Exclusion process

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Def For countable V, $\eta: V \to \{0,1\}$ exclusion such that for any $x \neq y$, $\eta(x)=1, \eta(y)=0 \Rightarrow \eta(x)=1, \eta(y)=0$ rate $p(x,y)$

Here $p(x,y) \geq 0$, $\sum_y p(x,y) = 1$, $\sum_y p(x,y) < \infty$

Interpretation: $\begin{cases} \eta(x)=1 & \text{particle at } x \\ \eta(x)=0 & x \text{ empty} \end{cases}$ exclusion: at most one particle at a location.

Question: stationary? Well understood in symmetric case ($p(x,y)=p(y,x)$); less understood in general settings.

- lattice, nearest neighbor; exact solvable; connections to random matrices

Other topics in 1D: growing, multi-species, grand coupling.

- Stationary measures.

Examples: ① If p is doubly stochastic; i.e. $\sum_x p(x,y) = \sum_y p(y,x) = 1$, then iid Bernoulli($\theta$) for any $p \in [0,1]$ is stationary.

② If p is reversible w.r.t. $\pi: V \to \mathbb{R}_{>0}$, i.e., $\pi(x)p(x,y) = \pi(y)p(y,x)$

then independent Bernoulli$\left(\frac{\pi(x)}{1+\pi(x)}\right)$ is stationary.

(proof: $\frac{\pi(x)}{1+\pi(x)} \cdot \frac{1}{1+\pi(y)} \cdot p(x,y) = \frac{\pi(y)}{1+\pi(y)} \cdot \frac{1}{1+\pi(x)} \cdot p(y,x)$)

For $\mathbb{Z}$ with $p(x,x+1) = \frac{1}{2}$, $p(x,x-1) = 1-\frac{1}{2}$ (simple exclusion process on $\mathbb{Z}$)

we take $\pi(x) = e\left(\frac{e-1}{2}\right)^x$, and get a family of stationary.

[$x \neq 0 \Rightarrow$ reduce to Φ; $\pi(0)$: $\mathbb{E}[\eta_0 \eta_1]$]

Symmetric case. $p(x,y) = p(y,x)$; self-dual

Think of as a swap process: $\mathbb{P}[\eta_t = 1 \text{ on } A_0] = \mathbb{P}[\eta_0 = 1 \text{ on } A_t]$

For all extremal stationary: consider harmonic functions $h: V \to [0,1]$, $\sum_y p(x,y)h(y) = h(x)$, $x \in V$.

Theorem μ_h = limit starting from independent Bernoulli($h(x)$) this limit exists; and $\{\mu_h\}_{h \text{ harmonic}}$ gives all the extremal stationary measures.

proof of convergence

take any $x_1, \dots, x_k$, and consider swap process with them as fixed

$H(t) = \mathbb{E}[\eta(x_1) \cdots \eta(x_k)] = \mathbb{P}[\eta(x_1) = \dots = \eta(x_k) = 1]$ is decreasing.

Moreover, $H(t)$ is martingale: $\begin{cases} b=1, & H(t) \text{ is martingale} \\ b=0, & H(t) \to \mathbb{E}[\eta(x_1) \cdots \eta(x_k)] \left(\frac{H(t)}{1-H(t)} - 1 + \frac{H(t)}{1-H(t)} - 1\right) \mathbb{1}(\eta(x_1) = \dots = \eta(x_k) = 0) \end{cases}$

$\Rightarrow H(t)$ converges as $t \to \infty$

- stationarity of μ_h: follows from convergence.
- Ergodicity: two cases.

Case 1. The independent walks (driving p) starting from any x, y ever hit each other, $\forall x, y$.

Case 2. The coupling case. i.e. $\exists$ some walks with positive prob. no collide.

(Case 1 implies uniqueness of translation invariant)

proof in Case 1. Lemma. $\mathbb{P}[\eta(x_1) = \dots = \eta(x_k) = 1]$ depends only on k.

Sketch proof. Consider swap process starting from $x_1, \dots, x_k, x_{k+1}$, they will "coalesce" after finite time, by condition above.

and from $x_1, \dots, x_k, x_{k+1}$

$\Rightarrow \mathbb{P}[\eta(x_1) = \dots = \eta(x_k) = 1] = \mathbb{P}[\eta(x_1) = \dots = \eta(x_{k+1}) = 1]$

by changing one-by-one

Therefore $\{\eta(x)\}_{x \in V}$ is an exchangeable distribution.

De Finetti's Theorem: any exchangeable distribution of $\{0,1\}^V$ is a mixture of Bernoulli(θ) iid.

Thus all extremal stationary dist iid Bernoulli

(Also in case 1, all bounded harmonic functions must be constant, by considering two coalescing random walks)

proof of De Finetti's Thm.

For an exchangeable sequence $X_1, X_2, \dots$, let $S_n = \sum_{i=1}^n X_i$; then $\mathbb{E}S_n^k$ converges as $n \to \infty$, for any $k \in \mathbb{N}$

$\Rightarrow S_n$ converge in distribution (to some measure μ, supp on $[0,1]$)

Consider the measure $\int \text{Bernoulli}(\theta)^{\mathbb{N}} d\mu(\theta)$; can check that $X_1, X_2, \dots$ has the same moments as μ, thus given by this law.

proof in Case 2. Key Lemma: For harmonic h, and measure μ, extremal starting from μ converges to μ_h iff

$$\sum_y h(x)p(x,y)\mu(\eta(y)=1) \to h(x) \quad \forall x$$

$$\sum h(x)p(x,y)\mu(\eta(x)=\eta(y)=1) \to h(x)^2 \quad \forall x \qquad (\mathbb{E}[\eta(x)\eta(y)] = \mathbb{E}[\eta(x)\eta(y)] \to h(x)h(y) \text{ in } \mu_h)$$

(Idea: doubling + coupling)

[Similar to Voter model, writing in class]

Then for $\mu_h = \alpha\mu^1 + (1-\alpha)\mu^0$, $\mu^1 \to \mu_h$ and $\mu^0 \to \mu_h$; if stationary. $\mu_h \to \mu_h$ $\Rightarrow \mu_h$ extremal

On the other hand, for any μ extremal, let $h(x) = \sum_y \mathbb{E}[\eta(y)]$; must be from μ_h.

Lemma. For μ extremal, $\mathbb{E}_\mu[\eta(x)\eta(y)] \leq \mathbb{E}_\mu[\eta(x)\eta(y)] \mathbb{E}_\mu[\eta(y)=1]$, $\forall x \neq y$.

This + reversible (assumption of case 2) $\Rightarrow$ condition of Key Lemma

proof. Let μ_1 be μ conditional on $\eta(x)=1$

μ_0 - - - - $\eta(x)=0$

$\Rightarrow \mu = h(x)\mu_1 + (1-h(x))\mu_0$

Also $\eta_t \mathbb{P}_\mu \to \mu$, $\eta_t \mathbb{P}_{\mu^1} \to \mu$, by extremality of μ

$\Rightarrow \sum_y \mathbb{E}[\eta(y,t)]p(\eta(x)=\eta(y)=1) \to h(x)h(y)$, as $t \to \infty$

Thus if we have independent walks for x, y (denoted by $X(t), Y(t)$)

$\mathbb{P}[\eta(X(t)) = \eta(Y(t)) = 1] \to h(x)h(y)$

on the other hand, if we have swap process for x, y (denoted by $\tilde{X}(t), \tilde{Y}(t)$)

we show that: $\mathbb{P}_\mu[\eta(\tilde{X}(t)) = \eta(\tilde{Y}(t)) = 1] \leq \mathbb{P}_\mu[\eta(X(t)) = \eta(Y(t)) = 1]$

by coupling, $\mathbb{P}_\mu[\eta(x)\eta(y)] \leq h(x)h(y)$.

So for, all in symmetric case: self-duality crucially used.

General case: on $\mathbb{Z}^d$, some translation invariant (i.e. $p(x,y) = p(x+z, y+z)$), and irreducible

(Then doubly stochastic since $\sum_x p(x,y) = \sum_x p(0, y-x) = 1$; iid Bernoulli($\theta$) are stationary)

Thm All translation invariant & stationary measures are iid Bernoulli(θ) for some θ.

(excludes Bernoulli$\left(\frac{\theta}{1+\theta}\right)$)

proof idea. Take μ_θ (iid Bernoulli(θ)) and any other trans inv. stationary μ.

Can couple them together; e.g. start with μ_θ and μ independently; two systems evolve the same Poisson clocks;

$\Rightarrow$ converge to coupled η, ξ, s.t. $\eta \sim \mu_\theta$, $\xi \sim \mu$, and $\mathbb{E}[\eta(x)=\xi(x)=1]$, $\mathbb{E}[\eta(x)=\xi(x)=0] \to 0$.

(Because we can consider $\mathbb{P}[\eta(x) \neq \xi(x)]$; observing the "other path" will further decrease)

Therefore μ_θ either dominates μ, or dominated by μ, direct study; if consider $\{\mu^p\}_{p \in [0,1]}$ all coupled together,

necessarily $\xi \sim \mu$ equals $\eta \sim \mu_p$ for some p.