Definitions

D1.1.2	A linear equation in n variables = $a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$
D1.1.4	If the linear equation is satisfied, a linear equation has infinitely many solutions unless n =1
D.1.1.6	A system of linear equations is a multiple combination of linear equations
D1.1.9	A system of linear equations has no solution (resp. at least one solution) if it's an inconsistent system (resp. consistent system)
D1.2.1	Linear system and augmented matrix are interchangeable
D1.2.4	Elementary row operations consist of 1. Multiply a row by a nonzero constant. 2. Interchange two rows. 3. Add a multiple of one row to another row.
D1.2.6	Two augmented matrices are row equivalent if one can be obtained from the other by a series of elementary row operations
D1.3.1	An augmented matrix is said to be in row-echelon form if it has: 1. any rows that consist of entirely of zeros are grouped at the bottom of the matrix. 2. In any two successive non-zero rows, the first nonzero umber in the lower row occurs farther to the right than the first nonzero number in the higher row An augmented matrix is said to be in reduced row-echelon form (RREF) if: 3. The leading entry of every nonzero row is 1. 4. In each pivot column, except the pivot point, all other entries are zeros.
D1.4.1	Gaussian Elimination is an algorithm to reduce an augmented matrix to a row-echelon form by using elementary row operations
D1.5.1	A system of linear equations is said to be homogeneous if it has all the constant terms to be zero
D2.2.8	(Matrix Multiplication) Let $\mathbf{A} = (a_{ij})_{mxp}$ and $\mathbf{B} = (b_{ij})_{pxn}$ be two matrices. The product **AB** is a mxn matrix. its (i,j) entry is $a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ip}b_{pj}$
D2.2.12	$\mathbf{A}^n = \mathbf{AA}\ldots\mathbf{A}$, $n \geq 1$; $\mathbf{A}^0 = \mathbf{I}$
D2.3.2	**A** is a square matrix of order n. **A** is invertible if there exists a square matrix **B** of order n such that **AB** = **I** and **BA** = **I**
D2.3.11	If **A** is invertible, $\mathbf{A}^{-n} = (\mathbf{A}^{-1})^n = \mathbf{A}^{-1}\mathbf{A}^{-1}\ldots\mathbf{A}^{-1}$
D2.4.2	A square matrix is called an elementary matrix if it can be obtained from an identity matrix by performing a single elementary row operation
D2.5.2	Let $\mathbf{A} = (a_{ij})$ be an nxn matrix. If $\mathbf{A}=(a_{11})$ is a 1x1 matrix, then $\det(\mathbf{A}) = a_{11}$ For n>1, let $\mathbf{M}_{1j}$ be the (n-1)x(n-1) matrix obtained from **A** by deleting the 1^{st} row and the j^{th} column. The determinant of **A** is defined to be: $\det(\mathbf{A}) = a_{11}\mathbf{A}_{11} + a_{12}\mathbf{A}_{12} + \ldots + a_{1n}\mathbf{A}_{1n}$ (cofactor expansion along row1)
D2.5.24	Let **A** be a square matrix of order n. The adjoint of **A** is the nxn matrix $\mathrm{adj}(\mathbf{A}) = \text{transpose of} \begin{bmatrix} A_{11} & A_{12} \cdots\cdots & A_{1n} \\ A_{21} & A_{22} \cdots\cdots & A_{2n} \\ A_{n1} & A_{n2} \cdots\cdots & A_{nn} \end{bmatrix}$ Where $\mathbf{A}_{ij}$ is the (i,j)-cofactor of **A** -> $(-1)^{i+j}\det(\mathbf{M}_{ij})$
D3.1.3	n-vector = $(u_1,u_2,\ldots u_i,\ldots u_n)$, where $u_1,u_2,\ldots,u_n$ are real numbers
D3.1.7	Euclidean n-space, $\mathbf{R}^n$, is the set of all n-vectors of real numbers
D3.2.1	$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_k\mathbf{u}_k$ is called a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$
D3.2.3	The set of all linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ is called the linear span of $\mathbf{u}_1, \mathbf{u}_2, \ldots., \mathbf{u}_k$ $\mathrm{span}\{\mathbf{u}_1,\mathbf{u}_2,\ldots\mathbf{u}_k\}$
D3.3.2	*V* is called a subspace of $\mathbf{R}^n$ provided there is a set $S = \{\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_k\}$ of $\mathbf{R}^n$ such that $V = \mathrm{span}(S)$
D3.4.2.1 D3.4.2.2*	[working definition] S is a linearly independent (resp. linearly dependent) set if the vector equation $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_k\mathbf{u}_k = \mathbf{0}$ has only the trivial solution (resp. non-trivial solution) i.e. the only possible scalars are $c_1=0,c_2=0,..,c_k=0$
D3.5.4	S is called a basis for $\mathbf{R}^n$ (resp. *V*) if 1. S is linearly independent and 2. S spans $\mathbf{R}^n$ (resp. *V*)
D3.5.8	$(V)_s = (c_1,c_2,\ldots,c_k)$ where $(V)_s$ is the coordinate vector of v relative to S and $c_1,c_2,\ldots,c_k$ are called the coordinates of *V* relative to the basis S
D3.6.3	dim(*V*) is the dimension of a vector space *V* and is the number of vectors in a basis for *V*
D3.7.3	$S = \{\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_k\}$ and $T=\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k\}$, two bases for a vector space *V*. Express each $\mathbf{u}_i$ as linear combination of $\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k\}$ 2. Form the column coordinate vectors w.r.t. T. 3. Form the matrix $\mathbf{P} = ([\mathbf{u}_1]_T[\mathbf{u}_2]_T\ldots[\mathbf{u}_k]_T)$ $\mathbf{P}[\mathbf{w}]_S=[\mathbf{w}]_T$ for any vector **w** in *V*

Theorems

T1.2.7
If augmented matrices of two linear systems are row equivalent, then the two systems have the same set of solutions.

T2.2.11
Associative Law **A**(**BC**) = (**AB**)**C**
Distributive Law **A**($\mathbf{B}_1$+$\mathbf{B}_2$) = $\mathbf{AB}_1$ + $\mathbf{AB}_2$; ($\mathbf{C}_1$+$\mathbf{C}_2$)**A** = $\mathbf{C}_1\mathbf{A}$ + $\mathbf{C}_2\mathbf{A}$
c(**AB**) = (c**A**)**B** = **A**(c**B**)
Let **A** be a mxn matrix
$\mathbf{A0}_{nxq} = \mathbf{0}_{mxq}$ and $\mathbf{0}_{pxm}\mathbf{A} = \mathbf{0}_{pxn}$
$\mathbf{AI}_n = \mathbf{I}_m\mathbf{A} = \mathbf{A}$

T2.2.22
Let **A** be a mxn matrix
$(\mathbf{A}^T)^T = \mathbf{A}$
If **B** is an mxn matrix, then
$(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$
If a is a scalar, then $(a\mathbf{A})^T = a\mathbf{A}^T$
If **B** is an nxp matrix, then $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$

T2.3.5
If **B** and **C** are inverses of a square matrix **A**, then **B**=**C**

T2.3.9
A,B: two invertible matrices of the same size
a: non-zero scalar

Matrix	Invertible?	Inverse	Det
a**A**	yes	$(a\mathbf{A})^{-1} = (1/a)\mathbf{A}^{-1}$	c^ndet(**A**)
$\mathbf{A}^T$	yes	$(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$	det(**A**)
$\mathbf{A}^{-1}$	yes	$(\mathbf{A}^{-1})^{-1} = \mathbf{A}$	$\det(\mathbf{A}^{-1})=\frac{1}{det(\boldsymbol{A})}$
AB	yes	$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$	det(**A**)det(**B**)

det(**A**+**B**) $\neq$ det(**A**) + det(**B**)

T2.4.7
T3.6.11
Let **A** be a square matrix. The following statements are equivalent
1. **A** is invertible
2. The linear system **Ax = 0** has only the trivial solution
3. The reduced row-echelon form of **A** is an identity matrix.
4. **A** can be expressed as a product of elementary matrices.
5. det(A)≠0
6. The rows and columns of A form a basis for $\mathbf{R}^n$

T2.4.12
Let **A**,**B** be square matrices of the same size. If **AB = I** then **BA = I**.
So **A** and **B** are invertible, $\mathbf{A}^{-1} = \mathbf{B}$, $\mathbf{B}^{-1} = \mathbf{A}$

T2.5.6
(Cofactor Expansions) det(**A**) can be expressed as a cofactor expansion using any row or column of **A**.
for any *i* = 1,2,...,n (cofactor expansion along row i)
det(**A**) = $a_{i1}\mathbf{A}_{i1} + a_{i2}\mathbf{A}_{i2} + ... + a_{in}\mathbf{A}_{in}$
for any *j* = 1,2,...,n (cofactor expansion along column j)
det(**A**) = $a_{1j}\mathbf{A}_{1j} + a_{2j}\mathbf{A}_{2j} + ... + a_{nj}\mathbf{A}_{nj}$

T2.5.8
If **A** is a triangular matrix, then the determinant of **A** is equal to the product of the diagonal entries of **A**.

T2.5.10
If **A** is a square matrix, then det(**A**) = det($\mathbf{A}^T$)

T2.5.12
The determinant of a square matrix with two identical rows is zero.
The determinant of a square matrix with two identical columns is zero.

T2.5.15

det(**E**)	E.R.O	Determinant
k	**A** --kR_i--> **B**	det(**B**) = k det(**A**)
-1	**A** --R_i<->R_j-->**B**	det(**B**) = -det(**A**)
1	**A** --R_j+kR_j-->**B**	det(**B**) = det(**A**)

T2.5.19
Square matrix **A** is invertible if and only if det(**A**) $\neq$ 0

T2.5.25
Let A be a square matrix. If A is invertible, then. $\boldsymbol{A}^{-1} = \frac{1}{det(\boldsymbol{A})}adj(\boldsymbol{A})$

T2.5.27 (Cramer's Rule)
Suppose **Ax=b** is a linear system where **A** is an nxn invertible matrix.
Let $\mathbf{A}_i$ be the matrix obtained from **A** by replacing the i^{th} column of **A** by **b**.

	Then the system has a unique solution $\boldsymbol{x} = \frac{1}{det(\boldsymbol{A})}\begin{pmatrix} det(\boldsymbol{A_1}) \\ det(\boldsymbol{A}_2) \\ \vdots \\ det(\boldsymbol{A}_n) \end{pmatrix}$
T3.2.7	Let S = {$\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_k$} be a set of vectors in $\mathbf{R}^n$. If k<n, then S cannot span $\mathbf{R}^n$
T3.2.9.1.	The zero vector 0 $\in$ span(S), any set of S
T3.2.9.2.	If $\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_r \in$ span(S) and $c_1,c_2,\ldots,c_r \in$ R, then $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_r\mathbf{v}_r \in$ span(S) If **u** and **v** $\in$ span(S), then **u** + **v** $\in$ span(S) <u>[Closure property under vector addition]</u> If **u** $\in$ span(S) and c $\in$ R, then c**u** span(S) <u>[Closure property under scalar multiplication]</u>
T3.2.10	span(S_1) $\subseteq$ span(S_2) if and only if each u_i is a linear combination of $v_1, v_2, \ldots, v_m$
T3.2.12	If $\mathbf{u}_k$ is a linear combination of $\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_{k-1}$, then span{ $\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_{k-1}$} = span{ $\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_{k-1}, \mathbf{u}_k$}
T3.3.6	The solution set of a homogeneous linear system in n variables is a subspace of $\mathbf{R}^n$
T3.4.4.1	S is linearly dependent if and only if at least one vector $\mathbf{u}_i$ in S can be written as a linear combination of the other vectors in S
T3.4.4.2	S is linearly independent if and only if no vector in S can be written as a linear combination of other vectors in S
T3.4.7	If S $\subseteq \mathbf{R}^n$ and S has more than n elements, then S is linearly dependent
T3.4.10	$\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_k$ are linearly independent. If $\mathbf{u}_{k+1}$ is not a linear combination of $\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_k$, then $\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_{k+1}$ are linearly independent
T3.5.7	Let S be a basis for a vector space V. Every vector **v** in V can be expressed in the form $\mathbf{v}=c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_k\mathbf{u}_k$ in exactly one way.
T3.5.11	Let S be a basis for a vector space V with |S| = k. Let $\mathbf{v}_1, \mathbf{v}_2, .., \mathbf{v}_r \subseteq$ V. Then 1. $\mathbf{v}_1, \mathbf{v}_2, .., \mathbf{v}_r$ are linearly dependent (resp. independent) in V if and only if $(\mathbf{v}_1)_s, (\mathbf{v}_2)_s,..,(\mathbf{v}_r)_s$ are linearly dependent (resp. independent in $\mathbf{R}^k$; 2. span{ $\mathbf{v}_1, \mathbf{v}_2, .., \mathbf{v}_r$} = V if and only if span{$(\mathbf{v}_1)_s, (\mathbf{v}_2)_s,..,(\mathbf{v}_r)_s$ } = $\mathbf{R}^k$
T3.6.1	Let V be a vector space which has a basis S = {$\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_k$} with k vectors. 1. Any subset of V with more than k vectors is always linearly dependent 2. Any subset of V with less than k vectors cannot span V
T3.6.7	Let *V* be a vector space of dimension k and S a subset of *V*. The following are equivalent: 1. S is a basis for *V*, 2. S is linearly independent and |S|=k, 3. S spans *V* and |S| =k
T3.6.9	Let *U* and *V* be subspaces of $\mathbf{R}^n$. We say: *U* is a subspace of *V*. i) If *U* $\subseteq$ *V*, then dim(*U*) ≤ dim(*V*) ii) If *U* $\subseteq$ *V* and *U* ≠ *V*, then dim(*U*) < dim(*V*)
T3.7.5	*S* and *T* are two bases of a vector space. **P** is the transition matrix from *S* to *T*. 1. **P** is invertible. 2. $\mathbf{P}^{-1}$ is the transition matrix from *S* to *T*

Methods

Given	Solve/Prove for	Method
Augmented matrices	Same set of solutions	Prove two linear systems are row equivalent
Linear systems	Solution sets	GJ elimination and investigate consistency, considering cases
Curve/plane equation and points on curve	Coefficient constants	Substitute (x,y) values and form linear systems
Supply equation and external demand	Solution set for supply	Supply = Internal Demand + External Demand $S_{3x1} = M_{3x3}\, x_{3x1} + D_{3x1}$
Matrix	Invertibility	1. Find matrix B s.t. AB = I or BA = I 2. det(A) $\neq$ 0 3. RREF of A is I
Matrix	Compute A^{-1}	1. (A|I) -GJ-> (I|A^{-1}) 2. $A^{-1} = \frac{1}{det(A)} adj(A)$ 3. Find matrix B s.t. AB = I or BA = I
Row operations	elementary matrix/ pre-multiply matrix	B = E x A
Matrix A	det(A)	1. Co-factor expansion 2. Convert to triangular matrix and $det(R) = det(E_k * \dots E_1) det(A)$ $det(A) = \frac{1}{det(E_k * \dots E_1)} det(R)$
Statement	Prove/Disprove	1. Proof by Contradiction -> assume outcome is false 2. Mathematic Induction -> assume p(1) and p(k) true
Implicit form of set	Explicit form of set	Solve for solution set Explicit: $\{(a_0,b_0,c_0) + t(a,b,c), t\in\mathbf{R}^3\}$ Implicit: {(x,y,z)|equations}
Linear systems	Equation for plane	$\{x_1a+y_1b+z_1c-d=0, x_2a+y_2b+z_2c-d=0, x_3a+y_3b+z_3c-d=0\}$
Point	Expressed point as linear combination of given set	Solve solution set of **:** $\mathbf{v} = a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_3$
$\mathbf{R}^n$ and span vector and span	if S spans the vector space/ another span	investigate if there is a (a,b,c) for any (x,y,z), check for consistency.
subset S	To show subspace	1. Express S as a linear span 2. Show that S is the solution set of a homogeneous system 3. Show that S represents a line or plane through origin (only for $\mathbf{R}^2$ and $\mathbf{R}^3$)
subset S	To show not subspace	1. Show that zero vector is not in S 2. Find u,v subset of S such that u+v not in subset S 3. Find v subset of S and scalar c such that cv not in S 4. Show that S is not a line or plane through origin (only for $\mathbf{R}^2$ and $\mathbf{R}^3$)
vector and vector space	Check for linear independence	If only trivial solution then two are linear independence