Lecture 6 - Comparative Statistics (tools)

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Section 1

Implicit Differentiation along Level Curve

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Implicity Differentiation along a Level Curve

$$y' = f'(x) = -\frac{\partial F(x,y)/\partial x}{\partial F(x,y)/\partial y}$$

Marginal Rate of Substitution

$$MRS = R_{yx} = \frac{\partial F(x,y)/\partial x}{\partial F(x,y)/\partial y}$$

- Used to investigate quantity of y needed to replace per unit of x
- e.g. number of sleep hours you are willing to forgo in exchange for one mark increment in exam

Subsection 1

Elasticity of Substitution

Elasticity of Substitution

Elasticity of Substitution

$$\sigma_{yx} = EL_{R_{yx}}\left(\frac{y}{x}\right) = \frac{R_{yx}}{y/x}\left[\frac{d(y/x)}{dR_{yx}}\right] = \frac{d\ln(y/x)}{d\ln(R_{yx})}$$

- Measures $\frac{\%\Delta(y/x)}{\%\Delta(R_{yx})}$, percentage change in $\frac{y}{x}$ when MRS increase by 1%
- Recall: $EL_x(f(x)) = \frac{x}{f(x)}f'(x)$

Section 2

Homogeneous and Homothetic Functions

Subsection 1

Homogeneous Functions

Homogeneous Functions

Homogeneity

Function f is homogeneous of degree k if for all (x, y) in domain

$$f(tx, ty) = t^k f(x, y), t > 0$$

- Return to scale: increasing inputs by t, will output increase more than/less than or equal to t
- Classic example: Cobb Douglas production function $F(K, L) = AK^\alpha L^\beta$
  - Cobb Douglas is homogeneous of degree $\alpha + \beta$
  - Constant return to scale: $\alpha + \beta = 1$
  - Increasing return to scale: $\alpha + \beta > 1$
  - Decreasing return to scale: $\alpha + \beta < 1$
- Two important results:
  1. Knowing one pair of $(x, y, f(x, y))$ provides knowledge of the whole function
  2. Knowing one level curve provides knowledge of all level curves

Subsection 2

Homothetic Functions

Homothetic Functions

Homothetic Functions

f is homothetic if for $(\mathbf{x}^{(1)}, \mathbf{x}^{(2)})$ in the domain with $f(\mathbf{x}^{(1)}) = f(\mathbf{x}^{(2)})$ we have

$$f(t\mathbf{x}^{(1)}) = f(t\mathbf{x}^{(2)}), t > 0$$

Theorem 12.7.2

Suppose $F(x) = H(f(x))$. If (1) H is strictly increasing and (2) f is homogeneous (of any degree) $\Rightarrow$ F is homothetic

- Homogeneous implies Homothetic ($P \Rightarrow Q$)
  - $Q \not\Rightarrow P$: Homothetic does not imply homogenous
  - $\neg P \not\Rightarrow Q$: Non-Homogeneous does not imply non Homothetic
  - $\neg Q \Rightarrow \neg P$: Non-Homothetic implies Non-Homogeneous

Subsection 3

Showing a function is homogenous and/or homothetic

Showing a function is homogenous and/or homothetic

1. Showing function is homogenous
   - Show $f(tx, ty) = t^k f(x, y)$
2. Showing function is homothetic
   1. Show by definition
      - Given: $f(\mathbf{x}^{(1)}) = f(\mathbf{x}^{(2)})$
      - Show: $f(t\mathbf{x}^{(1)}) = f(t\mathbf{x}^{(2)})$
   2. Use Thm 12.7.2
      1. Express function as composite function $F = H(f(x))$
      2. Show $H(\cdot)$ is strictly increasing function
      3. Show $f(\cdot)$ is homogenous function
      4. Invoke Theorem

Section 3

Differentials

Differentials

Differentials

$z := f(\mathbf{x}) = f(x_1, x_2, \cdots, x_n)$ the differential of z is

$$dz = \sum_{i=1}^{n} f_i'(\mathbf{x}) dx_i$$

- Graphically: tangent space
  - plane in 3 dimensional (R^3), line in 2 dimensional (R^2)
- Actual change (Δz) and approx change (dz):
  - $\Delta z = f(x + dx, y + dy) - f(x, y) \approx dz = f_1'(x, y)dx + f_2'(x, y)dy$

Chain rule for differentials

$$z = g(f(\mathbf{x})) \Rightarrow dz = g'(f(\mathbf{x})) \left(\sum_{i=1}^{n} f_i' dx_i \right), \mathbf{x} \in R^n$$

Subsection 1

Differentials for Systems of Equations

Differentials for Systems of Equations

System of Equations

$$
\begin{aligned}
f_1(\mathbf{x}) &= 0 \\
f_2(\mathbf{x}) &= 0 \\
&\vdots \\
f_m(\mathbf{x}) &= 0
\end{aligned}
$$

Solving differential for system of equations at a point P

1. Solve differentials
2. Substitution values of $\mathbf{x}$ at point P
3. Solve the system of equations
4. Partial derivatives is the coefficient of the differentials
   - e.g. $du = g(dx, dy) = 5dx - 3dy \Rightarrow \frac{\partial u}{\partial x} = 5, \frac{\partial u}{\partial y} = -3$

Section 4

Degree of Freedom

Degree of Freedom

Degrees of freedom

A system of equations with n variables is said to have k degrees of freedom if k variables can be freely chosen

- Remaining $n - k$ variables will be uniquely determined once k free variables have been assigned
- E.g. $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2$ (single equation)
    - Given $(y, x_1, x_2) = (-2, 1, 2) \Rightarrow \beta_0 + \beta_1 + 2\beta_2 = -2$
    - if $\beta_1 = 1, \beta_2 = -1 \Rightarrow \beta_0 = -1$
    - if $\beta_1 = -1, \beta_2 = 1 \Rightarrow \beta_0 = -3$
    - We can freely determine the value of β_1, β_2, but not the value of β_0.
    - We say this system has 2 degree of freedom
        - $n = 3, k = 2, n - k = 1$
    - This system is a least square linear regression with 3 parameters and 2 covariates
- If there are m independent equations: degree of freedom is $n - m$