$\sum_{k=1}^{n} c = nc, \sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$

$cos(0) = 1\ cos(\frac{\pi}{2}) = 0, cos(\pi) = -1$

$sin(0) = 0\ sin(\frac{\pi}{2}) = 1, sin(\frac{3\pi}{2}) = -1$

Definitions

1. (a) The **limit** of $f(x)$ equals number L as x approaches a, denoted by $\lim_{x\to a} f(x) = L$, if for any $\epsilon > 0$ there is a $\delta > 0$ such that $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon$.
   (b) $\lim_{x\to a^-} f(x) = L$: for any $\epsilon > 0$ there is a $\delta > 0$ such that $0 < a - x < \delta \Rightarrow |f(x) - L| < \epsilon$.
   $\lim_{x\to a^+} f(x) = L$: for any $\epsilon > 0$ there is a $\delta > 0$ such that $0 < x - a < \delta \Rightarrow |f(x) - L| < \epsilon$.
   (c) $\lim_{x\to a} f(x) = \infty$: for any $M > 0$ there is a $\delta > 0$ such that $0 < |x - a| < \delta \Rightarrow f(x) > M$.
   (d) $\lim_{x\to\infty} f(x) = L$: for any $\epsilon > 0$ there is an N such that $x > N \Rightarrow |f(x) - L| < \epsilon$.
   (e) $\lim_{x\to\infty} f(x) = \infty$: for any $M > 0$ there is an N such that $x > N \Rightarrow f(x) > M$.
2. A function f is **continuous** at a if $\lim_{x\to a} f(x)$ exists and equals $f(a)$.
   f is **continuous from the left** (resp. **from the right**) at a if $\lim_{x\to a^-} f(x)$ (resp. $\lim_{x\to a^+} f(x)$) exists and equals $f(a)$.
3. A function f is **differentiable** at a if $\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$ or equivalently $\lim_{x\to a} \frac{f(x) - f(a)}{x - a}$ exists. The limit, denoted by $f'(a)$, is called the **derivative** of f at a.
4. (a) Let f be a function with domain D. f has an **global maximum** (resp. **global minimum**) at c if $f(c) \geq f(x)$ (resp. $f(c) \leq f(x)$) for all $x \in D$, and $f(c)$ is called the **global maximum value** (resp. **global minimum value**) of f on D.
   (b) f has a **local maximum** (resp. **local minimum**) at c if $f(c) \geq f(x)$ (resp. $f(c) \leq f(x)$) for all x in an open interval containing c.
5. A **critical number** of a function f is an interior point c of the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.
6. The graph of a differentiable function f on an open interval I is **concave up** (resp. **concave down**) if it lies above (resp. below) all its tangent lines on I; or equivalently, if f' is increasing (resp. decreasing) on I.
7. An **inflection point** of a function f is a point at which f is continuous and changes concavity.
8. The **definite integral** of a continuous function f on the interval $[a, b]$ is defined as $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$, where $\Delta x = \frac{b-a}{n}$, $x_i = a + i\,\Delta x$, $i = 0, \ldots, n$, and $x_i^* \in [x_{i-1}, x_i]$, $i = 1, \ldots, n$.
9. An **antiderivative** of a continuous function f is a continuous function F such that $F'(x) = f(x)$ for all x. The **indefinite integral** $\int f(x)\,dx = F(x) + C$ is the class of all antiderivatives of f.
10. If f is continuous on $[a, \infty)$, then $\int_a^\infty f(x)\,dx = \lim_{c\to\infty} \int_a^c f(x)\,dx$. If f is continuous on $[a, b)$, then $\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx$. The **improper integral** is **convergent** if the corresponding limit exists, and it is **divergent** otherwise.

integration with discountinous at point c, or towards inf

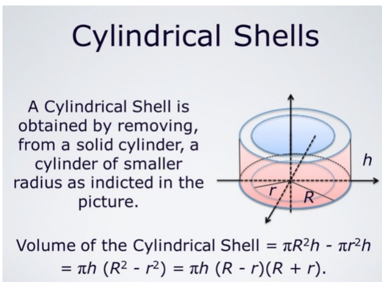
11. If f is a one-to-one function with domain A and range B, its **inverse function** f^{-1} is the function with domain B and range A such that $f^{-1}(y) = x \Leftrightarrow y = f(x)$ for all $x \in A$ and $y \in B$.
12. The **inverse trigonometric functions** are defined by
    (a) $\sin^{-1} x : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\sin^{-1} x = y \Leftrightarrow x = \sin y$;
    (b) $\cos^{-1} x : [-1, 1] \to [0, \pi]$, $\cos^{-1} x = y \Leftrightarrow x = \cos y$;
    (c) $\tan^{-1} x : (-\infty, \infty) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\tan^{-1} x = y \Leftrightarrow x = \tan y$;
    (d) $\sec^{-1} x : (-\infty, -1] \cup [1, \infty) \to \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$, $\sec^{-1} x = y \Leftrightarrow x = \sec y$.

Riemann Sum

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)\left(\frac{b-a}{n}\right)$$

$$\int_0^1 f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(1 + k\frac{1}{n}\right)\left(\frac{1}{n}\right)$$

13. The **natural logarithm function** is defined by $\ln x = \int_1^x \frac{1}{t}\,dt$ for $x > 0$. The **exponential function** is the inverse function of the natural logarithm function; that is, $\exp x = e^x = y \Leftrightarrow x = \ln y$.
14. The **hyperbolic trigonometric functions** are defined by
    (a) $\sinh x = \frac{e^x - e^{-x}}{2}$, (b) $\cosh x = \frac{e^x + e^{-x}}{2}$, (c) $\tanh x = \frac{\sinh x}{\cosh x}$.
15. A function f is **smooth** if f is differentiable and f' is continuous.



$\frac{d}{dx}a^u = a^u \ln(a)\frac{du}{dx}$ $\int \sec(x)\,dx = \ln|\tan(x) + \sec(x)|$

$\ln(ab) = \ln(a) + \ln(b)$, $\ln(a/b) = \ln(a) - \ln(b)$ $\int a^u du = \frac{a^u}{\ln(a)}$ $a^x = e^{x\ln(a)}$, $\lim_{x\to a}(f(x))^n = \left(\lim_{x\to a} f(x)\right)^n$ $\lim_{x\to\infty}\left[\left(1+\frac{1}{x}\right)^x\right]^n = e^n$

$\ln(a^2) = 2\ln(a) \neq (\ln(a))^2$

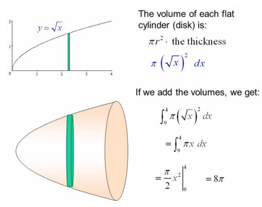
$y = f(x) \Rightarrow \ln(y) = \ln(f(x)) \Rightarrow \frac{dy}{dx} = \frac{f'(x)}{f(x)}y$

FORMULAS

$f(x)^n - 1 = (f(x) - 1)(f(x)^{n-1} + f(x)^{n-2} + \ldots + f(x) + 1)$

1. (a) $\csc x = \dfrac{1}{\sin x}$, (b) $\sec x = \dfrac{1}{\cos x}$, (c) $\tan x = \dfrac{\sin x}{\cos x}$, $\lim_{x\to\infty} x \times \sin\left(\frac{1}{x}\right) = \lim_{\frac{1}{x}\to 0}\dfrac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} = 1$
   (d) $\cot x = \dfrac{\cos x}{\sin x}$, (e) $\sin^2 x + \cos^2 x = 1$, (f) $1 + \tan^2 x = \sec^2 x$,
   (g) $1 + \cot^2 x = \csc^2 x$, (h) $\sin(-x) = -\sin x$, (i) $\cos(-x) = \cos x$, $\lim_{x\to 0}\dfrac{\sin(x^{-n})}{x^{-n}} = 0$
   (j) $\sin\left(\frac{\pi}{2} - x\right) = \cos x$, (k) $\cos\left(\frac{\pi}{2} - x\right) = \sin x$, (l) $\lim_{x\to 0}\dfrac{\sin x}{x} = 1$. $\lim_{x\to 0}\cos(x) = 1$

2. (a) $\sin 2x = 2\sin x\cos x$, (b) $\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$,
   (c) $\sin(x \pm y) = \sin x\cos y \pm \cos x\sin y$, (d) $\cos(x \pm y) = \cos x\cos y \mp \sin x\sin y$,
   (e) $\tan(x \pm y) = (\tan x \pm \tan y)/(1 \mp \tan x\tan y)$, (f) $\cosh^2 x - \sinh^2 x = 1$.



3. (a) $(c)' = 0$, (b) $(f(x) \pm g(x))' = f'(x) \pm g'(x)$,
   (c) $(cf(x))' = cf'(x)$, (d) $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$,
   (e) $(x^a)' = ax^{a-1}$, (f) $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$,
   (g) $(\sin x)' = \cos x$, (h) $(\tan x)' = \sec^2 x$, (i) $(\sec x)' = \sec x\tan x$,
   (j) $(\cos x)' = -\sin x$, (k) $(\cot x)' = -\csc^2 x$, (l) $(\csc x)' = -\csc x\cot x$,
   (m) $(\ln x)' = 1/x$, (n) $(e^x)' = e^x$, (o) $(a^x)' = a^x \ln a$, $\frac{d}{dx}(\log_a u) = \frac{1}{\ln(a)}\frac{1}{u}\frac{du}{dx}$
   (p) $(\sin^{-1} x)' = \dfrac{1}{\sqrt{1 - x^2}}$, (q) $(\tan^{-1} x)' = \dfrac{1}{1 + x^2}$, (r) $(\sec^{-1} x)' = \dfrac{1}{x\sqrt{x^2 - 1}}$,
   (s) $(\sinh x)' = \cosh x$, (t) $(\cosh x)' = \sinh x$.

4. (a) $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, (b) $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$,
   (c) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, (d) If $f(x) \geq g(x)$ on $[a, b]$, $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$, $u(x) \longrightarrow u'(x)$
   $v'(x) \longrightarrow v(x)$
   (e) **(Integration by parts)**. $\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$. $\int_a^b u(x)v'(x)dx = u(x)v(x)_a^b - \int_a^b v(x)u'(x)dx$

5. (a) **(Disk/washer method)**. If a solid is generated by rotating a region about a horizontal line, then its **volume** computed by the disk/washer method is a formula of the form $\int_a^b \pi(\text{radius})^2\,dx$ or

Radius w.r.t. to evolution
(If evolve around y-axis, use f(y))
(If evolve around x-axis, use f(x))

$\int_a^b \pi\left((\text{outer radius})^2 - (\text{inner radius})^2\right)\,dx$.

For instance, if a solid is obtained by rotating the region between the curve $y = f(x)$ and the x-axis from a to b ($a < b$) about the x-axis, then its volume is given by $\int_a^b \pi(f(x))^2\,dx$.

   (b) **(Cylindrical shell method)**. If a solid is generated by rotating a region about a vertical line, then its **volume** computed by the cylindrical shell method is a formula of the form $\int_a^b 2\pi(\text{radius})(\text{height})\,dx$.

Radius not w.r.t to evolution
(If evolve around y-axis, use f(x))
(If evolve around x-axis, use f(y))

For instance, if a solid is obtained by rotating the region between the curve $y = f(x)$ ($f(x) \geq 0$) and the x-axis from a to b ($0 \leq a < b$) about the y-axis, then its volume is given by $\int_a^b 2\pi x f(x)\,dx$.

   (c) The **arc length** of a smooth curve $y = f(x)$ from a to b ($a < b$) is $\int_a^b \sqrt{1 + (f'(x))^2}\,dx$.

   (d) The **surface area** of the revolution of a smooth curve $y = f(x)$ ($f(x) \geq 0$) from a to b ($a < b$) about the x-axis is given by $\int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx$.

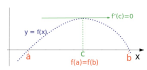
6. An **integrating factor** of the first-order linear differential equation $\dfrac{dy}{dx} + p(x)y = q(x)$ is $e^{\int p(x)\,dx}$.

Theorems

Center of mass

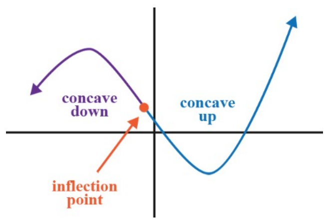
1. $dm = \delta \times dx \times (f(x) - g(x))$
2. $(\tilde{x}, \tilde{y}) = (x, \frac{f(x)+g(x)}{2})$
3. $\bar{x} = \frac{\int_a^b \tilde{x}\, dm}{\int_a^b dm}$, $\bar{y} = \frac{\int_a^b \tilde{y}\, dm}{\int_a^b dm}$

1. $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$.
2. If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then
   (a) $\lim_{x\to a} cf(x) = cL$, (b) $\lim_{x\to a}(f(x) \pm g(x)) = L \pm M$,
   (c) $\lim_{x\to a} f(x)g(x) = LM$, (d) $\lim_{x\to a}[f(x)/g(x)] = L/M$, $(M \neq 0)$.
3. Suppose $f(x) \leq g(x)$ for every x in an open interval containing a, except possibly at a. If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then $L \leq M$.
4. **(Squeeze/Sandwich theorem).** Suppose $f(x) \leq g(x) \leq h(x)$ for every x in an open interval containing a, except possibly at a. If $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x)$ exists and equals L. $-1 \leq cos(x) \leq 1$
5. Let f and g be functions which are continuous at a, and c a constant. Then the following functions are also continuous at a:
   (a) cf, (b) $f+g$, (c) $f-g$, (d) fg, (e) f/g, $(g(a) \neq 0)$.
6. If $\lim_{x\to a} g(x) = b$ and f is continuous at b, then $\lim_{x\to a} f(g(x)) = f(b)$.
7. If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.
8. **(Intermediate value theorem).** Let f be a continuous function on $[a,b]$, and let N be a number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number $c \in (a,b)$ such that $f(c) = N$.
9. If f is differentiable at a, then it is continuous at a.
10. **(Chain rule).** If g is differentiable at a and f is differentiable at $b = g(a)$, then $f \circ g$ is differentiable at a and $(f \circ g)'(a) = f'(b)g'(a)$. $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$
11. **(Extreme value theorem).** If f is continuous on a closed interval $[a,b]$, then f attains the global maximum $f(c)$ and the global minimum $f(d)$ at some numbers c and d in $[a,b]$.
12. **(Fermat's theorem).** If f has a local maximum or local minimum at an interior point c of its domain, and if f is differentiable at c, then $f'(c) = 0$.
13. **(Rolle's theorem).** Let f be a function which is continuous on $[a,b]$, differentiable on (a,b), and $f(a) = f(b)$. Then there exists a number $c \in (a,b)$ such that $f'(c) = 0$. If a=b, there must be a turning point where f'(c)=0



14. **(Mean value theorem).** Let f be a function which is continuous on $[a,b]$ and differentiable on (a,b). Then there exists a number $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. same gradient

    Consequences of MVT
    1. if f' = 0, f = Constant
    2. if f' = g' on all x, g' is a shift of f
15. **(Increasing/decreasing test).** Let f be a function which is continuous on $[a,b]$ and differentiable on (a,b). If $f'(x) > 0$ (resp. $f'(x) < 0$, $f'(x) = 0$) for all $x \in (a,b)$, then f is increasing (resp. decreasing, constant) on $[a,b]$.
16. **(First derivative test).** Let c be a critical number of a continuous function f. If f' changes from positive (resp. negative) to negative (resp. positive) at c, then f has a local maximum (resp. minimum) at c. If f' does not change sign at c, then f has neither a local maximum nor a local minimum at c.
17. **(Second derivative test).** If $f'(c) = 0$ and $f''(c) > 0$ (resp. $f''(c) < 0$), then f has a local minimum (resp. maximum) at c.
18. **(Concavity test).** Let f be a twice-differentiable function on an open interval I. If $f''(x) > 0$ (resp. $f''(x) < 0$) for all $x \in I$, then the graph of f is concave up (resp. concave down) on I.
19. If f has an inflection point at c, then either $f''(c) = 0$ or $f''(c)$ does not exist.



Proving roots:
1. Rolle's Theorem proves if >1 root is possible (i.e. when f(a) = f(b), there is a f' = 0)
2. Intermediate Value Theorem proves if there is root (i.e. when f(a)<0 and f(b)>0, there is a f(c) = 0)
3. Mean Value Theorem proves if there is root (i.e. when $\int_a^b f = 0$, there is a c s.t. $f(c) = 0$)

Procedure for Graphing $y = f$ of x

1. Identify the domain of f and any symmetries the curve may have.
2. Find the derivatives y' and y''.
3. Find the critical points of f, if any, and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes that may exist.
7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve together with any asymptotes that exist.

$$tangent\ line: \frac{dy_1}{dx_1} \div \frac{dy_2}{dx_2} = -1$$

Tangent line:
$L(x) = f(a) + f'(a)(x-a)$

Finding Absolute Extrema
1. Find critical points $f' = 0$ *or udefined*
2. Evaluate critical points & end points
3. Take largest/smallest points

20. **(L'Hôpital's rule)**.
    (a) Suppose $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\pm\infty$. If f and g are differentiable on an open interval I containing a and $g'(x) \neq 0$ on I except possibly at a, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$, provided that the limit on the right exists or equals $\pm\infty$.
    (b) Suppose $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$ or $\pm\infty$. If f and g are differentiable on an interval $I = (M, \infty)$ and $g'(x) \neq 0$ on I, then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$, provided that the limit on the right exists or equals $\pm\infty$.

21. **(Fundamental theorem of calculus)**.

$$y = \int_a^{f(x)} f(t)dt\,, y' = f(x)f'(x)$$

    I) If f is continuous on $[a,b]$, then $g(x) = \int_a^x f(t)\,dt$ is continuous on $[a,b]$ and differentiable on (a,b), and $g'(x) = f(x)$ for all $x \in (a,b)$.
    II) If f is continuous on $[a,b]$ and F is continuous on $[a,b]$ such that $F'(x) = f(x)$ for all $x \in (a,b)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

22. **(Substitution rule I)**. If g' is continuous on an interval and f is continuous on the range of $u = g(x)$, then $\int f(g(x))g'(x)\,dx = \int f(u)\,du$.

23. **(Substitution rule II)**. If f is continuous and $x = g(t)$ is smooth, then $\int f(x)\,dx = \int f(g(t))g'(t)\,dt$.

24. Let f be a one-to-one continuous function on an interval I. If f is differentiable at $a \in I$ and $f'(a) \neq 0$, then f^{-1} is differentiable at $b = f(a)$ and $(f^{-1})'(b) = \frac{1}{f'(a)}$.

Integration:

$$\int a^x dx = \frac{a^x}{\ln(a)} + C$$

$$\int \tan(x)dx = ln|\sec(x)| + C$$

$$\int \frac{dx}{\sqrt{a^x - x^2}} = sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} sec^{-1}\left|\frac{x}{a}\right| + C$$

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = sinh^{-1}\left(\frac{x}{a}\right) + C, a > 0$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = cosh^{-1}\left(\frac{x}{a}\right) + C, (x > a > 0)$$

$$\int_{-a}^{a} x^n = 0, n\ is\ odd$$

Reduction Formulas

$$\int tan^n\, xdx = \frac{1}{n-1} tan^{n-1}x - \int tan^{n-2}xdx$$

$$\int (lnx)^n dx = x(lnx)^n - n\int (lnx)^{n-1}dx$$

$$\int sin^n x cos^m xdx = -\frac{sin^{n-1}x cos^{m+1}x}{m+n} + \frac{n-1}{m+n}\int sin^{n-2}x cos^m xdx \quad (n \neq -m)$$

$$cos^2(x)sin^2(x) = \frac{1 - \cos(4x)}{8}$$

Partial fraction

$$(ax+b)^k = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \ldots + \frac{A_k}{(ax+b)^k}, k = 1,2,3\ldots$$