EC4302 Macroeconomics Analysis III

ling

2022-04-24

Contents

1 Basic Growth Facts

Central questions:

1. Why are some countries much poorer than others?
2. Why do countries grow?
3. What are the sources of recessions and booms?
4. Why is there unemployment and what determines its extent?

Questions 1 and 2 belongs to growth theory, question 3 belong to business cycle and question 4 belong to unemployment.

1.1 Evolution of input factors

Growth rate for $x(t)$ where $\dot{x} := \frac{dx(t)}{dt}$

$$g_x := \frac{\dot{x}}{x} = \frac{d\ln(x)}{dt} \Rightarrow X(t) = X_0 e^{g_x t}$$

Discrete-time growth rate:

$$g_x = \Delta \ln(x_t) = \ln(x_{t+1}) - \ln(x_t)$$

1.2 Equilibrium, steady-state, Balanced growth path, comparative statics, Golden-rule

- Equilibrium: k^* such that $\dot{k} = 0 \Rightarrow k(t+1) = k(t)$
  - An economy in equilibrium is called in steady-state
- Balanced growth path: how other variables behave when $k = k^*$
- Comparative statics: how changing hyper-parameter change variables
- Golden-rule of capital accumulation: k^* that maximises Consumption

2 Solow Growth Model

Variables	Symbol	Endogenous	growth rate
output	Y	yes	
capital	K	yes	
labor	L	no	n
knowledge	A	no	g

Equations and functions

1. Production functions: input factors to output
2. Saving, investment and capital accumulation over time

Assumes competitive market with no externalities:

1. Rent $r(t) = \frac{\partial F}{\partial K} = f'(k(t))$
2. Wage per person $W(t) = \frac{\partial F}{\partial AL} = A[f(k) - kf'(k)]$
   - Wage per unit of effective labour $w(t) = W(t)/A = f(k) - kf'(k)$

2.1 Neoclassical Production Function

$$Y(t) = F[K(t), A(t)L(t)] \Leftrightarrow y(t) = f(k(t))$$

- $A(t)L(t) :=$ efficiency units of labor
- $k(t) := \frac{K(t)}{L(t)A(t)}$, $f(k) := F(k, 1)$ intensive form (due to CRS)

Example: Cobb-Douglas function $F(K, AL) = K^\alpha (AL)^{1-\alpha}$

2.1.1 Critical assumptions

1. Constant returns to scale (CRS)
$$F(\lambda K, \lambda AL) = \lambda F(K, AL), \forall \lambda \geq 0$$
2. Diminishing marginal product (DMP)
$$\frac{\partial F}{\partial K} > 0, \frac{\partial^2 F}{\partial K^2} < 0$$

- same for labour, and capital per unit of effective labor

3. Inada conditions

$$\lim_{K\to 0}\frac{\partial F}{\partial K}=\infty,\quad \lim_{K\to\infty}\frac{\partial F}{\partial K}=0$$

2.2 Solow model: benchmark model with endogenous capital

Most important result: Differences in k cannot account for large income differences

Note: saving rate is exogenous

1. Evolution of capital
$$\dot{K}(t)=Y(t)-C(t)-\delta K(t)=sY(t)-\delta K(t)$$
2. Dynamics of k
$$\dot{k}=sy-(\delta+g+n)k$$
   - break-even investment: $(\delta+g+n)k$
3. Steady-state:
$$sf(k^*)=(\delta+g+n)k^*$$
4. Balanced growth path:
   - Capital $K=ALk^*\Rightarrow g_K=n+g$
   - Output $Y=f(k^*)\Rightarrow g_Y=n+g$
   - Capital per worker $=Ak^*\Rightarrow g_{Ak^*}=g$
   - Output per worker $=f(k^*)A\Rightarrow g_{f(k^*)A}=g$
   - Consumption per worker $=(1-s)f(k^*)A\Rightarrow g_{(1-s)f(k^*)A}=g$
5. Comparative statics of a change in s (saving rate)
$$\frac{\partial k^*}{\partial s}=\frac{f(k^*)k^*}{s[f(k^*)-k^*f'(k^*)]}>0$$
   - Effect on $(1)\dot{k},(2)k$, (3) $\frac{Y}{L}$, $(4)c=(1-s)f(k)$ (output per effective worker)
6. Golden-rule of capital accumulation
$$f'(k^*)=n+g+\delta$$
7. Long-run effect of s on output
$$EL_s(y^*)=\frac{\partial y^*/y^*}{\partial s/s}=\frac{\alpha_K(k^*)}{1-\alpha_k(k^*)}$$
   - $\alpha_K(k^*)=\frac{k^*f'(k^*)}{f(k^*)}$
8. Speed of convergence
$$k(t)-k^*\approx e^{-\lambda t}[k(0)-k^*]$$
   - $\lambda=-[sf'(k^*)-(\delta+g+n)]>0$

2.2.1 Solow in Discrete-time

Assume no population growth ($n=0$) and no technological process ($g=0$)

1. Law of motion for capital stock
$$k_{t+1}=(1-\delta)k_t+sf(k_t)$$
2. Steady-state
$$\frac{f(k^*)}{k^*}=\frac{\delta}{s}$$
3. Dynamics
   - All equilibrium converge to the positive steady-state k^*

3 Ramsey model: Solow with infinitely lived consumers determining saving rate

1. Evolution of capital
$$\dot{K}(t) = Y(t) - C(t)L(t)$$
2. Dynamics of c ($\dot{c}$ line in (k, c) space)
$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \rho - \theta g}{\theta}$$
   - $\dot{c}(t) = 0$ when $f'(k) = \rho + \theta g \Rightarrow k = k^*$
   - $\dot{c}(t) < 0$ when $f'(k) < \rho + \theta g \Rightarrow k > k^*$
   - $\dot{c}(t) > 0$ when $f'(k) > \rho + \theta g \Rightarrow k < k^*$
3. Dynamics of k ($\dot{k}$ curve in (k, c) space)
$$\dot{k}(t) = f(k(t)) - c(t) - (n + g)k(t)$$
   - $\dot{k}(t) = 0 \Rightarrow c(t) = f(k(t)) - (n + g)k(t)$
   - $k < k_{GR} \Rightarrow c$ is increasing
   - $k > k_{GR} \Rightarrow c$ is decreasing
4. Golden-rule of capital accumulation
$$f'(k) = (n + g)$$
   - Modified Golden-rule capital stock: $k_{KG} = n + g$
5. Equilibrium: a pair of functions $(c(t), k(t))$ s.t.
   1. Household: $\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \rho - \theta g}{\theta}$
   2. Capital: $\dot{k}(t) = f(k(t)) - c(t) - (n + g)k(t)$
   3. No-Ponzi-game: $\lim_{s\to\infty} e^{-R(s)}e^{(n+g)s}k(s) \geq 0$
   4. Non zero capital: $k(0) = k_0, k(t) \geq 0$
6. Welfare is maximised (by First welfare theorem)
   - Households utility is maximised
   - Planner face same technology as firm, thus obeying evolution of $k(t)$
7. Steady-state
$$\dot{c} = 0 \Rightarrow f'(k) = \rho + \theta g$$
$$\dot{k} = 0 \Rightarrow f(k) - (n + g)k = c$$
   - Saving rate: $\frac{y-c}{y}$ is constant (as c, k, y are constant)
8. Effects of a fall in the discount rate (ρ)
   - ρ decreases $\Rightarrow$ only c-locus is affected $\Rightarrow$ increase in k^*
9. Effects of a fall in growth rate of knowledge (g)
   - If $g > 0$, k^* increases with willingness to smooth consumption ($1/\theta$)
   - If $g = 0$, k^* is not affected by θ
10. Adding Government to the model

- Government buys at rate $G(t)$ per unit of effective labor per time
- $\dot{k}(t) = f(k(t)) - c(t) - G(t) - (n + g)k(t)$
- Higher G shifts $\dot{k}(t) = 0$ locus down
- Unexpected, permanent increase in $G \Rightarrow$ consumption immediately jumps to new saddle path
- Unanticipated, temporary increase in $G \Rightarrow$ effect on (c, k), $r(t)$, $c(t)$, $k(t)$

Most important result:

1. Optimising saving rate does not change Solow results
2. Saddle Path: There is a unique initial level of c that is consistent with the equilibrium conditions

3.1 Households

Note:

- $C(t) :=$ consumption per person
- $c(t) :=$ consumption per unit of effective labour
- $H :=$ number of households

- $\rho :=$ rate of time preference
- $\theta :=$ determines household's willingness $(1/\theta)$ to shift consumption between periods
  - $\theta = 0 \Rightarrow u(C(t)) = C(t)$ (more willing to shift)
  - $\theta = 1 \Rightarrow u(C(t)) = \ln(C(t))$ (less willing to shift)

1. Utility
$$U = \int_0^\infty e^{-\rho t} u(C(t)) \frac{L(t)}{H} dt$$
2. Instantaneous utility function
$$u(C(t)) = \frac{C(t)^{1-\theta}}{1-\theta}$$
   - $\theta > 0, \rho - n - (1-\theta)g > 0$
   - Constant relative risk aversion (CRRA) $\frac{-Cu''(C)}{u'(C)} = \theta$
3. Budget constraint
$$\int_0^\infty e^{-R(t)} C(t) \frac{L(t)}{H} dt \leq \frac{K(0)}{H} + \int_0^\infty e^{-R(t)} W(t) \frac{L(t)}{H} dt$$
   - $R(t) = \int 0^t r(\tau) d\tau :=$ growth rate of investment at time 0 to time t
4. No-Ponzi-game conditions (from budget constraint)
$$\lim_{s \to \infty} e^{-R(s)} \frac{K(s)}{H} \geq 0$$
   - Present value of the household's asset holdings cannot be negative in the limit
5. Household behavior (intensive form)
$$L = B \int_0^\infty e^{-\beta} t \frac{c(t)^{1-\theta}}{1-\theta} dt + \lambda \left[k(0) + \int_0^\infty e^{-R(t)} (w(t) - c(t)) e^{(n+g)t} dt \right]$$
   - $-\beta = -\rho + (1-\theta)g + n$
6. Euler equation (growth rate of intensive consumption)
$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho - \theta g}{\theta}$$
7. Consumption per worker
$$\frac{\dot{C}(t)}{C(t)} = \frac{r(t) - \rho}{\theta}$$

4 Diamond/OLG model: Discrete-time Solow with consumers living for two periods determining saving rate

Assumes

1. Discrete-time
2. No depreciation of capital stock $(\delta = 0)$

Results:

1. Evolution of Capital Stock
$$K_{t+1} = (w_t A_t - C_{1,t}) L_t$$
2. Dynamics of Capital stock
$$k_{t+1} = \frac{s(f'(k_{t+1}))}{(1+n)(1+g)} [f(k_t) - k_t f'(k_t)]$$
3. Equilibrium: type of sequences $\{w_t, r_t, k_t\}$ s.t.
   1. $r_t = f'(k_t)$
   2. $w_t = f(k_t) - k_t f'(k_t)$
   3. $k_{t+1} = \frac{s(f'(k_{t+1}))}{(1+n)(1+g)} [f(k_t) - k_t f'(k_t)]$
   4. $k_0 > 0$ (given)
4. Balanced growth path: not known in general, consider Cobb-Douglas and Log utility $(\theta = 1)$

Most important results:

1. Finitely lived consumers do not change Solow results
2. However, competitive equilibrium now is not Pareto optimal

4.1 Households

- L_{-1} := initial population
- L_{t-1} := individuals born in period $t-1$, the old
- $L_t := (1+n)L_{t-1}$ the young
- $A_0 k_0 = \frac{K_0}{L_0}$ initial capital stock owned by per production worker

1. Utility
$$U_t = u(C_{1,t}) + \frac{1}{1+\rho} u(C_{2,t+1})$$
2. Budget constraint
$$C_{2,t+1} = (1+r_{t+1})(w_t A_t - C_{1,t})$$
3. Household behavior
$$L = \frac{C_{1,t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho} \frac{C_{2,t+1}^{1-\theta}}{1-\theta} + \lambda \left[w_t A_t - \left(C_{1,t} + \frac{C_{2,t+1}}{1+r_{t+1}} \right) \right]$$
4. Euler equation
$$\frac{C_{2,t+1}}{C_{1,t}} = \left(\frac{1+r_{t+1}}{1+\rho} \right)^{\frac{1}{\theta}}$$
5. Saving rate
$$C_{1,t} = [1 - s(r_{t+1})] w_t A_t$$
$$S(r) = \frac{(1+r)^{\frac{1}{\theta}-1}}{(1+r)^{\frac{1}{\theta}-1} + (1+\rho)^{\frac{1}{\theta}}}$$
   - interest rate ($r(t)$) affect saving rate negatively (income effect) and positively (substition effect)
   - If $\theta = 1 \Rightarrow$ substition and income effect cancels
   - If $\theta < 1$, saving rate increase in interest rate (higher willingness for substition, substition effect dominates)

5 Growth accounting: Malthusian to Solow

6 Endogenous Growth

Key changes compare to previous growth models:

- country specific policies affect the permanent growth rate A_t
- endogenous growth is driven by innovative activities (R&D)

Sectors and dynamics:

- a_k := fraction of capital in used in R&D (and a_k for labour)
- $1 - a_k$:= fraction of capital in goods-producing (and $1 - a_l$ for labour)
- labor (L), capital (K), technology (A), output (Y)
- Assume no depreciation $\delta = 0$
- Continuous time
- Both sectors uses full stock of knowledge A (non rival)

1. Output (Y_t): combine inputs to produce final goods
$$Y(t) = [(1-a_k)K(t)]^{\alpha} [A(t)(1-a_L)L(t)]^{1-\alpha}$$
   - $0 \leq \alpha \leq 1$
2. R&D (A_t): innovate and produce new ideas
$$\dot{A}(t) = B [a_K K(t)]^{\beta} [a_L L(t)]^{\gamma} A(t)^{\theta}$$
   - $B > 0, \beta \geq 0, \gamma \geq 0$

3. Capital dynamics with exogenous saving rate
$$\dot{K}(t) = sY(t)$$
    - $0 < s < 1$
4. Exogenous population growth
$$\dot{L}(t) = nL(t)$$
    - $n \geq 0$

6.1 Simplified model - without capital

Simplified environment

- $Y(t) = A(t)(1 - a_L)L(t)$
- $\dot{A}(t) = B\left[a_L L(t)\right]^{\gamma} A(t)^{\theta}$

Dynamics of A

- $g_A(t) := \frac{\dot{A}(t)}{A(t)} = Ba_L^{\gamma} L(t)^{\gamma} A(t)^{\theta-1}$
- $\frac{\dot{g_A}(t)}{g_A(t)} = \gamma g_L + (\theta - 1)g_A(t) \Rightarrow \dot{g_A}(t) = \gamma n g_A(t) + (\theta - 1)[g_A(t)]^2$

Consider cases for $\dot{g_A}(t)$ vs g_A

1. $\theta < 1$: concave function
    - $g_A^* = \frac{\gamma}{1-\theta} n$: BGP
2. $\theta > 1$: convex function
    - No BGP, $\dot{g_A}$ increase to infinity as g_A increases
3. $\theta = 1$: linear function
    - If $n > 0$: No BGP, $\dot{g_A}$ increase to infinity as g_A increases
    - If $n = 0$: BGP with constant growth rate $g_A = g_Y = g_{Y/L} = BA_L^{\gamma}\bar{L}^{\gamma}$, a_L interpreted as the saving rate

Policy implications

- Higher population growth $g_L = n$ will lead to higher income growth
- However, it should actually be world population growth since knowledge is mobile and has learning or spillover effects

6.2 General model - with capital

Endogenous state variables: A, K

- State variable: stock variable, agent take as given in time t and does not change instantaneously
- Choice variable: decision variable, can be change instantaneously

Dynamics of Capital

- $\dot{K}(t) = s(1 - a_K)^{\alpha}(1 - a_L)^{1-\alpha} K(t)^{\alpha} A(t)^{1-\alpha} L(t)^{1-\alpha}$
- $c_K := s(1 - a_K)^{\alpha}(1 - a_L)^{1-\alpha}$
- $g_K(t) := \frac{\dot{K}(t)}{K(t)} = c_k \left[\frac{A(t)L(t)}{K(t)}\right]^{1-\alpha}$
- $\frac{\dot{g_K}(t)}{g_K(t)} = (1 - \alpha)[g_A(t) + n - g_K(t)]$

Dynamics of Knowledge

- $c_A := Ba_K^{\beta} a_L^{\gamma}$
- $g_A(t) := \frac{\dot{A}(t)}{A(t)} = c_A K(t)^{\beta} L(t)^{\gamma} A(t)^{\theta-1}$
- $\frac{\dot{g_A}(t)}{g_A(t)} = \beta g_K(t) + \gamma n + (\theta - 1)g_A(t)$

Locus of points where $\dot{g_K}, \dot{g_A} = 0$ is a constant:

- $\dot{g_K}(t) = 0 \Rightarrow g_K(t) = g_A(t) + n$
- $\dot{g_A}(t) = 0 \Rightarrow g_K(t) = -\frac{\gamma n}{\beta} + \frac{1-\theta}{\beta} g_A(t)$

Phase Diagram of g_A and g_K: solve $\dot{g_A} = 0, \dot{g_K}$

1. $\beta + \theta < 1$: unique BGP
    - $g_A^* = \frac{\beta+\gamma}{1-(\theta+\beta)} n$
    - $g_K^* = g_A^* + n$

- $g_Y^* = \alpha g_K^* + (1-\alpha)(g_A^* + n) = g_A^* + n$
- $g_{Y/L}^* = g_A^*$

2. $\beta + \theta = 1, n = 0$: unique BGP
   - $g_K(t) = g^* = c_K \bar{L}^{1-\alpha} \left[\frac{A(t)}{K(t)}\right]^{1-\alpha}$
   - $g_A(t) = g^* = c_A \bar{L}^{\gamma} \left[\frac{K(t)}{A(t)}\right]^{\beta}$
3. $\beta + \theta = 1, n > 0$: no BGP
   - similar outcome as $\theta = 1, n > 0$ or $\beta + \theta > 1$, $\theta > 1$ (without capital)

6.3 Simplified general model - learning by doing

All inputs engaged in goods production:

- $Y(t) = K(t)^{\alpha} \left[A(t)L(t)\right]^{1-\alpha}$

Learning occurs as side effect of new capital production

- $g_A = \phi g_K, \phi > 0$
- $\Rightarrow A(t) = BK(t)^{\phi}$
- $\Rightarrow Y(t) = K(t)^{\alpha} B^{1-\alpha} K(t)^{\phi(1-\alpha)} L(t)^{1-\alpha}$

Dynamics of capital

- $\dot{K}(t) = sY(t) = sB^{1-\alpha} K(t)^{\alpha+\phi(1-\alpha)} L(t)^{1-\alpha}$
- The BCP depends on how $\alpha + \phi(1-\alpha) compares to 1$

6.3.1 "Y=AK" models

Consider case where $\phi = 1 \Rightarrow \alpha + \phi(1-\alpha) = 1, n = 0$

- $Y(t) = bK(t)$, $b := B^{1-\alpha}\bar{L}^{1-\alpha}$
  - often represent b as A, thus the $Y = AK$ model
- $\dot{K}(t) = sbK(t)$
- BCP: $g_Y^* = g_K^* = sb$
  - saving rates affect long run growth because capital helps learning

7 Endogenous Growth - Romer Model

Key different from the endogenous growth model where allocation of resources to R&D sector is exogenous

- R&D is undertaken by profit-maximizing economic actors
- Ideas are excludable (e.g. patent)
- Monopolistic power of R&D firm (goods sector is still perfectly competitive)

Production Function (final goods and services)

- $Y = \left[\int_{i=0}^{A} L(i)^{\phi} di\right]^{1/\phi}$, $0 < \phi < 1$
  - $Z(i)$ denotes the specialised inputs
  - $L(i)$ denotes the quantity of labour used and the quantity of input used, $L(i) = Z(i)$
  - Available ideas extending continuously from 0 to A, A is a function of time but we look at a point in time here
- $Y = \left[A\left(\frac{L_Y}{A}\right)^{\phi}\right]^{1/\phi} = A^{(1-\phi)/\phi} L_Y$
  - $L_Y :=$ total number of workers producing inputs
  - Suppose the number of workers producing each available input is the same: $L(i) = L_Y/A$ for any $i \in (0, A)$
  - Constant returns to L_Y
  - Output is increasing in A

Market structure

- Monopolist patent holder
  - Hires workers in a competitive market and pay $w(t)$ to produce the inputs associated with his/her idea
  - Charges a constant price for each unit of the input
- Final output produced by a competitive firms who takes input prices as given

1. Cost minimization of output producer
   - $\min_{L(i)} \int_{i=0}^{A} p(i)L(i)di \ s.t. \left[\int_{i=0}^{A} L(i)^{\phi}di\right]^{1/\phi} = 1$
     - Normalise the budget constraint to 1
   - FOC: $L(i) : p(i) = \lambda(1/\phi) \left[\int_{i=0}^{A} L(i)^{\phi}di\right]^{1/\phi-1} = \lambda L(i)^{\phi-1}$
   - $L(i) = \left[\frac{\lambda}{p(i)}\right]^{1/(1-\phi)}$
     - Holder of idea patent faces an elasticity of demand of $\eta = \frac{1}{1-\phi}$
2. Economic Aggregates
   - Labour market clears: $L_A(t) + L_Y(t) = \bar{L}$
     - $L_A(t) :=$ total number of workers in R&D,
     - $L_Y(t) = \int_{i=0}^{A}(t)L(i,t)di :=$ total number of workers producing inputs
   - Production of new ideas: $\dot{A}(t) = BL_A(t)A(t)$
     - $B > 0, A > 0, A(0) > 0$
3. Utility maximising household
   - $\max_{C(t)} U = \int_{t=0}^{\infty} exp(-\rho t)\ln C(t)dt$ s.t. $\int_{t=0}^{\infty} exp(-rt)C(t)dt \leq X(0) + \int_{t=0}^{\infty} exp(-rt)w(t)dt$
   - $r :=$ interest rate, $X(0) :=$ initial wealth, $w(t) :=$ wage at time t
4. Free-entry condition in R&D:
   - Anyone can hire $\frac{1}{BA(t)}$ units of labour at the wage $w(t)$ and produce a new idea
   - $V(i,t) = \int_{\tau=t}^{\infty} exp(-r(\tau - t))\pi(i,\tau)d\tau = \int_{\tau=0}^{\infty} exp(-r\tau)\pi(i,t+\tau)d\tau$
     - $\pi(i,\tau) :=$ profits earned profits earned by the creator if idea i at time τ
   - $V(i,t) = \frac{w(t)}{BA(t)}$
     - present value of selling the idea-specific inputs $V(i,t)$ equals cost $w(t) \cdot \frac{1}{BA(t)}$
5. General equilibrium
   - Competitive labor markets imply that wages paid in R&D and input producers are equal
   - $C(t)\bar{L} = Y(t)$
     - Only use of output good is for consumption
     - All individuals are the same thus having identical consumption path

7.1 Solving the model

Steps:

1. Given a value of L_A, find the present value $V(i,t)$ and cost of creating an idea $\frac{w(t)}{BA(t)}$
2. Use the free-entry condition to solve for L_A
3. Verify that the L_A^* is constant over time
   - However, if L_A^* is time-varying, then there is not equilibrium solution

- Profits of a monopolist patent holder
  - Given: profit maximization price of a monopolist is $\frac{\eta}{\eta-1} \cdot MC$, where $\eta = \frac{1}{1-\phi}$
  - Therefore, patent holder charges charges price $p(i) = \frac{\eta}{\eta-1}w(t) = \frac{w(t)}{\phi}$
  - Since the price all inputs are the same (i.e. $p(i)$ is a constant), quantity of each input $L(i)$ is the same and labour used for each patent is $\frac{\bar{L}-L_A}{A(t)}$
  - The profit at time t (quantity times per quantity profit) is: $\pi(t) = \frac{\bar{L}-L_A}{A(t)}\left[\frac{w(t)}{\phi} - w(t)\right] = \frac{1-\phi}{\phi}\frac{\bar{L}-L_A}{A(t)}w(t)$
- Growth rates in equilibrium
  - When $L_A(t)$ is constant, $\dot{A}(t) = BL_A A(t) \Rightarrow g_A = BL_A$
  - When $L_Y(t)$ is constant, $g_Y = \frac{1-\phi}{\phi}g_A$
  - Since all outputs are consumed, $g_C = g_Y$
  - Since all payments goes to worker (no capital) and $L_Y(t)$ is constant $\Rightarrow w(t)$ grows at rate g_Y
- Interest rate $r(t)$
  - Solving FOC for household utility maximisation problem, $C(t) : exp(-\rho t)\frac{1}{C(t)} = \lambda exp(-r(t)t)$
  - Solving for growth rate of $C(t)$: $g_c = \frac{\dot{C}(t)}{C(t)} = r(t) - \rho$
  - Using the result of g_C from growth rates in equilibrium $g_C = \frac{1-\phi}{\phi}BL_A \Rightarrow r(t) = r = \rho + \frac{1-\phi}{\phi}BL_A$
- Present Value of discovering ideas
  - Note that $\pi(t) = \frac{1-\phi}{\phi}\frac{\bar{L}-L_A}{A(t)}w(t)$, which grows at $g_W - g_A$ and discount rate is r
  - Solving for $V(i,t) = \int_{\tau=0}^{\infty} exp(-r\tau)\pi(i,t+\tau)d\tau = \int_{\tau=0}^{\infty} exp(-r\tau)\pi(t)exp((g_w - g_A)\tau)d\tau = \pi(t)\int_{\tau=0}^{\infty} exp((g_w -$

$g_A - r)\tau)d\tau$
  - Given expression for $\pi(t)$ and $g_W - g_A - r = g_Y - g_A - (\rho + g_Y) = -(g_A + \rho) \Rightarrow V(i,t) = \pi(t) \int_0^\infty exp(-(g_A + \rho)\tau)d\tau = \pi(t) \cdot -\frac{1}{g_A+\rho} exp(-(g_A+\rho)\tau)|_0^\infty = \pi(t) \cdot \frac{1}{g_A+\rho}$
- Equilibrium L_A
  - Given $g_A = BL_A$, solution of $V(i,t)$ and equate to the cost: $\frac{1-\phi}{\phi}\frac{\bar{L}-L_A}{g_A+\rho}\frac{w(t)}{A(t)} = \frac{w(t)}{BA(t)} \Rightarrow L_A = (1-\phi)\bar{L} - \frac{\phi\rho}{B}$
  - Since $L_A < 0$ implies patent holder will gain negative profit $\Rightarrow L_A = \max\left\{(1-\phi)\bar{L} - \frac{\phi\rho}{B}, 0\right\}$
  - And growth rate of output: $g_Y = \frac{1-\phi}{\phi} BL_A = \max\left\{\frac{(1-\phi)^2}{\phi}B\bar{L} - (1-\phi)\rho, 0\right\}$
- Implications on long-run growth
  - increase ρ reduce growth due to lower patience
  - increase ϕ reduces growth: more substitutability among inputs, less monopoly power and lower L_A
  - Increase B increase growth: higher g_A
  - Increase population size $\bar{L}$ raises growth
- Equilibrium versus Optimal growth
  - $L_A^{EQ} = (1-\phi)L_A^{OPT} \Rightarrow L_A^{EQ}$ is always smaller than L_A^{OPT}. Therefore growth always inefficiently slow due to externalities

7.2 Empirical implications

Long-run growth

- Correlates of changes in long-run
  - From learning-by-doing model: $Y(t) = bK(t) \Rightarrow \frac{Y(t)}{L(t)} = b\frac{K(t)}{L(t)}$ and $g_{Y/L}(t) = g_K(t) - g_L(t)$
  - $g_K = \frac{sY(t)}{K(t)} - \delta$
  - Since s grows and $Y/K, \delta$ is stable $\Rightarrow g_{Y/L}$ increases
- Magnitudes of changes in long-run
  - Model prediction: with $Y/K = 0.4$, s increase 1 percentage point per decade
  - $\Rightarrow g_K and g_{Y/L}$ increase by 0.4 percentage points per decade
- Trend of growth rates in data
  - OLS: $g_{Y/L} = a + bt + e_t$
  - $b \in (-0.026, 0.028)$, does not contain the model prediction 0.04

Population growth

- Context
  - Population grew rapidly, but Y/L does not change much pre-industrial revolution, only after that period
  - $\Rightarrow$ technological progress mainly increase L instead of Y/L
  - $\Rightarrow$ endogenous growth model suggest rate of population growth should been rising
- Kremer framework
  - Output: $Y(t) = T^\alpha[A(t)L(t)]^{1-\alpha}$
  - Knowledge: $\dot{A}(t) = BL(t)A(t)^\theta$
  - Malthusian assumption: $\frac{Y(t)}{L(t)} = \bar{y}$, a fixed level
  - $\Rightarrow g_L \propto L(t)^\psi$, $\psi = 1 - \left[\frac{(1-\theta)\alpha}{1-\alpha}\right]$
  - Data shows indeed $g_L \propto L_t$
- Population prediction when regions are cut off from one another
  - No worldwide technological growth, model predicts population growth is proportional to land areas
  - $\Rightarrow$ regions with larger population have larger g_A thus higher g_L. Therefore, larger regions grow to have higher population densities before contact is reestablished
  - Data shows lesser land area proportional to smaller density, suggesting innovations are made by people
- g_L and $g_{Y/L}$ over very long run
  - When g_A is small, population adjusts rapidly enough to keep Y/L close to $\bar{y}$
  - However, this is no longer true when L is large enough and technological grows rapidly. Then population appears to be decreasing in y when y exceed some level
  - In cases where $\theta \le 1$, $\lim_{t\to\infty} g_A = \lim_{t\to\infty} L = 0$

8 Real Business Cycle

Key difference compared to other growth models:

- Focus on short-run fluctuations
- Allow shocks in technology and government sector
- Adds leisure choice to consumers

Model overview

- Built on Ramsey model
- Discrete time
- Consist of:
    1. Competitive identical, price-taking firms
    2. Identical, price-taking households
    3. Infinitely lived households
    4. Government collect non-distortionary taxes and makes purchases

Players

- Production, capital dynamics, wage and rent
    - $Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha}, 0 < \alpha < 1$
    - $K_{t+1} = K_t + I_t - \delta K_t = (1-\delta)K_t + Y_t - C_t - G_t$
    - $w_t = \frac{\partial Y_t}{\partial L_t} = (1-\alpha)\left(\frac{K_t}{A_t L_t}\right)^{\alpha}$
    - $r_t = \frac{\partial Y_t - \delta K_t}{\partial K_t} = \alpha\left(\frac{A_t L_t}{K_t}\right)^{1-\alpha} - \delta$
- Representative Household: maximise expected value of lifetime utility
    - $U = \sum_{t=0}^{\infty} exp(-\rho t) u(c_t, 1-l_t)\frac{N_t}{H}$
    - N_t := population, H := number of households, ρ := discount rate, l_t := labour supply
    - Budget constraint: $c_1 + \frac{1}{1+r}c_2 = w_1 l_1 + \frac{1}{1+r}w_2 l_2$
    - Population growth: $\ln N_t = \bar{N} + nt, n < \rho$ (growth rate $= n$)
    - $u_t = \ln c_t + b\ln(1 - I_t)$
    - $c_t := \frac{C}{N}$, consumption per household member
    - $1 - l_t := 1 - \frac{L}{N}$, leisure per household member
- Shocks: technology
    - growth rate g and random shock
    - $\ln A_t = \bar{A} + gt + \tilde{A}_t$
    - AR(1) disturbances: $\tilde{A}_t = \rho_A \tilde{A}_{t-1} + \epsilon_{A,t}$, $-1 < \rho_A < 1$
- Shocks: government purchases
    - per capital trend growth rate $g \Rightarrow$ growth rate $n + g$
    - $\ln G_t = \bar{G} + (g+n)t + \tilde{G}_t$
    - AR(1) disturbances: $\tilde{G}_t = \rho_G \tilde{G}_{t-1} + \epsilon_{G.t}$, $-1 < \rho_G < 1$

Household decisions

- Labor supply: FOC after solving Lagraian for utility maximisation
    - $(l_1) : \frac{b}{1-l_1} = \lambda w_1$
    - $(l_2) : \frac{exp(-\rho)b}{1-l_2} = \frac{1}{1+r}\lambda w_w$
- Elasticity of substitution between leisure
    - From $(l_1)/(l_2) : \frac{1-l_1}{1-l_2} = \frac{1}{exp(-\rho)(1+r)}\frac{w_2}{w_1}$ (intertemporal subsitution in labor supply)
    - elasticity of substitution between leisure in the two periods is 1
    - increase in r also increase first-period labor supply (l_1) relative to second-period supply (l_2)
- Optimization under uncertainty by changing consumption this period (inter-temporal)
    - Marginal cost of reducing consumption = expected marginal benefit of saving and consume in next period
    - Marginal utility in period $t : exp(-\rho t)\frac{N_t}{H}\frac{1}{c_t}$
    - Increase in consumption in $t+1 : exp(-n)(1 + r_{t+1})\Delta c$
    - Marginal utility in period $t+1 : exp(-\rho(t+1))\frac{N_{t+1}}{H}\frac{1}{c_{t+1}}$
    - $MC_t = MU_{t+1} \cdot \Delta c_{t+1}$: $\Rightarrow exp(-\rho t)\frac{N_t}{H}\frac{\Delta c}{c_t} = E_t\left[exp(-\rho(t+1))\frac{N_{t+1}}{H}\frac{1}{c_{t+1}}exp(-n)(1+r_{t+1})\right]\Delta c$
    - Simplified solution: $\frac{1}{c_t} = exp(-\rho)E_t\left[\frac{1}{c_{t+1}}(1+r_{t+1})\right]$
- Optimization by changing leisure this period (intra-temporal)
    - Marginal dis-utility of labour = marginal utility benefit of consumption
    - $exp(-\rho t)\frac{N_t}{H}\frac{b}{1-l_t}\Delta l = exp(-\rho t)\frac{N_t}{H}\frac{1}{c_t}w_t\Delta l$
    - Simplified solution: $\frac{c_t}{1-l_t} = \frac{w_t}{b}$

– Note that uncertainty does not affect decision between current labour and consumption

8.1 Simplified RBC - without government

Changes

- No government G_t
- Complete depreciation $\delta = 1$
- $K_{t+1} = Y_T - C_t$
- $1 + r_t = \alpha(\frac{A_t L_t}{K_t})^{1-\alpha}$
- Note consumption now is simply $C_t = (1-s)Y_t, c_t = (1-s)\frac{Y_t}{N_t}$

Simplified equations

- inter-temporal Optimization: $\frac{1}{c_t} = exp(-\rho)E_t\left[\frac{1+r_{t+1}}{(1-s_t)\frac{Y_t}{N_t}}\right]$ with $c_t = (1-s_t)\frac{Y_t}{N_t}$
- consider $1 + r_{t+1} = \alpha\frac{Y_{t+1}}{K_{t+1}}$, $K_{t+1} = s_t Y_t$, $\ln N_{t+1} = \ln N_t + n$
- intra-temporal Optimization simplifies to: $\frac{1}{c_t} = -\rho + \ln\alpha + \ln N_t + n - \ln s_t - \ln Y_t + \ln E_t\left[\frac{1}{1-s_{t+1}}\right]$
- Final equation: $\ln s_t - \ln(1-s_t) = -\rho + \ln\alpha + n + \ln E_t\left[\frac{1}{1-s_{t+1}}\right]$

Solving the equation with constant saving rate $\hat{s}$

- Consider $\hat{s} = s_t = S_{t+1}$
- Previous final equation reduced to $\ln\hat{s} = \ln\alpha + n - \rho \Rightarrow \hat{s} = \alpha exp(n-\rho)$
- Given $\hat{s}$, we can solve for constant labor supply $\hat{l}$, consider $\frac{c_t}{1-l_t} = \frac{w_t}{b}$, $c_t = \frac{C_t}{N_t} = \frac{(1-\hat{s})Y_t}{N_t}$, $w_t = (1-\alpha)\frac{Y_t}{L_t} = (1-\alpha)\frac{Y_t}{\hat{l}N_t}$
- constant labour supply: $\hat{l} = \frac{1-\alpha}{1-\alpha+b(1-\hat{s})}$

Consider Pareto optimum to solve for the model equilibrium

- Using the $\hat{s}, \hat{l}$, $K_t = \hat{s}Y_{t-1}, L_t = \hat{l}N_t$
- Production function: $\ln Y_t = \alpha\ln\hat{s} + \alpha\ln Y_{t-1} + (1-\alpha)(\bar{A} + gt + \tilde{A}_t) + (1-\alpha)(\ln\hat{l} + \bar{N} + nt)$
- Define $\ln Y_t^*$ as the value of $\ln Y_t$ if $\ln A_t$ equate to $\bar{A} + gt$ each period
- $\ln Y_t^* = \alpha\ln\hat{s} + \alpha\ln Y_{t-1} + (1-\alpha)(\bar{A} + gt) + (1-\alpha)(\ln\hat{l} + \bar{N} + nt)$
- $\tilde{Y}_t := \ln Y_t - \ln Y_t^* = \alpha\tilde{Y}_{t-1} + (1-\alpha)\tilde{A}_t$
- Since $\tilde{Y}_{t-1} = \alpha\tilde{Y}_{t-2} + (1-\alpha)\tilde{A}_{t-1} \Rightarrow \tilde{A}_{t-1} = \frac{1}{1-\alpha}(\tilde{Y}_{t-1} - \alpha\tilde{Y}_{t-2})$
- And $\tilde{A}_t = \rho_A\tilde{A}_{t-1} + \epsilon_{A,t}$
- Final expression: $\tilde{Y}_t = (\alpha + \rho_A)\tilde{Y}_{t-1} - \alpha\rho_A\tilde{Y}_{t-2} + (1-\alpha)\epsilon_{A,t}$ (AR2)
- Implication: since α is small ($\sim\frac{1}{3}$), dynamics determined largely by ρ_A and if $|\rho_A| < 1$, the disturbances does not translate into long-lasting output movements (stationary)

Issues with simplified equation

- Model saving rate $\hat{s}$ is constant: in reality investment varies much more than consumption and saving rate is low in recessions
- Model labour input $\hat{l}$ is constant: in reality employment and hours are strongly pro-cyclical
- Model wage $w_t = \frac{(1-\alpha)Y_t}{L_t}$ is proportional to output: in reality wages are not pro cyclical

8.2 General RBC - solution with log-linearization

Key trick:

- Converting agents' decision rule with Taylor approximation in the logs of the variables along path where shocks are not present
- E.g. $\tilde{Y}_t = \ln Y_t - \ln Y_t^*$

General model

- State variables (cannot be changed instantaneous): K_t, A_t, G_t
- Choice variables (can be changed instantaneous): C_t, L_t

Want to solve:

1. $\tilde{C}_t = a_{CK}\tilde{K}_t + a_{CA}\tilde{A}_t + a_{CG}\tilde{G}_t$
2. $\tilde{L}_t = a_{LK}\tilde{K}_t + a_{LA}\tilde{A}_t + a_{LG}\tilde{G}_t$

3. From $\tilde{C}_t, \tilde{L}_t$, derive $\tilde{K}_t$

Six equations

1. $a_{CK} + \left(\frac{l^*}{1-l^*} + \alpha\right) a_{LK} = \alpha$
2. $a_{CA} + \left(\frac{l^*}{1-l^*} + \alpha\right) a_{LA} = 1 - \alpha$
3. $a_{CG} + \left(\frac{l^*}{1-l^*} + \alpha\right) a_{LG} = 0$

- 3 more equations from $\frac{1}{c_t} = exp(-\rho)E_t\left[\frac{1}{c_{t+1}}(1 + r_{t+1})\right]$

9 Unemployment

Key investigation

- Long-run: determinants of average unemployment over extended periods and across countries
- Short-run: Cyclical properties of the labor market
- Question: why shifts in labor demand lead to large movement in unemployment but not wage (sticky wage)

9.1 Traditional approach - Supply and Demand of labour

Key implications:

- Unemployment is a result of a force that prevents wage from falling to the market clearing price (wage) and quantity

Model

- N number of identical competitive firms with $\pi = Y - wL$
- Critical assumption: effort depends positively on wage w the firm pays. $e = e(w), e'(\cdot) > 0$
- $Y = F(eL) :=$ output, $F'(\cdot) > 0, F''(\cdot) < 0$
- $L :=$ amount of labour hired, total of $\bar{L}$ identical workers, each supplies 1 unit of labor inelastically

Model result and interpretation

- Profit maximising firm solves :$\max_{L,w} F(e(w)L) - WL$
- FOC: $F'(e(w)L)e(w) - w = 0$, $F'(e(w)L)L'(w) - L = 0$
- Optimal wage w^* and labour L^*: $\frac{e(w)}{w} = e'(w)$
  - average effort per dollar $e(w)/w$
  - cost per unit of effective labor $w/e(w)$
- Graphically: line extending from origin = gradient

Unemployment

- N identical competitive firm choose the same (w^*, L^*)
- Total unconstrained labour demand is NL^*
  1. If labour supply $\bar{L} \geq NL^*$: firm choose w^*, employed NL^* and unemployment $\bar{L} - NL^*$
  2. If labour supply $\bar{L} < NL^*$: firm increase w^* such that $\bar{L} = NL^*$ and no unemployment

Implications

- Short-run implication: w^* does not respond to demand shocks, match the observation shifts in labour demand leads to large movement in employment and small changes in the real wage
- Long-run implication: Since w^* does not respond to demand shocks, economic growth implies the real wage remains unchanged and unemployment trends down (eventually to 0).
  - However, in reality no clear trend for unemployment over time, across countries unemployment increases with GDP per capita

9.2 Modern approach - Searching and matching of labour

Key difference:

- Heterogeneous workers and jobs

Environment

- Workers
  - Employed: produce output y, receive time varying wage $w(t)$ per unit time
  - Unemployed: receive benefit $b \geq 0$ per unit time
  - Risk neutral, discount rate $r > 0$, discount factor $\frac{1}{1+r}$
- Jobs
  - Filled: output y, cost firm $w(t)$
  - Vacant: no output for labour cost $w(t)$
  - Maintenance fee: $c > 0$
  - A filled job produce positive value when $y > w + c$, free-entry condition to create a vacant job and the number of jobs is endogenous

Absent of frictions

- All jobs are filled since $y > w + c$
- Workers earn marginal product $w(t) = y - c$ in this competitive market
- Shifts in labour demand y directly proportional to wage. But this contradict empirical observation with sticky wages

9.2.1 Matching problem

Search and Matching

- $E(t), U(t) :=$ number of employed and unemployment workers
- $F(t), V(t) :=$ number of filled and unfilled jobs
- $M(t) = M(U(t), V(t)) :=$ unemployed workers and vacant jobs arrange meetings
- $\dot{E}(t) = M(t) - \lambda E(t) :=$ employed workers leave existing jobs at exogenous separation rate λ

Matching

- Matching function
  - $\theta = \frac{V(t)}{U(t)} :=$ market tightness
  - $M(U(t), V(t)) = U(t)M(1, \frac{V(t)}{U(t)}) = U(t)m(\theta(t))$, assuming constant returns of $M(t)$
  - $a(t) = \frac{M(t)}{U(t)} = m(\theta(t)) :=$ job-finding rate
  - $\alpha(t) = \frac{M(t)}{V(t)} = \frac{m(\theta(t))}{\theta(t)} :=$ vacancy-filling rate
  - Assume Cobb Douglas $M(U, V) = kU^{1-\gamma}V^{\gamma}; m(\theta) = k\theta^{\gamma}$, $k > 0, 0 < \gamma < 1$
- Wage determination
  - Assume all match leads to hires since collectively utility is higher
  - wage is determined through Nash bargaining, $\phi\%$ of match surplus goes to worker and $1 - \phi\%$ goes to the firm
  - $\phi :=$ barging power

Value function

- Discrete time
  - Lifetime utility value of the agent: $V(c) = \sum_{t=0}^{\infty} \beta^t c$
  - $c_t = c :=$ consumption at any period t, $u(c) = c :=$ utility function, $\beta :=$ discount factor
  - Geometric sequence: $V(c) = \lim_{n\to\infty} \frac{1-\beta^n}{1-\beta} c = \frac{c}{1-\beta}$
- Continuous time
  - $rV(t) = R(t) + \dot{V}(t) + \lambda[P(t) - V(t)]$ equilibrium condition
  - $r :=$ discount rate, $R(t) :=$ rental income, $\dot{V}(t) :=$ change of asset value $\lambda :=$ probability of selling the house, $P(t) :=$ selling the price of house
  - Therefore, the return of owning the asset $rV(t)$ is the rental income $R(t)$ + potential return of owning house not in steady state $\dot{V}(t)$ + potential return of selling the house $P(t) - V(t)$

9.2.2 Equilibrium (from continuous time)

The expected lifetime utility from time t forward, discounted to time t of (1) worker who is employed $V_E(t)$, unemployed $V_U(t)$ and (2) job that is filled $V_F(t)$ and vacant $V_V(t)$

Four equilibrium

- $rV_E(t) = w(t) + \dot{V}_E(t) + \lambda[V_U(t) - V_E(t)]$
- $rV_U(t) = b + \dot{V}_U(t) + a(t)[V_E(t) - V_U(t)]$
- $rV_F(t) = [y - w(t) - c] + \dot{V}_F(t) + \lambda[V_V(t) - V_F(t)]$

- $rV_V(t) = -c + \dot{V}_V(t) + \alpha[V_F(t) - V_V(t)]$

Evolution of the number of workers who are employed

- $\dot{E}(t) = kU(t)^{1-\gamma}V(t)^{\gamma} - \lambda E(t)$

Nash bargaining assumption

- $\frac{V_E(t)-V_U(t)}{V_F(t)-V_V(t)} = \frac{\phi}{1-\phi}$

Free entry: new vacancies can be created and eliminated freely

- $V_V(t) = 0$ for all t

9.2.3 Steady-state

Steady-state equilibrium: $\dot{V}(t) = 0, \dot{E}(t) = 0$, $a(t), \alpha(t)$ are constant

- $rV_E = w + \lambda(V_U - V_E)$
- $rV_U = b + a(V_E - V_U)$
- $rV_F = y - w - c + \lambda(V_V - V_F)$
- $rV_V = -c + \alpha(V_F - V_V)$

Solution steps:

1. Use steady-state value functions and the Nash bargain condition to solve for $w(t)$
2. Use $w(t)$ and steady-state a, α to find V_V
3. Use free-entry condition $V_V = 0$ to find equilibrium E^* and U^*

Solving steady-state: E, V_V

- $V_E - V_U = \frac{w-b}{a+\lambda+r}$
- $V_F - V_V = \frac{y-w}{\alpha+\lambda+r}$
- Nash barging: $\frac{\phi}{1-\phi} = \frac{w-b}{\alpha+\lambda+r} \cdot \frac{\alpha+\lambda+r}{y-w} \Rightarrow w = b + \frac{(a+\lambda+r)\phi}{\phi a+(1-\phi)\alpha+\lambda+r}(y-b)$
- Free entry condtion: $rV_V = 0$ for all $t \Rightarrow rV_V = -c + \frac{(1-\phi)\alpha}{\phi a+(1-\phi)\alpha+\lambda+r}(y-b) = 0$
- In steady-state, $\dot{E}(t) = 0 \Rightarrow M(U, V) = \lambda E$
- Mass of workers is $1 \Rightarrow E + U = 1, U = 1 - E$
- Job finding rate: $a = \frac{M(U,V)}{U} = \frac{\lambda E}{1-E} = \frac{k(1-E)^{1-\gamma}V^{\gamma}}{1-E}$
- Vacancy filling rate: $\alpha = \frac{M(U,V)}{V} = k^{1/\gamma}\left(\frac{1-E}{\lambda E}\right)^{\frac{1-\gamma}{\gamma}}$

Solution

- $rV_V = 0 = -c + \frac{(1-\phi)\alpha(E)}{\phi a(E)+(1-\phi)\alpha(E)+\lambda+r}(y-b)$
- rV_V is decreasing in E
  - $E \to 1, a \to \infty, \alpha \to 0 \Rightarrow rVv \to -c < 0$
  - $E \to 0, a \to 0, \alpha \to \infty \Rightarrow rVv \to y - b - c > 0$
- Solving for the E^* and unemployment is $1 - E^* > 0$

9.2.4 Implications

Impact of a shift in labour demand

- Short-run implication: model cyclical changes due to shifts in y, but when y increases, E^* increases and unemployment decreases (pro-cyclical) which is observed empirically. However, wage rises more than proportionally with y due to a, α, which is not sticky wages.
- Long-run implication: proportional change in y, b, c does not affect the unemployment $1 - E^*$, and w^* changes the same proportion as y which matches the empirical observation

9.2.5 Efficiency

Externalities on firm's entry decisions:

- positive on unemployed worker
- negative on other firms looking for workers

Simplification: one off static setup

- Decision on creating V vacancies initially
- U unemployed workers
- hired workers $E = M(U,V) = kU^{1-\gamma}V^{\gamma}$
- Each vacancy has cost c and each worker produces y
- Unemployed workers receive no income ($b = 0$) and wage is simply ϕy (bargain power)

Equilibrium

- Market equilibrium
  - $V^{EQ} = \left[\frac{k(1-\phi)y}{c}\right]^{1/(1-\gamma)} U$
  - Free-entry condition and marginal benefit of filling jobs - cost of creating jobs
  - $V_V = 0 = \frac{M(U,V)}{V}(y-w) - c = k\left(\frac{U}{V}\right)^{1-\gamma}(1-\phi)y - c \Rightarrow V^{EQ}$
- Optimal allocation
  - $V^{OPT} = \left[\frac{k\gamma y}{c}\right]^{1/(1-\gamma)} U$
  - Social welfare: $E(y-w) - Vc + Ew = Ey - Vc = kU^{1-\gamma}V^{\gamma}y - Vc$
  - FOC $\frac{\partial}{\partial V}$: $\gamma\left(\frac{kU}{V}\right)^{1-\gamma} y - c = 0 \Rightarrow V^{OPT}$

Hosios condition $\gamma = 1 - \phi$

- Decentralised equilibrium achieve efficiency if $\gamma = 1 - \phi$
  - γ is the positive externality, which is the elasticity on matching, reflects which post gets how many matches
  - $1 - \phi$ is the negative externality, which is the firm bargain power