MA/ACM/IDS 140C: PROBABILITY (SPRING 2025)
PROBLEM SET 2

The difficulty of the problems may vary, so try to solve as many as you can. Most of the techniques have been covered in lectures, but not all. You are encouraged to consult the list of reference materials on the course website, as well as any other relevant textbooks. However, you should not use AI chatbots such as ChatGPT. Discussion with others is allowed, but you must not share intermediate work or final solutions. Feel free to come see me if you'd like to discuss any of the problems or get some hints.

(Due by the end of June 6. No extensions will be granted, as final grades are due on June 9.)

Problem 1: Moments and free convolution. Let X be an $n \times n$ real Wigner matrix with Rademacher entries, i.e., $\mathbb{P}[X_{ij} = 1] = \mathbb{P}[X_{ij} = -1] = \frac{1}{2}$ for all $1 \le i \le j \le n$, and $X_{ij} = X_{ji}$. Let Y be an $n \times n$ diagonal matrix, with $Y_{ii} = \sqrt{n}$ for $1 \le i < \frac{n}{2}$ and $Y_{ii} = -\sqrt{n}$ for $\frac{n}{2} \le i \le n$, and $Y_{ij} = 0$ for any $i \neq j$.
(i) For each $k = 1, 2, 3, 4$, show that

$$n^{-k/2-1}\mathrm{Tr}(X+Y)^k$$

converges in probability as $n \to \infty$, and find the limit.
(ii) Assuming convergence, find the limits using free convolution: for this, you need to compute the (limiting) R-transform of X and Y, to get the (limiting) R-transform and Stieltjes transform of $X + Y$; then expand it at ∞.

Problem 2: Steepest descent method and BBP transition. Consider the *spiked matrix* $X + Y$, where X is an $n \times n$ GUE matrix, and $Y = \mathrm{diag}(a, 0, \ldots, 0)$.

The eigenvalues of $X + Y$ are also determinantal, with kernel

$$K^{(n)}(x, y) = \frac{\exp\left(-\frac{x^2+y^2}{4}\right)}{(2\pi)^2} \oint_C dt \int_\Gamma ds \exp\left(-\frac{t^2}{2} + xt + \frac{s^2}{2} - ys\right) \frac{s-a}{t-a} \left(\frac{s}{t}\right)^{n-1} \frac{1}{t-s}.$$

Here the contour C of t is a circle enclosing 0 and a, and the contour Γ of s is a curve from $-\infty\mathbf{i}$ to $\infty\mathbf{i}$, intersecting the real axis to the right of 0 and a.

Take $a = \sqrt{n} + \hat{a}n^{1/6}$. For fixed $\hat{x}, \hat{y}, \hat{a} \in \mathbb{R}$, find the limit of

$$n^{-1/6}K^{(n)}\left(2\sqrt{n} + \hat{x}n^{-1/6}, 2\sqrt{n} + \hat{y}n^{-1/6}\right),$$

as $n \to \infty$.

Background: GUE plus any rank 1 matrix can be reduced to the form of $X + Y$, by a unitary conjugation; and $a/\sqrt{n}$ is sometimes referred to as the signal to noise ratio. *The Baik-Ben Arous-Péché (BBP) transition roughly refers to the following phenomenon: the largest eigenvalue of $X+Y$ would be significantly larger (than that of GUE X) if and only if the ratio is > 1. In other words, the rank 1 signal can be detected by looking at the largest eigenvalue, when the ratio is > 1.*

Problem 3: SI infection. Fix $d \geq 3$. Consider the random d-regular graph with n vertices. Initially, choose one vertex uniformly at random, make it infected; and all other vertices are susceptible. Then every infected vertex infects its susceptible neighbors with rate 1 independently. Every infected vertex remains infected forever.

Let T_n be the time until the whole graph is infected. Show that there exist $0 < c < C < \infty$, which may depend on d, such that $\mathbb{P}[c \log(n) < T_n < C \log(n)] \to 1$ as $n \to \infty$.

Hint: You may need to show that $\mathbb{P}[c' \log(n) < D_n < C' \log(n)] \to 1$ *for some* $0 < c' < C' < \infty$, *where* D_n *is the diameter of the graph.*

Problem 4: Exclusion process in a finite interval. For integers $1 \leq k < m$, consider the exclusion process on $\{1, \ldots, m\}$ with k particles: for $1 \leq i \leq m-1$, if there is a particle at i, and $i+1$ is empty, the particle jumps to $i+1$ with rate $p > 0$; and for $2 \leq i \leq m$, if there is a particle at i, and $i-1$ is empty, the particle jumps to $i-1$ with rate $q > 0$.

This is a continuous time irreducible Markov chain with finite state space, and has a unique stationary measure. Give an explicit expression of the stationary measure.

Hint: This exclusion process is reversible*. You may use the stationary measure in the reversible setting from lecture; but note that here the number of particles is finite and fixed.*

Bonus problem: Consider the following continuous time Markov chain whose state space is the set of all size m permutations (denoted by bijective functions $\sigma : \{1, \ldots, m\} \to \{1, \ldots, m\}$). For each $1 \leq i < m-1$, if $\sigma(i) < \sigma(i+1)$, swap $\sigma(i)$ and $\sigma(i+1)$ with rate $p > 0$; if $\sigma(i) > \sigma(i+1)$, swap $\sigma(i)$ and $\sigma(i+1)$ with rate $q > 0$. Find the stationary measure of this Markov chain (which is known as the *Mallows measure*).

Comment/Hint: For this Markov chain, the projection $i \mapsto \mathbb{1}[\sigma(i) \leq k]$ *precisely gives the previous exclusion process on* $\{1, \ldots, m\}$ *with* k *particles.*