EC2104 Lecture 5 - Multi variable functions

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Section 1

Multi variable functions

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Multi variable function

Given domain $D \in \mathbf{R}^n$, function f is a rule that maps $\mathbf{X} := x_1, \cdots, x_n$ to a specific number $\Rightarrow f(\mathbf{X}) = f(x_1, \cdots, x_n) = z$

Multi variable first and second order derivatives

We use a gradient vector $\nabla f(\mathbf{X})$ to denote all the partial derivatives of $f(\mathbf{X})$, and a Hessian matrix $\mathbf{H}_{f(\mathbf{X})}$ to denote all second order partial derivatives of $f(\mathbf{X})$ (more about matrix in Linear Algebra, after mid term)

$$\nabla f(\mathbf{X}) = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix} \quad \mathbf{H}_{f(\mathbf{X})} = \begin{pmatrix} \frac{\partial^2}{\partial x_1 \partial x_1} & \frac{\partial^2}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2}{\partial x_1 \partial x_n} \\ \frac{\partial^2}{\partial x_2 \partial x_1} & \frac{\partial^2}{\partial x_2 \partial x_2} & \cdots & \frac{\partial^2}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2}{\partial x_n \partial x_1} & \frac{\partial^2}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2}{\partial x_n \partial x_n} \end{pmatrix}$$

Subsection 1

Notation and identifying variables

Notation and identifying variables

Things you learnt in single variable calculus easily extends into multi variable calculus. However, some care needs to be taken.

Consider $f(\mathbf{X}) = ax_1x_2^c, x_1 \in \{1,2\}, x_2 \in \{4,5\}$

1. Identifying parameter and hyper-parameter
   - parameter (aka dependent or endogenous variables): x_1, x_2
   - hyper-parameter (aka independent or exogenous variables): a, c
   - hyper-parameter are determined outside of the system (or function), so take it as a given value (a constant), parameter are determined in the system, it is a variable that you can change
   - If question notation changed to $f(\mathbf{X}, a, c)$, now a, c becomes parameter.
2. Identify multi variable domain
   - Cartesian Product
     $(x_1, x_2) \in \{1,2\} \times \{4,5\} = \{(1,4),(1,5),(2,4),(2,5)\}$

Understanding parameter and hyper-parameter, you can easily extend single variable differentiation (and integration) onto partial differentiation and multi variable integration

Subsection 2

Young's Theorem

Young's Theorem

Sometimes differentiating w.r.t. one variable is easier than another. Young's theorem allows us to switch the sequence in multi variable differentiation

Young's Theorem

Suppose that all m^{th}-order partial derivatives of the function $f(\mathbf{X})$ are continuous. If any two of them involve differentiating w.r.t. each of the variable the same number of times, then they are necessarily equal

$$\Rightarrow \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j}\right)$$

Note:

- Although theorem require partial derivatives to be continuous, it has a wide range of applications.
- In most economics and statistics application, the partial derivatives are indeed continuous

Section 2

Level Curve

Level Curve

Level curves are useful in solving problems with 3 variables using graphical methods. A few important things to know:

1. Direction where level curve decrease
   - You can substitute values to check how does level curve value change
2. Shape of the level curve
   - Cobb Douglas (convex curve): $f(x, y) = Ax^{\alpha}y^{1-\alpha} = c$
   - Circle (similarity, Oval): $f(x, y) = x^2 + y^2 = c$
3. Identifying partial derivatives at a given point with level curve
   - Draw line parallel to the axis (x or y) to know the direction of change
   - To know the magnitude of change, approximate using $f_x'(x_1, y_1) \approx f_x(x_1 + 1, y_1) - f_x(x_1, y_1)$

Section 3

General Chain Rule

General Chain Rule

General Chain Rule

if $z = f(\mathbf{X})$ is continuously differentiable, and $x_i = g_i(\mathbf{t})$, for each $i = 1, 2, \cdots, n$ are all differentiable, then

$$\frac{\partial z}{\partial t_j} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial z}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots \frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial t_j} +$$

for each $j = 1, 2, \cdots, m$

Note:

- Think of it this way: to get the combined effect of a change in t_j, you need to add up how individual variable x_i change w.r.t t_j
- For example, if $Y = F(K, L)$, $Y :=$ GDP, $K :=$ capital and $L :=$ labour
  - To know how does increasing one year affects the GDP, you need to know how does increasing one year affects the capital and labour in order to know the combined effect on the GDP.