Interlacing adjacent levels of β-Jacobi corners processes

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Abstract. TODO

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1 Introduction

TODO

Acknowledgments

The second author is supported by Undergraduate Research Opportunity Program (UROP) in Massachusetts Institute of Technology.

2 Background and setup

2.1 β-Jacobi corners process

Definition 2.1. The *N-particle Jacobi ensemble* is a probability distribution on N-tuples of real numbers $0 \le x_1 < \cdots < x_N \le 1$ with density (with respect to Lebesgue measure) proportional to

$$\prod_{1\le i<j\le N}(x_i - x_j)^\beta \prod_{i=1}^{N} x_i^p (1-x_i)^q \tag{1}$$

where $\beta > 0, p, q > -1$ are real parameters.

This is the distribution of eigenvalues of the $N \times N$ MANOVA ensemble of random matrix, and is one of the three *classical* random matrix ensembles. Specifically, consider two infinite matrices X_{ij} and Y_{ij}, $i, j = 1, 2, \cdots$ where entries are i.i.d. real, complex, or quaternion Gaussian. For integers $A \ge B \ge N > 0$, let X^{AB} be the $A \times B$ top left corner of X, and Y^{NB} the $N \times B$ top left corner of Y. For the $B \times B$ matrix,

$$\mathcal{M}^{ANB} = (X^{AB})^* X^{AB} \left((X^{AB})^* X^{AB} + (Y^{NB})^* Y^{NB}\right)^{-1}, \tag{2}$$

almost surely N of its B eigenvalues are different from 0 and 1; if denote these eigenvalues as $x_1 \le \cdots \le x_N$, they are distributed as N-particle Jacobi ensemble, for $\beta = 1, 2, 4$ (respectively for real, complex, or quaternion entries), and $p = \frac{\beta}{2}(A - B + 1) - 1$, $q = \frac{\beta}{2}(B - N + 1) - 1$ [For10, Section 3.6].

For the joint distribution of eigenvalues in $\mathcal{M}^{AnB}$, $n = 1, \cdots, N$, we need the *β-Jacobi corners process*. Following [BG15], let's extend the corner process to infinitely many sequences. Let χ^M be the set of infinite families of sequences $x^1, x^2, \cdots$, where for each $N \ge 1$, x^N is an increasing sequence with length $\min(N, M)$:

$$0 \le x_1^N < \cdots < x_{\min(N,M)}^N \le 1 \tag{3}$$

and for each $N > 1$, x^N and x^{N-1} interlace:

$$x_1^N < x_1^{N-1} < x_2^N < \cdots. \tag{4}$$

Definition 2.2. Let $\mathbb{P}^{\alpha,M,\theta}$ be a distribution on χ^M, which is uniquely given by the sequence wise density and transition probabilities. The marginal distribution on a single sequence x^N has density (with respect to Lebesgue measure) proportional to

$$\prod_{1\le i<j\le \min(N,M)} (x_i^N - x_j^N)^{2\theta} \prod_{i=1}^{\min(N,M)} (x_i^N)^{\theta\alpha-1}(1-x_i^N)^{\theta(|M-N|+1)-1} \tag{5}$$

which is a $\min(N,M)$-particle Jacobi ensemble, with $\beta=2\theta$, $p=\theta\alpha-1$, and $q=\theta(|M-N|+1)-1$.

The conditional distribution of x^{N-1} given x^N has density proportional to

$$\begin{aligned}&\frac{\Gamma(N\theta)}{\Gamma(\theta)^N}\prod_{i=1}^{N}(x_i^N)^{(N-1)\theta}\prod_{1\le i<j<N}(x_j^{N-1}-x_i^{N-1})\prod_{1\le i<j\le N}(x_j^N-x_i^N)^{1-2\theta}\\ &\times\prod_{i=1}^{N-1}\prod_{j=1}^{N}|x_j^N-x_i^{N-1}|^{\theta-1}\prod_{i=1}^{N-1}\frac{1}{(x_i^{N-1})^{N\theta}}\end{aligned} \tag{6}$$

when $N\le M$ and

$$\begin{aligned}&\frac{\Gamma(N\theta)}{\Gamma(\theta)^M\Gamma(N\theta-M\theta)}\prod_{1\le i<j\le M}(x_i^{N-1}-x_j^{N-1})(x_i^N-x_j^N)^{1-2\theta}\\ &\times\prod_{j=1}^{M}(x_i^N)^{(N-1)\theta}(1-x_i^N)^{\theta(M-N-1)+1}\prod_{i,j=1}^{M}|x_j^N-x_i^{N-1}|^{\theta-1}\prod_{i=1}^{M}\frac{(1-x_i^{N-1})^{\theta(N-M)-1}}{(x_i^{N-1})^{N\theta}}\end{aligned} \tag{7}$$

when $N>M$. Here Γ denotes the well-known gamma function.

Remark 2.3. The proof that the distribution $\mathbb{P}^{\alpha,M,\theta}$ is well-defined (i.e., that the formulas (5), (6), and (7) agree with each other) can be found in [BG15, Proposition 2.7]. It is based on integral identities due to Dixon [Dix05] and Anderson [And91].

In [Sun15, Section 4], Yi Sun proved that the joint distribution of (different from 0, 1) eigenvalues in $\mathcal{M}^{AnB}$, $n=1,\cdots,N$ is the same as the first N rows of β-Jacobi corners process with $M=B$, $\alpha=A-B+1$, and $\theta=\frac{\beta}{2}$.

2.2 Diagram and signed measure

As in [Ker93] (also see [Buf13]), we use diagrams to describe interlacing sequences.

Definition 2.4. A *diagram* $w:\mathbb{R}\to\mathbb{R}$ is a function satisfying:

1. Lipschitz condition: $|w(u_1)-w(u_2)|\le|u_1-u_2|$, $\forall u_1,u_2\in\mathbb{R}$.

2. There is $u_0\in\mathbb{R}$, the *center* of w, such that $w(u)=|u-u_0|$ for $|u|$ large enough.

Any diagram w that is piecewise linear and $\frac{d}{du}w=\pm1$ (except for finitely many points) is called *rectangular*.

We draw connection between interlacing sequences and diagrams.

Definition 2.5. For any interlacing sequence $x_1 \le y_1 \le \cdots \le y_{n-1} \le x_n$, define its *corresponding diagram* $w : \mathbb{R} \to \mathbb{R}$ as following:

1. For $u \le x_1$ or $u > x_n$, let $w(u) = |u - u_0|$.
2. For $i = 1, \cdots, n$, let $w(x_i) = \sum_{i<j\le n} x_j - y_{j-1} + \sum_{1\le j<i} y_j - x_j$.
3. For $i = 1, \cdots, n-1$, let $w(y_i) = \sum_{i+1<j\le n} x_j - y_{j-1} + x_{i+1} + \sum_{1\le j<i} y_j - x_j$.
4. In all the intervals $[x_i, y_i]$ and $[y_i, x_{i+1}]$, w is linear.

Remark 2.6. It's easy to verify that the defined w is a rectangular diagram; to be more precise, it satisfies the following conditions:

1. $\frac{d}{du}w(u) = 1$, for any $u \in \left(\bigcup_{i=1}^{n-1}(x_i, y_i)\right) \bigcup (x_n, \infty)$.
2. $\frac{d}{du}w(u) = -1$, for any $u \in \left(\bigcup_{i=1}^{n-1}(y_i, x_{i+1})\right) \bigcup (-\infty, x_1)$.
3. The center of w is $u_0 = \sum_{i=1}^{n} x_i - \sum_{i=1}^{n-1} y_i$.

There is also a discrete signed measure on the $\mathbb{R}$.

Definition 2.7. Given an interlacing sequence $x_1 < y_1 < \cdots < y_{n-1} < x_n$, the corresponding *signed interlacing measure* $\mu^{\{x_i\},\{y_i\}}$ is defined as

$$\mu^{\{x_i\},\{y_i\}}(A) = \sum_{i=1}^{n} \mathbb{1}_{x_i\in A} - \sum_{i=1}^{n-1} \mathbb{1}_{y_i\in A}, \forall A \subset \mathbb{R}. \tag{8}$$

2.3 Pullback of Gaussian Free Field

In this section we briefly define a pullback of Gaussian Free Field, and review some results about the appearance of GFF in the asymptotic of β-Jacobi corners process. Many results presented here are proved in [BG15], or follow similar arguments as there.

We start by briefly recalling the 2-dimensional Gaussian Free Field. A detailed survey of Gaussian Free Field is given in [She07] or [Dub09, Section 4], and here we follow the definition in [BG15]. Informally, the Gaussian Free Field with Dirichlet boundary conditions in the upper half plane $\mathbb{H}$ is defined as the (generalized) Gaussian random field $\mathcal{G}$ on $\mathbb{H}$, whose covariance (for any $z, w \in \mathbb{H}$) is

$$\mathbb{E}(\mathcal{G}(z)\mathcal{G}(w)) = -\frac{1}{2\pi}\ln\left|\frac{z-w}{z-\bar{w}}\right|. \tag{9}$$

Now we consider the following pullback of $\mathcal{G}$, introduced in [BG15].

Definition 2.8. Let D be a subset of $[0,1] \times \mathbb{R}_{>0}$ defined by the following inequality

$$\left|x - \frac{\hat{M}\hat{N} + (\hat{M}+\hat{\alpha})(\hat{N}+\hat{\alpha})}{(\hat{N}+\hat{\alpha}+\hat{M})^2}\right| \le \frac{2\sqrt{\hat{M}\hat{N}(\hat{M}+\hat{\alpha})(\hat{N}+\hat{\alpha})}}{(\hat{N}+\hat{\alpha}+\hat{M})^2}. \tag{10}$$

Let $\Omega : D\bigcup\{\infty\} \to \mathbb{H}\bigcup\{\infty\}$ such that the horizontal section of D at height $\hat{N}$ is mapped to the half-plane part of the circle, centered at

$$\frac{\hat{N}(\hat{\alpha}+\hat{M})}{\hat{N}-\hat{M}} \tag{11}$$

with radius

$$\frac{\sqrt{\hat{M}\hat{N}(\hat{M}+\hat{\alpha})(\hat{N}+\hat{\alpha})}}{\left|\hat{N}-\hat{M}\right|} \tag{12}$$

(when $\hat{N}=\hat{M}$ the circle is replaced by the vertical line at $\frac{\hat{\alpha}}{2}$), and point $u \in \mathbb{H}$ is the image of

$$\left(\frac{u}{u+\hat{N}} \cdot \frac{u-\hat{\alpha}}{u-\hat{\alpha}-\hat{M}}, \hat{N}\right). \tag{13}$$

Its easy to check that Ω is a bijection between $D\bigcup\{\infty\}$ and $\mathbb{H}\bigcup\{\infty\}$.

Definition 2.9. Define $\mathcal{K} = \mathcal{G} \circ \Omega$ to be the pullback of $\mathcal{G}$ with respect to map Ω.

One can view both $\mathcal{G}$ and $\mathcal{K}$ as random variables in a certain functional space. The value of $\mathcal{G}$ or $\mathcal{K}$ at a given point in $\mathbb{H}$ or D is not well-defined since (9) blows up at $z = w$. However, $\mathcal{K}$ can be integrated with respect to certain types of measures; specifically, we have the following results.

Lemma 2.10. *For any positive integers $k_1, \cdots, k_h$ and $0 < \hat{N}_1 \le \cdots \le \hat{N}_h$, the following random vector*

$$\left(\int_0^1 u^{k_i}\mathcal{K}(u,\hat{N}_i)du\right)_{i=1}^h \tag{14}$$

is joint Gaussian, and the covariance between the ith and jth is

$$\frac{\theta^{-1}}{(2\pi\mathbf{i})^2(k_i+1)(k_j+1)}\oint\oint\frac{dv_1dv_2}{(v_1-v_2)^2} \\ \times\left(\frac{v_1}{v_1+\hat{N}_i}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_i+1}\left(\frac{v_2}{v_2+\hat{N}_j}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_j+1}, \tag{15}$$

where $|v_1| \ll |v_2|$, and the contours enclose $-\hat{N}_i$, $-\hat{N}_j$, but not $\hat{\alpha}+\hat{M}$.

This result is essentially proved in [BG15, Theorem 4.13]. Using the same computation there with [BG15, Lemma 4.5], we further obtain the following.

Lemma 2.11. *For any integers $k_1, \cdots, k_h$, and smooth functions $g_1, \cdots, g_h : [0,1] \to \mathbb{R}_{\ge 0}$, with each $g_i(1) = 0$, the joint distribution of the vector*

$$\left(\int_0^1\int_0^1 u^{k_i}g_i(y)\mathcal{K}(u,y)dudy\right)_{i=1}^h \tag{16}$$

is joint Gaussian, and the covariance between the ith and jth terms is

$$\int_0^1 \int_0^1 \frac{g_i(y_1)g_j(y_2)\theta^{-1}}{(2\pi\mathbf{i})^2(k_i+1)(k_j+1)} \oint\oint \frac{1}{(v_1-v_2)^2} \times \left(\frac{v_1}{v_1+y_1}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_i+1}\left(\frac{v_2}{v_2+y_2}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_j+1} dv_1dv_2dy_1dy_2, \quad (17)$$

where the inner contours enclose poles at $-y_1$ and $-y_2$, but not $\hat{\alpha}+\hat{M}$, and are nested: when $y_1 \le y_2$, v_2 is larger; when $y_1 \ge y_2$, v_1 is larger.

Remark 2.12. Let us emphasis that since the values of $\mathcal{K}$ are not defined, the expressions (14) and (16) are not conventional integrals, rather they are pairings of a generalized random functions $\mathcal{K}$ with certain measures, see [BG15, Section 4.5] for details. In fact, Lemma 2.10 and Lemma 2.11 can be taken as an alternative definition of $\mathcal{K}$.

3 Main results

We analyze the limit behavior of adjacent (interlacing) sequence in distribution $\mathbb{P}^{\alpha,M,\theta}$, thus in β-Jacobi corners process.

We have the following limit scheme. The parameters α, M, and N depend on a large auxiliary variable $L\to\infty$, in such a way that

$$\lim_{L\to\infty}\frac{\alpha}{L}=\hat{\alpha},\ \lim_{L\to\infty}\frac{N}{L}=\hat{N},\ \lim_{L\to\infty}\frac{M}{L}=\hat{M} \quad (18)$$

Let $(x^1,x^2,\cdots)\in\chi^M$ be distributed as $\mathbb{P}^{\alpha,M,\theta}$, and we denote

$$\mathfrak{P}_k(x^N)=\begin{cases}\sum_{i=1}^N (x_i^N)^k, & N\le M\\ \sum_{i=1}^M (x_i^N)^k+N-M, & N>M\end{cases} \quad (19)$$

which is a random variable whose distribution relies on $\mathbb{P}^{\alpha,M,\theta}$.

Theorem 3.1. *Under the limit scheme (18), the random variable $\mathfrak{P}_k(x^N)-\mathfrak{P}_k(x^{N-1})$ converges to a constant as $L\to\infty$, in the sense that the variance*

$$\mathbb{E}\left((\mathfrak{P}_k(x^N)-\mathfrak{P}_k(x^{N-1}))^2\right)-\left(\mathbb{E}(\mathfrak{P}_k(x^N)-\mathfrak{P}_k(x^{N-1}))\right)^2, \quad (20)$$

decays in the order of L^{-1}. The constant is given by the following contour integral:

$$\lim_{L\to\infty}\mathbb{E}\left(\mathfrak{P}_k(x^N)-\mathfrak{P}_k(x^{N-1})\right)=\frac{1}{2\pi\mathbf{i}}\oint\left(\frac{v}{v+\hat{N}}\cdot\frac{v-\hat{\alpha}}{v-\hat{\alpha}-\hat{M}}\right)^k\frac{1}{v+\hat{N}}dv, \quad (21)$$

where the integration contour encloses the pole at $-\hat{N}$ but not $\hat{\alpha}+\hat{M}$, and is positively oriented.

Let's consider the diagrams and signed interlacing measures corresponding to the interlacing sequences.

We denote

$$\tilde{x}^N=\begin{cases}x^N, & N\ge M\\ (x_1^N,\cdots,x_M^N,1,\cdots,1), & N<M\end{cases} \quad (22)$$

to be the sequence with length N.

Theorem 3.2. *Denote*

$$\begin{aligned}\gamma_1 &= \frac{\left(\sqrt{(\hat{\alpha}+\hat{M})(\hat{\alpha}+\hat{N})}-\sqrt{\hat{M}\hat{N}}\right)^2}{(\hat{N}+\hat{M}+\hat{\alpha})^2}\\ \gamma_2 &= \frac{\left(\sqrt{(\hat{\alpha}+\hat{M})(\hat{\alpha}+\hat{N})}+\sqrt{\hat{M}\hat{N}}\right)^2}{(\hat{N}+\hat{M}+\hat{\alpha})^2}.\end{aligned} \tag{23}$$

Then there is a unique diagram φ satisfying

$$\varphi''(u) = \begin{cases} \frac{\hat{M}-\hat{N}+(\hat{N}+\hat{M}+\hat{\alpha})(1-u)}{\pi(\hat{N}+\hat{M}+\hat{\alpha})(1-u)} \frac{1}{\sqrt{(\gamma_2-u)(u-\gamma_1)}}, & u \in (\gamma_1,\gamma_2)\\ 2C(\hat{M},\hat{N})\delta(u-1), & u \in (-\infty,\gamma_1]\bigcup[\gamma_2,\infty),\end{cases} \tag{24}$$

where

$$C(\hat{M},\hat{N}) = \begin{cases} 0, & \hat{M} > \hat{N}\\ \frac{1}{2}, & \hat{M} = \hat{N}\\ 1, & \hat{M} < \hat{N}\end{cases}. \tag{25}$$

Then the diagrams $w^{\tilde{x}^N,\tilde{x}^{N-1}}$ converge to φ under the limit scheme (18), in the sense that

$$\lim_{L\to\infty}\sup_{u\in\mathbb{R}}\left|w^{\tilde{x}^N,\tilde{x}^{N-1}}(u)-\varphi(u)\right| = 0, \tag{26}$$

in probability.

We can deduce the following result from Theorem 3.2.
TODO:FIX

Theorem 3.3. *Let γ_1, γ_2, and φ be defined as in Theorem 3.2. For any function $f : [0,1] \to \mathbb{R}$, such that f' exists almost everywhere, and $\int |f''|$ (f'' understood as distribution) is finite, the random variable*

$$\int f d\mu^{\tilde{x}^N,\tilde{x}^{N-1}} = \sum_{i=1}^{N} f(\tilde{x}_i^N) - \sum_{i=1}^{N-1} f(\tilde{x}_i^{N-1}) \tag{27}$$

converges (in probability) to a constant $\frac{1}{2}\int f(u)\varphi''(u)du$, under the limit scheme (18).

Remark 3.4. As each $\mu_{\tilde{x}^N,\tilde{x}^{N-1}}$ is not a positive measure, Theorem 3.3 does not hold for general f that is supported in $[0,1]$. For example, take any $A \subset [0,1]$, and let f be the indicator function of A; then (27) takes only integer values, and is impossible to converge to the given constant for almost every A. This implies that $d\mu_{\tilde{x}^N,\tilde{x}^{N-1}}$ does not weakly converge to the measure with density $\frac{\varphi''}{2}$.

Also, the measure with density $\frac{\varphi''}{2}$ is not necessarily positive (although it has total mass 1): when $\hat{M} < \hat{N}$ the density function can take negative values. This measure is an instance of the *interlacing measures*, which are studied in [Ker98]

Theorem 3.1, 3.2, and 3.3 have a remarkable limit as $\theta \to \infty$. They degenerate to statements about asymptotic separation of the roots of Jacobi orthogonal polynomials. See [Ker94] for similar arguments on Hermite or Tchebychef polynomials.

Theorem 3.5. *TODO: MOVE PROP 6.5*

Now we switch to fluctuations. We have the following central limit theorems.

Theorem 3.6. *For positive integers $k_1, \cdots, k_h$, $k'_1, \cdots, k'_{h'}$, and positive integers $N_1, \cdots, N_h$, $N'_1, \cdots, N'_{h'}$, in addition to the limit scheme (18) we also let*

$$\lim_{L\to\infty} \frac{N_i}{L} = \hat{N}_i, \quad 1 \le i \le h \tag{28}$$

and

$$\lim_{L\to\infty} \frac{N'_i}{L} = \hat{N}'_i, \quad 1 \le i \le h'. \tag{29}$$

The random vector

$$L^{\frac{1}{2}} \left(\mathfrak{P}_{k_i}(x^{N_i}) - \mathfrak{P}_{k_i}(x^{N_i-1}) - \mathbb{E}\left(\mathfrak{P}_{k_i}(x^{N_i}) - \mathfrak{P}_{k_i}(x^{N_i-1})\right)\right)_{i=1}^{h} \tag{30}$$

and

$$\left(\mathfrak{P}_{k'_i}(x^{N'_i}) - \mathbb{E}\left(\mathfrak{P}_{k'_i}(x^{N'_i})\right)\right)_{i=1}^{h'} \tag{31}$$

converge (as $L \to \infty$) jointly and weakly to a Gaussian random vector. The covariance is as following: within the vector (30), the covariance between the ith and jth term, if $\hat{N}_i = \hat{N}_j$, is

$$-\frac{k_i k_j}{k_i + k_j} \cdot \frac{\theta^{-1}}{2\pi\mathbf{i}} \oint \frac{1}{(v + \hat{N}_i)^2} \left(\frac{v}{v + \hat{N}_i} \cdot \frac{v - \hat{\alpha}}{v - \hat{\alpha} - \hat{M}}\right)^{k_i + k_j} dv \tag{32}$$

where the contour encloses $-\hat{N}_i$ but not $\hat{\alpha} + \hat{M}$; within the vector (31), the covariance between the ith and jth term is

$$\frac{\theta^{-1}}{(2\pi\mathbf{i})^2} \oint \oint \frac{1}{(v_1 - v_2)^2} \prod_{i=1}^{2} \left(\frac{v_i}{v_i + \hat{N}_i} \cdot \frac{v_i - \hat{\alpha}}{v_i - \hat{\alpha} - \hat{M}}\right)^{k_i} dv_i. \tag{33}$$

All other covariances are 0*. Especially, (30) and (31) are asymptotic independent.*

Remark 3.7. The Gaussianity of the vector (31) is actually [BG15, Theorem 4.1], and here we are more interested in its joint distribution with (30). The proof presented is this text can also be treated as an alternative proof for [BG15, Theorem 4.1].

When considering the asymptote of "weighted average" for a collection of interlacing chains, we have the following theorem.

Theorem 3.8. *Let $k_1, \cdots, k_h$ be integers, and $g_1, \cdots, g_h : [0,1] \to \mathbb{R}_{\ge 0}$ are functions, each in $L^1([0,1])$, bounded and continuous almost everywhere.*

Then under (18), the joint distribution of the vector

$$\left(L \int_0^1 g_i(y) \left(\mathfrak{P}_{k_i}(x^{\lfloor Ly \rfloor}) - \mathfrak{P}_{k_i}(x^{\lfloor Ly \rfloor - 1}) - \mathbb{E}\left(\mathfrak{P}_{k_i}(x^{\lfloor Ly \rfloor}) - \mathfrak{P}_{k_i}(x^{\lfloor Ly \rfloor - 1})\right)\right) dy\right)_{i=1}^{h} \tag{34}$$

converges weakly to joint Gaussian distribution, with covariance between the ith and jth term

$$\iint_{y_1<y_2} \frac{\theta^{-1}}{(2\pi\mathbf{i})^2} \oint\oint \frac{k_i k_j}{(v_1-v_2)^2(v_1+y_1)(v_2+y_2)}$$
$$\times \left(g_i(y_1)g_j(y_2) \left(\frac{v_1}{v_1+y_1} \cdot \frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}} \right)^{k_i} \left(\frac{v_2}{v_2+y_2} \cdot \frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}} \right)^{k_j} \right.$$
$$\left. + g_j(y_1)g_i(y_2) \left(\frac{v_1}{v_1+y_1} \cdot \frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}} \right)^{k_j} \left(\frac{v_2}{v_2+y_2} \cdot \frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}} \right)^{k_i} \right) dv_1 dv_2 dy_1 dy_2$$
$$- \int_0^1 \frac{\theta^{-1}}{2\pi\mathbf{i}} \oint \frac{g_i(y)g_j(y)k_i k_j}{(k_i+k_j)(v+y)^2} \left(\frac{v}{v+y} \cdot \frac{v-\hat{\alpha}}{v-\hat{\alpha}-\hat{M}} \right)^{k_i+k_j} dv dy, \quad (35)$$

where in the first integral, the contours are nested: $|v_1| \ll |v_2|$, and enclose $-y_1, -y_2$ but not $\hat{\alpha}+\hat{M}$; in the second integral, the contour encloses $-y$ but not $\hat{\alpha}+\hat{M}$.

Remark 3.9. It's worth noting that the scaling in (30) and (35) are different: for a single chain the scale is $L^{\frac{1}{2}}$, while for integral cross different levels the scale is L.

Theorems 3.6 and 3.8 have an interpretation in terms of GFF. For that we define (random) height functions.

Definition 3.10. Let sequences $x^1, x^2, \cdots$ be distributed as $\mathbb{P}^{\alpha,M,\theta}$. For any $(u,y) \in [0,1]\times\mathbb{R}_{>0}$, define $\mathcal{H}(u,y)$ to be the number of i such that $x_i^{\lfloor y \rfloor}$ is less than u. For $y>1$, let $\mathcal{W}(u,y) = \mathcal{H}(u,y) - \mathcal{H}(u,y-1)$.

In [BG15] the convergence of $\mathcal{H}$ to the random field $\mathcal{K}$ is proven. We point out that our central limit theorems imply the convergence of $\mathcal{W}$ to a derivative of the random field $\mathcal{K}$. For discrete levels, there is weak convergence to a "renormalized derivative" of the random field $\mathcal{K}$, in the following sense.

Theorem 3.11. *Under the limit scheme (18), for any integers $k_1, \cdots, k_h$, and $0 < \hat{N}_1 \le \cdots \le \hat{N}_h$, the distribution of the vector*

$$\left(L^{\frac{1}{2}} \int_0^1 u^{k_i} \left(\mathcal{W}(u, L\hat{N}_i) - \mathbb{E}\left(\mathcal{W}(u, L\hat{N}_i) \right) \right) du \right)_{i=1}^h \quad (36)$$

as $L\to\infty$ converges weakly to a joint Gaussian distribution, which is the same as the weak limit

$$\lim_{\delta\to 0} \delta^{-\frac{1}{2}} \left(\int_0^1 u^{k_i}\mathcal{K}(u,\hat{N}_i+\delta)du - \int_0^1 u^{k_i}\mathcal{K}(u,\hat{N}_i)du \right)_{i=1}^h. \quad (37)$$

Although there is such connection between the limit fields that $L^{\frac{1}{2}}\mathcal{W}$ and $\mathcal{H}$ converge to, passing $L\to\infty$ simultaneously for them gives independent fields. Specifically, we have the following result.

Lemma 3.12. *Under the limit scheme (18), for any integers $k_1, \cdots, k_h$, and $0 < \hat{N}_1 \le \cdots \le \hat{N}_h$, and $k'_1, \cdots, k'_{h'}$, and $0 < \hat{N}'_1 \le \cdots \le \hat{N}'_{h'}$, the random vectors*

$$\left(L^{\frac{1}{2}} \int_0^1 u^{k_i} \left(\mathcal{W}(u, L\hat{N}_i) - \mathbb{E}\left(\mathcal{W}(u, L\hat{N}_i) \right) \right) du \right)_{i=1}^h \quad (38)$$

and

$$\left(\int_0^1 u^{k_i'}\left(\mathcal{H}(u, L\hat{N}_i') - \mathbb{E}\left(\mathcal{H}(u, L\hat{N}_i')\right)\right) du\right)_{i=1}^{h'} \tag{39}$$

jointly converges (weakly) as $L \to \infty$, while the limit vectors are independent.

In contrast to Lemma 3.12, in the sense of 2-dimensional integrals the random height function converges to the derivative of $\mathcal{K}$, which is described in terms of its "integrals" against some functions. Namely, we define the pairings $\mathfrak{Z}_{g,k}$ of the y-derivative of the field with test functions $u^k g(y)$ through the following procedure based on integration by parts in y-direction.

Definition 3.13. For any smooth (i.e. C^∞) $g : [0,1] \to \mathbb{R}_{\geq 0}$, with $g(1) = 0$, define

$$\mathfrak{Z}_{g,k} = -\int_0^1 \int_0^1 u^k g'(y)\mathcal{K}(u, y) du dy. \tag{40}$$

Lemma 3.14. *For any $g \in L^2([0,1])$, and positive integer k, there exists a sequence of smooth functions $g_1, g_2, \cdots$, each $g_n(1) = 0$, that converges to g in $L^2([0,1])$, and the random variables $\mathfrak{Z}_{g_n,k}$ converge almost surely. If there is another such sequence $\tilde{g}_1, \tilde{g}_2, \cdots$, then the limits of $\mathfrak{Z}_{g_n,k}$ and $\mathfrak{Z}_{\tilde{g}_n,k}$ are almost surely the same.*

Definition 3.15. For any $g \in L^2([0,1])$ we define We define $\mathfrak{Z}_{g,k}$ to be the limit of Lemma 3.14.

Now we state the convergence of $\mathcal{W}$ to the y-derivative of $\mathcal{K}$ in the following sense.

Theorem 3.16. *Let $k_1, \cdots, k_h$ be integers and $g_1, \cdots, g_h : [0,1] \to \mathbb{R}_{\geq 0}$ be functions, each in $L^1([0,1])$, bounded and differentiable almost everywhere. The joint distribution of the vector*

$$\left(L \int_0^1 \int_0^1 u^{k_i} g_i(y) \left(\mathcal{W}(u, Ly) - \mathbb{E}\left(\mathcal{W}(u, Ly)\right)\right) du dy\right)_{i=1}^{h} \tag{41}$$

converges weakly to the joint distribution of the vector

$$\left(\mathfrak{Z}_{g_i,k_i}\right)_{i=1}^{h}. \tag{42}$$

Organization of remaining text

The remaining sections are devoted to proofs of the above stated results. In Section 4 we introduce integral identities, which are powerful tools in simplifying the computations. Section 5 gives the joint moments before passing to the limits.

TODO

4 Dimension reduction

In this section we discuss integral identities, which will be widely used in subsequent proofs. A special case ($m = 1$) of the following result was communicated to the authors by Alexei Borodin, and we present our own proof here.

For positive integer n, let σ_n be the cycle $(12 \cdots n)$, and let $S^{cyc}(n)$ be the n-element subgroup of symmetric group spanned by σ_n.

Theorem 4.1. *Let $n \geq 2$, and $f_1, \cdots, f_n : \mathbb{C} \to \mathbb{C}$ be meromorphic with possible poles at $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_m\}$. Then we have the identity*

$$\sum_{\sigma \in S^{cyc}(n)} \frac{1}{(2\pi\mathbf{i})^n} \oint \cdots \oint \frac{f_{\sigma(1)}(u_1) \cdots f_{\sigma(n)}(u_n)}{(u_2 - u_1) \cdots (u_n - u_{n-1})} du_1 \cdots du_n = \frac{1}{2\pi\mathbf{i}} \oint f_1(u) \cdots f_n(u) du, \tag{43}$$

where the contours in both sides are positively oriented, enclosing $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_m\}$, and for the left hand side we required $|u_1| \ll \cdots \ll |u_n|$.

Proof. Let $\mathfrak{C}_1, \cdots, \mathfrak{C}_{2n-1}$ be closed paths around $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_m\}$, and each $\mathfrak{C}_i$ is inside $\mathfrak{C}_{i+1}$, $1 \leq i \leq 2n-2$. Also, for the convenience of notations, set $f_{n+t} = f_t$ and $u_{n+t} = u_t$ for any $1 \leq t \leq n-1$. Then the left hand side of (43) can be written as

$$\sum_{t=0}^{n-1} \frac{1}{(2\pi\mathbf{i})^n} \oint_{\mathfrak{C}_1} \cdots \oint_{\mathfrak{C}_n} \frac{f_{1+t}(u_1) \cdots f_{n+t}(u_n)}{(u_2 - u_1) \cdots (u_n - u_{n-1})} du_n \cdots du_1. \tag{44}$$

When $n = 2$, we have

$$\begin{aligned}
&\frac{1}{(2\pi\mathbf{i})^2} \oint_{\mathfrak{C}_1} \oint_{\mathfrak{C}_2} \frac{f_1(u_1) f_2(u_2)}{u_2 - u_1} du_2 du_1 + \frac{1}{(2\pi\mathbf{i})^2} \oint_{\mathfrak{C}_1} \oint_{\mathfrak{C}_2} \frac{f_2(u_1) f_1(u_2)}{u_2 - u_1} du_2 du_1 \\
&\quad = \frac{1}{(2\pi\mathbf{i})^2} \oint_{\mathfrak{C}_1} \oint_{\mathfrak{C}_2} \frac{f_1(u_1) f_2(u_2)}{u_2 - u_1} du_2 du_1 + \frac{1}{(2\pi\mathbf{i})^2} \oint_{\mathfrak{C}_3} \oint_{\mathfrak{C}_2} \frac{f_1(u_1) f_2(u_2)}{u_1 - u_2} du_2 du_1 \\
&\quad = \frac{1}{(2\pi\mathbf{i})^2} \oint_{\mathfrak{C}_3 - \mathfrak{C}_1} \oint_{\mathfrak{C}_2} \frac{f_1(u_1) f_2(u_2)}{u_1 - u_2} du_2 du_1,
\end{aligned} \tag{45}$$

where $\oint_{\mathfrak{C}_3 - \mathfrak{C}_1}$ is a notation for the difference of integrals over $\mathfrak{C}_3$ and $\mathfrak{C}_1$. Further, by deforming the u_2 contour, we transform (45) into

$$\frac{1}{2\pi\mathbf{i}} \oint_{\mathfrak{C}_2} f_1(u) f_2(u) du, \tag{46}$$

since for any u_1, $\frac{f_1(u_1) f_2(u_2)}{u_1 - u_2}$ has a single pole at u_2 with residue $f_1(u_2) f_2(u_2)$, between $\mathfrak{C}_3$ and $\mathfrak{C}_1$. This proves the case of $n = 2$.

When $n \geq 3$, we argue by induction and assume that Theorem 4.1 is true for $n - 1$. For any $1 \leq t \leq n-1$, we have that

$$\begin{aligned}
&\frac{1}{(2\pi\mathbf{i})^n} \oint_{\mathfrak{C}_1} \cdots \oint_{\mathfrak{C}_n} \frac{f_{1+t}(u_1) \cdots f_{n+t}(u_n)}{(u_2 - u_1) \cdots (u_n - u_{n-1})} du_n \cdots du_1 \\
&\quad = \frac{1}{(2\pi\mathbf{i})^n} \oint_{\mathfrak{C}_{1+t}} \cdots \oint_{\mathfrak{C}_{n+t}} \frac{f_1(u_1) \cdots f_n(u_n)}{(u_{2+t} - u_{1+t}) \cdots (u_{n+t} - u_{n-1+t})} du_{n+t} \cdots du_{1+t} \\
&\quad = \frac{1}{(2\pi\mathbf{i})^n} \oint_{\mathfrak{C}_{n+1}} \cdots \oint_{\mathfrak{C}_{n+t}} \oint_{\mathfrak{C}_{t+1}} \cdots \oint_{\mathfrak{C}_n} \frac{f_1(u_1) \cdots f_n(u_n)}{(u_{2+t} - u_{1+t}) \cdots (u_{n+t} - u_{n-1+t})} du_n \cdots du_1.
\end{aligned} \tag{47}$$

Now we can move the contours of $u_1, \cdots, u_t$ from $\mathfrak{C}_{n+1}, \cdots, \mathfrak{C}_{n+t}$ to $\mathfrak{C}_1, \cdots, \mathfrak{C}_t$, respectively. We move the contours one by one starting from u_1, and each move is across $\mathfrak{C}_{t+1}, \cdots, \mathfrak{C}_n$. For

$u_1(=u_{n+1})$, the only pole between $\mathfrak{C}_{n+1}$ and $\mathfrak{C}_1$ is u_n; for any u_i, $1 < i \le t$, there is no pole between $\mathfrak{C}_{n+i}$ and $\mathfrak{C}_i$. Thus we have that

$$\begin{aligned}
\frac{1}{(2\pi\mathbf{i})^n}\oint_{\mathfrak{C}_{n+1}}\cdots\oint_{\mathfrak{C}_{n+t}}\oint_{\mathfrak{C}_{t+1}}\cdots\oint_{\mathfrak{C}_n}\frac{f_1(u_1)\cdots f_n(u_n)}{(u_{2+t}-u_{1+t})\cdots(u_{n+t}-u_{n-1+t})}du_n\cdots du_1 \\
= \frac{1}{(2\pi\mathbf{i})^n}\oint_{\mathfrak{C}_1}\cdots\oint_{\mathfrak{C}_n}\frac{f_1(u_1)\cdots f_n(u_n)}{(u_{2+t}-u_{1+t})\cdots(u_{n+t}-u_{n-1+t})}du_n\cdots du_1 \\
+\frac{1}{(2\pi\mathbf{i})^{n-1}}\oint_{\mathfrak{C}_1}\cdots\oint_{\mathfrak{C}_{n-1}}\frac{f_{1+t}(u_1)\cdots f_n(u_{n-t})f_{n+1}(u_{n-t})\cdots f_{n+t}(u_{n-1})}{(u_2-u_1)\cdots(u_{n-1}-u_{n-2})}du_{n-1}\cdots du_1.
\end{aligned} \tag{48}$$

Notice that (taking into account that $u_{n+t}=u_t$)

$$\sum_{t=0}^{n-1}\frac{f_1(u_1)\cdots f_n(u_n)}{(u_{2+t}-u_{1+t})\cdots(u_{n+t}-u_{n-1+t})}=0, \tag{49}$$

and by induction assumption (applied to $f_2,\cdots,f_{n-1},f_nf_1$), we have that

$$\begin{aligned}
\sum_{t=0}^{n-1}\frac{1}{(2\pi\mathbf{i})^n}\oint_{\mathfrak{C}_1}\cdots\oint_{\mathfrak{C}_n}\frac{f_{1+t}(u_1)\cdots f_{n+t}(u_n)}{(u_2-u_1)\cdots(u_n-u_{n-1})}du_n\cdots du_1 \\
=\sum_{t=1}^{n-1}\frac{1}{(2\pi\mathbf{i})^{n-1}}\oint_{\mathfrak{C}_1}\cdots\oint_{\mathfrak{C}_{n-1}}\frac{f_{1+t}(u_1)\cdots f_n(u_{n-t})f_{n+1}(u_{n-t})\cdots f_{n+t}(u_{n-1})}{(u_2-u_1)\cdots(u_{n-1}-u_{n-2})}du_{n-1}\cdots du_1 \\
=\frac{1}{2\pi\mathbf{i}}\oint f_1(u)\cdots f_n(u)du.
\end{aligned} \tag{50}$$

Then by the principle of induction we have shown Theorem 4.1 holds for any positive integer n. ◻

Theorem 4.1 immediately leads to the following result.

Corollary 4.2. *Let s be a positive integer. Let f, $g_1,\cdots,g_s$ be meromorphic functions with possible poles at $\{\mathfrak{p}_1,\cdots,\mathfrak{p}_m\}$. Then for $n\ge 2$,*

$$\begin{aligned}
\frac{1}{(2\pi\mathbf{i})^n}\oint\cdots\oint\frac{1}{(v_2-v_1)\cdots(v_n-v_{n-1})}\prod_{i=1}^n f(v_i)dv_i\prod_{i=1}^s\left(\sum_{j=1}^n g_i(v_j)\right) \\
=\frac{n^{s-1}}{2\pi\mathbf{i}}\oint f(v)^n\prod_{i=1}^s g_i(v)dv,
\end{aligned} \tag{51}$$

where the contours in both sides are around all of $\{\mathfrak{p}_1,\cdots,\mathfrak{p}_m\}$, and for the left hand side we required $|u_1|\ll\cdots\ll|u_n|$.

Proof. For disjoint sets $U_1,\cdots,U_n$, with $\bigcup_{i=1}^n U_i=\{1,\cdots,s\}$ (some of which might be empty), in Theorem 4.1 we let $f_i = f\prod_{j\in U_{\sigma(i)}} g_j$, $\forall 1\le i\le n$:

$$\sum_{\sigma\in S^{cyc}(n)} \frac{1}{(2\pi\mathbf{i})^n}\oint\cdots\oint \frac{1}{(v_2-v_1)\cdots(v_n-v_{n-1})}\prod_{i=1}^{n}\left(f(v_i)\prod_{j\in U_{\sigma(i)}} g_j(v_i)dv_i\right) = \frac{1}{2\pi\mathbf{i}}\oint f(v)^n\prod_{i=1}^{s} g_i(v)dv. \quad (52)$$

Summing over all n^s partitions $U_1,\cdots,U_n$ of $\{1,\cdots,s\}$ into n disjoint sets, we obtain (51). $\square$

5 Discrete joint moments

In this section we compute the joint moments in β-Jacobi corners processes. The main goal is to prove the following result.

Theorem 5.1. *Let $(x^1,x^2,\cdots)\in\chi^M$ be distributed as $\mathbb{P}^{\alpha,M,\theta}$, and let $\mathfrak{P}_k(x^N)$ be defined as (19). Let l, $N_1\le\cdots\le N_l$, and $k_1,\cdots,k_l$ be positive integers, satisfying $M>k_1+\cdots+k_l$.*

For any positive integers m, n, $\tilde m$, $\tilde n$, and variables $w_1,\cdots,w_m$, $\tilde w_1,\cdots,\tilde w_{\tilde m}$, denote

$$\mathfrak{I}(w_1,\cdots,w_m;\alpha,M,\theta,n) = \frac{1}{(w_2-w_1+1-\theta)\cdots(w_m-u_{m-1}+1-\theta)} \times\prod_{1\le i<j\le m}\frac{(w_j-w_i)(w_j-w_i+1-\theta)}{(w_j-w_i-\theta)(w_j-w_i+1)}\prod_{i=1}^{m}\frac{w_i-\theta}{w_i+(n-1)\theta}\cdot\frac{w_i-\theta\alpha}{w_i-\theta\alpha-\theta M}, \quad (53)$$

and

$$\mathfrak{L}(w_1,\cdots,w_m;\tilde w_1,\cdots,\tilde w_{\tilde m};\theta) = \prod_{1\le i\le\tilde m,1\le j\le m}\frac{(\tilde w_i-w_j)(\tilde w_i-w_j+1-\theta)}{(\tilde w_i-w_j-\theta)(\tilde w_i-w_j+1)}. \quad (54)$$

Then the expectation of higher moments $\mathfrak{P}_k(x^N)$ can be computed via

$$\mathbb{E}\left(\mathfrak{P}_{k_1}(x^{N_1})\cdots\mathfrak{P}_{k_l}(x^{N_l})\right) = \frac{(-\theta)^{-l}}{(2\pi\mathbf{i})^{k_1+\cdots+k_l}}\oint\cdots\oint\prod_{i=1}^{l}\mathfrak{I}(u_{i,1},\cdots,u_{i,k_i};\alpha,M,\theta,N_i) \times\prod_{i<j}\mathfrak{L}(u_{i,1},\cdots,u_{i,k_i};u_{j,1},\cdots,u_{j,k_j};\theta)\prod_{i=1}^{l}\prod_{i'=1}^{k_i}du_{i,i'}, \quad (55)$$

where for each $i=1,\cdots,l$, the contours of $u_{i,1},\cdots,u_{i,k_i}$ enclose $-\theta(N_i-1)$ but not $\theta(\alpha+M)$, and $|u_{i,1}|\ll\cdots\ll|u_{i,k_i}|$. For $1\le i<l$, we also require that $|u_{i,k_i}|\ll|u_{i+1,1}|$.

The main idea of the proof of Theorem 5.1 is to use the so called *Macdonald processes*, which is a distribution on infinite interlacing sequences of non-negative integers. Under certain limit transition it weakly converges to $\mathbb{P}^{\alpha,M,\theta}$. In turn, we compute the moments of Macdonald process by applying a remarkable family of difference operators coming from the work [Neg13] on the symmetric functions. A particular case ($N1 = \cdots = N_l$) of Theorem 5.1 was proven by one of the authors and Borodin in the appendix to [FD16].

We first introduce the Macdonald processes and operators on symmetric polynomials, and the asymptotic relations. Then we compute single moment in β-Jacobi corners processes to illustrate the approach. We will present the proof of Theorem 5.1 using these tools. We end this section by proving (21) in Theorem 3.1 as an application.

5.1 Macdonald processes and asymptotic relations

Let Λ_N denote the ring of symmetric polynomials in N variables, and Λ denote the ring of symmetric polynomials in countably many variables (see [Mac95, Chapter I, Section 2]). Let $\mathbb{Y}$ be the set of partitions, i.e. infinite non-increasing sequence of non-negative integers, which are eventually zero:

$$\mathbb{Y} = \{\lambda = (\lambda_1, \lambda_2, \cdots) \in \mathbb{Z}^\infty \,|\lambda_1 \geq \lambda_2 \geq \cdots \geq 0,\ \exists N \in \mathbb{Z}_+, \lambda_N = 0\},$$

and $\mathbb{Y}_N \subset \mathbb{Y}$ consists of sequences λ such that $\lambda_{N+1} = 0$. We can make $\mathbb{Y}$ a partially ordered set, declaring

$$\lambda \geq \mu \iff \lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i,\ \forall i = 1, 2, \cdots. \tag{56}$$

For any $\lambda \in \mathbb{Y}$, denote $P_\lambda(\cdot; q, t) \in \Lambda$ to be the Macdonald polynomial, which can be written as

$$P_\lambda(\cdot; q, t) = \sum_{\mu \leq \lambda} u_{\lambda\mu} m_\mu \tag{57}$$

where each m_μ is the sum of all $x_1^{\alpha_1} x_2^{\alpha_2} \cdots$, such that $(\alpha_1, \alpha_2, \cdots)$ is a permutation of $(\lambda_1, \lambda_2, \cdots)$, and $u_{\lambda\lambda} = 1$ (see [Mac95, Section VI.4]). Here q and t are real parameters, and we assume that $0 < q < 1$ and $0 < t < 1$. We also denote $Q_\lambda(\cdot; q, t) = b_\lambda(q, t) P_\lambda(\cdot; q, t)$, where $b_\lambda(q, t)$ is a constant uniquely defined by the identity 59 below and with explicit expression given in [Mac95, Chapter VI]. The collection

$$\left\{P_\lambda(\cdot; q, t) \middle| \lambda \in \mathbb{Y}\right\}$$

is a basis of Λ. We further define the skew Macdonald polynomials $P_{\lambda/\mu}$ and $Q_{\lambda/\mu}$, where $\lambda, \mu \in \mathbb{Y}$, to be the coefficients of the following expansions (see [Mac95, Chapter VI, (7.9)]):

$$\begin{aligned} P_\lambda(a_1, \cdots, a_N, b_1, \cdots, b_N; q, t) &= \sum_{\mu \in \mathbb{Y}_N} P_{\lambda/\mu}(a_1, \cdots, a_N; q, t) P_\mu(b_1, \cdots, b_N; q, t) \\ Q_\lambda(a_1, \cdots, a_N, b_1, \cdots, b_N; q, t) &= \sum_{\mu \in \mathbb{Y}_N} Q_{\lambda/\mu}(a_1, \cdots, a_N; q, t) Q_\mu(b_1, \cdots, b_N; q, t). \end{aligned} \tag{58}$$

Proposition 5.2. *For any finite sequences $a_1, \cdots, a_{M_1}$ and $b_1, \cdots, b_{M_2} \in \mathbb{C}$, with $|a_i b_j| < 1$, $\forall 1 \leq i \leq M_1$ for any $1 \leq j \leq M_2$, the Macdonald polynomials have the following identities:*

$$\sum_{\lambda \in \mathbb{Y}} P_\lambda(a_1, \cdots, a_{M_1}; q, t) Q_\lambda(b_1, \cdots, b_{M_2}; q, t) = \prod_{1 \leq i \leq M_1, 1 \leq j \leq M_2} \frac{\prod_{k=1}^\infty (1 - t a_i b_j q^{k-1})}{\prod_{k=1}^\infty (1 - a_i b_j q^{k-1})}, \tag{59}$$

$$\sum_{\lambda \in \mathbb{Y}} P_{\mu/\lambda}(a_1, \cdots, a_{M_1}; q, t) P_{\lambda/\nu}(b_1, \cdots, b_{M_2}; q, t) = P_{\mu/\nu}(a_1, \cdots, a_{M_1}, b_1, \cdots, b_{M_2}; q, t). \tag{60}$$

The identity (59) is usually referred to as the Cauchy identity. The deriving of both (59) and (60) can be found in [Mac95, Chapter VI].

Let Ψ^M be the set of all infinite families of sequences $\{\lambda^i\}_{i=1}^\infty$, which satisfy

1. For $N \geq 1$, $\lambda^N \in \mathbb{Y}_{\min\{M,N\}}$.
2. For $N \geq 2$, the sequences λ^N and λ^{N-1} interlace: $\lambda_1^N \geq \lambda_1^{N-1} \geq \lambda_2^N \geq \cdots$.

Definition 5.3. The *Macdonald process* with positive parameters $M \in \mathbb{Z}$, $\{a_i\}_{i=1}^\infty$, $\{b_i\}_{i=1}^M$, $0 < a_i < 1$, $0 < b_i < 1$, is the distribution on Ψ^M, such that the marginal distribution for λ^N is

$$\mathrm{Prob}(\lambda^N = \mu) = \prod_{1 \leq i \leq N, 1 \leq j \leq M} \frac{\prod_{k=1}^\infty (1 - a_i b_j q^{k-1})}{\prod_{k=1}^\infty (1 - t a_i b_j q^{k-1})} P_\mu(a_1, \cdots, a_N; q, t) Q_\mu(b_1, \cdots, b_M; q, t), \quad (61)$$

and $\{\lambda^N\}_{N \geq 1}$ is a trajectory of a Markov chain with (backward) transition probabilities

$$\mathrm{Prob}(\lambda^{N-1} = \mu | \lambda^N = \nu) = P_{\nu/\mu}(a_N; q, t) \frac{P_\mu(a_1, \cdots, a_{N-1}; q, t)}{P_\nu(a_1, \cdots, a_N; q, t)}. \quad (62)$$

Remark 5.4. The consistency of formulas (61) and (62) follows from properties of Macdonald polynomials. See [BC14], [BCGS16] for more details.

From this definition and (60), the following Proposition follows by simple induction.

Proposition 5.5. *Let $\{\lambda^N\}_{N \geq 1}$ distributed as a Macdonald process with positive parameters $M \in \mathbb{Z}$, $\{a_i\}_{i=1}^\infty$, $\{b_i\}_{i=1}^M$, $0 < a_i < 1$, $0 < b_i < 1$. For integers $0 < N_1 < \cdots < N_l$, and $\mu^1 \in \mathbb{Y}_1, \cdots, \mu^l \in \mathbb{Y}_l$, the joint distribution is*

$$\begin{aligned} \mathrm{Prob}(\lambda^{N_1} = \mu^1, \cdots, \lambda^{N_l} = \mu^l) = &\prod_{1 \leq i \leq N_l, 1 \leq j \leq M} \frac{\prod_{k=1}^\infty (1 - a_i b_j q^{k-1})}{\prod_{k=1}^\infty (1 - t a_i b_j q^{k-1})} \\ &\times P_{\mu^1}(a_1, \cdots, a_{N_1}; q, t) \left(\prod_{i=1}^{l-1} P_{\mu^{i+1}/\mu^i}(a_{N_i+1}, \cdots, a_{N_{i+1}}; q, t) \right) Q_{\mu^l}(b_1, \cdots, b_M; q, t). \end{aligned} \quad (63)$$

There is a limit transition which links Macdonald processes with $\mathbb{P}^{\alpha,M,\theta}$.

Theorem 5.6 ([BG15, Theorem 2.8])**.** *Given positive parameters $M \in \mathbb{Z}$, and α, θ. Let random family of sequences $\{\lambda^i\}_{i=1}^\infty$, which takes value in Ψ^M, be distributed according to Macdonald process with parameters M, $\{a_i\}_{i=1}^\infty$, $\{b_i\}_{i=1}^M$. For $\epsilon > 0$, set*

$$\begin{aligned} a_i &= t^{i-1}, \quad i = 1, 2, \cdots, \\ b_i &= t^{\alpha+i-1}, \quad i = 1, 2, \cdots, \\ q &= \exp(-\epsilon), \quad t = \exp(-\theta\epsilon) \\ x_j^i(\epsilon) &= \exp(-\epsilon\lambda_j^i) \quad i = 1, 2, \cdots, 1 \leq j \leq \min\{m, n\}, \end{aligned} \quad (64)$$

then as $\epsilon \to 0$, the distribution of $x^1, x^2, \cdots$ weakly converges to $\mathbb{P}^{\alpha,M,\theta}$.

5.2 Differential operator

We introduce operators acting on analytic symmetric functions. Such operators were originally defined to act on Λ, and more algebraic discussions of them can be found in [FHH+09] or [Neg13]. We will use them to extract moments of $\mathbb{P}^{\alpha,M,\theta}$.

Definition 5.7. Fix real parameters r, q, t, where $r > 0$ and $q, t \in [0, 1]$. Let $\mathcal{D}_{-n}^N$ be an operator acting on symmetric analytic functions defined on B_r^N, where $B_r = \{x \in \mathbb{C} : |x| < r\}$. For any $F : B_r^N \to \mathbb{C}$, if we expand

$$F(x_1, \cdots, x_N) = \sum_{\lambda \in \mathbb{Y}_N} c_\lambda P_\lambda(x_1, \cdots, x_N; q, t), \tag{65}$$

where c_λ are complex coefficients; then we set $\mathcal{D}_{-n}^N F : B_r^N \to \mathbb{C}$ to be also an analytic symmetric function on B_r^N, such that

$$\mathcal{D}_{-n}^N F(x_1, \cdots, x_N) \quad = \quad \sum_{\lambda \in \mathbb{Y}_N} c_\lambda \left((1 - t^{-n}) \sum_{i=1}^N (q^{\lambda_i} t^{-i+1})^n + t^{-Nn} \right) P_\lambda(x_1, \cdots, x_N; q, t). \tag{66}$$

Remark 5.8. It's worth noting that (66) indeed converges in B_r^N: this follows the fact that

$$\left((1 - t^{-n}) \sum_{i=1}^N (q^{\lambda_i} t^{-i+1})^n + t^{-Nn} \right) \tag{67}$$

is uniformly bounded, and the convergence of (65).

Obviously $\mathcal{D}_{-n}^N$ is linear; and its also "continuous", in the sense that uniform convergence ensures point wise convergence.

To see this, we need the following estimation on each term in the expression of (65).

Proposition 5.9. *For any $r, \delta > 0$, there is a constant $C > 0$ satisfying the following: for any symmetric analytic function $F : B_r^N \to \mathbb{C}$ given by (65), if $|F(x_1, \cdots, x_N)| \leq 1$ for any $x_1, \cdots, x_N \in B_r$, then for any $x_1, \cdots, x_N \in B_{r(1-\delta)}$, and $\lambda \in \mathbb{T}_N$, we have that $|c_\lambda P_\lambda(x_1, \cdots, x_N; q, t)| < (1 - \delta^3)^{|\lambda|}$, where $|\lambda| = \sum_{i=1}^N \lambda_i$.*

Proof. We first prove for $r = 1 + \delta$.

We define a scalar product for any two symmetric analytic functions f, g on $B_{1+\delta}^N$:

$$< f, g >= \frac{1}{N!} \int_T f(z_1, \cdots, z_N) \overline{g(z_1, \cdots, z_N)} \Delta(z_1, \cdots, z_N; q, t) dz_1 \cdots dz_N, \tag{68}$$

where T is the torus $T = \left\{(z_1, \cdots, z_N) \in \mathbb{C}^N : |z_i| = 1\right\}$, with $dz_1 \cdots dz_N$ the normalized Haar measure; and

$$\Delta(z_1, \cdots, z_N) = \prod_{i \neq j} \left(\prod_{r=0}^\infty \frac{1 - z_i z_j^{-1} q^r}{1 - t z_i z_j^{-1} q^r} \right). \tag{69}$$

This definition follows [Mac95, Section VI.9], where one can find more discussions. We immediately see that in T, $\Delta(z_1, \cdots, z_N)$ is always real and takes value in the interval $(\tau, 1)$, where $\tau > 0$ only relies on t and q.

By [Mac95, Chapter VI (9.5)], the Macdonald polynomials $P_\lambda(\cdot;q,t)$ are pairwise orthogonal with respect to this scalar product. Thus for $F: B_r^N \to \mathbb{C}$ given by (65), there is

$$< F, P_\lambda(\cdot;q,t) >= c_\lambda < P_\lambda(\cdot;q,t), P_\lambda(\cdot;q,t) > . \tag{70}$$

By Cauchy-Schwarz inequality, there is

$$\begin{aligned}< F, P_\lambda(\cdot;q,t) >^2 &= \left(\frac{1}{N!}\int_T F(z_1,\cdots,z_N)\overline{P_\lambda(z_1,\cdots,z_N;q,t)}\Delta(z_1,\cdots,z_N;q,t)dz_1\cdots dz_N\right)^2 \\ &\leq \left(\frac{1}{N!}\int_T |F(z_1,\cdots,z_N)|^2\Delta(z_1,\cdots,z_N;q,t)dz_1\cdots dz_N\right) \\ &\times \left(\frac{1}{N!}\int_T |P_\lambda(z_1,\cdots,z_N;q,t)|^2\Delta(z_1,\cdots,z_N;q,t)dz_1\cdots dz_N\right) \\ &=< F, F >< P_\lambda(\cdot;q,t), P_\lambda(\cdot;q,t) >,\end{aligned} \tag{71}$$

then

$$|c_\lambda| \leq \sqrt{\frac{< F, F >}{< P_\lambda(\cdot;q,t), P_\lambda(\cdot;q,t) >}}. \tag{72}$$

For $< F, F >$, since $|F|$ is bounded by 1 in B_2^N, there is $< F, F >\leq< 1, 1 >$.

Recall that $P_\lambda(z_1,\cdots,z_N;q,t) = \sum_{\mu\leq\lambda} u_{\lambda\mu}m_\mu$. Denote $\mathcal{N}_\mu$ to be the number of permutations of $(\mu_1,\cdots,\mu_N)$; then we have

$$< P_\lambda(\cdot;q,t), P_\lambda(\cdot;q,t) >\geq \frac{\tau}{N!}\int_T |P_\lambda(z_1,\cdots,z_N;q,t)|^2 dz_1\cdots dz_N = \frac{\tau}{N!}\sum_{\mu\leq\lambda}|m_\mu|^2\mathcal{N}_\mu. \tag{73}$$

For any $x_1,\cdots,x_N \in B_{1-\delta^2}$, we have

$$|P_\lambda(x_1,\cdots,x_N;q,t)| \leq \sum_{\mu\leq\lambda}|m_\mu|\mathcal{N}_\mu(1-\delta^2)^{|\lambda|}. \tag{74}$$

Then

$$\begin{aligned}|c_\lambda P_\lambda(x_1,\cdots,x_N;q,t)| &\leq \sqrt{\frac{< 1, 1 > N!}{\tau}}(1-\delta^2)^{|\lambda|}\frac{\sum_{\mu\leq\lambda}|m_\mu|\mathcal{N}_\mu}{\sqrt{\sum_{\mu\leq\lambda}|m_\mu|^2\mathcal{N}_\mu}} \\ &\leq \sqrt{\frac{< 1, 1 > N!}{\tau}}(1-\delta^2)^{|\lambda|}\sqrt{\sum_{\mu\leq\lambda}\mathcal{N}_\mu}.\end{aligned} \tag{75}$$

Note that $\sum_{\mu\leq\lambda}\mathcal{N}_\mu$ grows (with $|\lambda|$) in polynomial order. Then we conclude that there is constant C with $|c_\lambda P_\lambda(x_1,\cdots,x_N;q,t)| \leq C(1-\delta^3)^{|\lambda|}$

For the general case, consider the function $(x_1,\cdots,x_N) \mapsto F(\frac{1+\delta}{r}x_1,\cdots,\frac{1+\delta}{r}x_N)$, which is absolutely bounded by 1 in $B_{1+\delta}$; using the result for the special case of $r = 1+\delta$ we finish the proof. □

Lemma 5.10. *Let $\{F_i\}_{i=1}^{\infty}$ be a sequence of symmetric analytic functions defined on B_r^N, and as $i \to \infty$, F_i uniformly converges to another symmetric analytic function F_o. Then for any positive integer n, and $\delta > 0$, there is*

$$\lim_{i\to\infty} \mathcal{D}_{-n}^N F_i(x_1, \cdots, x_N) = \mathcal{D}_{-n}^N F_o(x_1, \cdots, x_N). \tag{76}$$

for any fixed $x_1, \cdots, x_N \in B_{r(1-\delta)}$.

Proof. Denote

$$F_i(x_1, \cdots, x_N) = \sum_{\lambda \in \mathbb{Y}_N} c_{i,\lambda} P_\lambda(x_1, \cdots, x_N; q, t), \quad i = o, 1, 2, \cdots. \tag{77}$$

From Proposition 5.9, the uniform convergence of F_i implies that, for any $x_1, \cdots, x_N \in B_{r(1-\delta)}$,

$$\lim_{i\to\infty} \left(1-\delta^3\right)^{-|\lambda|} c_{i,\lambda} P_\lambda(x_1, \cdots, x_N; q, t) = \left(1-\delta^3\right)^{-|\lambda|} c_{o,\lambda} P_\lambda(x_1, \cdots, x_N; q, t), \tag{78}$$

uniformly for each λ, which implies

$$\begin{aligned}
&\lim_{i\to\infty} \left((1-t^{-n}) \sum_{i=1}^N (q^{\lambda_i} t^{-i+1})^n + t^{-Nn} \right) \left(1-\delta^3\right)^{-|\lambda|} c_{i,\lambda} P_\lambda(x_1, \cdots, x_N; q, t) \\
&\qquad = \left((1-t^{-n}) \sum_{i=1}^N (q^{\lambda_i} t^{-i+1})^n + t^{-Nn} \right) \left(1-\delta^3\right)^{-|\lambda|} c_{o,\lambda} P_\lambda(x_1, \cdots, x_N; q, t),
\end{aligned} \tag{79}$$

uniformly for each λ. Multiplying by $\left(1-\delta^3\right)^{|\lambda|}$ and summing over all λ lead to (76), also using the power decay of $\left(1-\delta^3\right)^{|\lambda|}$. □

There is an integral form of the operators:

$$\begin{aligned}
\mathcal{D}_{-n}^N = &\frac{(-1)^{n-1}}{(2\pi\mathbf{i})^n} \oint \cdots \oint \frac{\sum_{i=1}^n \frac{z_n t^{n-i}}{z_i q^{n-i}}}{\left(1-\frac{tz_2}{qz_1}\right) \cdots \left(1-\frac{tz_n}{qz_{n-1}}\right)} \prod_{i<j} \frac{\left(1-\frac{z_i}{z_j}\right)\left(1-\frac{qz_i}{tz_j}\right)}{\left(1-\frac{z_i}{tz_j}\right)\left(1-\frac{qz_i}{z_j}\right)} \\
&\times \exp\left(\sum_{k=1}^\infty q^k (1-t^{-k}) \frac{z_1^{-k} + \cdots + z_n^{-k}}{k} p_k \right) \exp\left(\sum_{k=1}^\infty (z_1^k + \cdots z_n^k)(1-q^{-k}) \frac{\partial}{\partial p_k} \right) \prod_{i=1}^n \frac{dz_i}{z_i},
\end{aligned} \tag{80}$$

where the contours are nested, with $|z_1| \ll \cdots \ll |z_n|$. Here p_k is the factor of multiplying by p_k. By a change of notations this is a reformulation of [Neg13].

We apply this formula to functions that can be written as a multiplication of functions on single variable.

Proposition 5.11. *Let $f : B_r \to \mathbb{C}$ be analytic, with $f(x) = \exp(\sum_{k=1}^\infty c_k x^k)$ for any $x \in B_r$. The action of $\mathcal{D}_{-n}^N$ can be identified with the integral*

$$\begin{aligned}
\mathcal{D}_{-n}^N \prod_{i=1}^N f(a_i) = &\left(\prod_{i=1}^N f(a_i) \right) \frac{(-1)^{n-1}}{(2\pi\mathbf{i})^n} \oint \cdots \oint \frac{\sum_{i=1}^n \frac{z_n t^{n-i}}{z_i q^{n-i}}}{\left(1-\frac{tz_2}{qz_1}\right) \cdots \left(1-\frac{tz_n}{qz_{n-1}}\right)} \\
&\times \prod_{i<j} \frac{\left(1-\frac{z_i}{z_j}\right)\left(1-\frac{qz_i}{tz_j}\right)}{\left(1-\frac{z_i}{tz_j}\right)\left(1-\frac{qz_i}{z_j}\right)} \left(\prod_{i=1}^n \prod_{i'=1}^N \frac{z_i - t^{-1} q a_{i'}}{z_i - q a_{i'}} \right) \prod_{i=1}^n \frac{f(z_i)}{f(q^{-1} z_i)} \frac{dz_i}{z_i},
\end{aligned} \tag{81}$$

for any $a_1, \cdots, a_N \in B_r$. The contours are nested: all enclose 0 and $qa_{i'}$, and $|z_1| \ll \cdots \ll |z_N|$. The contours can be set in $B_{r'}$ for some $r' > r$, as long as the function $z \to \frac{f(z)}{f(q^{-1}z)}$ is analytic inside $B_{r'}$.

Proof. It suffices to identify the factors. Indeed, note that for any analytic function $g(x)$, the operator $\exp(c\frac{\partial}{\partial x}) = \sum_{i=0}^{\infty} \frac{c^i}{i!}\frac{\partial^i}{\partial x^i}$ acts on g as $\exp(c\frac{\partial}{\partial x})g(x) = g(x+c)$, as long as the series converges. For $\prod_{i=1}^{N} f(a_i)$, write it as $\exp\left(\sum_{k=0}^{\infty} c_k p_k\right)$. Then

$$\exp\left(\sum_{k=1}^{\infty}(z_1^k + \cdots z_n^k)(1-q^{-k})\frac{\partial}{\partial p_k}\right)\exp\left(\sum_{k=0}^{\infty} c_k p_k\right)$$
$$= \exp\left(c_0 + \sum_{k=1}^{\infty} c_k\left((z_1^k + \cdots z_n^k)(1-q^{-k}) + p_k\right)\right) = \prod_{i=1}^{n}\frac{f(z_i)}{f(q^{-1}z_i)}\prod_{i=1}^{N} f(a_i). \quad (82)$$

Also

$$\exp\left(\sum_{k=1}^{\infty} q^k(1-t^{-k})\frac{z_1^{-k} + \cdots + z_n^{-k}}{k}p_k\right)$$
$$= \prod_{i=1}^{n}\prod_{i'=1}^{N}\exp\left(\sum_{k=1}^{\infty} q^k(1-t^{-k})\frac{z_i a_{i'}}{k}\right) = \prod_{i=1}^{n}\prod_{i'=1}^{N}\frac{z_i - t^{-1}qa_{i'}}{z_i - qa_{i'}}. \quad (83)$$

Plugging these two identities back into the integrals finishes the proof. □

5.3 Joint higher order moments

In this subsection we present the proof of Theorem 5.1. The general idea is to apply the operators

$$\frac{1}{\theta k_l}\epsilon^{-1}(t^{-N_l k_l} - \mathcal{D}_{-k_l}^{N_l}), \quad \cdots, \quad \frac{1}{\theta k_1}\epsilon^{-1}(t^{-N_1 k_1} - \mathcal{D}_{-k_1}^{N_1}) \quad (84)$$

one by one to both sides of the Cauchy identity, and use Proposition 5.11 and (66) to evaluate the expressions. This is somewhat standard in the study of Macdonald processes, see [BC14], [BCGS16], and [BG15]. However, the operators $\mathcal{D}_{-n}$ are very different from the ones used in those articles. Finally we obtain the joint moments by passing to the limit.

Proposition 5.12. *Let $N_1 \le \cdots \le N_l, k_1, \cdots, k_l$ be positive integers, $a_1, \cdots, a_{N_l}$ and $b_1, \cdots, b_M$ be variables with each $|a_i b_j| < 1$, and $0 < q, t < 1$ be parameters. Then there is*

$$\prod_{i=1}^{l}\frac{1}{\theta k_i}(t^{-N_i k_i} - \mathcal{D}_{-k_i}^{N_i})\prod_{1\le i\le N_l, 1\le j\le M}\frac{\prod_{k=1}^{\infty}(1-ta_i b_j q^{k-1})}{\prod_{k=1}^{\infty}(1-a_i b_j q^{k-1})}$$
$$= \sum_{\mu^1\in\mathbb{Y}_{N_1},\cdots,\mu^l\in\mathbb{Y}_{N_l}}\prod_{i=1}^{l}\left(\frac{1}{\theta k_i}(t^{-k_i}-1)\sum_{j=1}^{N_i}(q^{\mu_j^i}t^{-j+1})^{k_i}\right)Q_{\mu^l}(b_1, \cdots, b_M; q, t)$$
$$\times P_{\mu^1}(a_1, \cdots, a_{N_1}; q, t)\prod_{1\le i<l}\mathfrak{T}_{i+1\to i}, \quad (85)$$

where the operator $\mathcal{D}_{-k_i}^{N_i}$ acts on variables $a_1, \cdots, a_{N_i}$, and

$$\mathfrak{T}_{i+1\to i} = \begin{cases} P_{\mu^{i+1}/\mu^i}(a_{N_i+1}, \cdots, a_{N_{i+1}}; q, t), & N_i < N_{i+1} \\ \mathbb{1}_{\mu^i=\mu^{i+1}}, & N_i = N_{i+1}. \end{cases} \tag{86}$$

Proof. The proof is similar to proofs in [BCGS16, Proposition 4.9], by induction on l. For $l = 1$, in Cauchy identity (59), set $M_1 = N_1$, $M_2 = M$. Apply the operator $\frac{1}{\theta k_1}\epsilon^{-1}(t^{-N_1 k_1} - \mathcal{D}_{-k_1}^{N_1})$, acting on variables $a_1, \cdots, a_{N_1}$, to both sides. Then we divide both sides by $\prod_{1\le i\le N_l, 1\le j\le M} \frac{\prod_{k=1}^{\infty}(1-ta_ib_jq^{k-1})}{\prod_{k=1}^{\infty}(1-a_ib_jq^{k-1})}$, and get the desired equation.

For general l, we assume that the statement is true for $l-1$; specifically, we have that

$$\begin{aligned} &\prod_{i=2}^{l} \frac{1}{\theta k_i}(t^{-N_i k_i} - \mathcal{D}_{-k_i}^{N_i}) \prod_{1\le i\le N_l, 1\le j\le M} \frac{\prod_{k=1}^{\infty}(1-ta_ib_jq^{k-1})}{\prod_{k=1}^{\infty}(1-a_ib_jq^{k-1})} \\ &\quad= \sum_{\mu^2\in\mathbb{Y}_{N_2}, \cdots, \mu^l\in\mathbb{Y}_{N_l}} \prod_{i=2}^{l} \left(\frac{1}{\theta k_i}(t^{-k_i}-1)\sum_{j=1}^{N_i}(q^{\mu^i_j}t^{-j+1})^{k_i} \right) Q_{\mu^l}(b_1, \cdots, b_M; q, t) \\ &\qquad\qquad \times P_{\mu^2}(a_1, \cdots, a_{N_2}; q, t) \prod_{2\le i<l} \mathfrak{T}_{i+1\to i}. \end{aligned} \tag{87}$$

If $N_1 = N_2$,

$$P_{\mu^2}(a_1, \cdots, a_{N_2}; q, t) = \sum_{\mu^1\in\mathbb{Y}_{N_1}} P_{\mu^1}(a_1, \cdots, a_{N_1}; q, t)\mathbb{1}_{\mu_r=\mu_2}; \tag{88}$$

If $N_1 < N_2$,

$$\begin{aligned} &P_{\mu^2}(a_1, \cdots, a_{N_2}; q, t) \\ &= \sum_{\mu^1\in\mathbb{Y}_{N_1}} P_{\mu^1}(a_1, \cdots, a_{N_1}; q, t) \sum_{\nu^1\in\mathbb{Y}_{N_1+1}, \cdots, \nu^{N_2-N_1}\in\mathbb{Y}_{N_2}} \prod_{1\le i\le N_2-N_1} P_{\nu^i/\nu^{i-1}}(a_{N_1+i}; q, t) \\ &= \sum_{\mu^1\in\mathbb{Y}_{N_1}} P_{\mu^1}(a_1, \cdots, a_{N_1}; q, t) P_{\mu^2/\mu^1}(a_{N_1+1}, \cdots, a_{N_2}; q, t), \end{aligned} \tag{89}$$

where $\nu^0 = \mu_r$ and $\nu^{N_2-N_1} = \mu_2$, and the last line follows from (60).

In either case we have

$$P_{\mu^2}(a_1, \cdots, a_{N_2}; q, t) = \sum_{\mu^1\in\mathbb{Y}_{N_1}} P_{\mu^1}(a_1, \cdots, a_{N_1}; q, t)\mathfrak{T}_{i+1\to i}. \tag{90}$$

Plug this into (87) and apply the operator $\frac{1}{\theta k_1}(t^{-N_1 k_1} - \mathcal{D}_{-k_1}^{N_1})$ to both sides, we immediately obtain (85). By principle of induction, (85) holds for any positive integer l. $\square$

Now we evaluate the same expression, in the special case where $b_j = t^{\alpha+j-1}$ for $1 \le j \le M$, by using Proposition 5.11 multiple times.

Proposition 5.13. *For any positive integer m, $\tilde{m}$, variables $w_1, \cdots, w_m$, $\tilde{w}_1, \cdots, \tilde{w}_{\tilde{m}}$, and parameters q, t, denote*

$$\mathfrak{B}(w_1, \cdots, w_m; q, t) = \frac{\sum_{i=1}^{m} \frac{w_m t^{m-i}}{w_i q^{m-i}}}{\left(1 - \frac{tw_2}{qw_1}\right) \cdots \left(1 - \frac{tw_m}{qw_{m-1}}\right)} \prod_{i<j} \frac{\left(1 - \frac{w_i}{w_j}\right)\left(1 - \frac{qw_i}{tw_j}\right)}{\left(1 - \frac{w_i}{tw_j}\right)\left(1 - \frac{qw_i}{w_j}\right)}, \tag{91}$$

$$\mathfrak{F}(w_1, \cdots, w_m; \tilde{w}_1, \cdots, \tilde{w}_{\tilde{m}}; q, t) = \prod_{i=1}^{m} \prod_{i'=1}^{\tilde{m}} \frac{w_i - t^{-1} q \tilde{w}_{i'}}{w_i - q\tilde{w}_{i'}}, \tag{92}$$

$$\mathfrak{C}(w_1, \cdots, w_m; \tilde{w}_1, \cdots, \tilde{w}_{\tilde{m}}; q, t) = \prod_{i=1}^{m} \prod_{i'=1}^{\tilde{m}} \frac{\left(1 - \frac{w_i}{\tilde{w}_{i'}}\right)\left(1 - \frac{qw_i}{t\tilde{w}_{i'}}\right)}{\left(1 - \frac{w_i}{t\tilde{w}_{i'}}\right)\left(1 - \frac{qw_i}{\tilde{w}_{i'}}\right)}. \tag{93}$$

Then for fixed real parameters $0 < q, t < 1$, $\alpha > 0$, positive integers $N_1 \leq \cdots \leq N_l, k_1, \cdots, k_l$, and $M > k_1 + \cdots + k_l$, and variables $a_1, \cdots, a_{N_l} \in B_1$, by letting each $\mathcal{D}_{-k_i}^{N_i}$ acting on variables $a_1, \cdots, a_{N_i}$ we have

$$\begin{aligned}
\prod_{i=1}^{l} \mathcal{D}_{-k_i}^{N_i} \prod_{i=1}^{N_l} \frac{\prod_{k=1}^{\infty}(1 - a_i t^{\alpha+M} q^{k-1})}{\prod_{k=1}^{\infty}(1 - a_i t^{\alpha} q^{k-1})} &= \prod_{i=1}^{N_l} \frac{\prod_{k=1}^{\infty}(1 - a_i t^{\alpha+M} q^{k-1})}{\prod_{k=1}^{\infty}(1 - a_i t^{\alpha} q^{k-1})} \\
\times \frac{(-1)^{k_1+\cdots+k_l-l}}{(2\pi\mathbf{i})^{k_1+\cdots+k_l}} \oint \cdots \oint &\prod_{i=1}^{l} \mathfrak{B}(z_{i,1}, \cdots, z_{i,k_i}; q, t) \prod_{i=1}^{l} \mathfrak{F}(z_{i,1}, \cdots, z_{i,k_i}; a_1, \cdots, a_{N_i}; q, t) \\
\times \prod_{i<j} \mathfrak{C}(z_{i,1}, \cdots, z_{i,k_i}; z_{j,1}, \cdots, z_{j,k_j}; q, t) &\prod_{i=1}^{l} \prod_{i'=1}^{k_i} \left(\frac{1 - q^{-1} t^{\alpha} z_{i,i'}}{1 - q^{-1} t^{\alpha+M} z_{i,i'}} \frac{dz_{i,i'}}{z_{i,i'}} \right),
\end{aligned} \tag{94}$$

where the contours are nested and satisfy the following: for each $1 \leq i \leq l$ and $1 \leq i' < k_i$ there is $|z_{i,i'}| < t|z_{i,i'+1}|$; and for each $1 \leq i < l$, there is $|z_{i,k_i}| < t|z_{i+1,1}|$; also, $q < |z_{1,1}|$, and $|z_{l,N_l}| < qt^{-\alpha-M}$.

Proof. We prove by induction on l.

For the base case where $l = 1$, we apply Proposition 5.11 to the function

$$f(x) = \frac{\prod_{k=1}^{\infty}(1 - xt^{\alpha+M} q^{k-1})}{\prod_{k=1}^{\infty}(1 - xt^{\alpha} q^{k-1})}. \tag{95}$$

Specifically, $f(x)$ can be expanded around 0 to a series, or written as $\exp(g(x))$ for some power series $g(x)$; both series converge in $B_{t^{-\alpha}}$. And $\frac{f(x)}{f(q^{-1}x)}$ is analytic in $B_{qt^{-\alpha-M}}$. For $a_1, \cdots, a_{N_1} \in B_1 \subset B_{t^{-\alpha}}$, we can construct contours of $z_{1,1}, \cdots, z_{1,k_1}$ such that $q < |z_{1,1}|$, $|z_{1,k_1}| < qt^{-\alpha-M}$, and $|z_{1,i}| < t|z_{1,i+1}|$ for each $1 \leq i < k_1$, which satisfies the requirements in Proposition 5.11. The expression given by Proposition 5.11 is precisely (94) for $l = 1$.

For more general $l \geq 2$, assume that the statement is true for $l - 1$; then we have

$$\begin{aligned}
&\prod_{i=2}^{l} \mathcal{D}_{-k_i}^{N_i} \prod_{i=1}^{N_l} \frac{\prod_{k=1}^{\infty}(1 - a_i t^{\alpha+M} q^{k-1})}{\prod_{k=1}^{\infty}(1 - a_i t^{\alpha} q^{k-1})} \\
&= \frac{(-1)^{k_2+\cdots+k_l-l}}{(2\pi\mathbf{i})^{k_2+\cdots+k_l}} \oint \cdots \oint \mathfrak{W}(a_1, \cdots, a_{N_1}; \{\{z_{i,i'}\}_{i'=1}^{k_i}\}_{i=2}^{l}) \prod_{i=2}^{l} \prod_{i'=1}^{k_i} dz_{i,i'},
\end{aligned} \tag{96}$$

for any $a_1, \cdots, a_{N_1} \in B_1$, where

$$\mathfrak{W}(a_1, \cdots, a_{N_1}; \{\{z_{i,i'}\}_{i'=1}^{k_i}\}_{i=2}^{l}) = \prod_{i=1}^{N_l} \frac{\prod_{k=1}^{\infty}(1 - a_i t^{\alpha+M} q^{k-1})}{\prod_{k=1}^{\infty}(1 - a_i t^{\alpha} q^{k-1})}$$
$$\times \prod_{i=2}^{l} \mathfrak{B}(z_{i,1}, \cdots, z_{i,k_i}; q, t) \prod_{i=2}^{l} \mathfrak{F}(z_{i,1}, \cdots, z_{i,k_i}; a_1, \cdots, a_{N_i}; q, t)$$
$$\times \prod_{1<i<j} \mathfrak{C}(z_{i,1}, \cdots, z_{i,k_i}; z_{j,1}, \cdots, z_{j,k_j}; q, t) \prod_{i=2}^{l} \prod_{i'=1}^{k_i} \left(\frac{1 - q^{-1} t^{\alpha} z_{i,i'}}{1 - q^{-1} t^{\alpha+M} z_{i,i'}} \frac{dz_{i,i'}}{z_{i,i'}} \right). \quad (97)$$

And in (96) the contours are constructed in the following way: for each $2 \le i \le l$ and $1 \le i' < k_i$ there is $|z_{i,i'}| < t|z_{i,i'+1}|$; and for each $2 \le i < l$, there is $|z_{i,k_i}| < t|z_{i+1,1}|$; also, $q < |z_{2,1}|$, and $|z_{l,N_l}| < qt^{-\alpha-M}$.

Now apply the operator $\mathcal{D}_{-k_1}^{N_1}$ to both sides of (96), acting on variables $a_1, \cdots, a_{N_1}$. We approximate the integral by finite sums. As $M > k_1 + \cdots + k_l$, we can fix the contours such that $|z_{2,1}| > qt^{-k_1-1}$. For each positive integer m, split each contour into m parts of equal measure, and by choosing one point from each contour we have a collection Θ_m of combinations $\{\{z_{i,i'}\}_{i'=1}^{k_i}\}_{i=2}^{l}$. Now define

$$F_m(a_1, \cdots, a_{N_1}) = \frac{(-1)^{k_2+\cdots+k_l-l}}{(2\pi \mathbf{i} m)^{k_2+\cdots+k_l}} \sum_{\{\{z_{i,i'}\}_{i'=1}^{k_i}\}_{i=2}^{l} \in \Theta_m} \mathfrak{W}(a_1, \cdots, a_{N_1}; \{\{z_{i,i'}\}_{i'=1}^{k_i}\}_{i=2}^{l}). \quad (98)$$

Then $F_m(a_1, \cdots, a_{N_1})$ converges to (96), uniformly for $a_1, \cdots, a_{N_1} \in B_1$. By Lemma 5.10 there is

$$\mathcal{D}_{-k_1}^{N_1} \frac{(-1)^{k_2+\cdots+k_l-l}}{(2\pi \mathbf{i})^{k_2+\cdots+k_l}} \oint \cdots \oint \mathfrak{W}(a_1, \cdots, a_{N_1}; \{\{z_{i,i'}\}_{i'=1}^{k_i}\}_{i=2}^{l}) \prod_{i=2}^{l} \prod_{i'=1}^{k_i} dz_{i,i'}$$
$$= \lim_{m\to\infty} \mathcal{D}_{-k_1}^{N_1} F_m(a_1, \cdots, a_{N_1}). \quad (99)$$

for any $a_1, \cdots, a_{N_1} \in B_1$. By linearity of $\mathcal{D}_{-k_1}^{N_1}$,

$$\mathcal{D}_{-k_1}^{N_1} F_m(a_1, \cdots, a_{N_1})$$
$$= \frac{(-1)^{k_2+\cdots+k_l-l}}{(2\pi \mathbf{i} m)^{k_2+\cdots+k_l}} \sum_{\{\{z_{i,i'}\}_{i'=1}^{k_i}\}_{i=2}^{l} \in \Theta_m} \mathcal{D}_{-k_1}^{N_1} \mathfrak{W}(a_1, \cdots, a_{N_1}; \{\{z_{i,i'}\}_{i'=1}^{k_i}\}_{i=2}^{l}). \quad (100)$$

Now evaluating the limit in (99) we conclude that

$$\mathcal{D}_{-k_1}^{N_1} \frac{(-1)^{k_2+\cdots+k_l-l}}{(2\pi \mathbf{i})^{k_2+\cdots+k_l}} \oint \cdots \oint \mathfrak{W}(a_1, \cdots, a_{N_1}; \{\{z_{i,i'}\}_{i'=1}^{k_i}\}_{i=2}^{l}) \prod_{i=2}^{l} \prod_{i'=1}^{k_i} dz_{i,i'}$$
$$= \frac{(-1)^{k_2+\cdots+k_l-l}}{(2\pi \mathbf{i})^{k_2+\cdots+k_l}} \oint \cdots \oint \mathcal{D}_{-k_1}^{N_1} \mathfrak{W}(a_1, \cdots, a_{N_1}; \{\{z_{i,i'}\}_{i'=1}^{k_i}\}_{i=2}^{l}) \prod_{i=2}^{l} \prod_{i'=1}^{k_i} dz_{i,i'}, \quad (101)$$

for any $a_1, \cdots, a_{N_1} \in B_1$.

Now it remains to evaluate $\mathcal{D}_{-k_1}^{N_1}\mathfrak{W}(a_1, \cdots, a_{N_1}; \{\{z_{i,i'}\}_{i'=1}^{k_i}\}_{i=2}^{l})$, for fixed $\{\{z_{i,i'}\}_{i'=1}^{k_i}\}_{i=2}^{l}$. We consider the following function

$$f(x) = \frac{\prod_{k=1}^{\infty}(1 - xt^{\alpha+M}q^{k-1})}{\prod_{k=1}^{\infty}(1 - xt^{\alpha}q^{k-1})} \prod_{i=2}^{l}\prod_{i'=1}^{k_i} \frac{z_{i,i'} - t^{-1}qx}{z_{i,i'} - qx}. \tag{102}$$

Such $f(x)$ can be expanded to a power series around 0, or written as $\exp(g(x))$ for some power series $g(x)$; both series converges in B_1. Noticing that as $|z_{i,i'}| > qt^{-k_1-1}$, for each $1 \leq i \leq l$ and $1 \leq i' \leq k_i$, the function $\frac{f(x)}{f(q^{-1}x)}$ is analytic inside $B_{qt^{-k_1}}$. By Proposition 5.11, there is

$$\begin{aligned}
&\mathcal{D}_{-k_1}^{N_1}\left(\prod_{i=1}^{N_l} \frac{\prod_{k=1}^{\infty}(1 - a_i t^{\alpha+M}q^{k-1})}{\prod_{k=1}^{\infty}(1 - a_i t^{\alpha}q^{k-1})} \prod_{i=2}^{l} \mathfrak{F}(z_{i,1}, \cdots, z_{i,k_i}; a_1, \cdots, a_{N_i}; q, t)\right) \\
&\qquad = \prod_{i=1}^{N_l} \frac{\prod_{k=1}^{\infty}(1 - a_i t^{\alpha+M}q^{k-1})}{\prod_{k=1}^{\infty}(1 - a_i t^{\alpha}q^{k-1})} \prod_{i=2}^{l} \mathfrak{F}(z_{i,1}, \cdots, z_{i,k_i}; a_1, \cdots, a_{N_i}; q, t) \\
&\qquad\times \frac{(-1)^{k_1-1}}{(2\pi\mathbf{i})^{k_1}} \oint \cdots \oint \mathfrak{B}(z_{1,1}, \cdots, z_{1,k_1}; q, t)\mathfrak{F}(z_{1,1}, \cdots, z_{1,k_1}; a_1, \cdots, a_{N_1}; q, t) \\
&\qquad\times \prod_{i=2}^{l} \mathfrak{C}(z_{1,1}, \cdots, z_{1,k_1}; z_{i,1}, \cdots, z_{i,k_i}; q, t) \prod_{i'=1}^{k_1}\left(\frac{1 - q^{-1}t^{\alpha}z_{1,i'}}{1 - q^{-1}t^{\alpha+M}z_{1,i'}} \frac{dz_{1,i'}}{z_{1,i'}}\right),
\end{aligned} \tag{103}$$

for any $a_1, \cdots, a_{N_1} \in B_1$, and the contours are constructed such that for each $1 \leq i < k_1$ there is $|z_{1,i}| < t|z_{1,i+1}|$, $q < |z_{1,1}|$, and $|z_{l,N_l}| < qt^{-k_1}$. Putting (96), (101), and (103) together we get exactly (94). $\square$

We next would like to perform limit transition (64) in formula (94). In the integral we would like to do a change of variable:

$$z_{i,i'} = \exp(\epsilon u_{i,i'}), \quad 1 \leq i \leq l, 1 \leq i' \leq k_i, \tag{104}$$

and all the contours $u_{i,i'}$ are nested in a certain way to give satisfied contours of $z_{i,i'}$.

However, as the contour of each $z_{i,i'}$ encloses 0, it is impossible to find desired contours of $u_{i,i'}$. The idea to solve this is to split each contour of $z_{i,i'}$ (94) into two: one enclosing 0 and another enclosing all of $qa_1, \cdots, qa_{N_l}$. It turns out that most terms with contours enclosing 0 are evaluated to zero or canceled out. The arguments below are similar to those in [FD16, Appendix A].

Formally stating, for each $a_{i,i'}$, we associate it with two contours $\mathfrak{U}_{i,i'}$ and $\mathfrak{V}_{i,i'}$, satisfying: for each $1 \leq i \leq l$ and $1 \leq i' < k_i$, $\mathfrak{U}_{i,i'}$ is inside $t\mathfrak{U}_{i,i'+1}$, $\mathfrak{V}_{i,i'}$ is inside $t\mathfrak{V}_{i,i'+1}$; for each $1 \leq i < l$, $\mathfrak{U}_{i,k_i}$ is inside $t\mathfrak{U}_{i+1,1}$, $\mathfrak{V}_{i,k_i}$ is inside $t\mathfrak{V}_{i+1,1}$. Also, each of $\mathfrak{U}_{i,i'}$ encloses 0 but none of $qa_1, \cdots, qa_{N_l}$, while each of $\mathfrak{V}_{i,i'}$ encloses $qa_1, \cdots, qa_{N_l}$ but not 0. All of these contours are inside $B_{qt^{-\alpha-M}}$. We state that such contours exist as long as $1 - t$ is small enough.

Let Π be the power set of $\{z_{i,i'} | 1 \leq i \leq l, 1 \leq i' \leq k_i\}$, then for each $\Upsilon \in \Pi$, denote $\Upsilon_{i,i'}$ to be $\mathfrak{U}_{i,i'}$ if $z_{i,i'} \in \Upsilon$, and $\mathfrak{V}_{i,i'}$ if $z_{i,i'} \notin \Upsilon$.

From now on we let

$$\mathfrak{Q}_\Upsilon = \oint_{\Upsilon_{1,1}} \cdots \oint_{\Upsilon_{l,k_l}} \prod_{i=1}^{l} \mathfrak{B}(z_{i,1}, \cdots, z_{i,k_i}; q, t) \prod_{i=1}^{l} \mathfrak{F}(z_{i,1}, \cdots, z_{i,k_i}; a_1, \cdots, a_{N_i}; q, t)$$
$$\times \prod_{i<j} \mathfrak{C}(z_{i,1}, \cdots, z_{i,k_i}; z_{j,1}, \cdots, z_{j,k_j}; q, t) \prod_{i=1}^{l} \prod_{i'=1}^{k_i} \left(\frac{1 - q^{-1}t^{\alpha} z_{i,i'}}{1 - q^{-1}t^{\alpha+M} z_{i,i'}} \frac{dz_{i,i'}}{z_{i,i'}} \right). \quad (105)$$

Then (94) can be written as

$$\frac{(-1)^{k_1+\cdots+k_l-l}}{(2\pi \mathbf{i})^{k_1+\cdots+k_l}} \prod_{i=1}^{N_l} \frac{\prod_{k=1}^{\infty}(1 - a_i t^{\alpha+M} q^{k-1})}{\prod_{k=1}^{\infty}(1 - a_i t^{\alpha} q^{k-1})} \sum_{\Upsilon \in \Pi} \mathfrak{Q}_\Upsilon. \quad (106)$$

The following Lemma (and the proof) is an extension of [FD16, Appendix A, Lemma 5].

Lemma 5.14. *Only subset Υ that satisfies the following can have $\mathfrak{Q}_\Upsilon \neq 0$: for each $1 \leq i \leq l$, $\Upsilon \bigcap \{z_{i,1}, \cdots, z_{i,k_i}\}$ is either empty, or of the form $\bigcup_{i=1}^{l} \{z_{i,s_i}, \cdots, z_{i,r_i}\}$, where $1 \leq s_i \leq r_i \leq k_i$.*

Proof. We compute the integral in (106) sequentially, from $z_{1,1}$ to z_{l,k_l}. For any $1 \leq i \leq l$, suppose that s_i is the smallest (if any) such that $z_{i,s_i} \in \Upsilon$. Integrating z_{i,s_i}, the residue is (106) setting $z_{i,s_i} = 0$, and the factor

$$\frac{\sum_{i'=1}^{k_i} \frac{z_{i,k_i} t^{k_i - i'}}{z_{i,i'} q^{k_i - i'}}}{\left(1 - \frac{tz_{i,2}}{qz_{i,1}}\right) \cdots \left(1 - \frac{tz_{i,k_i}}{qz_{i,k_i-1}}\right)} \quad (107)$$

is replaced by

$$\frac{-\frac{z_{i,k_i} t^{k_i - s_i - 1}}{z_{i,s_i+1} q^{k_i - s_i - 1}}}{\left(1 - \frac{tz_{i,2}}{qz_{i,1}}\right) \cdots \left(1 - \frac{tz_{i,s_i-1}}{qz_{i,s_i-2}}\right)\left(1 - \frac{tz_{i,s_i+2}}{qz_{i,s_i+1}}\right) \cdots \left(1 - \frac{tz_{i,k_i}}{qz_{i,k_i-1}}\right)}. \quad (108)$$

If $s_i < k_i$ and $z_{i,s_i+1} \notin \Upsilon$, then after integrating z_{i,s_i+1} there is no pole at 0 of $z_{i,i'}$, for any $s_i + 1 < i' \leq k_i$. If there is any $z_{i,i'} \in \Upsilon$, for $s_i + 1 < i' \leq k_i$, the whole integral gives 0.

If $s_i < k_i$ and $z_{i,s_i+1} \in \Upsilon$, integrating z_{i,s_i+1}, the residue is again (106), setting $z_{i,s_i} = z_{i,s_i+1} = 0$, and the factor (107) is replaced by

$$\frac{\frac{z_{i,k_i} t^{k_i - s_i - 2}}{z_{i,s_i+2} q^{k_i - s_i - 2}}}{\left(1 - \frac{tz_{i,2}}{qz_{i,1}}\right) \cdots \left(1 - \frac{tz_{i,s_i-1}}{qz_{i,s_i-2}}\right)\left(1 - \frac{tz_{i,s_i+3}}{qz_{i,s_i+2}}\right) \cdots \left(1 - \frac{tz_{i,k_i}}{qz_{i,k_i-1}}\right)}. \quad (109)$$

which still has the same form and we can continue in the same way. □

Lemma 5.15. *Let $\tilde{\Pi}$ be the collection of all Υ, where for each $1 \leq i \leq l$, $\Upsilon \bigcap \{z_{i,1}, \cdots, z_{i,k_i}\}$ is of the form $\bigcup_{i=1}^{l} \{z_{i,s_i}, \cdots, z_{i,r_i}\}$ for some $q \leq s_i \leq r_i \leq k_i$, or empty; but each $\{z_{i,1}, \cdots, z_{i,k_i}\} \not\subset \Upsilon$. Then we have*

$$\prod_{i=1}^{l} \left(\mathcal{D}_{-k_i}^{N_i} - t^{-N_i k_i}\right) \prod_{i=1}^{N_l} \frac{\prod_{k=1}^{\infty}(1 - a_i t^{\alpha+M} q^{k-1})}{\prod_{k=1}^{\infty}(1 - a_i t^{\alpha} q^{k-1})}$$
$$= \frac{(-1)^{k_1+\cdots+k_l-l}}{(2\pi \mathbf{i})^{k_1+\cdots+k_l}} \prod_{i=1}^{N_l} \frac{\prod_{k=1}^{\infty}(1 - a_i t^{\alpha+M} q^{k-1})}{\prod_{k=1}^{\infty}(1 - a_i t^{\alpha} q^{k-1})} \sum_{\Upsilon \in \tilde{\Pi}} \mathfrak{Q}_\Upsilon. \quad (110)$$

Proof. As seem above, while computing the integral in (106), if for some $1 \le w \le l$, $\{z_{w,1}, \cdots, z_{w,k_w}\} \subset \Upsilon$, integrating through $z_{w,1}, \cdots, z_{w,k_w}$ is essentially setting $z_{w,1} = \cdots = z_{w,k_w} = 0$.

Namely, for any $\Upsilon \in \Pi$, let $W \subset \{1, \cdots, l\}$ such that for any $w \in W$, there is $\{z_{w,1}, \cdots, z_{w,k_w}\} \subset \Upsilon$. Then we have

$$\prod_{w\in W} \frac{(-1)^{k_w-1}}{(2\pi\mathbf{i})^{k_w}} \mathfrak{Q}_\Upsilon = t^{-\sum_{w\in W} N_w k_w} \oint \cdots \oint_{\{\Upsilon_{1,1},\cdots,\Upsilon_{l,k_l}\}\setminus\bigcup_{w\in W}\{\Upsilon_{w,1},\cdots,\Upsilon_{w,k_w}\}}$$
$$\times \prod_{i\notin W} \mathfrak{B}(z_{i,1}, \cdots, z_{i,k_i}; q, t) \prod_{i\notin W} \mathfrak{F}(z_{i,1}, \cdots, z_{i,k_i}; a_1, \cdots, a_{N_i}; q, t)$$
$$\times \prod_{i<j, i,j\notin W} \mathfrak{C}(z_{i,1}, \cdots, z_{i,k_i}; z_{j,1}, \cdots, z_{j,k_j}; q, t) \prod_{i\notin W} \prod_{i'=1}^{k_i} \left(\frac{1 - q^{-1}t^\alpha z_{i,i'}}{1 - q^{-1}t^{\alpha+M} z_{i,i'}} \frac{dz_{i,i'}}{z_{i,i'}} \right). \quad (111)$$

Fix $W \subset \{1, \cdots, l\}$ and sum over all $\Upsilon \in \Pi$ with $\bigcup_{w\in W} \{z_{w,1}, \cdots, z_{w,k_w}\} \subset \Upsilon$, we get that

$$t^{-\sum_{i\in W} N_i k_i} \prod_{i\notin W} \mathcal{D}_{-k_i}^{N_i} \prod_{i=1}^{N_l} \frac{\prod_{k=1}^\infty (1 - a_i t^{\alpha+M} q^{k-1})}{\prod_{k=1}^\infty (1 - a_i t^\alpha q^{k-1})}$$
$$= \frac{(-1)^{k_1+\cdots+k_l-l}}{(2\pi\mathbf{i})^{k_1+\cdots+k_l}} \prod_{i=1}^{N_l} \frac{\prod_{k=1}^\infty (1 - a_i t^{\alpha+M} q^{k-1})}{\prod_{k=1}^\infty (1 - a_i t^\alpha q^{k-1})} \sum_{\substack{\Upsilon\in\Pi \\ \bigcup_{w\in W}\{z_{w,1},\cdots,z_{w,k_w}\}\subset\Upsilon}} \mathfrak{Q}_\Upsilon. \quad (112)$$

Multiplying by $(-1)^{|W|}$ and summing over all W finishes the proof. □

Now we consider the limit transition.

Lemma 5.16. *Set $a_i = t^{i-1}$, $t = \exp(-\theta\epsilon)$, $q = \exp(-\epsilon)$. For $\Upsilon \in \Pi$, if there is $1 \le w \le l$ such that $\{z_{w,1}, \cdots, z_{w,k_w}\} \bigcap \Upsilon = \{z_{w,s_w}, \cdots, z_{w,r_w}\}$, for some $1 < s_w \le r_w < k_w$, then*

$$\lim_{\epsilon\to 0} \epsilon^{-l} \mathfrak{Q}_\Upsilon = 0. \quad (113)$$

For $\Upsilon', \Upsilon'' \in \Pi$, if there is $1 \le w \le l$ such that $\{z_{w,1}, \cdots, z_{w,k_w}\} \bigcap \Upsilon' = \{z_{w,1}, \cdots, z_{w,r_w}\}$, $\{z_{w,1}, \cdots, z_{w,k_w}\} \bigcap \Upsilon'' = \{z_{w,s_w}, \cdots, z_{w,k_w}\}$, with $r_w = k_w - s_w + 1$; and for any $w' \ne w$ there is $\left\{z_{w',1}, \cdots, z_{w',k_{w'}}\right\} \bigcap \Upsilon' = \left\{z_{w',1}, \cdots, z_{w',k_w}\right\} \bigcap \Upsilon''$; then

$$\lim_{\epsilon\to 0} \epsilon^{-l} = 0. \quad (114)$$

Proof. By induction and using a similar argument in the proof of Lemma 5.14, for any $\Upsilon \in \Pi$, in the expression $\mathfrak{Q}_\Upsilon$ we integrate all variables in Υ and get

$$\mathfrak{Q}_\Upsilon = \oint \cdots \oint_{\{\Upsilon_{i,i'}: z_{i,i'}\notin W\}} \prod_{i=1}^l \mathfrak{Y}_i(\Upsilon) \prod_{i=1}^l \mathfrak{F}(\{z_{i,1}, \cdots, z_{i,k_i}\}\setminus\Upsilon; a_1, \cdots, a_{N_i}; q, t)$$
$$\times \prod_{i<j} \mathfrak{C}(\{z_{i,1}, \cdots, z_{i,k_i}\}\setminus\Upsilon; \{z_{j,1}, \cdots, z_{j,k_j}\}\setminus\Upsilon; q, t) \prod_{z_{i,i'}\notin\Upsilon} \left(\frac{1 - q^{-1}t^\alpha z_{i,i'}}{1 - q^{-1}t^{\alpha+M} z_{i,i'}} \frac{dz_{i,i'}}{z_{i,i'}} \right), \quad (115)$$

where

$$\mathfrak{Y}_i(\Upsilon) = (-1)^{k_i-s_i}\frac{t^{-N_i(k_i-s_i+1)}}{\left(1-\frac{tz_{i,2}}{qz_{i,1}}\right)\cdots\left(1-\frac{tz_{i,s_i-1}}{qz_{i,s_i-2}}\right)}, \tag{116}$$

if $\{z_{i,1},\cdots,z_{i,k_i}\}\bigcap\Upsilon$ is of the form $\bigcup_{i=1}^{l}\{z_{i,s_i},\cdots,z_{i,k_i}\}$, for some $1<s_i\leq k_i$;

$$\mathfrak{Y}_i(\Upsilon) = (-1)^{r_i}\frac{\frac{z_{i,k_i}t^{k_i-r_i-1}}{z_{i,r_i+1}q^{k_i-r_i-1}}t^{-N_ir_i}}{\left(1-\frac{tz_{i,r_i+2}}{qz_{i,r_i+1}}\right)\cdots\left(1-\frac{tz_{i,k_i}}{qz_{i,k_i-1}}\right)}, \tag{117}$$

if $\{z_{i,1},\cdots,z_{i,k_i}\}\bigcap\Upsilon$ is of the form $\bigcup_{i=1}^{l}\{z_{i,1},\cdots,z_{i,r_i}\}$, for some $1\leq r_i<k_i$;

$$\mathfrak{Y}_i(\Upsilon) = (-1)^{r_i-s_i+1}\frac{\frac{z_{i,k_i}t^{k_i-r_i-1}}{z_{i,r_i+1}q^{k_i-r_i-1}}t^{-N_i(r_i-s_i+1)}}{\left(1-\frac{tz_{i,2}}{qz_{i,1}}\right)\cdots\left(1-\frac{tz_{i,s_i-1}}{qz_{i,s_i-2}}\right)\left(1-\frac{tz_{i,r_i+2}}{qz_{i,r_i+1}}\right)\cdots\left(1-\frac{tz_{i,k_i}}{qz_{i,k_i-1}}\right)}, \tag{118}$$

if $\{z_{i,1},\cdots,z_{i,k_i}\}\bigcap\Upsilon$ is of the form $\bigcup_{i=1}^{l}\{z_{i,s_i},\cdots,z_{i,r_i}\}$, for some $1<s_i\leq r_i<k_i$.

As $\epsilon\to 0$, the length of each contour decays in the order of ϵ. For $\mathfrak{Y}_i(\Upsilon)$, the expression (116) grows in the order of $\epsilon^{-s_i+2}=\epsilon^{-|\{z_{i,1},\cdots,z_{i,k_i}\}\setminus\Upsilon|+1}$; and (117) grows in the order of $\epsilon^{-k_i+r_i+1}=\epsilon^{-|\{z_{i,1},\cdots,z_{i,k_i}\}\setminus\Upsilon|+1}$; while (118) grows in the order of $\epsilon^{-k_i+r_i-s_i+3}=\epsilon^{-|\{z_{i,1},\cdots,z_{i,k_i}\}\setminus\Upsilon|+2}$. All other factors in the integrand converge to constants. Then if there is at least one $\mathfrak{Y}_i(\Upsilon)$ in the case of (118), we obtain (113).

Now we consider a pair Υ' and Υ'' as described in the statement of this Lemma. Then for any $w'\neq w$, $\mathfrak{Y}_{w'}(\Upsilon')=\mathfrak{Y}_{w'}(\Upsilon'')$. For w, we can identify $z_{i,i'}$ in $\mathfrak{Y}_w(\Upsilon'')$ with $z_{i,i'+r_w}$ in $\mathfrak{Y}_w(\Upsilon')$; then we conclude (using the notation in $\mathfrak{Y}_w(\Upsilon')$)

$$\mathfrak{Y}_w(\Upsilon')+\mathfrak{Y}_w(\Upsilon'') = (-1)^{r_i}\frac{\left(\frac{z_{i,k_i}t^{k_i-r_i-1}}{z_{i,r_i+1}q^{k_i-r_i-1}}-1\right)t^{-N_ir_i}}{\left(1-\frac{tz_{i,r_i+2}}{qz_{i,r_i+1}}\right)\cdots\left(1-\frac{tz_{i,k_i}}{qz_{i,k_i-1}}\right)}, \tag{119}$$

and as $\epsilon\to 0$ this grows in the order of $\epsilon^{-k_i+r_i+2}=\epsilon^{-|\{z_{i,1},\cdots,z_{i,k_i}\}\setminus\Upsilon|+2}$. We thus conclude (114). □

Proposition 5.17. *Following the notation of Theorem 5.1 Let $N_1\leq\cdots\leq N_l, k_1,\cdots,k_l, M$ be positive integers, $M>k_1+\cdots+k_l$. Let $a_1,\cdots,a_{N_l}$ be variables, and q,t be parameters. Then under (64),*

$$\lim_{\epsilon\to 0}\frac{\epsilon^{-l}\prod_{i=1}^{l}(\mathcal{D}_{-k_i}^{N_i}-t^{-N_ik_i})\prod_{i=1}^{N_l}\frac{\prod_{k=1}^{\infty}(1-a_it^{\alpha+M}q^{k-1})}{\prod_{k=1}^{\infty}(1-a_it^{\alpha}q^{k-1})}}{\prod_{i=1}^{N_l}\frac{\prod_{k=1}^{\infty}(1-a_it^{\alpha+M}q^{k-1})}{\prod_{k=1}^{\infty}(1-a_it^{\alpha}q^{k-1})}} = \frac{\prod_{i=1}^{l}k_i}{(2\pi\mathbf{i})^{k_1+\cdots+k_l}}\oint\cdots\oint$$

$$\times\prod_{i=1}^{l}\mathfrak{I}(u_{i,1},\cdots,u_{i,k_i};\alpha,M,\theta,N_i)\prod_{i<j}\mathfrak{L}(u_{i,1},\cdots,u_{i,k_i};u_{j,1},\cdots,u_{j,k_j};\alpha,M,\theta)\prod_{i=1}^{l}\prod_{i'=1}^{k_i}du_{i,i'}, \tag{120}$$

where for each $i=1,\cdots,l$, the contours of $u_{i,1},\cdots,u_{i,k_i}$ enclose $-\theta(N_i-1)$ but not $\theta(\alpha+M)$, and $|u_{i,1}|\ll\cdots\ll|u_{i,k_i}|$. For $1\leq i\leq i+1\leq l$, we also require that $|u_{i,k_i}|\ll|u_{i+1,1}|$.

Proof. From Lemma 5.15 and Lemma 5.16 we see that

$$\lim_{\epsilon\to 0}\frac{\epsilon^{-l}\prod_{i=1}^{l}(\mathcal{D}_{-k_i}^{N_i}-t^{-N_ik_i})\prod_{i=1}^{N_l}\frac{\prod_{k=1}^{\infty}(1-a_it^{\alpha+M}q^{k-1})}{\prod_{k=1}^{\infty}(1-a_it^{\alpha}q^{k-1})}}{\prod_{i=1}^{N_l}\frac{\prod_{k=1}^{\infty}(1-a_it^{\alpha+M}q^{k-1})}{\prod_{k=1}^{\infty}(1-a_it^{\alpha}q^{k-1})}}=\lim_{\epsilon\to 0}\frac{(-1)^{k_1+\cdots+k_l-l}}{(2\pi\mathbf{i})^{k_1+\cdots+k_l}}\mathfrak{Q}_{\emptyset}. \tag{121}$$

Notice that in $\mathfrak{Q}_{\emptyset}$, each contour encloses all of $qa_1,\cdots,qa_{N_l}$ but not 0; then we can set each $z_{i,i'}=\exp(\epsilon u_{i,i'})$, with $u_{i,i'}$ independent of ϵ, satisfying the stated requirements. Evaluating the limit gives (120). □

With Proposition 5.12 and Proposition 5.17, we finish the proof of Theorem 5.1.

Proof of Theorem 5.1. With Proposition 5.5, the identity given by Proposition 5.12 can be interpreted as

$$\frac{\prod_{i=1}^{l}\frac{1}{\theta k_i}(t^{-N_ik_i}-\mathcal{D}_{-k_i}^{N_i})\prod_{i=1}^{N_l}\frac{\prod_{k=1}^{\infty}(1-a_it^{\alpha+M}q^{k-1})}{\prod_{k=1}^{\infty}(1-a_it^{\alpha}q^{k-1})}}{\prod_{i=1}^{N_l}\frac{\prod_{k=1}^{\infty}(1-a_it^{\alpha+M}q^{k-1})}{\prod_{k=1}^{\infty}(1-a_it^{\alpha}q^{k-1})}}$$
$$=\mathbb{E}\left(\prod_{i=1}^{l}\left(\frac{1}{\theta k_i}(t^{-N_ik_i}-1)\sum_{j=1}^{N_i}(q^{\lambda_j^i}t^{-j+1})^{k_i}\right)\right) \tag{122}$$

where the joint distribution of $\lambda^{N_1},\cdots,\lambda^{N_l}$ is as a Macdonald process with parameters M, $\{a_1,\cdots,a_{N_l},0,\cdots\}$, and $\{b_i\}_{i=1}^{M}$.

Under the limit transition (64) (and using Theorem 5.6) we have that

$$\lim_{\epsilon\to 0}\mathbb{E}\left(\prod_{i=1}^{l}\left(\frac{1}{\theta k_i}\epsilon^{-1}(t^{-N_ik_i}-1)\sum_{j=1}^{N_i}(q^{\lambda_j^i}t^{-j+1})^{k_i}\right)\right)=\mathbb{E}\left(\mathfrak{P}_{k_1}(x^{N_1})\cdots\mathfrak{P}_{k_l}(x^{N_l})\right). \tag{123}$$

Then we have

$$\mathbb{E}\left(\mathfrak{P}_{k_1}(x^{N_1})\cdots\mathfrak{P}_{k_l}(x^{N_l})\right)=\lim_{\epsilon\to 0}\frac{\epsilon^{-l}\prod_{i=1}^{l}\frac{1}{\theta k_i}(t^{-N_ik_i}-\mathcal{D}_{-k_i}^{N_i})\prod_{i=1}^{N_l}\frac{\prod_{k=1}^{\infty}(1-a_it^{\alpha+M}q^{k-1})}{\prod_{k=1}^{\infty}(1-a_it^{\alpha}q^{k-1})}}{\prod_{i=1}^{N_l}\frac{\prod_{k=1}^{\infty}(1-a_it^{\alpha+M}q^{k-1})}{\prod_{k=1}^{\infty}(1-a_it^{\alpha}q^{k-1})}}. \tag{124}$$

Plugging in Proposition 5.17 finishes the proof. □

5.4 First application: single moment of adjacent rows

We prove (21) in Theorem 3.1 as a direct application of Theorem 5.1. The decay of variance is proved as part of Theorem 3.6 in Section 7.

Partial of Theorem 3.1.

$$\mathbb{E}\left(\mathfrak{P}_k(x^N)-\mathfrak{P}_k(x^{N-1})\right)=\frac{(-\theta)^{-1}}{(2\pi\mathbf{i})^k}\oint\cdots\oint\frac{1}{(u_2-u_1+1-\theta)\cdots(u_k-u_{k-1}+1-\theta)}$$
$$\times\prod_{i<j}\frac{(u_j-u_i)(u_j-u_i+1-\theta)}{(u_j-u_i+1)(u_j-u_i-\theta)}\left(\prod_{i=1}^k\frac{u_i-\theta}{u_i+(N-1)\theta}-\prod_{i=1}^k\frac{u_i-\theta}{u_i+(N-2)\theta}\right)$$
$$\times\prod_{i=1}^k\frac{\theta\alpha-u_i}{\theta(\alpha+M)-u_i}du_i. \quad (125)$$

Send $L\to\infty$ under (18), setting $u_i\sim L\theta v_i$. Then

$$\lim_{L\to\infty}L\left(\prod_{i=1}^k\frac{u_i-\theta}{u_i+(N-1)\theta}-\prod_{i=1}^k\frac{u_i-\theta}{u_i+(N-2)\theta}\right)$$
$$=\lim_{L\to\infty}L\prod_{i=1}^k\frac{u_i-\theta}{u_i+(N-1)\theta}\left(1-\prod_{i=1}^k\frac{u_i+(N-1)\theta}{u_i+(N-2)\theta}\right)=-\prod_{i=1}^k\frac{v_i}{v_i+\hat{N}}\left(\sum_{i=1}^k\frac{1}{v_i+\hat{N}}\right). \quad (126)$$

Thus we have

$$\lim_{L\to\infty}\mathbb{E}\left(\mathfrak{P}_k(x^N)-\mathfrak{P}_k(x^{N-1})\right)=\frac{1}{(2\pi\mathbf{i})^k}\oint\cdots\oint\frac{1}{(v_2-v_1)\cdots(v_k-v_{k-1})}$$
$$\times\left(\prod_{i=1}^k\frac{v_i}{v_i+\hat{N}}\cdot\frac{\hat{\alpha}-v_i}{\hat{\alpha}+\hat{M}-v_i}dv_i\right)\left(\sum_{i=1}^k\frac{1}{v_i+\hat{N}}\right), \quad (127)$$

where the contours enclose $-\hat{N}$ but not $\hat{\alpha}+\hat{M}$, and $|v_1|\ll\cdots\ll|v_k|$. By Corollary 4.2, this is simplified to (21). $\square$

6 Limit of diagrams and signed measures

In this section we present the proof of Theorem 3.2 and Theorem 3.3. We also state that the set of approaches presented here can also be applied to analyzing the asymptotic behavior of roots of Jacobi polynomials. As an illustration, we discuss the limit behavior of roots of Jacobi polynomials under the scheme (18).

6.1 Convergence of diagrams

We need the following result to show the convergence of diagram $w^{\tilde{x}^N,\tilde{x}^{N-1}}$.

Lemma 6.1 ([IO02, Lemma 5.7])**.** *For any fixed interval $[a,b]\subset\mathbb{R}$, let Σ be the set of all functions $\rho:\mathbb{R}\to\mathbb{R}$, that are supported in $[a,b]$ and satisfy $|\rho(u_1)-\rho(u_2)|\le|u_1-u_2|$, $\forall u_1,u_2\in[a,b]$. Then the weak topology defined by the functionals*

$$\rho\to\int\rho(u)u^k du,\quad k=0,1,\cdots \quad (128)$$

coincides with the uniform topology given by the supremum norm $\|\rho\|=\sup|\rho(u)|$.

By this Lemma, the convergence of $w^{\tilde{x}^N,\tilde{x}^{N-1}}$ (in probability) under the uniform topology is equivalent to the convergence (in probability) of each moment. For Theorem 3.2, it suffices to prove the following Proposition.

Proposition 6.2. *Let φ be defined as in Theorem 3.2. For any nonnegative integer k, under the limit scheme (18) we have*

$$\lim_{L\to\infty}\int_{[0,1]} w^{\tilde{x}^N,\tilde{x}^{N-1}}(u)u^k du = \int_{[0,1]}\varphi(u)u^k du, \tag{129}$$

in probability.

We state the following identity.

Lemma 6.3. *There is*

$$\frac{1}{2\pi\mathbf{i}}\oint\left(\frac{v}{v+\hat{N}}\cdot\frac{v-\hat{\alpha}}{v-\hat{\alpha}-\hat{M}}\right)^k\frac{1}{v+\hat{N}}dv = \frac{1}{2}\int_{[0,1]}\varphi''(u)u^k du, \tag{130}$$

where the contour in the left hand side is around $-\hat{N}$ but not $\hat{\alpha}+\hat{M}$.

Proof. Fix the contour, let $\mathcal{Q}$ be a constant relying on the contour (and determined later), which satisfying

$$\frac{1}{\mathcal{Q}} \geq \max\left\{\sup\left|\frac{v}{v+\hat{N}}\cdot\frac{v-\hat{\alpha}}{v-\hat{\alpha}-\hat{M}}\right|, 1\right\}, \tag{131}$$

where the sup takes over all v in the contour. For any $z\in(0,\mathcal{Q})$, we have that

$$\begin{aligned}
&\sum_{k=0}^{\infty} z^k\frac{1}{2\pi\mathbf{i}}\oint\left(\frac{v}{v+\hat{N}}\cdot\frac{v-\hat{\alpha}}{v-\hat{\alpha}-\hat{M}}\right)^k\frac{1}{v+\hat{N}}dv\\
=&\frac{1}{2\pi\mathbf{i}}\oint\sum_{k=0}^{\infty}\left(z\cdot\frac{v}{v+\hat{N}}\cdot\frac{v-\hat{\alpha}}{v-\hat{\alpha}-\hat{M}}\right)^k\frac{1}{v+\hat{N}}dv\\
=&\frac{1}{2\pi\mathbf{i}}\oint\frac{v-\hat{\alpha}-\hat{M}}{(v-\hat{\alpha}-\hat{M})(v+\hat{N})-zv(v-\hat{\alpha})}dv.
\end{aligned} \tag{132}$$

Denote

$$\begin{aligned}
\mathcal{R}_1 &= \frac{(1-z)\hat{\alpha}+\hat{M}-\hat{N}-\sqrt{((1-z)\hat{\alpha}+\hat{M}+\hat{N})^2-4z\hat{M}\hat{N}}}{2(1-z)},\\
\mathcal{R}_2 &= \frac{(1-z)\hat{\alpha}+\hat{M}-\hat{N}+\sqrt{((1-z)\hat{\alpha}+\hat{M}+\hat{N})^2-4z\hat{M}\hat{N}}}{2(1-z)},
\end{aligned} \tag{133}$$

then we can take $\mathcal{Q}$ small enough such that the contour always encloses $\mathcal{R}_1$ but not $\mathcal{R}_2$. We thus have

$$\begin{aligned}
&\sum_{k=0}^{\infty} z^k\frac{1}{2\pi\mathbf{i}}\oint\left(\frac{v}{v+\hat{N}}\cdot\frac{v-\hat{\alpha}}{v-\hat{\alpha}-\hat{M}}\right)^k\frac{1}{v+\hat{N}}dv\\
=&\frac{1}{2\pi\mathbf{i}(1-z)}\oint\frac{v-\hat{\alpha}-\hat{M}}{(v-\mathcal{R}_1)(v-\mathcal{R}_2)}dv = \frac{1}{1-z}\cdot\frac{\mathcal{R}_1-\hat{\alpha}-\hat{M}}{\mathcal{R}_1-\mathcal{R}_2}.
\end{aligned} \tag{134}$$

On the other hand, we have that

$$
\begin{aligned}
&\sum_{k=0}^{\infty}\frac{1}{2}\int_{[0,1]}\varphi''(u)u^k z^k du\\
=&\frac{C(\hat{M},\hat{N})}{1-z}+\sum_{k=0}^{\infty}\frac{1}{2\pi}\int_{[\gamma_1,\gamma_2]}\frac{\hat{M}-\hat{N}+(\hat{N}+\hat{M}+\hat{\alpha})(1-u)}{(\hat{N}+\hat{M}+\hat{\alpha})(1-u)}\cdot\frac{z^k u^k}{\sqrt{(\gamma_2-u)(u-\gamma_1)}}du\\
=&\frac{C(\hat{M},\hat{N})}{1-z}+\frac{1}{2\pi}\int_{[\gamma_1,\gamma_2]}\frac{\hat{M}-\hat{N}+(\hat{N}+\hat{M}+\hat{\alpha})(1-u)}{(\hat{N}+\hat{M}+\hat{\alpha})(1-u)(1-zu)}\cdot\frac{1}{\sqrt{(\gamma_2-u)(u-\gamma_1)}}du\\
=&\frac{1}{2(1-z)}+\frac{\hat{M}+\hat{N}+\hat{\alpha}-2\hat{M}z-z\hat{\alpha}}{2(1-z)\sqrt{((1-z)\hat{\alpha}+\hat{M}+\hat{N})^2-4z\hat{M}\hat{N}}},
\end{aligned}
\tag{135}
$$

which coincides with (134). Namely, for any $z\in(0,\mathcal{Q})$, we have that

$$
\sum_{k=0}^{\infty}z^k\frac{1}{2\pi\mathbf{i}}\oint\left(\frac{v}{v+\hat{N}}\cdot\frac{v-\hat{\alpha}}{v-\hat{\alpha}-\hat{M}}\right)^k\frac{1}{v+\hat{N}}dv=\sum_{k=0}^{\infty}z^k\frac{1}{2}\int_{[0,1]}\varphi''(u)u^k du. \tag{136}
$$

Notice that

$$
\begin{aligned}
&\left|\left(\frac{1}{2\pi\mathbf{i}}\oint\left(\frac{v}{v+\hat{N}}\cdot\frac{v-\hat{\alpha}}{v-\hat{\alpha}-\hat{M}}\right)^k\frac{1}{v+\hat{N}}dv\right)-\left(\int_{[0,1]}\frac{1}{2}\varphi''(u)u^k du\right)\right|\\
&\qquad\le\left(\frac{1}{2\pi}\oint\left|\frac{v}{v+\hat{N}}\cdot\frac{v-\hat{\alpha}}{v-\hat{\alpha}-\hat{M}}\right|^k\left|\frac{1}{v+\hat{N}}\right|dv\right)+\left(\int_{[0,1]}\left|\frac{1}{2}\varphi''(u)\right|du\right)\le\mathcal{A}^k,
\end{aligned}
\tag{137}
$$

where $\mathcal{A}>1$ is a constant.

If there is a nonnegative integer κ, such that (130) does not hold for $k=\kappa$ but for any $k<\kappa$; take

$$
z=\left|\left(\frac{1}{2\pi\mathbf{i}}\oint\left(\frac{v}{v+\hat{N}}\cdot\frac{v-\hat{\alpha}}{v-\hat{\alpha}-\hat{M}}\right)^\kappa\frac{1}{v+\hat{N}}dv\right)-\left(\int_{[0,1]}\frac{1}{2}\varphi''(u)u^\kappa du\right)\right|\frac{\mathcal{A}^{-\kappa-1}}{2}; \tag{138}
$$

then

$$
\begin{aligned}
&z^\kappa\left|\left(\frac{1}{2\pi\mathbf{i}}\oint\left(\frac{v}{v+\hat{N}}\cdot\frac{v-\hat{\alpha}}{v-\hat{\alpha}-\hat{M}}\right)^\kappa\frac{1}{v+\hat{N}}dv\right)-\left(\int_{[0,1]}\frac{1}{2}\varphi''(u)u^\kappa du\right)\right|\\
&\qquad\ge\sum_{k=\kappa+1}^{\infty}z^k\left|\left(\frac{1}{2\pi\mathbf{i}}\oint\left(\frac{v}{v+\hat{N}}\cdot\frac{v-\hat{\alpha}}{v-\hat{\alpha}-\hat{M}}\right)^k\frac{1}{v+\hat{N}}dv\right)-\left(\int_{[0,1]}\frac{1}{2}\varphi''(u)u^k du\right)\right|,
\end{aligned}
\tag{139}
$$

and this contradicts with (136). □

Proof of Proposition 6.2. First, by Theorem 3.1 and Lemma 6.3, for any nonnegative integer k, under the limit scheme (18) we have

$$
\lim_{L\to\infty}\int_{[0,1]}w^{\tilde{x}^N,\tilde{x}^{N-1}\prime\prime}(u)u^k du=\lim_{L\to\infty}2\left(\mathfrak{P}_k(x^N)-\mathfrak{P}_k(x^{N-1})\right)=\int_{[0,1]}\varphi''(u)u^k du, \tag{140}
$$

in probability.

To show that (129) holds, we do integrating by parts for any $k = 0, 1, \cdots$:

$$\int_{[0,1]} w^{\tilde{x}^N,\tilde{x}^{N-1}}(u)u^k du = \frac{w^{\tilde{x}^N,\tilde{x}^{N-1}}(1)}{k+1} - \frac{1}{k+1}\int_{[0,1]} w^{\tilde{x}^N,\tilde{x}^{N-1}\prime}(u)u^{k+1}du$$
$$= \frac{w^{\tilde{x}^N,\tilde{x}^{N-1}}(1)}{k+1} - \frac{w^{\tilde{x}^N,\tilde{x}^{N-1}\prime}(1)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)}\int_{[0,1]} w^{\tilde{x}^N,\tilde{x}^{N-1}\prime\prime}(u)u^{k+2}du. \quad (141)$$

Since the center of $w^{\tilde{x}^N,\tilde{x}^{N-1}}$ is in $[0,1]$, there is $w^{\tilde{x}^N,\tilde{x}^{N-1}}(1)+w^{\tilde{x}^N,\tilde{x}^{N-1}}(0) = 1$ and $w^{\tilde{x}^N,\tilde{x}^{N-1}\prime}(1) = 1$. We thus have that

$$\int_{[0,1]} w^{\tilde{x}^N,\tilde{x}^{N-1}}(u)u^k du = \frac{w^{\tilde{x}^N,\tilde{x}^{N-1}}(1) - w^{\tilde{x}^N,\tilde{x}^{N-1}}(0)}{2(k+1)}$$
$$- \frac{1}{2(k+1)} + \frac{1}{k+2} + \frac{1}{(k+1)(k+2)}\int_{[0,1]} w^{\tilde{x}^N,\tilde{x}^{N-1}\prime\prime}(u)u^{k+2}du$$
$$= \frac{1}{k+2} - \frac{1}{2(k+1)}\int_{[0,1]} w^{\tilde{x}^N,\tilde{x}^{N-1}\prime\prime}(u)udu$$
$$+ \frac{1}{(k+1)(k+2)}\int_{[0,1]} w^{\tilde{x}^N,\tilde{x}^{N-1}\prime\prime}(u)u^{k+2}du. \quad (142)$$

Similarly, for any $k = 0, 1, \cdots$,

$$\int_{[0,1]} \varphi(u)u^k du = \frac{1}{k+2} - \frac{1}{2(k+1)}\int_{[0,1]} \varphi''(u)udu + \frac{1}{(k+1)(k+2)}\int_{[0,1]} \varphi''(u)u^{k+2}du, \quad (143)$$

and (140) implies (129). □

6.2 Convergence of discrete signed measures

Now we show how Theorem 3.2 implies Theorem 3.3.

For any function $f : [0,1] \to \mathbb{R}$, such that f' exists almost everywhere, and $\int |f''|$ is finite (understood as distribution), through integrating by parts we have

$$\int f d\mu^{\tilde{x}^N,\tilde{x}^{N-1}} = \int_{[0,1]} \frac{1}{2} f(u) w^{\tilde{x}^N,\tilde{x}^{N-1}\prime\prime}(u)du$$
$$= \frac{f(0)+f(1)}{2} - \int_{[0,1]} \frac{1}{2} f'(u) w^{\tilde{x}^N,\tilde{x}^{N-1}\prime}(u)du$$
$$= \frac{f(0)+f(1)}{2} - \frac{1}{2}\left(f'(1)w^{\tilde{x}^N,\tilde{x}^{N-1}}(1) - f'(0)w^{\tilde{x}^N,\tilde{x}^{N-1}}(0)\right)$$
$$+ \int_{[0,1]} \frac{1}{2} f''(u) w^{\tilde{x}^N,\tilde{x}^{N-1}}(u)du. \quad (144)$$

By Theorem 3.2, there is

$$\begin{aligned} \lim_{L\to\infty} w^{\tilde{x}^N,\tilde{x}^{N-1}}(0) &= \varphi(0), \\ \lim_{L\to\infty} w^{\tilde{x}^N,\tilde{x}^{N-1}}(1) &= \varphi(1), \end{aligned} \quad (145)$$

in probability. And since

$$\left|\int_{[0,1]} \frac{1}{2} f''(u) w^{\tilde{x}^N,\tilde{x}^{N-1}}(u) du - \int_{[0,1]} \frac{1}{2} f''(u)\varphi(u) du\right|$$
$$\leq \int_{[0,1]} \frac{1}{2} \left| f''(u) \left(w^{\tilde{x}^N,\tilde{x}^{N-1}}(u) - \varphi(u)\right)\right| du$$
$$\leq \sup_{u\in\mathbb{R}} \left| w^{\tilde{x}^N,\tilde{x}^{N-1}}(u) - \varphi(u)\right| \int_{[0,1]} \frac{1}{2} \left| f''(u)\right| du, \quad (146)$$

we have

$$\lim_{L\to\infty} \int_{[0,1]} \frac{1}{2} f''(u) w^{\tilde{x}^N,\tilde{x}^{N-1}}(u) du = \int_{[0,1]} \frac{1}{2} f''(u)\varphi(u) du, \quad (147)$$

in probability. Again through integrating by parts, we conclude

$$\lim_{L\to\infty} \int f d\mu^{\tilde{x}^N,\tilde{x}^{N-1}} = \frac{1}{2}\left(f(0) + f(1) - f'(1)\varphi(1) + f'(0)\varphi(0) + \int_{[0,1]} f''(u)\varphi(u) du \right)$$
$$= \frac{1}{2} \int_{[0,1]} f(u)\varphi''(u) du. \quad (148)$$

6.3 Asymptote of roots of Jacobi polynomials

Consider $\mathcal{F}_n^{p,q}$, the Jacobi orthogonal polynomials of degree n and corresponding to weight function $x^p(1-x)^q$. A more formal definition of Jacobi polynomials can be found in [Sze39].

There is a limit transition between the distribution $\mathbb{P}^{\alpha,M,\theta}$ on χ^M and the roots of $\mathcal{F}^{\alpha-1,|M-N|}_{\min(M,N)}$.

Theorem 6.4 ([BG15, Theorem 5.1])**.** *Let $(x^1, x^2, \cdots) \in \chi^M$ be distributed as $\mathbb{P}^{\alpha,M,\theta}$, and let $j_{M,N,\alpha,i}$ be the ith root (in increasing order) of $\mathcal{F}^{\alpha-1,|M-N|}_{\min(M,N)}$, for $1 \leq i \leq \min(M,N)$. Then there is*

$$\lim_{\theta\to\infty} x_i^N = j_{M,N,\alpha,i}, \quad (149)$$

in probability.

By this relationship, our approach on analyzing the limit behavior of the distribution $\mathbb{P}^{\alpha,M,\theta}$ on χ^M can be easily adapted to analyze the limit behavior of roots of Jacobi orthogonal polynomials. We just consider the same limit scheme (18) as above.

Since when α, M, θ are fixed, the sequences x_i^N and x_i^{N-1} interlaces, there is also an interlacing relationship for the roots:

$$j_{M,N,\alpha,1} \leq j_{M,N-1,\alpha,1} \leq j_{M,N,\alpha,2} \leq \cdots. \quad (150)$$

If we further denote $j_{M,N,\alpha,i} = 1$, for any fixed M, N, α, and $\min(M,N) < i \leq N$, there are rectangular diagrams (see Definition 2.5) defined by these interlacing sequences:

$$\iota_{M,N,\alpha} := w^{\{j_{M,N,\alpha,i}\},\{j_{M,N-1,\alpha,i}\}}. \quad (151)$$

As an analogue of Theorem 3.2, we show the convergence of the diagrams $\iota_{M,N,\alpha}$ under the same limit scheme (18).

Proposition 6.5. *Let φ be defined as in Theorem 3.2. Under the limit scheme (18), the diagrams $\iota_{M,N,\alpha}$ converge to φ in uniform topology.*

Proof. With Theorem 5.1, we can immediately compute the moments of the roots:

$$\sum_{i=1}^{N} j_{M,N,\alpha,i}^{k} = \frac{-1}{(2\pi\mathbf{i})^k} \oint \cdots \oint \frac{1}{(w_2 - w_1 - 1)\cdots(w_k - w_{k-1} - 1)} \times \prod_{i=1}^{k} \frac{w_i - 1}{w_i + N - 1} \cdot \frac{\alpha - w_i}{\alpha + M - w_i} dw_i, \quad (152)$$

where each contour encloses $-N+1$ but not $M+\alpha$, and $|w_1| \ll \cdots \ll |w_k|$, and k takes any positive integer.

Under (18), setting $w_i \sim L\upsilon_i$, there is

$$\begin{aligned}
&\lim_{L\to\infty} \sum_{i=1}^{N} j_{M,N,\alpha,i}^{k} - \sum_{i=1}^{N-1} j_{M,N-1,\alpha,i}^{k} \\
&\qquad = \lim_{L\to\infty} \frac{1}{(2\pi\mathbf{i})^k} \oint \cdots \oint \frac{1}{(w_2 - w_1 - 1)\cdots(w_k - w_{k-1} - 1)} \\
&\qquad \times \prod_{i=1}^{k} \frac{w_i - 1}{w_i + N - 1} \cdot \frac{\alpha - w_i}{\alpha + M - w_i} dw_i \left(\prod_{i=1}^{k} \frac{w_i + N - 1}{w_i + N - 2} - 1 \right) \\
&\qquad = \frac{1}{(2\pi\mathbf{i})^k} \oint \cdots \oint \frac{1}{(\upsilon_2 - \upsilon_1)\cdots(\upsilon_k - \upsilon_{k-1})} \\
&\qquad \times \prod_{i=1}^{k} \frac{\upsilon_i}{\upsilon_i + \hat{N}} \cdot \frac{\hat{\alpha} - \upsilon_i}{\hat{\alpha} + \hat{M} - \upsilon_i} d\upsilon_i \left(\sum_{i=1}^{k} \frac{1}{\upsilon_i + \hat{N}} \right) \qquad (153)
\end{aligned}$$

With Corollary 4.2, do dimension reduction as

$$\lim_{L\to\infty} \sum_{i=1}^{N} j_{M,N,\alpha,i}^{k} - \sum_{i=1}^{N-1} j_{M,N-1,\alpha,i}^{k} = \frac{1}{2\pi\mathbf{i}} \left(\frac{\upsilon}{\upsilon + \hat{N}} \cdot \frac{\upsilon - \hat{\alpha}}{\upsilon - \hat{\alpha} - \hat{M}} \right)^k \frac{1}{\upsilon + \hat{N}} d\upsilon, \qquad (154)$$

where the contour in the right hand side is around $-\hat{N}$ but not $\hat{\alpha} + \hat{M}$.

The right hand side now exactly fits Lemma 6.3, and we conclude that

$$\lim_{L\to\infty} \int_{[0,1]} \iota''_{M,N,\alpha}(u) u^k du = \lim_{L\to\infty} 2 \left(\sum_{i=1}^{N} j_{M,N,\alpha,i}^{k} - \sum_{i=1}^{N-1} j_{M,N-1,\alpha,i}^{k} \right) = \int_{[0,1]} \varphi''(u) u^k du. \quad (155)$$

Finally, we do integrating by parts, which leads to

$$\lim_{L\to\infty} \int_{[0,1]} \iota_{M,N,\alpha}(u) u^k du = \int_{[0,1]} \varphi(u) u^k du. \qquad (156)$$

With Lemma 6.1, the above is equivalent to the statement we wish to prove. □

7 Central limit theorem: gaussianity of fluctuation

In this section we focus on the fluctuation, and present the proof of Theorem 3.6 and Theorem 3.8. The general idea is to compute the limit joint higher moments, and check that they match Wick's formula (or Isserlis's theorem, see [Iss18]) .

We start by computing certain covariances in Section 7.1. Section 7.2 will be devoted to the proof of Theorem 3.6, where we explicitly check Wick's formula. For Theorem 3.8, in Section 7.3 we prove an identity, involving joint higher moments cross different levels, which is in the form of Wick's formula. Using these results we prove Theorem 3.8 in Section 7.4

7.1 Computation of covariance

Throughout this section, let k_1, k_2 and N_1, N_2 be positive integers. In addition to (18), we also let

$$\lim_{L\to} \frac{N_1}{L} = \hat{N}_1, \quad \lim_{L\to} \frac{N_2}{L} = \hat{N}_2, \tag{157}$$

where $\hat{N}_1$ and $\hat{N}_2$ are positive real numbers.

We state the following results.

Lemma 7.1. *Under the above limit scheme, for $\hat{N}_1 \leq \hat{N}_2$. there is*

$$\lim_{L\to\infty} L^2 \mathbb{E}\left(\prod_{i=1}^{2} \mathfrak{P}_{k_i}(x^{N_i}) - \mathbb{E}\left(\mathfrak{P}_{k_i}(x^{N_i})\right)\right)$$
$$= \frac{\theta^{-1}}{(2\pi\mathbf{i})^2} \oint \oint \frac{1}{(v_1 - v_2)^2} \prod_{i=1}^{2} \left(\frac{v_i}{v_i + \hat{N}_i} \cdot \frac{v_i - \hat{\alpha}}{v_i - \hat{\alpha} - \hat{M}}\right)^{k_i} dv_i, \tag{158}$$

where the contours enclose poles at $-\hat{N}_1$ and $-\hat{N}_2$, but not $\hat{\alpha} + \hat{M}$, and are nested with $|v_1| \ll |v_2|$.

This is the same as [BG15, Theorem 4.1]. Now we consider the difference of adjacent rows.

Lemma 7.2. *Under the same limit scheme, we have*

$$\lim_{L\to\infty} L\mathbb{E}\left(\left(\mathfrak{P}_{k_1}(x^{N_1}) - \mathfrak{P}_{k_1}(x^{N_1-1}) - \mathbb{E}\left(\mathfrak{P}_{k_1}(x^{N_1}) - \mathfrak{P}_{k_1}(x^{N_1-1})\right)\right)\left(\mathfrak{P}_{k_1}(x^{N_2}) - \mathbb{E}\left(\mathfrak{P}_{k_1}(x^{N_2})\right)\right)\right)$$
$$= -\frac{\theta^{-1}k_1}{(2\pi\mathbf{i})^2} \oint \oint \frac{1}{(v_1 - v_2)^2} \frac{1}{v_1 + \hat{N}_1} \prod_{i=1}^{2} \left(\frac{v_i}{v_i + \hat{N}_i} \cdot \frac{v_i - \hat{\alpha}}{v_i - \hat{\alpha} - \hat{M}}\right)^{k_i} dv_i, \tag{159}$$

where the contours enclose poles at $-\hat{N}_1$ and $-\hat{N}_2$, but not $\hat{\alpha} + \hat{M}$, and are nested: $|v_1| \ll |v_2|$ when there is $N_1 < N_2$ (for L large enough), and $|v_1| \gg |v_2|$ when there is $N_1 \geq N_2$ (for L large enough).

Proof. By Theorem 5.1 we have

$$\begin{aligned}
&\mathbb{E}\left(\left(\mathfrak{P}_{k_1}(x^{N_1})-\mathfrak{P}_{k_1}(x^{N_1-1})-\mathbb{E}\left(\mathfrak{P}_{k_1}(x^{N_1})-\mathfrak{P}_{k_1}(x^{N_1-1})\right)\right)\left(\mathfrak{P}_{k_1}(x^{N_2})-\mathbb{E}\left(\mathfrak{P}_{k_1}(x^{N_2})\right)\right)\right)\\
&\qquad=\frac{(-\theta)^{-2}}{(2\pi\mathbf{i})^{k_1+k_2}}\oint\cdots\oint\prod_{i=1}^{2}\Bigg(\frac{1}{(u_{i,2}-u_{i,1}+1-\theta)\cdots(u_{i,k_i}-u_{i,k_i-1}+1-\theta)}\\
&\quad\times\prod_{1\le i'<j'\le k_i}\frac{(u_{i,j'}-u_{i,i'})(u_{i,j'}-u_{i,i'}+1-\theta)}{(u_{i,j'}-u_{i,i'}-\theta)(u_{i,j'}-u_{i,i'}+1)}\prod_{i'=1}^{k_i}\frac{u_{i,i'}-\theta}{u_{i,i'}+(N_i-1)\theta}\cdot\frac{u_{i,i'}-\theta\alpha}{u_{i,i'}-\theta\alpha-\theta M}du_{i,i'}\Bigg)\\
&\qquad\times\left(1-\prod_{i'=1}^{k_1}\frac{u_{1,i'}+(N_1-1)\theta}{u_{1,i'}+(N_1-2)\theta}\right)\left(\prod_{\substack{1\le i'\le k_1,\\ 1\le j'\le k_2}}\frac{(u_{1,i'}-u_{2,j'})(u_{1,i'}-u_{2,j'}+1-\theta)}{(u_{1,i'}-u_{2,j'}-\theta)(u_{1,i'}-u_{2,j'}+1)}-1\right),
\end{aligned}\tag{160}$$

where the contours for $u_{i,k_1},\cdots,u_{i,k_i}$ enclose $-\theta(N_i-2)$ and $-\theta(N_i-1)$ but not $\theta(\alpha+M)$, for $i=1,2$. We also require that $|u_{1,1}|\ll\cdots\ll|u_{1,k_1}|$ and $|u_{2,1}|\ll\cdots\ll|u_{2,k_2}|$, and $|u_{1,k_1}|\ll|u_{2,1}|$ when $N_1\le N_2$, $|u_{2,k_2}|\ll|u_{1,1}|$ when $N_1>N_2$.

Set $u_{i,i'}=L\theta v_{i,i'}$ for $i=1,2$ and any $1\le i'\le k_i$. Send $L\to\infty$, the above integral becomes

$$\begin{aligned}
&\frac{\theta^{-1}}{(2\pi\mathbf{i})^{k_1+k_2}}\oint\cdots\oint\left(\sum_{\substack{1\le i'\le k_1,\\ 1\le j'\le k_2}}\frac{1}{(v_{1,i'}-v_{2,j'})^2}\right)\left(\sum_{i'=1}^{k_1}\frac{1}{v_{1,i'}+\hat{N}_1}\right)\\
&\qquad\times\prod_{i=1}^{2}\left(\frac{1}{(v_{i,2}-v_{i,1})\cdots(v_{i,k_i}-v_{i,k_i-1})}\left(\prod_{i'=1}^{k_i}\frac{v_{i,i'}}{v_{i,i'}+\hat{N}_i}\cdot\frac{v_{i,i'}-\hat{\alpha}}{v_{i,i'}-\hat{\alpha}-\hat{M}}dv_{i,i'}\right).\right)
\end{aligned}\tag{161}$$

Applying Corollary 4.2 to $v_{i,k_1},\cdots,v_{i,k_i}$ and $v_{j,k_1},\cdots,v_{j,k_j}$ respectively gives (159). ∎

Lemma 7.3. *When $\hat{N}_1<\hat{N}_2$, there is*

$$\begin{aligned}
&\lim_{L\to\infty}L^2\mathbb{E}\left(\prod_{i=1}^{2}\mathfrak{P}_{k_i}(x^{N_i})-\mathfrak{P}_{k_i}(x^{N_i-1})-\mathbb{E}\left(\mathfrak{P}_{k_i}(x^{N_i})-\mathfrak{P}_{k_i}(x^{N_i-1})\right)\right)\\
&\qquad=\frac{\theta^{-1}k_1k_2}{(2\pi\mathbf{i})^2}\oint\oint\frac{1}{(v_1-v_2)^2}\prod_{i=1}^{2}\frac{dv_i}{v_i+\hat{N}_i}\left(\frac{v_i}{v_i+\hat{N}_i}\cdot\frac{v_i-\hat{\alpha}}{v_i-\hat{\alpha}-\hat{M}}\right)^{k_i},
\end{aligned}\tag{162}$$

where the contours enclose poles at $-\hat{N}_1$ and $-\hat{N}_2$, but not $\hat{\alpha}+\hat{M}$, and are nested with $|v_1|\ll|v_2|$.

Proof. The proof is very similar to the proof of Lemma 7.2. By Theorem 5.1 we obtain that

$$\begin{aligned}
&\mathbb{E}\left(\left(\mathfrak{P}_{k_1}(x^{N_1})-\mathfrak{P}_{k_1}(x^{N_1-1})-\mathbb{E}\left(\mathfrak{P}_{k_1}(x^{N_1})-\mathfrak{P}_{k_1}(x^{N_1-1})\right)\right)\left(\mathfrak{P}_{k_1}(x^{N_2})-\mathbb{E}\left(\mathfrak{P}_{k_1}(x^{N_2})\right)\right)\right)\\
&\quad=\frac{(-\theta)^{-2}}{(2\pi\mathbf{i})^{k_1+k_2}}\oint\cdots\oint\prod_{i=1}^{2}\Bigg(\frac{1}{(u_{i,2}-u_{i,1}+1-\theta)\cdots(u_{i,k_i}-u_{i,k_i-1}+1-\theta)}\\
&\quad\times\prod_{1\le i'<j'\le k_i}\frac{(u_{i,j'}-u_{i,i'})(u_{i,j'}-u_{i,i'}+1-\theta)}{(u_{i,j'}-u_{i,i'}-\theta)(u_{i,j'}-u_{i,i'}+1)}\prod_{i'=1}^{k_i}\frac{u_{i,i'}-\theta}{u_{i,i'}+(N_i-1)\theta}\cdot\frac{u_{i,i'}-\theta\alpha}{u_{i,i'}-\theta\alpha-\theta M}du_{i,i'}\\
&\quad\times\left(1-\prod_{i'=1}^{k_i}\frac{u_{i,i'}+(N_i-1)\theta}{u_{i,i'}+(N_i-2)\theta}\right)\Bigg)\left(\prod_{\substack{1\le i'\le k_1,\\1\le j'\le k_2}}\frac{(u_{1,i'}-u_{2,j'})(u_{1,i'}-u_{2,j'}+1-\theta)}{(u_{1,i'}-u_{2,j'}-\theta)(u_{1,i'}-u_{2,j'}+1)}-1\right),
\end{aligned}\tag{163}$$

where the contours for $u_{i,k_1},\cdots,u_{i,k_i}$ enclose $-\theta(N_i-2)$ and $-\theta(N_i-1)$ but not $\theta(\alpha+M)$, for $i=1,2$. We also require that $|u_{1,1}|\ll\cdots\ll|u_{1,k_1}|\ll|u_{2,1}|\ll\cdots\ll|u_{2,k_2}|$.

Again set $u_{i,i'}=L\theta v_{i,i'}$ for $i=1,2$ and any $1\le i'\le k_i$. Sending $L\to\infty$ and applying Corollary 4.2 to $v_{i,k_1},\cdots,v_{i,k_i}$ and $v_{j,k_1},\cdots,v_{j,k_j}$, respectively, we eventually get (162). $\square$

Finally we consider the case where $\hat{N}_1=\hat{N}_2=\hat{N}$, for the same limit in Lemma 7.3.

Lemma 7.4. *Under the limit scheme (18), we have*

$$\begin{aligned}
&\lim_{L\to\infty}L\mathbb{E}\left(\prod_{i=1}^{2}\mathfrak{P}_{k_i}(x^N)-\mathfrak{P}_{k_i}(x^{N-1})-\mathbb{E}\left(\mathfrak{P}_{k_i}(x^N)-\mathfrak{P}_{k_i}(x^{N-1})\right)\right)\\
&\qquad=\frac{\theta^{-1}k_1k_2}{2\pi\mathbf{i}(k_1+k_2)}\oint\oint\frac{dv}{(v+\hat{N})^2}\left(\frac{v}{v+\hat{N}}\cdot\frac{v-\hat{\alpha}}{v-\hat{\alpha}-\hat{M}}\right)^{k_2+k_2},
\end{aligned}\tag{164}$$

where the contours enclose poles at $-\hat{N}$ but not $\hat{\alpha}+\hat{M}$.

Proof. We can write the expectation as

$$\begin{aligned}
&\mathbb{E}\left(\prod_{i=1}^{2}\mathfrak{P}_{k_i}(x^N)-\mathfrak{P}_{k_i}(x^{N-1})-\mathbb{E}\left(\mathfrak{P}_{k_i}(x^N)-\mathfrak{P}_{k_i}(x^{N-1})\right)\right)\\
&\quad=\mathbb{E}\left(\left(\mathfrak{P}_{k_1}(x^N)-\mathfrak{P}_{k_1}(x^{N-1})\right)\mathfrak{P}_{k_2}(x^N)\right)-\mathbb{E}\left(\mathfrak{P}_{k_2}(x^{N-1})\left(\mathfrak{P}_{k_1}(x^N)-\mathfrak{P}_{k_1}(x^{N-1})\right)\right)\\
&-\mathbb{E}\left(\mathfrak{P}_{k_1}(x^N)-\mathfrak{P}_{k_1}(x^{N-1})\right)\mathbb{E}\left(\mathfrak{P}_{k_2}(x^N)\right)+\mathbb{E}\left(\mathfrak{P}_{k_2}(x^{N-1})\right)\mathbb{E}\left(\mathfrak{P}_{k_1}(x^N)-\mathfrak{P}_{k_1}(x^{N-1})\right).
\end{aligned}\tag{165}$$

Now apply Lemma 7.2, and we obtain

$$\begin{aligned}
&\lim_{L\to\infty}L\mathbb{E}\left(\prod_{i=1}^{2}\mathfrak{P}_{k_i}(x^N)-\mathfrak{P}_{k_i}(x^{N-1})-\mathbb{E}\left(\mathfrak{P}_{k_i}(x^N)-\mathfrak{P}_{k_i}(x^{N-1})\right)\right)\\
&=\frac{k_1\theta^{-1}}{(2\pi\mathbf{i})^2}\oint\oint\left(\frac{v_2}{v_2+\hat{N}}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_1}\left(\frac{v_1}{v_1+\hat{N}}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_2}\frac{1}{(v_2+\hat{N})(v_2-v_1)^2}\\
&\quad-\left(\frac{v_1}{v_1+\hat{N}}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_1}\left(\frac{v_2}{v_2+\hat{N}}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_2}\frac{1}{(v_1+\hat{N})(v_2-v_1)^2}dv_1dv_2,
\end{aligned}\tag{166}$$

where the contours of v_1 and v_2 enclose $-\hat{N}$ but not $\hat{\alpha}+\hat{M}$; and $|v_1| \ll |v_2|$.

Interchanging k_1 and k_2, for the same limit we have

$$\frac{k_2\theta^{-1}}{(2\pi\mathbf{i})^2}\oint\oint\left(\frac{v_2}{v_2+\hat{N}}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_2}\left(\frac{v_1}{v_1+\hat{N}}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_1}\frac{1}{(v_2+\hat{N})(v_2-v_1)^2}$$
$$-\left(\frac{v_1}{v_1+\hat{N}}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_2}\left(\frac{v_2}{v_2+\hat{N}}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_1}\frac{1}{(v_1+\hat{N})(v_2-v_1)^2}dv_1dv_2, \quad (167)$$

where the contours of v_1 and v_2 enclose $-\hat{N}$ but not $\hat{\alpha}+\hat{M}$; and $|v_1| \ll |v_2|$.

Notice that

$$\frac{\theta^{-1}}{(2\pi\mathbf{i})^2}\oint\oint\left(\frac{v_2}{v_2+\hat{N}}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_1}\left(\frac{v_1}{v_1+\hat{N}}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_2}\frac{1}{(v_2+\hat{N})(v_2-v_1)^2}$$
$$-\left(\frac{v_1}{v_1+\hat{N}}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_1}\left(\frac{v_2}{v_2+\hat{N}}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_2}\frac{1}{(v_1+\hat{N})(v_2-v_1)^2}$$
$$+\left(\frac{v_2}{v_2+\hat{N}}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_2}\left(\frac{v_1}{v_1+\hat{N}}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_1}\frac{1}{(v_2+\hat{N})(v_2-v_1)^2}$$
$$-\left(\frac{v_1}{v_1+\hat{N}}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_2}\left(\frac{v_2}{v_2+\hat{N}}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_1}\frac{1}{(v_1+\hat{N})(v_2-v_1)^2}dv_1dv_2$$
$$=-\frac{\theta^{-1}}{(2\pi\mathbf{i})^2}\oint\oint\frac{1}{(v_1+\hat{N})(v_2+\hat{N})(v_2-v_1)}$$
$$\times\left(\left(\frac{v_2}{v_2+\hat{N}}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_1}\left(\frac{v_1}{v_1+\hat{N}}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_2}\right.$$
$$\left.+\left(\frac{v_1}{v_1+\hat{N}}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_1}\left(\frac{v_2}{v_2+\hat{N}}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_2}\right)dv_1dv_2$$
$$=-\frac{\theta^{-1}}{2\pi\mathbf{i}}\oint\frac{1}{(v+\hat{N})^2}\left(\frac{v}{v+\hat{N}}\cdot\frac{v-\hat{\alpha}}{v-\hat{\alpha}-\hat{M}}\right)^{k_1+k_2}dv \quad (168)$$

where the contour encloses $-\hat{N}$ but not $\hat{\alpha}+\hat{M}$. The last line is by Theorem 4.1, applied to v_1 and v_2. This directly implies (164). $\square$

7.2 Gaussianity of 1-dimensional integral

The Wick's formula to be checked is as following.

Theorem 7.5. *Let $k_1, \cdots, k_h$ and $N_1 \le \cdots \le N_h$ be positive integers.*

Now let $1 \le t_1 < \cdots t_s \le h$, such that $N_{t_j - 1} \le N_{t_j}$ for any $1 \le j \le s$ and $t_j > 1$. For any $i \in \{t_1, \cdots, t_s\}$, denote

$$\mathfrak{E}_i = \mathfrak{P}_{k_i}(x^{N_i}) - \mathfrak{P}_{k_i}(x^{N_i-1}) - \mathbb{E}\left(\mathfrak{P}_{k_i}(x^{N_i}) - \mathfrak{P}_{k_i}(x^{N_i-1})\right), \quad (169)$$

and for any $1 \le i \le h$, $i \notin \{t_1, \cdots, t_s\}$, denote

$$\mathfrak{E}_i = \mathfrak{P}_{k_i}(x^{N_i}) - \mathbb{E}\left(\mathfrak{P}_{k_i}(x^{N_i})\right). \quad (170)$$

For any positive integer d, when $h = 2d$, let Ξ be the set of all matchings of $\{1, \cdots, 2d\}$. Then under (18) and (28), when $h = 2d$ we have that

$$\lim_{L\to\infty} L^{\frac{s}{2}} \mathbb{E}\left(\prod_{i=1}^{2d} \mathfrak{E}_i\right) = \lim_{L\to\infty} L^{\frac{s}{2}} \sum_{\{(a_1,b_1),\cdots,(a_d,b_d)\}\in\Xi} \prod_{i=1}^{d} \mathbb{E}\left(\mathfrak{E}_{a_i}\mathfrak{E}_{b_i}\right); \tag{171}$$

when $h = 2d - 1$, we have that

$$\lim_{L\to\infty} L^{\frac{s}{2}} \mathbb{E}\left(\prod_{i=1}^{2d-1} \mathfrak{E}_i\right) = 0. \tag{172}$$

We prove by induction on d, using the following propositions, one following another.

Proposition 7.6. *Let Θ_h be the collection of all unordered partitions of $\{1, \cdots, h\}$:*

$$\Theta_h = \left\{ \{U_1, \cdots, U_t\} \middle| t \in \mathbb{Z}_+, \bigcup_{i=1}^{t} U_i = \{1, \cdots, h\}, U_i \bigcap U_j = \emptyset, U_i \neq \emptyset, \forall 1 \leq i \leq j \leq t \right\}. \tag{173}$$

For positive integers $k_1, \cdots, k_h$ and $N_1, \cdots, N_h$ with $h \geq 2$, under the limit scheme (18) and (28) we have that

$$\lim_{L\to\infty} L^{\eta} \sum_{\{U_1,\cdots,U_t\}\in\Theta_h} (-1)^{t-1}(t-1)! \prod_{i=1}^{t} \mathbb{E}\left(\mathfrak{F}(U_i) \prod_{\substack{j\in U_i,\\ j\geq 2}} \mathfrak{P}_{k_j}(x^{N_j})\right) = 0, \tag{174}$$

for any $\eta < h - 1$. Here $\mathfrak{F}(U_i) = \mathfrak{P}_{k_j}(x^{N_j}) - \mathfrak{P}_{k_j}(x^{N_j - 1})$ if $1 \in U_i$, and $\mathfrak{F}_i = 1$ if $1 \notin U_i$.

Proof. First assume that $N_2 \leq \cdots \leq N_h$, and $N_l < N_1 \leq N_{l+1}$, for some $0 \leq l \leq h$ (we assume that $N_0 = 0$ and $N_{h+1} = \infty$).

By Theorem 5.1, and using the notations there, we have that for any $\{U_1, \cdots, U_t\} \in \Theta_h$,

$$\begin{aligned}
&\prod_{i=1}^{t} \mathbb{E}\left(\mathfrak{F}(U_i) \prod_{\substack{j\in U_i,\\ j\geq 2}} \mathfrak{P}_{k_j}(x^{N_j})\right) \\
&\qquad = \frac{(-\theta)^{-h}}{(2\pi\mathbf{i})^{k_1+\cdots+k_h}} \oint \cdots \oint \prod_{i=2}^{h} \mathfrak{I}(u_{i,1}, \cdots, u_{i,k_i}; \alpha, M, \theta, N_i) \\
&\qquad \times \left(\mathfrak{I}(u_{1,1}, \cdots, u_{1,k_1}; \alpha, M, \theta, N_1) - \mathfrak{I}(u_{1,1}, \cdots, u_{1,k_1}; \alpha, M, \theta, N_1 - 1)\right) \\
&\qquad \times \prod_{r=1}^{t} \prod_{\substack{i<j,\\ i,j\in U_r}} \mathfrak{L}(u_{i,1}, \cdots, u_{i,k_i}; u_{j,1}, \cdots, u_{j,k_j}; \alpha, M, \theta) \prod_{i=1}^{h} \prod_{i'=1}^{k_i} du_{i,i'},
\end{aligned} \tag{175}$$

where the contours are nested in the following way: for each $1 \leq i \leq h$, we have $|u_{i,1}| \ll \cdots \ll |u_{i,k_i}|$; for $2 \leq i \leq h-1$, we have $|u_{i,k_i}| \ll |u_{i+1,1}|$; and $|u_{1,k_1}| \ll |u_{l+1,1}|$ if $l < h$, $|u_{l,k_l}| \ll |u_{1,1}|$ if $l > 0$.

For any $1 \leq i < j \leq h$, denote

$$\mathfrak{M}(i,j;\alpha,M,\theta) = \mathfrak{L}(u_{i,1},\cdots,u_{i,k_i};u_{j,1},\cdots,u_{j,k_j};\alpha,M,\theta) - 1. \tag{176}$$

Set $u_{i,i'} = L\theta v_{i,i'}$ for any $1 \leq i \leq h$, $1 \leq i' \leq k_i$, we have that

$$\lim_{L\to\infty} L^2 \mathfrak{M}(i,j;\alpha,M,\theta) < \infty, \forall 1 \leq i < j \leq h. \tag{177}$$

Then we have

$$\begin{aligned}
&\sum_{\{U_1,\cdots,U_t\}\in\Theta_h} (-1)^{t-1}(t-1)! \prod_{i=1}^{t} \mathbb{E}\left(\prod_{j\in U_i \bigcap \phi_o} \mathfrak{P}_{k_j}(x^{N_j-1}) \prod_{j\in U_i\bigcap\psi_o} \mathfrak{P}_{k_j}(x^{N_j}) \right) \\
&= \frac{(-\theta)^{-h}}{(2\pi\mathbf{i})^{k_1+\cdots+k_h}} \oint\cdots\oint \prod_{i\in\phi_o} \mathfrak{I}(u_{i,1},\cdots,u_{i,k_i};\alpha,M,\theta,N_i-1) \prod_{i\in\psi_o} \mathfrak{I}(u_{i,1},\cdots,u_{i,k_i};\alpha,M,\theta,N_i) \\
&\qquad \times \sum_{\{U_1,\cdots,U_t\}\in\Theta_h} (-1)^{t-1}(t-1)! \prod_{r=1}^{t} \prod_{\substack{i<j,\\ i,j\in U_r}} (\mathfrak{M}(i,j;\alpha,M,\theta)+1) \prod_{i=1}^{h}\prod_{i'=1}^{k_i} du_{i,i'}.
\end{aligned} \tag{178}$$

For any $\{U_1,\cdots,U_t\} \in \Theta_h$, denote

$$\mathcal{T}(\{U_1,\cdots,U_t\}) = \bigcup_{r=1}^{t} \{(i,j)\,|i<j, i,j\in U_r\}. \tag{179}$$

And thus we have

$$\begin{aligned}
&\sum_{\{U_1,\cdots,U_t\}\in\Theta_h} (-1)^{t-1}(t-1)! \prod_{r=1}^{t} \prod_{\substack{i<j,\\ i,j\in U_r}} (\mathfrak{M}(i,j;\alpha,M,\theta)+1) \\
&\qquad = \sum_{\{U_1,\cdots,U_t\}\in\Theta_h} (-1)^{t-1}(t-1)! \sum_{\Omega\subset\mathcal{T}(\{U_1,\cdots,U_t\})} \prod_{(i,j)\in\Omega} \mathfrak{M}(i,j;\alpha,M,\theta).
\end{aligned} \tag{180}$$

For each $\Omega \subset \mathcal{T}(\{\{1,\cdots,h\}\})$, we compute

$$\sum_{\substack{\{U_1,\cdots,U_t\}\in\Theta_h,\\ \Omega\subset\mathcal{T}(\{U_1,\cdots,U_t\})}} (-1)^{t-1}(t-1)!. \tag{181}$$

For any $\{U_1,\cdots,U_t\} \in \Theta_h$, $\Omega \subset \mathcal{T}(\{U_1,\cdots,U_t\})$ if and only if for any $(i,j) \in \Omega$, there is $1 \leq r \leq t$, such that $\{i,j\} \subset U_r$.

Consider the graph G_Ω, with vertex set $\{1,\cdots,h\}$, and edge set Ω. Assume that all connected components in G_Ω are $G_{\Omega,1},\cdots,G_{\Omega,s}$, each contains vertices $V_{\Omega,1},\cdots,V_{\Omega,s}$; then for any $\{U_1,\cdots,U_t\} \in \Theta_h$, $\Omega \subset \mathcal{T}(\{U_1,\cdots,U_t\})$ if and only if for any $1 \leq i \leq s$, there is $1 \leq r \leq t$, such that $V_{\Omega,i} \subset U_r$. For each fixed $1 \leq t \leq s$, there are

$$\frac{1}{t!}\sum_{i=0}^{t}(-1)^i \binom{t}{i} (t-i)^s \tag{182}$$

such $\{U_1,\cdots,U_t\}$. Then we have that

$$\begin{aligned}
\sum_{\substack{\{U_1,\cdots,U_t\}\in\Theta_h,\\ \Omega\subset\mathcal{T}(\{U_1,\cdots,U_t\})}}(-1)^{t-1}(t-1)! &= \sum_{t=1}^{s}(-1)^{t-1}(t-1)!\frac{1}{t!}\sum_{i=0}^{t}(-1)^i\binom{t}{i}(t-i)^s \\
&= \sum_{i+i'\leq s, i'>0}(-1)^{i'-1}\frac{(i+i'-1)!}{i!i'!}i'^s \\
&= \sum_{i'=1}^{s}(-1)^{i'-1}\binom{s}{s-i'}i'^{s-1}=0,
\end{aligned}\tag{183}$$

when $s>1$. Thus the only terms left in (180) are those Ω where G_Ω is connected, where $|\Omega|\geq h-1$.

For each Ω with $|\Omega|\geq h-1$, set $u_{i,i'}=L\theta v_{i,i'}$ for any $1\leq i\leq h$, $1\leq i'\leq k_i$; we then have that

$$\begin{aligned}
&\lim_{L\to\infty}L^\eta\oint\cdots\oint\prod_{i=2}^{h}\mathfrak{I}(u_{i,1},\cdots,u_{i,k_i};\alpha,M,\theta,N_i)\\
&\times\left(\mathfrak{I}(u_{1,1},\cdots,u_{1,k_1};\alpha,M,\theta,N_1)-\mathfrak{I}(u_{1,1},\cdots,u_{1,k_1};\alpha,M,\theta,N_1-1)\right)\\
&\times\prod_{(i,j)\in\Omega}\mathfrak{M}(i,j;\alpha,M,\theta)\prod_{i=1}^{h}\prod_{i'=1}^{k_i}du_{i,i'}\\
=&\lim_{L\to\infty}\theta^{k_1+\cdots+k_h}L^{\eta+h-2|\Omega|-1}\oint\cdots\oint\prod_{i=2}^{h}L^{k_i-1}\mathfrak{I}(u_{i,1},\cdots,u_{i,k_i};\alpha,M,\theta,N_i)\\
&\times L^{k_1}\left(\mathfrak{I}(u_{1,1},\cdots,u_{1,k_1};\alpha,M,\theta,N_1)-\mathfrak{I}(u_{1,1},\cdots,u_{1,k_1};\alpha,M,\theta,N_1-1)\right)\\
&\times\prod_{(i,j)\in\Omega}L^2\mathfrak{M}(i,j;\alpha,M,\theta)\prod_{i=1}^{h}\prod_{i'=1}^{k_i}dv_{i,i'}\\
=&0,
\end{aligned}\tag{184}$$

since $\eta+h-2|\Omega|-1\leq\eta+h-2(h-1)-1<0$.

□

Corollary 7.7. *Let Θ_h as defined in Proposition 7.6. Let $k_1,\cdots,k_h$ and $N_1,\cdots,N_h$ be positive integers with $h\geq 2$. For given $1\leq g\leq h$, let $\tau=\{1,\cdots,g\}$ and $\kappa=\{1,\cdots,h\}\backslash\tau$. Under the limit scheme (18) and (28) we have that*

$$\begin{aligned}
&\lim_{L\to\infty}L^\eta\sum_{\{U_1,\cdots,U_t\}\in\Theta_h}(-1)^{t-1}(t-1)!\\
&\qquad\times\prod_{i=1}^{t}\mathbb{E}\left(\prod_{j\in U_i\bigcap\tau}\left(\mathfrak{P}_{k_j}(x^{N_j})-\mathfrak{P}_{k_j}(x^{N_j-1})\right)\prod_{j\in U_i\bigcap\kappa}\mathfrak{P}_{k_j}(x^{N_j})\right)=0,
\end{aligned}\tag{185}$$

for any $\eta<h-1$.

Proof. The expression (185) can be written as a sum of 2^{g-1} expressions in the form of (174), and each equals 0. $\square$

Proposition 7.8. *Let Θ_h be defined as in Proposition 7.6. Set $k_1, \cdots, k_h$ and $N_1, \cdots, N_h$ to be positive integers, and $\phi \subset \{1, \cdots, h\}$ to be a proper subset. For given $1 \leq g \leq h$, let $\tau = \{1, \cdots, g\}$ and $\kappa = \{1, \cdots, h\} \backslash \tau$. Under the limit scheme (18) and (28) we have that*

$$\sum_{\{U_1, \cdots, U_t\} \in \Theta_\tau} (-1)^{t-1}(t-1)! \times \prod_{i=1}^{t} \mathbb{E}\left(\prod_{j \in U_i \bigcap \phi \bigcap \tau} \left(\mathfrak{P}_{k_j}(x^{N_j}) - \mathfrak{P}_{k_j}(x^{N_j-1})\right) \prod_{j \in U_i \bigcap \phi \bigcap \kappa} \mathfrak{P}_{k_j}(x^{N_j}) \right) = 0. \quad (186)$$

Proof. For $\{V_1, \cdots, V_s\} \in \Theta_\kappa$, a partition of ϕ, and $\{U_1, \cdots, U_t\} \in \Theta_h$, if for any $1 \leq i \leq t$, there is $1 \leq j \leq s$, such that $U_i \bigcap \phi = V_j$ or $\emptyset$, we denote that $\{V_1, \cdots, V_s\} \prec \{U_1, \cdots, U_t\}$.

Then it suffices to show that

$$\sum_{\substack{t \in \mathbb{Z}_+, \{U_1, \cdots, U_t\} \in \Theta_\tau, \\ \{V_1, \cdots, V_s\} \prec \{U_1, \cdots, U_t\}}} (-1)^{t-1}(t-1)! = 0, \quad (187)$$

for any fixed $\{V_1, \cdots, V_s\}$. For any t, $s \leq t \leq s + h - |\phi|$, there are

$$\frac{1}{(t-s)!} \sum_{i=0}^{t-s} (-1)^i \binom{t-s}{i} (t-i)^{h-|\phi|} \quad (188)$$

$\{U_1, \cdots, U_t\} \in \Theta_h$ such that $\{V_1, \cdots, V_s\} \prec \{U_1, \cdots, U_t\}$. Plug this back into (187) we conclude that

$$\begin{aligned}
&\sum_{\substack{t \in \mathbb{Z}_+, \{U_1, \cdots, U_t\} \in \Theta_h, \\ \{V_1, \cdots, V_s\} \prec \{U_1, \cdots, U_t\}}} (-1)^{t-1}(t-1)! \\
&= \sum_{t=s}^{s+h-|\phi|} \frac{(-1)^{t-1}(t-1)!}{(t-s)!} \left(\sum_{i=0}^{t-s} (-1)^i \binom{t-s}{i} (t-i)^{h-|\phi|} \right) \\
&= \sum_{j=0}^{h-|\phi|} \sum_{i=0}^{j} (-1)^{s+j-1}(s+j-1)! \frac{(-1)^i}{j!} \binom{j}{i} (s+j-i)^{h-|\phi|} \\
&= \sum_{i'=0}^{h-|\phi|} \sum_{i=0}^{h-|\phi|-i'} \binom{s+i+i'-1}{i} \frac{(s+i'-1)!}{i'!} (-1)^{s+i'-1} (s+i')^{h-|\phi|} \\
&= \sum_{i'=0}^{h-|\phi|} \binom{s+h-|\phi|}{h-|\phi|-i'} \frac{(s+i'-1)!}{i'!} (-1)^{s+i'-1} (s+i')^{h-|\phi|} \\
&= \sum_{i'=0}^{h-|\phi|} \frac{(s+h-|\phi|)!}{(h-|\phi|-i')! i'!} (s+i')^{h-|\phi|-1} = 0.
\end{aligned} \quad (189)$$

$\square$

Corollary 7.9. *Let $k_1, \cdots, k_h$, $N_1, \cdots, N_h$, and $\mathfrak{E}_1, \cdots, \mathfrak{E}_h$ be defined as in Theorem 7.5, and Θ_h be defined as in Proposition 7.6. Under the limit scheme (18) and (28), when $h > 2$ and $s > 1$ there is*

$$\lim_{L\to\infty} L^{\frac{s}{2}} \sum_{\{U_1,\cdots,U_t\}\in\Theta_h} (-1)^{t-1}(t-1)! \prod_{i=1}^{t} \mathbb{E}\left(\prod_{j\in U_i} \mathfrak{E}_j\right) = 0. \tag{190}$$

Proof. The expression (190) can be written as a sum of 1 term in the form of (185) and $2^h - 1$ terms of the form (186). Notice that when $h > 2$, $\frac{s}{2} \le \frac{h}{2} < h - 1$, and we conclude that all the terms vanish asymptotically. □

With this corollary we can prove Theorem 7.5 by induction.

Proof of Theorem 7.5. The case where $s = 0$ is proved in [BG15, Section 4]. Now consider $s > 0$, and we prove by inducing on d. The base case where $d = 1$ is trivial. For $d > 1$, from Corollary 7.9, we have that

$$\lim_{L\to\infty} L^{\frac{s}{2}} \mathbb{E}\left(\prod_{i=1}^{2d-1} \mathfrak{E}_i\right) = \lim_{L\to\infty} L^{\frac{s}{2}} \sum_{\{U_1,\cdots,U_t\}\in\Theta_{2d-1},t>1} (-1)^t(t-1)! \prod_{i=1}^{t} \mathbb{E}\left(\prod_{j\in U_i} \mathfrak{E}_j\right). \tag{191}$$

Since each term $\mathbb{E}\left(\prod_{j\in U_i} \mathfrak{E}_j\right)$ contains a U_i such that $|U_i|$ is odd, by the hypothesis of induction, the right hand side of (191) is zero.

For any $\{(a_1, b_1), \cdots, (a_d, b_d)\} \in \Xi$, and $\{U_1, \cdots, U_t\} \in \Theta_{2d}$, if each U_i is the union of one or more pairs in $\{(a_1, b_1), \cdots, (a_d, b_d)\}$, denote $\{(a_1, b_1), \cdots, (a_d, b_d)\} \dashv \{U_1, \cdots, U_t\}$.

Then by Corollary 7.9, and the hypothesis of induction,

$$\lim_{L\to\infty} L^{\frac{s}{2}} \mathbb{E}\left(\prod_{i=1}^{2d} \mathfrak{E}_i\right) = \lim_{L\to\infty} L^{\frac{s}{2}}$$
$$\times \sum_{\{U_1,\cdots,U_t\}\in\Theta_{2d},t>1} (-1)^t(t-1)! \left(\sum_{\{(a_1,b_1),\cdots,(a_d,b_d)\}\dashv\{U_1,\cdots,U_t\}} \prod_{i=1}^{d} \mathbb{E}\left(\mathfrak{E}_{a_i}\mathfrak{E}_{b_i}\right)\right). \tag{192}$$

Notice that for any fixed $\{(a_1, b_1), \cdots, (a_d, b_d)\} \in \Xi$,

$$\sum_{\{U_1,\cdots,U_t\}\in\Theta_{2d},\{(a_1,b_1),\cdots,(a_d,b_d)\}\dashv\{U_1,\cdots,U_t\}} (-1)^{t-1}(t-1)! = 0, \tag{193}$$

since for each $t \in \mathbb{Z}_+$,

$$|\{\{U_1, \cdots, U_t\} \in \Theta_{2d} \,|\{(a_1, b_1), \cdots, (a_d, b_d)\} \dashv \{U_1, \cdots, U_t\}\}| \frac{1}{t!} \sum_{i=0}^{t} (-1)^i \binom{t}{i} (t-i)^d. \tag{194}$$

Similar to (183), we have

$$\sum_{\{U_1,\cdots,U_t\}\in\Theta_{2d},\{(a_1,b_1),\cdots,(a_d,b_d)\}\dashv\{U_1,\cdots,U_t\}} (-1)^{t-1}(t-1)!$$
$$= \sum_{t=1}^{d} (-1)^{t-1}(t-1)! \frac{1}{t!} \sum_{i=0}^{t} (-1)^i \binom{t}{i} (t-i)^d = 0, \tag{195}$$

when $d > 1$.

Thus we obtain (171), which completes the induction. □

Now we obtain Theorem 3.6.

Proof of Theorem 3.6. Gaussianity follows Theorem 7.5, where we only need the case of $s = h$. The covariance is given by Lemma 7.1, Lemma 7.3, and Lemma 7.4, in slightly different notations. □

7.3 Gaussian type asymptote cross different levels

Starting from this section we will prove Theorem 3.8. In this subsection we will focus on an identity in the form of Wick's formula.

Theorem 7.10. *Let $k_1, \cdots, k_h$ and $N_1 \leq \cdots \leq N_h$ be positive integers.*

Now let $1 \leq t_1 < \cdots t_s \leq h$, such that $N_{t_j - 1} < N_{t_j}$ for any $1 \leq j \leq s$ and $t_j > 1$. For any $i \in \{t_1, \cdots, t_s\}$, denote

$$\mathfrak{E}_i = \mathfrak{P}_{k_i}(x^{N_i}) - \mathfrak{P}_{k_i}(x^{N_i - 1}) - \mathbb{E}\left(\mathfrak{P}_{k_i}(x^{N_i}) - \mathfrak{P}_{k_i}(x^{N_i - 1})\right), \tag{196}$$

and for any $1 \leq i \leq h$, $i \notin \{t_1, \cdots, t_s\}$, denote

$$\mathfrak{E}_i = \mathfrak{P}_{k_i}(x^{N_i}) - \mathbb{E}\left(\mathfrak{P}_{k_i}(x^{N_i})\right). \tag{197}$$

For any positive integer d, when $h = 2d$, let Ξ be the set of all matchings of $\{1, \cdots, 2d\}$.

Then under (18) and (28), when $h = 2d$ we have that

$$\lim_{L\to\infty} L^s \mathbb{E}\left(\prod_{i=1}^{2d} \mathfrak{E}_i\right) = \lim_{L\to\infty} L^s \sum_{\{(a_1,b_1),\cdots,(a_d,b_d)\}\in\Xi} \prod_{i=1}^{d} \mathbb{E}\left(\mathfrak{E}_{a_i}\mathfrak{E}_{b_i}\right); \tag{198}$$

when $h = 2d - 1$, we have that

$$\lim_{L\to\infty} L^s \mathbb{E}\left(\prod_{i=1}^{2d-1} \mathfrak{E}_i\right) = 0. \tag{199}$$

Remark 7.11. We point out a crucial point where it is not actual Wick's formula: $t_1, \cdots, t_s$ must be different; and for any $i \in \{t_1, \cdots, t_s\}$, the covariance of $\mathfrak{E}_i$ actually goes to infinity as $L \to \infty$.

Proof of Theorem 7.10. By Theorem 5.1, and using the notations there, we have that

$$\begin{aligned}\mathbb{E}\left(\prod_{i=1}^{h} \mathfrak{E}_i\right) = \frac{(-\theta)^{-h}}{(2\pi\mathbf{i})^{k_1+\cdots+k_h}} \oint \cdots \oint \prod_{i=1}^{h} \mathfrak{I}(u_{i,1}, \cdots, u_{i,k_i}; \alpha, M, \theta, N_i) \\ \times \left(\sum_{\Omega\subset\{1,\cdots,h\}} (-1)^{h-|\Omega|} \prod_{\substack{i,j\in\Omega,\\ i<j}} \mathfrak{L}(u_{i,1}, \cdots, u_{i,k_i}; u_{j,1}, \cdots, u_{j,k_j}; \alpha, M, \theta)\right) \\ \prod_{j=1}^{s}\left(1 - \prod_{i'=1}^{k_{t_j}} \frac{u_{t_j,i'} + (N_{t_j} - 1)\theta}{u_{t_j,i'} + (N_{t_j} - 2)\theta}\right) \prod_{i=1}^{h}\prod_{i'=1}^{k_i} du_{i,i'},\end{aligned} \tag{200}$$

where for each $i = 1, \cdots, l$, the contours of $u_{i,1}, \cdots, u_{i,k_i}$ enclose $-\theta(N_i - 1)$ but not $\theta(\alpha + M)$, and $|u_{i,1}| \ll \cdots \ll |u_{i,k_i}|$. For $1 \le i \le i+1 \le l$, we also require that $|u_{i,k_i}| \ll |u_{i+1,1}|$.

For any $1 \le i < j \le h$, denote

$$\mathfrak{M}(i,j;\alpha,M,\theta) = \mathfrak{L}(u_{i,1},\cdots,u_{i,k_i};u_{j,1},\cdots,u_{j,k_j};\alpha,M,\theta) - 1. \tag{201}$$

And for any $\Omega \subset \{1, \cdots, h\}$, denote

$$\Omega_p = \{(i,j)\,|i<j, i,j \in \Omega\}. \tag{202}$$

Then we have that

$$\begin{aligned}
&\sum_{\Omega\subset\{1,\cdots,h\}} (-1)^{h-|\Omega|} \prod_{\substack{i,j\in\Omega,\\ i<j}} \mathfrak{L}(u_{i,1},\cdots,u_{i,k_i};u_{j,1},\cdots,u_{j,k_j};\alpha,M,\theta)\\
&= \sum_{\Omega\subset\{1,\cdots,h\}} (-1)^{h-|\Omega|} \sum_{\Delta\subset\Omega_p} \prod_{(i,j)\in\Delta} \mathfrak{M}(i,j;\alpha,M,\theta)\\
&= \sum_{\Delta\subset\{(i,j)|1\le i<j\le h\}} \left(\sum_{\substack{\Omega\subset\{1,\cdots,h\},\\ \Delta\subset\Omega_p}} (-1)^{h-|\Omega|} \right) \prod_{(i,j)\in\Delta} \mathfrak{M}(i,j;\alpha,M,\theta)\\
&= \sum_{\substack{\Delta\subset\{(i,j)|1\le i<j\le h\}\\ \{1,\cdots,h\}\subset\bigcup_{(i,j)\in\Delta}\{i,j\}}} \prod_{(i,j)\in\Delta} \mathfrak{M}(i,j;\alpha,M,\theta)
\end{aligned} \tag{203}$$

Set $u_{i,i'} = L\theta v_{i,i'}$ for any $1 \le i \le \tau$, $1 \le i' \le k_i$, then we have that

$$\lim_{L\to\infty} L^2 \mathfrak{M}(i,j;\alpha,M,\theta) = \frac{1}{\theta} \sum_{1\le i'\le k_i, 1\le j'\le k_j} \frac{1}{(v_{i,i'} - v_{j,j'})^2}, \forall 1 \le i < j \le h, \tag{204}$$

$$\lim_{L\to\infty} L \left(1 - \prod_{i'=1}^{k_{t_j}} \frac{u_{t_j,i'} + (N_{t_j}-1)\theta}{u_{t_j,i'} + (N_{t_j}-2)\theta} \right) = -\sum_{i'=1}^{k_{t_j}} \frac{1}{v_{t_j,i'} + \hat{N}_{t_j}}, \forall 1 \le j \le s, \tag{205}$$

$$\begin{aligned}
&\lim_{L\to\infty} L^{k_i-1} \mathfrak{I}(u_{i,1},\cdots,u_{i,k_i};\alpha,M,\theta,N_i)\\
&\quad = \frac{\theta^{-(k_i-1)}}{(v_{i,2}-v_{i,1})\cdots(v_{i,k_i}-v_{i,k_i-1})} \prod_{i'=1}^{k_i} \frac{v_{i,i'}}{v_{i,i'}+\hat{N}} \cdot \frac{v_{i,i'} - \hat{\alpha}}{v_{i,i'} - \hat{\alpha} - \hat{M}}, \forall 1 \le i \le h.
\end{aligned} \tag{206}$$

Then for any $\Delta \subset \{(i,j)\,|1 \le i < j \le h\}$, $|\Delta| > \frac{h}{2}$, we have

$$\begin{aligned}
&\lim_{L\to\infty} \oint \cdots \oint L^s \prod_{i=1}^{h} \mathfrak{I}(u_{i,1},\cdots,u_{i,k_i};\alpha,M,\theta,N_i) \prod_{j=1}^{s} \left(1 - \prod_{i'=1}^{k_{t_j}} \frac{u_{t_j,i'} + (N_{t_j}-1)\theta}{u_{t_j,i'} + (N_{t_j}-2)\theta} \right)\\
&\qquad \times \prod_{(i,j)\in\Delta} \mathfrak{M}(i,j;\alpha,M,\theta) \prod_{i=1}^{h}\prod_{i'=1}^{k_i} du_{i,i'} = 0.
\end{aligned} \tag{207}$$

Thus we have

$$\lim_{L\to\infty} L^s \mathbb{E}\left(\prod_{i=1}^{h} \mathfrak{E}_i\right) = \lim_{L\to\infty} L^s \frac{(-\theta)^{-h}}{(2\pi\mathbf{i})^{k_1+\cdots+k_h}} \oint\cdots\oint \prod_{i=1}^{h} \mathfrak{I}(u_{i,1},\cdots,u_{i,k_i};\alpha,M,\theta,N_i)$$
$$\times \prod_{j=1}^{s}\left(1-\prod_{i'=1}^{k_{t_j}} \frac{u_{t_j,i'}+(N_{t_j}-1)\theta}{u_{t_j,i'}+(N_{t_j}-2)\theta}\right)$$
$$\times\left(\sum_{\substack{\Delta\subset\{(i,j)\mid 1\le i<j\le h\}\\ \{1,\cdots,h\}\subset\bigcup_{(i,j)\in\Delta}\{i,j\},\\ |\Delta|\le\frac{h}{2}}} \prod_{(i,j)\in\Delta} \mathfrak{M}(i,j;\alpha,M,\theta)\right)\prod_{i=1}^{h}\prod_{i'=1}^{k_i} du_{i,i'}. \quad (208)$$

For any $\Delta\subset\{(i,j)\,|1\le i<j\le h\}$, the only possibility that $|\Delta|\le\frac{h}{2}$ and $\{1,\cdots,h\}\subset\bigcup_{(i,j)\in\Delta}\{i,j\}$ is when $h=2d$ for some positive integer d, and $\Delta\in\Xi$, the set of matchings of $\{1,\cdots,2d\}$. Namely, we have that

$$\lim_{L\to\infty} L^{2d} \mathbb{E}\left(\prod_{i=1}^{2d} \mathfrak{E}_i\right) = \lim_{L\to\infty} L^{2d} \frac{(-\theta)^{-2d}}{(2\pi\mathbf{i})^{k_1+\cdots+k_{2d}}} \oint\cdots\oint \prod_{i=1}^{h} \mathfrak{I}(u_{i,1},\cdots,u_{i,k_i};\alpha,M,\theta,N_i)$$
$$\times\left(\sum_{\Delta\subset\Xi}\prod_{(i,j)\in\Delta}\left(\mathfrak{L}(u_{i,1},\cdots,u_{i,k_i};u_{j,1},\cdots,u_{j,k_j};\alpha,M,\theta)-1\right)\right)$$
$$\times \prod_{j=1}^{s}\left(1-\prod_{i'=1}^{k_{t_j}} \frac{u_{t_j,i'}+(N_{t_j}-1)\theta}{u_{t_j,i'}+(N_{t_j}-2)\theta}\right)\prod_{i=1}^{2d}\prod_{i'=1}^{k_i} du_{i,i'}, \quad (209)$$

$$\lim_{L\to\infty} L^{2d-1}\mathbb{E}\left(\prod_{i=1}^{2d-1}\mathfrak{E}_i\right)=0, \quad (210)$$

which lead to (198) and (199), respectively. □

Corollary 7.12. *Following notations above, there are constants $C(\hat{\alpha},\hat{M},k_1,\cdots,k_h)$ and $D(\hat{\alpha},\hat{M},k_1,\cdots,k_h)$, independent of L, $\hat{N}_1,\cdots,\hat{N}_h$, and $t_1,\cdots,t_s$ such that*

$$L^s\mathbb{E}\left(\prod_{i=1}^{h}\mathfrak{E}_i\right)\le C(\hat{\alpha},\hat{M},k_1,\cdots,k_h) \quad (211)$$

for any $L>D(\hat{\alpha},\hat{M},k_1,\cdots,k_h)$.

Proof. Set $u_{i,i'}=L\theta v_{i,i'}$ for any $1\le i\le\tau$, $1\le i'\le k_i$ in (200). We fix the contours for all $v_{i,i'}$, such that they are still nested and each encloses the line segment $[-1,0]$ but not $\hat{\alpha}+\hat{M}$. Then the integral is upper bounded by a constant relying on the chosen contours, and (211) follows. □

By a linear combination of terms in Theorem 7.10, we can also obtain the following result.

Corollary 7.13. *Let $k_1, \cdots, k_h$ be positive integers, and $\hat{N}_1 \le \cdots \le \hat{N}_h \in \mathbb{R}$ be positive. For any $1 \le i \le h$, denote*

$$\mathfrak{E}_i = \mathfrak{P}_{k_i}(x^{N_i}) - \mathfrak{P}_{k_i}(x^{N_i-1}) - \mathbb{E}\left(\mathfrak{P}_{k_i}(x^{N_i}) - \mathfrak{P}_{k_i}(x^{N_i-1})\right). \tag{212}$$

Now assume that there are s different numbers among $\hat{N}_1 \le \cdots \le \hat{N}_h$. For any positive integer d, when $h = 2d$, let Ξ be the set of all matchings of $\{1, \cdots, 2d\}$. Under (18) and (28), we have that

$$\lim_{L\to\infty} L^s \mathbb{E}\left(\prod_{i=1}^{2d} \mathfrak{E}_i\right) = \lim_{L\to\infty} L^s \sum_{\{(a_1,b_1),\cdots,(a_d,b_d)\}\in\Xi} \prod_{i=1}^{d} \mathbb{E}\left(\mathfrak{E}_{a_i}\mathfrak{E}_{b_i}\right). \tag{213}$$

When $h = 2d-1$, we have that

$$\lim_{L\to\infty} L^s \mathbb{E}\left(\prod_{i=1}^{2d-1} \mathfrak{E}_i\right) = 0. \tag{214}$$

7.4 Gaussianity of 2-dimensional integral

In this section we present the proof of Theorem 3.8, which largely relies on Theorem 7.10.

Proof. For the convenience of notations, denote

$$\mathfrak{C}_i(y) = \mathfrak{P}_{k_i}(x^{\lfloor y\rfloor}) - \mathfrak{P}_{k_i}(x^{\lfloor y\rfloor-1}) - \mathbb{E}\left(\mathfrak{P}_{k_i}(x^{\lfloor y\rfloor}) - \mathfrak{P}_{k_i}(x^{\lfloor y\rfloor-1})\right). \tag{215}$$

To show that (34) weakly converges to Gaussian, it suffices to show

$$\lim_{L\to\infty} L^{2d-1} \mathbb{E}\left(\prod_{i=1}^{2d-1} \int_0^1 g_i(y)\mathfrak{C}_i(Ly)dy\right) = 0 \tag{216}$$

for any positive integers d, $k_1, \cdots, k_{2d-1}$, and functions $g_1, \cdots, g_{2d-1} : [0,1] \to \mathbb{R}_{\ge 0}$, each in $L^1([0,1])$, bounded and continuous almost everywhere; and

$$\begin{aligned}&\lim_{L\to\infty} L^{2d} \mathbb{E}\left(\prod_{i=1}^{2d} \int_0^1 g_i(y)\mathfrak{C}_i(Ly)dy\right)\\ &= \lim_{L\to\infty} \sum_{\{(a_1,b_1),\cdots,(a_d,b_d)\}\in\Xi} L^{2d} \prod_{i=1}^{d} \mathbb{E}\left(\left(\int_0^1 g_{a_i}(y)\mathfrak{C}_{a_i}(Ly)dy\right)\left(\int_0^1 g_{b_i}(y)\mathfrak{C}_{a_i}(Ly)dy\right)\right)\end{aligned} \tag{217}$$

for any positive integers d, $k_1, \cdots, k_{2d}$, and functions $g_1, \cdots, g_{2d} : [0,1] \to \mathbb{R}_{\ge 0}$, each bounded and continuous almost everywhere; and Ξ is the set of all matchings of $\{1, \cdots, 2d\}$.

Then the Gaussianity of (34) follows from Wick's formula.

By a change of order of integral we have

$$L^h \mathbb{E}\left(\prod_{i=1}^{h} \int_0^1 g_i(y)\mathfrak{C}_i(Ly)dy\right) w = \int_0^1 \cdots \int_0^1 L^h \mathbb{E}\left(\prod_{i=1}^{h} \mathfrak{C}_i(Ly)\right) \prod_{i=1}^{h} g_i(y_i)dy_i, \tag{218}$$

for any positive integers d, $k_1, \cdots, k_h$, and functions $g_1, \cdots, g_h : [0,1] \to \mathbb{R}_{\geq 0}$, each bounded and continuous almost everywhere. We can split the integral into sum of the following form:

$$\int_0^1 \cdots \int_0^1 \mathbb{1}_{\lfloor Ly_1 \rfloor = \cdots = \lfloor Ly_{c_1} \rfloor < \lfloor Ly_{c_1+1} \rfloor = \cdots = \lfloor Ly_{c_2} \rfloor < \cdots < \lfloor Ly_{c_{s-1}+1} \rfloor = \cdots = \lfloor Ly_{c_s} \rfloor} \times L^h \mathbb{E}\left(\prod_{i=1}^h \mathfrak{C}_i(Ly)\right) \prod_{i=1}^h g_i(y_i) dy_i, \quad (219)$$

where $1 \leq s \leq h$ and $1 \leq c_1 < \cdots < c_s = h$. Notice that this integral is essentially a sum, and we can rewrite it as following:

$$\int_0^1 \cdots \int_0^1 \mathbb{1}_{\lfloor Lz_1 \rfloor < \cdots < \lfloor Lz_s \rfloor} \times L^s \mathbb{E}\left(\prod_{i=1}^h \mathfrak{E}_i\right) \prod_{i=1}^s \left(L^{c_i - c_{i-1}} \int_{\left[\frac{\lfloor Lz_i \rfloor}{L}, \min\{\frac{\lfloor Lz_i+1 \rfloor}{L}, 1\}\right]^{c_i - c_{i-1}}} \prod_{c_{i-1} < j \leq c_i} g_j \right) dz_i, \quad (220)$$

here $\mathfrak{E}_j = \mathfrak{C}_j(Lz_i)$, for any $1 \leq j \leq h$ and $c_{i-1} < j \leq c_i$ and $c_0 = 0$.

Since g_i is almost everywhere continuous, we have

$$\lim_{L \to \infty} L^{c_i - c_{i-1}} \int_{\left[\frac{\lfloor Lz_i \rfloor}{L}, \min\{\frac{\lfloor Lz_i+1 \rfloor}{L}, 1\}\right]^{c_i - c_{i-1}}} \prod_{c_{i-1} < j \leq c_i} g_j = \prod_{c_{i-1} < j \leq c_i} g_j(z_i) \quad (221)$$

for almost every z_i. Then the integrand in (220) then point-wisely converges to

$$\lim_{L \to \infty} \mathbb{1}_{z_1 < \cdots < z_s} L^s \mathbb{E}\left(\prod_{i=1}^h \mathfrak{E}_i\right) \prod_{i=1}^s \prod_{c_{i-1} < j \leq c_j} g_j(z_i), \quad (222)$$

which, by Corollary 7.13, equals

$$\lim_{L \to \infty} \mathbb{1}_{z_1 < \cdots < z_s} L^s \sum_{\{(a_1,b_1), \cdots, (a_d,b_d)\} \in \Xi} \prod_{i=1}^d \mathbb{E}\left(\mathfrak{E}_{a_i} \mathfrak{E}_{b_i}\right) \prod_{i=1}^s \prod_{c_{i-1} < j \leq c_j} g_j(z_i) \quad (223)$$

(Ξ is the set of all matchings of $\{1, \cdots, 2d\}$) for $h = 2d$; and 0 for $h = 2d - 1$.

By Corollary 7.10 the term

$$L^s \mathbb{E}\left(\prod_{i=1}^h \mathfrak{E}_i\right) \quad (224)$$

is bounded regardless of L and $z_1, \cdots, z_s$. Also, since each g_i is bounded, using Dominated Convergence Theorem we conclude that

$$\lim_{L \to \infty} \int_0^1 \cdots \int_0^1 \mathbb{1}_{\lfloor Lz_1 \rfloor < \cdots < \lfloor Lz_s \rfloor} L^s \times \mathbb{E}\left(\prod_{i=1}^h \mathfrak{E}_i\right) \prod_{i=1}^s \left(L^{c_i - c_{i-1}} \int_{\left[\frac{\lfloor Lz_i \rfloor}{L}, \min\{\frac{\lfloor Lz_i+1 \rfloor}{L}, 1\}\right]^{c_i - c_{i-1}}} \prod_{c_{i-1} < j \leq c_i} g_j \right) dz_i = 0 \quad (225)$$

for $h = 2d-1$, and

$$\lim_{L\to\infty}\int_0^1\cdots\int_0^1 \mathbb{1}_{\lfloor Lz_1\rfloor<\cdots<\lfloor Lz_s\rfloor} L^s \\ \times \mathbb{E}\left(\prod_{i=1}^h \mathfrak{E}_i\right)\prod_{i=1}^s\left(L^{c_i-c_{i-1}}\int_{\left[\frac{\lfloor Lz_i\rfloor}{L},\min\{\frac{\lfloor Lz_i+1\rfloor}{L},1\}\right]^{c_i-c_{i-1}}}\prod_{c_{i-1}<j\leq c_i} g_j\right)dz_i \\ = \int_0^1\cdots\int_0^1 \lim_{L\to\infty}\mathbb{1}_{z_1<\cdots<z_s}L^s\mathbb{E}\left(\prod_{i=1}^h\mathfrak{E}_i\right)\prod_{i=1}^s\left(\prod_{c_{i-1}<j\leq c_j} g_j(z_i)\right)dz_i \quad (226)$$

for $h = 2d$, and this equals

$$\lim_{L\to\infty}\sum_{\{(a_1,b_1),\cdots,(a_d,b_d)\}\in\Xi}\int_0^1\cdots\int_0^1 \mathbb{1}_{\lfloor Lz_1\rfloor<\cdots<\lfloor Lz_s\rfloor}L^s\prod_{i=1}^d\mathbb{E}\left(\mathfrak{E}_{a_i}\mathfrak{E}_{b_i}\right) \\ \times\prod_{i=1}^s\left(L^{c_i-c_{i-1}}\int_{\left[\frac{\lfloor Lz_i\rfloor}{L},\min\{\frac{\lfloor Lz_i+1\rfloor}{L},1\}\right]^{c_i-c_{i-1}}}\prod_{c_{i-1}<j\leq c_i} g_j\right)dz_i \\ = \int_0^1\cdots\int_0^1\lim_{L\to\infty}\mathbb{1}_{z_1<\cdots<z_s}L^s\left(\prod_{i=1}^d\mathbb{E}\left(\mathfrak{E}_{a_i}\mathfrak{E}_{b_i}\right)\right)\prod_{i=1}^s\left(\prod_{c_{i-1}<j\leq c_j} g_j(z_i)\right)dz_i. \quad (227)$$

Also by Corollary 211 and Dominated Convergence Theorem, we can now rewrite (227) as

$$\lim_{L\to\infty}\sum_{\{(a_1,b_1),\cdots,(a_d,b_d)\}\in\Xi}\int_0^1\cdots\int_0^1 \mathbb{1}_{\lfloor Ly_1\rfloor=\cdots=\lfloor Ly_{c_1}\rfloor<\lfloor Ly_{c_1+1}\rfloor=\cdots=\lfloor Ly_{c_2}\rfloor<\cdots<\lfloor Ly_{c_{s-1}+1}\rfloor=\cdots=\lfloor Ly_{c_s}\rfloor} \\ \times L^{2d}\prod_{i=1}^d\mathbb{E}\left(\mathfrak{E}_{a_i}(Ly_{a_i})\mathfrak{E}_{b_i}(Ly_{b_i})\right)\prod_{i=1}^{2d} g_i(y_i)dy_i. \quad (228)$$

Sum up over all such partitions of $1\leq c_1<\cdots<c_s=h$ we obtain (217).

Now we compute the covariance. Take $h=2$, the limit of covariance

$$\lim_{L\to\infty}L^2\mathbb{E}\left(\prod_{i=1}^2\int_0^1 g_i(y)\mathfrak{E}_i(Ly)dy\right) = \int_0^1\int_0^1 L^2\mathbb{E}\left(\prod_{i=1}^2\mathfrak{E}_i(Ly_i)\right)\prod_{i=1}^2 g_i(y_i)dy_i \quad (229)$$

can be written as

$$\lim_{L\to\infty}\iint_{\lfloor Ly_1\rfloor<\lfloor Ly_2\rfloor}L^2\mathbb{E}\left(\prod_{i=1}^2\mathfrak{E}_i(Ly_i)\right)\prod_{i=1}^2 g_i(y_i)dy_i \\ +\lim_{L\to\infty}\iint_{\lfloor Ly_2\rfloor<\lfloor Ly_1\rfloor}L^2\mathbb{E}\left(\prod_{i=1}^2\mathfrak{E}_i(Ly_i)\right)\prod_{i=1}^2 g_i(y_i)dy_i \\ +\lim_{L\to\infty}\int_0^1 L^3\left(\int_{\left[\frac{\lfloor Ly\rfloor}{L},\frac{\lfloor Ly+1\rfloor}{L}\right]^2} g_1g_2\right)\mathbb{E}\left(\prod_{i=1}^2\mathfrak{E}_i(Ly)\right)dy, \quad (230)$$

From boundedness of g_1, g_2, and the expectations computed in Section 7.1, the above can be written as

$$\iint_{y_1<y_2} \frac{\theta^{-1}g_1(y_1)g_2(y_2)}{(2\pi\mathbf{i})^2} \oint\oint \frac{k_1k_2}{(v_1-v_2)^2(v_1+y_1)(v_2+y_2)}$$
$$\times \left(\frac{v_1}{v_1+y_1}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_1}\left(\frac{v_2}{v_2+y_2}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_2} dv_1dv_2dy_1dy_2$$
$$+\iint_{y_1>y_2} \frac{\theta^{-1}g_1(y_1)g_2(y_2)}{(2\pi\mathbf{i})^2} \oint\oint \frac{k_1k_2}{(v_1-v_2)^2(v_1+y_1)(v_2+y_2)}$$
$$\times \left(\frac{v_1}{v_1+y_1}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_1}\left(\frac{v_2}{v_2+y_2}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_2} dv_1dv_2dy_1dy_2$$
$$-\int_0^1 \frac{\theta^{-1}g_1(y)g_2(y)}{2\pi\mathbf{i}} \oint \frac{k_1k_2}{(k_1+k_2)(v+y)^2}\left(\frac{v}{v+y}\cdot\frac{v-\hat{\alpha}}{v-\hat{\alpha}-\hat{M}}\right)^{k_1+k_2} dvdy, \quad (231)$$

where the contours of the first and second integrals are nested in different orders. In slightly different notation this is (35).

□

8 Connecting the limit field with Gaussian Free Field

In this section we interpret the Central Limit Theorems as convergence of the height function (see Definition 3.10) toward some Gaussian random field. The proofs of Theorem 3.11 and Theorem 3.16 presented here rely on Theorem 3.6 and Theorem 3.8, respectively.

Proof of Theorem 3.11. Denote $N_i = \lfloor L\hat{N}_i \rfloor$, for $i = 1, \cdots, h$. Do integrating by parts we obtain that

$$\int_0^1 u^{k_i}\mathcal{W}(u, L\hat{N}_i)du = \frac{1}{k_i+1}\left(\mathbb{1}_{N_i=M} - \sum_{j=1}^{\min\{N_i,M\}}\left(x_j^{N_i}\right)^{k_i+1} + \sum_{j=1}^{\min\{N_i-1,M\}}\left(x_j^{N_i}\right)^{k_i+1}\right), \quad (232)$$

and by Theorem 3.6 we immediately know that the limit of

$$\left(L^{\frac{1}{2}}\int_0^1 u^{k_i}\left(\mathcal{W}(u,L\hat{N}_i) - \mathbb{E}\left(\mathcal{W}(u,L\hat{N}_i)\right)\right)du\right)_{i=1}^h$$
$$= \left(\frac{L^{\frac{1}{2}}}{1+k_i}\left(\mathfrak{P}_{k_i+1}(x^{N_i}) - \mathfrak{P}_{k_i+1}(x^{N_i-1}) - \mathbb{E}\left(\mathfrak{P}_{k_i+1}(x^{N_i}) - \mathfrak{P}_{k_i+1}(x^{N_i-1})\right)\right)\right)_{i=1}^h \quad (233)$$

is a Gaussian vector. For $\hat{N}_i = \hat{N}_j$, the covariance of the ith and jth is

$$-\frac{1}{k_i+k_j+2}\cdot\frac{\theta^{-1}}{2\pi\mathbf{i}}\oint\frac{1}{(v+\hat{N}_i)^2}\left(\frac{v}{v+\hat{N}_i}\cdot\frac{v-\hat{\alpha}}{v-\hat{\alpha}-\hat{M}}\right)^{k_i+k_j+2}dv \quad (234)$$

where the contour encloses $-\hat{N}_i = -\hat{N}_j$ but not $\hat{\alpha}+\hat{M}$. For $\hat{N}_i \neq \hat{N}_j$, the covariance of the ith and jth is 0.

Now turn to (37). From Lemma 2.10, the distribution of (37) is Gaussian as well, with covariance of the ith and jth ($i < j$)

$$\begin{aligned}
&\frac{\delta^{-1}}{(k_i+1)(k_j+1)} \cdot \frac{\theta^{-1}}{(2\pi\mathbf{i})^2} \oint\oint \frac{dv_1 dv_2}{(v_1-v_2)^2} \\
&\qquad \times \left(\left(\frac{v_1}{v_1+\hat{N}_i+\delta} \cdot \frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}} \right)^{k_i+1} - \left(\frac{v_1}{v_1+\hat{N}_i} \cdot \frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}} \right)^{k_i+1} \right) \\
&\qquad\quad \times \left(\left(\frac{v_2}{v_2+\hat{N}_j+\delta} \cdot \frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}} \right)^{k_j+1} - \left(\frac{v_2}{v_2+\hat{N}_j} \cdot \frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}} \right)^{k_j+1} \right)
\end{aligned} \tag{235}$$

for $\hat{N}_i < \hat{N}_j$, where $|v_1| \ll |v_2|$, and the contours enclose $-\hat{N}_i$, $-\hat{N}_j$, $-\hat{N}_i-\delta$, and $-\hat{N}_j-\delta$, but not $\hat{\alpha}+\hat{M}$; and

$$\begin{aligned}
&\frac{\delta^{-1}}{(k_i+1)(k_j+1)} \cdot \frac{\theta^{-1}}{(2\pi\mathbf{i})^2} \oint\oint \frac{dv_1 dv_2}{(v_1-v_2)^2} \\
&\qquad \times \left(\left(\frac{v_1}{v_1+\hat{N}_i+\delta} \cdot \frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}} \right)^{k_i+1} \left(\frac{v_2}{v_2+\hat{N}_i+\delta} \cdot \frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}} \right)^{k_j+1} \right. \\
&\qquad\quad - \left(\frac{v_1}{v_1+\hat{N}_i} \cdot \frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}} \right)^{k_i+1} \left(\frac{v_2}{v_2+\hat{N}_i+\delta} \cdot \frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}} \right)^{k_j+1} \\
&\qquad\quad - \left(\frac{v_1}{v_1+\hat{N}_i} \cdot \frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}} \right)^{k_j+1} \left(\frac{v_2}{v_2+\hat{N}_i+\delta} \cdot \frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}} \right)^{k_i+1} \\
&\qquad\qquad \left. + \left(\frac{v_1}{v_1+\hat{N}_i} \cdot \frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}} \right)^{k_j+1} \left(\frac{v_2}{v_2+\hat{N}_i} \cdot \frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}} \right)^{k_i+1} \right)
\end{aligned} \tag{236}$$

for $\hat{N}_i = \hat{N}_j$, where $|v_1| \ll |v_2|$, and the contours enclose $-\hat{N}_i$ and $-\hat{N}_i-\delta$, but not $\hat{\alpha}+\hat{M}$.

Notice that by sending $\delta \to 0$, (37) converges to Gaussian as well, since the covariances converges: for (235) the limit is 0; for (236), the limit is

$$\begin{aligned}
&-\frac{1}{k_j+1} \cdot \frac{\theta^{-1}}{(2\pi\mathbf{i})^2} \oint\oint \frac{dv_1 dv_2}{(v_1-v_2)^2} \\
&\qquad \times \left(\frac{1}{v_1+\hat{N}_i} \left(\frac{v_1}{v_1+\hat{N}_i} \cdot \frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}} \right)^{k_i+1} \left(\frac{v_2}{v_2+\hat{N}_i} \cdot \frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}} \right)^{k_j+1} \right. \\
&\qquad \left. - \frac{1}{v_2+\hat{N}_i} \left(\frac{v_1}{v_1+\hat{N}_i} \cdot \frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}} \right)^{k_i+1} \left(\frac{v_2}{v_2+\hat{N}_i} \cdot \frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}} \right)^{k_j+1} \right) \\
&\qquad\qquad = \frac{1}{k_j+1} \cdot \frac{\theta^{-1}}{(2\pi\mathbf{i})^2} \oint\oint \frac{dv_1 dv_2}{(v_1-v_2)} \cdot \frac{1}{(v_1+\hat{N}_i)(v_2+\hat{N}_i)} \\
&\qquad\qquad\quad \times \left(\frac{v_1}{v_1+\hat{N}_i} \cdot \frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}} \right)^{k_i+1} \left(\frac{v_2}{v_2+\hat{N}_i} \cdot \frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}} \right)^{k_j+1}.
\end{aligned} \tag{237}$$

Simply switch k_i and k_j we obtain

$$\frac{1}{k_i+1}\cdot\frac{\theta^{-1}}{(2\pi\mathbf{i})^2}\oint\oint\frac{dv_1dv_2}{(v_1-v_2)}\cdot\frac{1}{(v_1+\hat{N}_i)(v_2+\hat{N}_i)} \times\left(\frac{v_1}{v_1+\hat{N}_i}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_j+1}\left(\frac{v_2}{v_2+\hat{N}_i}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_i+1}. \quad (238)$$

Applying Theorem 4.1 to a linear combination of (237) and (238) gives (234). □

Proof of Lemma 3.12. Through integrating by parts we are essentially considering the convergence of the following two vectors:

$$\left(L^{\frac{1}{2}}\left(\mathfrak{P}_{k_i}(x^{N_i})-\mathfrak{P}_{k_i}(x^{N_i-1})-\mathbb{E}\left(\mathfrak{P}_{k_i}(x^{N_i})-\mathfrak{P}_{k_i}(x^{N_i-1})\right)\right)\right)_{i=1}^{h} \quad (239)$$

and

$$\left(\left(\mathfrak{P}_{k_i'}(x^{N_i'})-\mathbb{E}\left(\mathfrak{P}_{k_i'}(x^{N_i'})\right)\right)\right)_{i=1}^{h'}. \quad (240)$$

The joint convergence and independence are given in Theorem 3.6. □

Let's consider the "derivative" of $\mathcal{K}$, where we first look at the random variables $\mathfrak{Z}_{g,k}$ for smooth g.

Proposition 8.1. *Let $k_1,\cdots,k_h$ be integers and $g_1,\cdots,g_h:[0,1]\to\mathbb{R}_{\geq 0}$ be smooth functions, with $g_i(1)=0$ for each $1\leq i\leq h$. Then the joint distribution of the vector*

$$(\mathfrak{Z}_{g_i,k_i})_{i=1}^{h} \quad (241)$$

is Gaussian, and the covariance between the ith and jth term is

$$\begin{aligned}&\iint_{y_1<y_2}\frac{g_i(y_2)g_j(y_1)\theta^{-1}}{(2\pi\mathbf{i})^2}\oint\oint\frac{1}{(v_1-v_2)^2(v_1+y_1)(v_2+y_2)}\\
&\quad\times\left(\frac{v_1}{v_1+y_1}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_j+1}\left(\frac{v_2}{v_2+y_2}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_i+1}dv_1dv_2dy_1dy_2\\
&\qquad+\iint_{y_1<y_2}\frac{g_i(y_1)g_j(y_2)\theta^{-1}}{(2\pi\mathbf{i})^2}\oint\oint\frac{1}{(v_1-v_2)^2(v_1+y_1)(v_2+y_2)}\\
&\quad\times\left(\frac{v_1}{v_1+y_1}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_i+1}\left(\frac{v_2}{v_2+y_2}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_j+1}dv_1dv_2dy_1dy_2\\
&\qquad-\int_0^1\frac{g_i(y)g_j(y)\theta^{-1}}{2\pi\mathbf{i}(k_i+k_j+2)}\oint\frac{1}{(v+y)^2}\left(\frac{v}{v+y}\cdot\frac{v-\hat{\alpha}}{v-\hat{\alpha}-\hat{M}}\right)^{k_i+k_j+2}dvdy, \quad (242)\end{aligned}$$

where for the first two summands, the contours enclose poles at $-y_1$ and $-y_2$, but not $\hat{\alpha}+\hat{M}$, and are nested with v_2 larger; for the last summand, the contour encloses pole at $-y$ but not $\hat{\alpha}+\hat{M}$.

Proof. By Lemma 2.11, the vector (241) is joint Gaussian, and the covariance between the ith and jth terms is

$$\int_0^1 \int_0^1 \frac{g_i'(y_1)g_j'(y_2)\theta^{-1}}{(2\pi\mathbf{i})^2(k_i+1)(k_j+1)} \oint\oint \frac{1}{(v_1-v_2)^2} \times \left(\frac{v_1}{v_1+y_1}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_i+1}\left(\frac{v_2}{v_2+y_2}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_j+1} dv_1dv_2dy_1dy_2, \quad (243)$$

where the inner contours enclose poles at $-y_1$ and $-y_2$, but not $\hat{\alpha}+\hat{M}$, and are nested: when $y_1 \le y_2$, v_2 is larger; when $y_1 \ge y_2$, v_1 is larger.

We first integrate by parts for y_1, for intervals $[0, y_2]$ and $[y_2, 1]$. The boundary term at 0 vanishes since the contour of v_1 encloses no pole when $y_1 = 0$; the boundary terms at y_2 cancels out, and the boundary terms at $y_1 = 0$ and 1 vanishes since $g_i(1) = 0$. Thus we obtain

$$\oint\oint\int_0^1\int_0^1 \frac{g_i(y_1)g_j'(y_2)\theta^{-1}}{(2\pi\mathbf{i})^2(k_j+1)}\cdot\frac{1}{(v_1-v_2)^2(v_1+y_1)} \times \left(\frac{v_1}{v_1+y_1}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_i+1}\left(\frac{v_2}{v_2+y_2}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_j+1} dy_1dy_2dv_1dv_2, \quad (244)$$

where the contours are nested: when $y_1 \le y_2$, v_2 is larger; when $y_1 \ge y_2$, v_1 is larger.

Then integrate by parts for y_2, again for two intervals respectively. For the area $y_1 < y_2$ we have

$$\begin{aligned}&\oint\oint\iint_{y_1<y_2} \frac{g_i(y_1)g_j(y_2)\theta^{-1}}{(2\pi\mathbf{i})^2}\cdot\frac{1}{(v_1-v_2)^2(v_1+y_1)(v_2+y_2)} \\ &\quad\times\left(\frac{v_1}{v_1+y_1}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_i+1}\left(\frac{v_2}{v_2+y_2}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_j+1} dy_1dy_2dv_1dv_2 \\ &\qquad - \oint\oint\int_0^1 \frac{g_i(y)g_j(y)\theta^{-1}}{(2\pi\mathbf{i})^2(k_j+1)}\cdot\frac{1}{(v_1-v_2)^2(v_1+y)} \\ &\quad\times\left(\frac{v_1}{v_1+y}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_i+1}\left(\frac{v_2}{v_2+y}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_j+1} dydv_1dv_2, \quad (245)\end{aligned}$$

where the contours are nested and v_2 is larger. The boundary term at $y_2 = 1$ vanishes since $g_j(1) = 0$.

For the integral in area $y_1 > y_2$ we exchange v_1 and v_2, y_1 and y_2, and add it to (245), and obtain

$$\oint\oint\iint_{y_1<y_2}\frac{g_i(y_2)g_j(y_1)\theta^{-1}}{(2\pi\mathbf{i})^2}\cdot\frac{1}{(v_1-v_2)^2(v_1+y_1)(v_2+y_2)}$$
$$\times\left(\frac{v_1}{v_1+y_1}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_j+1}\left(\frac{v_2}{v_2+y_2}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_i+1}dy_1dy_2dv_1dv_2$$
$$+\oint\oint\iint_{y_1<y_2}\frac{g_i(y_1)g_j(y_2)\theta^{-1}}{(2\pi\mathbf{i})^2}\cdot\frac{1}{(v_1-v_2)^2(v_1+y_1)(v_2+y_2)}$$
$$\times\left(\frac{v_1}{v_1+y_1}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_i+1}\left(\frac{v_2}{v_2+y_2}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_j+1}dy_1dy_2dv_1dv_2$$
$$+\oint\oint\int_0^1\frac{g_i(y)g_j(y)\theta^{-1}}{(2\pi\mathbf{i})^2(k_j+1)}\cdot\frac{1}{(v_1-v_2)^2(v_2+y)}$$
$$\times\left(\frac{v_1}{v_1+y}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_j+1}\left(\frac{v_2}{v_2+y}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_i+1}dydv_1dv_2$$
$$-\oint\oint\int_0^1\frac{g_i(y)g_j(y)\theta^{-1}}{(2\pi\mathbf{i})^2(k_j+1)}\cdot\frac{1}{(v_1-v_2)^2(v_1+y)}$$
$$\times\left(\frac{v_1}{v_1+y}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_i+1}\left(\frac{v_2}{v_2+y}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_j+1}dydv_1dv_2,\quad(246)$$

where the contours are nested and v_2 is larger.

The first and second summands above give the first and second summands in (242). For the third and last summands, if we first integrate by parts for y_2 then y_1, these two summands are instead

$$-\oint\oint\int_0^1\frac{g_i(y)g_j(y)\theta^{-1}}{(2\pi\mathbf{i})^2(k_i+1)}\cdot\frac{1}{(v_1-v_2)^2(v_1+y)}$$
$$\times\left(\frac{v_1}{v_1+y}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_j+1}\left(\frac{v_2}{v_2+y}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_i+1}dydv_1dv_2$$
$$+\oint\oint\int_0^1\frac{g_i(y)g_j(y)\theta^{-1}}{(2\pi\mathbf{i})^2(k_i+1)}\cdot\frac{1}{(v_1-v_2)^2(v_2+y)}$$
$$\times\left(\frac{v_1}{v_1+y}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_i+1}\left(\frac{v_2}{v_2+y}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_j+1}dydv_1dv_2,\quad(247)$$

where the contours also are nested and v_2 is larger. A linear combination of these two gives

$$\oint\oint\int_0^1\frac{g_i(y)g_j(y)\theta^{-1}}{(2\pi\mathbf{i})^2(k_i+k_j+2)}\cdot\frac{1}{(v_1-v_2)(v_1+y)(v_2+y)}$$
$$\times\left(\left(\frac{v_1}{v_1+y}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_j+1}\left(\frac{v_2}{v_2+y}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_i+1}\right.$$
$$\left.+\left(\frac{v_1}{v_1+y}\cdot\frac{v_1-\hat{\alpha}}{v_1-\hat{\alpha}-\hat{M}}\right)^{k_i+1}\left(\frac{v_2}{v_2+y}\cdot\frac{v_2-\hat{\alpha}}{v_2-\hat{\alpha}-\hat{M}}\right)^{k_j+1}\right)dydv_1dv_2,\quad(248)$$

where the contours are nested and v_2 is larger, and applying Theorem 4.1 to it gives the last summand in (242). $\square$

If we fix the contours in the expression (to enclose line segment $[-1,0]$ but not $\hat{\alpha}+\hat{M}$), we can bound the covariance for g uniformly.

Corollary 8.2. *There is constant $C(\hat{\alpha},\hat{M},k)$, such that for any smooth $g:[0,1]\to\mathbb{R}_{\geq 0}$, there is*

$$\mathbb{E}\left(\mathfrak{Z}_{g,k}^2\right) \leq C(\hat{\alpha},\hat{M},k)\|g\|_{L^2}^2. \tag{249}$$

Now we can show that $\mathfrak{Z}_{g,k}$ can be defined for any $g\in L^2([0,1])$.

Proof of Lemma 3.14. For the existence of such sequence, first one can always find a sequence of smooth $g_1,\cdots$, each $g_n(1)=0$, that converges to g in $L^2([0,1])$. Since smooth functions are dense in $L^2([0,1])$, we just need to ensure each $g_n(1)=0$; from any sequence of smooth functions that converge to g, we can just subtract by a sequence the converges to 0 in $L^2([0,1])$ and obtain what is desired.

We can also assume that for any n there is $\|g_n-g_{n+1}\|<2^{-n}$; otherwise we just take a subsequence. By Corollary 8.2, we have

$$\mathbb{E}\left(\left|\mathfrak{Z}_{g_n,k}-\mathfrak{Z}_{g_{n+1},k}\right|\right) \leq \mathbb{E}\left(\left(\mathfrak{Z}_{g_n,k}-\mathfrak{Z}_{g_{n+1},k}\right)^2\right)^{\frac{1}{2}} \leq C(\hat{\alpha},\hat{M},k)^{\frac{1}{2}}\|g_n-g_{n+1}\|_{L^2} \leq C(\hat{\alpha},\hat{M},k)^{\frac{1}{2}}2^{-n}, \tag{250}$$

and the random variable

$$\sum_{n=1}^{m}\left|\mathfrak{Z}_{g_n,k}-\mathfrak{Z}_{g_{n+1},k}\right| \tag{251}$$

converges almost surely, as $m\to\infty$. Then almost surely the limit

$$\lim_{n\to\infty}\mathfrak{Z}_{g_n,k} \tag{252}$$

exists (and is denoted as $\mathfrak{Z}_{g,k}$).

For the uniqueness, if there is another such sequence $\tilde{g}_1,\tilde{g}_2,\cdots$, there is

$$\begin{aligned}\mathbb{E}\left(\left|\mathfrak{Z}_{g,k}-\mathfrak{Z}_{\tilde{g}_n,k}\right|\right) &\leq \mathbb{E}\left(\left|\mathfrak{Z}_{g,k}-\mathfrak{Z}_{g_n,k}\right|\right)+\mathbb{E}\left(\left|\mathfrak{Z}_{g_n,k}-\mathfrak{Z}_{\tilde{g}_n,k}\right|\right)\\ &\leq \mathbb{E}\left(\left|\mathfrak{Z}_{g,k}-\mathfrak{Z}_{g_n,k}\right|\right)+C(\hat{\alpha},\hat{M},k)^{\frac{1}{2}}\|g_n-\tilde{g}_n\|_{L^2},\end{aligned} \tag{253}$$

and as $n\to\infty$ this goes to 0. Then $\mathfrak{Z}_{g_n,k}$ also converges almost surely to $\mathfrak{Z}_{g,k}$. $\square$

Now we finish the proof of Theorem 3.16.

Proof of Theorem 3.16. By integrating by parts in u direction, there is

$$\begin{aligned}&\int_0^1\int_0^1 u^{k_i}g_i(y)\mathcal{W}(u,L\hat{N}_i)dudy\\ &\quad=\int_0^1 g_i(y)\frac{1}{k_i+1}\left(\mathbb{1}_{N_i=M}-\sum_{j=1}^{\min\{N_i,M\}}\left(x_j^{N_i}\right)^{k_i+1}+\sum_{j=1}^{\min\{N_i-1,M\}}\left(x_j^{N_i}\right)^{k_i+1}\right)dy,\end{aligned} \tag{254}$$

and by Theorem 3.8 we immediately know that the limit of (41) converges weakly to Gaussian, and the covariance is given exactly by (242).

One other other hand, Proposition 8.1 can be extended to any $g_1, \cdots, g_h \in L^2([0,1])$ with each $g_i(1) = 0$, since the joint distribution of

$$(\mathfrak{Z}_{g_i,k_i})_{i=1}^{h} \tag{255}$$

is the weak limit of a sequence of vectors

$$\left((\mathfrak{Z}_{g_{n,i},k_i})_{i=1}^{h}\right)_{n=1}^{\infty} \tag{256}$$

where each $g_{n,i}$ is smooth with $g_{n,i}(1) = 0$, and $g_{n,i}$ converges to g_i in $L^2([0,1])$. Especially, when $g_1, \cdots, g_h$ are bounded and continuous almost everywhere, the limit distribution of (41), as $L \to \infty$, is exactly the distribution of (255). □

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