Problem Set #3

MACS 30250, Dr. Evans
Due Wednesday, May. 20 at 1:30pm

1. **Stochastic i.i.d. cake eating problem (5 points).** Assume a cake eating problem similar to the one in Section 5 of the dynamic programming introduction from last term in which cake size in the current period is W, cake size next period is W', current period consumption is the difference in the cake size $c = W - W'$, and the utility of consumption is a monotonically increasing concave function $u(c)$. Also assume that the individuals preferences fluctuate each period according to some i.i.d. shock ε. The Bellman equation can be written in the following way to incorporate the uncertainty,

$$\begin{aligned} V(W,\varepsilon) &= \max_{W'\in[0,W]} e^{\varepsilon}u(W-W') + \beta E_{\varepsilon'}\left[V(W',\varepsilon')\right] \\ &\text{where} \quad u(W-W') \equiv \frac{(W-W')^{1-\gamma}-1}{1-\gamma} \end{aligned} \tag{1}$$

where E is the unconditional expectations operator over all values in the support of ε. Assume that the preference shock ε can take on 5 possible values $\boldsymbol{\varepsilon}$ that have the following unconditional probability $\boldsymbol{\pi}$ each period.

$$\boldsymbol{\varepsilon} = \begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \varepsilon_4\\ \varepsilon_5\end{bmatrix} = \begin{bmatrix}-1.40\\ -0.55\\ 0.00\\ 0.55\\ 1.40\end{bmatrix} \qquad \boldsymbol{\pi} = \begin{bmatrix}\pi_1\\ \pi_2\\ \pi_3\\ \pi_4\\ \pi_5\end{bmatrix} = \begin{bmatrix}0.1\\ 0.2\\ 0.4\\ 0.2\\ 0.1\end{bmatrix}$$

   (a) Assume that $\beta = 0.9$ and $\gamma = 2.2$. Use value function iteration with interpolation over the expected value of the value function next period $E_{\varepsilon'}V(W',\varepsilon')$ to solve for the equilibrium solution value function $V(W,\varepsilon)$ and policy function $W' = \psi(W,\varepsilon)$ that solve the Bellman equation (1). For your grid, use 30 equally spaced points of cake size W between $W_{min} = 0.1$ and $W_{max} = 10.0$, and use the five points in the ε dimension listed above. [HINT: First integrate out the shock next period ε' just using the five-element stochastic vector $\boldsymbol{\pi}$. Then use a one-dimensional interpolant in the expected W' dimension.]

   (b) Display a plot of your equilibrium value function as five line plots $V(W,\varepsilon_j)$, one for each value of ε_j for $j = 1, 2, ...5$ with the value function $V(W,\varepsilon)$ on the y-axis and with cake size W on the x-axis. Make sure to include a legend to show which line belongs to which preference shock.

   (c) Display a plot of your equilibrium policy function as five line plots $W' = \psi(W,\varepsilon_j)$, one for each value of ε_j for $j = 1, 2, ...5$ with the policy function $\psi(W,\varepsilon)$ on the y-axis and with cake size W on the x-axis. Make sure to include a legend to show which line belongs to which preference shock.

2. **Persistent AR(1) stochastic cake eating problem (5 points).** Assume a cake eating problem similar to the one in Section 6 of the dynamic programming introduction from last term in which cake size in the current period is W, cake size next period is W', current period consumption is the difference in the cake size $c = W - W'$, and the utility of consumption is a monotonically increasing concave function $u(c)$. Also assume that the individuals preferences fluctuate each period according to some persistent AR(1) shock ε. The Bellman equation can be written in the following way to incorporate the uncertainty,

$$
\begin{aligned}
V(W,\varepsilon) = \max_{W'\in[0,W]} \; & e^{\varepsilon} u(W - W') + \beta E_{\varepsilon'|\varepsilon}\left[V(W',\varepsilon')\right] \\
\text{where} \quad u(W - W') \equiv \; & \frac{(W - W')^{1-\gamma} - 1}{1-\gamma}
\end{aligned}
\tag{2}
$$

where E is the conditional expectations operator over all values in the support of ε. Assume that the preference shock ε can take on 5 possible values $\boldsymbol{\varepsilon}$ that have the following Markov transition probability matrix $\boldsymbol{\pi}$ each period.

$$
\boldsymbol{\varepsilon} = \begin{bmatrix}\varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5\end{bmatrix}^T = \begin{bmatrix}-1.40 & -0.55 & 0.00 & 0.55 & 1.40\end{bmatrix}^T
$$

$$
\boldsymbol{\pi} = \pi_{j,k} = Pr(\varepsilon'_k|\varepsilon_j) = \begin{bmatrix}\pi_{1,1} & \pi_{1,2} & \pi_{1,3} & \pi_{1,4} & \pi_{1,5} \\ \pi_{2,1} & \pi_{2,2} & \pi_{2,3} & \pi_{2,4} & \pi_{2,5} \\ \pi_{3,1} & \pi_{3,2} & \pi_{3,3} & \pi_{3,4} & \pi_{3,5} \\ \pi_{4,1} & \pi_{4,2} & \pi_{4,3} & \pi_{4,4} & \pi_{4,5} \\ \pi_{5,1} & \pi_{5,2} & \pi_{5,3} & \pi_{5,4} & \pi_{5,5}\end{bmatrix} = \begin{bmatrix}0.40 & 0.28 & 0.18 & 0.10 & 0.04 \\ 0.20 & 0.40 & 0.20 & 0.13 & 0.07 \\ 0.10 & 0.20 & 0.40 & 0.20 & 0.10 \\ 0.07 & 0.13 & 0.20 & 0.40 & 0.20 \\ 0.04 & 0.10 & 0.18 & 0.28 & 0.40\end{bmatrix}
$$

where $\pi_{j,k}$ represents the probability of having preference shock ε'_k next period given current preference shock ε_j. So the rows of Markov transition matrix $\boldsymbol{\pi}$ represent the different values of ε in the current period, and the columns of $\boldsymbol{\pi}$ represent the different values of ε next period. The values in each row of $\boldsymbol{\pi}$ sum to one.

(a) Use value function iteration with interpolation over the expected value of the value function next period $E_{\varepsilon'|\varepsilon}V(W',\varepsilon')$ to solve for the equilibrium solution value function $V(W,\varepsilon)$ and policy function $W' = \psi(W,\varepsilon)$ that solve the Bellman equation (2). For your grid, use 30 equally spaced points of cake size W between $W_{min} = 0.1$ and $W_{max} = 10.0$, and use the five points in the ε dimension listed above. [HINT: First integrate out the shock next period ε' just using the 5×5 Markov transition matrix $\boldsymbol{\pi}$. In this case you will just use a different row of $\boldsymbol{\pi}$ corresponding to your current value of ε. Then use a one-dimensional interpolant in the expected W' dimension.]

(b) Display a plot of your equilibrium value function as five line plots $V(W,\varepsilon_j)$, one for each value of ε_j for $j = 1, 2, ...5$ with the value function $V(W,\varepsilon)$ on the y-axis and with cake size W on the x-axis. Make sure to include a legend to show which line belongs to which preference shock.

(c) Display a plot of your equilibrium policy function as five line plots $W' = \psi(W, \varepsilon_j)$, one for each value of ε_j for $j = 1, 2, ...5$ with the policy function $\psi(W, \varepsilon)$ on the y-axis and with cake size W on the x-axis. Make sure to include a legend to show which line belongs to which preference shock.