# Chapter 1. Solving Nonlinear Equation

I. Consider the bisection method starting with the initial interval $[1.5, 3.5]$. In the following questions "the interval" refers to the bisection interval whose width changes across different loops.

- What is the width of the interval at the $n$th step?
- What is the supremum of the distance between the root $r$ and the midpoint of the interval?

Initial interval $= [1.5, 3.5]$

$[a_0, b_0] \supset [a_1, b_1] \supset [a_2, b_2] \supset \cdots \supset [a_n, b_n]$

$$
\begin{aligned}
b_n - a_n &= \frac{1}{2}[b_{n-1} - a_{n-1}] \\
&= \frac{1}{2^n}[b_0 - a_0] \\
&= \frac{1}{2^n}(3.5 - 1.5) \\
&= 2^{1-n}
\end{aligned}
$$

$\therefore$ width of the interval at the nth step is $2^{1-n}$

$$
\begin{aligned}
|x_n - r| &\le b_n - a_n \\
&= \frac{1}{2^n}(b_0 - a_0) \\
&= \frac{1}{2^{n-1}}
\end{aligned}
$$

$\therefore$ the supremum of the distance is $\frac{1}{2^n}$.

II. In using the bisection algorithm with its initial interval as $[a_0, b_0]$ with $a_0 > 0$, we want to determine the root with its *relative* error no greater than $\epsilon$. Prove that this goal of accuracy is guaranteed by the following choice of the number of steps,

$$
n \ge \frac{\log(b_0 - a_0) - \log \epsilon - \log a_0}{\log 2} - 1.
$$

Initial interval $= [a_0, b_0]$

$$
\begin{aligned}
b_n - a_n &= \frac{1}{2^n}(b_0 - a_0) \\
\frac{|x_n - r|}{r} &\le \frac{b_n - a_n}{r} \\
&= \frac{1}{2^n r}(b_0 - a_0),
\end{aligned}
$$

as $r > a_0$

$$
\Rightarrow \frac{1}{2^n r}(b_0 - a_0) \le \frac{1}{2^n a_0}(b_0 - a_0) \le \varepsilon
$$

$$
\begin{aligned}
\log(b_0 - a_0) - n\log 2 - \log a_0 &\le \log \varepsilon \\
n \log 2 &\ge \log(b_0 - a_0) - \log \varepsilon - \log a_0 \\
n &\ge \frac{\log(b_0 - a_0) - \log \varepsilon - \log a_0}{\log 2}
\end{aligned}
$$

III. Perform four iterations of Newton's method for the polynomial equation $p(x) = 4x^3 - 2x^2 + 3 = 0$ with the starting point $x_0 = -1$. Use a hand calculator and organize results of the iterations in a table.

$$p(x) = 4x^3 - 2x^2 + 3 = 0, \quad x_0 = -1$$

$$p'(x) = 12x^2 - 4x$$

Newton's method : $x_{n+1} = x_n - \dfrac{p(x_n)}{p'(x_n)}$

| $n$ | $x$ |
| --- | --- |
| 1 | $-0.8125$ |
| 2 | $-0.770804$ |
| 3 | $-0.768832$ |
| 4 | $-0.768828$ |

IV. Consider a variation of Newton's method in which only the derivative at $x_0$ is used,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_0)}.$$

Find $C$ and $s$ such that

$$e_{n+1} = C e_n^s,$$

where $e_n$ is the error of Newton's method at step $n$, $s$ is a constant, and $C$ may depend on $x_n$, the given function $f$ and its derivatives.

$$\begin{aligned} e_{n+1} &= x_{n+1} - \alpha \\ &= x_n - \frac{f(x_n)}{f'(x_0)} - \alpha \\ &= e_n - \frac{f(x_n)}{f'(x_0)} \\ &= \frac{e_n f'(x_0) - f(x_n)}{f'(x_0)} \end{aligned}$$

By Taylor's Theorem,

$$f(\alpha) = f(x_n) + (\alpha - x_n) f'(x_n) + \frac{(\alpha - x_n)^2}{2} f''(\xi) = 0$$

$$\Rightarrow \quad f(x_n) - e_n f'(x_n) + \frac{1}{2} e_n^2 f''(\xi) = 0$$

$$e_{n+1} = \frac{e_n f'(x_0) - e_n f'(x_n) + \frac{1}{2} e_n^2 f''(\xi)}{f'(x_0)}$$

$$= e_n \left( \frac{f'(x_0) - f'(x_n) + \frac{1}{2} e_n f''(\xi)}{\quad} \right)$$

$$= e_n \left( \qquad \frac{\qquad}{f'(x_0)} \qquad \right)$$

assume that $x_0$ and $x_n$ are very close to the root $\alpha$.

$$e_{n+1} = e_n^2 \frac{f''(\alpha)}{2f'(\alpha)}$$

$$C = \frac{f''(\alpha)}{2f'(\alpha)} \quad . \ S = 2$$

OR

$$f(\alpha) = f(x_0) - e_0 f'(x_0) + \frac{1}{2} e_0^2 f''(\xi) = 0$$

$$\Rightarrow f'(x_0) = \frac{1}{e_0} f(x_0) + \frac{1}{2} e_0 f''(\xi)$$

$$e_{n+1} = \frac{\frac{e_n}{e_0} f(x_0) + \frac{1}{2} e_n e_0 f''(\xi) - f(x_n)}{f'(x_0)}$$

assume that $x_n$ and $x_0$ are close to the root $\alpha$

$$e_{n+1} = \frac{e_n e_0 f''(\alpha)}{2f'(\alpha)}$$

$$\therefore \ C = \frac{e_0 f''(\alpha)}{2f'(\alpha)} \ , \quad S = 1$$

V. Within $(-\frac{\pi}{2}, \frac{\pi}{2})$, will the iteration $x_{n+1} = \tan^{-1} x_n$ converge?

$$\begin{aligned} |g(x) - g(y)| &= |\arctan x - \arctan y| , \quad \text{by mean value theorem,} \\ &= |(\arctan' \xi)(x-y)| , \quad \xi \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) \\ &= \left| \frac{1}{1+\xi^2} (x-y) \right| \\ &\le \frac{1}{1+\xi^2} |x-y| \end{aligned}$$

obviously. $0 < \frac{1}{1+\xi^2} < 1$.

$\therefore$ function $g(x) = \arctan(x)$ is contraction and continuous in $(-\frac{\pi}{2}, \frac{\pi}{2})$.

By theorem 1.38, $g(x)$ has a unique fixed point $\alpha$ in $(-\frac{\pi}{2}, \frac{\pi}{2})$, the fixed point iteration converges to $\alpha$ for any $x_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$

VI. Let $p > 1$. What is the value of the following continued fraction?

$$x = \cfrac{1}{p + \cfrac{1}{p + \cfrac{1}{p + \cdots}}}$$

Prove that the sequence of values converges. (Hint: this can be interpreted as $x = \lim_{n\to\infty} x_n$, where $x_1 = \frac{1}{p}$, $x_2 = \frac{1}{p+\frac{1}{p}}$, $x_3 = \frac{1}{p+\frac{1}{p+\frac{1}{p}}}$, ..., and so forth. Formulate $x$ as a fixed point of some function.)

$$\frac{1}{x} = p + \cfrac{1}{p + \cfrac{1}{p + \cdots}}$$

$$= p + x$$

$$1 = px + x^2$$

$$x^2 + px - 1 = 0$$

$$x = \frac{-p \pm \sqrt{p^2+4}}{2}$$

$$\therefore \quad x = \frac{-p + \sqrt{p^2+4}}{2}$$

$$f(x_n) = \frac{1}{p+x_n} = x_{n+1}$$

$$|f(x_n) - f(y_n)| = \left|\frac{1}{p+x_n} - \frac{1}{p+y_n}\right|$$

$$= \left|\frac{x_n - y_n}{(p+x_n)(p+y_n)}\right|$$

$$\le \frac{1}{(p+x_n)(p+y_n)} |x_n - y_n|$$

$$= \lambda |x_n - y_n| , \quad 0 < \lambda < 1$$

$\therefore$ $f(x_n)$ is continuous contraction on $[0,1]$

By theorem 1.38. $f(x_n)$ has a unique fixed point $\alpha$ in $[0,1]$.

the fixed point iteration converges . so the sequence converges as well.

VII. What happens in problem II if $a_0 < 0 < b_0$? Derive an inequality of the number of steps similar to that in II. In this case, is the relative error still an appropriate measure?

Initial interval $= [a_0, b_0]$

$$b_n - a_n = \frac{1}{2^n}(b_0 - a_0)$$

$$\frac{|x_n - r|}{r} \le \frac{b_n - a_n}{r}$$

$$= \frac{1}{2^n r}(b_0 - a_0) \quad ,$$

as if $r \neq 0$ ,

$$(x_n - r) \leq \frac{1}{2}(b_n - a_n)$$

$$= \frac{1}{2^{n+1}}(b_0 - a_0)$$

$$x_n = \frac{1}{2^{n-1}}(b_0 + a_0)$$

$$\Rightarrow \quad \left(\frac{1}{2^{n-1}}(b_0 + a_0) - r\right) \leq \frac{1}{2^{n+1}}(b_0 - a_0)$$

$$\frac{2}{2^n}(b_0 + a_0) - r \leq \frac{1}{2^n \cdot 2}(b_0 - a_0)$$

$$r \geq -\frac{1}{2^n \cdot 2} b_0 + \frac{2}{2^n} b_0 + \frac{1}{2^n \cdot 2} a_0 + \frac{2}{2^n} a_0$$

$$= \frac{1}{2^n}\left(\frac{3}{2} b_0 + \frac{5}{2} a_0\right)$$

$$\log r \geq -n \log 2 + \log\left(\frac{3}{2} b_0 + \frac{5}{2} a_0\right)$$

$$n \geq \frac{\log\left(\frac{3}{2} b_0 + \frac{5}{2} a_0\right) - \log r}{\log 2}$$