EC2104 Lecture 3 - Single Variable Optimization

ling

Section 1

Implicit Differentiation

Implicit Differentiation

Solving $\frac{dy}{dx}$ without an explicit function

Steps:

1. Take derivatives of both side with respect to x
2. Solve for $\frac{dy}{dx}$

E.g. Solve $\frac{dy}{dx}$ for $y^2 + xy = 5$

1. $2y\frac{dy}{dx} + y + x\frac{dy}{dx} = 0$
2. $\frac{dy}{dx} = \frac{-y}{2y+x}$

Subsection 1

Differentiating Inverse Function

Differentiating Inverse Function

Common trick used in probability to retrieve transformed random variable

Differentiating Inverse Function

If x_0 is an interior point of interval I and $f'(x_0) \neq 0$, then g is differentiable at $y_0 = f(x_0)$ and

$$g'(y_0) = \frac{1}{f'(x_0)}$$

Section 2

Taylor Approximation

Taylor Approximation

nth-order Taylor polynomial

Approximate the function $f(x)$ around $x = a$

$$f(x) \approx \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

- Building block in many numerical approximation for solving optimization problems.
  - Besides economics, machine learning and operational research field heavily uses this approximation in their algorithms.
- Note that we are approximating $f(x)$ at a point a
  - When a change, the approximated value $\tilde{y}$ changes
- Error of approximation $(\tilde{y} - y)$ increase as x moves further away from a
- Note:
  - $0! = 1, n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$
  - $f^{(k)}(a)$ is the kth derivatives evaluated at a

Section 3

Convexity

Convexity

Key building blocks used in many techniques, including optimization

Convexity

A function is said to be convex if for any a, b and for any $t \in [0, 1]$

$$f(ta + (1 - t)b) \leq tf(a) + (1 - t)f(b)$$

- Strictly convex if $f(ta + (1 - t)b) > tf(a) + (1 - t)f(b)$
- Concave function when $f(ta + (1 - t)b) \geq tf(a) + (1 - t)f(b)$
  - Strictly concave when $f(ta + (1 - t)b) < tf(a) + (1 - t)f(b)$
- Note that a concave function is the negative of convex function
  - If $f(x)$ is concave, then $-f(x)$ is convex

Subsection 1

Testing Convexity with second derivative

Testing Convexity with second derivative

Second derivative test for convexity

Suppose that f is continuous in the interval I and twice differentiable is in the interior of I. Then:

- $f''(x) > 0 \Rightarrow f(x)$ is strictly convex in I
- $f''(x) \geq 0 \Rightarrow$ convex only
- $f''(x) < 0 \Rightarrow f(x)$ is strictly concave in I
- $f''(x) \leq 0 \Rightarrow$ concave only

Section 4

Single variable optimization

Subsection 1

Extreme Value Theorem

Extreme Value Theorem

Useful to know if extrema exist (note: but it does not tell you if extrema does not exist)

Extreme Value Theorem

Suppose that f is continuous function over a closed and bounded interval $[a, b]$. Then there exist point a minimum point d and a maximum point c

- Must be continuous
- Must be closed (point exist at the boundary)
  - Closed: $[1, 2]$
  - Open: $(1, 2), (1, 2], [1, 2)$
- Must be bounded
  - (skipping the definition) points must exist and not infinity

Subsection 2

Finding extrema using turning points

Subsection 3

Testing identify of turning points using First Derivative Test

Subsection 4

Testing identify of turning points using Second Derivative Test

Subsection 5

Determine point of inflection

Subsection 6

Note on Global vs Local Extrema

Note on Global vs Local Extrema

Extreme points are points where functions reach their highest/lowest values

Extrema

If $f(x)$ has domain D,

- $c \in D$ is maximum point for $f \Leftrightarrow f(x) \leq f(c), \forall x \in D$
- $d \in D$ is maximum point for $f \Leftrightarrow f(x) \geq f(c), \forall x \in D$

- Local extrema when the point is max/min point in a neighbourhood (within a interval of values)
- Global extrema when the point is max/min point across all the domain
- Generally, a function will have multiple local extrema
  - Only for a special subset of functions, the local extrema = global extreme (when functions are strictly convex/concave)
  - Determining global extrema remain an unsolved problem outside of convex functions. Best way is to compare the functional values

Subsection 7

Summary on finding the extrema of functions

Summary on finding the extrema of functions

Setup

Find maximum/minimum values of a differentiable function f defined on a closed, bounded interval $[a, b]$

Then the General step for finding the extrema:

1. Consider interior point
   - Using First Order Condition (FOC)
2. Consider boundary (end points)
   - In the case of the setup, $f(a), f(b)$
3. Consider non-differentiable points

Compare the functional values to determine global extrema

Determine extrema of unbounded functions

Value of extreme must be found through methods of limits