MA 17: HOW TO SOLVE IT
PROBLEM SET 5

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Due: 11:59PM of November 25. Late submissions will not be accepted.

Instructions:

- *Please submit solutions to at most three problems, written carefully. You are encouraged to work on more, but the TA will provide feedback on only three. I recommend selecting the three for which you most want feedback.*
- *Collaboration on homework before you have solved at least three problems independently is discouraged, though not strictly forbidden. You should make a serious effort to work through each problem independently before turning to classmates, the internet, or tools like ChatGPT.*
- *As long as your submission clearly shows attempts on at least three problems, you will earn 8 points toward the final grade. The correctness of your solutions will not affect your score, although feedback will be provided.*
- *Submission should be made through Canvas.*

Problem 1. Let A and B be $m \times m$ matrices with real entries satisfying

$$(AB - BA)^n = I_m$$

for some positive integer n. Prove that n is even.

Hint: You may consider the trace of these matrices.

Problem 2.
Let G be a finite set of real $n \times n$ matrices $\{M_i\}_{i=1}^r$ which form a group under matrix multiplication. Suppose that

$$\sum_{i=1}^r \mathrm{tr}(M_i) = 0,$$

where $\mathrm{tr}(A)$ denotes the trace of the matrix A. Prove that

$$\sum_{i=1}^r M_i$$

is the $n \times n$ zero matrix.

Hint: You may consider the sum of all the matrices in the group.

Problem 3. (Problem 3 in Handout 5) Consider bipartite graphs where the vertices on each side are labeled $\{1, 2, \ldots, n\}$. Find the number of such graphs for which there are an odd number of perfect matchings.

Problem 4.
A mansion has n rooms. Each room has a lamp and a switch connected to its lamp. However, switches may also be connected to lamps in other rooms, subject to the following condition: if the switch in room a is connected to the lamp in room b, then the switch in room b is also connected to the lamp in room a. Each switch, when flipped, changes the state (from on to off or vice versa) of each lamp connected to it. Suppose at some point the lamps are all off. Prove that no matter how the switches are wired, it is possible to flip some of the switches to turn all of the lamps on.

Hint: You may either use linear algebra over $\mathbb{F}_2$, or do induction.

Problem 5.
Given a point P_0 in the plane of the triangle $A_1A_2A_3$. Define $A_s = A_{s-3}$ for all $s \geq 4$. Construct a sequence of points $P_1, P_2, P_3, \ldots$ such that P_{k+1} is the image of P_k under a rotation centered at A_{k+1} through an angle 120° clockwise, for each $k = 0, 1, 2, \ldots$. Prove that if $P_{2025} = P_0$, then the triangle $A_1A_2A_3$ must be equilateral.

Problem 6.
Suppose $f(x)$ is a polynomial with real coefficients such that $f(x) \geq 0$ for all x. Show that there exist polynomials $g(x)$ and $h(x)$ with real coefficients such that

$$f(x) = g(x)^2 + h(x)^2.$$

Hint: f has no real root. For each complex root $x + iy$ of f, $x - iy$ is also a root.

Problem 7.
Let $v_1, v_2, \ldots, v_N$ be unit vectors in $\mathbb{R}^n$, such that for any $i \neq j$, $|v_i \cdot v_j| = \alpha$, for some $0 \leq \alpha < 1$. Show that $N \leq n(n+1)/2$.

Hint: For each $v_i = (v_{i,1}, v_{i,2}, \ldots, v_{i,n})$, consider the vector in $\mathbb{R}^{n(n+1)/2}$ whose entries are given by $v_{i,j}v_{i,k}$ for $1 \leq j \leq k \leq n$.

Problem 8. For the $n \times n$ matrix whose (i, j) entry is $1/(i + j - 1)$, prove that it is invertible, and its inverse has integer entries.

Hint: You consider the matrix whose (i, j) entry is $1/(x_i - y_j)$ for general $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$, and compute its determinant. Then compute the inverse of the given matrix as the ratio of determinants.

Problem 9.
Let n be a positive odd integer and let θ be a real number such that θ/π is irrational. Set $a_k = \tan(\theta + k\pi/n)$, $k = 1, 2, \ldots, n$. Prove that

$$\frac{a_1 + a_2 + \cdots + a_n}{a_1 a_2 \cdots a_n}$$

is an integer, and determine its value.

Hint: For i being the imaginary unit (i.e., $i^2 = -1$), write $a_k = \frac{xe^{ik\pi/n} - x^{-1}e^{-ik\pi/n}}{i(xe^{ik\pi/n} + x^{-1}e^{-ik\pi/n})}$, where $x = e^{i\theta}$. Then write $\frac{a_1+a_2+\cdots+a_n}{a_1a_2\cdots a_n}$ as the ratio of two polynomials in x, and show that they have the same roots.

Reminder Problem session: Nov 13, 7PM to 8PM (to discuss PSet 4). Next Class: Nov 18, 7PM-10PM. (Mock Putnam 3; you will be free to leave early, and there will be snacks.)