Chapter 5 Approximation

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Problem 1 We have to prove that

$$\langle u, v\rangle = \int_a^b \rho(x)u(x)\overline{v(x)}dx, \rho(x) > 0$$

satisfies the following axioms.
(1)real positive:

$$\begin{aligned}\forall u \in C[a,b], \langle u, u\rangle &= \int_a^b \rho(x)u(x)\overline{u(x)}dx \\ &= \int_a^b \rho(x)|u(x)|^2 dx \\ &\geq 0\end{aligned}$$

(2)definiteness:

$$\langle u, u\rangle = 0 \quad iff \quad u = 0$$

(3)additivity in the first slot:

$$\begin{aligned}\forall u, v, w \in C[a,b], \langle u + v, w\rangle &= \int_a^b \rho(x)(u(x) + v(x))\overline{w(x)}dx \\ &= \int_a^b \rho(x)u(x)\overline{w(x)}dx + \int_a^b \rho(x)v(x)\overline{w(x)}dx \\ &= \langle u, w\rangle + \langle v, w\rangle\end{aligned}$$

(4)homogeneity in the first slot:

$$\begin{aligned}\forall a \in \mathbb{C}, \forall v, w \in C[a,b], \langle av, w\rangle =& \int_a^b \rho(x)(a(v(x))\overline{w(x)}dx \\ =&a\int_a^b \rho(x)v(x)\overline{w(x)}dx \\ =&a\langle v, w\rangle\end{aligned}$$

(5)conjugate symmetry:

$$\forall v, w \in C[a,b], < u, v >= \int_a^b \rho(x)u(x)\overline{v(x)}dx$$

and to prove

$$\|u\|_2 = [\int_a^b \rho(x)|u(x)|^2 dx]^{\frac{1}{2}}$$

satisfies

(1)real positive:

$$\|u\|_2 = [\int_a^b \rho(x)|u(x)|^2 dx]^{\frac{1}{2}} \geq 0$$

(2)definiteness:

$$\langle u, v\rangle = 0 \quad iff \quad u = 0$$

(3)homogeneity in the first slot:

$$\begin{aligned}\forall a \in \mathbb{C}, \|au\|_2 =&[\int_a^b \rho(x)|au(x)|^2 dx]^{\frac{1}{2}} \\ =&|a|[\int_a^b \rho(x)|u(x)|^2 dx]^{\frac{1}{2}} \\ =&|a|\|u\|_2\end{aligned}$$

(4)Triangle inequality:

$$\begin{aligned}\forall u, v \in C[a,b], \|u+v\|_2 =&[\int_a^b \rho|u(x)+v(x)|^2 dx]^{\frac{1}{2}} \\ \leq&[\int_a^b \rho|u(x)|^2]^{\frac{1}{2}} + [\int_a^b \rho|u(x)|^2]^{\frac{1}{2}} \\ =&\|u\|_2 + \|u\|_2\end{aligned}$$

Problem 2 (a)The Chebyshev polynomials of the first kind is

$$T_n(x) = cos(n \arccos(x)), n = 0, 1, 2...$$

$$\begin{aligned}
\forall m, n \in [-1, 1], \langle T_m(x), T_n(x) \rangle &= \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_m(x)\overline{T_n(x)}dx \\
&= \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} cos(m \arccos(x))cos(n \arccos(x))dx \\
& let\ \theta = arccos(x) \\
&= \int_{0}^{\pi} cos(m\theta)cos(n\theta)d\theta \\
&= \frac{1}{2}\int_{0}^{\pi} [cos(m\theta + n\theta) + cos(m\theta - n\theta)]d\theta \\
&= \begin{cases} \pi, when\ m = n \\ 0, when\ m \neq n \end{cases}
\end{aligned}$$

thus, they are orthogonal.
(b)

$$\begin{aligned}
v_1 = 1, u_1^* = \frac{1}{\|v_1\|} &= \frac{1}{\sqrt{\pi}} \\
v_2 = x - \langle x, u_1 \rangle u_1 = x, u_2^* = \frac{1}{\|v_2\|} &= \sqrt{\frac{\pi}{2}}x \\
v_3 = 2x^2 - 1, u_3^* = \frac{1}{\|v_3\|} &= \sqrt{\frac{2}{\pi}}(2x^2 - 1)
\end{aligned}$$

Problem 3 (a)from the above question ,we know that

$$\begin{aligned}
u_1 &= \frac{1}{\sqrt{\pi}} \\
u_2 &= \sqrt{\frac{\pi}{2}}x \\
u_3 &= \sqrt{\frac{2}{\pi}}(2x^2 - 1)
\end{aligned}$$

thus ,the approximate function is

$$\begin{aligned} y =& \langle y, u_1^* \rangle u_1^* + \langle y, u_2^* \rangle u_2^* + \langle y, u_3^* \rangle u_3^* \\ =& \frac{2}{\pi} + 0 + \frac{4}{3\pi}(2x^2 - 1) \\ =& \frac{10}{3\pi} - \frac{8}{3\pi}x^2 \end{aligned}$$

(b)using the monomials$(1, x, x^2)$,we know that

$$\begin{aligned} G^T a =& c \\ G =& \begin{pmatrix} \langle 1,1 \rangle & \langle 1,x \rangle & \langle 1,x^2 \rangle \\ \langle x,1 \rangle & \langle x,x \rangle & \langle x,x^2 \rangle \\ \langle x^2,1 \rangle & \langle x^2,x \rangle & \langle x^2,x^2 \rangle \end{pmatrix} \\ =& \begin{pmatrix} \pi & 0 & \frac{\pi}{2} \\ 0 & \frac{\pi}{2} & 0 \\ \frac{\pi}{2} & 0 & \frac{3\pi}{8} \end{pmatrix} \\ c =& \begin{pmatrix} \langle y,1 \rangle \\ \langle y,x \rangle \\ \langle y,x^2 \rangle \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ \frac{2}{3} \end{pmatrix} \end{aligned}$$

thus, we have

$$a = \begin{pmatrix} \frac{10}{3\pi} \\ 0 \\ -\frac{8}{3\pi} \end{pmatrix}$$

since,we get

$$s = \frac{10}{3\pi} - \frac{8}{3\pi}x^2$$

Problem 4 (a)

$$u_1 = 1, v_1 = 1, \|v_1\|_2^2 = \sum_{i=1}^{12} v_1^2 = 12, u_1^* = \frac{1}{2\sqrt{3}}$$
$$u_2 = x, v_2 = x - \left\langle x, \frac{1}{2\sqrt{3}} \right\rangle \frac{1}{2\sqrt{3}} = x - \sum_{i=1}^{12} \frac{1}{2\sqrt{3}} i = x - \frac{13}{2}$$
$$\|v_2\|_2^2 = \sum_{i=1}^{12} (i - \frac{13}{2})^2 = \sqrt{143}, u_2^* = \frac{1}{\sqrt{143}}(x - \frac{13}{2})$$
$$u_3 = x^2, v_3 = x^2 - \left\langle x^2, \frac{1}{2\sqrt{3}} \right\rangle \frac{1}{2\sqrt{3}} - \left\langle x^2, \frac{1}{\sqrt{143}}(x - \frac{13}{2}) \right\rangle \frac{1}{\sqrt{143}}(x - \frac{13}{2}) = x^2 - 13x + \frac{91}{3}$$
$$\|v_3\|_2^2 = \frac{4004}{3}, u_3^* = \sqrt{\frac{3}{4004}}(x^2 - 13x + \frac{91}{3})$$

(b)by using Fourier expansion,

$$\langle y, u_1^* \rangle = \sum_{i=1}^{12} y(t_i) u_1^*(t_i) = \sum_{i=1}^{12} \frac{1}{2\sqrt{3}} y_i = 277\sqrt{3}$$
$$\langle y, u_2^* \rangle = \sum_{i=1}^{12} y(t_i) u_2^*(t_i) = \sum_{i=1}^{12} \frac{1}{\sqrt{143}}(x_i - \frac{13}{2}) y_i = \frac{589}{\sqrt{143}}$$
$$\langle y, u_3^* \rangle = \sum_{i=1}^{12} y(t_i) u_3^*(t_i) = \sum_{i=1}^{12} \sqrt{\frac{3}{4004}}(x^2 - 13x + \frac{91}{3}) y_i = 6004\sqrt{\frac{3}{1001}}$$

thus

$$\begin{aligned}
\phi(x) =& 277\sqrt{3}\frac{1}{2\sqrt{3}} + \frac{1}{\sqrt{143}}(x - \frac{13}{2})\frac{589}{\sqrt{143}} + \sqrt{\frac{3}{4004}}(x^2 - 13x + \frac{91}{3})6004\sqrt{\frac{3}{1001}} \\
=& \frac{277}{2} + \frac{589}{143}(x - \frac{13}{2}) + \frac{9051}{1001}(x^2 - 13x + \frac{91}{3}) \\
\approx& 9.04x^2 - 113.43x + 386
\end{aligned}$$

(c)If value of N and x_i's are same,but the value of y_i's are different ,from the above question we know that u_i^*, u_2^*, u_3^* are the same,but $\langle y, u_i^* \rangle$ have to recalculate.

Comparing the normal equation with orthonormal polynomials, normal equation have to recalculate the G and solving equation but for orthonormal polynomials just have to renew index of basis,so,orthonormal polynomials is more efficient.

Programming assignments

A This is the result of the discrete least square via normal equations and cond of G.
Thus, the approximation polynomial is

```
>> [a,s]=dls(x,y)

a =

    2.1757
    2.6704
   -0.2384


s =

   1.8981e+04
```

$$\phi(x) = -0.2384x^2 + 2.6704x + 2.1757$$

B This is the answer of the discrete least square via QR factorization and cond of R.
Thus, the approximation polynomial is

```
>> [b,c]=dsl_via_QR(x,y)

b =

    2.1757
    2.6704
   -0.2384


c =

  137.7712
```

$$\phi(x) = -0.2384x^2 + 2.6704x + 2.1757$$