EC2104 Tutorial 1 solution

ling

Section 1

Question 1: Finding limits

Before finding limits

Before you apply fancy L'Hopital rule or any other techniques

1. Check if limit exist (left limit = right limit)
   - There is no point trying to find something that does not exist
2. Check if the limit is in indeterminate form ($\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot 0, \infty \cdot \infty$)
   - If there is a solution simply by substituting the value of the limit (e.g. $\lim_{x \to 0} x + 4 = 0 + 4 = 4$), why bother wasting time
3. If limit is in indeterminate form and value tends towards infinity, check if denominator has higher rate of growth than numerator
   - Intuitively, if only the denominator is infinity, the fraction will tends towards 0
   - Example: solve question 1a) $\lim_{x \to 0} \frac{x}{e^x} = 0$ by eyeballing
   - Order of growth (when value tends towards infinity only): $O(1) < O(log(x)) < O(x^n) < O(e^x)$

Question 1b)

Q: Find $\lim_{x\to -4}\left(\frac{x^2-16}{x+4}\right)\ln|x|$

0. Identify: $\ln|-4| = ln(4)$ is a constant

Method 1: Simplification

1. Observe $x^2-16=(x+4)(x-4)$
2. Simplify question:
   $\lim_{x\to -4}\left(\frac{x^2-16}{x+4}\right)\ln|x| = \lim_{x\to -4}(x-4)\ln|x| = -8\ln(4)$

Method 2: L'H

1. Since the fraction is indeterminate, apply L'H:
   $\Rightarrow \lim_{x\to -4}\frac{\frac{d}{dx}x^2-16}{\frac{d}{dx}x+4}\ln|-x| = \lim_{x\to -4}\frac{2x}{1}\ln|-x| = -8\ln(4)$

Why bother with other methods when almost all questions can be solved by L'H?

- Solving differentiation can be challenging sometimes

Question 1c)

Q: Find $\lim_{x\to 1.5} \frac{2x^2-3x}{|2x-3|} = \lim_{x\to 1.5} \frac{x(2x-3)}{|2x-3|}$

0. Observe:
   - although fraction is indeterminate, absolute function is not differentiable.
   - Therefore, L'H cannot be used directly
   - and we cannot simplify the question directly.
   - We need to use piecewise function and find limit by finding the left limit and right limit
1. Left limit: $\lim_{x\to 1.5} \frac{x(2x-3)}{2x-3} = \lim_{x\to 1.5} x = 1.5$
2. Right limit: $\lim_{x\to 1.5} \frac{x(2x-3)}{-(2x-3)} = \lim_{x\to 1.5} -x = -1.5$
3. Observe: since left limit $\neq$ right limit, by definition of limits the limit does not exist

Question 1d): challenging but important

Q: Find $\lim_{x\to\infty} \sqrt{x^2+x} - x$

0. Observe:
   - The question is not in a fraction form, L'H cannot be applied directly.
   - Although $\sqrt{x^2+x} > 0$ but $-x < 0$, and it seems like $\sqrt{x^2+x} \approx x$. So it is hard to decide if the limit will diverge (positive or negative infinity) or it will converge
1. Use an important trick in solving limits question:
   $\lim_{x\to\infty} \sqrt{x^2+x} - x = \lim_{x\to\infty} \frac{(\sqrt{x^2+x}-x)(\sqrt{x^2+x}+x)}{(\sqrt{x^2+x}+x)}$
   - Note: since $\frac{(\sqrt{x^2+x}+x)}{(\sqrt{x^2+x}+x)} = 1$, we can multiply any functions with 1 without changing the function.
   - Note: furthermore, we simplify the function using identity: $(a+b)(a-b) = a^2 - b^2$
2. Therefore, $\lim_{x\to\infty} \sqrt{x^2+x} - x = \lim_{x\to\infty} \frac{x^2+x-x^2}{\sqrt{x^2+x}+x}$
   - to be continued in the next slide

Question 1d) cont

Now, we solve $\lim_{x\to\infty} \frac{x^2+x-x^2}{\sqrt{x^2+x}+x} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+x}+x}$

3. Reapplying the trick in step (1):
   - $\lim_{x\to\infty} \frac{x}{\sqrt{x^2+x}+x} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+x}+x} \cdot \frac{1/x}{1/x} = \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x^2}}+1}$
4. Deduce: $\lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x^2}}+1} = \frac{1}{2}$

Note: We skipped some detailed working here, you can refer to the step by step procedure in tutorial solution or work it out yourself.

Question 1e)

Similar idea as question 1d), refer to tutorial solution for detailed steps.

Question 1f)

Q: Find a, b such that $\lim_{x\to 0} \frac{\sqrt{ax+b}-2}{x} = 1$

0. For L'Hopital to work, assume $\lim_{x\to 0} \sqrt{ax+b} - 2 = 0 \Rightarrow \sqrt{b} = 2 \Rightarrow b = 4$
1. Applying L'Hopital rule: $\Rightarrow \lim_{x\to 0} \frac{a}{2\sqrt{ax+b}} = 1$
2. Substitute $x = 0$: $\Rightarrow \frac{a}{2\sqrt{b}} = 1$
3. Find relationship between a, b: $\Rightarrow a = 2\sqrt{b}$
4. Since from step (0): $b = 4 \Rightarrow a = 4$

Section 2

Question 2: Finding equilibrium

Question 2: Finding equilibrium

Important things to know:

- Competitive market always clears
  - quantity demanded = quantity supplied, $Q_S = Q_D$
  - Assumes no regulations and firms, buyers have no market power (perfect competition)
- To solve the equilibrium (the easier way):
  - $1, 2$ variables: substitution
  - ≥ 2 variable: matrix operations (linear algebra)
- A shift in line when intercept changes, a rotation in line when gradient changes
  - $Q_S = 10P_J - 5P_A \Rightarrow P_J = \frac{Q_S}{10} + \frac{1}{2}P_A$
  - Note in plotting demand and supply curve, we usually plot price as the function output (inverse demand and supply) $\Leftrightarrow P_J(Q_S) = f(Q_S) = \frac{Q_S}{10} + \frac{1}{2}P_A$
- When asked about price ceiling (max price), price floor (min price), check if the policy is binding first

Section 3

Question 3: Finding inverse

Question 3a

Q: $f(x) = \frac{1}{2}(e^x - e^{-x})$

1. Solve for x: $y = \frac{1}{2}(e^x - e^{-x})$
2. Multiply both side by e^x: $ye^x = \frac{1}{2}(e^{2x} - 1)$
3. Let $z = e^x$, solve for: $yz - \frac{1}{2}z^2 + \frac{1}{2} = 0$
   - $\Leftrightarrow$ solve $z^2 - 1 - 2yz = 0$
4. $\Rightarrow z = y + \sqrt{y^2 + 1}$
   - Reject $z = y - \sqrt{y^2 + 1}$ since $z = e^x > 0$
5. Take ln: $y = \ln(x + \sqrt{x^2 + 1})$

- Key trick: substituting the function of x

Question 3b

Q: $f(x) = 40x - 4x^2, x \in [0, 10]$

1. Show function is not valid (not one-to-one) with numeric example: $f(4) = f(6)$

- Key learning point: if function is not one-to-one, the inverse function does not exist

Section 4

Question 4: Finding first derivatives

Question 4a

Show f is differentiable at $x = 1$

$$f(x) = \begin{cases} 2 + \sqrt{x}, & x \geq 1 \\ \frac{1}{2}x + \frac{5}{2}, & x < 1 \end{cases}$$

- Key: show by definition (limit of Newton's quotient exist)

1. Write down the definition

$$f'(1) = \lim_{\Delta x \to 0} \frac{f(1 + \Delta x) - f(1)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(1 + \Delta x) - 3}{\Delta x}$$

2. Test if limit exist (left limit = right limit)
   - Left limit: $\lim_{\Delta x \to 0} \frac{2 + \sqrt{1 + \Delta x} - 3}{\Delta x} = \frac{1}{2}$
   - Right limit: $\lim_{\Delta x \to 0} \frac{\frac{1}{2}(1 + \Delta x) + \frac{5}{2} - 3}{\Delta x} = \frac{1}{2}$
3. Since limit exist, f is differentiable at $x = 1$ and $f'(1) = \frac{1}{2}$

Question 4b

Determine if f is differentiable at $x = 2$, $f(x) = \begin{cases} \frac{1}{4}x^3 - \frac{1}{2}x^2, & x \geq 2 \\ \frac{-6x-6}{x^2+2}, & x < 2 \end{cases}$

- Note that only continuous function are differentiable

0. For a quick check, try if $f(2)$ from the left equals to the right
1. More formally:
   - Left limit $(x \geq 2)$: $\lim_{x \to 2^+} f(x) = \frac{1}{4}2^3 - \frac{1}{2}2^2 = 0$
   - Right limit $(x < 2)$: $\lim_{x \to 2^+} f(x) = \frac{-6 \times 2 - 6}{2^2+2} = -3$
2. Since limit does not exist (left limit not equal to right limit), function is not continuous at $x = 2$ and is not differentiable

Section 5

Question 5: Finding roots with Intermediate Value Theorem

Question 5a

Q: Use IVT to show $2^x = \frac{10}{x}$ for some $x > 0$

0. Express function in a more familiar way: $\Rightarrow 2^x - \frac{10}{x} = 0$
1. Check if function is continuous in the domain
   - Note: although function is not continuous in the whole real line, when restricted to only positive values the function is continuous
2. Find alternating functional value. For example: $f(2) < 0, f(3) > 0$
3. Invoke IVT
   - By IVT, there exists at least one value c in $[2, 3]$ such that $f(c) = 2^c - \frac{10}{c} = 0$

Question 5b

Q: Can we use IVT to conclude function has no roots in the interval?

- No, IVT only allows us to conclude there exists at least one root.
- In Math and Logic, one way deduction is different from two way deduction ($P \Rightarrow Q \neq P \Leftarrow Q$)
  - If a definition/theorem works two ways, we say it is both a necessary and sufficient condition, or say P if and only if Q $P \Leftrightarrow Q$
- Example: $f(x) = (x-1)^2$
  - $f(0) = 1 = f(2) > 0$, but $f(1) = 0 \Rightarrow$ there is at least one root

Section 6

Question 6: Elasticity

Question 6: Elasticity

Given:

- $P_{t-1} = \$20$
- $Q_{t-1} = 100$ mil
- PED $= \epsilon_D = -0.25$
- PES $= \epsilon_S = 0.5$
- demand function: $Q_d = a - bP$
- supply function: $Q_s = c + dP$

Q6a: Solve for demand and supply function

1. Demand function: $\epsilon_D = \frac{dQ_d}{dP}\left(\frac{P}{Q_d}\right) = -b\frac{20}{100} = -0.25 \Rightarrow b = 1.25$
   - Since $Q_d = a - bP \Rightarrow 100 = a - 1.25 \cdot 20 \Rightarrow a = 125$
   - Therefore: $Q_d = 125 - 1.25P$
2. Supply function: $\epsilon_S = \frac{dQ_s}{dP}\left(\frac{P}{Q_s}\right) = d\frac{20}{100} = 0.5 \Rightarrow d = 2.5$
   - Since $Q_s = c - dP \Rightarrow 100 = c - 2.5 \cdot 20 \Rightarrow c = 50$
   - Therefore: $Q_S = 50 + 2.5P$

Question 6b

- Demand and Supply have same slope, different intercepts
  - Shift in demand and supply
- Name any factor that shifted demand and supply