EC3312 Game theory
Lingjie, September 26, 2020

Game theory basics

GT does not state how players *do* but *how* should behave

Nash Theorem: in n-player normal-form game G, if n is finite and S_i is finite $\forall i$ $\exists$ at least one NE, possibly involving mixed strategy

	Complete information	**Incomplete information**
Static	Normal-form games (NE)	Bayesian games (Bayesian NE)
Dynamic	PI games (Subgame perfect NE)	II games (Perfect Bayesian NE)

NE = Nash equilibrium
PI game = Extensive games with perfect information
II game = Extensive games with imperfect information
Complete info = each players' payoff function is common knowledge among all the players

Steps in solving the game:

1. set up (player, strategy, payoff)
2. simplify game through $IEDS$
3. play edge cases, balance game & A dominate B
4. solve game by double underline or FOC
5. play the game to see if it makes logical sense

Notation for best strategy

$R_i(q_i)$ player i's best-response *function*
$r_i^*(q_i)$ player i's best-response *correspondence*

Strictly Dominated Strategy

s_i' is strictly dominated by $s_i'' \Leftrightarrow u_i(s_i', s_{-i}) < u_i(s_i'', s_{-i})$
Rational player may not exclude playing a weakly dominated strategy but will never play a strictly dominated strategy

Information set/ Contigency

a collection of decision nodes satisfying

1. players need to move at every node
2. players do not know which node he's in
$\Rightarrow$ each node player must have same set of feasible actions. In singleton, players knows that the only node has been reached

perfect information : every information set is singleton
imperfect information : $\exists$ at least one nonsingleton information set (dotted line)
: can't differentiate the payoffs between the sets

1. Set up: Representation

Normal-form representation

$G = \{S_i, u_i \mid i \in [1, n]\}$
S_i is the strategy space and u_i is the payoff function of player i

Extensive-form representation

1. players

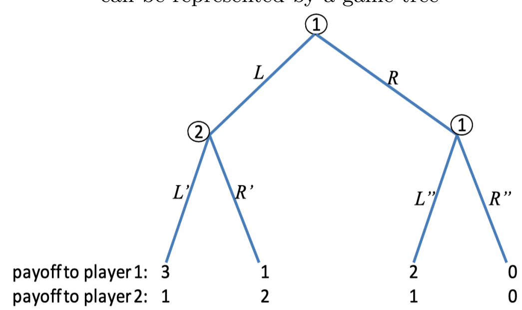
2a. timing for moves

2b. strategy at each timing

2c. information players know

3. payoffs from each possible combination of moves

can be represented by a game tree



Subgame in extensive-form

1. begins at a decision node n that is a singleton information set (but not first decision node)
2. includes all decision and terminal nodes following n in the game tree
3. does not cut any information sets

1. Set up: Strategy

Strategy

Complete plan of actions, specifies feasible actions for each player in every contingency (or information set)
f(contingency1, contingency2) $\rightarrow$ (action1, action2)

Mixed Strategies

mixed strategy for player i is a probability distribution
$p_i = \{p_{ik}\}$ $\forall k \in [1, K]$ for corresponding $S_i = \{s_{ik}\}$
$0 \le p_{ik} \le 1$ and $\sum_1^K p_ik = 1$

1. Set up: Payoff

Payoff

$u_i(s_i, s_{-i})$

Expected payoffs

$v_i(p_i, p_{-i}) = \sum_{j=1}^{J} \sum_{k=1}^{K} [p_{1j} p_{2k} u_i(s_{1j}, s_{2k})]$

2. Simplify

Iterated Elimination of Strictly Dominated Strategies

$IESD$ eliminate dominated strategies by **pure** or **mixed strategy**
NE always survive $IESD$ and unique strategy from $IESD \rightarrow NE$

4. Solution: Equilibrium

Nash Equilibrium (Pure strategy)

s_i^* is player i's best response to strategies specified for other players

$$u_i(s_i^*, s_{-i}^*) \ge u_i(s_i, s_{-i}^*) \Leftrightarrow max_{s_i \in S_i} u_i(s_i, s_{-i}^*)$$

NE can exist as a set of values $e.g.\{s^* \mid s_1^* + s_2^* = 1\} \cup \{(1, 1)\}$

Nash Equilibrium (Mixed Strategies)

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Backward-Induction Outcome

every finite game with perfect information has an SPNE, which can be found by backwards induction

1. start with last players: each of last players choose strategy maximising their payoffs
2. now with second last players: take last players' strategy as given then select their strategy
3. continue until the beginning of game

The BI outcome is $(a_1^*, R_2(a_1^*))$ but subgame perfect NE is $(a_1^*, R_2(a_1))$

Subgame-Perfect Nash Equilibrium

A Nash equilibrium is subgame-perfect if the players' strategies constitute a Nash equilibrium in every (proper) subgame
Subgame perfect outcome is $(a_1^*, a_2^*, a_3^*(a_1^*, a_2^*), a_4^*(a_1^*, a_2^*))$ but subgame perfect NE is $(a_1^*, a_2^*, a_3^*(a_1, a_2), a_4^*(a_1, a_2))$

Solving Static Games of Complete Information with Pure Strategy

first form normal-form game, then solve by

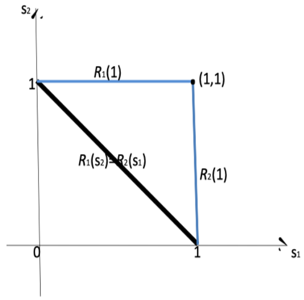
1. maximisation problem
$max_{p_i \in P_i} u_i(p_i, p_{-i}^*) \Rightarrow \frac{d}{dp_i} = 0$ then solve n players' optimal strategies simultaneously
2. graphically
plot S_i against S_j to find intersection
3. $IESD$
eliminate the impossible strategies by taking the first step in game, then repeat till equilibrium is reached

Splitting a pie

2 players choose to split a dollar $S_i = [0, 1]$
NE: $\{s^* | s_1^* + s_2^* = 1, s_1^*, s_2^* \geq 0\} \cup \{(1, 1)\}$

$$u_i(s_i, s_{-i}) = \begin{cases} s_i, & s_1 + s_2 \leq 1 \\ 0, & s_1 + s_2 > 1 \end{cases}$$

$$R_i(s_{-i}) = \begin{cases} 1 - s_{-i}, & s_2 < 1 \\ [0, 1], & s_{-i} = 1 \end{cases}$$

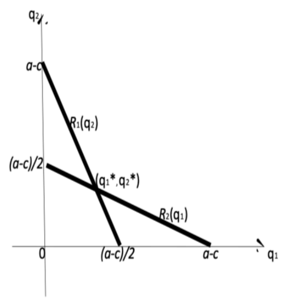


Cournot Model of Duopoly

2 firms choose **quantity**, $S_i = [0, \infty), q_i \geq 0$
$NE : (q_1^*, q_2^*) = (\frac{a-c}{3}, \frac{a-c}{3})$

$$u_i(s_1, s_2) : \pi_i(q_1, q_2) = q_i[a - (q_1 + q_2) - c]$$

$$R_i(q_j) : q_i = \frac{1}{2}(a - c - q_j)$$



Bertrand Model of Duopoly

2 firms choose **price**, $S_i = [0, \infty), p_i \geq 0$
$NE : (p_1^*, p_2^*) = (\frac{a+c}{2-b}, \frac{2+c}{2-b})$

$$u_i(s_1, s_2) : \pi_i(p_1, p_2) = (a - p_i + bp_j)(p_i - c)$$

$$R_i(p_j) : p_i = \frac{1}{2}(a + bp_j + c)$$

$b > 0$ for substitute products and marginal cost $c : 0 < c < a$

Voting (2 players)

NE: $(s_1, s_2) = (\frac{1}{2}, \frac{1}{2})$
Player 1, 2 decide of a location on line $s_i \in [0, 1]$

$$u_i(s_i, s_{-i}) = \begin{cases} \frac{s_1+s_2}{2}, & s_i \leq s_j \\ 1 - \frac{s_1+s_2}{2}, & s_j \leq s_i \end{cases}$$

if both players do not choose the same platform, assume $s_i < s_j$, then i can improve payoff by choosing $s_i + \epsilon$
if both players do not choose 0.5 platform, assume they choose $x > 0.5$, then each can improve payoff by choosing $s_i - \epsilon$
Note: for 3 players voting there is no pure strategy NE

Final-Offer Arbitration

Firm offer w_f and Union offer w_u, arbitrator with ideal x wage

$$w = \begin{cases} w_f, & x < (w_f + w_u)/2 \\ w_u, & x > (w_f + w_u)/2 \end{cases}$$

$$\begin{aligned} E(w) &= P(w = w_f)w_f + P(w = w_u)w_u \\ &= F_X(w_f + w_u)/2)w_f + [1 - F_X(w_f + w_u)/2)]\, w_u \end{aligned}$$

solving for $\max_{w_f} E(w)$ and $\max_{w_u} E(w)$
$\Rightarrow (w_u^* - w_f^*) \cdot \frac{1}{2} f_X(\frac{w_f^* + w_u^*}{2}) = F(\frac{w_f^* + w_u^*}{2}) = 1 - F(\frac{w_f^* + w_u^*}{2})$
$\Rightarrow F(\frac{w_f^* + w_u^*}{2}) = \frac{1}{2}$ and $w_u^* - w_f^* = \frac{1}{f_X(\frac{w_f^* + w_u^*}{2})}$
Assume $x \sim N(m, \sigma^2)$, from $F(\frac{w_f^* + w_u^*}{2}) = \frac{1}{2} \Rightarrow \frac{w_f^* + w_u^*}{2} = m$
$w_u^* - w_f^* = \frac{1}{f_X(\frac{w_f^* + w_u^*}{2})} = \sqrt{2\pi\sigma^2}$
Solving the two equations, NE:
$(w_u^*, w_f^*) = (m + \sqrt{\frac{\pi\sigma^2}{2}}, m - \sqrt{\frac{\pi\sigma^2}{2}})$

The Problems of the Commons

farms $i \in [1, n]$ keeps goat g_i, $G = \sum g_i$, grazing benefit $v(G)$

$$u_i = g_i v(G) - c g_i$$

$\max_{g_i} u_i \Rightarrow g_i v'(G) + v(G) - c = 0 = \frac{1}{n} G^* v'(G^*) + v(G^*) - c$
but for society: $\max_G u \Rightarrow G^{**} v'(G^{**}) + v(G^{**}) - c$
since $G^* > G^{**}$, common resource is overutilised

Solving Static Games of Complete Information with Mixed Strategy

Matching Pennies

	heads	tails
heads	-1,1	1,-1
tails	1,-1	-1,1

MSNE: $(p_1^*, p_2^*) = (\frac{1}{2}H + \frac{1}{2}T, \frac{1}{2}H + \frac{1}{2}T)$
$p_1 = (r, 1 - r), p_2 = (q, 1 - q)$

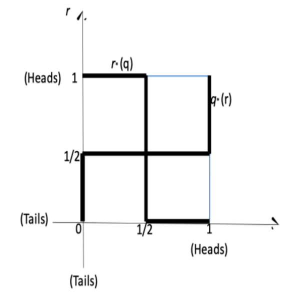
$$\begin{aligned} v_1(p_1, p_2) =& r \cdot q \cdot (-1) + (1 - r) \cdot q \cdot (1) \\ &+ r \cdot (1 - q) \cdot (1) + (1 - r) \cdot (1 - q) \cdot (-1) \\ =& -4rq + 2r + 2q - 1 \end{aligned}$$

$$R_2(r) : q(r) = \frac{1}{2}$$

graphically

$$v_1(s_{11}, p_2) = q \cdot (-1) + (1 - q) \cdot 1 = 1 - 2q$$
$$v_1(s_{12}, p_2) = q \cdot 1 + (1 - q) \cdot (-1) = -1 + 2q$$

$$r(q) = \begin{cases} 1, & q < \frac{1}{2} \\ [0, 1], & q = \frac{1}{2} \\ 0, & q > \frac{1}{2} \end{cases}$$

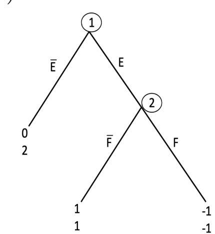


Solving Dynamic Games of Complete and Perfect Information

Besides (player, strategy, payoff), **timing** is required
Theory: Backwards Induction

Entry Deterrence

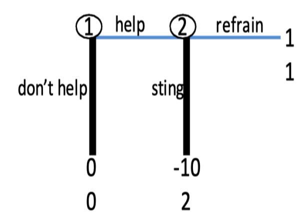
BI outcome: $(E, \bar{F})$



Payer 1 is an entrant; Payer 2 is an incumbent
E = enter, Ē = don't enter, F = fight, F̄ = don't fight

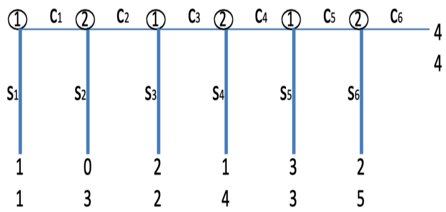
The Frog and the Scorpion

BI outcome: (don't help, sting)



Centipede game

BI outcome: $(s_1, s_2, s_3, s_4, s_5, s_6)$

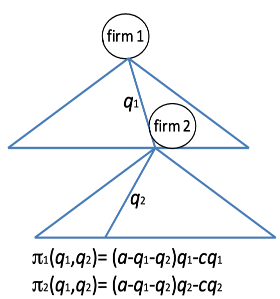


Stackelberg Duopoly

a leader and follower deciding on output to produce
timing:
1. leader $q_1 \geq 0$
2. follower observe then $q_2 \geq 0$

BI outcome: $(q_1^*, q_2^*) = (\frac{a-c}{2}, \frac{a-c}{4})$

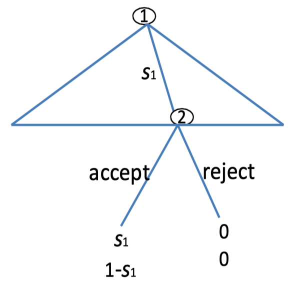
$$u_i : \pi_i(q_i, q_j) = P(Q)q_i - cq_i$$
$$R_2(q_1) = \frac{1}{2}(a - c - q_1)$$
$$\pi_1(q_1, R_2(q_1)) = \frac{1}{2}(a - c - q_1)q_1$$
$$R_1(q_2) = \frac{a-c}{2}$$



Sequential Bargaining (1-Period)

player 1, 2 splitting pie
timing:
1. player 1 propose s_1, leave $1 - s_1$ for player 2
2. player 2 then accept/reject

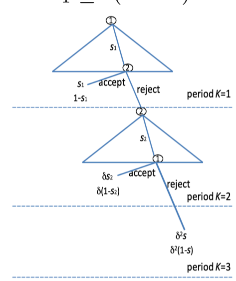
BI outcome: $(s_1^* = 1, R_2(s_1^*) = \text{accept})$



Sequential Bargaining (3-Period)

player 1, 2 splitting pie
timing:
1. player 1 propose s_1, leave $1 - s_1$ for player 2
2. player 2 then accept/reject, if reject then continue
3. game repeat till period 3

BI outcome: 1 offer $(s_1^*, 1 - s_1^*)$ and 2 accept immediately
$(s_2^*, 1 - s_2^*) = (\delta s, 1 - \delta s)$
player 1 accepts iif $s_2 \geq \delta s$
$(s_1^*, 1 - s_1^*) = (1 - \delta(1 - \delta s), \delta(1 - \delta s))$
player 2 accepts iif $1 - s_1 \geq \delta(1 - \delta s)$



The players discount payoffs received a period later by a factor δ ∈(0,1)

Note: they are only comparing between this period and next

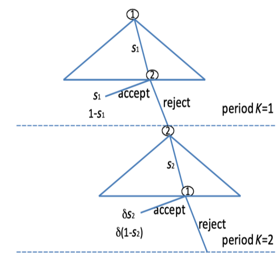
Rubinstein's Model (Infinite-Period)

player 1, 2 splitting pie
timing:
1. player 1 propose s_1, leave $1 - s_1$ for player 2
2. player 2 then accept/reject, if reject then continue
3. game repeat infinity

BI outcome: player 1 offer $(s^*, 1 - s^*) = (\frac{1}{1+\delta}, \frac{\delta}{1+\delta})$
Since when 1 offers $(f(s), 1 - f(s))$ 2 will accept immediately,

$$f(s) = 1 - \delta(1 - \delta s)$$

derive answer through G.S.



The first two periods of the alternative-offer bargaining game

Note: they are only comparing between this period and next

Solving Two-Stage Games of Complete but Imperfect Information

Theory: Subgame Perfection

Bank run

NE:
1. (withdraw, withdraw),
2. [(don't, don't), (withdraw, withdraw)]

		withdraw	don't
date 1	withdraw	40,40	50,30
	don't	30,50	next stage

		withdraw	don't
date 2	withdraw	100,100	150,50
	don't	50,150	100,100

work backwards: solve for date 2, then date 1