Lecture 12 - Inequality Constraints Optimization

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Section 1

Lagrange Multiplier Method (Inequality)

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Solving inequality constraint optimization

$$\max_{\mathbf{x}} f(\mathbf{x}) \ s.t. \ h_i(\mathbf{x}) - c_i \leq 0, i = 1, \cdots, m$$

Solve Lagrangian function instead

$$\max_{\mathbf{x},\mu} \mathcal{L}(\mathbf{x}, \mu) = f(\mathbf{x}) - \sum_{i=1}^{m} \mu_i [g_i(\mathbf{x}) - c_i], \mu_i \geq 0$$

- Note the constraint presentation in order to ensure $\mu_i \geq 0$

Note the difference between equality constraint:

- λ_i is unconstrained ($\in R$), μ_i has to be non-negative (≥ 0).
- Representation here is for maximisation problem, note that for minimisation problem the form will be different.

Complementary slackness

Solving inequality optimization problems is more challenging than equality optimization problems. We need to consider cases when

1. Constraint is binding: $h_i - m_i = 0$
2. Constraint is not binding: $h_i - m_i < 0$

Therefore, to summarises the possible cases, we introduce complementary slackness condition

Complementary Slackness

1. $\mu_i > 0,\ h_i - m_i = 0$
2. $\mu_i = 0,\ h_i - m_i < 0$

Solving Inequality Lagrangian

We need find KKT points (i.e. points that satisfied the following conditions)

1. FOC: $\frac{\partial \mathcal{L}(\mathbf{x},\mu)}{x_i} = 0$
2. Complementary Slackness: $\mu_i \geq 0$, $\mu_i = 0$ when $h_i - m_i < 0$
3. All constraints must be satisfied

Note:

- In general, solving inequality constraint problems require a lot of patience to work out each possible cases.
- For m constraints, there are $2m$ different cases to consider
- Note that Lagrangian method failed when $h(\mathbf{x}) = c$ and $h'_1 = h'_2 = 0$ (for two variable case)

FOC and SOC

First Order Necessary Condition (FOC)

- Unconstrained
  - Single variable: $f'(x^*) = 0$
  - Multiple variables: $\nabla f(\mathbf{x}^*) = \mathbf{0}$
- Equality constraint
  - Lagrangian: $\nabla \mathcal{L}(\mathbf{x}^*, \lambda^*) = \mathbf{0}$
- Inequality constraint
  - Lagrangian: $\frac{\partial \mathcal{L}(x_i^*, mu_i^*)}{x_i} = 0$
  - Complementary Slackness: $\mu_i \geq 0$, $\mu_i = 0$ when $h_i(\mathbf{x}) - c_i < 0$

Second Order Necessary and Sufficient Condition (SOC)

- Unconstrained (local min)
  - Single variable: $f''(x^*) > 0$
  - Multiple variables: $H_f(\mathbf{x}^*) \succ 0$ (positive definite)
- Equality constraint (local min)
  - Lagrangian: $H_{\mathcal{L}}(\mathbf{x}^*, \lambda^*) \succ 0$ (positive definite)

Sufficient Condition for global extreme point

Remark

We are testing if function is convex. Therefore, require the following to hold true for all $\mathbf{x}$ in domain, not just the minimizer $\mathbf{x}^*$

- Unconstrained (global min)
  - Single variable: $f''(x) \geq 0$
  - Multiple variable: $H_f(\mathbf{x}) \succeq \mathbf{0}$ (positive semi-definite)
- Equality constraint (global min)
  - Lagrangian: $H_{\mathcal{L}}(\mathbf{x}, \lambda) \succeq \mathbf{0}$ (positive semi-definite)

General comments on global extreme points

1. Closed and bounded domain will have global extreme points
   - Prove by Weierstrass Extreme Value Theorem
   - Compare objective values (including interior, boundary and non-differentiable points)
   - Consider $\max_{x_1,x_2} -x_1^2 + x_2 \; s.t. \; x_1^2 + x_2^2 = 1$
2. Unbounded domain might not have global extreme points
   - Consider $\max_{x_1,x_2} x_1^2 + x_2^2$