Lecture 11 - Equality Constraints Optimization

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Section 1

Lagrange Multiplier Method

Lagrange Multiplier Method

Solving equality constraint optimization

$$\min_{\mathbf{x}} f(\mathbf{x}) \ s.t. \ m_i - g_i(\mathbf{x}) = 0, i = 1, \cdots, m$$

Solve Lagrangian function instead

$$\min_{\mathbf{x},\lambda} \mathcal{L}(\mathbf{x}, \lambda) = f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i [m_i - g_i(\mathbf{x})]$$

- Note the constraint presentation in order to preserve λ_i interpretation
- $\lambda_i = \frac{\partial f^*(m_i)}{\partial m_i}$, value function rate of change when constraint m_i changes

Require Non-Degenerate Constraint Qualification (NDCQ)

- $rank(\mathbf{J_g}(\mathbf{x}^*)) = m$
- $\mathbf{J_g}(\mathbf{x}^*)$ is the Jacobian of the constraints at the minimizer
- Rank of matrix **A** is the number of linearly independent rows/columns

FOC and SOC

First Order Necessary Condition (FOC)

- Unconstrained
  - Single variable: $f'(x^*) = 0$
  - Multiple variables: $\nabla f(\mathbf{x}^*) = \mathbf{0}$
- Equality constraint
  - Lagrangian: $\nabla \mathcal{L}(\mathbf{x}^*, \lambda^*) = \mathbf{0}$

Second Order Necessary and Sufficient Condition (SOC)

- Unconstrained (local min)
  - Single variable: $f''(x^*) > 0$
  - Multiple variables: $H_f(\mathbf{x}^*) \succ 0$ (positive definite)
- Equality constraint (local min)
  - Lagrangian: $H_{\mathcal{L}}(\mathbf{x}^*, \lambda^*) \succ 0$ (positive definite)

Sufficient Condition for global extreme point

Remark

We are testing if function is convex. Therefore, require the following to hold true for all $\mathbf{x}$ in domain, not just the minimizer $\mathbf{x}^*$

- Unconstrained (global min)
  - Single variable: $f''(x) \geq 0$
  - Multiple variable: $H_f(\mathbf{x}) \succeq \mathbf{0}$ (positive semi-definite)
- Equality constraint (global min)
  - Lagrangian: $H_{\mathcal{L}}(\mathbf{x}, \lambda) \succeq \mathbf{0}$ (positive semi-definite)

General comments on global extreme points

1. Closed and bounded domain will have global extreme points
   - Prove by Weierstrass Extreme Value Theorem
   - Compare objective values (including interior, boundary and non-differentiable points)
   - Consider $\max_{x_1,x_2} -x_1^2 + x_2 \; s.t. \; x_1^2 + x_2^2 = 1$
2. Unbounded domain might not have global extreme points
   - Consider $\max_{x_1,x_2} x_1^2 + x_2^2$