Solving Models in Sequence Space
Economics 210C

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Outline

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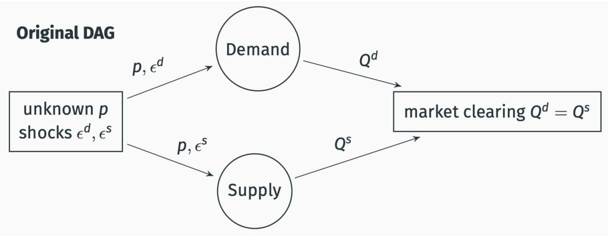
Introduction

- Today will be heavy.
- We will learn how to solve models in sequence space.
- Basic idea: organize models into "blocks" that represent behavior of (possibly heterogeneous) agents, and interact in GE via a small set of aggregates.
- We will arrange these blocks into Directed Acyclic Graph ("DAG"). Helpful to solve model, think about causality in GE, do decompositions, etc.



Outline

Equilibrium Definition

An equilibrium is an allocation $\{C_{t+s}, N_{t+s}, Y_{t+s}, B_{t+s}\}_{s=0}^{\infty}$, a set of prices $\{W_{t+s}, P_{t+s}, Q_{t+s}\}_{s=0}^{\infty}$, an exogenous processes $\{A_{t+s}, Tr_{t+s}, M_{t+s}\}_{s=0}^{\infty}$ and initial conditions for bonds and capital B_{t-1} such that:

1. Households maximize utility subject to budget constraints.
2. Firms maximize profits given their technology.
3. The government satisfies its budget constraint.
4. Markets clear:
    - 3.1 Labor demanded equals labor supplied.
    - 3.2 Bond issuance by the government equals bond holding by households.
    - 3.3 Money issuance by the government equals money holdings by households.
    - 3.4 Output equals consumption plus investment.

Equilibrium Equations

$$
\begin{aligned}
Y_t &= A_t N_t \\
\frac{W_t}{P_t} &= A_t \\
\frac{W_t}{P_t} &= \frac{\chi N_t^{\varphi}}{C_t^{-\gamma}} \\
Y_t &= C_t \\
1 &= \beta E_t \left\{ R_{t+1} \frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}} \right\}
\end{aligned}
$$

LINEARIZED EQUILIBRIUM EQUATIONS

- Notation: $\hat{x} = \frac{X_t - \bar{X}}{\bar{X}} \approx \ln \frac{X_t}{\bar{X}}$

$$\begin{aligned}
\hat{y}_t &= \hat{a}_t + \hat{n}_t \\
\hat{w}_t - \hat{p}_t &= \hat{a}_t \\
\hat{w}_t - \hat{p}_t &= \varphi \hat{n}_t + \gamma \hat{c}_t \\
\hat{y}_t &= \hat{c}_t \\
0 &= E_t \left\{ \hat{r}_{t+1} - \gamma(\hat{c}_{t+1} - \hat{c}_t) \right\}
\end{aligned}$$

General Sequence Space Representation

- Equilibrium is a solution to an equation

$$\mathbf{H}(\mathbf{U},\mathbf{Z}) = 0$$

where

  - $\mathbf{U}$ represents the time path $U_0, U_1, ...,$ of unknown aggregate sequences (e.g., quantities, prices).
  - $\mathbf{Z}$ represents the time path $Z_0, Z_1, ...,$ of known exogenous shocks.

- Totally differentiate and evaluate at steady state to get:

$$d\mathbf{U} = -\mathbf{H}_{\mathbf{U}}(\bar{\mathbf{U}},\bar{\mathbf{Z}})^{-1}\mathbf{H}_{\mathbf{Z}}(\bar{\mathbf{U}},\bar{\mathbf{Z}})d\mathbf{Z}$$

- Solution requires finding the sequence-space Jacobians $\mathbf{H}_{\mathbf{U}}$ and $\mathbf{H}_{\mathbf{Z}}$.

Outline

Solving in Sequence Space

- Organize the model in blocks.

1. Firm block:

$$\hat{y}_t = \hat{a}_t + \hat{n}_t$$
$$\hat{w}_t - \hat{p}_t = \hat{a}_t$$

2. Household block:

$$\hat{w}_t - \hat{p}_t = \varphi \hat{n}_t + \gamma \hat{c}_t$$

3. Market clearing block:

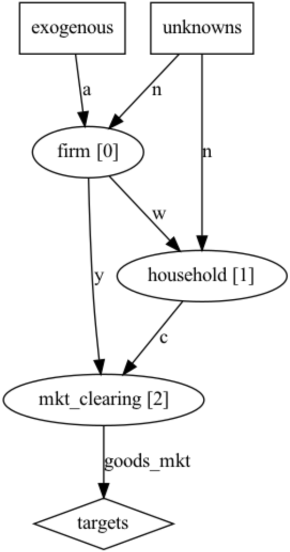
$$0 = \hat{c}_t - \hat{y}_t$$

Sequence Space Algorithm Heuristics

1. Start with an initial guess of the sequences $(\hat{\mathbf{n}}) = \{\hat{n}_t\}_{t=0}^{\infty}$.
   - $\hat{\mathbf{a}} = \{a_t\}_{t=0}^{\infty}$ are given to us.
2. Solve the firm block for $\hat{\mathbf{y}}$ and $\hat{\mathbf{w}} - \hat{\mathbf{p}}$.
3. Solve the household block for $\hat{\mathbf{c}}$.
4. Check that market clears. If not update guess $\hat{\mathbf{n}}$
   - In turns out we do not need to guess, but can solve the entire system with linear algebra in one step.
   - Approach follows equilibrium definition: find sequences such that everyone optimizes and markets clear.

DAG Representation

- Effectively what we have done is organized our model in a Directed Acyclical Graph (DAG).



DAG Rules

- There are no cycles in a DAG—we travel in one direction only.
- A model has many DAG representations.
  - One representation is to treat every endogenous variable as unknown and have single block.
  - Another representation is to treat each equation as an individual block.
  - These are often not the most useful representation.

DAG Suggestions

- A good DAG minimizes the number of unknowns.
- Generally useful to organize blocks by agent: household, firm, union, government, market clearing.
- Blocks make updating the model easy. Often we change just one problem (e.g., firm for NK model) and leave others untouched.
- Logical check for each block: are the number of variables to solve for equal to the number of equations?
- The computer can organize our model in a DAG and substitute for us.
- Today we will see what the computer does under the hood.

Outline

Solving in Sequence Space

- We want to clear markets at all points in time:

$$\mathbf{H} = \begin{pmatrix} \hat{c}_0 - \hat{y}_0 \\ \vdots \\ \hat{c}_T - \hat{y}_T \end{pmatrix} = \left(\Phi_{gm,c}\hat{\mathbf{c}} + \Phi_{gm,y}\hat{\mathbf{y}}\right) = \mathbf{0}$$

  where

$$\Phi_{gm,c} = I_T$$
$$\Phi_{gm,y} = -I_T$$

  and I_T is the $T \times T$ identity matrix.

- How does adjusting the sequences $\mathbf{U} = (\hat{\mathbf{n}})$ change the target?

$$\mathbf{H}_{\mathbf{U}} = \left(\Phi_{gm,c}\frac{\partial \hat{\mathbf{c}}}{\partial \hat{\mathbf{n}}} + \Phi_{gm,y}\frac{\partial \hat{\mathbf{y}}}{\partial \hat{\mathbf{n}}}\right)$$

Solving in Sequence Space

- A simpler and equivalent expression to work with is

$$\mathbf{H_U} = \begin{pmatrix} \Phi_{gm,c} & \Phi_{gm,y} & \mathbf{0}_T \end{pmatrix} \begin{pmatrix} \frac{\partial \hat{\mathbf{c}}}{\partial \hat{\mathbf{n}}} \\ \frac{\partial \hat{\mathbf{y}}}{\partial \hat{\mathbf{n}}} \\ \frac{\partial \hat{\mathbf{w}} - \hat{\mathbf{p}}}{\partial \hat{\mathbf{n}}} \end{pmatrix} \equiv \frac{\partial \mathbf{H}}{\partial \mathbf{Y}} \frac{\partial \mathbf{Y}}{\partial \mathbf{U}}$$

- Now we move along using the chain rule:

$$\frac{\partial \mathbf{Y}}{\partial \mathbf{U}} = \begin{pmatrix} \frac{\partial \hat{\mathbf{c}}}{\partial \mathbf{U}} \\ \frac{\partial (\hat{\mathbf{y}}, (\hat{\mathbf{w}} - \hat{\mathbf{p}}))}{\partial \mathbf{U}} \end{pmatrix}$$

where we partitioned based on the two blocks we looked at earlier.

Solving in Sequence Space

- We start with the firm block:

$$\hat{y}_t = \hat{a}_t + \hat{n}_t$$
$$\hat{w}_t - \hat{p}_t = \hat{a}_t$$

- In matrix notation:

$$\hat{\mathbf{y}} = \Phi_{y,a}\hat{\mathbf{a}} + \Phi_{y,n}\hat{\mathbf{n}}$$
$$\hat{\mathbf{w}} - \hat{\mathbf{p}} = \Phi_{wp,a}\hat{\mathbf{a}}$$

- The matrices are:

$$\Phi_{y,a} = I_T$$
$$\Phi_{y,n} = I_T$$
$$\Phi_{wp,a} = I_T$$

Solving in Sequence Space

- With $\hat{\mathbf{w}} - \hat{\mathbf{p}}$ and $\hat{\mathbf{n}}$ we solve for $\hat{\mathbf{c}}$ using the household FOC for labor supply:

$$\hat{c}_t = \gamma^{-1}(\hat{w}_t - \hat{p}_t) - \gamma^{-1}\varphi \hat{n}_t$$

- In matrix notation:

$$\hat{\mathbf{c}} = \Phi_{c,wp}(\hat{\mathbf{w}} - \hat{\mathbf{p}}) + \Phi_{c,n}\hat{\mathbf{n}}$$

- The matrices are:

$$\begin{aligned} \Phi_{c,wp} &= \gamma^{-1} I_T \\ \Phi_{c,n} &= -\gamma^{-1}\varphi I_T \end{aligned}$$

Solving in Sequence Space

- From the firm block we have:

$$\frac{\partial(\hat{\mathbf{y}},(\hat{\mathbf{w}}-\hat{\mathbf{p}}))}{\partial \mathbf{U}} = \begin{pmatrix} \Phi_{y,n} \\ \Phi_{wp,n} \end{pmatrix}$$

- From the household block we have:

$$\begin{aligned} \frac{\partial \hat{\mathbf{c}}}{\partial \mathbf{U}} &= \left(\Phi_{c,n} + \Phi_{c,wp} \frac{d\hat{\mathbf{w}}-\hat{\mathbf{p}}}{\partial \hat{\mathbf{n}}}\right) \\ &= \left(\Phi_{c,n}\right) \end{aligned}$$

- We solved for $\mathbf{H_U}$:

$$\mathbf{H_U} = \begin{pmatrix} \Phi_{gm,c} & -I_T & \mathbf{0}_T \end{pmatrix} \times \begin{pmatrix} \Phi_{c,n} \\ \Phi_{y,n} \\ \Phi_{wp,n} \end{pmatrix} = \begin{pmatrix} \Phi_{gm,c}\Phi_{c,n} - \Phi_{y,n} \end{pmatrix}$$

Solving in Sequence Space

- Solving for $\mathbf{H_Z}$ is a bit more straightforward:

$$\mathbf{H_Z} = \frac{\partial \mathbf{H}}{\partial \mathbf{Y}} \frac{\partial \mathbf{Y}}{\partial \mathbf{Z}}$$

  We already know the first derivative.

- From the firm block we have:

$$\frac{\partial(\hat{\mathbf{y}}, (\hat{\mathbf{w}} - \hat{\mathbf{p}}))}{\partial \mathbf{Z}} = \begin{pmatrix} I_T \\ I_T \end{pmatrix}$$

- From the household block we have:

$$\frac{\partial \hat{\mathbf{c}}}{\partial \mathbf{Z}} = \left(\Phi_{c,wp} \frac{\partial \hat{\mathbf{w}} - \hat{\mathbf{p}}}{\partial \hat{\mathbf{a}}} \right) = \left(\Phi_{c,wp} \right)$$

Solving in Sequence Space

- We solved for $\mathbf{H_Z}$:

$$\mathbf{H_Z} = \begin{pmatrix} \Phi_{gm,c} & -I_T & \mathbf{0}_T \end{pmatrix} \times \begin{pmatrix} \Phi_{c,wp} \\ I_T \\ I_T \end{pmatrix} = \begin{pmatrix} \Phi_{gm,c}\Phi_{c,wp} - I_T \end{pmatrix}$$

- We now have the solution to the model:

$$d\mathbf{U} = -\mathbf{H_U}^{-1}\mathbf{H_Z}\,d\mathbf{Z}$$

and calculate the remaining sequences

$$d\mathbf{Y} = \frac{\partial \mathbf{Y}}{\partial \mathbf{U}} d\mathbf{U} + \frac{\partial \mathbf{Y}}{\partial \mathbf{Z}} d\mathbf{Z} = \left(\frac{\partial \mathbf{Y}}{\partial \mathbf{U}}\mathbf{H_U}^{-1}\mathbf{H_Z} + \frac{\partial \mathbf{Y}}{\partial \mathbf{Z}}\right) d\mathbf{Z}$$

Solving in Sequence Space

- If you plug in:

$$\begin{aligned} d\mathbf{U} &= -\mathbf{H}_{\mathbf{U}}^{-1}\mathbf{H}_{\mathbf{Z}} d\mathbf{Z} \\ &= -\left(\Phi_{gm,c}\Phi_{c,n} - \Phi_{y,n}\right)^{-1}\left(\Phi_{gm,wp}\Phi_{c,wp} - I_T\right) d\mathbf{Z} \\ &= (1+\gamma^{-1}\varphi)^{-1}(\gamma^{-1}-1)\mathbf{Z} \\ &= (\gamma+\varphi)^{-1}(1-\gamma)\mathbf{Z} \\ &= \hat{\mathbf{n}} \end{aligned}$$

- Then solve for the other sequences.

Lessons

- Can solve any linearized dynamic model using linear algebra.
- Extremely fast once matrices are created.
  - If we tell the computer that the matrices are sparse (mostly 0s).
- Replaced substitution with matrix multiplication.
- Strategy follows equilibrium definition: looking for sequence such that everyone optimizes and markets clear.
- Everyone should do this once by hand. Then let the computer do the work for you.

Outline

More Sequence Space

- Our model had a simple static solution, so we did not need the sequence space.
- We now add capital to the model, which makes the solution dynamic.
- We will see how to use the sequence space in this more complex environment.

Capital

- The production function:

$$Y_t = A_t N_t^{1-\alpha} K_{t-1}^{\alpha}$$

- Capital depreciates at rate δ

$$K_t = (1-\delta)K_{t-1} + I_t$$

- The real rental rate of capital is the marginal product of labor.

$$R_t^k = \alpha A_t N_t^{-\alpha} K_{t-1}^{\alpha}$$

- The household has to be indifferent between investing in a bond or capital

$$R_{t+1} = R_{t+1}^k + 1 - \delta$$

FOC WITH CAPITAL

$$
\begin{aligned}
Y_t &= A_t N_t^{1-\alpha} K_{t-1}^{\alpha} \\
R_t^k &= \alpha A_t N_t^{-\alpha} K_{t-1}^{\alpha} \\
\frac{W_t}{P_t} &= (1-\alpha) A_t N_t^{-\alpha} K_{t-1}^{\alpha} \\
\frac{W_t}{P_t} &= \frac{\chi N_t^{\varphi}}{C_t^{-\gamma}} \\
Y_t &= C_t + I_t \\
1 &= \beta R_{t+1} \frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}} \\
1 &= \beta (R_{t+1}^k + 1 - \delta) \frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}} \\
K_t &= (1-\delta) K_{t-1} + I_t
\end{aligned}
$$

Linearized FOC with Capital

$$
\begin{aligned}
\hat{y}_t &= \hat{a}_t + (1-\alpha)\hat{n}_t + \alpha\hat{k}_{t-1} \\
\hat{r}_t^k &= \hat{a}_t + (1-\alpha)\hat{n}_t + (\alpha-1)\hat{k}_{t-1} \\
\hat{w}_t - \hat{p}_t &= \hat{a}_t - \alpha\hat{n}_t + \alpha\hat{k}_{t-1} \\
\hat{w}_t - \hat{p}_t &= \varphi\hat{n}_t + \gamma\hat{c}_t \\
\hat{y}_t &= s_c\hat{c}_t + (1-s_c)\hat{\imath}_t \\
0 &= \hat{r}_{t+1} - \gamma(\hat{c}_{t+1} - \hat{c}_t) \\
0 &= (1-\beta(1-\delta))\hat{r}_{t+1}^k - \gamma(\hat{c}_{t+1} - \hat{c}_t) \\
\hat{k}_t &= (1-\delta)\hat{k}_{t-1} + \delta\hat{\imath}_t
\end{aligned}
$$

OUTLINE

Solving in Sequence Space

- Organize the model in blocks.
  1. Firm block:

$$\begin{aligned} \hat{y}_t &= \hat{a}_t + (1-\alpha)\hat{n}_t + \alpha \hat{k}_{t-1} \\ \hat{r}^k_t &= \hat{a}_t + (1-\alpha)\hat{n}_t + (\alpha - 1)\hat{k}_{t-1} \\ \hat{w}_t - \hat{p}_t &= \hat{a}_t - \alpha \hat{n}_t + \alpha \hat{k}_{t-1} \end{aligned}$$

  2. Household block:

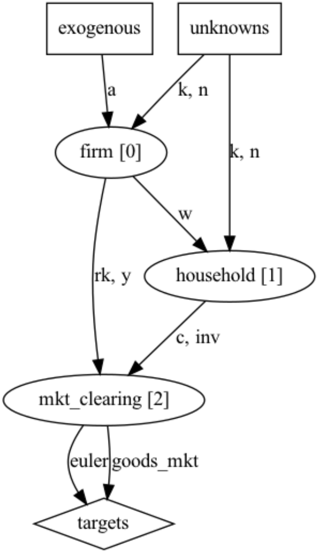
$$\begin{aligned} \hat{w}_t - \hat{p}_t &= \varphi \hat{n}_t + \gamma \hat{c}_t \\ \hat{k}_t &= (1-\delta)\hat{k}_{t-1} + \delta \hat{\imath}_t \end{aligned}$$

  3. Market clearing block:

$$\begin{aligned} 0 &= s_c \hat{c}_t + (1-s_c)\hat{\imath}_t - \hat{y}_t \\ 0 &= (1-\beta(1-\delta))\hat{r}^k_{t+1} - \gamma(\hat{c}_{t+1} - \hat{c}_t) \end{aligned}$$

DAG Representation

- We have a new Directed Acyclical Graph (DAG) for our model with capital.



Solving in Sequence Space

- We start with the market clearing block:

$$0 = s_c \hat{c}_t + (1 - s_c)\hat{\imath}_t - \hat{y}_t$$
$$0 = (1 - \beta(1 - \delta))\hat{r}^k_{t+1} - \gamma(\hat{c}_{t+1} - \hat{c}_t)$$

- In matrix notation:

$$\mathbf{0} = \Phi_{gm,c}\hat{\mathbf{c}} + \Phi_{gm,\imath}\hat{\imath} - \hat{\mathbf{y}}$$
$$\mathbf{0} = \Phi_{eul,rk}\hat{\mathbf{r}}^{\mathbf{k}} + \Phi_{eul,c}\hat{\mathbf{c}}$$

- The matrices are:

$$\Phi_{gm,c} = s_c I_T \qquad \Phi_{gm,\imath} = (1 - s_c) I_T$$
$$\Phi_{eul,rk} = (1 - \beta(1 - \delta)) I_T \qquad \Phi_{eul,c} = -\gamma I_T$$

- I_T is the $T \times T$ identity matrix.
- Note timing of $\hat{\mathbf{r}}^{\mathbf{k}} = \{\hat{r}^k_{t+1}, ..., \hat{r}^k_{T+1}\}$ sequence.

Solving in Sequence Space

- We want to clear markets at all points in time:

$$\mathbf{H} = \begin{pmatrix} s_c \hat{c}_0 + (1-s_c)\hat{\imath}_0 - \hat{y}_0 \\ \vdots \\ s_c \hat{c}_T + (1-s_c)\hat{\imath}_T - \hat{y}_T \\ \hat{r}^k_1 - \gamma(\hat{c}_{T+1} - \hat{c}_T) \\ \vdots \\ \hat{r}^k_T - \gamma(\hat{c}_T - \hat{c}_{T-1}) \\ \gamma \hat{c}_T \end{pmatrix} = \begin{pmatrix} \Phi_{gm,c}\hat{\mathbf{c}} + \Phi_{gm,\imath}\hat{\boldsymbol{\imath}} - \hat{\mathbf{y}} \\ \Phi_{eul,rk}\hat{\mathbf{r}}^{\mathbf{k}} + \Phi_{eul,c}\hat{\mathbf{c}} \end{pmatrix} = \mathbf{0}$$

- How does adjusting the sequences $\mathbf{U} = (\hat{\mathbf{k}}, \hat{\mathbf{n}})$ change the target?

$$\mathbf{H}_{\mathbf{U}} = \begin{pmatrix} \Phi_{gm,c}\frac{\partial \hat{\mathbf{c}}}{\partial \hat{\mathbf{k}}} + \Phi_{gm,\imath}\frac{\partial \hat{\boldsymbol{\imath}}}{\partial \hat{\mathbf{k}}} - \frac{\partial \hat{\mathbf{y}}}{\partial \hat{\mathbf{k}}} & \Phi_{gm,c}\frac{\partial \hat{\mathbf{c}}}{\partial \hat{\mathbf{n}}} + \Phi_{gm,\imath}\frac{\partial \hat{\boldsymbol{\imath}}}{\partial \hat{\mathbf{n}}} - \frac{\partial \hat{\mathbf{y}}}{\partial \hat{\mathbf{n}}} \\ \Phi_{eul,rk}\frac{\partial \hat{\mathbf{r}}^{\mathbf{k}}}{\partial \hat{\mathbf{k}}} + \Phi_{eul,c}\frac{\partial \hat{\mathbf{c}}}{\partial \hat{\mathbf{k}}} & \Phi_{eul,rk}\frac{\partial \hat{\mathbf{r}}^{\mathbf{k}}}{\partial \hat{\mathbf{n}}} + \Phi_{eul,c}\frac{\partial \hat{\mathbf{c}}}{\partial \hat{\mathbf{n}}} \end{pmatrix}$$

Solving in Sequence Space

- A simpler and equivalent expression to work with is

$$\mathbf{H_U} = \begin{pmatrix} \Phi_{gm,c} & \Phi_{gm,\imath} & -I_T & \mathbf{0}_T & \mathbf{0}_T \\ \Phi_{eul,c} & \mathbf{0}_T & \mathbf{0}_T & \Phi_{eul,rk} & \mathbf{0}_T \end{pmatrix} \begin{pmatrix} \frac{\partial \hat{\mathbf{c}}}{\partial \hat{\mathbf{k}}} & \frac{\partial \hat{\mathbf{c}}}{\partial \hat{\mathbf{n}}} \\ \frac{\partial \hat{\imath}}{\partial \hat{\mathbf{k}}} & \frac{\partial \hat{\imath}}{\partial \hat{\mathbf{n}}} \\ \frac{\partial \hat{\mathbf{y}}}{\partial \hat{\mathbf{k}}} & \frac{\partial \hat{\mathbf{y}}}{\partial \hat{\mathbf{n}}} \\ \frac{\partial \hat{\mathbf{r}}^{\mathbf{k}}}{\partial \hat{\mathbf{k}}} & \frac{\partial \hat{\mathbf{r}}^{\mathbf{k}}}{\partial \hat{\mathbf{n}}} \\ \frac{\partial \hat{\mathbf{w}}-\hat{\mathbf{p}}}{\partial \hat{\mathbf{k}}} & \frac{\partial \hat{\mathbf{w}}-\hat{\mathbf{p}}}{\partial \hat{\mathbf{n}}} \end{pmatrix} \equiv \frac{\partial \mathbf{H}}{\partial \mathbf{Y}} \frac{\partial \mathbf{Y}}{\partial \mathbf{U}}$$

- Now we move along using the chain rule:

$$\frac{\partial \mathbf{Y}}{\partial \mathbf{U}} = \begin{pmatrix} \frac{\partial (\mathbf{c},\imath)}{\partial \mathbf{U}} \\ \frac{\partial (\hat{\mathbf{y}},\hat{\mathbf{r}}^{\mathbf{k}},(\hat{\mathbf{w}}-\hat{\mathbf{p}}))}{\partial \mathbf{U}} \end{pmatrix}$$

where we partitioned based on the two blocks we looked at earlier.

Solving in Sequence Space

- Then from production function we know the firm produces output:

$$\hat{y}_t = \hat{a}_t + (1-\alpha)\hat{n}_t + \alpha \hat{k}_{t-1}$$

- In matrix notation:

$$\hat{\mathbf{y}} = \hat{\mathbf{a}} + \Phi_{y,n}\hat{\mathbf{n}} + \Phi_{y,k}\hat{\mathbf{k}} + \Phi_{y,k_{-1}}\hat{k}_{-1}$$

- The matrices are:

$$\Phi_{y,n} = (1-\alpha)I_T,$$

$$\Phi_{y,k} = \alpha \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \ \Phi_{y,k_{-1}} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

where I_T is the $T \times T$ identity matrix.

Solving in Sequence Space

- From the firm FOC we can also compute the sequence of prices:

$$
\begin{aligned}
\hat{r}^k_{t+1} &= \hat{a}_{t+1} + (1-\alpha)\hat{n}_{t+1} + (\alpha-1)\hat{k}_t \\
\hat{w}_t - \hat{p}_t &= \hat{a}_t - \alpha\hat{n}_t + \alpha\hat{k}_{t-1}
\end{aligned}
$$

- We shift capital return one period forward since that is what we need in the Euler equation.

- In matrix notation:

$$
\begin{aligned}
\hat{\mathbf{r}}^{\mathbf{k}} &= \Phi_{rk,a}\hat{\mathbf{a}} + \Phi_{rk,n}\hat{\mathbf{n}} + \Phi_{rk,k}\hat{\mathbf{k}} \\
\hat{\mathbf{w}} - \hat{\mathbf{p}} &= \Phi_{wp,a}\hat{\mathbf{a}} + \Phi_{wp,n}\hat{\mathbf{n}} + \Phi_{wp,k}\hat{\mathbf{k}} + \Phi_{wp,k_{-1}}\hat{k}_{-1}
\end{aligned}
$$

Solving in Sequence Space

- The matrices are

$$\Phi_{rk,a} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

$$\Phi_{rk,n} = (1-\alpha)\Phi_{rk,a}$$

$$\Phi_{rk,k} = -(1-\alpha)I_T$$

$$\Phi_{wp,a} = I_T$$

$$\Phi_{wp,n} = -\alpha I_T$$

$$\Phi_{wp,k} = \Phi_{y,k}$$

Solving in Sequence Space

- With $\hat{\mathbf{w}} - \hat{\mathbf{p}}$ and $(\hat{\mathbf{k}}, \hat{\mathbf{n}})$ we solve for $\hat{\mathbf{c}}$ and $\hat{\imath}$ using the household FOC for labor supply and the capital accumulation equation.

$$\hat{c}_t = \gamma^{-1}(\hat{w}_t - \hat{p}_t) - \gamma^{-1}\varphi \hat{n}_t$$
$$\hat{k}_t = (1-\delta)\hat{k}_{t-1} + \delta \hat{\imath}_t$$

- In matrix notation:

$$\hat{\mathbf{c}} = \Phi_{c,wp}(\hat{\mathbf{w}} - \hat{\mathbf{p}}) + \Phi_{c,n}\hat{\mathbf{n}}$$
$$\hat{\imath} = \Phi_{\imath,k}\hat{\mathbf{k}} + \Phi_{\imath,k_{-1}}\hat{k}_{-1}$$

- The matrices are $\Phi_{c,wp} = \gamma^{-1} I_T$, $\Phi_{c,n} = -\gamma^{-1}\varphi I_T$ and

$$\Phi_{\imath,k} = \frac{1}{\delta}\begin{pmatrix} 1 & 0 & \dots & 0 \\ \delta - 1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & \delta - 1 & 1 \end{pmatrix}$$

- From the firm block we have:

$$\frac{\partial(\hat{\mathbf{y}}, \hat{\mathbf{r}}^{\mathbf{k}}, (\hat{\mathbf{w}} - \hat{\mathbf{p}}))}{\partial \mathbf{U}} = \begin{pmatrix} \Phi_{y,k} & \Phi_{y,n} \\ \Phi_{rk,k} & \Phi_{rk,n} \\ \Phi_{wp,k} & \Phi_{wp,n} \end{pmatrix}$$

- From the household block we have:

$$\frac{\partial(\hat{\mathbf{c}}, \hat{\imath})}{\partial \mathbf{U}} = \begin{pmatrix} \Phi_{c,wp} \frac{\partial \hat{\mathbf{w}} - \hat{\mathbf{p}}}{\partial \hat{\mathbf{k}}} & \Phi_{c,n} + \Phi_{c,wp} \frac{d \hat{\mathbf{w}} - \hat{\mathbf{p}}}{\partial \hat{\mathbf{n}}} \\ \Phi_{\imath,k} & \mathbf{0} \end{pmatrix}$$

$$= \begin{pmatrix} \Phi_{c,wp} \Phi_{wp,k} & \Phi_{c,n} + \Phi_{c,wp} \Phi_{wp,n} \\ \Phi_{\imath,k} & \mathbf{0} \end{pmatrix}$$

Solving in Sequence Space

- We solved for $\mathbf{H_U}$:

$$\mathbf{H_U} = \begin{pmatrix} \Phi_{gm,c} & \Phi_{gm,\imath} & -I_T & \mathbf{0}_T & \mathbf{0}_T \\ \Phi_{eul,c} & \mathbf{0}_T & \mathbf{0}_T & \Phi_{eul,rk} & \mathbf{0}_T \end{pmatrix} \times$$

$$\times \begin{pmatrix} \Phi_{c,wp}\Phi_{wp,k} & \Phi_{c,n} + \Phi_{c,wp}\Phi_{wp,n} \\ \Phi_{\imath,k} & \mathbf{0} \\ \Phi_{y,k} & \Phi_{y,n} \\ \Phi_{rk,k} & \Phi_{rk,n} \\ \Phi_{wp,k} & \Phi_{wp,n} \end{pmatrix}$$

Solving in Sequence Space

- Solving for $\mathbf{H_Z}$ is a bit more straightforward:

$$\mathbf{H_Z} = \frac{\partial \mathbf{H}}{\partial \mathbf{Y}} \frac{\partial \mathbf{Y}}{\partial \mathbf{Z}}$$

  We already know the first derivative.

- From the firm block we have:

$$\frac{\partial(\hat{\mathbf{y}}, \hat{\mathbf{r}}^{\mathbf{k}}, (\hat{\mathbf{w}} - \hat{\mathbf{p}}))}{\partial \mathbf{Z}} = \begin{pmatrix} I_T \\ \Phi_{rk,a} \\ I_T \end{pmatrix}$$

- From the household block we have:

$$\frac{\partial(\hat{\mathbf{c}}, \hat{\imath})}{\partial \mathbf{U}} = \begin{pmatrix} \Phi_{c,wp} \frac{\partial \hat{\mathbf{w}} - \hat{\mathbf{p}}}{\partial \hat{\mathbf{a}}} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \Phi_{c,wp} \\ \mathbf{0} \end{pmatrix}$$

Solving in Sequence Space

- We solved for $\mathbf{H_Z}$:

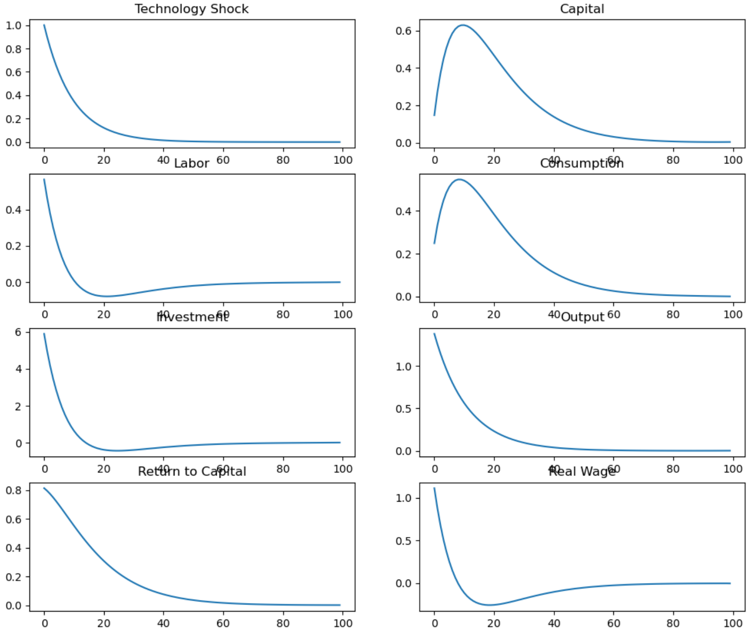
$$\mathbf{H_Z} = \begin{pmatrix} \Phi_{0,c} & \Phi_{0,\imath} & -I_T & \mathbf{0}_T & \mathbf{0}_T \\ \Phi_{0,c} & \mathbf{0}_T & \mathbf{0}_T & \Phi_{0,rk} & \mathbf{0}_T \end{pmatrix} \times \begin{pmatrix} \Phi_{c,wp} \\ \mathbf{0} \\ I_T \\ \Phi_{rk,a} \\ I_T \end{pmatrix}$$

- We now have the solution to the model:

$$d\mathbf{U} = \mathbf{H_U}^{-1} \mathbf{H_Z} d\mathbf{Z}$$

and calculate the remaining sequences

$$d\mathbf{Y} = \frac{\partial \mathbf{Y}}{\partial \mathbf{U}} d\mathbf{U} + \frac{\partial \mathbf{Y}}{\partial \mathbf{Z}} d\mathbf{Z} = \left(\frac{\partial \mathbf{Y}}{\partial \mathbf{U}} \mathbf{H_U}^{-1} \mathbf{H_Z} + \frac{\partial \mathbf{Y}}{\partial \mathbf{Z}}\right) d\mathbf{Z}$$



Technology Shock
Capital
Labor
Consumption
Investment
Output
Return to Capital
Real Wage