MA 17: HOW TO SOLVE IT
PROBLEM SET 1

Instructor Lingfu Zhang (lingfuz@caltech) **Office** Linde Hall 358
TA Minghao Pan (mpan2@caltech)

Due: 11:59PM of October 8. Late submissions will not be accepted.

Instructions:

- *Please submit solutions to at most three problems, written carefully. You are encouraged to work on more, but the TA will provide feedback on only three. I recommend selecting the three for which you most want feedback.*
- *Collaboration on homework before you have solved at least three problems independently is discouraged, though not strictly forbidden. You should make a serious effort to work through each problem independently before turning to classmates, the internet, or tools like ChatGPT.*
- *As long as your submission clearly shows attempts on at least three problems, you will earn 8 points toward the final grade. The correctness of your solutions will not affect your points, although feedback will be provided.*
- *Submission should be made through Canvas.*

Problem 1.
Given a positive integer n, what is the largest k such that the numbers $1, 2, \ldots, n$ can be put into k boxes so that the sum of the numbers in each box is the same? [When $n = 8$, the example $\{1, 2, 3, 6\}, \{4, 8\}, \{5, 7\}$ shows that the largest k is *at least* 3.]

Problem 2.
Let $x_1, x_2, x_2, \ldots$ be a sequence of integers, such that $1 = x_1 < x_2 < x_3 < \cdots$, and $x_{n+1} \le 2n$ for each $n = 1, 2, 3, \ldots$.
Show that every positive integer k is equal to $x_i - x_j$ for some i and j.

Problem 3.
Fifteen chairs are evenly placed around a circular table on which there are name cards for fifteen guests. The guests fail to notice these cards until after they sit down, and it turns out that no one is sitting in front of his own card. Prove that the table can be rotated so that at least two of the guests are simultaneously correctly seated.

Problem 4.
Prove that among any seven real numbers $y_1, \ldots, y_7$, there exist two y_i and y_j with

$$0 \le \frac{y_i - y_j}{1 + y_i y_j} \le \frac{1}{\sqrt{3}}.$$

Problem 5.
Let $A = \{a_1, a_2, \ldots, a_{100}\}$ be a set of 100 numbers, with $a_1 < a_2 < \cdots < a_{100}$. Suppose that

$\phi : A \to A$ is a bijection from A to itself, satisfying

$$a_1 + \phi(a_1) < a_2 + \phi(a_2) < \cdots < a_{100} + \phi(a_{100}).$$

Show that ϕ must be the identity map, i.e., $\phi(a_i) = a_i$ for each $i = 1, 2, \ldots, 100$.

Problem 6. (Problem 6 in Handout 1)
Prove that any positive integer can be represented as $\pm 1^2 \pm 2^2 \pm \cdots \pm n^2$ for some positive integer n and some choice of the signs.

Problem 7.
Assign to each positive real number a color, either red or blue. Let D be the set of all distances $d > 0$ such that there are two points of the same color at distance d apart. Recolor the positive reals so that the numbers in D are red and the numbers not in D are blue. If we iterate this recoloring process, will we always end up with all the numbers red after a finite number of steps?

Problem 8.
Define a sequence by $a_0 = 1$, together with the rules $a_{2n+1} = a_n$ and $a_{2n+2} = a_n + a_{n+1}$ for each integer $n \geq 0$. Prove that every positive rational number appears in the set

$$\left\{ \frac{a_{n-1}}{a_n} : n \geq 1 \right\} = \left\{ \frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \cdots \right\}.$$

Reminder Next class: Oct 7, 7PM to 9PM.