Lecture 8 - Linear Algebra

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Section 1

Matrix, Vectors

Matrix, Vectors

Vector: an array of values

Row vectors:

$$\mathbf{a}_{1\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{pmatrix}$$

Column vectors:

$$\mathbf{a}_{m\times 1} = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}$$

Matrix: an array of vectors

$$\mathbf{A}_{m\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Subsection 1

Definitions and properties

Definitions and properties

- Dimension: $dim(\mathbf{A}) = m \times n$
  - m rows and n columns
- Linear independence in a matrix
  - Vectors are not a multiple of each other
  - Vectors are not a linear combination of each other
- Rank: $rank(\mathbf{A})$
  - Rank describes the number of linearly independent rows and columns
  - $rank(\mathbf{A}) \leq \min(m, n)$
  - Column rank = row rank
- Transpose: $\mathbf{A}^T$ or $\mathbf{A}'$
  - $a_{ij} = a_{ji}$

Section 2

Matrix operations

Matrix operations

- Matrix addition: (two matrices must be of the same dimension)

$$\mathbf{A}_{m\times m}+\mathbf{B}_{m\times m}=\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n}\\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn}\end{pmatrix}$$

- Inner Product/ Dot product: (vectors must be of the same length)
  $\mathbf{a}=(a_1,a_2,\cdots,a_n),\mathbf{b}=(b_1,b_2,\cdots,b_n)$
  $\mathbf{a}\cdot\mathbf{b}=a_1b_1+a_2b_2+\cdots+a_nb_n$

- Scaler multiplication: (scalar is a 1×1 matrix)

$$\lambda\mathbf{A}_{m\times n}=\begin{pmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1n}\\ \lambda a_{21} & \lambda a_{22} & \cdots & \lambda a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ \lambda a_{m1} & \lambda a_{m2} & \cdots & \lambda a_{mn}\end{pmatrix}$$

Subsection 1

Matrix multiplication

Matrix multiplication

$$\mathbf{A}_{m\times n} \times \mathbf{B}_{n\times k} = \begin{pmatrix} \mathbf{a_1} \\ \mathbf{a_2} \\ \vdots \\ \mathbf{a_m} \end{pmatrix} \times \begin{pmatrix} \mathbf{b_1} & \mathbf{b_2} & \cdots & \mathbf{b_k} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{a_1}\cdot\mathbf{b_1} & \mathbf{a_1}\cdot\mathbf{b_2} & \cdots & \mathbf{a_1}\cdot\mathbf{b_k} \\ \mathbf{a_2}\cdot\mathbf{b_1} & \mathbf{a_2}\cdot\mathbf{b_2} & \cdots & \mathbf{a_2}\cdot\mathbf{b_k} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a_m}\cdot\mathbf{b_1} & \mathbf{a_m}\cdot\mathbf{b_2} & \cdots & \mathbf{a_m}\cdot\mathbf{b_k} \end{pmatrix}_{m\times k}$$

Note:

- Matrix multiplication only exist if the inside dimension are the same
- The resultant matrix is the outside dimension
- Power of matrices is a matrix multiplication operations
  - e.g. $\mathbf{A}^3 = \mathbf{A} \times \mathbf{A} \times \mathbf{A}$
- Matrix operations are not commutative: $\mathbf{AB} \neq \mathbf{BA}$

Subsection 2

Converting system of linear equations into matrix form

Converting system of linear equations into matrix form

We illustrate an example here

Given:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

Express in matrix form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \Leftrightarrow \mathbf{Ax} = \mathbf{b}$$

Subsection 3

Special matrices

Special matrices

- Square matrices: $\mathbf{A}_{n \times n}$
- Diagonal matrices: $a_{ij} \neq 0, i = j$, $a_{ij} = 0, i \neq j$
- Symmetric matrices: $a_{ij} = a_{ji}$, for all $i \in [1, m], j \in [1, n]$
- Identify matrices: $a_{ij} = 1, i = j$, $a_{ij} = 0, i \neq j$
- Null matrices: $a_{ij} = 0$, for all $i \in [1, m], j \in [1, n]$
- Idempotent matrices: $\mathbf{A}^2 = \mathbf{AA} = \mathbf{A}$

Section 3

Gauss-Jordan Elimination technique

Gauss-Jordan Elimination technique

- Leading entry: first non-zero element in each row
- Row echelon form (ref):
  - Leading entry is non-zero
  - Rows below leading entries are 0
- Reduced row echelon form (rref):
  - In REF form and
  - Leading entries are 1
  - Non-leading entries are 0
- Operations:
  - row switching
  - row scalar multiplication
  - row addition

Section 4

Solving system of linear equations with G-J Elimination

Solving system of linear equations with G-J Elimination

Given:

$$\mathbf{A}_{m\times n}\mathbf{x}_{n\times 1} = \mathbf{b}_{n\times 1}$$

1. Form an augmented matrix: $[\mathbf{A}|\mathbf{b}]$
2. Express the augmented matrix in RREF form
   - $\Leftrightarrow [\mathbf{I}|\mathbf{A}^{-1}\mathbf{b}] = [\mathbf{A}^{-1}\mathbf{A}|\mathbf{A}^{-1}\mathbf{b}]$
3. Deduce solution with
   - Unique solution: $\Leftrightarrow rank(\mathbf{A}) = rank(\mathbf{A}|\mathbf{b}) = \min(m, n)$
   - Infinitely many solutions: $\Leftrightarrow rank(\mathbf{A}) = rank(\mathbf{A}|\mathbf{b}) < \min(m, n)$
   - No solution: $\Leftrightarrow rank(\mathbf{A}) < rank(\mathbf{A}|\mathbf{b})$