EC4301 Macroeconomic Analysis III

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Contents

1 Agency Theory

Players:

- Principal (**P**)
  - Designs contract, screen types
  - Does not know about **A**'s type
- Agent (**A**)
  - Perform action, react based on contract
  - Signal type to **P**

Incentive problems:

1. Hidden information (HI) $\rightarrow$ Adverse selection
   - Hiding smoker information from insurance agent

2. Hidden action (HA) $\rightarrow$ Moral hazard
   - Coming to office but does no work, my action is not observed

1.1 Adverse selection

1. Screening problem
   - **P** does not know about **A** type, **P** needs to screen for different types of **A**

   - Bank offer loan to house buyer but does not know buyer's future income
2. Signalling problem
   - **A** takes an action for **P** to inference
   - Job applicants chooses education to signal ability

1.2 Moral hazard

P owns firm to produce output, designs contract and gives agent reward w to maximise expected profit.

A product output y based on agent's effort a and random factor ϵ, accepts contract to maximise expected payoff (wage - effort cost)

Timeline

1. **P** and **A** sign a contract $w = s + bp$
2. **A** choose action $\mathbf{a} = (a_1, a_2)$ which is not observed by **P**
3. Actions and noise (ϵ, ϕ) determined **A** output (y) and performance (p), **P** observe p (or y) and **A** receive compensation specified by contract.

1.2.1 Risk-neutral principal and risk-neutral agent

1.2.1.1 Pay for performance

Consider $y = a + \epsilon$

1. [Case 1] effort a is not observed, but output y is observed
   - Consider: linear contract $w = s + by$, $s \in R, b \in (0, 1]$
   - Agent either accept or reject contract, if agent accept contract, exert effort a to maximise expected payoff

     $$\max_a E[s + b \cdot (a + \epsilon)] - c(a)$$

     The optimal effort is when marginal benefit = marginal cost

     $$b = c'(a^*)$$

   - Implications: as b increase, **A** exert higher effort. However, **P** will not assign $b = 1$ unless $s < 0$. Production efficiency occur when $b = 1$ since **A** exert the optimal effort to maximise output.
2. [Case 2] effort a is not observed, but performance p is observed
   - Consider: $w = s + bp$
   - Performance is based on effort and noise $p = \mathbf{a} + \phi, E(\phi) = 0$
   - There could be multiple efforts $\mathbf{a} = (a_1, a_2)$
   - If $y = a_1 + a_2, p = a_1$, then contract based on p cannot incentives a_2
   - If $y = a_1, p = a_1 + a_2$, then contract based on p could incentive **A** to exert a_2 but not a_1.

1.2.1.2 Multi-task agency model

Solving for [Case 2] effort a is not observed, but performance p is observed.

Consider $y = f_1 a_1 + f_2 a_2 + \epsilon$, $p = g_1 a_1 + g_2 a_2 + \phi$, $w = s + bp$, $c(a_1, a_2) = \frac{1}{2}a_1^2 + \frac{1}{2}a_2^2$

- **A** solves

  $$\max_{a_1, a_2} s + b(g_1 a_1 + g_2 a_2) - \frac{1}{2}a_1^2 - \frac{1}{2}a_2^2$$

  yield

  $$a_1^*(b) = g_1 b, a_2^*(b) = g_2 b$$

- **A** expected utility $E(u) = E(w) - c(a_1, a_2)$

  $$E(u) = s + b[g_1 a_1^* + g_2 a_2^*] - \frac{1}{2}(a_1^*)^2 - \frac{1}{2}(a_2^*)^2$$

- **P** expected profit $E(\pi) = E(y - w)$
$$E(\pi) = f_1 a_1^* + f_2 a_2^* - s - b g_1 a_1^* - b g_2 a_2^*$$
solving for best bonus $b_{\mathbf{P}}^* = arg\max_b E(\pi)$
$$b_{\mathbf{P}}^* = \frac{f_1 g_1 + f_2 g_2}{2(g_1^2 + g_2^2)}$$
- Total expected surplus $E(\pi + u) = E(y) - c(a_1, a_2)$
$$E(\pi + u) = (f_1 g_1 + f_2 g_2) b - \frac{1}{2}(g_1^2 + g_2^2) b^2$$
- Socially (second-best) efficient bonus: $b_{social}^* = arg\max_b E(\pi + u)$
$$b_{social}^* = \frac{f_1 g_1 + f_2 g_2}{g_1^2 + g_2^2}$$
- Therefore, **A**'s optimal decisions do not lead to productive efficiency.

Interpreting the b^*

- Scaling and alignment with b^*
$$b^* = \frac{\sqrt{f_1^2 + f_2^2}}{\sqrt{g_1^2 + g_2^2}} \cos(\theta)$$
derived from cosine distance: $\cos(\theta) = \frac{\mathbf{f} \cdot \mathbf{g}}{||\mathbf{f}|| \cdot ||\mathbf{g}||} = \frac{f_1 g_1 + f_2 g_2}{\sqrt{f_1^2 + f_2^2}\sqrt{g_1^2 + g_2^2}}$, $b^* = \frac{f_1 g_1 + f_2 g_2}{g_1^2 + g_2^2} = \cos(\theta) \frac{\sqrt{f_1^2 + f_2^2}}{\sqrt{g_1^2 + g_2^2}}$

  Scaling factor: $\frac{\sqrt{f_1^2 + f_2^2}}{\sqrt{g_1^2 + g_2^2}}$, alignment factor: $\cos(\theta)$
- Scaling issue: $g_1 >> f_1, g_2 >> f_2$
  - Performance will be inflated: high p but low y
  - **P** design a weak incentive: $b^* < 1$
  - However, if p is still align with y (i.e. $g_1 = k f_1, g_2 = k f_2$), then $b^* = \frac{1}{k}$ can be used to correct the scaling effect
- Alignment issue: $f_1 = 0, g_2 = 0$
  - Performance will be badly align ($y = f_2 a_2, p = g_1 a_1$), incentive b^* is useless to planner
- Scaling and alignment issue: $g_1 = 2 f_1, g_2 = \frac{3}{2} f_2$
  - Performance will be inflated, and **A** put in disproportional effort on a_1
- Note: if p and y are correlated, i.e. $cov(\epsilon, \phi) \neq 0$, then random noise can result in high performance but no impact on output
  - For example, business cycle may increase stock price but fundamental is still not strong

1.2.2 Risk-neutral principal and risk-adverse agent

Main issue: under risk-aversion, who should bear the risk and how will risk-aversion shape the optimal (second-best) contract? Examples:

- Tenant (**A**) who cultivates a land with no other means is risk adverse as he does not want to end up in a ruin. Landlord (**P**) who is rich and can absorb risks is risk neural.
- Innovator (**A**) who wants to start a company is assumed to be small player while bank (**P**) is a large organisation that is risk-neutral.

Consider an outcome $q \in \{0, 1\}$ with $q = 1$ denotes success, $P(q = 1|a) = p(a)$, $p(\cdot)$ is strictly increasing and concave (i.e. $p(0) = 0, p(\infty) = 1, p'(\cdot) > 0$). **P** is risk neutral with utility $V(q - w) = q - w$, $V'(\cdot) > 0, V''(\cdot) = 0$ and **A** is risk averse with utility $u(w) - \Psi(a)$, $u'(\cdot) > 0, u''(\cdot) < 0, \Psi'(\cdot) > 0, \Psi''(\cdot) \geq 0$.

1.2.2.1 Solving first-best

Consider $\Psi(a) = a$, assume **P** can ensure any choice of effort **a** subject to compensating **A** for his cost.

- Principal maximises utility $V(q - w)$
$$\max_{a, w_1, w_0} p(a)(1 - w_1) + (1 - p(a))(-w_0)$$
subject to **A**'s utility equal or higher than outside option. (here assume $\bar{u} = 0$)
$$s.t.\ p(a)u(w_1) + (1 - p(a))u(w_0) - a \geq \bar{u}$$

- Solving FB problem with Lagrangian

$$\max_{a,w_1,w_0,\lambda} L = p(a)(1-w_1) + (1-p(a))(-w_0) + \lambda[p(a)u(w_1) + (1-p(a))u(w_0) - a]$$

  Key solutions from $\nabla_L(a, w_1, w_0, \lambda) = 0$
  - Borch rule: $\lambda = \frac{1}{u'(w_1)} = \frac{1}{w'(w_0)}$ and $w_0 = w_1 = w^*$
  - Unique a^* such that: $u(w^*) = a^*$, $p'(a^*) = \frac{1}{u'(w^*)}$
- Intuition behind FB solution: marginal product of effort = **A**'s marginal utility loss due to effort

$$\frac{dEq(a)}{da} = \frac{du^{-1}(a)}{da} \Rightarrow p'(a) = \frac{1}{u'(w)}$$

  where $E(q(a)) :=$ expected output, $u^{-1}(a) :=$ utility loss due to effort cost and $u(w) = a$

1.2.2.2 Solving second-best (optimal) Consider **P** cannot observe for effort **a** and must induce **A** through compensation package (w_0, w_1)

- Principal maximises utility $V(q-w)$

$$\max_{w_1,w_0} p(a)(1-w_1) + (1-p(a))(-w_0)$$

  subject to **A**'s maximising utility

$$\max_a p(a)u(w_1) + (1-p(a))u(w_0) - a$$

  and participation constraint (IR) (here assume $\bar{u} = 0$)

$$p(a)u(w_1) + (1-p(a))u(w_0) - a \geq 0$$

- **A**'s incentive compatibility (IC)

$$p'(a)[u(w_1) - u(w_0)] = 1$$

  A's marginal expected utility from effort must equal to his marginal cost of effort
- Simplified Principal utility maximisation

$$\max_{w_1,w_0} p(a)(1-w_1) + (1-p(a))(-w_0)$$

  subject to participation constraint (IR)

$$s.t.\ p(a)u(w_1) + (1-p(a))u(w_0) - a \geq 0$$

  and subject to **A**'s incentive compatibility (IC)

$$s.t.\ p'(a)[u(w_1) - u(w_0)] = 1$$

  The Lagrangian

$$\max_{w_1,w_0,\lambda_1,\lambda_2} p(a)(1-w_1) + (1-p(a))(-w_0) + \lambda_1[p(a)u(w_1) + (1-p(a))u(w_0) - a] + \lambda_2[p'(a)[u(w_1) - u(w_0)] - 1]$$

  Key solutions from $\nabla_L(w_1, w_0, \lambda_1, \lambda_2)$
  - Contrast to Borch rule: $\frac{1}{u'(w_1)} = \lambda_1 + \lambda_2\frac{p'(a)}{p(a)}$, $\frac{1}{u'(w_2)} = \lambda_1 - \lambda_2\frac{p'(a)}{1-p(a)}$
- Interpreting observation
  - Compared FB solution that eliminated any risk for the agent **A**, optimal insurance is distorted by asking **A** to bear some risks.
  - **A** gets reward for good performance, larger share of surplus relative to Borch rule when high performance, and get penalty for bad performance.
  - Here we have $\lambda_1, \lambda_2 \geq 0$ due to the interpretation of Lagrange multiplier

1.2.2.3 Application: Optimal debt financing under moral hazard and limited liability Assume both **P** and **A** are risk-neutral, entrepreneur **A** require funding for project and market **P** provides the capital. Subject to funding **A** exert effort **a** resulting in revenues q with density function $f(q|a)$ and distribution function $F(q|a)$ where $q \in [0, \bar{q}]$

Assume $f(q|a)$ satisfies MLRP (monotone likelihood ratio property): $\Rightarrow \frac{d}{dq}\frac{f_a(q|a)}{f(q|a)} \geq 0$, for two effort case where $\mathbf{a} = (a_1, a_2)$: $\Rightarrow \frac{d}{dq}\frac{f(q|a_L)}{f(q|a_H)} \geq 0$. MLRP implies density has higher mass on higher values of q as **a** increase.

Result: debt contract is optimal under the assumption that the agent is subject to limited liability (LL). A debt contract pays **P** amount $r(q) = D, q \geq D$ and $r(q) = q, q < D$. This makes **A** residual claimant for the revenues above the promised debt D.

Setup: **A** maximises utility $v(w,a) = w - \Psi(a)$ with $w :=$ wealth and $\Psi(0) = 0$, $\Psi'(\cdot) > 0$, $\Psi''(\cdot) > 0$ (convex effort cost). Assume $\Psi(a) = \frac{1}{2}a^2$. Limited liability means **A** can be asked to repay **P** at most his revenue: $0 \leq r(q) \leq q$, further assumes **A**'s repayment never goes down with revenue $r'(q) \geq 0$.

- **A** maximise utility $\int_0^{\bar{q}} v(w,a)dq$

$$\max_{r(q),a} \int_0^{\bar{q}} [q - r(q)] f(q|a) dq - \Psi(a)$$

subject to **A**'s optimal effort decision (IC), i.e. marginal benefit = marginal cost

$$\int_0^{\bar{q}} [q - r(q)] f_a(q|a) dq = \Psi'(a)$$

and subject to zero profit of the lender (IR)

$$\int_0^{\bar{q}} r(q) f(q|a) dq = I$$

and limited liability (LL)

$$0 \leq r(q) \leq q$$

- Solving the Lagrangian

$$\max_{r(q),a,\lambda_1,\lambda_2} L = \int_0^{\bar{q}} [q - r(q)] f(q|a) dq - \Psi(a) + \lambda_1 \left[\int_0^{\bar{q}} r(q) f(q|a) dq - I \right] + \lambda_2 \left[\int_0^{\bar{q}} [q - r(q)] f_a(q|a) dq - \Psi'(a) \right]$$

Rearranging the Lagrangian and realise Lagrangian is linear in $r(q)$

$$L = \int_0^{\bar{q}} r(q) \left[\lambda_1 - \lambda_2 \frac{f_a(q|a)}{f(q|a)} - 1 \right] f(q|a) dq + \int_0^{\bar{q}} q \left[1 + \lambda_2 \frac{f_a(q|a)}{f(q|a)} \right] f(q|a) dq - \Psi(a) - \lambda_2 \Psi'(a) - \lambda_1 I$$

Solution for optimal repayment schedule where **A** is published ($r^*(q) = q$) and rewarded ($r^*(q) = 0$) by the revenue generated. Solution is arrived by setting $\lambda_1 - \lambda_2 \frac{f_a(q|a)}{f(q|a)} - 1 > 0$ and $\max(r(q))$ to max L

$$r^*(q) = \begin{cases} q & \lambda_1 > 1 + \lambda_2 \frac{f_a(q|a)}{f(q|a)} \\ 0 & \lambda_1 \leq 1 + \lambda_2 \frac{f_a(q|a)}{f(q|a)} \end{cases}$$

Therefore, reward **A** when

$$\frac{f_q(q|a)}{f(q|a)} > \frac{\lambda_1 - 1}{\lambda_2}$$

- Designing the debt contract by payout function with a threshold revenue

$$r^*(q) = \begin{cases} 0 & q > Z \\ q & q \leq Z \end{cases}$$

- Practical implementation require **A** to payback **P** the loan amount, therefore modify the debt contract

$$r^*(q) = \begin{cases} D & q > D \\ q & q \leq D \end{cases}$$

and the minimum value of D is achieved by solving the total expected repayment is equal to the investment amount (zero profit)

$$\int_0^D q f(q|a^*) dq + [1 - F(D|a^*)] D = I$$

where a^* is given by IC

$$\int_D^q (q - D) f_a(q|a^*) dq = \Psi'(a^*)$$

This modification ensures **P** gets the loan amount back while high residual claimant for **A** in the high revenue state.

1.2.3 Relational contracts

Relational contract is a case of no explicit contract between **P** and **A**. In various employment not all contingencies can be written in a contract, making formal contract that require verifiable outcomes unrealistic. Therefore, some contracts are incomplete. The key idea here is trigger strategies.

Consider output y can be observed but cannot be verified. Therefore, no formal contract available between **P** and **A** but strategies can be conditioned on y, implying a subjective and not objective assessment of y.

A exerts effort costly effort $a \in [0,1]$, with effort cost $c(a)$, $c'(a) > 0$ and $P(y = H) = a$. **P**'s payoff is $y - w$ and **A**'s payoff is $w - c(a)$, **A** has an outside option w_a.

1.2.3.1 One-shot interaction

1. **A** expects **P** to not pay a bonus even if $y = H$
2. Therefore, **A** exert $a = 0$ and $y = L$
3. If $L < w_a$, then **P** and **A** failed to form a relationship contract. Note that it will be more efficient if they have a contract.

1.2.3.2 Repeated Game: Relational Contract

- If **A** believes **P** will pay a bonus b if $y = H$ (honor relational contract), **A** solves

  $$\max_a s + ab - c(a)$$

  optimal a^* is when marginal cost = marginal benefit

  $$c'(a^*) = b$$

  The resultant **P**'s expected profit per period $E(\pi) = E(y - w)$

  $$E(\pi) = a^*(a)H + (1 - a^*(b))L - [s + a^*(b)b] = L + a^*(b)[H - L] - [s + a^*(b)b]$$

  The resultant **A**'s expected payoff per period $E(s + ab - c(a))$

  $$E(s + a^*b - c(a^*)) = s + a^*(b)b - c(a^*(b)) \geq w_a$$

  If **P** provide the minimal bonus $\hat{b}$ (optimal bonus) to meet **A**'s participation constraint.

  $$s + a^*(b)b = w_a + c(a^*(b)) \Rightarrow \hat{b} = \frac{w_a + c(a^*(b)) - s}{a^*(b)}$$

  The final **P**'s expected profit per period

  $$E[\pi(\hat{b})] = L + a^*(\hat{b})[H - L] - c(a^*(\hat{b})) - w_a$$

- Trigger strategies: **P** considers if to honour the relational contract

  1. If **P** does not honour the relational contract when $y = H$, the current payoff is: $H - s$
  2. If **P** honour the relational contract, the current payoff is $H - s - \hat{b}$ and $E(\pi(\hat{b}))$ thereafter.
  3. **P** consider the lifetime utility and pay bonus if and only if (considering only the excess profit)

     $$(H - s - \hat{b}) + \frac{E(\pi(\hat{b}))}{r} \geq (H - s)$$

     Therefore, **P** pays bonus $\hat{b}$ if and only if $E(\pi(\hat{b})) \geq r\hat{b}$

- interpretation: relational contract depends on interest rate

  - If r is low, it is easy to satisfy the requirement for efficient contracting as future profit is equally valuable.
  - If r is high, then requirement cannot be fulfilled as futture profit is not valued.

1.2.4 Application: Credit Rationing

Credit Rationing: Banks (**P**) are selective in offering credits to loaners (**A**). Although demand for credit exceeds supply, bank is not willing to increase the supply nor the interest rate.

1.2.4.1 Setup and issues

- Profit maximising lender considers

  1. Interest rate
  2. Riskiness of loans

  However, higher interest rate increase the riskiness of loans

- Adverse selection & incentive effects

  - Adverse selection: affects the type of borrower
    - Higher interest rate attracts more risky borrowers
    - More risky borrowers are willing to take a higher risk project to compensate for the higher cost of borrowing
  - Incentive effects: affects the type of projects

* Higher interest rate results in more risky projects due to limited liability
* Between two mean-preserving projects, select the project with higher payoff & lower chance of success since the maximum pay back is the collateral

Therefore, there exist an optimal interest rate that maximises the bank's expected payoff. The bank will not monotonically increase interest rate or increase supply of loans to address the excess demand for loans.

1.2.4.2 Rationing: select amongst identical borrowers

- Screening with interest rate
- Project return & firm profit
- Expected return & interest rate

1.2.4.3 Results and Intuition

- Results & intuitions
- Non-monotonicity of bank's return in borrower's risk type
- Optimal interest rate & Market equilibrium
- Credit Rationing Result

2 Auctions

2.1 Market Design and Auction market

2.1.1 Market Design

A set of training rules that governs

1. Collection of objects of heterogeneous qualities
2. Two sides - potential buyers and sellers
3. Allocation or matching to achieve certain objective(s)
4. Rules/institutions to carry out all the relevant transactions between the parties

Note that sometimes trade happens even without a formal format. Such as negotiation between 2 parties in a non competitive market.

2.1.2 Standard Auction: Rules for Selling a single object

1. Potential buyers
2. Auction rules
   1. Sealed bid: First-price auction
      - Bid: $E(V_{1:n-1}|V_{1:n-1} < v_i)$
      - Highest bidder pays at price equal to his bid

   2. Sealed-bid: Second-price auction
      - Bid: private value v_i
      - Highest bidder wins at price to the second-highest bid
   3. Ascending price: English auction
      - Bid: private value v_i
      - Price stops increasing when all but one bidder drops out
   4. Descending price: Dutch auction
      - Bid: $E(V_{1:n-1}|V_{1:n-1} < v_i)$
      - Price stops decreasing when one bidder takes his finder off the button

Note:

- first-price and Dutch, second-price and English are similar in decision, while all auctions are similar in outcome
- All auctions end with efficiency V_{highest} or $V_{1:n}$
- All winning bidder pays $V_{2:n}$
  - Second-price or English: $V_{2:n}$
  - First-price or Dutch: $E(V_{2:n})$

2.2 Complete information bidding strategies

Assume an item on sale with 5 bidders bidding $\{v_1 = 10, v_2 = 5, v_3 = 7, v_4 = 2, v_5 = 8\}$, the ordered statistics are:

$$\{v_{1:5} = 10, v_{2:5} = 8, v_{3:5} = 7, v_{4:5} = 5, v_{5:5} = 2\}$$

The bidding strategies under different auction system

- First-price and Dutch: expected private value of the second highest bidder
  - $b_1 = 8.01, b_2 = 5, b_3 = 7, b_4 = 2, b_5 = 8$
  - bidder 1 pays 8.01
- Second-price and English: private value
  - $b_1 = 10, b_2 = 5, b_3 = 7, b_4 = 2, b_5 = 8$
  - bidder 1 pays 8

2.3 Asymmetric information bidding strategies

The Auction Model (IPV: Independent Private Values)

- Single object for sale
- n bidders
- Buyers have private value v_i for the object
- A buyer knows his own value, while the others think of it as a random variable
- The private value follow a uniform distribution over [a, b]

2.3.1 Second-price and English Auction bidding strategy

- Bidder decide at each point if to continue bidding or not
- Bidder select a point to drop out from auction
- Bidder i with private value v_i has to decide if to continue the auction when
  1. $p < v_i$: stay in bidding
  2. $p = v_i$: drop out of bidding

Therefore, the bidding strategy is to stay in auction until private value

$$b_i = \beta(v_i) = v_i$$

2.3.2 First-price and Dutch Auction bidding strategy

- Bidder i knows his own private value v_i but treats the other bidders private value as random variable
- The only private value of concern is the largest ordered statistics $v_{1:n-1}$
  1. If $v_i < v_{1:n-1}$, bidder i's bid is irrelevant
  2. If $v_i > v_{1:n-1}$, bidder i bid $v_{1:n-1}$

Therefore, the bidding strategy is to stay in auction or bid the second highest private value

$$b_i = \beta(v_i) = E(v_{1:n-1}|v_{1:n-1} < v_i) = v_i - \frac{v_i}{N}$$

2.4 Reserve price

Seller set reserve price r in the auction and bids below r will not be considered

Bidding strategy

- SPA: $b(v) = v$
- FPA: $b(v) = E\left(\max\{V_{1:n-1}, r\} \middle| \max\{V_{1:n-1}, r\} < v \right)$

2.4.1 Optimal Reserve

A binding reserve is when reserve r is set between a, b, it prevents sale with a positive probability.

The reserve insures a case where the second bid is too low relative to the first bid

The level below which bidders do not participate is called the screening level, the screening level $v^* = r$

To calculate the optimal reserve, consider maximise expected revenue. The expected revenue is the total probability of the reserve price between the top 2 valuation and lower than second highest valuation.

$$\begin{aligned}
&\max_r r \cdot P(V_{2:n} < r < V_{1:n}) + E(V_{2:n}|V_{2:n} > r) \cdot P(V_{2:n} > r) \\
&= rn(1-F(r))F(r)^{n-1} + \int_r^b xn(n-1)F(x)^{n-2}(1-F(x))f(x)dx
\end{aligned}$$

The first order condition

$$n(1-F(r))F(r)^{n-1} - rnf(r)F(r)^{n-1} + rn(n-1)(1-F(r))F(r)^{n-2}f(r) - rn(n-1)F(r)^{n-2}(1-F(r))f(r) = 0$$

With the optimal reserve price

$$r = \frac{1-F(r)}{f(r)} \propto \frac{1}{\text{Hazard rate}}$$

2.5 Participation/Bidding/Entry Fee

Bidding strategy if seller charge a bidding fee ϕ

- If bidder has a low private value, he should not bid
- If a bidder decide to bid then
  - SPA: bid private value
  - FPA: bid the expected min price to win

2.6 First Price Sealed Bid Auction derivation

- $v :=$ private value, a random distribution assumed to be uniform with cdf F
- $b = \beta(v)$: $\beta(v)$ is the bid function for all bidders while $b :=$ is the bid value

2.6.1 Deriving the utility function

The utility of bidder i is the difference between private value and bid value $(v-b)$ given that individual has won the bid, i.e. $b > \beta(v_j)$ for all the other bidders.

$$\begin{aligned}
U_i(v,b) &= (v-b)\prod_{j=1}^{N-1} P(b > \beta(v_j)) \\
&= (v-b)\prod_{j=1}^{N-1} P(v_j < \beta^{-1}(b)) \\
&= (v-b)F^{N-1}(\beta^{-1}(b)) \\
&\because v_j \sim Unif(0,1) \\
\Rightarrow U_i(v,b) &= (v-b)(\beta^{-1}(b))^{N-1}
\end{aligned}$$

2.6.2 Solving for the best bid value

Bidder i maximises the utility by changing the bid price

$$\begin{aligned}
&\max_b \ U_i(v,b) = (v-b)(\beta^{-1}(b))^{N-1} \\
&\frac{\partial U_i(v,b)}{\partial b} = -\left(\beta^{-1}(b)\right)^{N-1} + (v-b)(N-1)\left(\beta^{-1}(b)\right)^{N-2}\frac{1}{\beta'(\beta^{-1}(b))} = 0 \ (FOC) \\
&\because b = \beta(v) \Rightarrow v = \beta^{-1}(b) \\
&\Rightarrow -v^{N-1} + (N-1)(v-\beta(v))\frac{v^{N-2}}{\beta'(v)} = 0
\end{aligned}$$

Note: deriving $\frac{d}{db}\beta^{-1}(b) = \frac{1}{\beta'(\beta^{-1}(b))}$

$$
\begin{aligned}
b &= \beta(v) \\
\frac{db}{dv} &= \beta'(v) \\
\Rightarrow \frac{dv}{db} = \frac{d}{db}\beta^{-1}(b) &= \frac{1}{\beta'(v)} = \frac{1}{\beta'(\beta^{-1}(b))}
\end{aligned}
$$

Solving for the best bid value by rearranging the previous equation

$$
\begin{aligned}
v^{N-1}\beta'(v) + (N-1)v^{N-2}\beta(v) &= (N-1)v^{N-1} \\
\Leftrightarrow \frac{d}{dv}v^{N-1}\beta(v) &= (N-1)v^{N-1} \\
\Rightarrow v^{N-1}\beta(v) &= \int_0^v (N-1)s^{N-1}ds \\
v^{N-1}\beta(v) &= (N-1)\frac{v^N}{N} \\
\Rightarrow \beta(v) &= \frac{N-1}{N}v = v - \frac{v}{N}
\end{aligned}
$$

General solution when v is not uniform distribution: $b = \beta(v) = v - \frac{1}{F^{N-1}(v)}\int_{\underline{v}}^{v} F^{N-1}(s)ds$