Homework 1 Answers

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(a)

The household's maximization problem is

$$\max_{\{C_{t+s},N_{t+s},B_{t+s},M_{t+s}\}_{s=0}^{\infty}} \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \left(\frac{X_{t+s}^{1-\gamma}-1}{1-\gamma} - \chi \frac{N_{t+s}^{1+\varphi}}{1+\varphi} \right)$$

where

$$X_t = \left[(1-\theta) C_t^{1-\nu} + \theta \left(\frac{M_t}{P_t} \right)^{1-\nu} \right]^{\frac{1}{1-\nu}}$$

s.t.

$$P_{t+s}C_{t+s} + B_{t+s} + M_{t+s} \leq W_{t+s}N_{t+s} + Q_{t+s-1}B_{t+s-1} + M_{t+s-1} + P_{t+s}(TR_{t+s} + PR_{t+s})$$

The associated Lagrangian is

$$\begin{aligned}
\mathcal{L} = \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \Bigg\{ & \left(\frac{X_{t+s}^{1-\theta} C_{t+s}^{1-\theta}}{1-\theta} + \chi \frac{N_{t+s}^{1+\varphi}}{1+\varphi} \right) \\
& + \lambda_{t+s} \left[W_{t+s}N_{t+s} + Q_{t+s}B_{t+s-1} + M_{t+s-1} + P_{t+s}(TR_{t+s} + PR_{t+s}) \right. \\
& - \left. P_{t+s}C_{t+s} - B_{t+s} - M_{t+s} \right] \Bigg\}
\end{aligned}$$

*Solutions were discussed with Jiahui Shui.

FOCs are as follows:

$$
\begin{aligned}
[C_{t+s}]: &\quad (1-\theta) X_{t+s}^{\nu-\gamma} C_{t+s}^{-\nu} = P_{t+s}\lambda_{t+s} \\
[N_{t+s}]: &\quad \chi N_{t+s}^{\varphi} = \lambda_{t+s} W_{t+s} \\
[B_{t+s}]: &\quad \lambda_{t+s} = \beta \mathbb{E}_t \left[\lambda_{t+s+1} Q_{t+s}\right] \\
[N_{t+s}]: &\quad X_{t+s}^{\nu-\gamma}\theta \left(\frac{M_{t+s}}{P_{t+s}}\right)^{-\nu} \frac{1}{P_{t+s}} = \beta \mathbb{E}_t \left[\lambda_{t+s+1}\right]
\end{aligned}
$$

Combining these FOCs, we can solve for

$$
1 = \mathbb{E}_t \left[\beta Q_{t+s} \left(\frac{P_{t+s}}{P_{t+s+1}}\right) \left(\frac{X_{t+s+1}}{X_{t+s}}\right)^{\nu-\gamma} \left(\frac{C_{t+s+1}}{C_{t+s}}\right)^{-\theta}\right] \quad \text{(Euler's equation)} \tag{1}
$$

$$
\frac{W_{t+s}}{P_{t+s}} = \chi \frac{N_{t+s}^{\varphi}}{X_{t+s}^{\nu-\gamma} C_{t+s}^{\theta}} \quad \text{(Labor-leisure condition)} \tag{2}
$$

$$
1 = \beta \mathbb{E}_t \left[\left(\frac{X_{t+s+1}}{X_{t+s}}\right)^{\nu-\gamma} \left(\frac{C_{t+s+1}}{C_{t+s}}\right)^{-\nu} \frac{P_{t+s}}{P_{t+s+1}}\right] + \frac{\theta}{1-\theta} \left(\frac{M_{t+s}}{P_{t+s}}\right)^{-\nu} \left(\frac{1}{C_{t+s}}\right)^{-\nu} \tag{3}
$$

$$
\frac{M_{t+s}}{P_{t+s}} = \left(1 - \frac{1}{Q_{t+s}}\right)^{-\frac{1}{\nu}} \left(1 - \frac{\theta}{1-\theta}\right)^{-\frac{1}{\nu}} C_{t+s} \quad \text{(Money demand)} \tag{4}
$$

(b)

The firm's problem is:

$$
\begin{aligned}
\max_{N_{t+s}} &\quad PR_{t+s} = Y_{t+s} - \frac{W_{t+s}}{P_{t+s}} N_{t+s} \\
\text{where} &\quad Y_{t+s} = A_{t+s} N_{t+s}
\end{aligned}
$$

FOC with respect to N_{t+s} yields:

$$
A_{t+s} = \frac{W_{t+s}}{P_{t+s}}
$$

Government's budget constraint in real terms can be expressed as

$$
\frac{B_{t+s}}{P_{t+s}} + \frac{M_{t+s}}{P_{t+s}} = TR_{t+s} + \frac{Q_{t+s-1} - B_{t+s-1}}{P_{t+s}} + \frac{M_{t+s-1}}{P_{t+s}}
$$

Combining labor demand with labor supply, we have

$$\frac{W_{t+s}}{P_{t+s}} = A_{t+s} = \frac{\chi N_{t+s}^{\varphi}}{(X_{t+s})^{\nu-\gamma}(1+\theta)C_{t+s}^{-\nu}}$$

when labor market clears.

Notice that X_{t+s} is a function of M_{t+s}, so money is neutral if $(X_{t+s})^{\nu-\gamma} = 1 \iff \nu = \gamma$.

(c)

Assume $A = 1$. In steady state, $\frac{W}{P} = A = 1 = \frac{\chi N^{\varphi}}{X^{\nu-\gamma}(1-\theta)C^{-\nu}}$ by labor market clearing condition, which implies that

$$(1-\theta)X^{\nu-\gamma} = \chi N^{\varphi} C^{\nu} \tag{5}$$

Plug the output market clearing condition that $Y = N = C$ back in the equation (5), we can derive that

$$(1-\theta)X^{\nu-\gamma} = \chi C^{\varphi+\nu} \tag{6}$$

In steady state (SS), the gross real interest rate is a constant $R = R_{t+s} \forall s$. Combining it with Euler's equation (1), $1 = \beta R \Rightarrow R = \frac{1}{\beta}$.

We also have constant gross nominal interest rate $Q = Q_{t+s} \forall s$, so the inflation rate $\pi_{t+s} = \frac{P_{t+s+1}}{P_{t+s}} = Q\beta = \pi$ is also a constant.

Then the money demand can be rewritten as:

$$\frac{M}{P} = (1 - \frac{\beta}{\pi})^{-\frac{1}{\gamma}} \left(\frac{1-\theta}{\theta}\right)^{-\frac{1}{\gamma}} C \tag{7}$$

$$\begin{aligned} X &= \left\{(1-\theta)C^{1-\nu} + \theta\left(1-\frac{\beta}{\pi}\right)^{-\frac{\nu-1}{\nu}}\left(\frac{1-\theta}{\theta}\right)^{-\frac{\nu-1}{\nu}} C^{1-\nu}\right\}^{\frac{1}{1-\nu}} \\ &= \left\{(1-\theta) + \theta\left(1-\frac{\beta}{\pi}\right)^{-\frac{\nu-1}{\nu}}\left(\frac{1-\theta}{\theta}\right)^{-\frac{\nu-1}{\nu}}\right\}^{\frac{1}{1-\nu}} C \end{aligned} \tag{8}$$

Together with (6),

$$C_{ss}=\left\{\frac{1-\theta}{\chi}\left[(1-\theta)+\theta\left(1-\frac{\beta}{\pi}\right)^{\frac{\nu-1}{\nu}}\left(\frac{1-\theta}{\theta}\right)^{\frac{\nu-1}{\nu}}\right]^{\frac{\nu-\gamma}{\nu-1}}\right\}^{\frac{1}{\varphi+\gamma}}. \tag{9}$$

which is expressed in terms of model parameters.

And we can further express $Y_{ss}=N_{ss}=C_{ss}$ and $\frac{W_{ss}}{P_{ss}}$ as functions of C_{ss} and model parameters.

(d)

The algorithm takes Q and M_t as given. After specifying the model parameters, the algorithm can solve for $\pi=Q\beta$ and C_{ss} using equation (9). Then Y_{ss} and N_{ss} are identifiable since they are functions of C_{ss} and model parameters. Also, because we treat M_t as exogenous, by steady state money demand $\frac{M_t}{P_t}=\frac{M}{P}$, the price level P_t can be pinned down.

(e)

Suppose we are given the model parameters, by steady state expressions of $\frac{M_{ss}}{P_{ss}}$ and C_{ss}, we have

$$\frac{M_{ss}}{P_{ss}}=\left(1-\frac{\beta}{\pi}\right)^{-\frac{1}{\nu}}\left(\frac{1-\theta}{\theta}\right)^{-\frac{1}{\nu}}\left\{\frac{1-\theta}{\chi}\left[(1-\theta)+\theta\left(1-\frac{\beta}{\pi}\right)^{\frac{\nu-1}{\nu}}\left(\frac{1-\theta}{\theta}\right)^{\frac{\nu-1}{\nu}}\right]^{\frac{\nu-\gamma}{\nu-1}}\right\}^{\frac{1}{\varphi+\gamma}}$$

Using the equation above, we can calibrate θ given ν.

(f)

Once we have $P=1$ in steady state,

$$M_{ss}=\left(1-\frac{\beta}{\pi}\right)^{-\frac{1}{\nu}}\left(\frac{1-\theta}{\theta}\right)^{-\frac{1}{\nu}}\left\{\frac{1-\theta}{\chi}\left[(1-\theta)+\theta\left(1-\frac{\beta}{\pi}\right)^{\frac{\nu-1}{\nu}}\left(\frac{1-\theta}{\theta}\right)^{\frac{\nu-1}{\nu}}\right]^{\frac{\nu-\gamma}{\nu-1}}\right\}^{\frac{1}{\varphi+\gamma}}$$

conditional on we know other parameters.

(g)

Equilibrium equations of this system, with slight modifications on time subscripts, are

$$
\begin{aligned}
Y_t &= A_t N_t \\
\frac{W_t}{P_t} &= A_t \\
\frac{W_t}{P_t} &= \frac{\chi N_t^{\varphi}}{X_t^{\nu-\gamma}(1-\theta)C_t^{-\nu}} \\
X_t &= \left[(1-\theta)C_t^{1-\nu} + \theta\left(\frac{M_t}{P_t}\right)^{1-\nu}\right]^{\frac{1}{1-\nu}}, \\
Y_t &= C_t, \\
1 &= \mathbb{E}_t\left[\beta Q_t \frac{P_t}{P_{t+1}}\left(\frac{X_{t+1}}{X_t}\right)^{\nu-\gamma}\left(\frac{C_{t+1}}{C_t}\right)^{-\nu}\right]
\end{aligned}
$$

Define log-deviations around steady state as $\hat{z}_t = \log \frac{Z_t}{Z_{ss}}$. Then the log-linearization of the system is

$$
\begin{aligned}
\hat{y}_t &= \hat{a}_t + \hat{n}_t \\
\hat{w}_t - \hat{p}_t &= \hat{a}_t \\
\hat{w}_t - \hat{p}_t &= \varphi\hat{n}_t - (\nu-\gamma)\hat{x}_t + \nu\hat{c}_t \\
\hat{x}_t &= \frac{(1-\theta)C_{ss}^{1-\nu}}{X_{ss}}\hat{c}_t + \frac{\theta\frac{M_{ss}}{P_{ss}}^{1-\nu}}{X_{ss}}(\hat{m}_t - \hat{p}_t) \\
\hat{y}_t &= \hat{c}_t \\
0 &= \mathbb{E}_t\left[\hat{q}_t + \hat{p}_t - \hat{p}_{t+1} + (\nu-\gamma)(\hat{x}_{t+1} - \hat{x}_t) - \nu(\hat{c}_{t+1} - \hat{c}_t)\right]
\end{aligned}
$$

where steady state values are solved in previous questions.

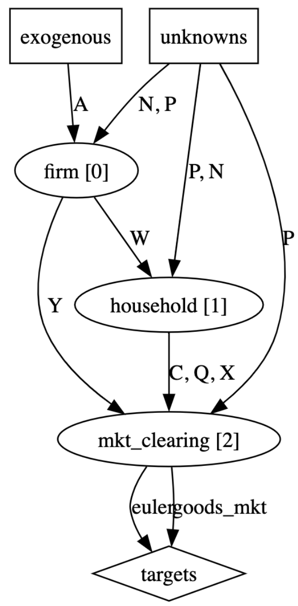

Figure 1: DAG

(h)

Figure 1 represents the DAG. Figure 2 shows the impulse response functions of consumption, prices and norminal interest rate.

(i)

ν is the reciprocal of the elasticity of substitution between consumption and real balances. An increase in ν decreases the extent of an individual's trade-off between C_t and $\frac{M_t}{P_t}$.

When $\nu < 1$, the exponents of consumption and real balances in X_t is positive, indicating that an increase in money supply decreases the marginal utility of consumption and labor supply. As a result, production and consumption diminish. When money supply goes up, the prices also go up. To support a hightened money demand, the interest rate declines.

When $\nu = 1$, an increase in money supply does not change the marginal utility of consumption, so the household maintains their labor supply and production and consumption remain unchanged.

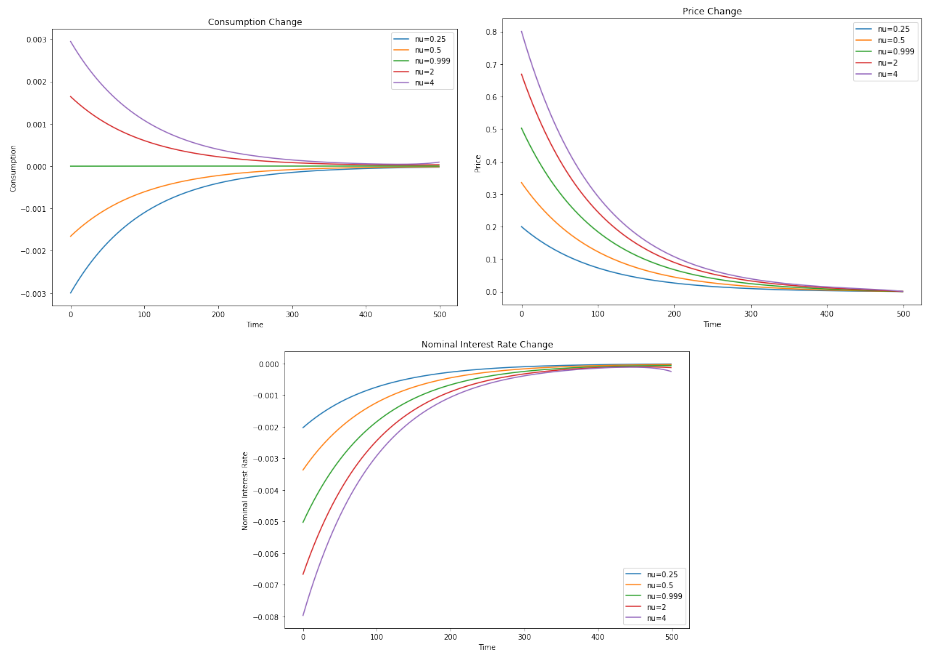

Figure 2: Impulse responses for C_t, P_t and Q_t

When $\nu > 1$, the exponents of consumption and real balances in X_t is negative. So, an increase in money supply leads to an increase in the marginal utility of consumption, compelling the household to supply more labor and subsequently boosts production and consumption. A higher ν also corresponds to an intertemporal substitution from money to consumption, resulting in higher prices and interest rates.

(j)

Based on discussions in part (i), we need $\nu > 1$ to support the evidence that an increase in the money supply increases consumption.

(k)

Please refer to `hw1_sequence_space.py`.