Chapter 3 Splines

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Problem 1 From the given conditions,We know that
$s(0) = 0 = f(0)$, $s(1) = 1 = f(1)$, $s(2) = 0 = f(2)$, $\mu_2 = \frac{1}{2}$, $\lambda_2 = \frac{1}{2}$, $S''(1) = 6 = M_2$, $s''(2) = 0 = M_3$. From lemma 3.4,we have

$$M_1 + 4M_2 + M_3 = -12 \tag{1}$$

and get $M_2 = -36$, $s'(0) = 12$,so

$$p(x) = 12x - 18x^2 + 7x^3 \tag{2}$$

As $s''(0) = -36 \neq 0$ so $s(x)$ not a natural cubic splines.

Problem 2 (a) s is a quadratic spline,assume $s(x) = ax^2 + bx + c$,we have three unknown in each interval,so there need three equation to solve out the a,b,c in each interval, but we only have two condition given for each interval,there miss one condition,that's why we need an additional condition to determine s.
(b)As $s(x)$ is a quadratic spline, we have

$$s'(x) = m_i + \frac{m_{i+1} - m_i}{x_{i+1} - x_i}(x - x_i) \tag{3}$$

Integrate the equation (3),we have

$$s(x) = f_i + m_i(x - x_i) + \frac{m_{i+1} - m_i}{2(x_{i+1} - x_i)}(x - x_i)^2 \tag{4}$$

Substitute $x = x_{i+1}$ into the equation (4),

$$s(x_{i+1}) = f_i + m_i(x_{i+1} - x_i) + \frac{1}{2}(m_{i+1} - m_i)(x_{i+1} - x_i)$$

$$m_{i+1} = \frac{f_{i+1} - f_i}{x_{i+1} - x_i} - m_i \tag{5}$$

so we can get

$$p_i = f_i + m_i(x - x_i) + \frac{f_{i+1} - f_i - m_i(x_{i+1} - x_i)}{(x_{i+1} - x_i)^2}(x - x_i)^2 \tag{6}$$

(c)From equation (5) and equation given,we have

$$\begin{cases} m_1 = f''(a) \\ m_i + m_{i+1} = 2f[x_i, x_{i+1}] \end{cases}$$

thus,

$$m_i = 2\sum_{k=1}^{i-1}(-1)^{k+1} f[x_k, x_{k+1}] + f'(a)$$

Problem 3 As $s(x)$ is a natural cubic spline,we have $s''(-1) = 0 = M_1$, $s''(1) = 0 = M_2$,and from the given condition we have $s(-1) = 1 = f(-1)$, $s(0) = 1 + c$, $\mu_2 = \frac{1}{2} = \lambda_2$.
From Taylor expansion of s(x) at -1,yields

$$s_1(x) = 1 + s'(-1)(x+1) + \frac{M_1}{2}(x+1)^2 + \frac{M_2 - M_1}{6}(x+1)^3 \tag{7}$$

where $x \in [-1, 0]$. Compare equation (7) with $s_1(x) = 1 + c(x+1)^3$,get $M_2 = 6c$. From lemma 3.4,we have

$$M_1 + 4M_2 + M_3 = 6(f(1) - 1 - 2c) \tag{8}$$

get $f(1) = 1 + 6c$,so we have $s'(0) = 3c$ and

$$s_2(x) = 1 + c + 3cx + 3cx^2 - cx^3 \tag{9}$$

When $s_2(1) = -1$,$c = -\frac{1}{3}$,thus

$$s_2(x) = \frac{2}{3} - x - x^2 + \frac{1}{3}x^3 \tag{10}$$

Problem 4 (a) From the given conditions,We know that $s''(-1) = 0 = M_1$, $s''(1) = 0 = M_3$, $\mu_2 = \frac{1}{2} = \lambda_2$.From lemma 3.4,we have

$$M_1 + 4M_2 + M_3 = -12 \tag{11}$$

thus get $M_2 = -3$, $s_1'(-1) = \frac{3}{2}$, $s_2'(0) = 0$ so the natural cubic splines is

$$s(x) = \begin{cases} s_1(x) = \frac{3}{2}(x+1) - \frac{1}{2}(x+1)^3 & x \in [-1, 0] \\ s_2(x) = 1 - \frac{3}{2}x^2 + \frac{1}{2}x^3 & x \in [0, 1] \end{cases} \tag{12}$$

(b)(i)From Newton Interpolation

$$g(x) = (x+1) - x(x+1) \tag{13}$$

thus g"$(x) = -2$,so

$$\int_1^{-1} (g''(x))^2 dx = \int_1^{-1} 4dx = 8$$
$$\int_1^{-1} (s''(x))^2 dx = 6$$

thus we get

$$\int_1^{-1} (s''(x))^2 dx \leq \int_1^{-1} (g''(x))^2 dx \tag{14}$$

(b)(ii)

$$\int_1^{-1} (g''(x))^2 dx = (\frac{\pi^4}{16}) \int_1^{-1} (cos(\frac{\pi}{2}x))^2 dx = \frac{\pi^4}{16}$$

from(b)(i) we know that$\int_1^{-1} (s''(x))^2 dx = 6$ thus we still get

$$\int_1^{-1} (s''(x))^2 dx \leq \int_1^{-1} (g''(x))^2 dx \tag{15}$$

Problem 5 (a)https://www.overleaf.com/project/63621c49e75047059fcc74e7

$$B_i^2(x) = \frac{x - t_{i-1}}{t_{i+1} - T_{i-1}} B_i^1(x) + \frac{t_{i+2} - x}{t_{i+2} - t_i} B_{i+1}^1(x)$$

From theorem 3.34,

$$\frac{d}{dx} B_i^2(x) = \frac{2}{t_{i+1} - t_{i-1}} B_i^1(x) + \frac{2}{t_{i+2} - t_i} B_{i+1}^1(x) \tag{16}$$

and we know that

$$
\begin{aligned}
B_i^1(x) =& \frac{x-t_{i-1}}{t_i-t_{i-1}} B_i^0(x) + \frac{t_{i+1}-x}{t_{i+1}-t_i} B_{i+1}^0(x) \\
=& \begin{cases} \frac{x-t_{i-1}}{t_i-t_{i-1}}, x \in (t_{i-1}, t_i] \\ \frac{t_{i+1}-x}{t_{i+1}-t_i}, x \in (t_i, t_{i+1}] \\ 0, Otherwise \end{cases} \\
B_{i+1}^1(x) =& \frac{x-t_i}{t_{i+1}-t_i} B_{i+1}^0(x) + \frac{t_{i+2}-x}{t_{i+2}-t_{i+1}} B_{i+2}^0(x) \\
=& \begin{cases} \frac{x-t_i}{t_{i+1}-t_i}, x \in (t_i, t_{i+1}] \\ \frac{t_{i+2}-x}{t_{i+2}-t_{i+1}}, x \in (t_{i+1}, t_{i+2}] \\ 0, Otherwise \end{cases}
\end{aligned}
$$

Substitute the above equation to equation(16),get

$$
\frac{d}{dx} B_i^2(x) = \begin{cases} \frac{2(x-t_{i-1})}{(t_i-t_{i-1})(t_{i+1}-t_{i-1})}, x \in (t_{i-1}, t_i] \\ \frac{2(t_{i+1}-x)}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} - \frac{2(x-t_i)}{(t_{i+2}-t_i)(t_{i+1}-t_i)}, x \in (t_i, t_{i+1}] \\ \frac{2(x-t_{i+2})}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})}, x \in (t_{i+1}, t_{i+2}] \\ 0, Otherwise \end{cases}
$$

(b)

$$
\begin{aligned}
\lim_{x\to t_i^+}\frac{d}{dx}B_i^2(x) &= \lim_{x\to t_i^+}\left(\frac{2(t_{i+1}-x)}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)}-\frac{2(x-t_i)}{(t_{i+1}-t_i)(t_{i+1}-t_i)}\right)\\
&=\frac{2}{t_{i+1}-t_{i-1}}\\
\lim_{x\to t_i^-}\frac{d}{dx}B_i^2(x) &= \lim_{x\to t_i^-}\frac{2(x-t_{i-1})}{(t_i-t_{i-1})(t_{i+1}-t_{i-1})}\\
&=\frac{2}{t_{i+1}-t_{i-1}}\\
\frac{d}{dx}B_i^2(t_i) &=\frac{2}{t_{i+1}-t_{i-1}}\\
\lim_{x\to t_{i+1}^+}\frac{d}{dx}B_i^2(x) &= \lim_{x\to t_{i+1}^+}\frac{2(x-t_{i+2})}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})}\\
&=-\frac{2}{t_{i+2}-t_i}\\
\lim_{x\to t_{i+1}^-}\frac{d}{dx}B_i^2(x) &= \lim_{x\to t_{i+1}^-}\frac{2(t_{i+1}-x)}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)}-\frac{2(x-t_i)}{(t_{i+1}-t_i)(t_{i+1}-t_i)}\\
&=-\frac{2}{t_{i+2}-t_i}\\
\frac{d}{dx}B_i^2(t_{i+1}) &=-\frac{2}{t_{i+2}-t_i}
\end{aligned}
$$

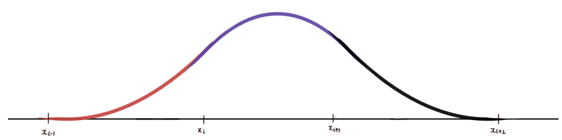
thus, $\frac{d}{dx}B_i^2(x)$ is continuous at t_i and t_{i+1}.

(c)When $x\in(t_{i-1},t_i]$, we have $\frac{d}{dx}B_i^2(t_i)>0$, when $x\in(t_i,t_{i+1}]$, $\frac{d}{dx}B_i^2(t_{i+1})<0$, thus, $\exists x^*\in(t_i,t_{i+1})$ satisfies $\frac{d}{dx}B_i^2(x^*)=0$

$$
\begin{aligned}
\frac{d}{dx}B_i^2(x^*) &=0\\
\frac{2(t_{i+1}-x^*)}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)}-\frac{2(x^*-t_i)}{(t_{i+2}-t_i)(t_{i+1}-t_i)} &=0\\
(t_{i+1}-x^*)(t_{i+2}-t_i)-(x^*-t_i)(t_{i+1}-t_{i-1}) &=0\\
x^*=\frac{t_{i+1}t_{i+2}-t_it_{i-1}}{t_{i+2}-t_i+t_{i+1}-t_{i-1}}
\end{aligned}
$$

(d)We know that the boundary point was at t_{i-1} , t_{i+2} and the extreme point was at x^*,since we have $B_i^2(t_{i-1}) = 0 = B_i^2(t_{i+2})$, $B_i^2(x^*) < 1$, thus $B_i^2(x^*) \in [0, 1)$

(e)



Problem 6

$$
\begin{aligned}
LHS =&[t_i, t_{i+1}, t_{i+2}](t-x)_+^2 - [t_{i-1}, t_i, t_{i+1}](t-x)_+^2 \\
=&\frac{[t_{i+1}, t_{i+2}](t-x)_+^2 - [t_i, t_{i+1}](t-x)_+^2}{(t_{i+2}-t_i)} - \frac{[t_i, t_{i+1}](t-x)_+^2 - [t_{i-1}, t_i](t-x)_+^2}{(t_{i+1}-t_i-1)} \\
=&\frac{(t_{i+2}-x)_+^2 - (t_{i+1}-x)_+^2}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})} - \frac{(t_{i+1}-x)_+^2 - (t_i-x)_+^2}{(t_{i+2}-t_i)(t_{i+1}-t_i)} - \\
&\frac{(t_{i+1}-x)_+^2 - (t_i-x)_+^2}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{(t_i-x)_+^2 - (t_{i-1}-x)_+^2}{(t_{i+1}-t_{i-1})(t_i-t_{i-1})} \\
=&\begin{cases}
\frac{(t_{i+2}-x)^2-(t_{i+1}-x)^2}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})} - \frac{(t_{i+1}-x)^2-(t_i-x)^2}{(t_{i+2}-t_i)(t_{i+1}-t_i)} - \frac{(t_{i+1}-x)^2-(t_i-x)^2}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{(t_i-x)^2-(t_{i-1}-x)^2}{(t_{i+1}-t_{i-1})(t_i-t_{i-1})} & x \in (t_{i-1}, t_i] \\
\frac{(t_{i+2}-x)^2-(t_{i+1}-x)^2}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})} - \frac{(t_{i+1}-x)^2-(t_i-x)^2}{(t_{i+2}-t_i)(t_{i+1}-t_i)} - \frac{(t_{i+1}-x)^2-(t_i-x)^2}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} & x \in (t_i, t_{i+1}] \\
\frac{(t_{i+2}-x)^2}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})} & x \in (t_{i+1}, t_{i+2}] \\
0 & \text{otherwise}
\end{cases} \\
=&\begin{cases}
\frac{(t_{i-1}-x)^2}{(t_{i+1}-t_{i-1})(t_i-t_{i-1})} & x \in (t_{i-1}, t_i] \\
\frac{(x-t_{i-1})(t_{i+1}-x)}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{(x-t_i)(t_{i+2}-x)}{(t_{i+2}-t_i)(t_{i+1}-t_i)} & x \in (t_i, t_{i+1}] \\
\frac{(t_{i+2}-x)^2}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})} & x \in (t_{i+1}, t_{i+2}]
\end{cases} \\
=&B_i^2(x) \\
=&RHS
\end{aligned}
$$

Problem 7 When n=0,

$$\frac{1}{t_i - t_{i-1}} \int_{t_i-1}^{t_i} B_i^0(x)dx = 1$$

The statement is true when n=0.Assume that the statement is true when n=k-1.

$$\frac{1}{t_{i+k-1} - t_{i-1}} \int_{t_i-1}^{t_{i+k-1}} B_i^{k-1}(x)dx = \frac{1}{k} \tag{17}$$

when n=k,

$$\begin{aligned}
\frac{1}{t_{i+k} - t_{i-1}} \int_{t_i-1}^{t_{i+k}} B_i^k(x)dx =& \frac{1}{t_{i+k} - t_{i-1}} [xB_i^k(x)|_{i+k t_{i-1}}^{t} - \int_{t_i-1}^{t_{i+k}} x\frac{d}{dx}B_i^k(x)dx] \\
=& -\frac{n}{t_{i+k} - t_{i-1}} \int_{t_{i-1}}^{t_{i+k}} (\frac{xB_i^{k-1}(x)}{t_{i+n-1} - t_{i-1}} - \frac{xB_{i+1}^{k-1}(x)}{t_{i+n} - t_i})dx
\end{aligned}$$

From the definition of B-splines,we have

$$\int_{i-1}^{i+k} B_i^k(x) = \int_{t_{i-1}}^{t_{i+k-1}} \frac{x - t_{i-1}}{t_{i+k} - t_{i-1}} B_i^{k-1}(x)dx + \int_{t_i}^{t_{i+k}} \frac{t_{i+k} - x}{t_{i+k+1} - t_i} B_{i+1}^{k-1}(x)dx \tag{18}$$

substitute equation(18),

$$\begin{aligned}
\frac{1}{t_{i+k} - t_{i-1}} \int_{t_i-1}^{t_{i+k}} B_i^k(x)dx =& -\frac{k}{t_{i+k} - t_{i-1}} [\int_{t_i-1}^{t_{i+k}} B_i^k(x)dx + \frac{t_{i-1}}{t_{i+k-1} - t_{i-1}} \int_{t_{i-1}}^{t_{i+k-1}} B_i^{k-1}(x)dx - \\
& \frac{t_{i+k}}{t_{i+k} - t_i} \int_{t_i}^{t_{i+k}} B_{i+1}^{k-1}(x)dx] \\
\frac{1+k}{t_{i+k} - t_{i-1}} \int_{t_i-1}^{t_{i+k}} B_i^k(x)dx =& \frac{kt_{i+k}}{t_{i+k} - t_{i-1}} (\frac{1}{t_{i+k} - t_i} \int_{t_i}^{t_{i+k}} B_{i+1}^{k-1}(x)dx) - \\
& \frac{kt_{i-1}}{t_{i+k} - t_{i-1}} (\frac{1}{t_{i+k-1} - t_{i-1}} \int_{t_{i-1}}^{t_{i+k-1}} B_i^{k-1}(x)dx) \\
=& \frac{kt_{i+k}}{t_{i+k} - t_{i-1}} (\frac{1}{k}) - \frac{kt_{i-1}}{t_{i+k} - t_{i-1}} (\frac{1}{k}) \\
\frac{1}{t_{i+k} - t_{i-1}} \int_{t_i-1}^{t_{i+k}} B_i^k(x)dx =& \frac{1}{1+k}
\end{aligned}$$

The statement also true when n=k.

Problem 8 (a)Table of difference quotient:

x_1	x_1^4		
x_2	x_2^4	$x_1^3+x_2^3+x_1x_2^2+x_1^2x_2$	
x_3	x_3^4	$x_2^3+x_3^3+x_2x_3^2+x_2^2x_3$	$x_1^2+x_2^2+x_3^3+x_1x_2+x_1x_3+x_2x_3$

thus,we have

$$\begin{aligned}\tau_2(x_1,x_2,x_3)=&x_1^2+x_2^2+x_3^3+x_1x_2+x_1x_3+x_2x_3\\=&[x_1,x_2,x_3]x^4\end{aligned}$$

(b)From Lemma 3.45,we have

$$\begin{aligned}\tau_{k+1}(x_1,......,x_n,x_{n+1})=&\tau_{k+1}(x_1,.....,x_n)+x_{n+1}\tau_k(x_1,.....,x_n,x_{n+1})\\\tau_{k+1}(x_1,......,x_n,x_{n+1})=&\tau_{k+1}(x_1,.....,x_n)+x_{n+1}\tau_k(x_1,.....,x_n,x_{n+1})\\&-x_1\tau_k(x_1,.....,x_n,x_{n+1})+x_1\tau_k(x_1,.....,x_n,x_{n+1})\\(x_{n+1}-x_1)\tau_k(x_1,.....,x_n,x_{n+1})=&\tau_{k+1}(x_1,......,x_n,x_{n+1})-\tau_{k+1}(x_1,.....,x_n)-x_1\tau_k(x_1,.....,x_n,x_{n+1})\\=&\tau_{k+1}(x_2,......,x_{n+1},x_1)-\tau_{k+1}(x_1,.....,x_n)-x_1\tau_k(x_1,.....,x_n,x_{n+1})\\=&\tau_{k+1}(x_2,......,x_{n+1})+x_1\tau_{k+1}(x_2,......,x_{n+1},x_1)-\tau_{k+1}(x_1,.....,x_n)-\\&x_1\tau_k(x_1,.....,x_n,x_{n+1})\\=&\tau_{k+1}(x_2,......,x_{n+1})-\tau_{k+1}(x_1,.....,x_n)\end{aligned}$$

when n=0,

$$\tau_m(x_i)=x_i^m=[x_i]x^m$$

The statement is true when n=0.Assume that the statement is true when n=k-1,we have

$$\tau_{m-k+1}(x_i,......x_{i+k-1})=[x_i,......x_{i+k-1}]x^m$$

When n=k,

$$\begin{aligned}\tau_{m-k}(x_i,......x_{i+k})=&\frac{1}{x_{n+1}-x_1}\tau_{m-k+1}(x_2,......,x_{n+1})-\tau_{m-k+1}(x_1,.....,x_n)\\=&\frac{[x_2,......,x_{n+1}]x^m-[x_1,.....,x_n]x^m}{x_{n+1}-x_1}\\=&[x_1,......,x_{n+1}]x^m\end{aligned}$$

The statement is also true when n=k.