MA 17: HOW TO SOLVE IT
HANDOUT 6: DISCUSSIONS OF PSET 5 AND MOCK 3

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Selected problems from PSet 5:

Problem 2. Let G be a finite set of real $n \times n$ matrices $\{M_i\}_{i=1}^r$ which form a group under matrix multiplication. Suppose that

$$\sum_{i=1}^{r} \operatorname{tr}(M_i) = 0,$$

where $\operatorname{tr}(A)$ denotes the trace of the matrix A. Prove that

$$\sum_{i=1}^{r} M_i$$

is the $n \times n$ zero matrix.

Problem 6. Suppose $f(x)$ is a polynomial with real coefficients such that $f(x) \geq 0$ for all x. Show that there exist polynomials $g(x)$ and $h(x)$ with real coefficients such that

$$f(x) = g(x)^2 + h(x)^2.$$

Problem 8. For the $n \times n$ matrix whose (i, j) entry is $1/(i+j-1)$, prove that it is invertible, and its inverse has integer entries.

Some more problems (analysis, inequality, and sums/integrals):

Problem 1. Let $a_1, a_2, \ldots, a_n$ be real numbers. Show that

$$\min_{i<j}(a_i - a_j)^2 \;\leq\; \frac{12}{n(n^2-1)}\,(a_1^2 + a_2^2 + \cdots + a_n^2).$$

Problem 2. Show that if $r_1, \ldots, r_n$ are nonnegative real numbers and $x_1, \ldots, x_n$ are real numbers, then

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \min(r_i, r_j)\, x_i x_j \;\geq\; 0.$$

Problem 4. (Putnam 2017, A3)
Let a and b be real numbers with $a < b$, and let f and g be continuous functions from $[a, b]$ to $(0, \infty)$ such that $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$ but $f \neq g$. For every positive integer n, define

$$I_n = \int_a^b \frac{(f(x))^{n+1}}{(g(x))^n}\,dx.$$

Show that $I_1, I_2, I_3, \ldots$ is an increasing sequence with $\lim_{n\to\infty} I_n = \infty$.

Consider using symmetry in the next two problems.

Problem 5. Compute $\int_{-1}^{1} \frac{\cos(x)}{e^x+1} dx$.

Problem 6. (Putnam 2020, B4)
Let n be a positive integer, and let V_n be the set of integer $(2n+1)$-tuples $\mathbf{v} = (s_0, s_1, \cdots, s_{2n-1}, s_{2n})$ for which $s_0 = s_{2n} = 0$ and $|s_j - s_{j-1}| = 1$ for $j = 1, 2, \cdots, 2n$. Define

$$q(\mathbf{v}) = 1 + \sum_{j=1}^{2n-1} 3^{s_j},$$

and let $M(n)$ be the average of $\frac{1}{q(\mathbf{v})}$ over all $\mathbf{v} \in V_n$. Evaluate $M(2020)$.

Another problem on rational/irrational numbers:

Problem 7. (Putnam 2017, B3)
Suppose that $f(x) = \sum_{i=0}^{\infty} c_i x^i$ is a power series for which each coefficient c_i is 0 or 1. Show that if $f(2/3) = 3/2$, then $f(1/2)$ must be irrational.

Reminder PSet 6 to be released today, and due by the end of Dec 1 (on Canvas). No problem session this week, and happy thanksgiving!