EC2104 Lecture 2 - continuity & differentiation

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Section 1

Continuity

Continuity

Definition

A function is continuous at $x = a \Leftrightarrow$

$$\lim_{x \to a} f(x) = f(a)$$

Conditions for function to be continuous

1. Function f must be defined at $x = a$
2. $\lim_{x \to a} f(x)$ exist
   - By definition of limits: $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$
3. $\lim_{x \to a} f(x) = L = f(a)$

Subsection 1

Intermediate Value Theorem (IVT)

Intermediate Value Theorem (IVT)

IVT

Let f be a function which is continuous in the interval $[a, b]$. If $f(a), f(b)$ have different sign, then there is at least one c in (a, b) such that $f(c) = 0$

Note:

- function must be continuous
- $f(a), f(b)$ must be different sign
  - Intuitively, if the function has both positive and negative y value, then 0 passes through the function at least once
- useful to check if there is at least one root
  - but does not inform us on the number of roots (could be more than 1) or the value of the root
  - Furthermore, failing IVT does not ensure root DO NOT exist

Subsection 2

Checking if equation has solution

Checking if equation has solution

Note: we are using the first definition of IVT. Therefore, we need to ensure the function is in the form $f(c) = 0$

- E.g. if given $g(x) = e^x = 6$, we convert to $f(x) = e^x - 6$ before applying IVT

Steps:

1. Ensure $f(x)$ is continuous
   - Exponential functions are continuous
2. Check for sign reversal
   - $f(1)e - 6 < 0, f(100) = e^{100} - 6 > 0$
3. Invoke IVT
   - By IVT, there is at least c in $(1, 100)$ such that $f(c) = 0$

By narrowing down the intervals (a, b) we can find a numerical solution for the roots. This is known as the bisection method.

Section 2

Differentiability

Differentiability

Differentiability

A function $f(x)$ is differentiable at $x = a$ as long as the following limit exist

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

- The limit comes as a result of secant line formula: $\frac{\Delta y}{\Delta x}$
  - by decreasing the change in x we get tangent line
- Limit exist when left limit = right limit
- Non-continuous function are not differentiable $P \Rightarrow Q$
  - but continuous does not mean differentiable $P \not\Leftarrow Q$
  - Note that in mathematics (and logic), statement P implies Q does not mean statement Q implies P. Very important.

Subsection 1

L'Hopital Rule

L'Hopital Rule

One key method used to solve limits

L'Hopital Rule

If $f(a) = g(a) = 0$ or ∞, and $g'(a)$ is non-zero

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Subsection 2

Increasing/ Decreasing Functions

Increasing/ Decreasing Functions

Derivative	slope	function value
$f'(x) > 0$	upward sloping	increasing
$f'(x) < 0$	downward sloping	decreasing
$f'(x) = 0$	0 slope (flat)	turning point

Note:

- If function is 1) monotonically increasing/ decreasing, combined with 2) Intermediate Value Theorem we can show the equation has a unique solution
    - monotonically increasing/ decreasing means function value only increase/ decrease
    - monotone can be shown by showing derivatives is always greater than/ less than 0

Section 3

Applications

Subsection 1

Cost Curves (MC, AC)

Cost Curves (MC, AC)

$$\frac{d}{dx}\frac{C(x)}{x} = \frac{1}{x}\left(C'(x) - \frac{C(x)}{x}\right) \Rightarrow \begin{cases} > 0, & C'(x) > \frac{C(x)}{x} \\ < 0, & C'(x) < \frac{C(x)}{x} \\ = 0, & C'(x) = \frac{C(x)}{x} \end{cases}$$

- marginal cost (MC): $\frac{d}{dx}C(x)$, cost of producing one more unit of goods x
- average cost (AC): $\frac{C(x)}{x}$, the average cost of production
- firm production quantity Q^*: when MC cuts AC from below
    - average cost is lowered with an additional production $\frac{d}{dx}\frac{C(x)}{x} < 0$

Subsection 2

National Income Model

National Income Model

$$\frac{dY}{d\bar{I}} = \frac{1}{[1 - f'(Y)]} > 0, \because f'(Y) \in (0,1)$$

- $Y(C) = C + \bar{I} = f(Y) + \bar{I} \Rightarrow \frac{dY}{d\bar{I}} = f'(Y)\frac{dY}{d\bar{I}} + 1$
- Income is a function of consumption
  - The more you spend, the more income is for another person
- Consumption is a function of income
  - How much you spend depends on how much you have
- We are interested in knowing how a change in investment will affect national income
  - investment $\bar{I}$ is assumed to be exogenous (determiend outside of the system, or taken to be a given value)

Subsection 3

Elasticities

Elasticities

How certain variable change in response to changes in other variables

Elasticity of f with respect to x

$$\epsilon_x f(x) = \frac{df(x)}{dx}\left[\frac{x}{f(x)}\right] = f'(x)\left[\frac{x}{f(x)}\right]$$

- An equivalent expression:

$$\frac{d\ln f(x)}{d\ln(x)}$$

  - most of the time it's easier to differentiate the log transformed function

Subsection 4

Risk Aversion

Risk Aversion

Arrow-Pratt measure of absolute risk aversion

$A(w) = -\frac{u''(w)}{u'(w)}$

Arrow-Pratt measure of absolute risk aversion

$R(w) = wA(w) = -w\frac{u''(w)}{u'(w)}$

- $u(w) :=$ utility based on the level of wealth w