is replaced by the vertical line at $\frac{\hat{\alpha}}{2}$), and point $u \in \mathbb{H}$ is the image of

$$\left(\frac{u}{u+\hat{N}} \cdot \frac{u-\hat{\alpha}}{u-\hat{\alpha}-\hat{M}}, \hat{N} \right). \tag{17}$$

One proves (see [BG15, Section 4.6]) that Ω is a bijection between $D \bigcup \{\infty\}$ and $\mathbb{H} \bigcup \{\infty\}$.

Definition 2.8. $\mathcal{K}$ is a generalized Gaussian random field in $[0,1] \times \mathbb{R}_{\ge 0}$ which is 0 outside D and is equal to $\mathcal{G} \circ \Omega$ (i.e. the pullback of $\mathcal{G}$ with respect to map Ω) inside D.

Again, the value of $\mathcal{K}$ at a given point in D is not well-defined, but it can be integrated with respect to certain types of measures; specifically, we have the following results, which can essentially be taken as an alternative definition of $\mathcal{K}$, cf. [BG15, Sections 4.6].

Lemma 2.9. *For any positive integers $k_1, \cdots, k_h$ and $0 < \hat{N}_1 \le \cdots \le \hat{N}_h$, the following random vector*

$$\left(\int_0^1 u^{k_i} \mathcal{K}(u, \hat{N}_i) du \right)_{i=1}^{h} \tag{18}$$

is jointly centered Gaussian, and the covariance between the ith and jth is

$$\frac{\theta^{-1}}{(2\pi\mathbf{i})^2(k_i+1)(k_j+1)} \oint\oint \frac{dv_1 dv_2}{(v_1-v_2)^2} \times \left(\frac{v_1}{v_1+\hat{N}_i} \cdot \frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}} \right)^{k_i+1} \left(\frac{v_2}{v_2+\hat{N}_j} \cdot \frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}} \right)^{k_j+1}, \tag{19}$$

where $|v_1| \ll |v_2|$, and the contours enclose $-\hat{N}_i$, $-\hat{N}_j$, but not $\hat{\alpha}+\hat{M}$.

Lemma 2.10. *For any integers $k_1, \cdots, k_h$, and $g_1, \cdots, g_h \in C^\infty([0,1])$, the joint distribution of the vector*

$$\left(\int_0^1 \int_0^1 u^{k_i} g_i(y) \mathcal{K}(u,y) du dy \right)_{i=1}^h \tag{20}$$

is centered Gaussian, and the covariance between the ith and jth component is

$$\int_0^1 \int_0^1 \frac{g_i(y_1) g_j(y_2) \theta^{-1}}{(2\pi \mathbf{i})^2 (k_i+1)(k_j+1)} \oint \oint \frac{1}{(v_1 - v_2)^2} \\ \times \left(\frac{v_1}{v_1 + y_1} \cdot \frac{v_1 - \hat{\alpha}}{v_1 - \hat{\alpha} - \hat{M}} \right)^{k_i+1} \left(\frac{v_2}{v_2 + y_2} \cdot \frac{v_2 - \hat{\alpha}}{v_2 - \hat{\alpha} - \hat{M}} \right)^{k_j+1} dv_1 dv_2 dy_1 dy_2, \tag{21}$$

where the inner contours enclose poles at $-y_1$ and $-y_2$, but not $\hat{\alpha} + \hat{M}$, and are nested: when $y_1 \le y_2$, v_2 is larger; when $y_1 \ge y_2$, v_1 is larger.

Let us emphasize once again that since the values of $\mathcal{K}$ are not defined, the expressions (18) and (20) are not conventional integrals, rather they are pairings of a generalized random function $\mathcal{K}$ with certain measures. The formula (21) is obtained from (19) integrating in y–direction.

3 Main results

We proceed to statements of our asymptotic theorems.

In our limit regime the parameters α, M of the β–Jacobi corners process and level N depend on a large auxiliary variable $L \to \infty$, in such a way that

$$\lim_{L\to\infty} \frac{\alpha}{L} = \hat{\alpha}, \quad \lim_{L\to\infty} \frac{N}{L} = \hat{N}, \quad \lim_{L\to\infty} \frac{M}{L} = \hat{M}. \tag{22}$$

For a random $\mathbb{P}^{\alpha,M,\theta}$–distributed sequence $(x^1, x^2, \cdots) \in \chi^M$, we introduce random variables

$$\tilde{x}^N = \begin{cases} x^N, & N \le M \\ (x_1^N, \cdots, x_M^N, \underbrace{1, \cdots, 1}_{N-M}), & N > M \end{cases} \tag{23}$$

and

$$\mathfrak{P}_k(x^N) = \begin{cases} \sum_{i=1}^N (x_i^N)^k, & N \le M \\ \sum_{i=1}^M (x_i^N)^k + N - M, & N > M. \end{cases} \tag{24}$$

The following theorems give the $L \to \infty$ Law of Large Numbers for the pair (x^{N-1}, x^N) in three different forms.

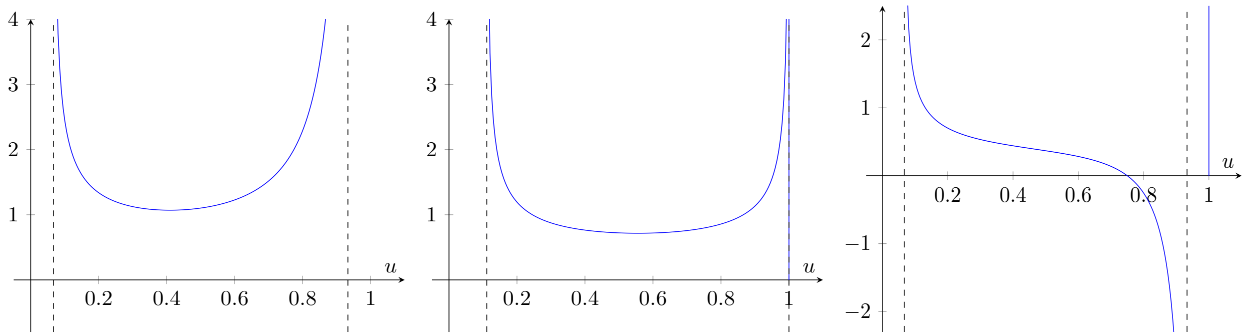
Theorem 3.1. *In the limit regime (22), the random variable $\mathfrak{P}_k(x^N) - \mathfrak{P}_k(x^{N-1})$ converges to a constant as $L \to \infty$, in the sense that the variance*

$$\mathbb{E}\left[(\mathfrak{P}_k(x^N) - \mathfrak{P}_k(x^{N-1})) - \mathbb{E}(\mathfrak{P}_k(x^N) - \mathfrak{P}_k(x^{N-1})) \right]^2, \tag{25}$$

decays in $O(L^{-1})$. The constant is given by the following contour integral:

$$\lim_{L\to\infty} \mathbb{E}\left(\mathfrak{P}_k(x^N) - \mathfrak{P}_k(x^{N-1}) \right) = \frac{1}{2\pi \mathbf{i}} \oint \left(\frac{v}{v + \hat{N}} \cdot \frac{v - \hat{\alpha}}{v - \hat{\alpha} - \hat{M}} \right)^k \frac{1}{v + \hat{N}} dv, \tag{26}$$

where the integration contour encloses the pole at $-\hat{N}$ but not $\hat{\alpha} + \hat{M}$, and is positively oriented.



$\hat{M} = 2,\ \hat{N} = \hat{\alpha} = 1$ $\qquad$ $\hat{M} = \hat{N} = \hat{\alpha} = 1$ $\qquad$ $\hat{N} = 2,\ \hat{M} = \hat{\alpha} = 1$

Figure 2: Plots of the density $\frac{\varphi''}{2}$.

Theorem 3.2. *Let $w^{\tilde{x}^N,\tilde{x}^{N-1}}$ be the interlacing diagram of the sequence $\tilde{x}_1^N \le \tilde{x}_1^{N-1} \le \dots \le \tilde{x}_{N-1}^{N-1} \le \tilde{x}_N^N$. Then it converges to a deterministic diagram φ under the limit scheme* (22)*, in the sense that*

$$\lim_{L\to\infty} \sup_{u\in\mathbb{R}} \left| w^{\tilde{x}^N,\tilde{x}^{N-1}}(u) - \varphi(u) \right| = 0, \tag{27}$$

in probability. φ is a unique diagram satisfying

$$\varphi''(u) = \begin{cases} \frac{\hat{M}-\hat{N}+(\hat{N}+\hat{M}+\hat{\alpha})(1-u)}{\pi(\hat{N}+\hat{M}+\hat{\alpha})(1-u)} \frac{1}{\sqrt{(\gamma_2-u)(u-\gamma_1)}}, & u \in (\gamma_1,\gamma_2) \\ 2C(\hat{M},\hat{N})\delta(u-1), & u \in (-\infty,\gamma_1] \bigcup [\gamma_2,\infty), \end{cases} \tag{28}$$

where

$$\gamma_1 = \frac{\left(\sqrt{(\hat{\alpha}+\hat{M})(\hat{\alpha}+\hat{N})} - \sqrt{\hat{M}\hat{N}}\right)^2}{(\hat{N}+\hat{M}+\hat{\alpha})^2}, \qquad \gamma_2 = \frac{\left(\sqrt{(\hat{\alpha}+\hat{M})(\hat{\alpha}+\hat{N})} + \sqrt{\hat{M}\hat{N}}\right)^2}{(\hat{N}+\hat{M}+\hat{\alpha})^2}, \tag{29}$$

$$C(\hat{M},\hat{N}) = \begin{cases} 0, & \hat{M} > \hat{N} \\ \frac{1}{2}, & \hat{M} = \hat{N} \\ 1, & \hat{M} < \hat{N} \end{cases}. \tag{30}$$

Theorem 3.3. *Let φ be defined as in Theorem 3.2. Take a function $f : [0,1] \to \mathbb{R}$, such that f' exists almost everywhere, and is of finite variation. Then the random variable*

$$\int_0^1 f d\mu^{\tilde{x}^N,\tilde{x}^{N-1}} = \sum_{i=1}^{N} f(\tilde{x}_i^N) - \sum_{i=1}^{N-1} f(\tilde{x}_i^{N-1}) \tag{31}$$

converges (in probability) as $L \to \infty$ in the limit regime (22) *to a constant $\frac{1}{2}\int_0^1 f(u)\varphi''(u)du$.*

Remark 3.4. As each $\mu_{\tilde{x}^N,\tilde{x}^{N-1}}$ is not a positive measure, Theorem 3.3 does not hold for general f. For example, take any $A \subset [0,1]$, and let f be the indicator function of A; then (31) takes only integer values, and can not converge to a non-integer constant. This implies that the measures $d\mu_{\tilde{x}^N,\tilde{x}^{N-1}}$ *do not* weakly converge to the measure with density $\frac{\varphi''}{2}$.

Also, the measure with density $\frac{\varphi''}{2}$ is not necessarily positive (although it has total mass 1): when $\hat{M} < \hat{N}$ the density function can take negative values, cf. Figure 2. This measure is an instance of the *interlacing measures*, which were introduced and studied by Kerov (see [Ker98, Section 1.3]).

Theorem 3.1, 3.2, and 3.3 have a remarkable limit as $\theta \to \infty$. They degenerate to statements about asymptotic separation of the roots of Jacobi orthogonal polynomials.

Let $\mathcal{F}_n^{p,q}$ be the Jacobi orthogonal polynomials of degree n with weight function $x^p(1-x)^q$ on $[0,1]$, see e.g. [Sze39]. Let $j_{M,N,\alpha,i}$ be the ith root (in increasing order) of $\mathcal{F}_{\min(M,N)}^{\alpha-1,|M-N|}$, for $1 \le i \le \min(M,N)$. And we further denote $j_{M,N,\alpha,i} = 1$, for any fixed M, N, α, and $\min(M,N) < i \le N$.

Theorem 3.5. *There is an interlacing relationship for the roots:*

$$j_{M,N,\alpha,1} \le j_{M,N-1,\alpha,1} \le j_{M,N,\alpha,2} \le \cdots. \tag{32}$$

Let $\iota_{M,N,\alpha}$ be the diagram corresponding to this interlacing sequence, as in Definition 2.5, and let φ be defined as in Theorem 3.2. Under the limit scheme (22)*, the diagrams $\iota_{M,N,\alpha}$ converge to φ in uniform topology.*

Kerov in [Ker94] proved similar statements on Hermite and Chebyshev polynomials.

Now we switch to the Central Limit Theorems. The first result describes the asymptotic behavior of fluctuations for the individual $\mathfrak{P}_{k_i}(x^N) - \mathfrak{P}_{k_i}(x^{N-1})$.

Theorem 3.6. *For positive integers $k_1, \cdots, k_h$, $k'_1, \cdots, k'_{h'}$, $N_1, \cdots, N_h$ and $N'_1, \cdots, N'_{h'}$, in addition to the limit scheme (22) we also let*

$$\lim_{L\to\infty} \frac{N_i}{L} = \hat{N}_i, \quad 1 \le i \le h, \qquad\qquad \lim_{L\to\infty} \frac{N'_i}{L} = \hat{N}'_i, \quad 1 \le i \le h'. \tag{33}$$

The random vectors

$$L^{\frac{1}{2}} \left(\mathfrak{P}_{k_i}(x^{N_i}) - \mathfrak{P}_{k_i}(x^{N_i-1}) - \mathbb{E}\left(\mathfrak{P}_{k_i}(x^{N_i}) - \mathfrak{P}_{k_i}(x^{N_i-1})\right)\right)_{i=1}^{h} \tag{34}$$

and

$$\left(\mathfrak{P}_{k'_i}(x^{N'_i}) - \mathbb{E}\left(\mathfrak{P}_{k'_i}(x^{N'_i})\right)\right)_{i=1}^{h'} \tag{35}$$

converge (as $L \to \infty$) jointly in distribution to centered Gaussian random vectors of the following covariance: the vectors (34) *and* (35) *are asymptptotically independent; within the vector* (34)*, the covariance between the ith and jth component becomes,*

$$-\mathbb{1}_{\hat{N}_i = \hat{N}_j} \cdot \frac{k_i k_j}{k_i + k_j} \cdot \frac{\theta^{-1}}{2\pi\mathbf{i}} \oint \frac{1}{(v+\hat{N}_i)^2} \left(\frac{v}{v+\hat{N}_i} \cdot \frac{v-\hat{\alpha}}{v-\hat{\alpha}-\hat{M}}\right)^{k_i+k_j} dv, \tag{36}$$

where the contour encloses $-\hat{N}_i$ but not $\hat{\alpha}+\hat{M}$; within the vector (35)*, the covariance between the ith and jth component is*

$$\frac{\theta^{-1}}{(2\pi\mathbf{i})^2} \oint\oint \frac{1}{(v_1-v_2)^2} \prod_{i=1}^{2} \left(\frac{v_i}{v_i+\hat{N}_i} \cdot \frac{v_i-\hat{\alpha}}{v_i-\hat{\alpha}-\hat{M}}\right)^{k_i} dv_i. \tag{37}$$

Remark 3.7. The Gaussianity of the vector (35) is actually [BG15, Theorem 4.1], and here we are more interested in its joint distribution with (34). The proof presented in this text also gives an alternative proof for [BG15, Theorem 4.1].

The asymptotic behavior is different for the weighted averages (in N) of $\mathfrak{P}_{k_i}(x^N) - \mathfrak{P}_{k_i}(x^{N-1})$.

Theorem 3.8. *Let $k_1, \cdots, k_h$ be integers, and $g_1, \cdots, g_h \in L^\infty([0,1])$ continuous almost everywhere. Under (22), the random vector*

$$\left(L \int_0^1 g_i(y) \left(\mathfrak{P}_{k_i}(x^{\lfloor Ly \rfloor}) - \mathfrak{P}_{k_i}(x^{\lfloor Ly \rfloor - 1}) - \mathbb{E}\left(\mathfrak{P}_{k_i}(x^{\lfloor Ly \rfloor}) - \mathfrak{P}_{k_i}(x^{\lfloor Ly \rfloor - 1}) \right) \right) dy \right)_{i=1}^{h} \tag{38}$$

converges jointly in distribution to a centered Gaussian vector, with covariance between the ith and jth component

$$\begin{aligned} \iint_{0 \le y_1 < y_2 \le 1} & \frac{\theta^{-1}}{(2\pi \mathbf{i})^2} \oint \oint \frac{k_i k_j}{(v_1 - v_2)^2 (v_1 + y_1)(v_2 + y_2)} \\ & \times \left(g_i(y_1) g_j(y_2) \left(\frac{v_1}{v_1 + y_1} \cdot \frac{v_1 - \hat{\alpha}}{v_1 - \hat{\alpha} - \hat{M}} \right)^{k_i} \left(\frac{v_2}{v_2 + y_2} \cdot \frac{v_2 - \hat{\alpha}}{v_2 - \hat{\alpha} - \hat{M}} \right)^{k_j} \right. \\ & \left. + g_j(y_1) g_i(y_2) \left(\frac{v_1}{v_1 + y_1} \cdot \frac{v_1 - \hat{\alpha}}{v_1 - \hat{\alpha} - \hat{M}} \right)^{k_j} \left(\frac{v_2}{v_2 + y_2} \cdot \frac{v_2 - \hat{\alpha}}{v_2 - \hat{\alpha} - \hat{M}} \right)^{k_i} \right) dv_1 dv_2 dy_1 dy_2 \\ & - \int_0^1 \frac{\theta^{-1}}{2\pi \mathbf{i}} \oint \frac{g_i(y) g_j(y) k_i k_j}{(k_i + k_j)(v + y)^2} \left(\frac{v}{v + y} \cdot \frac{v - \hat{\alpha}}{v - \hat{\alpha} - \hat{M}} \right)^{k_i + k_j} dv dy, \end{aligned} \tag{39}$$

where in the first integral, the contours are nested: $|v_1| \ll |v_2|$, and enclose $-y_1, -y_2$ but not $\hat{\alpha} + \hat{M}$; in the second integral, the contour encloses $-y$ but not $\hat{\alpha} + \hat{M}$.

Remark 3.9. Let us emphasize that the scalings in (34) and (39) are different: for a single difference the scale is $L^{\frac{1}{2}}$, while for the weighted average the scale is L.

Theorems 3.6 and 3.8 have an interpretation in terms of the Gaussian Free Field. For that we define (random) height functions, following [Bor14], [BG15].

Definition 3.10. Let sequences $x^1, x^2, \cdots$ be distributed as $\mathbb{P}^{\alpha, M, \theta}$. For any $(u, y) \in [0,1] \times \mathbb{R}_{>0}$, define $\mathcal{H}(u, y)$ to be the number of i such that $x_i^{\lfloor y \rfloor}$ is less than u. For $y > 1$, let $\mathcal{W}(u, y) = \mathcal{H}(u, y) - \mathcal{H}(u, y-1)$.

In [BG15] the convergence of $\mathcal{H}$ to the random field $\mathcal{K}$ of Definition 2.8 is proven. Our Central Limit Theorems imply the convergence of $\mathcal{W}$ to a derivative of the random field $\mathcal{K}$. In more details, Theorem 3.6 leads to the weak convergence to a "renormalized derivative" of the random field $\mathcal{K}$, in the following sense.

Theorem 3.11. *Under the limit scheme (22), for any integers $k_1, \cdots, k_h$, and real numbers $0 < \hat{N}_1 \le \cdots \le \hat{N}_h$, the distribution of the vector*

$$\left(L^{\frac{1}{2}} \int_0^1 u^{k_i} \left(\mathcal{W}(u, L\hat{N}_i) - \mathbb{E}\left(\mathcal{W}(u, L\hat{N}_i) \right) \right) du \right)_{i=1}^{h} \tag{40}$$

as $L \to \infty$ converges weakly to a joint Gaussian distribution, which is the same as the weak limit

$$\lim_{\delta \to 0_+} \delta^{-\frac{1}{2}} \left(\int_0^1 u^{k_i} \mathcal{K}(u, \hat{N}_i + \delta) du - \int_0^1 u^{k_i} \mathcal{K}(u, \hat{N}_i) du \right)_{i=1}^{h}. \tag{41}$$

In words, Theorem 3.11 means that the limiting field for $1d$–slices of $L^{\frac{1}{2}} \mathcal{W}$ (in u–direction), is the same as the renormalized y–derivative of the limiting field for $\mathcal{H}$. However, passing $L \to \infty$ simultaneously for $1d$ slices of $\mathcal{W}$ and $\mathcal{H}$ one gets independent fields. Specifically, we have the following result.

Lemma 3.12. *Under the limit scheme (22), for any integers $k_1, \cdots, k_h$ and $k'_1, \cdots, k'_{h'}$, real numbers $0 < \hat{N}_1 \leq \cdots \leq \hat{N}_h$ and $0 < \hat{N}'_1 \leq \cdots \leq \hat{N}'_{h'}$, the vector (40) and*

$$\left(\int_0^1 u^{k'_i} \left(\mathcal{H}(u, L\hat{N}'_i) - \mathbb{E}\left(\mathcal{H}(u, L\hat{N}'_i) \right) \right) du \right)_{i=1}^{h'} \tag{42}$$

jointly converges (weakly) as $L \to \infty$, while the limit vectors are independent.

In contrast to Lemma 3.12, when we deal with $2d$–integrals of $\mathcal{W}$ and $\mathcal{H}$, then the limiting fields turn out to be much closer related. Namely, we define the pairings $\mathfrak{z}_{g,k}$ of the y-derivative of the field $\mathcal{K}$ with test functions $u^k g(y)$ through the following procedure based on integration by parts in y-direction.

Definition 3.13. For any $g \in C^\infty([0,1])$, with $g(1) = 0$, define

$$\mathfrak{z}_{g,k} = \int_0^1 \int_0^1 u^k g'(y) \mathcal{K}(u, y) du dy. \tag{43}$$

Lemma 3.14. *For any $g \in L^2([0,1])$, and positive integer k, there exists a sequence of functions $g_1, g_2, \cdots$, satisfying that*

1. *Each $g_n \in C^\infty([0,1])$, and $g_n(1) = 0$.*
2. *$\lim_{n\to\infty} g_n = g$ in $L^2([0,1])$.*
3. *The sequence of random variables $\mathfrak{z}_{g_1,k}, \mathfrak{z}_{g_2,k}, \cdots$ converges almost surely.*

If there is another sequence $\tilde{g}_1, \tilde{g}_2, \cdots$ satisfying the same conditions, then the limits $\lim_{n\to\infty} \mathfrak{z}_{g_n,k}$ and $\lim_{n\to\infty} \mathfrak{z}_{\tilde{g}_n,k}$ are almost surely the same.

With this we can extend the definition of $\mathfrak{z}_{g,k}$ to any $g \in L^2([0,1])$.

Definition 3.15. For any $g \in L^2([0,1])$ we define $\mathfrak{z}_{g,k}$ to be the limit in Lemma 3.14.

Now we state the convergence of $\mathcal{W}$ to the y-derivative of $\mathcal{K}$ in the following sense.

Theorem 3.16. *Let $k_1, \cdots, k_h$ be positive integers and $g_1, \cdots, g_h \in L^\infty([0,1])$, each continuous almost everywhere. As $L \to \infty$, the distribution of the vector*

$$\left(L \int_0^1 \int_0^1 u^{k_i} g_i(y) \left(\mathcal{W}(u, Ly) - \mathbb{E}\left(\mathcal{W}(u, Ly) \right) \right) du dy \right)_{i=1}^{h} \tag{44}$$

converges weakly to the distribution of the vector $(\mathfrak{z}_{g_i,k_i})_{i=1}^h$. Moreover, take differentiable functions $\tilde{g}_1, \cdots, \tilde{g}_{h'} \in L^\infty([0,1])$, such that $\tilde{g}_i(1) = 0$ and $\tilde{g}'_i \in L^\infty([0,1])$ for each $1 \leq i \leq h'$, and positive integers $k'_1, \cdots, k'_{h'}$. Then the distribution of the vector

$$\left(\int_0^1 \int_0^1 -u^{k'_i} \tilde{g}'_i(y) \left(\mathcal{H}(u, Ly) - \mathbb{E}\left(\mathcal{H}(u, Ly) \right) \right) du dy \right)_{i=1}^{h'} \tag{45}$$

converges weakly to the distribution of the vector $\left(\mathfrak{z}_{\tilde{g}_i, k'_i} \right)_{i=1}^{h'}$, as $L \to \infty$; and the convergence of (44) and (45) are joint.

Organization of remaining text

The remaining sections are devoted to proofs of the above stated results.

In Section 4 we introduce integral identities, which are powerful tools in simplifying the computations. Section 5 presents the formulas for the expectations of the joint moments of β–Jacobi corners processes, using Macdonald processes and difference operators.

The last three sections finish the proofs. The proofs of the Law of Large Numbers (Theorem 3.1) and the related convergence of diagrams and measures (Theorems 3.2, 3.3) can be found in Section 6, except that the decay of variance in Theorem 3.1 is left for Section 7, which have the proofs of the Central Limit Theorems (Theorems 3.6, 3.8). Section 8 has the proofs of Theorems 3.11 and 3.16, and Lemma 3.12, 3.14.

4 Dimension reduction

In this section we discuss integral identities, which will be widely used in subsequent proofs. A special case ($m = 1$) of the following result was communicated to the authors by Alexei Borodin, and we present our own proof here.

For positive integer n, let σ_n denote the cycle $(12\cdots n)$, and let $S^{cyc}(n)$ denote the n-element subgroup of symmetric group spanned by σ_n.

Theorem 4.1. *Let $n \geq 2$, and $f_1, \cdots, f_n : \mathbb{C} \to \mathbb{C}$ be meromorphic with possible poles at $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_m\}$. Then we have the identity*

$$\sum_{\sigma \in S^{cyc}(n)} \frac{1}{(2\pi\mathbf{i})^n} \oint \cdots \oint \frac{f_{\sigma(1)}(u_1) \cdots f_{\sigma(n)}(u_n)}{(u_2 - u_1) \cdots (u_n - u_{n-1})} du_1 \cdots du_n = \frac{1}{2\pi\mathbf{i}} \oint f_1(u) \cdots f_n(u) du, \tag{46}$$

where the contours in both sides are positively oriented, enclosing $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_m\}$, and for the left hand side we required $|u_1| \ll \cdots \ll |u_n|$.

Proof. Let $\mathfrak{C}_1, \cdots, \mathfrak{C}_{2n-1}$ be closed paths around $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_m\}$, and each $\mathfrak{C}_i$ is inside $\mathfrak{C}_{i+1}$, $1 \leq i \leq 2n-2$. Also, for the convenience of notations, set $f_{n+t} = f_t$ and $u_{n+t} = u_t$ for any $1 \leq t \leq n-1$. Then the left hand side of (46) can be written as

$$\sum_{t=0}^{n-1} \frac{1}{(2\pi\mathbf{i})^n} \oint_{\mathfrak{C}_1} \cdots \oint_{\mathfrak{C}_n} \frac{f_{1+t}(u_1) \cdots f_{n+t}(u_n)}{(u_2 - u_1) \cdots (u_n - u_{n-1})} du_n \cdots du_1. \tag{47}$$

When $n = 2$, we have

$$\begin{aligned}
&\frac{1}{(2\pi\mathbf{i})^2} \oint_{\mathfrak{C}_1} \oint_{\mathfrak{C}_2} \frac{f_1(u_1) f_2(u_2)}{u_2 - u_1} du_2 du_1 + \frac{1}{(2\pi\mathbf{i})^2} \oint_{\mathfrak{C}_1} \oint_{\mathfrak{C}_2} \frac{f_2(u_1) f_1(u_2)}{u_2 - u_1} du_2 du_1 \\
&\quad = \frac{1}{(2\pi\mathbf{i})^2} \oint_{\mathfrak{C}_1} \oint_{\mathfrak{C}_2} \frac{f_1(u_1) f_2(u_2)}{u_2 - u_1} du_2 du_1 + \frac{1}{(2\pi\mathbf{i})^2} \oint_{\mathfrak{C}_3} \oint_{\mathfrak{C}_2} \frac{f_1(u_1) f_2(u_2)}{u_1 - u_2} du_2 du_1 \\
&\quad = \frac{1}{(2\pi\mathbf{i})^2} \oint_{\mathfrak{C}_3 - \mathfrak{C}_1} \oint_{\mathfrak{C}_2} \frac{f_1(u_1) f_2(u_2)}{u_1 - u_2} du_2 du_1,
\end{aligned} \tag{48}$$

where $\oint_{\mathfrak{C}_3 - \mathfrak{C}_1}$ is a notation for the difference of integrals over $\mathfrak{C}_3$ and $\mathfrak{C}_1$. Further, (48) is equals to

$$\frac{1}{2\pi\mathbf{i}} \oint_{\mathfrak{C}_2} f_1(u) f_2(u) du, \tag{49}$$

since as a function of u_1, $\frac{f_1(u_1)f_2(u_2)}{u_1-u_2}$ has a single pole at u_2 between $\mathfrak{C}_3$ and $\mathfrak{C}_1$; and the residue at this pole equals $f_1(u_2)f_2(u_2)$. This proves the case of $n=2$.

When $n \geq 3$, we argue by induction and assume that Theorem 4.1 is true for $n-1$. For any $1 \leq t \leq n-1$, we have that

$$\begin{aligned}\frac{1}{(2\pi\mathbf{i})^n}&\oint_{\mathfrak{C}_1}\cdots\oint_{\mathfrak{C}_n}\frac{f_{1+t}(u_1)\cdots f_{n+t}(u_n)}{(u_2-u_1)\cdots(u_n-u_{n-1})}du_n\cdots du_1\\&=\frac{1}{(2\pi\mathbf{i})^n}\oint_{\mathfrak{C}_{1+t}}\cdots\oint_{\mathfrak{C}_{n+t}}\frac{f_1(u_1)\cdots f_n(u_n)}{(u_{2+t}-u_{1+t})\cdots(u_{n+t}-u_{n-1+t})}du_{n+t}\cdots du_{1+t}\\&=\frac{1}{(2\pi\mathbf{i})^n}\oint_{\mathfrak{C}_{n+1}}\cdots\oint_{\mathfrak{C}_{n+t}}\oint_{\mathfrak{C}_{t+1}}\cdots\oint_{\mathfrak{C}_n}\frac{f_1(u_1)\cdots f_n(u_n)}{(u_{2+t}-u_{1+t})\cdots(u_{n+t}-u_{n-1+t})}du_n\cdots du_1. \quad (50)\end{aligned}$$

Now we can move the contours of $u_1,\cdots,u_t$ from $\mathfrak{C}_{n+1},\cdots,\mathfrak{C}_{n+t}$ to $\mathfrak{C}_1,\cdots,\mathfrak{C}_t$, respectively. We move the contours one by one starting from u_1, and each move is across $\mathfrak{C}_{t+1},\cdots,\mathfrak{C}_n$. For $u_1(=u_{n+1})$, the only pole between $\mathfrak{C}_{n+1}$ and $\mathfrak{C}_1$ is u_n; for any u_i, $1<i\leq t$, there is no pole between $\mathfrak{C}_{n+i}$ and $\mathfrak{C}_i$. Thus we have that

$$\begin{aligned}\frac{1}{(2\pi\mathbf{i})^n}&\oint_{\mathfrak{C}_{n+1}}\cdots\oint_{\mathfrak{C}_{n+t}}\oint_{\mathfrak{C}_{t+1}}\cdots\oint_{\mathfrak{C}_n}\frac{f_1(u_1)\cdots f_n(u_n)}{(u_{2+t}-u_{1+t})\cdots(u_{n+t}-u_{n-1+t})}du_n\cdots du_1\\&=\frac{1}{(2\pi\mathbf{i})^n}\oint_{\mathfrak{C}_1}\cdots\oint_{\mathfrak{C}_n}\frac{f_1(u_1)\cdots f_n(u_n)}{(u_{2+t}-u_{1+t})\cdots(u_{n+t}-u_{n-1+t})}du_n\cdots du_1\\&+\frac{1}{(2\pi\mathbf{i})^{n-1}}\oint_{\mathfrak{C}_1}\cdots\oint_{\mathfrak{C}_{n-1}}\frac{f_{1+t}(u_1)\cdots f_n(u_{n-t})f_{n+1}(u_{n-t})\cdots f_{n+t}(u_{n-1})}{(u_2-u_1)\cdots(u_{n-1}-u_{n-2})}du_{n-1}\cdots du_1. \quad (51)\end{aligned}$$

Notice that (taking into account that $u_{n+t}=u_t$)

$$\sum_{t=0}^{n-1}\frac{f_1(u_1)\cdots f_n(u_n)}{(u_{2+t}-u_{1+t})\cdots(u_{n+t}-u_{n-1+t})}=0, \quad (52)$$

and by induction assumption (applied to $f_2,\cdots,f_{n-1},f_nf_1$), we have that

$$\begin{aligned}\sum_{t=0}^{n-1}&\frac{1}{(2\pi\mathbf{i})^n}\oint_{\mathfrak{C}_1}\cdots\oint_{\mathfrak{C}_n}\frac{f_{1+t}(u_1)\cdots f_{n+t}(u_n)}{(u_2-u_1)\cdots(u_n-u_{n-1})}du_n\cdots du_1\\&=\sum_{t=1}^{n-1}\frac{1}{(2\pi\mathbf{i})^{n-1}}\oint_{\mathfrak{C}_1}\cdots\oint_{\mathfrak{C}_{n-1}}\frac{f_{1+t}(u_1)\cdots f_n(u_{n-t})f_{n+1}(u_{n-t})\cdots f_{n+t}(u_{n-1})}{(u_2-u_1)\cdots(u_{n-1}-u_{n-2})}du_{n-1}\cdots du_1\\&=\frac{1}{2\pi\mathbf{i}}\oint f_1(u)\cdots f_n(u)du. \quad (53)\end{aligned}$$

□

Corollary 4.2. *Let s be a positive integer. Let f, $g_1,\cdots,g_s$ be meromorphic functions with possible poles at $\{\mathfrak{p}_1,\cdots,\mathfrak{p}_m\}$. Then for $n\geq 2$,*

$$\begin{aligned}\frac{1}{(2\pi\mathbf{i})^n}&\oint\cdots\oint\frac{1}{(v_2-v_1)\cdots(v_n-v_{n-1})}\prod_{i=1}^n f(v_i)dv_i\prod_{i=1}^s\left(\sum_{j=1}^n g_i(v_j)\right)\\&=\frac{n^{s-1}}{2\pi\mathbf{i}}\oint f(v)^n\prod_{i=1}^s g_i(v)dv, \quad (54)\end{aligned}$$

where the contours in both sides are around all of $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_m\}$*, and for the left hand side we required* $|u_1| \ll \cdots \ll |u_n|$.

Proof. Take disjoint sets $U_1, \cdots, U_n$, with $\bigcup_{i=1}^n U_i = \{1, \cdots, s\}$ (some of which might be empty). In Theorem 4.1 we let $f_i = f \prod_{j \in U_i} g_j$ for each $1 \le i \le n$, and get

$$\sum_{\sigma \in S^{cyc}(n)} \frac{1}{(2\pi\mathbf{i})^n} \oint \cdots \oint \frac{1}{(v_2 - v_1) \cdots (v_n - v_{n-1})} \prod_{i=1}^n \left(f(v_i) \prod_{j \in U_{\sigma(i)}} g_j(v_i) dv_i \right)$$
$$= \frac{1}{2\pi\mathbf{i}} \oint f(v)^n \prod_{i=1}^s g_i(v) dv. \quad (55)$$

Summing over all n^s partitions $U_1, \cdots, U_n$ of $\{1, \cdots, s\}$ into n disjoint sets, we obtain (54). $\square$

5 Discrete joint moments

In this section we compute the joint moments in β-Jacobi corners processes. The main goal is to prove the following result.

Theorem 5.1. *Let* $(x^1, x^2, \cdots) \in \chi^M$ *be distributed as* $\mathbb{P}^{\alpha, M, \theta}$*, and let* $\mathfrak{P}_k(x^N)$ *be defined as (24). Let* l*,* $N_1 \le \cdots \le N_l$*, and* $k_1, \cdots, k_l$ *be positive integers, satisfying* $M > k_1 + \cdots + k_l$*.*

For any positive integers m*,* n*,* $\tilde{m}$*,* $\tilde{n}$*, and variables* $w_1, \cdots, w_m$*,* $\tilde{w}_1, \cdots, \tilde{w}_{\tilde{m}}$*, denote*

$$\mathfrak{I}(w_1, \cdots, w_m; \alpha, M, \theta, n) = \frac{1}{(w_2 - w_1 + 1 - \theta) \cdots (w_m - u_{m-1} + 1 - \theta)}$$
$$\times \prod_{1 \le i < j \le m} \frac{(w_j - w_i)(w_j - w_i + 1 - \theta)}{(w_j - w_i - \theta)(w_j - w_i + 1)} \prod_{i=1}^m \frac{w_i - \theta}{w_i + (n-1)\theta} \cdot \frac{w_i - \theta\alpha}{w_i - \theta\alpha - \theta M}, \quad (56)$$

and

$$\mathfrak{L}(w_1, \cdots, w_m; \tilde{w}_1, \cdots, \tilde{w}_{\tilde{m}}; \theta) = \prod_{1 \le i \le \tilde{m}, 1 \le j \le m} \frac{(\tilde{w}_i - w_j)(\tilde{w}_i - w_j + 1 - \theta)}{(\tilde{w}_i - w_j - \theta)(\tilde{w}_i - w_j + 1)}. \quad (57)$$

Then the expectation of higher moments $\mathfrak{P}_k(x^N)$ *can be computed via*

$$\mathbb{E}\left(\mathfrak{P}_{k_1}(x^{N_1}) \cdots \mathfrak{P}_{k_l}(x^{N_l})\right) = \frac{(-\theta)^{-l}}{(2\pi\mathbf{i})^{k_1 + \cdots + k_l}} \oint \cdots \oint \prod_{i=1}^l \mathfrak{I}(u_{i,1}, \cdots, u_{i,k_i}; \alpha, M, \theta, N_i)$$
$$\times \prod_{i<j} \mathfrak{L}(u_{i,1}, \cdots, u_{i,k_i}; u_{j,1}, \cdots, u_{j,k_j}; \theta) \prod_{i=1}^l \prod_{i'=1}^{k_i} du_{i,i'}, \quad (58)$$

where for each $i = 1, \cdots, l$*, the contours of* $u_{i,1}, \cdots, u_{i,k_i}$ *enclose* $-\theta(N_i - 1)$ *but not* $\theta(\alpha + M)$*, and* $|u_{i,1}| \ll \cdots \ll |u_{i,k_i}|$*. For* $1 \le i < l$*, we also require that* $|u_{i,k_i}| \ll |u_{i+1,1}|$*.*

The proof of Theorem 5.1 relies on the formalism of *Macdonald processes*. Under certain limit transition it weakly converges to $\mathbb{P}^{\alpha, M, \theta}$. In turn, we compute the moments of Macdonald process by applying a remarkable family of difference operators coming from the work [Neg13] on the symmetric functions. A particular case ($N1 = \cdots = N_l$) of Theorem 5.1 was proven by one of the authors and Borodin in the appendix to [FD16].

We remark that a *different* contour integral expression for the left-hand side of (58) is given in [BG15, Section 3]. The authors are not aware of a direct way to match the two expressions.

5.1 Macdonald processes and asymptotic relations

Let Λ_N denote the ring of symmetric polynomials in N variables, and Λ denote the ring of symmetric polynomials in countably many variables (see [Mac95, Chapter I, Section 2]). Let $\mathbb{Y}$ be the set of partitions, i.e. infinite non-increasing sequence of non-negative integers, which are eventually zero:

$$\mathbb{Y} = \{\lambda = (\lambda_1, \lambda_2, \cdots) \in \mathbb{Z}^\infty \,|\lambda_1 \geq \lambda_2 \geq \cdots \geq 0,\ \exists N \in \mathbb{Z}_+, \lambda_N = 0\},$$

and $\mathbb{Y}_N \subset \mathbb{Y}$ consists of sequences λ such that $\lambda_{N+1} = 0$. We can make $\mathbb{Y}$ a partially ordered set, declaring

$$\lambda \geq \mu \iff \lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i,\ \forall i = 1, 2, \cdots. \tag{59}$$

For any $\lambda \in \mathbb{Y}$, denote $P_\lambda(\cdot; q, t) \in \Lambda$ to be the normalized Macdonald polynomial,

$$P_\lambda(\cdot; q, t) = m_\lambda + \sum_{\mu<\lambda} u_{\lambda\mu} m_\mu \tag{60}$$

where each m_μ is the monomial symmetric polynomials (see [Mac95, Section VI.4]). Here q and t are real parameters, and we assume that $0 < q < 1$ and $0 < t < 1$. We also denote $Q_\lambda(\cdot; q, t) = b_\lambda(q, t)P_\lambda(\cdot; q, t)$, where $b_\lambda(q, t)$ is a constant uniquely defined by the identity (62) below and with explicit expression given in [Mac95, Chapter VI]. The collection

$$\left\{P_\lambda(\cdot; q, t)\middle|\lambda \in \mathbb{Y}\right\}$$

is a basis of Λ. We further define the skew Macdonald polynomials $P_{\lambda/\mu}$ and $Q_{\lambda/\mu}$, where $\lambda, \mu \in \mathbb{Y}$, to be the coefficients of the following expansions (see [Mac95, Chapter VI, (7.9)]):

$$\begin{aligned} P_\lambda(a_1, \cdots, a_N, b_1, \cdots, b_N; q, t) &= \sum_{\mu\in\mathbb{Y}_N} P_{\lambda/\mu}(a_1, \cdots, a_N; q, t)P_\mu(b_1, \cdots, b_N; q, t) \\ Q_\lambda(a_1, \cdots, a_N, b_1, \cdots, b_N; q, t) &= \sum_{\mu\in\mathbb{Y}_N} Q_{\lambda/\mu}(a_1, \cdots, a_N; q, t)Q_\mu(b_1, \cdots, b_N; q, t). \end{aligned} \tag{61}$$

Proposition 5.2 (see [Mac95, Chapter VI]). *For any finite sequences $a_1, \cdots, a_{M_1}$ and $b_1, \cdots, b_{M_2} \in \mathbb{C}$, with $|a_ib_j| < 1$, $\forall 1 \leq i \leq M_1$ for any $1 \leq j \leq M_2$, the Macdonald polynomials have the following identities:*

$$\sum_{\lambda\in\mathbb{Y}} P_\lambda(a_1, \cdots, a_{M_1}; q, t)Q_\lambda(b_1, \cdots, b_{M_2}; q, t) = \prod_{1\leq i\leq M_1, 1\leq j\leq M_2} \frac{\prod_{k=1}^\infty(1 - ta_ib_jq^{k-1})}{\prod_{k=1}^\infty(1 - a_ib_jq^{k-1})}, \tag{62}$$

$$\sum_{\lambda\in\mathbb{Y}} P_{\mu/\lambda}(a_1, \cdots, a_{M_1}; q, t)P_{\lambda/\nu}(b_1, \cdots, b_{M_2}; q, t) = P_{\mu/\nu}(a_1, \cdots, a_{M_1}, b_1, \cdots, b_{M_2}; q, t). \tag{63}$$

Let Ψ^M be the set of all infinite families of sequences $\{\lambda^i\}_{i=1}^\infty$, which satisfy

1. For $N \geq 1$, $\lambda^N \in \mathbb{Y}_{\min\{M,N\}}$.
2. For $N \geq 2$, the sequences λ^N and λ^{N-1} interlace: $\lambda_1^N \geq \lambda_1^{N-1} \geq \lambda_2^N \geq \cdots$.

Definition 5.3. The infinite ascending *Macdonald process* with positive parameters $M \in \mathbb{Z}$, $\{a_i\}_{i=1}^\infty$, $\{b_i\}_{i=1}^M$, $0 < a_i < 1$, $0 < b_i < 1$, is the distribution on Ψ^M, such that the marginal distribution for λ^N is

$$\text{Prob}(\lambda^N = \mu) = \prod_{1\leq i\leq N, 1\leq j\leq M} \frac{\prod_{k=1}^\infty(1 - a_ib_jq^{k-1})}{\prod_{k=1}^\infty(1 - ta_ib_jq^{k-1})} P_\mu(a_1, \cdots, a_N; q, t)Q_\mu(b_1, \cdots, b_M; q, t), \tag{64}$$

and $\{\lambda^N\}_{N\geq 1}$ is a trajectory of a Markov chain with (backward) transition probabilities

$$\mathrm{Prob}(\lambda^{N-1}=\mu|\lambda^N=\nu)=P_{\nu/\mu}(a_N;q,t)\frac{P_\mu(a_1,\cdots,a_{N-1};q,t)}{P_\nu(a_1,\cdots,a_N;q,t)}. \tag{65}$$

Remark 5.4. The consistency of formulas (64) and (65) follows from properties of Macdonald polynomials. See [BC14], [BCGS16] for more details.

From this definition and (63), the following Proposition follows by simple induction.

Proposition 5.5. *Let $\{\lambda^N\}_{N\geq 1}$ distributed as a Macdonald process with positive parameters $M\in\mathbb{Z}$, $\{a_i\}_{i=1}^\infty$, $\{b_i\}_{i=1}^M$, $0<a_i<1$, $0<b_i<1$. For integers $0<N_1<\cdots<N_l$, and $\mu^1\in\mathbb{Y}_1,\cdots,\mu^l\in\mathbb{Y}_l$, the joint distribution is*

$$\begin{aligned}\mathrm{Prob}(\lambda^{N_1}=\mu^1,\cdots,\lambda^{N_l}=\mu^l)=&\prod_{1\leq i\leq N_l,1\leq j\leq M}\frac{\prod_{k=1}^\infty(1-a_ib_jq^{k-1})}{\prod_{k=1}^\infty(1-ta_ib_jq^{k-1})}\\ &\times P_{\mu^1}(a_1,\cdots,a_{N_1};q,t)\left(\prod_{i=1}^{l-1}P_{\mu^{i+1}/\mu^i}(a_{N_i+1},\cdots,a_{N_{i+1}};q,t)\right)Q_{\mu^l}(b_1,\cdots,b_M;q,t).\end{aligned} \tag{66}$$

There is a limit transition which links Macdonald processes with $\mathbb{P}^{\alpha,M,\theta}$.

Theorem 5.6. *[BG15, Theorem 2.8] Given positive parameters $M\in\mathbb{Z}$, and α, θ. Let random family of sequences $\{\lambda^i\}_{i=1}^\infty$, which takes value in Ψ^M, be distributed according to Macdonald process with parameters M, $\{a_i\}_{i=1}^\infty$, $\{b_i\}_{i=1}^M$. For $\epsilon>0$, set*

$$\begin{aligned}a_i&=t^{i-1}, && i=1,2,\cdots,\\ b_i&=t^{\alpha+i-1}, && i=1,2,\cdots,\\ q&=\exp(-\epsilon), && t=\exp(-\theta\epsilon)\\ x_j^i(\epsilon)&=\exp(-\epsilon\lambda_j^i) && i=1,2,\cdots,1\leq j\leq\min\{m,n\},\end{aligned} \tag{67}$$

then as $\epsilon\to 0_+$, the distribution of $x^1,x^2,\cdots$ weakly converges to $\mathbb{P}^{\alpha,M,\theta}$.

5.2 Differential operator

We introduce operators acting on analytic symmetric functions. Such operators were originally defined to act on Λ, and more algebraic discussions of them can be found in [FHH+09] or [Neg13]. We will use them to extract moments of $\mathbb{P}^{\alpha,M,\theta}$.

Definition 5.7. Fix real parameters r,q,t, where $r>0$ and $q,t\in[0,1]$. Let $\mathcal{D}_{-n}^N$ be an operator acting on symmetric analytic functions defined on B_r^N, where $B_r=\{x\in\mathbb{C}:|x|<r\}$. For any $F:B_r^N\to\mathbb{C}$, if we expand

$$F(x_1,\cdots,x_N)=\sum_{\lambda\in\mathbb{Y}_N}c_\lambda P_\lambda(x_1,\cdots,x_N;q,t), \tag{68}$$

where c_λ are complex coefficients, then we set $\mathcal{D}_{-n}^N F:B_r^N\to\mathbb{C}$ to the sum of series

$$\mathcal{D}_{-n}^N F(x_1,\cdots,x_N)=\sum_{\lambda\in\mathbb{Y}_N}c_\lambda\left((1-t^{-n})\sum_{i=1}^N(q^{\lambda_i}t^{-i+1})^n+t^{-Nn}\right)P_\lambda(x_1,\cdots,x_N;q,t). \tag{69}$$

Proposition 5.8. *The series (69) converges uniformly on compact subsets of B_r^N. Thus the defined $\mathcal{D}_{-n}^N$ is a linear operator, which is continuous in the following sense: for a sequence $\{F_i\}_{i=1}^\infty$ of symmetric analytic functions converging to 0 uniformly on every compact subset of B_r^N, then so is the sequence $\{\mathcal{D}_{-n}^N F_i\}_{i=1}^\infty$.*

We need the following Lemma in the proof.

Lemma 5.9. *For any $r, \delta > 0$, there is a constant $C > 0$ satisfying the following: for any symmetric analytic function $F : B_r^N \to \mathbb{C}$ given by (68), if $|F(x_1, \cdots, x_N)| \leq 1$ for every $x_1, \cdots, x_N \in B_r$, then for every $x_1, \cdots, x_N \in B_{r(1-\delta)}$, and $\lambda \in \mathbb{Y}_N$, there is $|c_\lambda P_\lambda(x_1, \cdots, x_N; q, t)| < (1-\delta^3)^{|\lambda|} C$, where $|\lambda| = \sum_{i=1}^N \lambda_i$.*

Proof. By rescaling $x_1, \cdots, x_N$ it suffices to consider the case where $r = 1 + \delta$.

We define a scalar product for any two symmetric analytic functions f, g on $B_{1+\delta}^N$:

$$\langle f, g \rangle = \frac{1}{N!} \int_T f(z_1, \cdots, z_N) \overline{g(z_1, \cdots, z_N)} \Delta(z_1, \cdots, z_N; q, t) d\vec{z}, \tag{70}$$

where T is the torus $T = \left\{(z_1, \cdots, z_N) \in \mathbb{C}^N : |z_i| = 1\right\}$, and $d\vec{z}$ is the uniform measure on T.

$$\Delta(z_1, \cdots, z_N) = \prod_{i \neq j} \left(\prod_{r=0}^{\infty} \frac{1 - z_i z_j^{-1} q^r}{1 - t z_i z_j^{-1} q^r} \right). \tag{71}$$

This definition follows [Mac95, Section VI.9], where one can find more discussions. We immediately see that in T, $\Delta(z_1, \cdots, z_N)$ is always real and takes value in the interval $(\tau, 1)$, where $\tau > 0$ depends on t and q.

By [Mac95, Chapter VI (9.5)], the Macdonald polynomials $P_\lambda(\cdot; q, t)$ are pairwise orthogonal with respect to this scalar product. Thus, for $F : B_r^N \to \mathbb{C}$ given by (68), there is

$$\langle F, P_\lambda(\cdot; q, t) \rangle = c_\lambda \langle P_\lambda(\cdot; q, t), P_\lambda(\cdot; q, t) \rangle. \tag{72}$$

By Cauchy-Schwarz inequality, we have

$$|\langle F, P_\lambda(\cdot; q, t) \rangle|^2 \leq \langle F, F \rangle \langle P_\lambda(\cdot; q, t), P_\lambda(\cdot; q, t) \rangle, \tag{73}$$

then

$$|c_\lambda| \leq \sqrt{\frac{\langle F, F \rangle}{\langle P_\lambda(\cdot; q, t), P_\lambda(\cdot; q, t) \rangle}}. \tag{74}$$

For $\langle F, F \rangle$, since $|F|$ is bounded by 1 in $B_{1+\delta}^N$, there is $\langle F, F \rangle \leq \langle 1, 1 \rangle$.

Recall that $P_\lambda(z_1, \cdots, z_N; q, t) = \sum_{\mu \leq \lambda} u_{\lambda\mu} m_\mu$, here $u_{\lambda\lambda} = 1$. Denote $\mathcal{N}_\mu$ to be the number of permutations of $(\mu_1, \cdots, \mu_N)$, then we have

$$\langle P_\lambda(\cdot; q, t), P_\lambda(\cdot; q, t) \rangle \geq \frac{\tau}{N!} \int_T |P_\lambda(z_1, \cdots, z_N; q, t)|^2 \, d\vec{z} = \frac{(2\pi)^N \tau}{N!} \sum_{\mu \leq \lambda} |u_{\lambda\mu}|^2 \mathcal{N}_\mu. \tag{75}$$

For any $x_1, \cdots, x_N \in B_{1-\delta^2}$, we have

$$|P_\lambda(x_1, \cdots, x_N; q, t)| \leq \sum_{\mu \leq \lambda} |u_{\lambda\mu}| \mathcal{N}_\mu (1-\delta^2)^{|\lambda|}. \tag{76}$$

Then

$$|c_\lambda P_\lambda(x_1,\cdots,x_N;q,t)| \leq \sqrt{\frac{\langle 1,1\rangle N!}{\tau(2\pi)^N}}(1-\delta^2)^{|\lambda|}\frac{\sum_{\mu\leq\lambda}|u_{\lambda\mu}|\mathcal{N}_\mu}{\sqrt{\sum_{\mu\leq\lambda}|u_{\lambda\mu}|^2\mathcal{N}_\mu}}$$
$$\leq \sqrt{\frac{\langle 1,1\rangle N!}{\tau(2\pi)^N}}(1-\delta^2)^{|\lambda|}\sqrt{\sum_{\mu\leq\lambda}\mathcal{N}_\mu}. \quad (77)$$

Note that each $\mathcal{N}_\mu$ is bounded by $N!$, and the number of $\mu \leq \lambda$ grows in polynomial order of $|\lambda|$. Then we conclude that there is constant C with $|c_\lambda P_\lambda(x_1,\cdots,x_N;q,t)| \leq (1-\delta^3)^{|\lambda|}C$. □

Proof of Proposition 5.8. The uniform convergence of (69) follows from the fact that

$$\left((1-t^{-n})\sum_{i=1}^{N}(q^{\lambda_i}t^{-i+1})^n + t^{-Nn}\right) \quad (78)$$

is uniformly bounded, and the absolute and uniform convergence of (68) in any compact subset of B_r^N.

For the continuity, expand

$$F_i(x_1,\cdots,x_N) = \sum_{\lambda\in\mathbb{Y}_N} c_{i,\lambda}P_\lambda(x_1,\cdots,x_N;q,t), \quad i = o,1,2,\cdots. \quad (79)$$

By Lemma 5.9, for any small $\delta > 0$ and $x_1,\cdots,x_N \in B^N_{r(1-\delta)}$, we have

$$\left|\mathcal{D}^N_{-n}F_i(x_1,\cdots,x_N)\right| \leq \sup|F_i|\sum_{\lambda\in\mathbb{Y}_N}\left|(1-t^{-n})\sum_{j=1}^{N}(q^{\lambda_j}t^{-j+1})^n + t^{-Nn}\right|(1-\delta^3)^{|\lambda|}C, \quad (80)$$

and this converges to 0 as $i\to\infty$. □

We can evaluate the action of $\mathcal{D}^N_{-n}$ on a special class of functions as a nested contour integral.

Proposition 5.10. *Let $f : B_r \to \mathbb{C}$ be analytic, such that $f(0) \neq 0$; and $g : B_{r'} \to \mathbb{C}$ be analytic, with $r' > r$, such that $g(z)f(q^{-1}z) = f(z)$ for any $z \in B_r$. The action of $\mathcal{D}^N_{-n}$ can be identified with the integral*

$$\mathcal{D}^N_{-n}\prod_{i=1}^{N}f(a_i) = \left(\prod_{i=1}^{N}f(a_i)\right)\frac{(-1)^{n-1}}{(2\pi\mathbf{i})^n}\oint\cdots\oint\frac{\sum_{i=1}^{n}\frac{z_nt^{n-i}}{z_iq^{n-i}}}{\left(1-\frac{tz_2}{qz_1}\right)\cdots\left(1-\frac{tz_n}{qz_{n-1}}\right)}$$
$$\times\prod_{i<j}\frac{\left(1-\frac{z_i}{z_j}\right)\left(1-\frac{qz_i}{tz_j}\right)}{\left(1-\frac{z_i}{tz_j}\right)\left(1-\frac{qz_i}{z_j}\right)}\left(\prod_{i=1}^{n}\prod_{i'=1}^{N}\frac{z_i-t^{-1}qa_{i'}}{z_i-qa_{i'}}\right)\prod_{i=1}^{n}\frac{g(z_i)dz_i}{z_i}, \quad (81)$$

for any $a_1,\cdots,a_N \in B_r$. The contours are in $B_{r'}$ and nested: all enclose 0 and $qa_{i'}$, and $|z_i| < |tz_{i+1}|$ for each $1 \leq i \leq n-1$.

Proof. We start by introducing the algebraic version of the operator $\mathcal{D}^N_{-n}$.

Define $\mathbf{Z}$ to be the algebra of complex coefficient Laurent power series in variables $\frac{z_1}{z_2}, \frac{z_2}{z_3}, \cdots \frac{z_{n-1}}{z_n}$, and z_n. Namely, it contains all elements in the form $\sum_{m\in\mathbb{Z}^n} d_m \prod_{i=1}^{n-1}\left(\frac{z_i}{z_{i+1}}\right)^{m_i} z_n^{m_n}$, such that for some $M\in\mathbb{Z}$, all coefficients d_m with $\min_i m_i < M$ vanishes. Let $\mathrm{Res}:\mathbf{Z}\to\mathbb{C}$ be a ring homomorphism, sending every Laurent power series to its coefficient of $\prod_{i=1}^n z_i^{-1}$. This is an analogue of doing contour integrals around 0, requiring $|z_1|\ll\cdots\ll|z_n|$.

Define $\tilde{\Lambda}$ to be the ring of symmetric formal power series with complex coefficients in countably many variables $x_1, x_2, \cdots$, and $\tilde{\Lambda}[\mathbf{Z}]$ to be the ring of symmetric formal power series in $x_1, x_2, \cdots$, with coefficients in $\mathbf{Z}$. For any $F\in\tilde{\Lambda}[\mathbf{Z}]$, it can be uniquely written as

$$F = \sum_{\lambda\in\mathbb{Y}} c_\lambda P_\lambda(\cdot;q,t), \tag{82}$$

where each $c_\lambda\in\mathbf{Z}$. We can also define Res as an operator $\tilde{\Lambda}[\mathbf{Z}]\to\tilde{\Lambda}$, by acting on each coefficient.

Let $\mathbf{D}_{-n}:\tilde{\Lambda}\to\tilde{\Lambda}$, such that

$$\mathbf{D}_{-n}\left(\sum_{\lambda\in\mathbb{Y}} c_\lambda P_\lambda(\cdot;q,t)\right) := \sum_{\lambda\in\mathbb{Y}} c_\lambda\left((1-t^{-n})\sum_{i=1}^\infty (q^{\lambda_i}t^{-i+1})^n\right) P_\lambda(\cdot;q,t). \tag{83}$$

For each $k\in\mathbb{Z}_+$, denote $p_k=\sum_{i=1}^\infty x_i^k\in\tilde{\Lambda}$. Note that any element in $\tilde{\Lambda}$ can be uniquely written as a formal power series in $p_1, p_2, \cdots$; then $\frac{\partial}{\partial p_k}$ defines an operator from $\tilde{\Lambda}$ to itself. Now we present the "integral form" of the operator $\mathbf{D}_{-n}$, which is a reformulation of [Neg13, Theorem 1.2]:

$$\begin{aligned}\mathbf{D}_{-n} = \frac{(-1)^{n-1}}{(2\pi\mathbf{i})^n}\,\mathrm{Res}\Bigg[&\frac{\sum_{i=1}^n \frac{z_n t^{n-i}}{z_i q^{n-i}}}{\left(1-\frac{tz_2}{qz_1}\right)\cdots\left(1-\frac{tz_n}{qz_{n-1}}\right)}\prod_{i<j}\frac{\left(1-\frac{z_i}{z_j}\right)\left(1-\frac{qz_i}{tz_j}\right)}{\left(1-\frac{z_i}{tz_j}\right)\left(1-\frac{qz_i}{z_j}\right)}\\ &\times\exp\left(\sum_{k=1}^\infty q^k(1-t^{-k})\frac{z_1^{-k}+\cdots+z_n^{-k}}{k}p_k\right)\exp\left(\sum_{k=1}^\infty (z_1^k+\cdots z_n^k)(1-q^{-k})\frac{\partial}{\partial p_k}\right)\prod_{i=1}^n z_i^{-1}\Bigg],\end{aligned} \tag{84}$$

where the factors $\left(1-\frac{tz_{i+1}}{qz_i}\right)^{-1}$ and $\left(1-\frac{z_i}{tz_j}\right)\left(1-\frac{qz_i}{z_j}\right)^{-1}$ are elements in $\mathbf{Z}$, by expanding in z_i/z_{i+1} for $1\le i\le n-1$, and in z_i/z_j for $1\le i<j\le n$. The exponents are operators from $\tilde{\Lambda}$ to $\tilde{\Lambda}[\mathbf{Z}]$, by expanding into a power series in the usual way (where p_k is the operator of multiplying by p_k).

Now we evaluate the operator $\mathbf{D}_{-n}$ on a power series. First we rewrite

$$\begin{aligned}&\exp\left(\sum_{k=1}^\infty q^k(1-t^{-k})\frac{z_1^{-k}+\cdots+z_n^{-k}}{k}p_k\right)\\ &\qquad = \prod_{i=1}^n\prod_{i'=1}^\infty \exp\left(\sum_{k=1}^\infty q^k(1-t^{-k})\frac{z_i x_{i'}}{k}\right) = \prod_{i=1}^n\prod_{i'=1}^\infty \frac{1-t^{-1}q\frac{x_{i'}}{z_i}}{1-q\frac{x_{i'}}{z_i}}.\end{aligned} \tag{85}$$

Take any complex coefficient formal power series $f(x)=\sum_{i=0}^\infty s_i x^i$, with $s_0=1$. Using the expansion $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots$, we define $\sum_{k=1}^\infty w_k x^k := \ln(f(x))$, and we have $f(x)=\exp(\sum_{i=1}^\infty w_i x_i)$. Note that for any $k\in\mathbb{Z}_+$, any power series $h(p_k)$ of p_k, and any $C\in\mathbf{Z}$, by expanding the operators one can check that $\exp\left(\frac{C\partial}{\partial p_k}\right)h(p_k)=h(p_k+C)$. Therefore,

$$\begin{aligned}\exp\left(\sum_{k=1}^{\infty}(z_1^k+\cdots z_n^k)(1-q^{-k})\frac{\partial}{\partial p_k}\right)\prod_{i=1}^{\infty}f(x_i)\\
&=\prod_{k=1}^{\infty}\left(\exp\left((z_1^k+\cdots z_n^k)(1-q^{-k})\frac{\partial}{\partial p_k}\right)\exp\left(w_kp_k\right)\right)\\
&=\exp\left(\sum_{k=1}^{\infty}w_k\left((z_1^k+\cdots z_n^k)(1-q^{-k})+p_k\right)\right)=\prod_{i=1}^{n}\frac{f(z_i)}{f(q^{-1}z_i)}\prod_{i=1}^{\infty}f(x_i),\end{aligned}\tag{86}$$

where $f(q^{-1}x)^{-1}$ is understood as the power series $\sum_{j=0}^{\infty}\left(-\sum_{i=1}^{\infty}s_iq^{-i}x^i\right)^j$.

Applying both sides of (84) to $\prod_{i=1}^{\infty}f(x_i)\in\tilde{\Lambda}$, we obtain the following formula:

$$\begin{aligned}\mathbf{D}_{-n}\prod_{i=1}^{\infty}f(x_i)=\frac{(-1)^{n-1}}{(2\pi\mathbf{i})^n}\prod_{i=1}^{\infty}f(x_i)\,\mathrm{Res}\left[\frac{\sum_{i=1}^{n}\frac{z_nt^{n-i}}{z_iq^{n-i}}}{\left(1-\frac{tz_2}{qz_1}\right)\cdots\left(1-\frac{tz_n}{qz_{n-1}}\right)}\prod_{i<j}\frac{\left(1-\frac{z_i}{z_j}\right)\left(1-\frac{qz_i}{tz_j}\right)}{\left(1-\frac{z_i}{tz_j}\right)\left(1-\frac{qz_i}{z_j}\right)}\right.\\
\left.\times\prod_{i=1}^{n}\left(\prod_{i'=1}^{\infty}\frac{1-t^{-1}q\frac{x_{i'}}{z_i}}{1-q\frac{x_{i'}}{z_i}}\cdot\frac{f(z_i)}{f(q^{-1}z_i)}z_i^{-1}\right)\right].\end{aligned}\tag{87}$$

Now we pass to N variables. Define $\tilde{\Lambda}_N$ to be the ring of symmetric formal power series (with complex coefficients) in N variables $x_1,\cdots,x_N$, and $\tilde{\Lambda}_N[\mathbf{Z}]$ to be the ring of symmetric formal power series in $x_1,\cdots,x_M$, with coefficients in $\mathbf{Z}$. Then Res can also be defined to act on $\tilde{\Lambda}_N[\mathbf{Z}]$, with image in $\tilde{\Lambda}_N$. Let $\pi_N:\tilde{\Lambda}\to\tilde{\Lambda}_N$ be the projection setting $0=x_{N+1}=x_{N+2}=\cdots$; and $\iota_N:\tilde{\Lambda}_N\to\tilde{\Lambda}$ be an embedding, sending each $P_\lambda(x_1,\cdots,x_N;q,t)$, $\lambda\in\mathbb{Y}_N$, to $P_\lambda(\cdot;q,t)$. Then $\pi_N\circ\iota_N$ is the identity map of $\tilde{\Lambda}_N$.

We claim that for any $F\in\tilde{\Lambda}$, if $\pi_N(F)=0$, then $\pi_N(\mathbf{D}_{-n}(F))=0$. Indeed, write $F=\sum_{\lambda\in\mathbb{Y}}u_\lambda P_\lambda(\cdot;q,t)$; by $\pi(F)=0$, $u_\lambda=0$ for any $\lambda\in\mathbb{Y}_N$. Since $P_\lambda(\cdot;q,t)$ are eigenvectors of $\mathbf{D}_{-n}$, the coefficient of $P_\lambda(\cdot;q,t)$ in $\mathbf{D}_{-n}(F)$ is zero for any $\lambda\in\mathbb{Y}_N$. As π_N sends every $P_\lambda(\cdot;q,t)$ to 0 for $\lambda\in\mathbb{Y}\backslash\mathbb{Y}_N$, we conclude $\pi_N(\mathbf{D}_{-n}(F))=0$.

Define $\mathbf{D}_{-n}^N:\tilde{\Lambda}_N\to\tilde{\Lambda}_N$ by $\mathbf{D}_{-n}^N=\pi_N\circ\mathbf{D}_{-n}\circ\iota_N$. Then $\mathbf{D}_{-n}^N\circ\pi_N=\pi_N\circ\mathbf{D}_{-n}$, since for any $F\in\tilde{\Lambda}$, $\pi_N(\iota_N\circ\pi_N(F)-F)=\pi_N\circ\iota_N\circ\pi_N(F)-\pi_N(F)=0$, thus $\mathbf{D}_{-n}^N\circ\pi_N(F)-\pi_N\circ\mathbf{D}_{-n}(F)=0$.

It's also easy to check that for each $\lambda\in\mathbb{Y}_N$, $P_\lambda(x_1,\cdots,x_N;q,t)$ is an eigenvector of $\mathbf{D}_{-n}^N$, with eigenvalue $\left((1-t^{-n})\sum_{i=1}^{N}(q^{\lambda_i}t^{-i+1})^n+t^{-Nn}\right)$. For any power series in $\tilde{\Lambda}$ that converges on B_r^N, so does its image under $\mathbf{D}_{-n}^N$. Then the action of $\mathbf{D}_{-n}^N$ is the same as the action of $\mathcal{D}_{-n}^N$.

Note that $\pi_N\prod_{i=1}^{\infty}f(x_i)=\prod_{i=1}^{N}f(x_i)$; hence there is $\mathbf{D}_{-n}^N\prod_{i=1}^{N}f(x_i)=\pi_N\circ\mathbf{D}_{-n}\prod_{i=1}^{\infty}f(x_i)$, which, by (87), equals

$$\begin{aligned}\frac{(-1)^{n-1}}{(2\pi\mathbf{i})^n}\prod_{i=1}^{N}f(x_i)\,\mathrm{Res}\left[\frac{\sum_{i=1}^{n}\frac{z_nt^{n-i}}{z_iq^{n-i}}}{\left(1-\frac{tz_2}{qz_1}\right)\cdots\left(1-\frac{tz_n}{qz_{n-1}}\right)}\prod_{i<j}\frac{\left(1-\frac{z_i}{z_j}\right)\left(1-\frac{qz_i}{tz_j}\right)}{\left(1-\frac{z_i}{tz_j}\right)\left(1-\frac{qz_i}{z_j}\right)}\right.\\
\left.\times\prod_{i=1}^{n}\left(\prod_{i'=1}^{N}\frac{1-t^{-1}q\frac{x_{i'}}{z_i}}{1-q\frac{x_{i'}}{z_i}}\cdot\frac{f(z_i)}{f(q^{-1}z_i)}z_i^{-1}\right)\right].\end{aligned}\tag{88}$$

When $f:B_r\to\mathbb{C}$ is an analytic function with $f(0)=1$, the action of $\mathcal{D}_{-n}^N$ can be computed as (88), by expanding $f(z_i)$ and $f(q^{-1}z_i)$ as power series. The same is true for any analytic $f:B_r\to\mathbb{C}$

such that $f(0) \neq 0$, by multiplying a constant. Further, the map Res can be identified with contour integrals of $z_1, \cdots, z_n$, with the part inside Res being the integrand, and the contours must be taken in a way such that: first, the coefficient for each $\prod_{i=1}^{N} x_i^{m_i}$, which is a series in $\mathbf{Z}$, converges; second, the power series for $x_1, \cdots, x_N$ converges. It suffices to ensure that each $|tz_{i+1}| > |qz_i|$ for $1 \leq i \leq n-1$, $|z_i| < |tz_j|$ for $1 \leq i < j \leq n$, $|qx_{i'}| < |z_i|$ for $1 \leq i \leq n$, $1 \leq i' \leq N$, and each power series $\frac{f(z_i)}{f(q^{-1}z_i)}$ converges. This is guaranteed by the conditions given in Proposition 5.10. $\square$

5.3 Joint higher order moments

In this subsection we present the proof of Theorem 5.1. The general idea is to apply the operators

$$\frac{1}{\theta k_l}\epsilon^{-1}(t^{-N_l k_l} - \mathcal{D}_{-k_l}^{N_l}), \quad \cdots, \quad \frac{1}{\theta k_1}\epsilon^{-1}(t^{-N_1 k_1} - \mathcal{D}_{-k_1}^{N_1}) \tag{89}$$

one by one to both sides of (62), and use Proposition 5.10 and (69) to evaluate the expressions. This is somewhat standard in the study of Macdonald processes, see [BC14], [BCGS16], and [BG15]. However, the operators $\mathcal{D}_{-n}$ are very different from the ones used in those articles. Finally we obtain the joint moments by passing to the limit $\epsilon \to 0_+$.

Proposition 5.11. *Let $N_1 \leq \cdots \leq N_l, k_1, \cdots, k_l$ be positive integers, $a_1, \cdots, a_{N_l}$ and $b_1, \cdots, b_M$ be variables with each $|a_i b_j| < 1$, and $0 < q, t < 1$ be parameters. Then*

$$\begin{aligned}\prod_{i=1}^{l} \frac{1}{\theta k_i}(t^{-N_i k_i} - \mathcal{D}_{-k_i}^{N_i}) &\prod_{1\leq i\leq N_l, 1\leq j\leq M} \frac{\prod_{k=1}^{\infty}(1 - t a_i b_j q^{k-1})}{\prod_{k=1}^{\infty}(1 - a_i b_j q^{k-1})} \\ = \sum_{\mu^1\in\mathbb{Y}_{N_1},\cdots,\mu^l\in\mathbb{Y}_{N_l}} &\prod_{i=1}^{l}\left(\frac{1}{\theta k_i}(t^{-k_i}-1)\sum_{j=1}^{N_i}(q^{\mu_j^i}t^{-j+1})^{k_i}\right) Q_{\mu^l}(b_1,\cdots,b_M;q,t) \\ &\times P_{\mu^1}(a_1,\cdots,a_{N_1};q,t)\prod_{1\leq i<l}\mathfrak{T}_{i+1\to i},\end{aligned} \tag{90}$$

where the operator $\mathcal{D}_{-k_i}^{N_i}$ acts on variables $a_1, \cdots, a_{N_i}$, and

$$\mathfrak{T}_{i+1\to i} = \begin{cases} P_{\mu^{i+1}/\mu^i}(a_{N_i+1},\cdots,a_{N_{i+1}};q,t), & N_i < N_{i+1} \\ \mathbb{1}_{\mu^i=\mu^{i+1}}, & N_i = N_{i+1}. \end{cases} \tag{91}$$

Proof. The proof is similar to proofs in [BCGS16, Proposition 4.9], by induction on l. For $l = 1$, in Cauchy identity (62), set $M_1 = N_1$, $M_2 = M$. Apply the operator $\frac{1}{\theta k_1}\epsilon^{-1}(t^{-N_1 k_1} - \mathcal{D}_{-k_1}^{N_1})$, acting on variables $a_1, \cdots, a_{N_1}$, to both sides. Then we divide both sides by $\prod_{1\leq i\leq N_l, 1\leq j\leq M} \frac{\prod_{k=1}^{\infty}(1-t a_i b_j q^{k-1})}{\prod_{k=1}^{\infty}(1-a_i b_j q^{k-1})}$, and get the desired equation. The condition $|a_i b_j| < 1$ is used to ensure that the left hand side of (90) expands into a convergent symmetric power series in $\{a_i\}_{i=1}^{N_l}$ and $\{b_j\}_{j=1}^{M}$.

For general l, we assume that the statement is true for $l-1$; specifically, we have that

$$\begin{aligned}\prod_{i=2}^{l} \frac{1}{\theta k_i}(t^{-N_i k_i} - \mathcal{D}_{-k_i}^{N_i}) &\prod_{1\leq i\leq N_l, 1\leq j\leq M} \frac{\prod_{k=1}^{\infty}(1 - t a_i b_j q^{k-1})}{\prod_{k=1}^{\infty}(1 - a_i b_j q^{k-1})} \\ = \sum_{\mu^2\in\mathbb{Y}_{N_2},\cdots,\mu^l\in\mathbb{Y}_{N_l}} &\prod_{i=2}^{l}\left(\frac{1}{\theta k_i}(t^{-k_i}-1)\sum_{j=1}^{N_i}(q^{\mu_j^i}t^{-j+1})^{k_i}\right) Q_{\mu^l}(b_1,\cdots,b_M;q,t) \\ &\times P_{\mu^2}(a_1,\cdots,a_{N_2};q,t)\prod_{2\leq i<l}\mathfrak{T}_{i+1\to i}.\end{aligned} \tag{92}$$

If $N_1 = N_2$,

$$P_{\mu^2}(a_1, \cdots, a_{N_2}; q, t) = \sum_{\mu^1 \in \mathbb{Y}_{N_1}} P_{\mu^1}(a_1, \cdots, a_{N_1}; q, t) \mathbb{1}_{\mu_r = \mu_2}; \tag{93}$$

If $N_1 < N_2$,

$$\begin{aligned} &P_{\mu^2}(a_1, \cdots, a_{N_2}; q, t) \\ &\quad = \sum_{\mu^1 \in \mathbb{Y}_{N_1}} P_{\mu^1}(a_1, \cdots, a_{N_1}; q, t) \sum_{\nu^1 \in \mathbb{Y}_{N_1+1}, \cdots, \nu^{N_2-N_1} \in \mathbb{Y}_{N_2}} \prod_{1 \le i \le N_2 - N_1} P_{\nu^i/\nu^{i-1}}(a_{N_1+i}; q, t) \\ &\quad = \sum_{\mu^1 \in \mathbb{Y}_{N_1}} P_{\mu^1}(a_1, \cdots, a_{N_1}; q, t) P_{\mu^2/\mu^1}(a_{N_1+1}, \cdots, a_{N_2}; q, t), \end{aligned} \tag{94}$$

where $\nu^0 = \mu_r$ and $\nu^{N_2-N_1} = \mu_2$, and the last line follows from (63).

In either case we have

$$P_{\mu^2}(a_1, \cdots, a_{N_2}; q, t) = \sum_{\mu^1 \in \mathbb{Y}_{N_1}} P_{\mu^1}(a_1, \cdots, a_{N_1}; q, t) \mathfrak{T}_{i+1 \to i}. \tag{95}$$

Plug this into (92) and apply the operator $\frac{1}{\theta k_1}(t^{-N_1 k_1} - \mathcal{D}^{N_1}_{-k_1})$ to both sides, we immediately obtain (90). By principle of induction, (90) holds for any positive integer l. □

Now we evaluate the same expression, in the special case where $b_j = t^{\alpha+j-1}$ for $1 \le j \le M$, by using Proposition 5.10 multiple times.

Proposition 5.12. *For any positive integer m, $\tilde{m}$, variables $w_1, \cdots, w_m$, $\tilde{w}_1, \cdots, \tilde{w}_{\tilde{m}}$, and parameters q, t, denote*

$$\mathfrak{B}(w_1, \cdots, w_m; q, t) = \frac{\sum_{i=1}^m \frac{w_m t^{m-i}}{w_i q^{m-i}}}{\left(1 - \frac{tw_2}{qw_1}\right) \cdots \left(1 - \frac{tw_m}{qw_{m-1}}\right)} \prod_{i<j} \frac{\left(1 - \frac{w_i}{w_j}\right)\left(1 - \frac{qw_i}{tw_j}\right)}{\left(1 - \frac{w_i}{tw_j}\right)\left(1 - \frac{qw_i}{w_j}\right)}, \tag{96}$$

$$\mathfrak{F}(w_1, \cdots, w_m; \tilde{w}_1, \cdots, \tilde{w}_{\tilde{m}}; q, t) = \prod_{i=1}^m \prod_{i'=1}^{\tilde{m}} \frac{w_i - t^{-1} q \tilde{w}_{i'}}{w_i - q \tilde{w}_{i'}}, \tag{97}$$

$$\mathfrak{C}(w_1, \cdots, w_m; \tilde{w}_1, \cdots, \tilde{w}_{\tilde{m}}; q, t) = \prod_{i=1}^m \prod_{i'=1}^{\tilde{m}} \frac{\left(1 - \frac{w_i}{\tilde{w}_{i'}}\right)\left(1 - \frac{qw_i}{t\tilde{w}_{i'}}\right)}{\left(1 - \frac{w_i}{t\tilde{w}_{i'}}\right)\left(1 - \frac{qw_i}{\tilde{w}_{i'}}\right)}. \tag{98}$$

Then for fixed real parameters $0 < q, t < 1$, $\alpha > 0$, positive integers $N_1 \le \cdots \le N_l, k_1, \cdots, k_l$, and $M > k_1 + \cdots + k_l$, and variables $a_1, \cdots, a_{N_l} \in B_1$, by letting each $\mathcal{D}^{N_i}_{-k_i}$ acting on variables $a_1, \cdots, a_{N_i}$ we have

$$\begin{aligned} &\prod_{i=1}^l \mathcal{D}^{N_i}_{-k_i} \prod_{i=1}^{N_l} \frac{\prod_{k=1}^\infty (1 - a_i t^{\alpha+M} q^{k-1})}{\prod_{k=1}^\infty (1 - a_i t^\alpha q^{k-1})} = \prod_{i=1}^{N_l} \frac{\prod_{k=1}^\infty (1 - a_i t^{\alpha+M} q^{k-1})}{\prod_{k=1}^\infty (1 - a_i t^\alpha q^{k-1})} \\ &\quad \times \frac{(-1)^{k_1+\cdots+k_l-l}}{(2\pi\mathbf{i})^{k_1+\cdots+k_l}} \oint \cdots \oint \prod_{i=1}^l \mathfrak{B}(z_{i,1}, \cdots, z_{i,k_i}; q, t) \prod_{i=1}^l \mathfrak{F}(z_{i,1}, \cdots, z_{i,k_i}; a_1, \cdots, a_{N_i}; q, t) \\ &\quad \times \prod_{i<j} \mathfrak{C}(z_{i,1}, \cdots, z_{i,k_i}; z_{j,1}, \cdots, z_{j,k_j}; q, t) \prod_{i=1}^l \prod_{i'=1}^{k_i} \left(\frac{1 - q^{-1} t^\alpha z_{i,i'}}{1 - q^{-1} t^{\alpha+M} z_{i,i'}} \frac{dz_{i,i'}}{z_{i,i'}} \right), \end{aligned} \tag{99}$$

where the contours are nested and satisfy the following: for each $1 \leq i \leq l$ and $1 \leq i' < k_i$ there is $|z_{i,i'}| < t|z_{i,i'+1}|$; and for each $1 \leq i < l$, there is $|z_{i,k_i}| < t|z_{i+1,1}|$; also, $q < |z_{1,1}|$, and $|z_{l,N_l}| < qt^{-\alpha-M}$.

Proof. We prove by induction on l.

For the base case where $l = 1$, we apply Proposition 5.10 to the function

$$f(x) = \frac{\prod_{k=1}^{\infty}(1 - xt^{\alpha+M}q^{k-1})}{\prod_{k=1}^{\infty}(1 - xt^{\alpha}q^{k-1})}. \tag{100}$$

Specifically, $f(x)$ is analytic in $B_{t^{-\alpha}}$, with $f(0) = 1$. And $\frac{f(x)}{f(q^{-1}x)}$ is analytic in $B_{qt^{-\alpha-M}}$. For $a_1, \cdots, a_{N_1} \in B_1 \subset B_{t^{-\alpha}}$, we can construct contours of $z_{1,1}, \cdots, z_{1,k_1}$ such that $q < |z_{1,1}|$, $|z_{1,k_1}| < qt^{-\alpha-M}$, and $|z_{1,i}| < t|z_{1,i+1}|$ for each $1 \leq i < k_1$, which satisfies the requirements in Proposition 5.10. The expression given by Proposition 5.10 is precisely (99) for $l = 1$.

For more general $l \geq 2$, assume that the statement is true for $l - 1$; then we have

$$\begin{aligned}
&\prod_{i=2}^{l} \mathcal{D}_{-k_i}^{N_i} \prod_{i=1}^{N_l} \frac{\prod_{k=1}^{\infty}(1 - a_i t^{\alpha+M}q^{k-1})}{\prod_{k=1}^{\infty}(1 - a_i t^{\alpha}q^{k-1})} \\
&\qquad = \frac{(-1)^{k_2+\cdots+k_l-l}}{(2\pi\mathbf{i})^{k_2+\cdots+k_l}} \oint \cdots \oint \mathfrak{W}(a_1, \cdots, a_{N_1}; \{\{z_{i,i'}\}_{i'=1}^{k_i}\}_{i=2}^{l}) \prod_{i=2}^{l}\prod_{i'=1}^{k_i} dz_{i,i'},
\end{aligned} \tag{101}$$

for any $a_1, \cdots, a_{N_1} \in B_1$, where

$$\begin{aligned}
\mathfrak{W}(a_1, \cdots, a_{N_1}; \{\{z_{i,i'}\}_{i'=1}^{k_i}\}_{i=2}^{l}) &= \prod_{i=1}^{N_l} \frac{\prod_{k=1}^{\infty}(1 - a_i t^{\alpha+M}q^{k-1})}{\prod_{k=1}^{\infty}(1 - a_i t^{\alpha}q^{k-1})} \\
&\times \prod_{i=2}^{l} \mathfrak{B}(z_{i,1}, \cdots, z_{i,k_i}; q, t) \prod_{i=2}^{l} \mathfrak{F}(z_{i,1}, \cdots, z_{i,k_i}; a_1, \cdots, a_{N_i}; q, t) \\
&\times \prod_{1<i<j} \mathfrak{C}(z_{i,1}, \cdots, z_{i,k_i}; z_{j,1}, \cdots, z_{j,k_j}; q, t) \prod_{i=2}^{l}\prod_{i'=1}^{k_i} \left(\frac{1 - q^{-1}t^{\alpha}z_{i,i'}}{1 - q^{-1}t^{\alpha+M}z_{i,i'}} \frac{dz_{i,i'}}{z_{i,i'}} \right).
\end{aligned} \tag{102}$$

And in (101) the contours are constructed in the following way: for each $2 \leq i \leq l$ and $1 \leq i' < k_i$ there is $|z_{i,i'}| < t|z_{i,i'+1}|$; and for each $2 \leq i < l$, there is $|z_{i,k_i}| < t|z_{i+1,1}|$; also, $q < |z_{2,1}|$, and $|z_{l,N_l}| < qt^{-\alpha-M}$.

Now apply the operator $\mathcal{D}_{-k_1}^{N_1}$ to both sides of (101), acting on variables $a_1, \cdots, a_{N_1}$. In the right hand side of (101), $\mathcal{D}_{-k_1}^{N_1}$ can be directly applied to the integrand, by its linearity and continuity stated in Proposition 5.8. Then we just need to consider the following function

$$f(x) = \frac{\prod_{k=1}^{\infty}(1 - xt^{\alpha+M}q^{k-1})}{\prod_{k=1}^{\infty}(1 - xt^{\alpha}q^{k-1})} \prod_{i=2}^{l}\prod_{i'=1}^{k_i} \frac{z_{i,i'} - t^{-1}qx}{z_{i,i'} - qx}. \tag{103}$$

This $f(x)$ is analytic in B_1, with $f(1) = 0$. Also note that, as $|z_{i,i'}| > qt^{-k_1-1}$ for each $1 \leq i \leq l$

and $1 \le i' \le k_i$, the function $\frac{f(x)}{f(q^{-1}x)}$ is analytic inside $B_{qt^{-k_1}}$. By Proposition 5.10, there is

$$
\begin{aligned}
&\mathcal{D}_{-k_1}^{N_1}\left(\prod_{i=1}^{N_l}\frac{\prod_{k=1}^{\infty}(1-a_i t^{\alpha+M}q^{k-1})}{\prod_{k=1}^{\infty}(1-a_i t^{\alpha}q^{k-1})}\prod_{i=2}^{l}\mathfrak{F}(z_{i,1},\cdots,z_{i,k_i};a_1,\cdots,a_{N_i};q,t)\right)\\
&\qquad=\prod_{i=1}^{N_l}\frac{\prod_{k=1}^{\infty}(1-a_i t^{\alpha+M}q^{k-1})}{\prod_{k=1}^{\infty}(1-a_i t^{\alpha}q^{k-1})}\prod_{i=2}^{l}\mathfrak{F}(z_{i,1},\cdots,z_{i,k_i};a_1,\cdots,a_{N_i};q,t)\\
&\qquad\times\frac{(-1)^{k_1-1}}{(2\pi\mathbf{i})^{k_1}}\oint\cdots\oint\mathfrak{B}(z_{1,1},\cdots,z_{1,k_1};q,t)\mathfrak{F}(z_{1,1},\cdots,z_{1,k_1};a_1,\cdots,a_{N_1};q,t)\\
&\qquad\qquad\times\prod_{i=2}^{l}\mathfrak{C}(z_{1,1},\cdots,z_{1,k_1};z_{i,1},\cdots,z_{i,k_i};q,t)\prod_{i'=1}^{k_1}\left(\frac{1-q^{-1}t^{\alpha}z_{1,i'}}{1-q^{-1}t^{\alpha+M}z_{1,i'}}\frac{dz_{1,i'}}{z_{1,i'}}\right),
\end{aligned}
\tag{104}
$$

for any $a_1,\cdots,a_{N_1}\in B_1$, and the contours are constructed such that for each $1\le i<k_1$ there is $|z_{1,i}|<t|z_{1,i+1}|$, $q<|z_{1,1}|$, and $|z_{l,N_l}|<qt^{-k_1}$.

Putting (101) and (104) together we get exactly (99). □

We next perform limit transition (67) in formula (99). In the integral we want to change variables:

$$
z_{i,i'}=\exp(\epsilon u_{i,i'}),\quad 1\le i\le l, 1\le i'\le k_i,
\tag{105}
$$

and all the contours $u_{i,i'}$ are nested in a certain way to give satisfied contours of $z_{i,i'}$.

There is a difficulty in doing this: (105) implies that as $\epsilon\to 0_+$, $z_{i,i'}$ approaches to 1. However, originally each $z_{i,i'}$ encloses 0. The idea to resolve this is to split each contour of $z_{i,i'}$ (99) into two: one enclosing 0 and another enclosing all of $qa_1,\cdots,qa_{N_l}$. It turns out that most terms with contours enclosing 0 are evaluated to zero or cancel out.

Formally, for each $z_{i,i'}$, we associate it with two contours $\mathfrak{U}_{i,i'}$ and $\mathfrak{V}_{i,i'}$, satisfying: for each $1\le i\le l$ and $1\le i'<k_i$, $\mathfrak{U}_{i,i'}$ is inside $t\mathfrak{U}_{i,i'+1}$, $\mathfrak{V}_{i,i'}$ is inside $t\mathfrak{V}_{i,i'+1}$; for each $1\le i<l$, $\mathfrak{U}_{i,k_i}$ is inside $t\mathfrak{U}_{i+1,1}$, $\mathfrak{V}_{i,k_i}$ is inside $t\mathfrak{V}_{i+1,1}$. Also, each of $\mathfrak{U}_{i,i'}$ encloses 0 but none of $qa_1,\cdots,qa_{N_l}$, while each of $\mathfrak{V}_{i,i'}$ encloses $qa_1,\cdots,qa_{N_l}$ but not 0. All of these contours are inside $B_{qt^{-\alpha-M}}$. Such contours exist as long as $1-t$ is small enough.

Let Π be the power set of $\{z_{i,i'}|1\le i\le l,1\le i'\le k_i\}$, then for each $\Upsilon\in\Pi$, denote $\Upsilon_{i,i'}$ to be $\mathfrak{U}_{i,i'}$ if $z_{i,i'}\in\Upsilon$, and $\mathfrak{V}_{i,i'}$ if $z_{i,i'}\notin\Upsilon$. Let

$$
\begin{aligned}
\mathfrak{Q}_{\Upsilon}=&\oint_{\Upsilon_{1,1}}\cdots\oint_{\Upsilon_{l,k_l}}\prod_{i=1}^{l}\mathfrak{B}(z_{i,1},\cdots,z_{i,k_i};q,t)\prod_{i=1}^{l}\mathfrak{F}(z_{i,1},\cdots,z_{i,k_i};a_1,\cdots,a_{N_i};q,t)\\
&\times\prod_{i<j}\mathfrak{C}(z_{i,1},\cdots,z_{i,k_i};z_{j,1},\cdots,z_{j,k_j};q,t)\prod_{i=1}^{l}\prod_{i'=1}^{k_i}\left(\frac{1-q^{-1}t^{\alpha}z_{i,i'}}{1-q^{-1}t^{\alpha+M}z_{i,i'}}\frac{dz_{i,i'}}{z_{i,i'}}\right).
\end{aligned}
\tag{106}
$$

Then (99) can be written as

$$
\frac{(-1)^{k_1+\cdots+k_l-l}}{(2\pi\mathbf{i})^{k_1+\cdots+k_l}}\prod_{i=1}^{N_l}\frac{\prod_{k=1}^{\infty}(1-a_i t^{\alpha+M}q^{k-1})}{\prod_{k=1}^{\infty}(1-a_i t^{\alpha}q^{k-1})}\sum_{\Upsilon\in\Pi}\mathfrak{Q}_{\Upsilon}.
\tag{107}
$$

The following Lemma is an extension of [FD16, Appendix A, Lemma 5].

Lemma 5.13. $\mathfrak{Q}_{\Upsilon}=0$ *unless for each* $1\le i\le l$, $\Upsilon\bigcap\{z_{i,1},\cdots,z_{i,k_i}\}$ *is either empty or of the form* $\bigcup_{i=1}^{l}\{z_{i,s_i},z_{i,s_i+1},\cdots,z_{i,r_i}\}$, *where* $1\le s_i\le r_i\le k_i$.

Proof. Let us order the variables $z_{i,i'}$ as in the nesting of the contours, from inner to outer: $z_{1,1}, z_{1,2}, \cdots, z_{1,k_1}, z_{2,1}, \cdots, z_{l,k_l}$. Fix some $1 \le i \le l$, take (i, s_i) (if any) such that z_{i,s_i} is the smallest one belonging to Υ. Integrating z_{i,s_i}, the residue is of similar form to (106), but fewer variable, with factor

$$\frac{\sum_{i'=1}^{k_i} \frac{z_{i,k_i} t^{k_i - i'}}{z_{i,i'} q^{k_i - i'}}}{\left(1 - \frac{tz_{i,2}}{qz_{i,1}}\right) \cdots \left(1 - \frac{tz_{i,k_i}}{qz_{i,k_i-1}}\right)} \tag{108}$$

replaced by

$$\frac{-\frac{z_{i,k_i} t^{k_i - s_i - 1}}{z_{i,s_i+1} q^{k_i - s_i - 1}}}{\left(1 - \frac{tz_{i,2}}{qz_{i,1}}\right) \cdots \left(1 - \frac{tz_{i,s_i-1}}{qz_{i,s_i-2}}\right)\left(1 - \frac{tz_{i,s_i+2}}{qz_{i,s_i+1}}\right) \cdots \left(1 - \frac{tz_{i,k_i}}{qz_{i,k_i-1}}\right)}. \tag{109}$$

If $s_i < k_i$ and $z_{i,s_i+1} \notin \Upsilon$, then after integrating z_{i,s_i+1} there is no pole at 0 of $z_{i,i'}$, for any $s_i + 1 < i' \le k_i$. If there is any $z_{i,i'} \in \Upsilon$, for $s_i + 1 < i' \le k_i$, the whole integral gives 0.

If $s_i < k_i$ and $z_{i,s_i+1} \in \Upsilon$, integrating z_{i,s_i+1}, the residue is again of similar form to (106), with fewer variables, and the factor (108) is replaced by

$$\frac{\frac{z_{i,k_i} t^{k_i - s_i - 2}}{z_{i,s_i+2} q^{k_i - s_i - 2}}}{\left(1 - \frac{tz_{i,2}}{qz_{i,1}}\right) \cdots \left(1 - \frac{tz_{i,s_i-1}}{qz_{i,s_i-2}}\right)\left(1 - \frac{tz_{i,s_i+3}}{qz_{i,s_i+2}}\right) \cdots \left(1 - \frac{tz_{i,k_i}}{qz_{i,k_i-1}}\right)}. \tag{110}$$

which still has the same form and we can continue in the same way. □

Lemma 5.14. *Let $\tilde{\Pi}$ be the collection of all Υ, such that for each $1 \le i \le l$, $\Upsilon \bigcap \{z_{i,1}, \cdots, z_{i,k_i}\}$ is either empty or of the form $\bigcup_{i=1}^{l} \{z_{i,s_i}, z_{i,s_i+1} \cdots, z_{i,r_i}\}$ for some $1 \le s_i \le r_i \le k_i$, but $(s_i, r_i) \ne (1, k_i)$, i.e. we prohibit $\{z_{i,1}, \cdots, z_{i,k_i}\} \subset \Upsilon$. Then we have*

$$\prod_{i=1}^{l} \left(\mathcal{D}_{-k_i}^{N_i} - t^{-N_i k_i}\right) \prod_{i=1}^{N_l} \frac{\prod_{k=1}^{\infty}(1 - a_i t^{\alpha+M} q^{k-1})}{\prod_{k=1}^{\infty}(1 - a_i t^{\alpha} q^{k-1})}$$
$$= \frac{(-1)^{k_1+\cdots+k_l-l}}{(2\pi\mathbf{i})^{k_1+\cdots+k_l}} \prod_{i=1}^{N_l} \frac{\prod_{k=1}^{\infty}(1 - a_i t^{\alpha+M} q^{k-1})}{\prod_{k=1}^{\infty}(1 - a_i t^{\alpha} q^{k-1})} \sum_{\Upsilon \in \tilde{\Pi}} \mathfrak{Q}_{\Upsilon}. \tag{111}$$

Proof. When we compute the integral in (107), if for some $1 \le w \le l$, $\{z_{w,1}, \cdots, z_{w,k_w}\} \subset \Upsilon$, integrating through $z_{w,1}, \cdots, z_{w,k_w}$ we get $t^{-N_w k_w}$. Therefore, (99) and (107) lead to (111). □

Now we can send $\epsilon \to 0_+$.

Lemma 5.15. *Set $a_i = t^{i-1}$, $t = \exp(-\theta\epsilon)$, $q = \exp(-\epsilon)$. For $\Upsilon \in \Pi$, if there is $1 \le w \le l$ such that $\{z_{w,1}, \cdots, z_{w,k_w}\} \bigcap \Upsilon = \{z_{w,s_w}, \cdots, z_{w,r_w}\}$, for some $1 < s_w \le r_w < k_w$, then*

$$\lim_{\epsilon \to 0_+} \epsilon^{-l} \mathfrak{Q}_{\Upsilon} = 0. \tag{112}$$

For $\Upsilon', \Upsilon'' \in \Pi$, if there is $1 \le w \le l$ such that $\{z_{w,1}, \cdots, z_{w,k_w}\} \bigcap \Upsilon' = \{z_{w,1}, \cdots, z_{w,r_w}\}$, $\{z_{w,1}, \cdots, z_{w,k_w}\} \bigcap \Upsilon'' = \{z_{w,s_w}, \cdots, z_{w,k_w}\}$, with $r_w = k_w - s_w + 1$; and for any $w' \ne w$ there is $\{z_{w',1}, \cdots, z_{w',k_{w'}}\} \bigcap \Upsilon' = \{z_{w',1}, \cdots, z_{w',k_w}\} \bigcap \Upsilon''$; then

$$\lim_{\epsilon \to 0_+} \epsilon^{-l} \mathfrak{Q}_{\Upsilon'} + \mathfrak{Q}_{\Upsilon''} = 0. \tag{113}$$

Proof. By induction and using a similar argument in the proof of Lemma 5.13, for any $\Upsilon \in \Pi$, in the expression $\mathfrak{Q}_\Upsilon$ we integrate all variables in Υ and get

$$\mathfrak{Q}_\Upsilon = \oint \cdots \oint_{\{\Upsilon_{i,i'}: z_{i,i'} \notin \Upsilon\}} \prod_{i=1}^{l} \mathfrak{Y}_i(\Upsilon) \prod_{i=1}^{l} \mathfrak{F}(\{z_{i,1}, \cdots, z_{i,k_i}\} \backslash \Upsilon; a_1, \cdots, a_{N_i}; q, t)$$
$$\times \prod_{i<j} \mathfrak{C}(\{z_{i,1}, \cdots, z_{i,k_i}\} \backslash \Upsilon; \{z_{j,1}, \cdots, z_{j,k_j}\} \backslash \Upsilon; q, t) \prod_{z_{i,i'} \notin \Upsilon} \left(\frac{1 - q^{-1} t^{\alpha} z_{i,i'}}{1 - q^{-1} t^{\alpha+M} z_{i,i'}} \frac{dz_{i,i'}}{z_{i,i'}} \right), \quad (114)$$

where

$$\mathfrak{Y}_i(\Upsilon) = (-1)^{k_i - s_i} \frac{t^{-N_i(k_i - s_i + 1)}}{\left(1 - \frac{tz_{i,2}}{qz_{i,1}}\right) \cdots \left(1 - \frac{tz_{i,s_i-1}}{qz_{i,s_i-2}}\right)}, \quad (115)$$

if $\{z_{i,1}, \cdots, z_{i,k_i}\} \bigcap \Upsilon$ is of the form $\bigcup_{i=1}^{l} \{z_{i,s_i}, \cdots, z_{i,k_i}\}$, for some $1 < s_i \le k_i$;

$$\mathfrak{Y}_i(\Upsilon) = (-1)^{r_i} \frac{\frac{z_{i,k_i} t^{k_i - r_i - 1}}{z_{i,r_i+1} q^{k_i - r_i - 1}} t^{-N_i r_i}}{\left(1 - \frac{tz_{i,r_i+2}}{qz_{i,r_i+1}}\right) \cdots \left(1 - \frac{tz_{i,k_i}}{qz_{i,k_i-1}}\right)}, \quad (116)$$

if $\{z_{i,1}, \cdots, z_{i,k_i}\} \bigcap \Upsilon$ is of the form $\bigcup_{i=1}^{l} \{z_{i,1}, \cdots, z_{i,r_i}\}$, for some $1 \le r_i < k_i$;

$$\mathfrak{Y}_i(\Upsilon) = (-1)^{r_i - s_i + 1} \frac{\frac{z_{i,k_i} t^{k_i - r_i - 1}}{z_{i,r_i+1} q^{k_i - r_i - 1}} t^{-N_i(r_i - s_i + 1)}}{\left(1 - \frac{tz_{i,2}}{qz_{i,1}}\right) \cdots \left(1 - \frac{tz_{i,s_i-1}}{qz_{i,s_i-2}}\right) \left(1 - \frac{tz_{i,r_i+2}}{qz_{i,r_i+1}}\right) \cdots \left(1 - \frac{tz_{i,k_i}}{qz_{i,k_i-1}}\right)}, \quad (117)$$

if $\{z_{i,1}, \cdots, z_{i,k_i}\} \bigcap \Upsilon$ is of the form $\bigcup_{i=1}^{l} \{z_{i,s_i}, \cdots, z_{i,r_i}\}$, for some $1 < s_i \le r_i < k_i$.

Now we make the change of variables (105) and send $\epsilon \to 0_+$. There is $dz_{i,i'} = \epsilon \exp(\epsilon u_{i,i'}) du_{i,i'}$. Therefore, each integration variable produces an ϵ factor. For $\mathfrak{Y}_i(\Upsilon)$, the expression (115) grows in the order of $\epsilon^{-s_i+2} = \epsilon^{-|\{z_{i,1},\cdots,z_{i,k_i}\}\backslash\Upsilon|+1}$; and (116) grows in the order of $\epsilon^{-k_i+r_i+1} = \epsilon^{-|\{z_{i,1},\cdots,z_{i,k_i}\}\backslash\Upsilon|+1}$; while (117) grows in the order of $\epsilon^{-k_i+r_i-s_i+3} = \epsilon^{-|\{z_{i,1},\cdots,z_{i,k_i}\}\backslash\Upsilon|+2}$. All other factors in the integrand converge to constants. Then if there is at least one $\mathfrak{Y}_i(\Upsilon)$ in the case of (117), we obtain (112).

Now we consider a pair Υ' and Υ'' as described in the statement of this Lemma. Then for any $w' \neq w$, $\mathfrak{Y}_{w'}(\Upsilon') = \mathfrak{Y}_{w'}(\Upsilon'')$. For w, we can identify $z_{i,i'}$ in $\mathfrak{Y}_w(\Upsilon'')$ with $z_{i,i'+r_w}$ in $\mathfrak{Y}_w(\Upsilon')$; then we conclude (using the notation in $\mathfrak{Y}_w(\Upsilon')$)

$$\mathfrak{Y}_w(\Upsilon') + \mathfrak{Y}_w(\Upsilon'') = (-1)^{r_i} \frac{\left(\frac{z_{i,k_i} t^{k_i - r_i - 1}}{z_{i,r_i+1} q^{k_i - r_i - 1}} - 1\right) t^{-N_i r_i}}{\left(1 - \frac{tz_{i,r_i+2}}{qz_{i,r_i+1}}\right) \cdots \left(1 - \frac{tz_{i,k_i}}{qz_{i,k_i-1}}\right)}, \quad (118)$$

and as $\epsilon \to 0_+$ this grows in the order of $\epsilon^{-k_i+r_i+2} = \epsilon^{-|\{z_{i,1},\cdots,z_{i,k_i}\}\backslash\Upsilon|+2}$. We thus conclude (113). □

Proposition 5.16. *Following the notation of Theorem 5.1, let $N_1 \le \cdots \le N_l, k_1, \cdots, k_l, M$ be positive integers, $M > k_1 + \cdots + k_l$. Let $a_1, \cdots, a_{N_l}$ be variables, and q, t be parameters. Then* *under (67),*

$$\lim_{\epsilon\to 0_+}\frac{\epsilon^{-l}\prod_{i=1}^{l}(\mathcal{D}_{-k_i}^{N_i}-t^{-N_ik_i})\prod_{i=1}^{N_l}\frac{\prod_{k=1}^{\infty}(1-a_it^{\alpha+M}q^{k-1})}{\prod_{k=1}^{\infty}(1-a_it^{\alpha}q^{k-1})}}{\prod_{i=1}^{N_l}\frac{\prod_{k=1}^{\infty}(1-a_it^{\alpha+M}q^{k-1})}{\prod_{k=1}^{\infty}(1-a_it^{\alpha}q^{k-1})}}=\frac{\prod_{i=1}^{l}k_i}{(2\pi\mathbf{i})^{k_1+\cdots+k_l}}\oint\cdots\oint$$

$$\times\prod_{i=1}^{l}\mathfrak{I}(u_{i,1},\cdots,u_{i,k_i};\alpha,M,\theta,N_i)\prod_{i<j}\mathfrak{L}(u_{i,1},\cdots,u_{i,k_i};u_{j,1},\cdots,u_{j,k_j};\alpha,M,\theta)\prod_{i=1}^{l}\prod_{i'=1}^{k_i}du_{i,i'},\quad(119)$$

where for each $i=1,\cdots,l$, the contours of $u_{i,1},\cdots,u_{i,k_i}$ enclose $-\theta(N_i-1)$ but not $\theta(\alpha+M)$, and $|u_{i,1}|\ll\cdots\ll|u_{i,k_i}|$. For $1\leq i\leq i+1\leq l$, we also require that $|u_{i,k_i}|\ll|u_{i+1,1}|$.

Proof. From Lemma 5.14 and Lemma 5.15 we see that

$$\lim_{\epsilon\to 0_+}\frac{\epsilon^{-l}\prod_{i=1}^{l}(\mathcal{D}_{-k_i}^{N_i}-t^{-N_ik_i})\prod_{i=1}^{N_l}\frac{\prod_{k=1}^{\infty}(1-a_it^{\alpha+M}q^{k-1})}{\prod_{k=1}^{\infty}(1-a_it^{\alpha}q^{k-1})}}{\prod_{i=1}^{N_l}\frac{\prod_{k=1}^{\infty}(1-a_it^{\alpha+M}q^{k-1})}{\prod_{k=1}^{\infty}(1-a_it^{\alpha}q^{k-1})}}=\lim_{\epsilon\to 0_+}\frac{(-1)^{k_1+\cdots+k_l-l}}{(2\pi\mathbf{i})^{k_1+\cdots+k_l}}\mathfrak{Q}_{\emptyset}.\quad(120)$$

Notice that in $\mathfrak{Q}_{\emptyset}$, each contour encloses all of $qa_1,\cdots,qa_{N_l}$ but not 0; then we can set each $z_{i,i'}=\exp(\epsilon u_{i,i'})$, with $u_{i,i'}$ independent of ϵ, satisfying the stated requirements. Evaluating the limit gives (119). □

Proof of Theorem 5.1. With Proposition 5.5, the identity given by Proposition 5.11 can be interpreted as

$$\frac{\prod_{i=1}^{l}\frac{1}{\theta k_i}(t^{-N_ik_i}-\mathcal{D}_{-k_i}^{N_i})\prod_{i=1}^{N_l}\frac{\prod_{k=1}^{\infty}(1-a_it^{\alpha+M}q^{k-1})}{\prod_{k=1}^{\infty}(1-a_it^{\alpha}q^{k-1})}}{\prod_{i=1}^{N_l}\frac{\prod_{k=1}^{\infty}(1-a_it^{\alpha+M}q^{k-1})}{\prod_{k=1}^{\infty}(1-a_it^{\alpha}q^{k-1})}}$$

$$=\mathbb{E}\left(\prod_{i=1}^{l}\left(\frac{1}{\theta k_i}(t^{-N_ik_i}-1)\sum_{j=1}^{N_i}(q^{\lambda_j^i}t^{-j+1})^{k_i}\right)\right)\quad(121)$$

where the joint distribution of $\lambda^{N_1},\cdots,\lambda^{N_l}$ is as a Macdonald process with parameters M, $\{a_1,\cdots,a_{N_l},0,\cdots\}$, and $\{b_i\}_{i=1}^{M}$.

Under the limit transition (67) (and using Theorem 5.6) we have that

$$\lim_{\epsilon\to 0_+}\mathbb{E}\left(\prod_{i=1}^{l}\left(\frac{1}{\theta k_i}\epsilon^{-1}(t^{-N_ik_i}-1)\sum_{j=1}^{N_i}(q^{\lambda_j^i}t^{-j+1})^{k_i}\right)\right)=\mathbb{E}\left(\mathfrak{P}_{k_1}(x^{N_1})\cdots\mathfrak{P}_{k_l}(x^{N_l})\right).\quad(122)$$

Then we have

$$\mathbb{E}\left(\mathfrak{P}_{k_1}(x^{N_1})\cdots\mathfrak{P}_{k_l}(x^{N_l})\right)=\lim_{\epsilon\to 0_+}\frac{\epsilon^{-l}\prod_{i=1}^{l}\frac{1}{\theta k_i}(t^{-N_ik_i}-\mathcal{D}_{-k_i}^{N_i})\prod_{i=1}^{N_l}\frac{\prod_{k=1}^{\infty}(1-a_it^{\alpha+M}q^{k-1})}{\prod_{k=1}^{\infty}(1-a_it^{\alpha}q^{k-1})}}{\prod_{i=1}^{N_l}\frac{\prod_{k=1}^{\infty}(1-a_it^{\alpha+M}q^{k-1})}{\prod_{k=1}^{\infty}(1-a_it^{\alpha}q^{k-1})}}.\quad(123)$$

Plugging in Proposition 5.16 finishes the proof. □

6 Law of large numbers: proofs of Theorems 3.1, 3.2, 3.3, 3.5

In this section we present the proofs of Theorems 3.3, 3.2, and 3.3. We also explain how these results lead to Theorem 3.5 under the limit $\theta\to\infty$.

6.1 First moment of adjacent rows

Proof of Theorem 3.1. Taking $l = 1$ in Theorem 5.1 we get

$$\mathbb{E}\left(\mathfrak{P}_k(x^N) - \mathfrak{P}_k(x^{N-1})\right) = \frac{(-\theta)^{-1}}{(2\pi\mathbf{i})^k} \oint \cdots \oint \frac{1}{(u_2 - u_1 + 1 - \theta) \cdots (u_k - u_{k-1} + 1 - \theta)}$$
$$\times \prod_{i<j} \frac{(u_j - u_i)(u_j - u_i + 1 - \theta)}{(u_j - u_i + 1)(u_j - u_i - \theta)} \left(\prod_{i=1}^k \frac{u_i - \theta}{u_i + (N-1)\theta} - \prod_{i=1}^k \frac{u_i - \theta}{u_i + (N-2)\theta} \right)$$
$$\times \prod_{i=1}^k \frac{\theta\alpha - u_i}{\theta(\alpha + M) - u_i} du_i. \quad (124)$$

Send $L \to \infty$ under (22), setting $u_i \sim L\theta v_i$. Then

$$\lim_{L\to\infty} L \left(\prod_{i=1}^k \frac{u_i - \theta}{u_i + (N-1)\theta} - \prod_{i=1}^k \frac{u_i - \theta}{u_i + (N-2)\theta} \right)$$
$$= \lim_{L\to\infty} L \prod_{i=1}^k \frac{u_i - \theta}{u_i + (N-1)\theta} \left(1 - \prod_{i=1}^k \frac{u_i + (N-1)\theta}{u_i + (N-2)\theta} \right) = -\left(\prod_{i=1}^k \frac{v_i}{v_i + \hat{N}} \right) \left(\sum_{i=1}^k \frac{1}{v_i + \hat{N}} \right). \quad (125)$$

Thus we have

$$\lim_{L\to\infty} \mathbb{E}\left(\mathfrak{P}_k(x^N) - \mathfrak{P}_k(x^{N-1})\right) = \frac{1}{(2\pi\mathbf{i})^k} \oint \cdots \oint \frac{1}{(v_2 - v_1) \cdots (v_k - v_{k-1})}$$
$$\times \left(\prod_{i=1}^k \frac{v_i}{v_i + \hat{N}} \cdot \frac{\hat{\alpha} - v_i}{\hat{\alpha} + \hat{M} - v_i} dv_i \right) \left(\sum_{i=1}^k \frac{1}{v_i + \hat{N}} \right), \quad (126)$$

where the contours enclose $-\hat{N}$ but not $\hat{\alpha} + \hat{M}$, and $|v_1| \ll \cdots \ll |v_k|$. By Corollary 4.2, this is simplified to (26).

The decay of variance will be proved as a special case of Lemma 7.5. $\square$

6.2 Convergence of diagrams

For Theorem 3.2, we first prove the following Proposition.

Proposition 6.1. *Let φ be defined as in Theorem 3.2. For any nonnegative integer k, under the limit scheme (22) we have*

$$\lim_{L\to\infty} \int_0^1 w^{\tilde{x}^N, \tilde{x}^{N-1}}(u) u^k du = \int_0^1 \varphi(u) u^k du, \quad (127)$$

in probability.

The proof of Proposition 6.1 relies on the following identity.

Lemma 6.2. *For φ as in Theorem 3.2, we have*

$$\frac{1}{2\pi\mathbf{i}} \oint_\Gamma \left(\frac{v}{v + \hat{N}} \cdot \frac{v - \hat{\alpha}}{v - \hat{\alpha} - \hat{M}} \right)^k \frac{1}{v + \hat{N}} dv = \frac{1}{2} \int_0^1 \varphi''(u) u^k du, \quad (128)$$

where Γ is a positive oriented contour enclosing $-\hat{N}$ but not $\hat{\alpha} + \hat{M}$.

Proof. Fix the contour Γ, and let $\mathcal{Q}$ be a constant

$$\mathcal{Q} := \inf_{v\in\Gamma}\left|\frac{v+\hat{N}}{v}\cdot\frac{v-\hat{\alpha}-\hat{M}}{v-\hat{\alpha}}\right|. \tag{129}$$

For any $z \in \mathbb{C}$, $|z| < Q$, summing the geometric series we get

$$\begin{aligned}
&\sum_{k=0}^{\infty} z^k \frac{1}{2\pi\mathbf{i}}\oint_{\Gamma}\left(\frac{v}{v+\hat{N}}\cdot\frac{v-\hat{\alpha}}{v-\hat{\alpha}-\hat{M}}\right)^k\frac{1}{v+\hat{N}}dv\\
&\quad=\frac{1}{2\pi\mathbf{i}}\oint_{\Gamma}\sum_{k=0}^{\infty}\left(z\cdot\frac{v}{v+\hat{N}}\cdot\frac{v-\hat{\alpha}}{v-\hat{\alpha}-\hat{M}}\right)^k\frac{1}{v+\hat{N}}dv=\frac{1}{2\pi\mathbf{i}}\oint_{\Gamma}\frac{v-\hat{\alpha}-\hat{M}}{(1-z)(v-\mathcal{R}_1)(v-\mathcal{R}_2)}dv,
\end{aligned} \tag{130}$$

where

$$\begin{aligned}
\mathcal{R}_1&=\frac{(1-z)\hat{\alpha}+\hat{M}-\hat{N}-\sqrt{((1-z)\hat{\alpha}+\hat{M}+\hat{N})^2-4z\hat{M}\hat{N}}}{2(1-z)},\\
\mathcal{R}_2&=\frac{(1-z)\hat{\alpha}+\hat{M}-\hat{N}+\sqrt{((1-z)\hat{\alpha}+\hat{M}+\hat{N})^2-4z\hat{M}\hat{N}}}{2(1-z)}.
\end{aligned} \tag{131}$$

The definition of the contour Γ implies that there exists $0 < \mathcal{Q}' < \mathcal{Q}$, such that Γ encloses $\mathcal{R}_1$, but not $\mathcal{R}_2$, for all $z \in \mathbb{C}$, $|z| < Q'$. Then (130) is evaluated as a residue at $\mathcal{R}_1$; more precisely,

$$\begin{aligned}
&\sum_{k=0}^{\infty} z^k \frac{1}{2\pi\mathbf{i}}\oint_{\Gamma}\left(\frac{v}{v+\hat{N}}\cdot\frac{v-\hat{\alpha}}{v-\hat{\alpha}-\hat{M}}\right)^k\frac{1}{v+\hat{N}}dv\\
&\qquad=\frac{1}{2\pi\mathbf{i}(1-z)}\oint_{\Gamma}\frac{v-\hat{\alpha}-\hat{M}}{(v-\mathcal{R}_1)(v-\mathcal{R}_2)}dv=\frac{1}{1-z}\cdot\frac{\mathcal{R}_1-\hat{\alpha}-\hat{M}}{\mathcal{R}_1-\mathcal{R}_2}.
\end{aligned} \tag{132}$$

On the other hand, for $|z| < 1$ we have that

$$\begin{aligned}
&\sum_{k=0}^{\infty}\frac{1}{2}\int_0^1\varphi''(u)u^k z^k du\\
&\quad=\frac{C(\hat{M},\hat{N})}{1-z}+\sum_{k=0}^{\infty}\frac{1}{2\pi}\int_{\gamma_1}^{\gamma_2}\frac{\hat{M}-\hat{N}+(\hat{N}+\hat{M}+\hat{\alpha})(1-u)}{(\hat{N}+\hat{M}+\hat{\alpha})(1-u)}\cdot\frac{z^k u^k}{\sqrt{(\gamma_2-u)(u-\gamma_1)}}du\\
&\quad=\frac{C(\hat{M},\hat{N})}{1-z}+\frac{1}{2\pi}\int_{\gamma_1}^{\gamma_2}\frac{\hat{M}-\hat{N}+(\hat{N}+\hat{M}+\hat{\alpha})(1-u)}{(\hat{N}+\hat{M}+\hat{\alpha})(1-u)(1-zu)}\cdot\frac{1}{\sqrt{(\gamma_2-u)(u-\gamma_1)}}du,
\end{aligned} \tag{133}$$

where γ_1 and γ_2 are defined as in Theorem 3.2. The last integral can be evaluated in the following way: define $\eta : (-\infty,\gamma_1]\bigcup[\gamma_2,\infty) \to \mathbb{C}$, as

$$\eta(w)=\frac{\hat{M}-\hat{N}+(\hat{N}+\hat{M}+\hat{\alpha})(1-w)}{(\hat{N}+\hat{M}+\hat{\alpha})(1-w)(1-zw)}\cdot\frac{1}{\sqrt{(w-\gamma_1)(w-\gamma_2)}}, \tag{134}$$

and then extend the definition of η to $\mathbb{C}\backslash[\gamma_1,\gamma_2]\bigcup\{1\}\bigcup\{z^{-1}\}$, by taking the analytic continuation. The integral in the last line of (133) is equal to one half of the contour integral of η around $[\gamma_1,\gamma_2]$. Then it suffices to compute the residues of $\eta(w)$ at 1 and z^{-1}, which are

$$\frac{1}{1-z}\left(\frac{1}{2}-C(\hat{M},\hat{N})\right),\quad \frac{\hat{M}+\hat{N}+\hat{\alpha}-2\hat{M}z-z\hat{\alpha}}{2(1-z)\sqrt{((1-z)\hat{\alpha}+\hat{M}+\hat{N})^2-4z\hat{M}\hat{N}}} \tag{135}$$

respectively. We thus conclude that (133) equals

$$\frac{1}{2(1-z)}+\frac{\hat{M}+\hat{N}+\hat{\alpha}-2\hat{M}z-z\hat{\alpha}}{2(1-z)\sqrt{((1-z)\hat{\alpha}+\hat{M}+\hat{N})^2-4z\hat{M}\hat{N}}}, \tag{136}$$

which coincides with (132).

In other words, for any $z\in\mathbb{C}$, $|z|<\min\{\mathcal{Q}',1\}$, we have

$$\sum_{k=0}^{\infty}z^k\frac{1}{2\pi\mathbf{i}}\oint\left(\frac{v}{v+\hat{N}}\cdot\frac{v-\hat{\alpha}}{v-\hat{\alpha}-\hat{M}}\right)^k\frac{1}{v+\hat{N}}dv=\sum_{k=0}^{\infty}z^k\frac{1}{2}\int_0^1\varphi''(u)u^kdu. \tag{137}$$

The uniqueness of Taylor series expansion then implies (128). □

Proof of Proposition 6.1. First, by Remark 2.6, Theorem 3.1, and Lemma 6.2, for any nonnegative integer k, under the limit scheme (22) we have

$$\lim_{L\to\infty}\int_0^1\frac{d^2}{du^2}w^{\tilde{x}^N,\tilde{x}^{N-1}}(u)u^kdu=\lim_{L\to\infty}2\left(\mathfrak{P}_k(x^N)-\mathfrak{P}_k(x^{N-1})\right)=\int_0^1\varphi''(u)u^kdu, \tag{138}$$

in probability.

To show that (127) holds, we integrate by parts for any $k=0,1,\cdots$:

$$\begin{aligned}\int_0^1 w^{\tilde{x}^N,\tilde{x}^{N-1}}(u)u^kdu&=\frac{w^{\tilde{x}^N,\tilde{x}^{N-1}}(1)}{k+1}-\frac{1}{k+1}\int_0^1\frac{d}{du}w^{\tilde{x}^N,\tilde{x}^{N-1}}(u)u^{k+1}du\\&=\frac{w^{\tilde{x}^N,\tilde{x}^{N-1}}(1)}{k+1}-\frac{\frac{d}{du}w^{\tilde{x}^N,\tilde{x}^{N-1}}(1)}{(k+1)(k+2)}+\frac{1}{(k+1)(k+2)}\int_0^1\frac{d^2}{du^2}w^{\tilde{x}^N,\tilde{x}^{N-1}}(u)u^{k+2}du.\end{aligned} \tag{139}$$

Since the center of $w^{\tilde{x}^N,\tilde{x}^{N-1}}$ is $\sum_{i=1}^N\tilde{x}_i^N-\sum_{i=1}^{N-1}\tilde{x}_i^{N-1}\in[0,1]$, we have that

$$w^{\tilde{x}^N,\tilde{x}^{N-1}}(1)+w^{\tilde{x}^N,\tilde{x}^{N-1}}(0)=1, \tag{140}$$

and $\frac{d}{du}w^{\tilde{x}^N,\tilde{x}^{N-1}}(1)=1$. Therefore

$$\begin{aligned}\int_0^1 w^{\tilde{x}^N,\tilde{x}^{N-1}}(u)u^kdu&=\frac{1}{k+2}-\frac{1}{2(k+1)}+\frac{w^{\tilde{x}^N,\tilde{x}^{N-1}}(1)-w^{\tilde{x}^N,\tilde{x}^{N-1}}(0)}{2(k+1)}\\&\quad+\frac{1}{(k+1)(k+2)}\int_0^1\frac{d^2}{du^2}w^{\tilde{x}^N,\tilde{x}^{N-1}}(u)u^{k+2}du\\&=\frac{1}{k+2}-\frac{1}{2(k+1)}\int_0^1\frac{d^2}{du^2}w^{\tilde{x}^N,\tilde{x}^{N-1}}(u)udu\\&\qquad+\frac{1}{(k+1)(k+2)}\int_0^1\frac{d^2}{du^2}w^{\tilde{x}^N,\tilde{x}^{N-1}}(u)u^{k+2}du.\end{aligned} \tag{141}$$

Similarly, for any $k=0,1,\cdots$,

$$\int_0^1\varphi(u)u^kdu=\frac{1}{k+2}-\frac{1}{2(k+1)}\int_0^1\varphi''(u)udu+\frac{1}{(k+1)(k+2)}\int_0^1\varphi''(u)u^{k+2}du, \tag{142}$$

and (138) implies (127). □

The following result connects Proposition 6.1 and Theorem 3.2.

Lemma 6.3. *[IO02, Lemma 5.7] For any fixed interval $[a,b]\subset\mathbb{R}$, let Σ be the set of all functions $\rho:\mathbb{R}\to\mathbb{R}$, that are supported in $[a,b]$ and satisfy $|\rho(u_1)-\rho(u_2)|\leq|u_1-u_2|$, $\forall u_1,u_2\in[a,b]$. Then the weak topology defined by the functionals*

$$\rho\to\int\rho(u)u^k du,\quad k=0,1,\cdots \tag{143}$$

coincides with the uniform topology given by the supremum norm $\|\rho\|=\sup|\rho(u)|$.

Proof of Theorem 3.2. By Lemma 6.3, the convergence of $w^{\tilde{x}^N,\tilde{x}^{N-1}}$ (in probability) under the uniform topology is equivalent to the convergence (in probability) of each moment, and the later is precisely Proposition 6.1. $\square$

6.3 Convergence of discrete signed measures

Now we show how Theorem 3.2 implies Theorem 3.3.

Proof of Theorem 3.3. It suffices to prove the case where the function $f:[0,1]\to\mathbb{R}$ is nondecreasing, since each function of finite variation can be written as the difference of two nondecreasing functions. Now since f' is nondecreasing, we can define $g:[0,1]\to\mathbb{R}$ such that $g(u)$ is the right limit of f' at u, for any $u\in[0,1)$; and $g(1)$ the left limit of f' at 1. Then g is also nondecreasing and bounded, and right continuous. Also $g=f'$ almost everywhere.

Thus we can define η to be a measure on $[0,1]$, such that $\eta([0,u])=g(u)$, for any $u\in[0,1]$.

Integrating by parts (and with Remark 2.6) we have

$$\begin{aligned}\int_0^1 f d\mu^{\tilde{x}^N,\tilde{x}^{N-1}} &= \frac{-f(0)\frac{d}{du}w^{\tilde{x}^N,\tilde{x}^{N-1}}(0)+f(1)\frac{d}{du}w^{\tilde{x}^N,\tilde{x}^{N-1}}(1)}{2}-\frac{1}{2}\int_0^1 f'(u)\frac{d}{du}w^{\tilde{x}^N,\tilde{x}^{N-1}}(u)du\\ &= \frac{f(0)+f(1)}{2}-\frac{1}{2}\int_0^1 g(u)\frac{d}{du}w^{\tilde{x}^N,\tilde{x}^{N-1}}(u)du\\ &= \frac{f(0)+f(1)}{2}-\frac{1}{2}\left(g(1)w^{\tilde{x}^N,\tilde{x}^{N-1}}(1)-g(0)w^{\tilde{x}^N,\tilde{x}^{N-1}}(0)\right)+\frac{1}{2}\int_0^1 w^{\tilde{x}^N,\tilde{x}^{N-1}}(u)d\eta.\end{aligned} \tag{144}$$

By Theorem 3.2, there is

$$\lim_{L\to\infty} w^{\tilde{x}^N,\tilde{x}^{N-1}}(0)=\varphi(0),\quad \lim_{L\to\infty} w^{\tilde{x}^N,\tilde{x}^{N-1}}(1)=\varphi(1), \tag{145}$$

in probability; and since

$$\begin{aligned}\left|\frac{1}{2}\int_0^1 w^{\tilde{x}^N,\tilde{x}^{N-1}}(u)d\eta-\frac{1}{2}\int_0^1\varphi(u)d\eta\right| &\leq \frac{1}{2}\int_0^1\left|\left(w^{\tilde{x}^N,\tilde{x}^{N-1}}(u)-\varphi(u)\right)\right|d\eta\\ &\leq \frac{1}{2}\sup_{u\in\mathbb{R}}\left|w^{\tilde{x}^N,\tilde{x}^{N-1}}(u)-\varphi(u)\right|\eta([0,1]),\end{aligned} \tag{146}$$

we have

$$\lim_{L\to\infty}\frac{1}{2}\int_0^1 w^{\tilde{x}^N,\tilde{x}^{N-1}}(u)d\eta=\frac{1}{2}\int_0^1\varphi(u)d\eta, \tag{147}$$

in probability. Again through integrating by parts, we conclude

$$\lim_{L\to\infty}\int_0^1 f d\mu^{\tilde{x}^N,\tilde{x}^{N-1}} = \frac{1}{2}\left(f(0)+f(1)-g(1)\varphi(1)+f'(0)\varphi(0)+\int_0^1 \varphi(u)d\eta\right)$$
$$= \frac{1}{2}\int_0^1 f(u)\varphi''(u)du, \quad (148)$$

which is precisely (31). □

6.4 Asymptote of roots of Jacobi polynomials

We prove Theorem 3.5 by utilizing a limit transition between the distribution $\mathbb{P}^{\alpha,M,\theta}$ on χ^M and the roots of $\mathcal{F}^{\alpha-1,|M-N|}_{\min(M,N)}$.

Theorem 6.4. *[BG15, Theorem 5.1] Let $(x^1,x^2,\cdots)\in\chi^M$ be distributed as $\mathbb{P}^{\alpha,M,\theta}$, and let $j_{M,N,\alpha,i}$ be the ith root (in increasing order) of $\mathcal{F}^{\alpha-1,|M-N|}_{\min(M,N)}$, for $1\le i\le \min(M,N)$. Then there is*

$$\lim_{\theta\to\infty} x_i^N = j_{M,N,\alpha,i}, \quad (149)$$

in probability.

Proof of Theorem 3.5. The interlacing relationship for the roots immediately follows Theorem 6.4 and the interlacing relationship for the sequences x_i^N and x_i^{N-1}.

Sending $\theta\to\infty$ in Theorem 5.1, and using Theorem 6.4 we compute for $k=1,2,\cdots$:

$$\sum_{i=1}^N j^k_{M,N,\alpha,i} = \frac{-1}{(2\pi\mathbf{i})^k}\oint\cdots\oint \frac{1}{(w_2-w_1-1)\cdots(w_k-w_{k-1}-1)}$$
$$\times\prod_{i=1}^k \frac{w_i-1}{w_i+N-1}\cdot\frac{\alpha-w_i}{\alpha+M-w_i}dw_i, \quad (150)$$

where each contour encloses $-N+1$ but not $M+\alpha$, and $|w_1|\ll\cdots\ll|w_k|$.

Under (22), setting $w_i\sim L\upsilon_i$, we have

$$\lim_{L\to\infty}\left(\sum_{i=1}^N j^k_{M,N,\alpha,i} - \sum_{i=1}^{N-1} j^k_{M,N-1,\alpha,i}\right)$$
$$= \lim_{L\to\infty}\frac{1}{(2\pi\mathbf{i})^k}\oint\cdots\oint\frac{1}{(w_2-w_1-1)\cdots(w_k-w_{k-1}-1)}$$
$$\times\prod_{i=1}^k\frac{w_i-1}{w_i+N-1}\cdot\frac{\alpha-w_i}{\alpha+M-w_i}dw_i\left(\prod_{i=1}^k\frac{w_i+N-1}{w_i+N-2}-1\right)$$
$$= \frac{1}{(2\pi\mathbf{i})^k}\oint\cdots\oint\frac{1}{(\upsilon_2-\upsilon_1)\cdots(\upsilon_k-\upsilon_{k-1})}\prod_{i=1}^k\frac{\upsilon_i}{\upsilon_i+\hat{N}}\cdot\frac{\hat{\alpha}-\upsilon_i}{\hat{\alpha}+\hat{M}-\upsilon_i}d\upsilon_i\left(\sum_{i=1}^k\frac{1}{\upsilon_i+\hat{N}}\right). \quad (151)$$

We apply Corollary 4.2 to (151) to do dimension reduction, and get

$$\lim_{L\to\infty}\sum_{i=1}^N\left(j^k_{M,N,\alpha,i}-\sum_{i=1}^{N-1}j^k_{M,N-1,\alpha,i}\right) = \frac{1}{2\pi\mathbf{i}}\oint\left(\frac{\upsilon}{\upsilon+\hat{N}}\cdot\frac{\upsilon-\hat{\alpha}}{\upsilon-\hat{\alpha}-\hat{M}}\right)^k\frac{1}{\upsilon+\hat{N}}d\upsilon, \quad (152)$$

where the contour in the right hand side is around $-\hat{N}$ but not $\hat{\alpha}+\hat{M}$.

The right hand side now exactly fits Lemma 6.2, and we conclude that

$$\lim_{L\to\infty}\int_{[0,1]}\iota''_{M,N,\alpha}(u)u^k du = \lim_{L\to\infty} 2\left(\sum_{i=1}^{N} j^k_{M,N,\alpha,i} - \sum_{i=1}^{N-1} j^k_{M,N-1,\alpha,i}\right) = \int_{[0,1]}\varphi''(u)u^k du. \tag{153}$$

Finally, we integrate by parts, which leads to

$$\lim_{L\to\infty}\int_{[0,1]}\iota_{M,N,\alpha}(u)u^k du = \int_{[0,1]}\varphi(u)u^k du. \tag{154}$$

With Lemma 6.3, the above is equivalent to the statement of Theorem 3.5. □

7 Central limit theorem and gaussianity of fluctuation: proofs of Theorems 3.6 and 3.8

In this section we present the proofs of Theorem 3.6 and Theorem 3.8. To prove the weak convergence of the joint distribution of a vector to Gaussian, it suffices to check that the joint higher moments converges to the corresponding ones of the limit Gaussian.

The following criterion is known for Gaussianity of a random vector.

Lemma 7.1. *Let $\mathbf{v}=\{v_i\}_{i=1}^h\in\mathbb{R}^h$ be a Gaussian random vector, and let Θ_h be the collection of all unordered partitions of $\{1,\cdots,h\}$:*

$$\Theta_h=\left\{\{U_1,\cdots,U_t\}\,\middle|\, t\in\mathbb{Z}_+,\bigcup_{i=1}^{t}U_i=\{1,\cdots,h\},U_i\bigcap U_j=\emptyset,U_i\neq\emptyset,\forall 1\leq i\leq j\leq t\right\}. \tag{155}$$

Then when $h>2$, there is

$$\sum_{\{U_1,\cdots,U_t\}\in\Theta_h}(-1)^{t-1}(t-1)!\prod_{i=1}^{t}\mathbb{E}\left[\prod_{j\in U_i}v_j\right]=0. \tag{156}$$

Conversely, given a random vector $\mathbf{u}=\{u_i\}_{i=1}^w\in\mathbb{R}^w$ such that each moment is finite. If for any $h>2$, and $v_1,\cdots,v_h\in\{u_1,\cdots,u_w\}$, (156) holds, then $\mathbf{u}$ is (almost surely) Gaussian.

Proof. Given Gaussian random vector $\mathbf{v}=\{v_i\}_{i=1}^h$, let $M:\mathbb{R}^h\to\mathbb{R}$ be the moment generating function:

$$M(\mathbf{w}):=\mathbb{E}\left[\exp(\mathbf{w}\cdot\mathbf{v})\right],\quad \mathbf{w}=\{w_i\}_{i=1}^h\in\mathbb{R}^h. \tag{157}$$

Since $\mathbf{v}$ is Gaussian, we have that $M(\mathbf{w})=Z\exp\big((\mathbf{w}-\mathbf{c})^T\Sigma(\mathbf{w}-\mathbf{c})\big)$ for some non-negative definite Σ, constant vector $\mathbf{c}$, and $Z>0$. Then $\ln(M(\mathbf{w}))$ is a degree 2 polynomial on $w_1,\cdots,w_h$.

On the other hand, as the moment generating function, M can be expanded as

$$M(\mathbf{w})=\sum_{d=0}^{\infty}\sum_{\substack{p_1,\cdots,p_h\in\mathbb{Z}_{\geq 0}:\\ p_1+\cdots+p_h=d}}\frac{\prod_{i=1}^h w_i^{p_i}}{\prod_{i=1}^h p_i!}\mathbb{E}\left[\prod_{i=1}^h v_i^{p_i}\right], \tag{158}$$

and

$$\ln(M(\mathbf{w}))=\sum_{t=1}^{\infty}\frac{(-1)^{t-1}}{t}\left(\sum_{d=1}^{\infty}\sum_{\substack{p_1,\cdots,p_h\in\mathbb{Z}_{\geq 0}:\\ p_1+\cdots+p_h=d}}\frac{\prod_{i=1}^h w_i^{p_i}}{\prod_{i=1}^h p_i!}\mathbb{E}\left[\prod_{i=1}^h v_i^{p_i}\right]\right)^t. \tag{159}$$

The right hand side should be a degree 2 polynomial on $w_1, \cdots, w_h$ as well. Note that the coefficient for $w_1 \cdots w_h$ is precisely

$$\sum_{\{U_1,\cdots,U_t\}\in\Theta_h} (-1)^{t-1}(t-1)! \prod_{i=1}^{t} \mathbb{E}\left[\prod_{j\in U_i} v_j\right], \tag{160}$$

which equals 0 for $h \geq 3$.

For a given random vector $\mathbf{u} = \{u_i\}_{i=1}^w$, define $\tilde{\mathbf{u}} = \{\tilde{u}_i\}_{i=1}^w$ to be the Gaussian vector with the same expectation and covariances as $\mathbf{u} = \{u_i\}_{i=1}^w$. Since (156) holds for any $h > 2$, $v_1, \cdots, v_h \in \{u_1, \cdots, u_w\}$, and the corresponding $v_1, \cdots, v_h \in \{\tilde{u}_1, \cdots, \tilde{u}_w\}$, by induction each joint moment of $\mathbf{u} = \{u_i\}_{i=1}^w$ and $\tilde{\mathbf{u}} = \{\tilde{u}_i\}_{i=1}^w$ matches. Then almost surely these two vectors have the same distribution. □

7.1 Computation of covariance

The first step of our proof of Theorems 3.6, 3.8 is the covariance computation, presented in this section.

Throughout this section, let k_1, k_2 and N_1, N_2 be positive integers. In addition to (22), we also let

$$\lim_{L\to\infty} \frac{N_1}{L} = \hat{N}_1, \quad \lim_{L\to\infty} \frac{N_2}{L} = \hat{N}_2, \tag{161}$$

where $\hat{N}_1$ and $\hat{N}_2$ are positive real numbers.

Lemma 7.2. *[BG15, Theorem 4.1] Under the limit scheme (22), (161), for $\hat{N}_1 \leq \hat{N}_2$, we have*

$$\lim_{L\to\infty} \mathbb{E}\left[\prod_{i=1}^{2} \mathfrak{P}_{k_i}(x^{N_i}) - \mathbb{E}\left(\mathfrak{P}_{k_i}(x^{N_i})\right)\right] \\ = \frac{\theta^{-1}}{(2\pi\mathbf{i})^2} \oint\oint \frac{1}{(v_1-v_2)^2} \prod_{i=1}^{2} \left(\frac{v_i}{v_i+\hat{N}_i} \cdot \frac{v_i-\hat{\alpha}}{v_i-\hat{\alpha}-\hat{M}}\right)^{k_i} dv_i, \tag{162}$$

where the contours enclose poles at $-\hat{N}_1$ and $-\hat{N}_2$, but not $\hat{\alpha}+\hat{M}$, and are nested with $|v_1| \ll |v_2|$.

Proof. By Theorem 5.1 there is

$$\mathbb{E}\left[\prod_{i=1}^{2} \mathfrak{P}_{k_i}(x^{N_i}) - \mathbb{E}\left(\mathfrak{P}_{k_i}(x^{N_i})\right)\right] \\ = \frac{(-\theta)^{-2}}{(2\pi\mathbf{i})^{k_1+k_2}} \oint\cdots\oint \prod_{i=1}^{2} \left[\frac{1}{(u_{i,2}-u_{i,1}+1-\theta)\cdots(u_{i,k_i}-u_{i,k_i-1}+1-\theta)}\right. \\ \times \prod_{1\leq i'<j'\leq k_i} \frac{(u_{i,j'}-u_{i,i'})(u_{i,j'}-u_{i,i'}+1-\theta)}{(u_{i,j'}-u_{i,i'}-\theta)(u_{i,j'}-u_{i,i'}+1)} \prod_{i'=1}^{k_i} \left.\frac{u_{i,i'}-\theta}{u_{i,i'}+(N_i-1)\theta} \cdot \frac{u_{i,i'}-\theta\alpha}{u_{i,i'}-\theta\alpha-\theta M} du_{i,i'}\right] \\ \times \left(\prod_{\substack{1\leq i'\leq k_1,\\ 1\leq j'\leq k_2}} \frac{(u_{1,i'}-u_{2,j'})(u_{1,i'}-u_{2,j'}+1-\theta)}{(u_{1,i'}-u_{2,j'}-\theta)(u_{1,i'}-u_{2,j'}+1)} - 1\right), \tag{163}$$

where the contours for $u_{i,k_1},\cdots,u_{i,k_i}$ enclose $-\theta(N_i-1)$ but not $\theta(\alpha+M)$, for $i=1,2$. We also require that $|u_{1,1}|\ll\cdots\ll|u_{1,k_1}|\ll|u_{2,1}|\ll\cdots\ll|u_{2,k_2}|$.

Set $u_{i,i'}=L\theta v_{i,i'}$ for $i=1,2$ and any $1\le i'\le k_i$. Sending $L\to\infty$, note that

$$\prod_{\substack{1\le i'\le k_1,\\ 1\le j'\le k_2}}\frac{(u_{1,i'}-u_{2,j'})(u_{1,i'}-u_{2,j'}+1-\theta)}{(u_{1,i'}-u_{2,j'}-\theta)(u_{1,i'}-u_{2,j'}+1)}-1=L^{-2}\cdot\left(\sum_{\substack{1\le i'\le k_1,\\ 1\le j'\le k_2}}\frac{\theta^{-1}}{(v_{1,i'}-v_{2,j'})^2}\right)+O(L^{-4}). \quad (164)$$

Then (163) converges to

$$\frac{\theta^{-1}}{(2\pi\mathbf{i})^{k_1+k_2}}\oint\cdots\oint\left(\sum_{\substack{1\le i'\le k_1,\\ 1\le j'\le k_2}}\frac{1}{(v_{1,i'}-v_{2,j'})^2}\right)$$
$$\times\prod_{i=1}^{2}\left(\frac{1}{(v_{i,2}-v_{i,1})\cdots(v_{i,k_i}-v_{i,k_i-1})}\left(\prod_{i'=1}^{k_i}\frac{v_{i,i'}}{v_{i,i'}+\hat{N}_i}\cdot\frac{v_{i,i'}-\hat{\alpha}}{v_{i,i'}-\hat{\alpha}-\hat{M}}dv_{i,i'}\right)\right). \quad (165)$$

Finally, applying Theorem 4.1 to $v_{i,k_1},\cdots,v_{i,k_i}$ and $v_{j,k_1},\cdots,v_{j,k_j}$ respectively, we get (162). $\square$

Lemma 7.3. *Under the limit scheme (22) and (161), let us assume additionally either I) $N_1\le N_2$ for all L large enough, or II) $N_1>N_2$ for all L large enough. Then*

$$\lim_{L\to\infty}L\cdot\mathbb{E}\left[\left(\mathfrak{P}_{k_1}(x^{N_1})-\mathfrak{P}_{k_1}(x^{N_1-1})\right)\mathfrak{P}_{k_2}(x^{N_2})\right]-L\cdot\mathbb{E}\left[\mathfrak{P}_{k_1}(x^{N_1})-\mathfrak{P}_{k_1}(x^{N_1-1})\right]\mathbb{E}\left[\mathfrak{P}_{k_2}(x^{N_2})\right]$$
$$=-\frac{\theta^{-1}k_1}{(2\pi\mathbf{i})^2}\oint\oint\frac{1}{(v_1-v_2)^2}\frac{1}{v_1+\hat{N}_1}\prod_{i=1}^{2}\left(\frac{v_i}{v_i+\hat{N}_i}\cdot\frac{v_i-\hat{\alpha}}{v_i-\hat{\alpha}-\hat{M}}\right)^{k_i}dv_i, \quad (166)$$

where the contours enclose poles at $-\hat{N}_1$ and $-\hat{N}_2$, but not $\hat{\alpha}+\hat{M}$, and are nested: $|v_1|\ll|v_2|$ in case I, and $|v_1|\gg|v_2|$ in case II.

Proof. By Theorem 5.1 we have

$$\mathbb{E}\left[\left(\mathfrak{P}_{k_1}(x^{N_1})-\mathfrak{P}_{k_1}(x^{N_1-1})\right)\mathfrak{P}_{k_2}(x^{N_2})\right]-\mathbb{E}\left[\mathfrak{P}_{k_1}(x^{N_1})-\mathfrak{P}_{k_1}(x^{N_1-1})\right]\mathbb{E}\left[\mathfrak{P}_{k_2}(x^{N_2})\right]$$
$$=\frac{(-\theta)^{-2}}{(2\pi\mathbf{i})^{k_1+k_2}}\oint\cdots\oint\prod_{i=1}^{2}\Bigg[\frac{1}{(u_{i,2}-u_{i,1}+1-\theta)\cdots(u_{i,k_i}-u_{i,k_i-1}+1-\theta)}$$
$$\times\prod_{1\le i'<j'\le k_i}\frac{(u_{i,j'}-u_{i,i'})(u_{i,j'}-u_{i,i'}+1-\theta)}{(u_{i,j'}-u_{i,i'}-\theta)(u_{i,j'}-u_{i,i'}+1)}\prod_{i'=1}^{k_i}\frac{u_{i,i'}-\theta}{u_{i,i'}+(N_i-1)\theta}\cdot\frac{u_{i,i'}-\theta\alpha}{u_{i,i'}-\theta\alpha-\theta M}du_{i,i'}\Bigg]$$
$$\times\left(1-\prod_{i'=1}^{k_1}\frac{u_{1,i'}+(N_1-1)\theta}{u_{1,i'}+(N_1-2)\theta}\right)\left(\prod_{\substack{1\le i'\le k_1,\\ 1\le j'\le k_2}}\frac{(u_{1,i'}-u_{2,j'})(u_{1,i'}-u_{2,j'}+1-\theta)}{(u_{1,i'}-u_{2,j'}-\theta)(u_{1,i'}-u_{2,j'}+1)}-1\right), \quad (167)$$

where the contours for $u_{i,k_1},\cdots,u_{i,k_i}$ enclose $-\theta(N_i-2)$ and $-\theta(N_i-1)$ but not $\theta(\alpha+M)$, for $i=1,2$. We also require that $|u_{1,1}|\ll\cdots\ll|u_{1,k_1}|$; and $|u_{2,1}|\ll\cdots\ll|u_{2,k_2}|$, and $|u_{1,k_1}|\ll|u_{2,1}|$ when $N_1\le N_2$, $|u_{2,k_2}|\ll|u_{1,1}|$ when $N_1>N_2$.

Set $u_{i,i'} = L\theta v_{i,i'}$ for $i = 1, 2$, and any $1 \le i' \le k_i$, and send $L \to \infty$. In addition to (164), observe that

$$1 - \prod_{i'=1}^{k_1} \frac{u_{1,i'} + (N_1 - 1)\theta}{u_{1,i'} + (N_1 - 2)\theta} = L^{-1} \cdot \left(\sum_{i'=1}^{k_1} \frac{1}{v_{1,i'} + \hat{N}_1} \right) + O(L^{-2}). \tag{168}$$

Therefore, (167) multiplied by L converges to

$$\frac{\theta^{-1}}{(2\pi\mathbf{i})^{k_1+k_2}} \oint \cdots \oint \left(\sum_{\substack{1 \le i' \le k_1, \\ 1 \le j' \le k_2}} \frac{1}{(v_{1,i'} - v_{2,j'})^2} \right) \left(\sum_{i'=1}^{k_1} \frac{1}{v_{1,i'} + \hat{N}_1} \right)$$
$$\times \prod_{i=1}^{2} \left(\frac{1}{(v_{i,2} - v_{i,1}) \cdots (v_{i,k_i} - v_{i,k_i-1})} \left(\prod_{i'=1}^{k_i} \frac{v_{i,i'}}{v_{i,i'} + \hat{N}_i} \cdot \frac{v_{i,i'} - \hat{\alpha}}{v_{i,i'} - \hat{\alpha} - \hat{M}} dv_{i,i'} \right) \right). \tag{169}$$

Applying Corollary 4.2 to $v_{i,k_1}, \cdots, v_{i,k_i}$ and $v_{j,k_1}, \cdots, v_{j,k_j}$, respectively, we get (166). □

Lemma 7.4. *Assume that $N_1 < N_2$ for L large enough, then*

$$\lim_{L \to \infty} L^2 \mathbb{E} \left[\prod_{i=1}^{2} \mathfrak{P}_{k_i}(x^{N_i}) - \mathfrak{P}_{k_i}(x^{N_i - 1}) - \mathbb{E}\left(\mathfrak{P}_{k_i}(x^{N_i}) - \mathfrak{P}_{k_i}(x^{N_i-1}) \right) \right]$$
$$= \frac{\theta^{-1} k_1 k_2}{(2\pi\mathbf{i})^2} \oint \oint \frac{1}{(v_1 - v_2)^2} \prod_{i=1}^{2} \frac{dv_i}{v_i + \hat{N}_i} \left(\frac{v_i}{v_i + \hat{N}_i} \cdot \frac{v_i - \hat{\alpha}}{v_i - \hat{\alpha} - \hat{M}} \right)^{k_i}, \tag{170}$$

where the contours enclose poles at $-\hat{N}_1$ and $-\hat{N}_2$, but not $\hat{\alpha} + \hat{M}$, and are nested with $|v_1| \ll |v_2|$.

Proof. The proof is very similar to the proof of Lemma 7.3. By Theorem 5.1 we obtain that

$$\mathbb{E} \left[\prod_{i=1}^{2} \mathfrak{P}_{k_i}(x^{N_i}) - \mathfrak{P}_{k_i}(x^{N_i-1}) - \mathbb{E}\left(\mathfrak{P}_{k_i}(x^{N_i}) - \mathfrak{P}_{k_i}(x^{N_i-1}) \right) \right]$$
$$= \frac{(-\theta)^{-2}}{(2\pi\mathbf{i})^{k_1+k_2}} \oint \cdots \oint \prod_{i=1}^{2} \left[\frac{1}{(u_{i,2} - u_{i,1} + 1 - \theta) \cdots (u_{i,k_i} - u_{i,k_i-1} + 1 - \theta)} \right.$$
$$\times \prod_{1 \le i' < j' \le k_i} \frac{(u_{i,j'} - u_{i,i'})(u_{i,j'} - u_{i,i'} + 1 - \theta)}{(u_{i,j'} - u_{i,i'} - \theta)(u_{i,j'} - u_{i,i'} + 1)} \prod_{i'=1}^{k_i} \frac{u_{i,i'} - \theta}{u_{i,i'} + (N_i - 1)\theta} \cdot \frac{u_{i,i'} - \theta\alpha}{u_{i,i'} - \theta\alpha - \theta M} du_{i,i'}$$
$$\left. \times \left(1 - \prod_{i'=1}^{k_i} \frac{u_{i,i'} + (N_i - 1)\theta}{u_{i,i'} + (N_i - 2)\theta} \right) \right] \left(\prod_{\substack{1 \le i' \le k_1, \\ 1 \le j' \le k_2}} \frac{(u_{1,i'} - u_{2,j'})(u_{1,i'} - u_{2,j'} + 1 - \theta)}{(u_{1,i'} - u_{2,j'} - \theta)(u_{1,i'} - u_{2,j'} + 1)} - 1 \right), \tag{171}$$

where the contours for $u_{i,k_1}, \cdots, u_{i,k_i}$ enclose $-\theta(N_i - 2)$ and $-\theta(N_i - 1)$ but not $\theta(\alpha + M)$, for $i = 1, 2$. We also require that $|u_{1,1}| \ll \cdots \ll |u_{1,k_1}| \ll |u_{2,1}| \ll \cdots \ll |u_{2,k_2}|$.

Again set $u_{i,i'} = L\theta v_{i,i'}$ for $i = 1, 2$ and any $1 \le i' \le k_i$. Sending $L \to \infty$, using (168) and (164), and applying Corollary 4.2 to $v_{i,k_1}, \cdots, v_{i,k_i}$ and $v_{j,k_1}, \cdots, v_{j,k_j}$, respectively, we eventually get (170). □

Lemma 7.5. *Under the limit scheme (22), we have*

$$\lim_{L\to\infty} L\cdot\mathbb{E}\left[\prod_{i=1}^{2}\left(\mathfrak{P}_{k_i}(x^N)-\mathfrak{P}_{k_i}(x^{N-1})-\mathbb{E}\left(\mathfrak{P}_{k_i}(x^N)-\mathfrak{P}_{k_i}(x^{N-1})\right)\right)\right]$$
$$=\frac{\theta^{-1}k_1k_2}{2\pi\mathbf{i}(k_1+k_2)}\oint\frac{dv}{(v+\hat{N})^2}\left(\frac{v}{v+\hat{N}}\cdot\frac{v-\hat{\alpha}}{v-\hat{\alpha}-\hat{M}}\right)^{k_2+k_2}, \quad (172)$$

where the contours enclose poles at $-\hat{N}$ but not $\hat{\alpha}+\hat{M}$.

Proof. We can write the expectation as

$$\mathbb{E}\left[\prod_{i=1}^{2}\left(\mathfrak{P}_{k_i}(x^N)-\mathfrak{P}_{k_i}(x^{N-1})-\mathbb{E}\left(\mathfrak{P}_{k_i}(x^N)-\mathfrak{P}_{k_i}(x^{N-1})\right)\right)\right]$$
$$=\mathbb{E}\left[\left(\mathfrak{P}_{k_1}(x^N)-\mathfrak{P}_{k_1}(x^{N-1})\right)\mathfrak{P}_{k_2}(x^N)\right]-\mathbb{E}\left[\mathfrak{P}_{k_2}(x^{N-1})\left(\mathfrak{P}_{k_1}(x^N)-\mathfrak{P}_{k_1}(x^{N-1})\right)\right]$$
$$-\mathbb{E}\left(\mathfrak{P}_{k_1}(x^N)-\mathfrak{P}_{k_1}(x^{N-1})\right)\mathbb{E}\left(\mathfrak{P}_{k_2}(x^N)\right)+\mathbb{E}\left(\mathfrak{P}_{k_2}(x^{N-1})\right)\mathbb{E}\left(\mathfrak{P}_{k_1}(x^N)-\mathfrak{P}_{k_1}(x^{N-1})\right). \quad (173)$$

Now apply Lemma 7.3, and we obtain

$$\lim_{L\to\infty} L\cdot\mathbb{E}\left[\prod_{i=1}^{2}\left(\mathfrak{P}_{k_i}(x^N)-\mathfrak{P}_{k_i}(x^{N-1})-\mathbb{E}\left(\mathfrak{P}_{k_i}(x^N)-\mathfrak{P}_{k_i}(x^{N-1})\right)\right)\right]$$
$$=\frac{k_1\theta^{-1}}{(2\pi\mathbf{i})^2}\oint\oint\left[\left(\frac{v_2}{v_2+\hat{N}}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_1}\left(\frac{v_1}{v_1+\hat{N}}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_2}\frac{1}{(v_2+\hat{N})(v_2-v_1)^2}\right.$$
$$\left.-\left(\frac{v_1}{v_1+\hat{N}}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_1}\left(\frac{v_2}{v_2+\hat{N}}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_2}\frac{1}{(v_1+\hat{N})(v_2-v_1)^2}\right]dv_1dv_2, \quad (174)$$

where the contours of v_1 and v_2 enclose $-\hat{N}$ but not $\hat{\alpha}+\hat{M}$; and $|v_1|\ll|v_2|$.

Interchanging k_1 and k_2, for the same limit we have

$$\frac{k_2\theta^{-1}}{(2\pi\mathbf{i})^2}\oint\oint\left[\left(\frac{v_2}{v_2+\hat{N}}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_2}\left(\frac{v_1}{v_1+\hat{N}}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_1}\frac{1}{(v_2+\hat{N})(v_2-v_1)^2}\right.$$
$$\left.-\left(\frac{v_1}{v_1+\hat{N}}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_2}\left(\frac{v_2}{v_2+\hat{N}}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_1}\frac{1}{(v_1+\hat{N})(v_2-v_1)^2}\right]dv_1dv_2, \quad (175)$$

where the contours of v_1 and v_2 enclose $-\hat{N}$ but not $\hat{\alpha}+\hat{M}$; and $|v_1|\ll|v_2|$.

Notice that (174) $\times\frac{1}{k_1}$ + (175) $\times\frac{1}{k_2}$ equals

$$-\frac{\theta^{-1}}{(2\pi\mathbf{i})^2}\oint\oint\frac{1}{(v_1+\hat{N})(v_2+\hat{N})(v_2-v_1)}$$
$$\times\left(\left(\frac{v_2}{v_2+\hat{N}}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_1}\left(\frac{v_1}{v_1+\hat{N}}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_2}\right.$$
$$\left.+\left(\frac{v_1}{v_1+\hat{N}}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_1}\left(\frac{v_2}{v_2+\hat{N}}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_2}\right)dv_1dv_2$$
$$=-\frac{\theta^{-1}}{2\pi\mathbf{i}}\oint\frac{1}{(v+\hat{N})^2}\left(\frac{v}{v+\hat{N}}\cdot\frac{v-\hat{\alpha}}{v-\hat{\alpha}-\hat{M}}\right)^{k_1+k_2}dv, \quad (176)$$

where the contour encloses $-\hat{N}$ but not $\hat{\alpha}+\hat{M}$. The last line is by Theorem 4.1, applied to v_1 and v_2. Finally, (176) directly implies (172). $\square$

7.2 Gaussian type asymptote of joint moments

In this section we present a Gaussian-type formula for the asymptotic of joint moments.

Proposition 7.6. *Let $k_1, \cdots, k_h$ and $N_1 \leq \cdots \leq N_h$ be positive integers, and let $D \subset \{1, \cdots, h\}$ be a subset of indices, such that for any $1 \leq i < j \leq h$, and $j \in D$, $N_i < N_j$.*

For any $i \in D$, denote

$$\mathfrak{E}_i = \mathfrak{P}_{k_i}(x^{N_i}) - \mathfrak{P}_{k_i}(x^{N_i-1}) - \mathbb{E}\left(\mathfrak{P}_{k_i}(x^{N_i}) - \mathfrak{P}_{k_i}(x^{N_i-1})\right), \tag{177}$$

and for any $i \notin D$, denote

$$\mathfrak{E}_i = \mathfrak{P}_{k_i}(x^{N_i}) - \mathbb{E}\left(\mathfrak{P}_{k_i}(x^{N_i})\right). \tag{178}$$

Let Θ_h be defined as in Lemma 7.1. Under the limit scheme (22) and (33), for any $\eta < h-2+|D|$,

$$\lim_{L\to\infty} L^{\eta} \sum_{\{U_1,\cdots,U_t\}\in\Theta_h} (-1)^{t-1}(t-1)! \prod_{i=1}^{t} \mathbb{E}\left[\prod_{j\in U_i} \mathfrak{E}_j\right] = 0. \tag{179}$$

Remark 7.7. We should emphasize that this is a "Gaussian type" formula instead of (156): to show that a random vector $\{v_i\}_{i=1}^h$ is Gaussian, we need to check that each of its moment coincides with the one of a Gaussian. To be precise, for any $w > 2$, and $1 \leq i_1 \leq \cdots \leq i_w \leq h$, we must check that the random vector $\{v_{i_j}\}_{j=1}^w$ satisfies (156). Here we only show the formula for specific higher moments: those satisfying that for any $i < j$, and $i, j \in D$, $N_i < N_j$.

In the special case where $D = \emptyset$, Proposition 7.6 indeed implies Gaussianity. As illustrated in the proof of Lemma 7.1, if we take $k_1 = \cdots = k_h = k$ and $N_1 = \cdots = N_h = N$, then $\sum_{\{U_1,\cdots,U_t\}\in\Theta_h}(-1)^{t-1}(t-1)!\prod_{i=1}^t \mathbb{E}\left[\prod_{j\in U_i}\mathfrak{E}_j\right]$ is the h-th order cumulant of $\mathfrak{P}_k(x^N) - \mathbb{E}\left(\mathfrak{P}_k(x^N)\right)$. Here we see that it decays in the order of $O(L^{2-h+\epsilon})$ for any $\epsilon > 0$.

The following lemma explains that (177) and (178) can be replaced by non-centered versions.

Lemma 7.8. *Let Θ_h be defined as in Lemma 7.1. For any h random variables $r_1, \cdots, r_h$, and constants $c_1, \cdots, c_h$, there is*

$$\begin{aligned} &\sum_{\{U_1,\cdots,U_t\}\in\Theta_h} (-1)^{t-1}(t-1)! \prod_{i=1}^{t} \mathbb{E}\left[\prod_{j\in U_i} r_j\right] \\ &\qquad = \sum_{\{U_1,\cdots,U_t\}\in\Theta_h} (-1)^{t-1}(t-1)! \prod_{i=1}^{t} \mathbb{E}\left[\prod_{j\in U_i} (r_j - c_j)\right]. \end{aligned} \tag{180}$$

Proof. Denote $\varphi = \{1, \cdots, h-1\}$. The c_r derivative of (180) is

$$\sum_{\{U_1,\cdots,U_t\}\in\Theta_h} (-1)^{t}(t-1)! \prod_{i=1}^{t} \mathbb{E}\left[\prod_{j\in U_i\bigcap\varphi} (r_j - c_j)\right]. \tag{181}$$

For any set partition $\{V_1, \cdots, V_s\}$ of φ, and $\{U_1, \cdots, U_t\} \in \Theta_h$, consider the unordered collection $\{U_1 \bigcap \varphi, \cdots, U_t \bigcap \varphi\}$. After removing empty sets it is the same as $\{V_1, \cdots, V_s\}$, in each of the following two cases:

1. $t = s$, and up to a permutation of indices, $U_1 = V_1 \bigcup \{h\}$, $U_i = V_i$ for all $2 \leq i \leq t$.

2. $t = s+1$, and up to a permutation of indices, $U_1 = \{h\}$, $U_i = V_{i-1}$ for all $2 \leq i \leq t$.

Therefore there are s partitions $\{U_1, \cdots, U_s\} \in \Theta_h$, such that $\{U_1 \bigcap \varphi, \cdots, U_s \bigcap \varphi\} = \{V_1, \cdots, V_s\}$; and 1 partition $\{U_1, \cdots, U_{s+1}\} \in \Theta_h$, such that $\{U_1 \bigcap \varphi, \cdots, U_{s+1} \bigcap \varphi\} = \{V_1, \cdots, V_s\}$. Then the coefficient of $\prod_{i=1}^s \mathbb{E}\left[\prod_{j \in V_i} (r_j - c_j)\right]$ in (181) is $s(-1)^{s-1}(s-1)! + (-1)^s s! = 0$; thus (181) is 0. We conclude that (181) does not depend on the choice of c_h. Similarly it does not depend on $c_1, \cdots, c_{h-1}$. $\square$

Proof of Proposition 7.6. By Lemma 7.8, the limit in (179) equals

$$\lim_{L\to\infty} L^\eta \sum_{\{U_1,\cdots,U_t\}\in\Theta_h} (-1)^{t-1}(t-1)! \times \prod_{i=1}^t \mathbb{E}\left[\prod_{j\in U_i \bigcap D} \left(\mathfrak{P}_{k_j}(x^{N_j}) - \mathfrak{P}_{k_j}(x^{N_j-1})\right) \prod_{j\in U_i\setminus D} \mathfrak{P}_{k_j}(x^{N_j})\right]. \quad (182)$$

For any fixed $\{U_1, \cdots, U_t\} \in \Theta_h$, we apply Theorem 5.1 to expectations in the following form:

$$\mathbb{E}\left[\prod_{j\in U_i\setminus D} (-1)^{\lambda_j} \mathfrak{P}_{k_j}(x^{N_j-\lambda_j}) \prod_{j\in U_i\bigcap D} \mathfrak{P}_{k_j}(x^{N_j})\right], \quad (183)$$

where $1 \leq i \leq t$, and each $\lambda_j \in \{0, 1\}$. We multiply over all i, and sum over all choices of $\{\lambda_j\}_{j\in D}$. Note that for any $j \in D$, $j > 1$, we have assumed $N_{j-1} \leq N_j - 1 < N_j$; thus the nesting order of contour integrals are the same for different choices of $\{\lambda_j\}_{j\in D}$. To be more precise, there is

$$\begin{aligned}
&\prod_{i=1}^t \mathbb{E}\left[\prod_{j\in U_i\bigcap D} \left(\mathfrak{P}_{k_j}(x^{N_j}) - \mathfrak{P}_{k_j}(x^{N_j-1})\right) \prod_{j\in U_i\setminus D} \mathfrak{P}_{k_j}(x^{N_j})\right] \\
&= \frac{(-\theta)^{-h}}{(2\pi\mathbf{i})^{k_1+\cdots+k_h}} \oint\cdots\oint \prod_{j\in U_i\setminus D} \mathfrak{I}(u_{j,1}, \cdots, u_{j,k_j}; \alpha, M, \theta, N_j) \\
&\times \prod_{j\in U_i\bigcap D} \left(\mathfrak{I}(u_{j,1}, \cdots, u_{j,k_j}; \alpha, M, \theta, N_j) - \mathfrak{I}(u_{j,1}, \cdots, u_{j,k_j}; \alpha, M, \theta, N_j - 1)\right) \\
&\times \prod_{r=1}^t \prod_{\substack{i<j,\\ i,j\in U_r}} \mathfrak{L}(u_{i,1}, \cdots, u_{i,k_i}; u_{j,1}, \cdots, u_{j,k_j}; \alpha, M, \theta) \prod_{i=1}^h \prod_{i'=1}^{k_i} du_{i,i'}, \quad (184)
\end{aligned}$$

where the contours are nested such that for each $1 \leq i \leq h$, we have $|u_{i,1}| \ll \cdots \ll |u_{i,k_i}|$; for $1 \leq i \leq h-1$, we have $|u_{i,k_i}| \ll |u_{i+1,1}|$.

For each $1 \leq i < j \leq h$, denote

$$\mathfrak{M}_{i,j} = \mathfrak{L}(u_{i,1}, \cdots, u_{i,k_i}; u_{j,1}, \cdots, u_{j,k_j}; \alpha, M, \theta) - 1. \quad (185)$$

Now sum (184) over all $\{U_1, \cdots, U_t\} \in \Theta_h$, and rewrite the expression with the notation $\mathfrak{M}_{i,j}$:

$$\sum_{\{U_1,\cdots,U_t\}\in\Theta_h} (-1)^{t-1}(t-1)! \prod_{i=1}^{t} \mathbb{E}\left[\prod_{j\in U_i\bigcap D} \left(\mathfrak{P}_{k_j}(x^{N_j}) - \mathfrak{P}_{k_j}(x^{N_j-1})\right) \prod_{j\in U_i\setminus D} \mathfrak{P}_{k_j}(x^{N_j})\right]$$
$$= \frac{(-\theta)^{-h}}{(2\pi\mathbf{i})^{k_1+\cdots+k_h}} \oint\cdots\oint \prod_{j\in U_i\setminus D} \mathfrak{I}(u_{j,1},\cdots,u_{j,k_j};\alpha,M,\theta,N_j)$$
$$\times \prod_{j\in U_i\bigcap D} \left(\mathfrak{I}(u_{j,1},\cdots,u_{j,k_j};\alpha,M,\theta,N_j) - \mathfrak{I}(u_{j,1},\cdots,u_{j,k_j};\alpha,M,\theta,N_j-1)\right)$$
$$\times \sum_{\{U_1,\cdots,U_t\}\in\Theta_h} (-1)^{t-1}(t-1)! \prod_{r=1}^{t} \prod_{\substack{i<j,\\ i,j\in U_r}} (\mathfrak{M}_{i,j}+1) \prod_{i=1}^{h}\prod_{i'=1}^{k_i} du_{i,i'}. \quad (186)$$

We use a combinatorial argument to simplify this expression. Consider the complete graph G with vertex set $\{1,\cdots,h\}$, then any $\{U_1,\cdots,U_t\} \in \Theta_h$ corresponds to a partition of the vertices. Denote

$$\mathcal{T}(\{U_1,\cdots,U_t\}) = \bigcup_{r=1}^{t} \{(i,j)\in G \,|\, i<j; i,j\in U_r\}, \quad (187)$$

be the set of edges connecting two vertices in the same component. In particular, $\mathcal{T}_0 = \mathcal{T}(\{\{1,\cdots,h\}\})$ is the collection of all edges in G. Then there is

$$\sum_{\{U_1,\cdots,U_t\}\in\Theta_h} (-1)^{t-1}(t-1)! \prod_{r=1}^{t} \prod_{\substack{i<j,\\ i,j\in U_r}} (\mathfrak{M}_{i,j}+1)$$
$$= \sum_{\{U_1,\cdots,U_t\}\in\Theta_h} (-1)^{t-1}(t-1)! \sum_{\Omega\subset\mathcal{T}(\{U_1,\cdots,U_t\})} \prod_{(i,j)\in\Omega} \mathfrak{M}_{i,j}$$
$$= \sum_{\Omega\subset\mathcal{T}_0} \left(\sum_{\substack{\{U_1,\cdots,U_t\}\in\Theta_h:\\ \Omega\subset\mathcal{T}(\{U_1,\cdots,U_t\})}} (-1)^{t-1}(t-1)! \right) \prod_{(i,j)\in\Omega} \mathfrak{M}_{i,j}. \quad (188)$$

For each $\Omega \subset \mathcal{T}_0$, consider the graph G_Ω, with vertex set $\{1,\cdots,h\}$, and edge set Ω. We claim that if G_Ω is not connected, then

$$\sum_{\substack{\{U_1,\cdots,U_t\}\in\Theta_h:\\ \Omega\subset\mathcal{T}(\{U_1,\cdots,U_t\})}} (-1)^{t-1}(t-1)! = 0. \quad (189)$$

Indeed, for any $\{U_1,\cdots,U_t\} \in \Theta_h$, $\Omega \subset \mathcal{T}(\{U_1,\cdots,U_t\})$ if and only if for any $(i,j)\in\Omega$, there is $1 \le r \le t$, such that $\{i,j\} \subset U_r$. Assume that all connected components in G_Ω are $G_{\Omega,1},\cdots,G_{\Omega,s}$, $s \ge 2$, and each contains vertices $V_{\Omega,1},\cdots,V_{\Omega,s}$; then for any $\{U_1,\cdots,U_t\} \in \Theta_h$, $\Omega \subset \mathcal{T}(\{U_1,\cdots,U_t\})$ if and only if for any $1 \le i \le s$, there is $1 \le r \le t$, such that $V_{\Omega,i} \subset U_r$. For each fixed $1 \le t \le s$, the number of such $\{U_1,\cdots,U_t\}$ is

$$\frac{1}{t!}\sum_{i=0}^{t}(-1)^i\binom{t}{i}(t-i)^s. \quad (190)$$

Then we have that

$$\sum_{\substack{\{U_1,\cdots,U_t\}\in\Theta_h,\\ \Omega\subset\mathcal{T}(\{U_1,\cdots,U_t\})}}(-1)^{t-1}(t-1)! = \sum_{t=1}^{s}(-1)^{t-1}(t-1)!\frac{1}{t!}\sum_{i=0}^{t}(-1)^i\binom{t}{i}(t-i)^s$$

$$= \sum_{i+i'\le s, i'>0}(-1)^{i'-1}\frac{(i+i'-1)!}{i!i'!}i'^s = \sum_{i'=1}^{s}(-1)^{i'-1}\binom{s}{s-i'}i'^{s-1} = 0. \quad (191)$$

This proves (189).

When G_Ω is connected, the only partition $\{U_1,\cdots,U_t\}\in\Theta_h$ such that $\Omega\subset\mathcal{T}(\{U_1,\cdots,U_t\})$ is when $t=1$ and $U_1=\{1,\cdots,h\}$. Therefore

$$\sum_{\substack{\{U_1,\cdots,U_t\}\in\Theta_h:\\ \Omega\subset\mathcal{T}(\{U_1,\cdots,U_t\})}}(-1)^{t-1}(t-1)! = 1. \quad (192)$$

Now we see that (186) multiplied by L^η is simplified to

$$L^\eta\sum_{\{U_1,\cdots,U_t\}\in\Theta_h}(-1)^{t-1}(t-1)!\prod_{i=1}^{t}\mathbb{E}\left[\prod_{j\in U_i\bigcap D}\left(\mathfrak{P}_{k_j}(x^{N_j})-\mathfrak{P}_{k_j}(x^{N_j-1})\right)\prod_{j\in U_i\setminus D}\mathfrak{P}_{k_j}(x^{N_j})\right]$$
$$= \frac{(-\theta)^{-h}L^\eta}{(2\pi\mathbf{i})^{k_1+\cdots+k_h}}\oint\cdots\oint\prod_{j\in U_i\setminus D}\mathfrak{I}(u_{j,1},\cdots,u_{j,k_j};\alpha,M,\theta,N_j)$$
$$\times\prod_{j\in U_i\bigcap D}\left(\mathfrak{I}(u_{j,1},\cdots,u_{j,k_j};\alpha,M,\theta,N_j)-\mathfrak{I}(u_{j,1},\cdots,u_{j,k_j};\alpha,M,\theta,N_j-1)\right)$$
$$\times\sum_{\substack{\Omega\subset\mathcal{T}_0:\\ G_\Omega\text{ is connected}}}\prod_{(i,j)\in\Omega}\mathfrak{M}_{i,j}\prod_{i=1}^{h}\prod_{i'=1}^{k_i}du_{i,i'}. \quad (193)$$

Set $u_{i,i'}=L\theta v_{i,i'}$ for any $1\le i\le h$, $1\le i'\le k_i$; under a change of notations (193) becomes

$$\frac{(-1)^{-h}L^{\eta-(h-2+|D|)}}{(2\pi\mathbf{i})^{k_1+\cdots+k_h}}\oint\cdots\oint\prod_{j\in U_i\setminus D}\mathcal{S}_j\prod_{j\in U_i\bigcap D}\mathcal{R}_j$$
$$\times\sum_{\substack{\Omega\subset\mathcal{T}_0:\\ G_\Omega\text{ is connected}}}L^{2(h-1-|\Omega|)}\prod_{(i,j)\in\Omega}\mathfrak{N}_{i,j}\prod_{i=1}^{h}\prod_{i'=1}^{k_i}dv_{i,i'}, \quad (194)$$

where

$$\mathcal{S}_j = \frac{1}{(v_{j,2}-v_{j,1}+(\theta^{-1}-1)L^{-1})\cdots\left(v_{j,k_j}-v_{j,k_j-1}+(\theta^{-1}-1)L^{-1}\right)}$$
$$\times\prod_{1\le i<i'\le m}\frac{(v_{j,i'}-v_{j,i})(v_{j,i'}-v_{j,i}+(\theta^{-1}-1)L^{-1})}{(v_{j,i'}-v_{j,i}-L^{-1})(v_{j,i'}-v_{j,i}+\theta^{-1}L^{-1})}\prod_{i=1}^{k_j}\frac{v_{j,i}-L^{-1}}{v_{j,i}+\hat{N}_j-L^{-1}}\cdot\frac{v_{j,i}-\hat{\alpha}}{v_{j,i}-\hat{\alpha}-\hat{M}}, \quad (195)$$

$$\mathcal{R}_j = \frac{1}{(v_{j,2}-v_{j,1}+(\theta^{-1}-1)L^{-1})\cdots\left(v_{j,k_j}-v_{j,k_j-1}+(\theta^{-1}-1)L^{-1}\right)}$$
$$\times \prod_{1\le i<i'\le m} \frac{(v_{j,i'}-v_{j,i})(v_{j,i'}-v_{j,i}+(\theta^{-1}-1)L^{-1})}{(v_{j,i'}-v_{j,i}-L^{-1})(v_{j,i'}-v_{j,i}+\theta^{-1}L^{-1})}$$
$$\times L\left(1-\prod_{i=1}^{k_j}\frac{v_{j,i}+\hat{N}_j-L^{-1}}{v_{j,i}+\hat{N}_j-2L^{-1}}\right)\prod_{i=1}^{k_j}\frac{v_{j,i}-L^{-1}}{v_{j,i}+\hat{N}_j-L^{-1}}\cdot\frac{v_{j,i}-\hat{\alpha}}{v_{j,i}-\hat{\alpha}-\hat{M}}, \quad (196)$$

and

$$\mathfrak{N}_{i,j} = L^2\left(\prod_{1\le i'\le k_i, 1\le j'\le k_j}\frac{(v_{i,i'}-v_{j,j'})(v_{i,i'}-v_{j,j'}+(\theta^{-1}-1)L^{-1})}{(v_{i,i'}-v_{j,j'}-L^{-1})(v_{i,i'}-v_{j,j'}+\theta^{-1}L^{-1})}-1\right). \quad (197)$$

Also, the contours in (194) are nested, such that for any $1\le i\le h$, we have $|v_{i,1}|\ll\cdots\ll|v_{i,k_i}|$; for $1\le i\le h-1$, we have $|u_{i,k_i}|\ll|u_{i+1,1}|$; and all contours enclose all $-\hat{N}_i$, but not $\hat{\alpha}+\hat{M}$. Notice that

$$L\left(1-\prod_{i=1}^{k_j}\frac{v_{j,i}+\hat{N}_j-L^{-1}}{v_{j,i}+\hat{N}_j-2L^{-1}}\right) = -\sum_{i=1}^{k_j}\frac{1}{v_{j,i}+\hat{N}_j-2L^{-1}}+O(L^{-1}) \quad (198)$$

and

$$\mathfrak{N}_{i,j} = \prod_{1\le i'\le k_i, 1\le j'\le k_j}\frac{\theta^{-1}}{(v_{i,i'}-v_{j,j'}-L^{-1})(v_{i,i'}-v_{j,j'}+\theta^{-1}L^{-1})}+O(L^{-2}), \quad (199)$$

so if we send $L\to\infty$, each $\mathcal{S}_j$, $\mathcal{R}_j$, and $\mathfrak{N}_{i,j}$ converges.

Since for any $\Omega\subset\mathcal{T}_0$, if G_Ω is connected, $|\Omega|\ge h-1$, and $\eta<h-2+|D|$, we will get 0 if we send $L\to\infty$ for (194). This finishes the proof. □

The expression (194) also implies that the convergence to 0 is uniform for $\hat{N}_1,\cdots,\hat{N}_h$.

Corollary 7.9. *There is a constant $C(\hat{\alpha},\hat{M},k_1,\cdots,k_h)$, independent of L, $\hat{N}_1,\cdots,\hat{N}_h$, and D, such that for any $0\le\hat{N}_1\le\cdots\le\hat{N}_h\le 1$, satisfying $\hat{N}_i<\hat{N}_j$ for any $1\le i<j\le h$, $j\in D$, there is*

$$L^\eta\left|\sum_{\{U_1,\cdots,U_t\}\in\Theta_h}(-1)^{t-1}(t-1)!\prod_{i=1}^{t}\mathbb{E}\left[\prod_{j\in U_i}\mathfrak{E}_j\right]\right|\le C(\hat{\alpha},\hat{M},k_1,\cdots,k_h) \quad (200)$$

for any $L>C(\hat{\alpha},\hat{M},k_1,\cdots,k_h)$.

Proof. In (194), we can fix the contours for all $v_{i,i'}$, such that they are nested and each encloses the line segment $[-1,0]$ but not $\hat{\alpha}+\hat{M}$. Then each of $|\mathcal{S}_j|$, $|\mathcal{R}_j|$, and $|\mathfrak{N}_{i,j}|$ is upper bounded by a constant relying on the chosen contours, for L large enough; thus so is the integral. Then we conclude (200). □

By a linear combination of (179), we can also obtain the following result, where, instead of imposing strict ordering, we just consider the number of different values among $\hat{N}_1,\cdots,\hat{N}_h$.

Corollary 7.10. *Let $k_1,\cdots,k_h$ and $N_1\le\cdots\le N_h$ be positive integers. For each $1\le i\le h$, denote*

$$\mathfrak{E}_i = \mathfrak{P}_{k_i}(x^{N_i})-\mathfrak{P}_{k_i}(x^{N_i-1})-\mathbb{E}\left(\mathfrak{P}_{k_i}(x^{N_i})-\mathfrak{P}_{k_i}(x^{N_i-1})\right). \quad (201)$$

Let Θ_h be defined as in Lemma 7.1. Under the limit scheme (22), (33), and assuming that there are s different values among $\hat{N}_1, \cdots, \hat{N}_h$; we have for any $\eta < h - 2 + s$,

$$\lim_{L\to\infty} L^{\eta} \sum_{\{U_1,\cdots,U_t\}\in\Theta_h} (-1)^{t-1}(t-1)! \prod_{i=1}^{t} \mathbb{E}\left[\prod_{j\in U_i} \mathfrak{E}_j\right] = 0. \tag{202}$$

7.3 Gaussianity of discrete levels: proof of Theorem 3.6

Proof of Theorem 3.6. To show that (34) and (35) jointly weakly converge to Gaussian, it suffices to show that each joint moment converges to the same joint moment of the Gaussian vector with covariances specified in Theorem 3.6.

The first two orders of moments are trivial: all the first order moments are zero, and the second order moments (covariances) are given by Lemmas 7.2, 7.4, and 7.5.

For higher moments, by Lemma 7.1 we need to show that, for any positive integers $k_1, \cdots, k_h$ and $N_1 \le \cdots \le N_h$, and $D \subset \{1, \cdots, h\}$ a subset of indices, under the limit scheme (22), (33) there is

$$\lim_{L\to\infty} L^{\frac{|D|}{2}} \sum_{\{U_1,\cdots,U_t\}\in\Theta_h} (-1)^{t-1}(t-1)! \prod_{i=1}^{t} \mathbb{E}\left[\prod_{j\in U_i} \mathfrak{E}_j\right] = 0, \tag{203}$$

where Θ_h is defined as in Lemma 7.1; and we denote

$$\mathfrak{E}_i = \mathfrak{P}_{k_i}(x^{N_i}) - \mathfrak{P}_{k_i}(x^{N_i-1}) - \mathbb{E}\left(\mathfrak{P}_{k_i}(x^{N_i}) - \mathfrak{P}_{k_i}(x^{N_i-1})\right), \tag{204}$$

for any $i \in D$; and for any $i \notin D$,

$$\mathfrak{E}_i = \mathfrak{P}_{k_i}(x^{N_i}) - \mathbb{E}\left(\mathfrak{P}_{k_i}(x^{N_i})\right). \tag{205}$$

Note that compared with Proposition 7.6, here we do not impose any strict ordering for $N_1 \le \cdots \le N_h$.

Now let us show that (203) holds. The case where $|D| = 0$ is covered by Proposition 7.6, where we take $\eta = h - 3 + |D| \ge 0 = \frac{|D|}{2}$ for $h \ge 3$. When $|D| \ge 1$, pick $i_o \in D$. Denote $D_1 = \{i_o\}$ and $D_2 = D \backslash D_1$. By Proposition 7.6 there is

$$\begin{aligned}
&\lim_{L\to\infty} L^{h-\frac{3}{2}} \sum_{\{U_1,\cdots,U_t\}\in\Theta_h} (-1)^{t-1}(t-1)! \\
&\qquad \times \prod_{i=1}^{t} \mathbb{E}\left[\prod_{j\in U_i\bigcap D_1} \left(\mathfrak{P}_{k_j}(x^{N_j}) - \mathfrak{P}_{k_j}(x^{N_j-1}) - \mathbb{E}\left(\mathfrak{P}_{k_j}(x^{N_j}) - \mathfrak{P}_{k_j}(x^{N_j-1})\right)\right)\right. \\
&\left.\prod_{j\in U_i\bigcap D_2} (-1)^{\lambda_j}\left(\mathfrak{P}_{k_j}(x^{N_j-\lambda_j}) - \mathbb{E}\left(\mathfrak{P}_{k_i}(x^{N_j-\lambda_j})\right)\right) \prod_{j\in U_i\backslash D} \left(\mathfrak{P}_{k_j}(x^{N_j}) - \mathbb{E}\left(\mathfrak{P}_{k_j}(x^{N_j})\right)\right)\right] = 0,
\end{aligned} \tag{206}$$

where for each $j \in D_2$, there is $\lambda_j \in \{0, 1\}$. We sum over all $2^{|D_2|}$ possible choices of λ_j; and since $h - \frac{3}{2} \ge \frac{h}{2} \ge \frac{|D|}{2}$, for $h \ge 3$, we obtain (203). $\square$

7.4 Gaussianity of integral over levels: proof of Theorem 3.8

In this section we present the proof of Theorem 3.8.

Proof of Theorem 3.8. For the convenience of notations, denote

$$\mathfrak{C}_i(y) = \mathfrak{P}_{k_i}(x^{\lfloor y \rfloor}) - \mathfrak{P}_{k_i}(x^{\lfloor y \rfloor - 1}) - \mathbb{E}\left(\mathfrak{P}_{k_i}(x^{\lfloor y \rfloor}) - \mathfrak{P}_{k_i}(x^{\lfloor y \rfloor - 1})\right). \tag{207}$$

By Lemma 7.1, to show that (38) is asymptotically Gaussian, it suffices to check the covariances match, and show that for any positive integers $h, k_1, \cdots, k_h$, $N_1 \leq \cdots \leq N_h$, and any almost everywhere continuous functions $g_1, \cdots, g_h \in L^\infty([0,1])$, and Θ_h defined as in Lemma 7.1, we have

$$\lim_{L\to\infty} L^h \sum_{\{U_1,\cdots,U_t\}\in\Theta_h} (-1)^{t-1}(t-1)! \prod_{i=1}^{t} \mathbb{E}\left[\prod_{j\in U_i} \int_0^1 g_j(y)\mathfrak{C}_j(Ly)dy\right] = 0. \tag{208}$$

Changing the order of integration, the left hand side of (208) equals

$$\lim_{L\to\infty} \int_0^1 \cdots \int_0^1 L^h \sum_{\{U_1,\cdots,U_t\}\in\Theta_h} (-1)^{t-1}(t-1)! \prod_{i=1}^{t} \mathbb{E}\left[\prod_{j\in U_i} \mathfrak{C}_j(Ly_j)\right] \prod_{i=1}^{h} g_i(y_i)dy_i. \tag{209}$$

The expression (209) can be split into (finite) sum of integrals in the form of

$$\int_0^1 \cdots \int_0^1 \mathbb{1}_{\lfloor Ly_1\rfloor=\cdots=\lfloor Ly_{c_1}\rfloor<\lfloor Ly_{c_1+1}\rfloor=\cdots=\lfloor Ly_{c_2}\rfloor<\cdots<\lfloor Ly_{c_{s-1}+1}\rfloor=\cdots=\lfloor Ly_{c_s}\rfloor}$$
$$\times L^h \left| \sum_{\{U_1,\cdots,U_t\}\in\Theta_h} (-1)^{t-1}(t-1)! \prod_{i=1}^{t} \mathbb{E}\left[\prod_{j\in U_i} \mathfrak{C}_j(Ly_j)\right]\right| \prod_{i=1}^{h} g_i(y_i)dy_i, \tag{210}$$

where $1 \leq s \leq h$ and $1 \leq c_1 < \cdots < c_s = h$. Since each g_i is bounded, there is a constant K, such that (210) is bounded from above by

$$\int_0^1 \cdots \int_0^1 K\mathbb{1}_{\lfloor Ly_1\rfloor=\cdots=\lfloor Ly_{c_1}\rfloor<\lfloor Ly_{c_1+1}\rfloor=\cdots=\lfloor Ly_{c_2}\rfloor<\cdots<\lfloor Ly_{c_{s-1}+1}\rfloor=\cdots=\lfloor Ly_{c_s}\rfloor}$$
$$\times L^h \sum_{\{U_1,\cdots,U_t\}\in\Theta_h} (-1)^{t-1}(t-1)! \prod_{i=1}^{t} \mathbb{E}\left[\prod_{j\in U_i} \mathfrak{C}_j(Ly_j)\right] \prod_{i=1}^{h} dy_i. \tag{211}$$

Note that each $\mathfrak{C}_j(Ly_j)$ is constant when $y_j \in \left[\frac{m}{L}, \frac{m+1}{L}\right)$, for any $0 \leq m \leq L-1$, we can do a change of variable: $z_i = y_{c_{i-1}+1} = \cdots = y_{c_i}$, for $i = 1, \cdots, s$ and $c_0 = 0$. Thus (211) becomes

$$\int_0^1 \cdots \int_0^1 K\mathbb{1}_{\lfloor Lz_1\rfloor<\cdots<\lfloor Lz_s\rfloor} L^s \sum_{\{U_1,\cdots,U_t\}\in\Theta_h} (-1)^{t-1}(t-1)! \prod_{i=1}^{t} \mathbb{E}\left[\prod_{j\in U_i} \mathfrak{W}_j\right] \prod_{i=1}^{s} dz_i, \tag{212}$$

here $\mathfrak{W}_j = \mathfrak{C}_j(Lz_i)$, for any $1 \leq j \leq h$ and $c_{i-1} < j \leq c_i$ and $c_0 = 0$.

By Corollary 7.10, the integrand of (212) converges to 0 pointwise; and by Corollary 7.9 the integrand is bounded regardless of L and $z_1, \cdots, z_s$. Using Dominated Convergence Theorem we conclude that (212) converges to 0, as $L \to \infty$. This implies (208).

It remains to match the covariances. For any almost everywhere continuous functions $g_1, g_2 \in L^\infty([0,1])$, there is

$$
\begin{aligned}
&\lim_{L\to\infty} L^2 \mathbb{E}\left(\prod_{i=1}^{2}\int_0^1 g_i(y)\mathfrak{C}_i(Ly)dy\right) = \lim_{L\to\infty}\int_0^1\int_0^1 L^2\mathbb{E}\left(\prod_{i=1}^{2}\mathfrak{C}_i(Ly_i)\right)\prod_{i=1}^{2} g_i(y_i)dy_i \\
&= \lim_{L\to\infty}\iint_{\lfloor Ly_1\rfloor<\lfloor Ly_2\rfloor} L^2\mathbb{E}\left(\prod_{i=1}^{2}\mathfrak{C}_i(Ly_i)\right)\prod_{i=1}^{2} g_i(y_i)dy_i \\
&\quad + \lim_{L\to\infty}\iint_{\lfloor Ly_2\rfloor<\lfloor Ly_1\rfloor} L^2\mathbb{E}\left(\prod_{i=1}^{2}\mathfrak{C}_i(Ly_i)\right)\prod_{i=1}^{2} g_i(y_i)dy_i \\
&\quad + \lim_{L\to\infty}\int_0^1 L^3\left(\iint_{\left[\frac{\lfloor Ly\rfloor}{L},\frac{\lfloor Ly+1\rfloor}{L}\right]^2}\prod_{i=1}^{2} g_i(y_i)dy_i\right)\mathbb{E}\left(\prod_{i=1}^{2}\mathfrak{C}_i(Ly)\right)dy.
\end{aligned}
\tag{213}
$$

As g_1, g_2 are almost everywhere continuous, for y at almost everywhere there is

$$
\lim_{L\to\infty} L^2\iint_{\left[\frac{\lfloor Ly\rfloor}{L},\frac{\lfloor Ly+1\rfloor}{L}\right]^2}\prod_{i=1}^{2} g_i(y_i)dy_i = g_1(y)g_2(y). \tag{214}
$$

From the expectations computed in Section 7.1, the integrands in (213) converges pointwise. Since g_1 and g_2 are bounded, and with Corollary 7.9, the integrands in (213) are also bounded. Thus the integrals also converge, and (213) equals

$$
\begin{aligned}
&\iint_{0\le y_1<y_2\le 1}\frac{\theta^{-1}g_1(y_1)g_2(y_2)}{(2\pi\mathbf{i})^2}\oint\oint\frac{k_1k_2}{(v_1-v_2)^2(v_1+y_1)(v_2+y_2)} \\
&\qquad\times\left(\frac{v_1}{v_1+y_1}\cdot\frac{v_1-\hat\alpha}{v_1-\hat\alpha-\hat M}\right)^{k_1}\left(\frac{v_2}{v_2+y_2}\cdot\frac{v_2-\hat\alpha}{v_2-\hat\alpha-\hat M}\right)^{k_2}dv_1dv_2dy_1dy_2 \\
&\qquad+\iint_{0\le y_2<y_1\le 1}\frac{\theta^{-1}g_1(y_1)g_2(y_2)}{(2\pi\mathbf{i})^2}\oint\oint\frac{k_1k_2}{(v_1-v_2)^2(v_1+y_1)(v_2+y_2)} \\
&\qquad\times\left(\frac{v_1}{v_1+y_1}\cdot\frac{v_1-\hat\alpha}{v_1-\hat\alpha-\hat M}\right)^{k_1}\left(\frac{v_2}{v_2+y_2}\cdot\frac{v_2-\hat\alpha}{v_2-\hat\alpha-\hat M}\right)^{k_2}dv_1dv_2dy_1dy_2 \\
&\qquad-\int_0^1\frac{\theta^{-1}g_1(y)g_2(y)}{2\pi\mathbf{i}}\oint\frac{k_1k_2}{(k_1+k_2)(v+y)^2}\left(\frac{v}{v+y}\cdot\frac{v-\hat\alpha}{v-\hat\alpha-\hat M}\right)^{k_1+k_2}dvdy,
\end{aligned}
\tag{215}
$$

where the contours of the first and second integrals are nested in different orders. In slightly different notation this is precisely (39). □

8 Connecting the limit field with Gaussian Free Field: proofs of Theorems 3.11 and 3.16

In this section we interpret Theorem 3.6 and 3.8 as convergence of the height function (see Definition 3.10) toward the Gaussian random field.

8.1 Identification of 1-dimensional integral

Proof of Theorem 3.11. Denote $N_i = \left\lfloor L\hat{N}_i \right\rfloor$, for $i = 1, \cdots, h$. Integrating by parts we transform the integrals of $\mathcal{W}(u, L\hat{N}_i)$ as follows:

$$\int_0^1 u^{k_i} \mathcal{W}(u, L\hat{N}_i) du = \frac{1}{k_i+1} \left(\min\{N_i, M\} - \sum_{j=1}^{\min\{N_i, M\}} \left(x_j^{N_i}\right)^{k_i+1} \right) - \frac{1}{k_i+1} \left(\min\{N_i - 1, M\} - \sum_{j=1}^{\min\{N_i-1, M\}} \left(x_j^{N_i-1}\right)^{k_i+1} \right), \quad (216)$$

and by Theorem 3.6 we conclude that

$$\lim_{L\to\infty} \left(L^{\frac{1}{2}} \int_0^1 u^{k_i} \left(\mathcal{W}(u, L\hat{N}_i) - \mathbb{E}\left(\mathcal{W}(u, L\hat{N}_i) \right) \right) du \right)_{i=1}^h$$
$$= \lim_{L\to\infty} \left(-\frac{L^{\frac{1}{2}}}{k_i+1} \left(\mathfrak{P}_{k_i+1}(x^{N_i}) - \mathfrak{P}_{k_i+1}(x^{N_i-1}) - \mathbb{E}\left(\mathfrak{P}_{k_i+1}(x^{N_i}) - \mathfrak{P}_{k_i+1}(x^{N_i-1}) \right) \right) \right)_{i=1}^h \quad (217)$$

is a Gaussian vector. For $\hat{N}_i = \hat{N}_j$, the covariance of the ith and jth component of (217) is

$$-\frac{1}{k_i+k_j+2} \cdot \frac{\theta^{-1}}{2\pi\mathbf{i}} \oint \frac{1}{(v+\hat{N}_i)^2} \left(\frac{v}{v+\hat{N}_i} \cdot \frac{v-\hat{\alpha}}{v-\hat{\alpha}-\hat{M}} \right)^{k_i+k_j+2} dv \quad (218)$$

where the contour encloses $-\hat{N}_i = -\hat{N}_j$ but not $\hat{\alpha} + \hat{M}$. For $\hat{N}_i \neq \hat{N}_j$, the covariance of the ith and jth component is 0.

Now let's turn to (41). From Lemma 2.9, for $\delta > 0$ the distribution of

$$\delta^{-\frac{1}{2}} \left(\int_0^1 u^{k_i} \mathcal{K}(u, \hat{N}_i + \delta) du - \int_0^1 u^{k_i} \mathcal{K}(u, \hat{N}_i) du \right)_{i=1}^h, \quad (219)$$

is also Gaussian, and the covariance of the ith and jth ($i < j$) component for $\delta < \hat{N}_j - \hat{N}_i$ is

$$\frac{\delta^{-1}}{(k_i+1)(k_j+1)} \cdot \frac{\theta^{-1}}{(2\pi\mathbf{i})^2} \oint \oint \frac{dv_1 dv_2}{(v_1-v_2)^2}$$
$$\times \left(\left(\frac{v_1}{v_1+\hat{N}_i+\delta} \cdot \frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}} \right)^{k_i+1} - \left(\frac{v_1}{v_1+\hat{N}_i} \cdot \frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}} \right)^{k_i+1} \right)$$
$$\times \left(\left(\frac{v_2}{v_2+\hat{N}_j+\delta} \cdot \frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}} \right)^{k_j+1} - \left(\frac{v_2}{v_2+\hat{N}_j} \cdot \frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}} \right)^{k_j+1} \right), \quad (220)$$

where $|v_1| \ll |v_2|$, and the contours enclose $-\hat{N}_i$, $-\hat{N}_j$, $-\hat{N}_i - \delta$, and $-\hat{N}_j - \delta$, but not $\hat{\alpha} + \hat{M}$; for

$\hat{N}_i = \hat{N}_j$ the covariance is

$$\frac{\delta^{-1}}{(k_i+1)(k_j+1)} \cdot \frac{\theta^{-1}}{(2\pi\mathbf{i})^2} \oint\oint \frac{dv_1 dv_2}{(v_1-v_2)^2}$$
$$\times \Bigg(\left(\frac{v_1}{v_1+\hat{N}_i+\delta} \cdot \frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}} \right)^{k_i+1} \left(\frac{v_2}{v_2+\hat{N}_i+\delta} \cdot \frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}} \right)^{k_j+1}$$
$$- \left(\frac{v_1}{v_1+\hat{N}_i} \cdot \frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}} \right)^{k_i+1} \left(\frac{v_2}{v_2+\hat{N}_i+\delta} \cdot \frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}} \right)^{k_j+1}$$
$$- \left(\frac{v_1}{v_1+\hat{N}_i} \cdot \frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}} \right)^{k_j+1} \left(\frac{v_2}{v_2+\hat{N}_i+\delta} \cdot \frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}} \right)^{k_i+1}$$
$$+ \left(\frac{v_1}{v_1+\hat{N}_i} \cdot \frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}} \right)^{k_j+1} \left(\frac{v_2}{v_2+\hat{N}_i} \cdot \frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}} \right)^{k_i+1} \Bigg), \quad (221)$$

where $|v_1| \ll |v_2|$, and the contours enclose $-\hat{N}_i$ and $-\hat{N}_i - \delta$, but not $\hat{\alpha} + \hat{M}$.

Notice that by sending $\delta \to 0_+$, (219) converges to Gaussian as well, since the covariances converges: for (220) the limit is 0; for (221), the limit is

$$-\frac{1}{k_j+1} \cdot \frac{\theta^{-1}}{(2\pi\mathbf{i})^2} \oint\oint \frac{dv_1 dv_2}{(v_1-v_2)^2}$$
$$\times \left(\frac{1}{v_1+\hat{N}_i} \left(\frac{v_1}{v_1+\hat{N}_i} \cdot \frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}} \right)^{k_i+1} \left(\frac{v_2}{v_2+\hat{N}_i} \cdot \frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}} \right)^{k_j+1} \right.$$
$$\left. - \frac{1}{v_2+\hat{N}_i} \left(\frac{v_1}{v_1+\hat{N}_i} \cdot \frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}} \right)^{k_i+1} \left(\frac{v_2}{v_2+\hat{N}_i} \cdot \frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}} \right)^{k_j+1} \right)$$
$$= \frac{1}{k_j+1} \cdot \frac{\theta^{-1}}{(2\pi\mathbf{i})^2} \oint\oint \frac{dv_1 dv_2}{(v_1-v_2)} \cdot \frac{1}{(v_1+\hat{N}_i)(v_2+\hat{N}_i)}$$
$$\times \left(\frac{v_1}{v_1+\hat{N}_i} \cdot \frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}} \right)^{k_i+1} \left(\frac{v_2}{v_2+\hat{N}_i} \cdot \frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}} \right)^{k_j+1}. \quad (222)$$

We switch k_i and k_j to obtain the equivalent form

$$\frac{1}{k_i+1} \cdot \frac{\theta^{-1}}{(2\pi\mathbf{i})^2} \oint\oint \frac{dv_1 dv_2}{(v_1-v_2)} \cdot \frac{1}{(v_1+\hat{N}_i)(v_2+\hat{N}_i)}$$
$$\times \left(\frac{v_1}{v_1+\hat{N}_i} \cdot \frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}} \right)^{k_j+1} \left(\frac{v_2}{v_2+\hat{N}_i} \cdot \frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}} \right)^{k_i+1}. \quad (223)$$

Applying Theorem 4.1 to the sum of (222) multiplied by $\frac{k_j+1}{k_i+k_j+2}$ and (223) multiplied by $\frac{k_i+1}{k_i+k_j+2}$, we get (218). ☐

Proof of Lemma 3.12. Integrating by parts we reduce (40) and (42) to the following vectors:

$$\left(L^{\frac{1}{2}} \left(\mathfrak{P}_{k_i}(x^{N_i}) - \mathfrak{P}_{k_i}(x^{N_i-1}) - \mathbb{E}\left(\mathfrak{P}_{k_i}(x^{N_i}) - \mathfrak{P}_{k_i}(x^{N_i-1}) \right) \right) \right)_{i=1}^{h} \quad (224)$$

and

$$\left(\left(\mathfrak{P}_{k'_i}(x^{N'_i}) - \mathbb{E}\left(\mathfrak{P}_{k'_i}(x^{N'_i}) \right) \right) \right)_{i=1}^{h'}. \quad (225)$$

The joint convergence and independence are then given in Theorem 3.6. ☐

8.2 Identification of 2-dimensional integral

Now let us discuss the random variable $\mathfrak{Z}_{g,k}$ from Definition 3.13.

Proposition 8.1. *Let $k_1, \cdots, k_h$ be integers and let $g_1, \cdots, g_h \in C^\infty([0,1])$, satisfying $g_i(1) = 0$ for each $1 \le i \le h$. The joint distribution of the vector $(\mathfrak{Z}_{g_i,k_i})_{i=1}^h$ is centered Gaussian, and the covariance between the ith and jth component is*

$$\begin{aligned}
&\iint_{0\le y_1<y_2\le 1} \frac{g_i(y_2)g_j(y_1)\theta^{-1}}{(2\pi\mathbf{i})^2} \oint\oint \frac{1}{(v_1-v_2)^2(v_1+y_1)(v_2+y_2)} \\
&\qquad \times \left(\frac{v_1}{v_1+y_1}\cdot\frac{v_1-\hat\alpha}{v_1-\hat\alpha-\hat M}\right)^{k_j+1}\left(\frac{v_2}{v_2+y_2}\cdot\frac{v_2-\hat\alpha}{v_2-\hat\alpha-\hat M}\right)^{k_i+1} dv_1dv_2dy_1dy_2 \\
&\qquad\quad + \iint_{0\le y_1<y_2\le 1} \frac{g_i(y_1)g_j(y_2)\theta^{-1}}{(2\pi\mathbf{i})^2} \oint\oint \frac{1}{(v_1-v_2)^2(v_1+y_1)(v_2+y_2)} \\
&\qquad \times \left(\frac{v_1}{v_1+y_1}\cdot\frac{v_1-\hat\alpha}{v_1-\hat\alpha-\hat M}\right)^{k_i+1}\left(\frac{v_2}{v_2+y_2}\cdot\frac{v_2-\hat\alpha}{v_2-\hat\alpha-\hat M}\right)^{k_j+1} dv_1dv_2dy_1dy_2 \\
&\qquad\qquad - \int_0^1 \frac{g_i(y)g_j(y)\theta^{-1}}{2\pi\mathbf{i}(k_i+k_j+2)} \oint \frac{1}{(v+y)^2}\left(\frac{v}{v+y}\cdot\frac{v-\hat\alpha}{v-\hat\alpha-\hat M}\right)^{k_i+k_j+2} dvdy, \qquad (226)
\end{aligned}$$

where for the first two summands, the contours enclose poles at $-y_1$ and $-y_2$, but not $\hat\alpha+\hat M$, and are nested with v_2 larger; for the last summand, the contour encloses pole at $-y$ but not $\hat\alpha+\hat M$.

Proof. By Lemma 2.10, the vector $(\mathfrak{Z}_{g_i,k_i})_{i=1}^h$ is centered Gaussian, and the covariance between the ith and jth component is

$$\begin{aligned}
&\int_0^1\int_0^1 \frac{g_i'(y_1)g_j'(y_2)\theta^{-1}}{(2\pi\mathbf{i})^2(k_i+1)(k_j+1)} \oint\oint \frac{1}{(v_1-v_2)^2} \\
&\qquad \times \left(\frac{v_1}{v_1+y_1}\cdot\frac{v_1-\hat\alpha}{v_1-\hat\alpha-\hat M}\right)^{k_i+1}\left(\frac{v_2}{v_2+y_2}\cdot\frac{v_2-\hat\alpha}{v_2-\hat\alpha-\hat M}\right)^{k_j+1} dv_1dv_2dy_1dy_2, \qquad (227)
\end{aligned}$$

where the inner contours enclose poles at $-y_1$ and $-y_2$, but not $\hat\alpha+\hat M$, and are nested: when $y_1 \le y_2$, v_2 is larger; when $y_1 \ge y_2$, v_1 is larger.

We fix the contours when $y_1 \le y_2$ and $y_2 < y_1$, respectively, and switch the order of contour integrals of y_1, y_2:

$$\begin{aligned}
&\oint\oint\iint_{0\le y_1\le y_2\le 1} \frac{g_i'(y_1)g_j'(y_2)\theta^{-1}}{(2\pi\mathbf{i})^2(k_i+1)(k_j+1)} \frac{1}{(v_1-v_2)^2} \\
&\qquad \times \left(\frac{v_1}{v_1+y_1}\cdot\frac{v_1-\hat\alpha}{v_1-\hat\alpha-\hat M}\right)^{k_i+1}\left(\frac{v_2}{v_2+y_2}\cdot\frac{v_2-\hat\alpha}{v_2-\hat\alpha-\hat M}\right)^{k_j+1} dv_1dv_2dy_1dy_2 \\
&\qquad\qquad + \oint\oint\iint_{0\le y_2<y_1\le 1} \frac{g_i'(y_1)g_j'(y_2)\theta^{-1}}{(2\pi\mathbf{i})^2(k_i+1)(k_j+1)} \frac{1}{(v_1-v_2)^2} \\
&\qquad \times \left(\frac{v_1}{v_1+y_1}\cdot\frac{v_1-\hat\alpha}{v_1-\hat\alpha-\hat M}\right)^{k_i+1}\left(\frac{v_2}{v_2+y_2}\cdot\frac{v_2-\hat\alpha}{v_2-\hat\alpha-\hat M}\right)^{k_j+1} dv_1dv_2dy_1dy_2, \qquad (228)
\end{aligned}$$

where in the first summand the contour of v_2 is larger; and for the second summand the contour of v_1 is larger.

We then integrate by parts for y_1. The first summand in (228) has boundary terms at 0 and y_2, while the second summand has boundary terms at y_2 and 1. The boundary term at 0 vanishes since the contour of v_1 encloses no pole when $y_1 = 0$; and the boundary term at 1 vanishes since $g_i(1) = 0$. Both boundary terms at y_2 are contour integrals for y_2, with same expressions and different contours, but both contours enclose exactly the same poles; then they cancel out. Thus we obtain

$$\begin{aligned}
&\oint\oint\iint_{0\le y_1\le y_2\le 1} \frac{g_i(y_1)g_j'(y_2)\theta^{-1}}{(2\pi\mathbf{i})^2(k_j+1)}\cdot\frac{1}{(v_1-v_2)^2(v_1+y_1)} \\
&\quad\times\left(\frac{v_1}{v_1+y_1}\cdot\frac{v_1-\hat\alpha}{v_1-\hat\alpha-\hat M}\right)^{k_i+1}\left(\frac{v_2}{v_2+y_2}\cdot\frac{v_2-\hat\alpha}{v_2-\hat\alpha-\hat M}\right)^{k_j+1} dy_1dy_2dv_1dv_2 \\
&\qquad+\oint\oint\iint_{0\le y_2< y_1\le 1} \frac{g_i(y_1)g_j'(y_2)\theta^{-1}}{(2\pi\mathbf{i})^2(k_j+1)}\cdot\frac{1}{(v_1-v_2)^2(v_1+y_1)} \\
&\quad\times\left(\frac{v_1}{v_1+y_1}\cdot\frac{v_1-\hat\alpha}{v_1-\hat\alpha-\hat M}\right)^{k_i+1}\left(\frac{v_2}{v_2+y_2}\cdot\frac{v_2-\hat\alpha}{v_2-\hat\alpha-\hat M}\right)^{k_j+1} dy_1dy_2dv_1dv_2 \qquad (229)
\end{aligned}$$

where the contours are nested: in the first summand the contour of v_2 is larger; and for the second summand the contour of v_1 is larger. Then we integrate by parts for y_2. For the first summand in (229) we get

$$\begin{aligned}
&\oint\oint\iint_{0\le y_1\le y_2\le 1} \frac{g_i(y_1)g_j(y_2)\theta^{-1}}{(2\pi\mathbf{i})^2}\cdot\frac{1}{(v_1-v_2)^2(v_1+y_1)(v_2+y_2)} \\
&\quad\times\left(\frac{v_1}{v_1+y_1}\cdot\frac{v_1-\hat\alpha}{v_1-\hat\alpha-\hat M}\right)^{k_i+1}\left(\frac{v_2}{v_2+y_2}\cdot\frac{v_2-\hat\alpha}{v_2-\hat\alpha-\hat M}\right)^{k_j+1} dy_1dy_2dv_1dv_2 \\
&\qquad-\oint\oint\int_0^1 \frac{g_i(y)g_j(y)\theta^{-1}}{(2\pi\mathbf{i})^2(k_j+1)}\cdot\frac{1}{(v_1-v_2)^2(v_1+y)} \\
&\quad\times\left(\frac{v_1}{v_1+y}\cdot\frac{v_1-\hat\alpha}{v_1-\hat\alpha-\hat M}\right)^{k_i+1}\left(\frac{v_2}{v_2+y}\cdot\frac{v_2-\hat\alpha}{v_2-\hat\alpha-\hat M}\right)^{k_j+1} dydv_1dv_2, \qquad (230)
\end{aligned}$$

where the contours are nested and v_2 is larger. The boundary term at $y_2 = 1$ vanishes since $g_j(1) = 0$. For the second summand in (229) we exchange v_1 and v_2, y_1 and y_2. Adding them together, we obtain that (227) is equal to $A + B$, where

$$\begin{aligned}
A = &\oint\oint\iint_{0\le y_1\le y_2\le 1} \frac{g_i(y_2)g_j(y_1)\theta^{-1}}{(2\pi\mathbf{i})^2}\cdot\frac{1}{(v_1-v_2)^2(v_1+y_1)(v_2+y_2)} \\
&\quad\times\left(\frac{v_1}{v_1+y_1}\cdot\frac{v_1-\hat\alpha}{v_1-\hat\alpha-\hat M}\right)^{k_j+1}\left(\frac{v_2}{v_2+y_2}\cdot\frac{v_2-\hat\alpha}{v_2-\hat\alpha-\hat M}\right)^{k_i+1} dy_1dy_2dv_1dv_2 \\
&\qquad+\oint\oint\iint_{0\le y_1\le y_2\le 1} \frac{g_i(y_1)g_j(y_2)\theta^{-1}}{(2\pi\mathbf{i})^2}\cdot\frac{1}{(v_1-v_2)^2(v_1+y_1)(v_2+y_2)} \\
&\quad\times\left(\frac{v_1}{v_1+y_1}\cdot\frac{v_1-\hat\alpha}{v_1-\hat\alpha-\hat M}\right)^{k_i+1}\left(\frac{v_2}{v_2+y_2}\cdot\frac{v_2-\hat\alpha}{v_2-\hat\alpha-\hat M}\right)^{k_j+1} dy_1dy_2dv_1dv_2, \qquad (231)
\end{aligned}$$

and

$$B = \oint\oint\int_0^1 \frac{g_i(y)g_j(y)\theta^{-1}}{(2\pi\mathbf{i})^2(k_j+1)} \cdot \frac{1}{(v_1-v_2)^2(v_2+y)} \times \left(\frac{v_1}{v_1+y}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_j+1}\left(\frac{v_2}{v_2+y}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_i+1} dydv_1dv_2 - \oint\oint\int_0^1 \frac{g_i(y)g_j(y)\theta^{-1}}{(2\pi\mathbf{i})^2(k_j+1)} \cdot \frac{1}{(v_1-v_2)^2(v_1+y)} \times \left(\frac{v_1}{v_1+y}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_i+1}\left(\frac{v_2}{v_2+y}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_j+1} dydv_1dv_2, \quad (232)$$

where the contours in both A and B are nested and v_2 is larger. Note that A equals the first and second summands in (226).

If we first integrate by parts for y_2 then y_1, we instead have that (227) is equal to $A'+B'$, where $A' = A$, and

$$B' = -\oint\oint\int_0^1 \frac{g_i(y)g_j(y)\theta^{-1}}{(2\pi\mathbf{i})^2(k_i+1)} \cdot \frac{1}{(v_1-v_2)^2(v_1+y)} \times \left(\frac{v_1}{v_1+y}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_j+1}\left(\frac{v_2}{v_2+y}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_i+1} dydv_1dv_2 + \oint\oint\int_0^1 \frac{g_i(y)g_j(y)\theta^{-1}}{(2\pi\mathbf{i})^2(k_i+1)} \cdot \frac{1}{(v_1-v_2)^2(v_2+y)} \times \left(\frac{v_1}{v_1+y}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_i+1}\left(\frac{v_2}{v_2+y}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_j+1} dydv_1dv_2, \quad (233)$$

where the contours also are nested and v_2 is larger. There is

$$B = B' = \frac{(k_j+1)B + (k_i+1)B'}{k_i+k_j+2} = \oint\oint\int_0^1 \frac{g_i(y)g_j(y)\theta^{-1}}{(2\pi\mathbf{i})^2(k_i+k_j+2)} \cdot \frac{1}{(v_1-v_2)(v_1+y)(v_2+y)} \times \left(\left(\frac{v_1}{v_1+y}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_j+1}\left(\frac{v_2}{v_2+y}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_i+1} + \left(\frac{v_1}{v_1+y}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_i+1}\left(\frac{v_2}{v_2+y}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_j+1}\right) dydv_1dv_2, \quad (234)$$

where the contours are nested and v_2 is larger, and applying Theorem 4.1 to it gives the last summand in (226). □

Using this Proposition we can bound the variance uniformly:

Corollary 8.2. *There is constant $C(\hat{\alpha}, \hat{M}, k)$, such that for any $g_1, g_2 \in C^\infty([0,1])$, we have*

$$\mathbb{E}\left(\mathfrak{Z}_{g_1,k}\mathfrak{Z}_{g_2,k}\right) \le C(\hat{\alpha}, \hat{M}, k)\|g_1\|_{L^2}\|g_2\|_{L^2}. \quad (235)$$

Proof. By Proposition 8.1, since the random variables $\mathfrak{Z}_{g_1,k}$ and $\mathfrak{Z}_{g_2,k}$ are centered Gaussian, $\mathbb{E}\left(\mathfrak{Z}_{g_1,k}\mathfrak{Z}_{g_2,k}\right)$ is just the covariance given by (226). We fix the contours in (226) to enclose line segment $[-1,0]$ but not $\hat{\alpha}+\hat{M}$; then it is bounded by

$$C\left(\int_0^1\int_0^1 |g_1(y_1)g_2(y_2)|\,dy_1dy_2 + \int_0^1 |g_1(y)g_2(y)|\,dy\right) \\ \leq C\left(\|g_1\|_{L^1}\|g_2\|_{L^1} + \|g_1\|_{L^2}\|g_2\|_{L^2}\right) \leq 2C\|g_1\|_{L^2}\|g_2\|_{L^2}, \quad (236)$$

for some constant C independent of g_1, g_2. Setting $C(\hat{\alpha},\hat{M},k)=2C$ finishes the proof. □

Now we show that $\mathfrak{Z}_{g,k}$ can be defined for any $g\in L^2([0,1])$.

Proof of Lemma 3.14. Since smooth functions are dense in $L^2([0,1])$, there is a sequence $h_1, h_2, \cdots \in C^\infty([0,1])$ which converges to g in $L^2([0,1])$. We further a sequence $\lambda_1, \lambda_2, \cdots \in C^\infty([0,1])$, where each $0\leq\lambda_n\leq 1$, $\lambda_n(1)=1$, and each $\|\lambda_n\|_{L^2} < 2^{-n}\|h_n\|_{L^2}^{-1}$. Set $g_n=(1-\lambda_n)k_n$, then each $g_n\in C^\infty([0,1])$ satisfies $g_n(1)=0$, and $\|g_n-g\|_{L^2}\leq\|h_n-g\|_{L^2}+\|h_n\|_{L^2}\|\lambda_n\|_{L^2}$. Then the sequence $g_1, g_2, \cdots$ converges to g in L^2.

Passing to a subsequence if necessary, we can assume that for any n there is $\|g_n-g_{n+1}\|_{L^2}<2^{-n}$. By Corollary 8.2, we have

$$\mathbb{E}\left(\left|\mathfrak{Z}_{g_n,k}-\mathfrak{Z}_{g_{n+1},k}\right|\right) \leq \mathbb{E}\left(\left|\mathfrak{Z}_{g_n,k}-\mathfrak{Z}_{g_{n+1},k}\right|^2\right)^{\frac{1}{2}} = \mathbb{E}\left(\left|\mathfrak{Z}_{g_n-g_{n+1},k}\right|^2\right)^{\frac{1}{2}} \\ \leq C(\hat{\alpha},\hat{M},k)^{\frac{1}{2}}\|g_n-g_{n+1}\|_{L^2} \leq 2^{-n}C(\hat{\alpha},\hat{M},k)^{\frac{1}{2}}. \quad (237)$$

Then by Dominated Convergence Theorem, the limit

$$\lim_{m\to\infty}\mathfrak{Z}_{g_m,k} = \lim_{m\to\infty}\sum_{n=0}^{m-1}\left(\mathfrak{Z}_{g_{n+1},k}-\mathfrak{Z}_{g_n,k}\right), \text{ where } \mathfrak{Z}_{g_0,k}=0, \quad (238)$$

exists almost surely. Denoted it as $\mathfrak{Z}_{g,k}$.

For the uniqueness of $\mathfrak{Z}_{g,k}$, if there is another such sequence $\tilde{g}_1, \tilde{g}_2, \cdots$, there is

$$\mathbb{E}\left(\left|\mathfrak{Z}_{g,k}-\mathfrak{Z}_{\tilde{g}_n,k}\right|\right) \leq \mathbb{E}\left(\left|\mathfrak{Z}_{g,k}-\mathfrak{Z}_{g_n,k}\right|\right) + \mathbb{E}\left(\left|\mathfrak{Z}_{g_n,k}-\mathfrak{Z}_{\tilde{g}_n,k}\right|\right) \\ \leq \mathbb{E}\left(\left|\mathfrak{Z}_{g,k}-\mathfrak{Z}_{g_n,k}\right|\right) + C(\hat{\alpha},\hat{M},k)^{\frac{1}{2}}\|g_n-\tilde{g}_n\|_{L^2}, \quad (239)$$

which goes to 0 as $n\to\infty$. Then $\mathfrak{Z}_{\tilde{g}_n,k}$ also converges almost surely to $\mathfrak{Z}_{g,k}$. □

With this we can extend Proposition 8.1 to functions in $L^2([0,1])$.

Proposition 8.3. *Let $k_1, \cdots, k_h$ be integers and $g_1, \cdots, g_h \in L^2([0,1])$. Then the joint distribution of the vector $\left(\mathfrak{Z}_{g_i,k_i}\right)_{i=1}^h$ is Gaussian, and the covariance between the ith and jth component is given by the same expression (226).*

Proof. For each $1\leq i\leq h$, take a sequence $g_{1,i}, g_{2,i}, \cdots$ such that each is in $C^\infty([0,1])$, and $g_{n,i}(1)=0$ for each positive integer n, and $g_{n,i}$ converges to g_i in $L^2([0,1])$. The joint distribution of $\left(\mathfrak{Z}_{g_i,k_i}\right)_{i=1}^h$ is the limit

$$\lim_{n\to\infty}\left(\mathfrak{Z}_{g_{n,i},k_i}\right)_{i=1}^h, \quad (240)$$

in the sense that this vector almost surely converges. By Proposition 8.1, for each n $\left(\mathfrak{Z}_{g_{n,i},k_i}\right)_{i=1}^h$ is jointly Gaussian, then so is $\left(\mathfrak{Z}_{g_i,k_i}\right)_{i=1}^h$; and the covariances are given by passing the corresponding covariances to the limit $n\to\infty$. □

Finally we finish the proof of Theorem 3.16.

Proof of Theorem 3.16. Integrating by parts in u direction, we obtain

$$\int_0^1 \int_0^1 u^{k_i} g_i(y) \mathcal{W}(u, L\hat{N}_i) du dy = \int_0^1 \frac{g_i(y)}{k_i+1} \left(\min\{N_i, M\} - \min\{N_i - 1, M\}\right) dy \\ - \int_0^1 \frac{g_i(y)}{k_i+1} \left(\sum_{j=1}^{\min\{N_i,M\}} \left(x_j^{N_i}\right)^{k_i+1} - \sum_{j=1}^{\min\{N_i-1,M\}} \left(x_j^{N_i}\right)^{k_i+1} \right) dy. \tag{241}$$

Thus, (44) equals

$$\left(-L \int_0^1 \frac{g_i(y)}{k_i+1} \left(\mathfrak{P}_{k_i+1}(x^{N_i}) - \mathfrak{P}_{k_i+1}(x^{N_i-1}) - \mathbb{E}\left(\mathfrak{P}_{k_i+1}(x^{N_i}) - \mathfrak{P}_{k_i+1}(x^{N_i-1})\right)\right) dy \right)_{i=1}^{h}, \tag{242}$$

which, by Theorem 3.8, is asymptotically Gaussian, and the covariances are given by (226).

On the other hand, by Proposition 8.3, the joint distribution of $(\mathfrak{Z}_{g_i,k_i})_{i=1}^h$ is also Gaussian, with covariances also given by (226).

For (45), it suffices to show that for each $1 \le i \le h'$ the random variable

$$\int_0^1 \int_0^1 u^{k_i'} \tilde{g}_i'(y) \left(\mathcal{H}(u, Ly) - \mathbb{E}\left(\mathcal{H}(u, Ly)\right)\right) + L u^{k_i'} \tilde{g}_i(y) \left(\mathcal{W}(u, Ly) - \mathbb{E}\left(\mathcal{W}(u, Ly)\right)\right) du dy \tag{243}$$

weakly converges to 0 as $L \to \infty$.

Take L_0 be the largest integer strictly less than L. Note that $\mathcal{H}(u, Ly) - \mathbb{E}\left(\mathcal{H}(u, Ly)\right)$ is a piecewise constant function for y: specifically, fixing u, it is constant for $y \in \left[\frac{n-1}{L}, \frac{n}{L}\right)$ for any positive integer n. By integrating in y direction there is

$$\int_0^1 \int_0^1 u^{k_i'} \tilde{g}_i'(y) \left(\mathcal{H}(u, Ly) - \mathbb{E}\left(\mathcal{H}(u, Ly)\right)\right) du dy \\ = - \int_0^1 u^{k_i'} \sum_{n=1}^{L_0} \tilde{g}_i\left(\frac{n}{L}\right) \left(\mathcal{W}(u, n) - \mathbb{E}\left(\mathcal{W}(u, n)\right)\right) du. \tag{244}$$

This implies that the absolute value of (243) is bounded by

$$L \cdot \sup_{a,b\in[0,1], |a-b|\le L^{-1}} |\tilde{g}_i(a) - \tilde{g}_i(b)| \int_0^1 \int_0^1 u^{k_i'} \left|\mathcal{W}(u, Ly) - \mathbb{E}\left(\mathcal{W}(u, Ly)\right)\right| du dy \\ \le \|\tilde{g}_i'\|_{L^\infty} \int_0^1 \int_0^1 u^{k_i'} \left|\mathcal{W}(u, Ly) - \mathbb{E}\left(\mathcal{W}(u, Ly)\right)\right| du dy. \tag{245}$$

By Theorem 3.11, (245) weakly converges to 0 as $L \to \infty$. Then (243) weakly converges to 0 as $L \to \infty$. □

References

[AGZ10] G. W. Anderson, A. Guionnet, and O. Zeitouni. *An Introduction to Random Matrices*, volume 118. Cambridge university press, 2010.

[And91] G. W. Anderson. A short proof of Selberg's generalized beta formula. *Forum Math.*, 3(4):415–418, 1991.

[BBNY16] R. Bauerschmidt, P. Bourgade, M. Nikula, and H. T. Yau. The two-dimensional Coulomb plasma: quasi-free approximation and central limit theorem. 2016, arXiv:1609.08582.

[BC14] A. Borodin and I. Corwin. Macdonald processes. *Probab. Theory Related Fields*, 158(1-2):225–400, 2014, arXiv:1111.4408.

[BCGS16] A. Borodin, I. Corwin, V. Gorin, and S. Shakirov. Observables of Macdonald processes. *Trans. Amer. Math. Soc.*, 368(3):1517–1558, 2016, arXiv:1111.4408.

[BG15] A. Borodin and V. Gorin. General β-Jacobi corners process and the Gaussian free field. *Comm. Pure Appl. Math.*, 68(10):1774–1844, 2015, arXiv:1305.3627.

[Bor14] A. Borodin. CLT for spectra of submatrices of Wigner random matrices. *Mosc. Math. J.*, 14(1):29–38, 2014, arXiv:1010.0898.

[Buf13] A. Bufetov. Kerov's interlacing sequences and random matrices. *J. Math. Phys.*, 54(11), Nov 2013, arXiv:1211.1507.

[CS14] J. G. Conlon and T. Spencer. A strong central limit theorem for a class of random surfaces. *Comm. Math. Phys.*, 325(1):1–15, 2014, arXiv:1105.2814.

[Dix05] A. L. Dixon. Generalization of Legendre's formula. *Proc. Lond. Math. Soc.*, 2(1):206–224, 1905.

[DP14] I. Dumitriu and E. Paquette. Spectra of overlapping Wishart matrices and the Gaussian Free Field. 2014, arXiv:1410.7268.

[Dub09] J. Dubédat. SLE and the free field: partition functions and couplings. *J. Amer. Math. Soc.*, 22(4):995–1054, 2009, arXiv:0712.3018.

[ES16] L. Erdős and D. Schröder. Fluctuations of rectangular Young diagrams of interlacing Wigner eigenvalues. 2016, arXiv:1608.05163.

[FD16] Y. Fyodorov and P. Doussal. Moments of the position of the maximum for GUE characteristic polynomials and for log-correlated Gaussian processes. *J. Stat. Phys.*, pages 1–51, 2016, arXiv:1511.04258.

[FHH+09] B. Feigin, K. Hashizume, A. Hoshino, J. Shiraishi, and S. Yanagida. A commutative algebra on degenerate $\mathbb{CP}^1$ and Macdonald polynomials. *J. Math. Phys.*, 50(9):095215, 2009, arXiv:0904.2291.

[For10] P. Forrester. *Log-Gases and Random Matrices.* London Mathematical Society Monographs. Princeton University Press, 2010.

[GP14] S. Ganguly and S. Pal. The random transposition dynamics on random regular graphs and the Gaussian Free Field. 2014, arXiv:1409.7766.

[IO02] V. Ivanov and G. Olshanski. *Kerov's Central Limit Theorem for the Plancherel Measure on Young Diagrams*, pages 93–151. Springer, Dordrecht, 2002, arXiv:0304010.

[JP14] T. Johnson and S. Pal. Cycles and eigenvalues of sequentially growing random regular graphs. *Ann. Probab.*, 42(4):1396–1437, 2014, arXiv:1203.1113.

[Ker93] S. Kerov. Transition probabilities for continual Young diagrams and the Markov moment problem. *Funct. Anal. Appl.*, 27(2):104–117, 1993.

[Ker94] S. Kerov. Asymptotics of the separation of roots of orthogonal polynomials. *St. Petersburg Math Journal*, pages 925–941, 1994.

[Ker98] S. Kerov. Interlacing measures. *Amer. Math. Soc. Transl.*, pages 35–84, 1998.

[Mac95] I. G. Macdonald. *Symmetric Functions and Hall Polynomials.* Oxford Mathematical Monographs. Oxford University Press, USA, 2 edition, 1995.

[Neg13] A. Negut. Operators on symmetric polynomials. 2013, arXiv:1310.3515.

[Neg14] A. Negut. The shuffle algebra revisited. *Int. Math. Res. Not. IMRN*, 2014(22):6242–6275, 2014, arXiv:1209.3349.

[PS11] L. Pastur and M. Shcherbina. *Eigenvalue Distribution of Large Random Matrices*, volume 171. American Mathematical Society, 2011.

[She07] S. Sheffield. Gaussian free fields for mathematicians. *Probab. Theory Related Fields*, 139(3-4):521–541, 2007, arXiv:0312099.

[Sod16] S. Sodin. Fluctuations of interlacing sequences. 2016, arXiv:1610.02690.

[Sun16] Y. Sun. Matrix models for multilevel Heckman-Opdam and multivariate Bessel measures. 2016, arXiv:1609.09096.

[Sze39] G. Szegő. *Orthogonal Polynomials.* Colloquium Publications Colloquium Publications Amer Mathematical Soc. American Mathematical Society, 4th edition, 1939.

[Wer14] W. Werner. Topics on the two-dimensional Gaussian Free Field. 2014, lecture notes, available online at `https://people.math.ethz.ch/~wewerner/GFFln.pdf`.