MA 17: HOW TO SOLVE IT
HANDOUT 5: DISCUSSIONS AFTER MOCK 2

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Elements in the theory of matrix: trace, determinant, eigenvalues, eigenvectors, characteristic equation/polynomial.

The Cayley–Hamilton Theorem. Any $n \times n$ matrix A satisfies its characteristic equation, which means that if $P_A(\lambda) = \det(\lambda I_n - A)$, then $P_A(A) = O_n$.

The Perron–Frobenius Theorem. Any square matrix with positive entries has a unique eigenvector with positive entries (up to multiplication by a positive scalar), and the corresponding eigenvalue has multiplicity one and is strictly greater than the absolute value of any other eigenvalue.

Some more problems:

Problem 1. (Putnam 1986, B6)
Suppose that A, B, C, D are $n \times n$ matrices (with entries in some field), such that $AB^\top$ and $CD^\top$ are symmetric, and $AD^\top - BC^\top = I$. Prove that $A^\top D - C^\top B = I$.

Hint: find a more "matricial" interpretation of the condition $AD^\top - BC^\top = I$.

Problem 2. (Putnam 1995, A5)
Let $x_1, x_2, \ldots, x_n$ be differentiable (real-valued) functions of a single variable f which satisfy

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
\frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
&\vdots \qquad\qquad\qquad\qquad \vdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n
\end{aligned}$$

for some constants $a_{ij} > 0$. Suppose that for all i, $x_i(t) \to 0$ as $t \to \infty$. Are the functions $x_1, x_2, \ldots, x_n$ necessarily linearly dependent?

Problem 3. Consider bipartite graphs where the vertices on each side are labeled $\{1, 2, \ldots, n\}$. Find the number of such graphs for which there are an odd number of perfect matchings.

Problem 4. (Putnam 1999, B5)
For an integer $n \geq 3$, let $\theta = 2\pi/n$. Evaluate the determinant of the $n \times n$ matrix $I + A$, where I is the $n \times n$ identity matrix and $A = (a_{jk})$ has entries $a_{jk} = \cos(j\theta + k\theta)$ for all j, k.

Hint: find the eigenvectors, and use complex numbers.

Another problems on complex numbers:

Problem 5. (Putnam 1991, B2)
Suppose f and g are non-constant, differentiable, real-valued functions defined on $(-\infty, \infty)$. Furthermore, suppose that for each pair of real numbers x and y,

$$f(x+y) = f(x)f(y) - g(x)g(y),$$
$$g(x+y) = f(x)g(y) + g(x)f(y).$$

If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all x.

Reminder PSet 5 to be released tomorrow, and due by the end of Nov 19 (on Canvas). The fourth problem session (to discuss PSet 4) will be on Nov 13, 7-8 PM, at the same location.