

**MA 17: HOW TO SOLVE IT**  
**HANDOUT 6: DISCUSSIONS OF PSET 5 AND MOCK 3**

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*Selected problems from PSet 5:*

**Problem 2.** Let  $G$  be a finite set of real  $n \times n$  matrices  $\{M_i\}_{i=1}^r$  which form a group under matrix multiplication. Suppose that

$$\sum_{i=1}^r \text{tr}(M_i) = 0,$$

where  $\text{tr}(A)$  denotes the trace of the matrix  $A$ . Prove that

$$\sum_{i=1}^r M_i$$

is the  $n \times n$  zero matrix.

**Problem 6.** Suppose  $f(x)$  is a polynomial with real coefficients such that  $f(x) \geq 0$  for all  $x$ . Show that there exist polynomials  $g(x)$  and  $h(x)$  with real coefficients such that

$$f(x) = g(x)^2 + h(x)^2.$$

**Problem 8.** For the  $n \times n$  matrix whose  $(i, j)$  entry is  $1/(i+j-1)$ , prove that it is invertible, and its inverse has integer entries.

*Some more problems (analysis, inequality, and sums/integrals):*

**Problem 1.** Let  $a_1, a_2, \dots, a_n$  be real numbers. Show that

$$\min_{i < j} (a_i - a_j)^2 \leq \frac{12}{n(n^2 - 1)} (a_1^2 + a_2^2 + \dots + a_n^2).$$

**Problem 2.** Show that if  $r_1, \dots, r_n$  are nonnegative real numbers and  $x_1, \dots, x_n$  are real numbers, then

$$\sum_{i=1}^n \sum_{j=1}^n \min(r_i, r_j) x_i x_j \geq 0.$$

**Problem 3.** Find all differentiable functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$f'(x) = \frac{f(x+n) - f(x)}{n}$$

for all real numbers  $x$  and all positive integers  $n$ .

**Problem 4.** (Putnam 2017, A3)

Let  $a$  and  $b$  be real numbers with  $a < b$ , and let  $f$  and  $g$  be continuous functions from  $[a, b]$  to  $(0, \infty)$  such that  $\int_a^b f(x) dx = \int_a^b g(x) dx$  but  $f \neq g$ . For every positive integer  $n$ , define

$$I_n = \int_a^b \frac{(f(x))^{n+1}}{(g(x))^n} dx.$$

Show that  $I_1, I_2, I_3, \dots$  is an increasing sequence with  $\lim_{n \rightarrow \infty} I_n = \infty$ .

*Consider using symmetry in the next two problems.*

**Problem 5.** Compute  $\int_{-1}^1 \frac{\cos(x)}{e^x + 1} dx$ .

**Problem 6.** (Putnam 2020, B4)

Let  $n$  be a positive integer, and let  $V_n$  be the set of integer  $(2n+1)$ -tuples  $\mathbf{v} = (s_0, s_1, \dots, s_{2n-1}, s_{2n})$  for which  $s_0 = s_{2n} = 0$  and  $|s_j - s_{j-1}| = 1$  for  $j = 1, 2, \dots, 2n$ . Define

$$q(\mathbf{v}) = 1 + \sum_{j=1}^{2n-1} 3^{s_j},$$

and let  $M(n)$  be the average of  $\frac{1}{q(\mathbf{v})}$  over all  $\mathbf{v} \in V_n$ . Evaluate  $M(2020)$ .

*Another problem on rational/irrational numbers:*

**Problem 7.** (Putnam 2017, B3)

Suppose that  $f(x) = \sum_{i=0}^{\infty} c_i x^i$  is a power series for which each coefficient  $c_i$  is 0 or 1. Show that if  $f(2/3) = 3/2$ , then  $f(1/2)$  must be irrational.

**Reminder** PSet 6 to be released today, and due by the end of Dec 1 (on Canvas). No problem session this week, and happy thanksgiving!