MA 17: HOW TO SOLVE IT
PROBLEM SET 6

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Due: 11:59PM of December 1. Late submissions will not be accepted.

Instructions:

- *Please submit solutions to at most three problems, written carefully. You are encouraged to work on more, but the TA will provide feedback on only three. I recommend selecting the three for which you most want feedback.*
- *Collaboration on homework before you have solved at least three problems independently is discouraged, though not strictly forbidden. You should make a serious effort to work through each problem independently before turning to classmates, the internet, or tools like ChatGPT.*
- *As long as your submission clearly shows attempts on at least three problems, you will earn 8 points toward the final grade. The correctness of your solutions will not affect your score, although feedback will be provided.*
- *Submission should be made through Canvas.*

Problem 1. Compute $\sum_{k=0}^{\infty} \frac{2^k}{5^{2^k}+1}$.

Hint: Try to find a function f such that $\frac{1}{1+x} = 2f(x^2) - f(x)$.

Problem 2. If $a_n > 0$ for $n = 1, 2, \ldots$, show that

$$\sum_{n=1}^{\infty} \sqrt{n}\, a_1 a_2 \cdots a_n \;\le\; e^{\sum_{n=1}^{\infty} a_n},$$

provided that $\sum_{n=1}^{\infty} a_n$ converges.

Problem 3. Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) = \frac{f(x+n) - f(x)}{n}$$

for all real numbers x and all positive integers n.

Problem 4. Prove that $\int_0^1 x^{-x} dx = \sum_{n=1}^{\infty} n^{-n}$.

Hint: Consider Taylor expansion.

Problem 5. A rectangle in $\mathbb{R}^2$ is called *great* if either its width or its height is an integer. Prove that if a rectangle X can be dissected into great rectangles, then the rectangle X is itself great.

Hint: Consider the integral of $e^{2\pi i(x+y)}$.

Problem 6. Let $f(x)$ be a continuous real-valued function defined on the interval $[0, 1]$. Show that

$$\int_0^1 \int_0^1 |f(x) + f(y)|\, dx\, dy \;\geq\; \int_0^1 |f(x)|\, dx.$$

Hint: Separate the integral into parts where $f \geq 0$ and $f < 0$.

Problem 7. Show that

$$\frac{1}{(n-1)!} \int_n^\infty w(t) e^{-t}\, dt < \frac{1}{(e-1)^n},$$

where t is real, n is a positive integer, and

$$w(t) = (t-1)(t-2)\cdots(t-n+1).$$

Hint: Split the left-hand side into integrals in intervals $[k, k+1]$ for each $k \geq n$. Expand the right-hand side as a power series of e^{-1}.

Problem 8. Show that all solutions of the differential equation

$$y'' + e^x y = 0$$

remain bounded as $x \to \infty$.

Hint: Consider $E(x) = (y'(x))^2 + e^x (y(x))^2$.

Reminder No problem session this week. Next (and the last) Class: Dec 2, 7PM-9PM.