

MA 17: HOW TO SOLVE IT
MOCK PUTNAM 3

C1. Let a, b, x_0 be positive integers. For each $n = 1, 2, 3, \dots$, define

$$x_n = ax_{n-1} + b.$$

Prove that there are infinitely many numbers in the sequence x_1, x_2, x_3, \dots that are not prime numbers.

C2. Suppose there are $2n$ people arranged in a circle. The first n of them are *good guys*, and the last n are *bad guys*. Show that there always exists an integer m (depending on n) such that, if we go around the circle executing every m th person, then all of the bad guys are eliminated before any good guy is executed. (For example, when $n = 3$ we may take $m = 5$, and when $n = 4$ we may take $m = 30$.)

C3. Let $P(x)$ be a polynomial of degree n such that

$$P(x) = Q(x) P''(x),$$

where $Q(x)$ is a quadratic polynomial and $P''(x)$ denotes the second derivative of $P(x)$. Show that if $P(x)$ has at least two distinct roots, then it must have n distinct roots.

C4. A checker is placed at every lattice point (x, y) with $y \leq 0$. A move consists of jumping one checker over an adjacent checker onto an empty lattice point, and then removing the jumped checker. Prove that no checker can ever reach a point (x, y) with $y \geq 5$.

C5. Let n and k be positive integers. Cathy is playing the following game. There are n marbles and k boxes, with the marbles labeled 1 through n . Initially, all marbles are placed inside one box. Each turn, Cathy chooses a box and then moves the marble with the smallest label in that box, say i , to either any empty box or to the box containing marble $i + 1$. Cathy wins if at any point there is a box containing only marble n . Determine all pairs of integers (n, k) such that Cathy can win this game.

C6. Find all continuously differentiable functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that for every rational number q , the number $f(q)$ is rational and has the same denominator as q . (The denominator of a rational number q is the unique positive integer b such that $q = a/b$ for some integer a with $\gcd(a, b) = 1$.) (Note: gcd means greatest common divisor.)

Reminder The fifth problem session will be this week, on Nov 20, from 7-8 PM, at the same location.