Analysis

[Matrix] $c^Tc = ||c||^2 = c_1^2 + \cdots + c_k^2$, cc^T is $k \times k$ matrix with (i,j)th element as $c_i c_j$,
[Max, Min] $\max(a,b) = \frac{1}{2}(a+b+|a-b|)$, $\min(a,b) = \frac{1}{2}(a+b-|a-b|)$

Probability

[Deduce $X=0$] If $X \geq 0$ a.s. and $EX = 0$ then $X = 0$ a.s.
[Variance, Covariance] $Var(X) = E[(X-EX)(X-EX)^T]$, $Cov(X,Y) = E[(X-EX)(Y-EY)^T]$, $Corr(X,Y) = Cov(X,Y)/(\sigma_X\sigma_Y)$, $E(a^TX) = a^TEX$, $Var(a^TX) = a^TVar(X)a$
[CHF] $\phi_X(t) = E\left[exp(\sqrt{-1}t^TX\right] = E\left[\cos(t^TX) + \sqrt{-1}\sin(t^TX)\right]\ \forall\, t \in \mathcal{R}^d$, well defined with $|\phi_X| \leq 1$
[MGF] $\psi_X(t) = E\left[exp(t^TX)\right]\ \forall\, t \in \mathcal{R}^d$,
[MGF properties] $\psi_{-X}(t) = \psi_X(-t)$, if $\psi(t) < \infty\ \forall\ ||t|| < \delta \Rightarrow E|X|^a < \infty\ \forall\ a > 1$ and $\phi_X(t) = \psi_X(\sqrt{-1}t)$
[Conditional Exp] $f_{X|Y}(x|Y=y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{f_{Y|X}(y|X=x)f_X(x)}{f_Y(y)}$

Integration

[MCT] $0 \leq f_1 \leq f_2 \leq \cdots \leq f_n$ and $\lim_n f_n = f$ a.e. $\Rightarrow \int \lim_n f_n d\nu = \lim_n \int f_n d\nu$
[Fatou] $f_n \geq 0 \Rightarrow \int \liminf_n f_n d\nu \leq \liminf_n \int f_n d\nu$
[DCT] $\lim_{n\to\infty} f_n = f$ and $|f_n| \leq g$ a.e. $\Rightarrow \int \lim_n f_n d\nu = \lim_n \int f_n d\nu$. g is an integrable function.
[Interchange Diff and Int] ① $\partial f(\omega,\theta)/\partial\theta$ exists in (a,b) ② $|\partial f(\omega,\theta/\partial\theta| \leq g(\omega)$ a.e. $\Rightarrow$
① $\partial f(\omega,\theta)/\partial\theta$ integrable in (a,b) ② $\frac{d}{d\theta}\int f(\omega,\theta)d\nu(\omega) = \int \frac{\partial f(\omega,\theta)}{\partial\theta}d\nu(\omega)$
[Change of Var] $Y = g(X), X = g^{-1}(Y) = h(Y)$ and A_i disjoint, $f_Y(y) = \sum_{j:1\leq j\leq m, y\in g(A_j)}\left|\det\left(\frac{\partial h_j(y)}{\partial y}\right)\right| f_X(h_j(y))$. Simple version: $f_Y(y) = |det(\partial h(y)/\partial y)| f_X(h(y))$

Inequalities

[Cauchy-Schewarz] $Cov(X,Y)^2 \leq Var(X)Var(Y)$, and $E^2[XY] \leq EX^2EY^2$
[Jensen] φ is convex $\Rightarrow \varphi(EX) \leq E\varphi(X)$ e.g. $(EX)^{-1} < E(X^{-1})$ and $E(logX) < log(EX)$
[Chebyshev] If $\varphi(-x) = \varphi(x)$, and φ non-decreasing on $[0,\infty) \Rightarrow \varphi(t)P(|X| \geq t) \leq \int_{\{|X|\geq t\}}\varphi(X)dP \leq E\varphi(X) \forall\, t \geq 0$. e.g. $P(|X-\mu| \geq t) \leq \frac{\sigma_X^2}{t^2}$ and $P(|X| \geq t) \leq \frac{E|X|}{t}$
[Hölder] $p, q > 0$ and $1/p + 1/q = 1$ or $q = p/(p-1) \Rightarrow E|XY| \leq (E|X|^p)^{1/p}(E|Y|^q)^{1/q}$. Equality $\Leftrightarrow |X|^p$ and $|Y|^q$ linearly dependent
[Young] $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$, equality $\Leftrightarrow a^p = b^q$
[Minkowski] $p \geq 1$, $(E|X+Y|^p)^{1/p} \leq (E|X|^p)^{1/p} + (E|Y|^p)^{1/p}$
[Lyapunov] for $0 < s < t$, $(E|X|^s)^{1/2} \leq (E|X|^t)^{1/t}$
[KL] $K(f_0, f_1) = E_0 \log\frac{f_0(X)}{f_1(X)} = \int \log\left(\frac{f_0(x)}{f_1(x)}\right) f_0(x)d\nu(x) \geq 0$ equality $\Leftrightarrow f_1(\omega) = f_0(\omega)$

Convergence

[a.s] $X_n \xrightarrow{\text{a.s.}} X$ if $P\left(\lim_{n\to\infty} X_n = X\right) = 1$. Can show $\forall\ \epsilon > 0, \sum_{i=1}^{\infty} P(|X_n - X| > \epsilon) < \infty$ via BC lemma
[Infinity often] $\{A_n\ i.o.\} = \cap_{n\geq 1}\cup_{j\geq n} A_j := \limsup_{n\to\infty} A_n$
[Borel-Cantelli lemmas]
(First BC) If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(A_n\ i.o.) = 0$
(Second BC) Given pairwisely independent events $\{A_n\}_{n=1}^{\infty}$, if $\sum_{n=1}^{\infty} P(A_n) = \infty$, then $P(A_n\ i.o.) = 1$
[L^p] $X_n \xrightarrow{L_p} X$ if $\lim_{n\to\infty} E|X_n - X|^p = 0$, given $p > 0$, $E|X|^p < \infty$ and $E|X_n|^p < \infty$
[Probability] $X_n \xrightarrow{P} X$ if $\forall\ \epsilon > 0 \lim_{n\to\infty} P(|X_n - X| > \epsilon) = 0$. Can show $E(X_n) = X$, $\lim_{n\to\infty} Var(X_n) = 0$
[Distribution] $X_n \xrightarrow{D} X$ if $\lim_{n\to} F_n(x) = F(x)$ for every $x \in \mathcal{R}$ at which F is continuous
[Relationships between convergence]
① $L^p \Rightarrow L^q \Rightarrow P$ ② $a.s. \Rightarrow P$, $P \Rightarrow D$ ③ $X_n \to_D C \Rightarrow X_n \to_P C$ ④ If $X_n \to_P X \Rightarrow \exists$ sub-seq s.t. $X_{n_j} \to_{\text{a.s.}} X$.
[Continuous mapping] If $g: \mathcal{R}^k \to \mathcal{R}$ is continuous and $X_n \xrightarrow{*} X$, then $g(X_n) \xrightarrow{*} g(X)$, where $*$ is either ⓐ a.s. ⓑ P ⓒ D.
[Convengence properties]
① Unique in limit: $X = Y$ if $X_n \to X$ and $X_n \to Y$ for ⓐ a.s., ⓑ P, ⓒ L^p. ⓓ If $F_n \to F$ and $F_n \to G$, then $F(t) = G(t)\ \forall\ t$
② Concatenation: $(X_n, Y_n) \to (X, Y)$ when ⓐ P ⓑ a.s. ⓒ $(X_n, Y_n) \xrightarrow{D} (X, c)$ only when c is constant.
③ Linearity: $(aX_n + bY_n) \to aX + bY$ when ⓐ a.s. ⓑ P ⓒ L^p ⓓ NOT for distribution.
④ Cramér-Wold device: for k-random vectors, $X_n \xrightarrow{D} X \Leftrightarrow c^TX_n \xrightarrow{D} c^TX$ for every $c \in \mathcal{R}^k$
[Lévy continuity] $X_n \xrightarrow{D} X \Leftrightarrow \phi_{X_n} \to \phi_X$ pointwise
[Scheffés theorem] If $\lim_{n\to\infty} f_n(x) = f(x) \Rightarrow \lim_{n\to\infty}\int |f_n(x) - f(x)|d\nu = 0$ and $P_{f_n} \to P_f$. Useful to check pdf converge in distribution.
[Slutsky's theorem] If $X_n \xrightarrow{D} X$ and $Y_n \xrightarrow{D} c$ for constant c. Then $X_n + Y_n \xrightarrow{D} X + c$, $X_nY_n \xrightarrow{D} cX$, $X_n/Y_n \xrightarrow{D} X/c$ if $c \neq 0$
[Skorohod's theorem] If $X_n \xrightarrow{D} X$, then $\exists\, Y, Y_1, Y_2, \cdots$ s.t. $P_{Y_n} = P_{X_n}$, $P_Y = P_X$ and $Y_n \xrightarrow{\text{a.s.}} Y$
[δ-method - first order] If $\{a_n\} > 0$ and $\lim_{n\to\infty} a_n = \infty$ and $a_n(X_n - c) \xrightarrow{D} Y$ and $c \in \mathcal{R}$ and $g'(c)$ exists at c, then $a_n[g(X_n) - g(c)] \xrightarrow{D} g'(c)Y$
[δ-method - higher order] If $g^{(j)}(c) = 0$ for all $1 \leq j \leq m-1$ and $g^{(m)}(c) \neq 0$. Then $a_n^m[g(X_n) - g(c)] \xrightarrow{D} \frac{1}{m!}g^{(m)}(c)Y^m$
[δ-method - multivariate] If X_i, Y are k-vectors rvs and $c \in \mathcal{R}^k$ and $a_n[g(X_n) - g(c)] \xrightarrow{D} \nabla g(c)^TY$
[Stochastic order - Real] for a constant $c > 0$ and all n, ① $a_n = O(b_n) \Leftrightarrow |a_n| \leq c|b_n|$ ② $a_n = o(b_n) \Leftrightarrow \lim_{n\to\infty} a_n/b_n = 0$
[Stochastic order - RV] ① $X_n = O_{\text{a.s.}}(Y_n) \Leftrightarrow P\{|X_n| = O(|Y_n|)\} = 1$ ② $X_n = o_{\text{a.s.}}(Y_n) \Leftrightarrow X_n/Y_n \xrightarrow{\text{a.s.}} 0$, ③ $\forall\ \epsilon > 0, \exists C_\epsilon > 0, n_\epsilon \in \mathcal{N} s.t.$ $X_n = O_P(Y_n) \Leftrightarrow \sup_{n\geq n_\epsilon} P\left(\{\omega \in \Omega : |X_n(\omega) \geq C_\epsilon|Y_n(\omega)|\}\right) < \epsilon$ ④ If $X_n = O_P(1)$, $\{X_n\}$ is bounded in probability. ⑤ $X_n = o_P(Y_n) \Leftrightarrow X_n/Y_n \xrightarrow{P} 0$
[Stochastic Order Properties] ① If $X_n \xrightarrow{\text{a.s.}} X$, then $\{\sup_{n\geq k}|X_n|\}_k$ is $O_p(1)$. ② If $X_n \xrightarrow{D} X$ for a rvs, then $X_n = O_P(1)$ (tightness). ③ If $E|X_n| = O(a_n)$, then $X_n = O_P(a_n)$ ④ If $E|X_n| = o(a_n)$, then $X_n = o_P(a_n)$
[SLLN, iid] $E|X_1| < \infty \Leftrightarrow \frac{1}{n}\sum_{i=1}^n X_1 \xrightarrow{\text{a.s.}} EX_1$
[SLLN, non-idential but independent] If $\exists\, p \in [1,2]$ s.t. $\sum_{i=1}^{\infty}\frac{E|X_i|^p}{i^p} < \infty$, then $\frac{1}{n}\sum_{i=1}^n (X_i - EX_i) \xrightarrow{\text{a.s.}} 0$

[USLLN, idd] Suppose ① $U(x,\theta)$ is continuous in θ for any fixed x ② for each θ, $\mu(\theta) = EU(X,\theta)$ is finite ③ Θ is compact ④ There exists function $M(x)$ s.t. $EM(X) < \infty$ and $|U(x,\theta) \le M(x)|$ for all x, θ. Then $P\left\{\lim_{n\to\infty}\sup_{\theta\in\Theta}\left|\frac{1}{n}\sum_{i=1}^n U(X_j,\theta) - \mu(\theta)\right| = 0\right\} = 1$

[WLLN, iid] $a_n = E(X_1 I_{\{|X_1|\le n\}}) \in [-n,n]$ $nP(|X_1| > n) \to 0 \Leftrightarrow \frac{1}{n}\sum_{i=1}^n X_i - a_n \xrightarrow{P} 0$

[WLLN, non-identical but independent] If $\exists\, p \in [1,2]$ s.t. $\lim_{n\to\infty}\frac{1}{n^p}\sum_{i=1}^n E|X_i|^p = 0$, then $\frac{1}{n}\sum_{i=1}^n (X_i - EX_i) \xrightarrow{P} 0$

[Weak Convergency] $\int f d\nu_n \to \int f d\nu$ for every bounded and continous real function f. $X_n \xrightarrow{D} X \Leftrightarrow E[h(X_n)] \to E[h(X)]$

[CLT, iid] Suppose $\Sigma = Var X_1 < \infty$, then $\frac{1}{\sqrt{n}}\sum_{i=1}^n (X_i - EX_i) \xrightarrow{D} N(0,\Sigma)$

[CLT, non-identical but independent] Suppose ① $k_n \to \infty$ as $n \to \infty$ ② (Lindeberg's condition) $0 < \sigma_n^2 = Var\left(\sum_{j=1}^{k_n} X_{nj}\right) < \infty$. ③ If for any $\epsilon > 0$, $\frac{1}{\sigma_n^2}\sum_{j=1}^{k_n} E\left\{(X_{nj} - EX_{nj})^2 I_{\{|X_{nj}-EX_{nj}|>\epsilon\sigma_n\}}\right\} \to 0$. Then $\frac{1}{\sigma_n}\sum_{j=1}^{k_n}(X_{nj} - EX_{nj}) \xrightarrow{D} N(0,1)$

[Check Lindeberg condition] Option ① (Lyapunov condition) $\frac{1}{\sigma_n^{2+\delta}}\sum_{j=1}^{k_n} E|X_{nj} - EX_{nj}|^{2+\delta} \to 0$ for some $\delta > 0$

Option ② (Uniform boundedness) If $|X_{nj}| \le M$ for all n and j and $\sigma_n^2 = \sum_{j=1}^{k_n} Var(X_{nj}) \to \infty$

[Feller's condition] Ensures Lindeberg's condition is sufficient and necessary (else only sufficient). $\lim_{n\to\infty}\max_{j\le k_n}\frac{Var(X_{nj})}{\sigma_n^2} = 0$

Exponential Families

[NEF] $f_\eta(X) = \exp\left\{\eta^T T(X) - \mathcal{C}(\eta)\right\} h(x)$, where $\eta = \eta(\theta)$ and $\mathcal{C}(\eta) = \log\left\{\int_\Omega \exp\left\{\eta^T T(X)\right\} h(X) dX\right\}$. NEF is full rank if Ξ contains open set in $\mathcal{R}^p$, $\Xi = \{\eta(\theta) : \theta \in \Theta\} \subset \mathcal{R}^p$. Suppose $X_i \sim f_i$ independently with f_i Exp Fam, then joint distribution X is also Exp Fam.

[Showing non Exp Fam] For an exp fam P_θ, there is nonzero measure λ s.t. $\frac{dP_\theta}{d\lambda}(\omega) > 0$ λ-a.e. and for all θ. Consider $f = \frac{dP_\theta}{d\lambda} I_{(t,\infty)}(x)$, $\int f d\lambda = 0, f \ge 0 \Rightarrow f = 0$. Since $\frac{dP_\theta}{d\lambda} > 0$ by assumption, then $I_{(t,\infty)}(x) = 0 \Rightarrow v([t,\infty)) = 0$. Since t is arbitary, consider $v(\mathcal{R}) = 0$ (contradiction)

[NEF MGF] Suppose η_0 is interior point on Ξ, then $\psi_{\eta_0}(t) = \exp\left\{\mathcal{C}(\eta_0 + t) - \mathcal{C}(\eta_0)\right\}$ and is finite in neighborhood of $t = 0$.

[NEF Moments] Let $A(\theta) = \mathcal{C}(\eta_0(\theta))$, $\frac{dA(\theta)}{d\theta} = \frac{d\mathcal{C}(\eta_0(\theta))}{d\eta_0(\theta)}\cdot\frac{d\eta_0(\theta)}{d\theta}$, $T(x) = \frac{\partial\mathcal{C}(\eta)}{\partial\eta}$ ⓐ $E_{\eta_0}T = \frac{d\psi_{\eta_0}}{dt}|_{t=0} = \frac{d\mathcal{C}}{d\eta_0} = \frac{A'(\theta)}{\eta_0'(\theta)}$, ⓑ $E_{\eta_0}T^2 = \mathcal{C}''(\eta_0) + \mathcal{C}'(\eta_0)^2$, ⓒ $Var(T) = \mathcal{C}''(\eta_0) = \frac{A''(\theta)}{[\eta_0(\theta)]^2} - \frac{\eta_0(\theta)''A'(\theta)}{[\eta_0(\theta)']^3} = \frac{\partial^2\mathcal{C}(\eta)}{\partial\eta\partial\eta^T}$

[NEF Differential] $G(\eta) := E_\eta(g) = \int g(\omega)\exp\left\{\eta^T T(\omega) - \mathcal{C}(\eta)\right\} h(\omega) d\nu(\omega)$ for η in interior of Ξ_g ① G is continuous and has continuous derivatives of all orders. ② Derivatives can be computed by differentiation under the integral sign. $\frac{dG(\eta)}{d\eta} = E_\eta\left[g(\omega)\left(T(\omega) - \frac{\partial}{\partial\eta}\xi(\eta)\right)\right]$ where Ξ_g is set η such that $\int |g(\omega)|\exp\left\{\eta^T T(\omega) - \mathcal{C}(\eta)\right\} h(\omega) d\nu(\omega) < \infty$

[NEF Min Suff] ① If there exists $\Theta_0 = \{\theta_0, \theta_1, \cdots, \theta_p\} \subset \Theta$ s.t. vectors $\eta_i = \eta(\theta_i) - \eta(\theta_0), i \in [1,p]$ are linearly independent in $\mathcal{R}^p$, then T is also minimal sufficient. Check $det([\eta_1, \cdots, \eta_p])$ is non-zero ② $\Xi = \{\eta(\theta) : \theta \in \Theta\}$ contains $(p+1)$ points that do not lie on the same hyperplane ③ Ξ is full rank.

[NEF complete and sufficient] If $\mathcal{P}$ is NEF of full rank then $T(X)$ is complete and sufficient for $\eta \in \Xi$

[NEF MLE] $\hat{\theta} = \eta^{-1}(\hat{\eta})$ or solution of $\frac{\partial\eta(\theta)}{\partial\theta}T(x) = \frac{\partial\xi(\theta)}{\partial\theta}$

[NEF Fisher Info] If $\underline{I}(\eta)$ is fisher info natural parameter η, then $Var(T) = \underline{I}(\eta)$. Let $\psi = E[T(X)]$. Suppose $\bar{I}(\psi)$ is fisher info matrix for parameter ψ, then $Var(T) = [\bar{I}(\psi)]^{-1}$

Statistics

[Sufficiency] $T(X)$ is sufficient for $P \in \mathcal{P} \Leftrightarrow P_X(x|Y = y)$ is known and does not depend on P. T sufficient for $\mathcal{P}_0$ but not ncessarily $\mathcal{P}_1$, $\mathcal{P}_0 \subset \mathcal{P} \subset \mathcal{P}_1$.

[Factorization theorem] $T(X)$ is sufficient for $P \in \mathcal{P} \Leftrightarrow$ there are non-negative Borel functions h with ① $h(x)$ does not depend on P ② $g_P(t)$ which depends on P s.t. $\frac{dP}{d\nu}(x) = g_P(T(x))h(x)$

[Minimal sufficiency] T is minimal sufficient $\Leftrightarrow T = \psi(S)$ for any other sufficient statistics S

[Min Suff-Method 1] (Theorem A) Suppose $\mathcal{P}_0 \subset \mathcal{P}$ and $\mathcal{P}_0$-a.s. implies $\mathcal{P}$-a.s. If T is sufficient for $P \in \mathcal{P}$ and minimal sufficient for $P \in \mathcal{P}_0$, then T is minimal sufficient for $P \in \mathcal{P}$ (Theorem B) Suppose $\mathcal{P}$ contains PDFs $f_0, f_1, \cdots$ w.r.t a σ-finite measure. ⓐ Define $f_\infty(x) = \sum_{i=0}^\infty c_i f_i(x)$ and $T_i(x) = f_i(x)/f_\infty(x)$, then $T(X) = (T_0(X), T_1(X), \cdots)$ is minimal sufficient for $\mathcal{P}$. Where $c_i > 0, \sum_{i=0}^\infty c_i = 1, f_\infty(x) > 0$. ⓑ If $\{x : f_i(x) > 0\} \subset \{x : f_0(x) > 0\}$ for all i, then $T(X) = (f_1(x)/f_0(x), f_2(x)/f_0(x), \cdots$ is minimal sufficient for $\mathcal{P}$

[Min Suff-Method 2] (Theorem C) If ⓐ $T(X)$ is sufficient, and ⓑ $\exists\,\phi$ s.t. for $\forall\, x, y$. $f_P(x) = f_P(y)\phi(x,y)\ \forall\, P \in \mathcal{P} \Rightarrow T(x) = T(y)$. Then $T(X)$ is minimal sufficient for $\mathcal{P}$

[Ancillary statistics] A statistics $V(X)$ is ancillary for $\mathcal{P}$ if its distribution does not depend on population $P \in \mathcal{P}$ (First-order ancillary) if $E_P[V(X)]$ does not depend on $P \in \mathcal{P}$

[Completeness] $T(X)$ is complete for $P \in \mathcal{P} \Leftrightarrow$ for any Borel function g, $E_P g(T) = 0$ implies $g(T) = 0$, boundedly complete $\Leftrightarrow g$ is bounded. Completeness + Sufficiency $\Rightarrow$ Minimal Sufficiency

[Basu's theorem] If V is ancillary and T is boundedly complete and sufficient, then V and T are independent w.r.t any $P \in \mathcal{P}$

Fisher information $I(\theta) = E\left(\frac{\partial}{\partial\theta}\log f_\theta(X)\right)^2 = \int\left(\frac{\partial}{\partial\theta}\log f_\theta(X)\right)^2 f_\theta(X) d\nu(x) = E\left\{\frac{\partial}{\partial\theta}\log f_\theta(X)\left[\frac{\partial}{\partial\theta}\log f_\theta(X)\right]^T\right\}$

[Parameterization] If $\theta = \psi(\eta)$ and ψ' exists, $\tilde{I}(\eta) = \psi'(\eta)^2 I(\psi(\eta))$

[Twice differentiable] Suppose f_θ is twice differentiable in θ and $\int\frac{\partial^2}{\partial\theta^2}f_\theta(x)I_{f_\theta(x)>0}d\nu = 0$, then $I(\theta) = -E\left[\frac{\partial^2}{\partial\theta\partial\theta^T}\log f_\theta(X)\right]$

[Independent samples] If $\int\frac{\partial}{\partial\theta}f_\theta(x)d\nu = 0$ holds, then $I_{(X,Y)}(\theta) = I_X(\theta) + I_Y(\theta)$, and $I_{(X_1,\cdots,X_n)}(\theta) = nI_{X_1}(\theta)$

Comparing decision rules

[Compare decision rules] ⓐ as good as if $R_{T_1}(P) \le P_{T_2}(P)$. $\forall\, P \in \mathcal{P}$ ⓑ better if $R_{T_1}(P) < R_{T_2}(P)$ for some $P \in \mathcal{P}$ (and T_2 is dominated by T_1). ⓒ equivalent if $R_{T_2}(P) = R_{T_2}(P)$ for all $P \in \mathcal{P}$

[Optimal] T_* is $\mathcal{J}$-optimal if T_* is as good as any other rule in $\mathcal{J}$,

[Admissibility] $T \in \mathcal{J}$ is $\mathcal{J}$-admissible if no $S \in \mathcal{J}$ is better than T in terms of the risk.

[Minimaxity] $T_* \in \mathcal{J}$ is $\mathcal{J}$-minimax if $\sup_{P\subset\mathcal{P}} R_{T_*}(P) \le \sup_{P\subset\mathcal{P}} R_T(P)$ for any $T \in \mathcal{J}$

[Bayes Risk] A form of averaging $R_T(P)$ over $P \in \mathcal{P}$. Bayes risk $r_T(\Pi) = \int_{\mathcal{P}} R_T(P) d\Pi(P)$, $R_T(\Pi)$ is Bayes risk of T wrt a known probability measure Π.

[Bayes rule] T_* is $\mathcal{J}$-Bayes rule wrt Π if $r_{T_*}(\Pi) \le r_T(\Pi)$ for any $T \in \mathcal{J}$.

[Finding Bayes rule] Let $\tilde{\theta} \sim \pi$, $X|\tilde{\theta} \sim P_{\tilde{\theta}}$, then $r_\pi(T) = E\left[L(\tilde{\theta}, T(X)\right] = E\left[E\left[L(\tilde{\theta}, T(X)\right]|X\right]$ where E is taken jointly over $(\tilde{\theta}, X)$.

Then find $T_*(x)$ that minimises the conditional risk.

[Rao-Blackwell] ⓐ Suppose $L(P,a)$ is convex and T is sufficient and S_0 is decision rule satisfying $E_P|||S_0|| < \infty$ for all $P \in \mathcal{P}$. Let $S_1 = E[S_0(X)|T]$, then $R_{S_1}(P) \leq R_{S_0}(P)$. ⓑ If $L(P,a)$ is strictly convex in a, and S_0 is not a funciton of T, then S_0 is inadmissible and dominated by S_1.

MLE

[MLE Consistency] Suppose ① Θ is compact ② $f(x|\theta)$ is continuous in θ for all x ③ There exists a function $M(x)$ s.t. $E_{\theta_0}[M(X)] < \infty$ and $|\log f(x|\theta) - \log f(x|\theta_0)| \leq M(x)$ for all x, θ ④ identifiability holds $f(x|\theta) = f(x|\theta_0)$ ν-a.e. $\Rightarrow \theta = \theta_0$. Then MLE estimate $\hat{\theta}_n \xrightarrow{\text{a.s}} \theta_0$

[RLE Consistency]

[RLE Asymptotic normality]

Unbiased Estimators

[UMVUE] $T(X)$ is UMVUE for $\theta \Leftrightarrow Var(T(X) \leq Var(U(X))$ for any $P \in \mathcal{P}$ and any other unbiased estimator $U(X)$ of θ

[Lehmann-Scheffé] If $T(X)$ is sufficient and complete for θ. If θ is estimable, then there is a unique unbiased estimator of θ that is of the form $h(T)$.

[UMVUE method1] Using Lehmann-Scheffé, suppose T is sufficient and complete manipulate $E(h(T)) = \theta$ to get $\hat{\theta}$.

[UMVUE method2] Using Rao-Blackwellization. Find ① unbiased estimator of $\theta = U(X)$ ② sufficient and complete statistics $T(X)$ ③ then $E(U|T)$ is the UMVUE of θ by Lehmann-Scheffé.

[UMVUE method3] Useful when no complete and sufficient statistics. Can use to find UMVUE, check if estimator is UMVUE, show nonexistence of UMVUE. $T(X)$ is UMVUE $\Leftrightarrow E[T(X)U(X)] = 0$

ⓐ T is unbiased estimator of η with finite variance, $\mathcal{U}$ is set of all unbiased estimators of 0 with finite variances. ⓑ $T = h(S)$, where S is sufficient and h is Borel function, $\mathcal{U}_S$ is subset of $\mathcal{U}$ consisting of Borel functions of S.

[Using method3] ① Find $U(x)$ via $E[U(x)] = 0$ ② Construct $T = h(S)$ s.t. T is unbiased ③ Find T via $E[TU] = 0$

[Corollary] If T_j is UMVUE of η_j with finite variances, then $T = \sum_{j=1}^k c_j T_j$ is UMVUE of $\eta = \sum_{j=1}^k c_j \eta_j$. If T_1, T_2 are UMVUE of η with finite variances, then $T_1 = T_2$ a.s. $P, P \in \mathcal{P}$

[Cramér-Rao Lower Bound] Suppose ① Θ is an open set and P_θ has pdf f_θ ② f_θ is differentiable and $\frac{\partial}{\partial\theta}\int f_\theta(x)d\nu = \int \frac{\partial}{\partial\theta} f_\theta(x)d\nu = 0$. ③ $g(\theta)$ is differentiable and $T(X)$ is unbiased estimator of $g(\theta)$ s.t. $g'(\theta) = \frac{\partial}{\partial\theta}\int T(x) f_\theta(x)d\nu = \int T(x)\frac{\partial}{\partial\theta} f_\theta(x)d\nu, \theta \in \Theta$. Then $Var(T(X)) \geq \frac{g'(\theta)^2}{I(\theta)} = \left[\frac{\partial}{\partial\theta}g(\theta)\right]^T [I(\theta)]^{-1}\frac{\partial}{\partial\theta}g(\theta)$

[CR LB for biasd estimator] $Var(T) \geq \frac{[g'(\theta)+b'(\theta)]^2}{I(\theta)}$

[CR LB iff] CR achieve equality ⓐ $\Leftrightarrow T = \left[\frac{g'(\theta)}{I(\theta)}\right]\frac{\partial}{\partial\theta}\log f_\theta(X) + g(\theta)$ ⓑ $\Leftrightarrow f_\theta(X) = \exp(\eta(\theta)T(x) - \xi(\theta))h(x)$, s.t. $\xi'(\theta) = g(\theta)\eta'(\theta)$ and $I(\theta) = \eta'(\theta)g'(\theta)$

Asymptotics [Consistency of point estimators] $X = (X_1, \cdots, X_n)$ is sample from $P \in \mathcal{P}$ and $T_n(X)$ be estimator of θ for P. (consistent) $\Leftrightarrow T_n(X) \to^P \theta$ (strongly consistent) $\Leftrightarrow T_n(X) \to^{\text{a.s.}} \theta$ (a_n-consistent) $\Leftrightarrow a_n(T_n(X) - \theta) = O_P(1)$, $\{a_n\} > 0$ and diverge to ∞ (L_r-consistent) $T_n(X) \to^{L^P} \theta$ for some fixed $r > 0$ A combination of LLN, CLT, Slustky's, continuous mapping, δ-method are used. If T_n is (strongly) consistent for θ and g is continuous at θ then $g(T_n)$ is (strongly) consistent for $g(\theta)$

[Affine estimator] Consider $T_n = \sum_{i=1}^n c_{ni}X_i$ (1) If $c_{ni} = c_i/n$ satisfy (1) $\frac{1}{n}\sum_{i=1}^n c_i \to 1$ and $\sup_i |c_i| < \infty$ then T_n is strongly consistent. (2) If population variance is finite, then T_n is consistent in mse $\Leftrightarrow \sum_{i=1}^n c_{ni} \to 1$ and $\sum_{i=1}^n c_{ni}^2 \to 0$

[Asympotics bias, variance, MSE] (Approximate unbiased) Estimator $T_n(X)$ for θ is approximately unbiased if $b_{T_n}(P) \to 0$ as $n \to \infty$, $b_{T_n}(P) := ET_n(X) - \theta$ When estimator's expectations or second moment are not well defined, we need asymptotic behaviours. (Asymptotic statistics conditions) $\{a_n\} > 0$ and either (a) $a_n \to \infty$ or (b) $a_n \to a > 0$. If

$$a_n(T_n - \theta) \to^D Y$$

(Asymptotic expectation) If $a_n\xi_n \to^D \xi$, $E|\xi| < \infty$, then asymptotic expectation of ξ_n is $E\xi/a_n$ (Asymptotic bias) $\tilde{b}_{T_n} = EY/a_n$, asymptotically unbiased if $\lim_{n\to\infty}\tilde{b}_{T_n}(P) = 0$ for any $P \in \mathcal{P}$. (Asymptotic MSE) amse is the asymptotic expectation of $(T_n - \theta)^2$ or $\text{amse}_{T_n}(P) = EY^2/a_n^2$ (Asymptotic Variance) $\sigma^2_{T_n}(P) = Var(Y)/a_n^2$ (Remark) $EY^2 \leq \liminf_{n\to\infty} E[a_n^2(T_n - v)^2]$ (amse is no greater than exact mse)

[Asym Relative Efficiency] $e_{T_{1n},T_{2n}} = amse_{T_{2n}(P)}/amse_{T_{1n}(P)}$. Note efficiency of estimator T refers to $1/[I(\theta)MSE_T(\theta)]$

[δ-method corollary] If $a_n \to \infty$, g is differentiable at θ, $U_n = g(T_n)$. Then amse of U_n is $[g'(\theta)^2EY^2]/a_n^2$, asym var of U_n is $[g'(\theta)^2Var(Y)]/a_n^2$

[Properties of MOM] θ_n is unique if h^{-1} exists. Strongly consistent if h^{-1} is continuous via SLLN and continuous mapping. If h^{-1} is differentiable and $E|X_1|^{2k} < \infty$ then by CLT and δ-method. V_μ is $k \times k$ with $(i,j) = \mu_{i+j} - \mu_i\mu_j$ $\sqrt{n}(\hat{\theta}_n - \theta) \to_D N(0, [\nabla g]^T V_\mu \nabla g)$ MOM is $\sqrt{n}$-consistent, and if $k = 1$ $amse_{\hat{\theta}_n}(\theta) = g'(\mu_1)^2\sigma^2/n$, $\sigma^2 = \mu_2 - \mu_1^2$

[Asym Properties of UMVUE] Typically consistent, exactly unbiased, ratio of mse over Cramér-Rao LB converges to 1 (asym they are the same).

[Asym sample quantiles] $X_1, X_2, \cdots$ iid rvs with CDF F, $\gamma \in (0,1)$, $\hat{\theta}_n := \lfloor \gamma n \rfloor$-th order statistics. Suppose $F(\theta) = \gamma$ and $F'(\theta) > 0$ and exists. $\sqrt{n}(\hat{\theta}_n - \theta) \to^D N\left(0, \frac{\gamma(1-\gamma)}{[F'(\theta)]^2}\right)$

[Cons and Asym eff MLEs, RLEs]

[Continuous in θ] Suppose (1) Θ is compact (2) $f(x|\theta)$ is continuous in θ for all x (3) there exists a function $M(x)$ s.t. $E_{\theta_0}|M(X)| < \infty$ and $|\log f(x|\theta) - log f(x|\theta_0)| \leq M(x)$ for all x and θ (4) identifiable $f(x|\theta) = f(x|\theta_0)$ ν-a.e. $\Rightarrow \theta = \theta_0$. Then for any sequence of MLE $\hat{\theta}_n \to_{\text{a.s.}} \theta_0$

[Upper semi-continuous (usc)] $\lim_{\rho\to 0}\left\{\sup_{||\theta'-\theta||<\rho} f(x|\theta')\right\} = f(x|\theta)$

[USC in θ] Suppose (1) Θ is compact with metric $d(\cdot,\cdot)$ (2) $f(x|\theta)$ is usc in θ and for all x (3) there exists a function $M(x)$ s.t. $E_{\theta_0}|M(X)| < \infty$ and $\log f(x|\theta) - \log f(x|\theta_0) \leq M(x)$ for all x and θ (4) for all $\theta \in \Theta$ and sufficiency small $\rho > 0$, $\sup_{d(\theta',\theta)<\rho} f(x|\theta')$ is measurable in x (5) identifiable $f(x|\theta) = f(x|\theta_0)$ ν-a.e. $\Rightarrow \theta = \theta_0$. Then $d(\hat{\theta}_n, \theta_0) \to_{\text{a.s.}} 0$

[M-estimators] General method to find $\hat{\theta}_n$ maximises criterion function $S_\theta(x)$, for MLE $s_\theta(x) = \log f(x|\theta)$. $E_{\theta_0}s_\theta(X) < E_{\theta_0}s_{\theta_0}(X)$ $\forall \theta \neq \theta_0$. $\theta \mapsto S_n(\theta) = \frac{1}{n}\sum_{i=1}^n s_\theta(X_i)$

[Consistency of M-estimators] $S_n(\theta)$ is random function while $S(\theta)$ is fixed s.t. $\sup_{\theta\in\Theta}|S_n(\theta) - S(\theta)| \to_P 0$ and for every $\rho > 0$ $\sup_{\theta: d(\theta,\theta_0)>\rho} S(\theta) < S(\theta_0)$. Then any sequence of estimators $\hat{\theta}_n$ with $S_n(\hat{\theta}_n) \geq S_n(\theta_0) - o_P(1)$ converges in probability to θ_0

[RLE] [Roots of the Likelihood Equation] θ that solves $\frac{\partial}{\partial\theta}\log L_n(\theta) = 0$

[Basic Regularity conditions] Suppose (1) Θ is open subset of $\mathcal{R}^k$ (2) $f(x|\theta)$ is twice continuously differentiable in θ for all x, and $\frac{\partial}{\partial\theta}\int f(x|\theta)d\nu = \int \frac{\partial}{\partial\theta} f(x|\theta)d\nu$, $\frac{\partial}{\partial\theta}\int \frac{\partial}{\partial\theta^T} f(x|\theta)d\nu = \int \frac{\partial^2}{\partial\theta\partial\theta^T} f(x|\theta)d\nu$. (3) $\Psi(x,\theta) = \frac{\partial^2}{\partial\theta\partial\theta^T}\log f(x|\theta)$, there exists a constant c and non-negative function H s.t. $EH(X) < \infty$ and $\sup_{||\theta-\theta_*||<c} ||\Psi(x,\theta)|| \leq H(x)$. (4) Identifiable

[Consistency of RLEs] Under basic regularity conditions, there exists a sequence of $\hat{\theta}_n$ s.t. $\frac{\partial}{\partial\theta}\log L_n(\hat{\theta}_n) = 0$ and $\hat{\theta}_n \to_{\text{a.s.}} \theta_*$. More useful if likelihood is concave or unique.

[Asymptotic Normality of RLEs] Assume basic regularity conditions, and $I(\theta) = \int \frac{\partial}{\partial\theta}\log f(x|\theta)\left[\frac{\partial}{\partial\theta}\log f(x|\theta)\right]^T d\nu(x)$ is positive definite and $\theta = \theta_*$. Then any consistent sequence $\{\tilde{\theta_n}\}$ of RLE it holds $\sqrt{n}(\tilde{\theta_n} - \theta_*) \to_D N\left(0, \frac{1}{I(\theta_*)}\right)$

[NEF RLEs] Basic regularity condition (1, 2, 3, 4) holds due to proposition 3.2 and theorem 2.1, and result for (3). Only need to check condition on Fisher Info, then when n is large, there exists $\hat{\eta}_n$ s.t. $g(\hat{\eta}_n) = \hat{\mu}_n$ and $\hat{\eta}_n \to_{\text{a.s.}} \eta$ $\sqrt{n}(\hat{\eta}_n - \eta) \to_D N\left(0, \left[\frac{\partial^2}{\partial\eta\partial\eta^T}\mathcal{C}(\eta)\right]^{-1}\right)$

Where $g(\eta) = \frac{\partial\mathcal{C}(\eta)}{\partial\eta}$ and $\hat{\mu}_n = \frac{1}{n}\sum_{i=1}^n T(X_i)$

[Asym Covariance Matrix] $V_n(\theta)$ is $k\times k$ positive definite matrix called asym covariance matrix. $V_n(\theta)$ is usually in form of $n^{-\delta}V(\theta)$, higher δ means faster convergence. $[V_n(\theta)]^{-1/2}(\hat{\theta}_n - \theta) \to_D N_k(0, I_k)$

[Information Inequalities] $A \preccurlyeq B$ means $B - A$ is positive semi-definite. Suppose two estimators $\hat{\theta}_{1n}, \hat{\theta}_{2n}$ satisfy asym covariance matrix with $V_{1n}(\theta), V_{2n}(\theta)$. $\hat{\theta}_{1n}$ is asym more efficient thant $\hat{\theta}_{2n}$ if (1) $V_{1n}(\theta) \preccurlyeq V_{2n}(\theta)$ for all $\theta\in\Theta$ and all large n (2) $V_{1n}(\theta) \prec V_{2n}(\theta)$ for at least one $\theta\in\Theta$ But note $\hat{\theta}_n$ is asym unbiased but CR LB might not hold even if regularity condition is satisfied.

[Hodges' estimator] $X_i \sim N(\theta, 1)$, $\hat{\theta}_n = \bar{X}_n$ if $\bar{X}_n \geq n^{-1/4}$ and $t\bar{X}_n$ otherwise. $V_n(\theta) = 1/n$ if $\theta \neq 0$ and t^2/n otherwise. if $\theta\neq 0$: $\sqrt{n}(\hat{\theta}-\theta) = \sqrt{n}(\bar{X}_n - \theta) - (1-t)\sqrt{n}\bar{X}_n I_{|\bar{\theta}_n|<n^{-1/4}}$ if $\theta = 0$: $= t\sqrt{n}(\bar{X}_n - \theta) + (1-t)\sqrt{n}\bar{X}_n I_{|\bar{X}_n|\geq n^{-1/4}}$

[Super-efficiency] Point where UMVUE failed Hodeges' estiamtor in information inequality (2). But under the basic regularity condition and if Fisher Information is positive definite at $\theta = \theta_*$, if $\hat{\theta}_n$ satisfies Asym covariance matrix, then there is a $\Theta_0 \subset \Theta$ with Lebesgue measure 0 s.t. information inequality (2) holds for any $\theta \notin \Theta_0$

[Asym efficiency] Assume Fisher Info $I_n(\theta)$ is well-defined and positive definite for every n, seq of estimators $\{\hat{\theta}_n\}$ satisfies asym cov matrix is asym efficient or asym optimal if and only if $V_n(\theta) = [I_n(\theta)]^{-1}$.

[One-step MLE] Often asym efficient, useful to adjust an non asym efficient estimators provided $\hat{\theta}_n^{(0)}$ is $\sqrt{n}$-consistent. $\hat{\theta}_n^{(1)} = \hat{\theta}_n^{(0)} - \left[\nabla s_n(\hat{\theta}_n^{(0)})\right]^{-1} s_n(\hat{\theta}_n^{(0)})$

Hypo testing

[Hypothesis tests] Let $\mathcal{P}$ be a family of distributions, $\mathcal{P}_0 \subset \mathcal{P}, \mathcal{P}_1 = \mathcal{P}\backslash\mathcal{P}_0$. Hypothesis testing decides between $H_0: P\in\mathcal{P}_0, H_1: P\in\mathcal{P}_1$. Action space $\mathcal{A} = \{0,1\}$, decision rule is called a test $T: \mathcal{X}\to\{0,1\} \Rightarrow T(X) = I_C(X)$ for some $C\subset\mathcal{X}$. C is called the region/critical region.

[$0-1$ loss] Common loss function for hypo test, $L(P,j) = 0$ for $P\in\mathcal{P}_j$ and $= 1$ for $P\in\mathcal{P}_{1-j}, j\in\{0,1\}$ Risk $R_T(P) = P(T(X)=1) = P(X\in C)$ if $P\in\mathcal{P}_0$ or $P(T(X)=0) = P(X\notin C)$ if $P\in\mathcal{P}_1$

[Type I and II errors] Type I: H_0 is rejected when H_0 is true. Error rate: $\alpha_T(P) = P(T(X)=1), P\in\mathcal{P}_0$ Type II: H_0 is accepted when H_0 is false. Error rate: $1-\alpha_T(P) = P(T(X)=1), P\in\mathcal{P}_1$

[Power function of T] $\alpha_T(P)$, Type I and Type II error rates cannot be minimized simultaneously.

[Significance level] Under Neyman-Pearson framework, assign pre-specified bound α (significance level of test): $\sup_{P\subset\mathcal{P}_0} P(T(X)=1) \leq \alpha$

[size of test] α' is the size of the test $\sup_{P\subset\mathcal{P}_0} P(T(X)=1) = \alpha'$

NP Test (Steps) (1) Find joint distribution $f(X_1,\cdots,X_n)$ - MLR/NEF (2) Hypothesis H_0, H_1 - simple/composite, must be θ and not $f(\theta)$ (3) Form N-P test structure T_* (4) Find test dist, rejection/acceptance region. (Type I error) reject H_0 when H_0 is correct. $\beta_T(\theta_0) = E_{H_0}(T) \leq \alpha$ (within controlled with size α) (Type II error) do not reject H_0 when H_1 is correct. $1-\beta_T(\theta)$ for $\theta\in\Theta_1$ (N-P lemma) NP test has non-trival power $\alpha < \beta_{H_1}(T)$ unless $P_0 = P_1$, and is unique up to γ (randomised test) (Show T_* is UMP) UMP when $E_1[T_*] - E_1[T] \geq 0$, *key equation*: $(T_*-T)(f_1 - cf_0) \geq 0$. $\Rightarrow \int(T_*-T)(f_1-cf_0) = \beta_{H_1}(T_*) - \beta_{H_1}(T) \geq 0$. (Composite hypothesis) Simple $\Rightarrow$ Composite when $\beta_T(\theta_0) \geq \beta_T(\theta\in H_0)$ and/or $\beta_T(\theta_0) \leq \beta_T(\theta\in H_1)$ (or does not depend on θ. For MLR this is satisfied, others need to check.

[Monoton Likelihood] $\theta_2 > \theta_2$, increasing likelihood ratio in Y if $g(Y) = \frac{f_{\theta_2}(Y)}{f_{\theta_1}(Y)} > 1$ or $g'(Y) > 0$. For NEF, check $\eta'(\theta) > 0$.

[UMP] (1) $H_0: P = p_0$ $H_1: P = p_1 \Rightarrow T(X) = I(p_1(X) > cp_0(X))$, $\beta_T(p_0) = \alpha$ (2) $H_0: \theta\leq\theta_0$ $H_1: \theta > \theta_0 \Rightarrow T(Y) = I(Y > c)$, $\beta_T(\theta_0) = \alpha$ (3) $H_0: \theta \leq \theta_1$ or $\theta\geq\theta_2$ $H_1: \theta_1 < \theta < \theta_2, \Rightarrow T(Y) = I(c_1 < Y < c_2)$, $\beta_T(\theta_1) = \beta_T(\theta_2) = \alpha$ (No UMP) $H_0: \theta = \theta_1, H_1: \theta\neq\theta_1$ and $H_0: \theta\in(\theta_1,\theta_2)$ $H_1: \theta\notin(\theta_1,\theta_2)$

[UMP Exp fam] ($\eta(\theta)$ increasing, $H_0: \theta\leq\theta_0$) ($\eta(\theta)$ decreasing, $H_0: \theta\geq\theta_0$) Same UMP $T(Y) = I(Y < c)$ ($\eta(\theta)$ increasing, $H_0: \theta\geq\theta_0$) ($\eta(\theta)$ decreasing, $H_0: \theta\leq\theta_0$) Reverse inequalities $T(Y) = I(Y > c)$

[Normal results] $X_i \sim N(\mu,\sigma^2)$, under $H_0: \sigma^2 = \sigma_0^2$, note $S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$ independent to $\bar{X}$ $V = \frac{1}{\sigma_0^2}\sum_{i=1}^n (X_i-\bar{X})^2 = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$ $t = \frac{\sqrt{n}(\bar{X}-\mu)/\sigma}{\sqrt{V/(n-1)}} = \frac{Z}{\sqrt{V/(n-1)}} \sim t_{(n-1)}$ [(only if $X_i\sim N$]

[Simultaneous] (Bonferroni) adjust each paramter level to $\alpha_t = \alpha/k$ (Bootstrap) Monte Carlo percentile estimate

[UMPU NEF $\eta(\theta) = \theta$] Require: (1) suff stat Y for θ (2) suff and complete U for φ (2a) U complete when φ to be full-rank (1) $H_0: \theta\leq\theta_1$ or $\theta\geq\theta_2$ $H_1: \theta_1 < \theta < \theta_2 \Rightarrow T(Y,U) = I(c_1(U) < Y < c_2(U))$, $E_{\theta_1}[T(Y,U)|U=u] = E_{\theta_2}[T(Y,U)|U=u] = \alpha$ (2) $H_0: \theta_1\leq\theta\leq\theta_2$ $H_1: \theta < \theta_1$ or $\theta > \theta_2 \Rightarrow T(Y,U) = I(Y < c_1(U)$ or $Y > c_2(U))$, $E_{\theta_1}[T(Y,U)|U=u] = E_{\theta_2}[T(Y,U)|U=u] = \alpha$ (3) $H_0: \theta = \theta_0$ $H_1: \theta\neq\theta_0 \Rightarrow T(Y,U) = I(Y < c_1(U)$ or $Y > c_2(U))$, $E_{\theta_0}[T_*(Y,U)|U=u] = \alpha$ and $E_{\theta_0}[T_*(Y,U)Y|U=u] = \alpha E_{\theta_0}(Y|U=u)$ (4) $H_0: \theta\leq\theta_0$ $H_1: \theta > \theta_0 \Rightarrow T(Y,U) = I(Y > c(U))$, $E_{\theta_0}[T(Y,U)|U=u] = \alpha$

[UMPU Normal] Assume $V(Y,U)$ independent of U under H_0 (1) $H_0: \theta\leq\theta_1$ or $\theta\geq\theta_2$ $H_1: \theta_1 < \theta < \theta_2$ Require V to be increasing in Y. $\Rightarrow T(V) = I(c_1 < V < c_2)$, $E_{\theta_1}[T(V)] = E_{\theta_2}[T(V)] = \alpha$ (2) $H_0: \theta_1\leq\theta\leq\theta_2$ $H_1: \theta < \theta_1$ or $\theta > \theta_2$ Require V to be increasing in Y. $\Rightarrow T(V) = I(V < c_1$ or $V > c_2)$, $E_{\theta_1}[T(V)] = E_{\theta_2}[T(V)] = \alpha$ (3) $H_0: \theta = \theta_0$ $H_1: \theta\neq\theta_0$ Require $V(Y,U) = a(u)Y + bU$ $\Rightarrow T(V) = I(V < c_1$ or $V > c_2)$, $E_{\theta_0}[T(V)] = \alpha$, $E_{\theta_0}[T(V)V] = \alpha E_{\theta_0}(V)$ (4) $H_0: \theta\leq\theta_0$ $H_1: \theta > \theta_0$ Require V to be increasing in Y. $\Rightarrow T(V) = I(V > c)$, $E_{\theta_0}[T(V)] = \alpha$

LR test $\lambda(X) = \frac{\sup_{\theta\in\theta_0}\ell(\theta)}{\sup_{\theta\in\Theta}\ell(\theta)}$ Rejects $H_0 \Leftrightarrow \lambda(X) < c\in[0,1]$. *1-param Exp Fam LR test is also UMP.*

Asym test Assume MLE regularity condition, under H_0, $-2\log\lambda(X) \to \chi^2_r$, where $r := dim(\theta)$ $T(X) = I\left[\lambda(X) < \exp(-\chi^2_{r,1-\alpha}/2)\right]$ where $\chi^2_{r,1-\alpha}$ is the $(1-\alpha)$th quantile of χ^2_r.
[Asymptotic Tests] $H_0 : R(\theta) = 0$, $\lim_{n\to\infty} W_n, Q_n \sim \chi^2_r$, $T(X) = I(W_n > \chi^2_{r,1-\alpha})$ or $I(Q_n > \chi^2_{r,1-\alpha})$ (Wald's test) $W_n = R(\hat{\theta})^T\{C(\hat{\theta})^T I_n^{-1}(\hat{\theta})C(\hat{\theta})\}^{-1}R(\hat{\theta})$ $C(\theta) = \partial R(\theta)/\partial\theta$, $I_n(\theta)$ is fisher info for $X_1, \cdots, X_n$, $\hat{\theta}$ is unrestricted MLE/RLE of θ. if $H_0 : \theta = \theta_0$ $\Rightarrow R(\theta) = \theta - \theta_0$, and $W_n = (\hat{\theta} - \theta_0)^T I_n(\hat{\theta})(\hat{\theta} - \theta_0)$ (Rao's score test) $Q_n = s_n(\tilde{\theta})^T I_n^{-1}(\tilde{\theta})s_n(\tilde{\theta})$. $s_n(\theta) = \partial\log\ell(\theta)/\partial\theta$ is score function, $\tilde{\theta}$ is MLE/RLE of θ under $H_0 : R(\theta) = 0$ (under H_0).

Non-param tests

[Sign test]
$X_i \sim^{iid} F$, u is fixed constant, $p = F(u)$, $\triangle_i = I(X_i - u \le 0)$, $P(\triangle_i = 1) = p$, $p_0 \in (0,1)$ $H_0 : p \le p_0$ $H_1 : p > p_0 \Rightarrow T(Y) = I(Y > m)$, $Y = \sum_{i=1}^n \triangle_i \sim Bin(n,p)$, m, γ s.t. $\alpha = E_{p_0}[T(Y)]$ $H_0 : p = p_0$ $H_1 : p \ne p_0 \Rightarrow T(Y) = I(Y < c_1$ or $Y > c_2)$, $E_{p_0}[T] = \alpha$ and $E_{p_0}[TY] = \alpha n p_0$
[Permutation test] $X_{i1}, \cdots, X_{in_i} \sim^{iid} F_i$, $i = 1,2$ $H_0 : F_1 = F_2$ $H_1 : F_1 \ne F_2$, $\Rightarrow T(X)$ with $\frac{1}{n!}\sum_{z\in\pi(x)} T(z) = \alpha$ $\pi(x)$ is set of $n!$ points obtained from x by permuting components of x E.g. $T(X) = I(h(X) > h_m)$, $h_m := (m+1)^{th}$ largest $\{h(z : z \in \pi(x)\}$ e.g $h(X) = |\bar{X}_1 - \bar{X}_2|$ or $|S_1 - S_2|$
[Rank test] $X_i \sim^{iid} F$, $Rank(X_i) = \#\{X_j : X_j \le X_i\}$, $H_0 : F$ symm ard 0, $H_1 : H_0$ false, R^o_+ vector of ordered R_+. (Wilcoxon) $T(X) = I[W(R^o_+) < c_1$ or $W(R^o_+ > c_2)]$, $W(R^o_+) = J(R^o_{+1}/n) + \cdots + J(R^o_{+n_*}/n)$ c_1, c_2 are $(m+1)^{th}$ smallest/largest of $\{W(y) : y \in \mathcal{Y}\}$, $\gamma = \alpha 2^n/2 - m$
[KS test] $X_i \sim^{iid} F$ $H_0 : F = F_0$, $H_1 : F \ne F_0$, $\Rightarrow T(X) = I(D_n(F_0) > c)$, $D_n(F) = \sup_{x\in\mathcal{R}} |F_n(x) - F(x)|$ With F_n Emp CDF, and for any $d, n > 0$, $P(D_n(F) > d) \le 2\exp(-2nd^2)$,
[Cramer-von test] Modified KS with $T(X) = I(C_n(F_0) > c)$, $C_n(F) = \int\{F_n(x) - F(x)\}^2 dF(x)$ $nC_n(F_0) \xrightarrow{D} \sum_{j=1}^\infty \lambda_j \chi^2_{1j}$, with $\chi^2_{1j} \sim \chi^2_1$ and $\lambda_j = j^{-2}\pi^{-2}$
[Empirical LR] $X_i \sim^{iid} F$, $H_0 : \Lambda(F) = t_0$ $H_1 : \Lambda(F) \ne t_0$, $\Rightarrow T(X) = I(ELR_n(X) < c)$ $ELR_n(X) = \frac{\ell(\hat{F}_0)}{\ell(\hat{F})}$, $\ell(G) = \prod_{i=1}^n P_G(\{x_i\})$, $G \in \mathcal{F}$. ($\mathcal{F} :=$ collection of CDFs, $P_G :=$ measure induced by CDF G)
Confidence set $C(X) : X \to \mathcal{B}(\Theta)$, Require $\inf_{P\in\mathcal{P}} P(\theta \in C(X)) \ge 1 - \alpha$. Conf coeff more than level (via pivotal qty) $C(X) = \{\theta : c_1 \le \mathcal{R}(X,\theta) \le c_2\}$, *not dependent on P*, common pivotal qty: $(X_i - \mu)/\sigma$ (invert accept region) $C(X) = \{\theta : x \in A(\theta)\}$, Acceptance region $A(\theta) = \{x : T_{\theta_0}(x) \ne 1\}$. $H_0 : \theta = \theta_0$, H_1 any
[Shortest CI] (unimodal) $f'(x_0) = 0$ $f'(x) < 0, x < x_0$ and $f'(X) > 0, x > x_0$ (Pivotal $(T-\theta)/U$, f unimodal at x_0) $[T - b_*U, T - a_*U]$, shortest when $f(a_*) = f(b_*) > 0$ $a_* \le x_0 \le b_*$ (Pivotal T/θ, $x^2 f(x)$ unimodal at x_0) $[b_*^{-1}T, a_*^{-1}T_*]$ shortest when $a_*^2 f(a_*) = b_*^2 f(b_*) > 0$ $a_* \le x_0 \le b_*$ (General) Suppose $f > 0$, integrable, unimodal at x_0, want: $\min b - a$ s.t. $\int_a^b f(x)dx$ and $a \le b$ sol: a_*, b_* satisfy (1) $a_* \le x_0 \le b_*$ (2) $f(a_*) = f(b_*) > 0$ (3) $\int_{a_*}^{b_*} f(x)dx = 1 - \alpha$
[asym] require $\lim_{n\to} P(\theta \in C(X)) \ge 1 - \alpha$, (asym pivotal) $\mathcal{R}_n(X,\theta) = \hat{V}_n^{-1/2}(\hat{\theta}_n - \theta)$ does not depend on P in limit (LR) $C(X) = \left\{\theta : \ell(\theta,\hat{\varphi}) \ge exp(-\chi^2_{r,1-\alpha} - \alpha/2)\ell(\hat{\theta})\right\}$ (Wald) $C(X) = \left\{\theta : (\hat{\theta} - \theta)^T\left[C^T\left(I_n(\hat{\theta})\right)^{-1}C\right]^{-1}(\hat{\theta} - \theta) \le \chi^2_{r,1-\alpha}\right\}$ (Rao) $C(X) = \Big\{\theta : [s_n(\theta, \varphi$

Bayesian

[Method] (Bayes formula) $\frac{dP_{\theta|X}}{d\Pi} = \frac{f_\theta(X)}{m(X)}$. (Bayes action $\delta(x)$) $\arg\min_a E[L(\theta,a)|X = x]$, when $L(\theta,a) = (\theta - a)^2$, $\delta(x) = E(\theta|X = x)$. (Generalised Bayes action) $\arg\min_a \int_\Theta L(\theta,a)f_\theta(x)d\Pi$, works for improper prior where $\Pi(\Theta) \ne 1$ (Interval estimation - Credible sets) $P_{\theta|x}(\theta \in C) = \int_C p_x(\theta)d\lambda \ge 1 - \alpha$ (HPD (highest posterior dentsity)) $C(x) = \{\theta : p_x(\theta) \ge c_\alpha\}$, often shortest length credible set. Is a horizontal line in the posterior density plot. Might not have confidence level $1 - \alpha$. (Hierachical Bayes) With hyper-priors as hyper-parameters on the priors.
[Empirical Bayes] Estimate hyper-paramter via data using MoM (no MLE as not independent). $X_i \sim N(\mu,\sigma^2)$, $\mu|\xi \sim N(\mu_0,\sigma_0^2)$, σ^2 known, $\xi = (\mu_0,\sigma_0^2)$, Using MoM $E_\xi(X|\xi) = E_\xi(E[X|\mu,\xi]) = E_\xi(\mu|\xi) = \mu_0 \approx \bar{X}$, $E_\xi(X^2|\xi) = E_\xi(\mu^2 + \sigma^2|\xi) = \sigma^2 + \mu_0^2 + \sigma_0^2 \approx \frac{1}{n}\sum X_i^2$ $\Rightarrow \sigma_0^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2 - \sigma^2$
[Normal posterior] Normal posterior with prior unknown μ and known σ^2 $N(\mu_*(x), c^2)$: $\mu_*(x) = \frac{\sigma^2}{\sigma^2 + n\sigma_0^2}\mu_0 + \frac{n\sigma_0^2}{\sigma^2 + n\sigma_0^2}\bar{x}$, $c^2 = \frac{\sigma_0^2\sigma^2}{n\sigma_0^2 + \sigma^2}$ $C(x) = [\mu_*(x) - cz_{1-\alpha/2},\ \mu_*(x) + cz_{1-\alpha/2}]$.
[Decision theory] (Admissibility) (1) $\delta(X)$ unique $\Rightarrow$ admissible, (2, 3) $r_\delta(\Pi) < \infty$, $\Pi(\theta) > 0$ for all θ and δ is Bayes action with respect to $\Pi \Rightarrow$ admissible. *Not true for improper priors*, Improper priors require excessive risk ignorable, take limit and observe if risk is admissible. (Bias) Under squared error loss, $\delta(X)$ is biased unless $r_\delta(\Pi) = 0$. *No applicable to improper priors.* (Minimax) If T is (unique) Bayes estimator under Π and $R_T(\theta) = \sup_{\theta'} R_T(\theta')$ π-a.e., , then T is (unique) minimax. *Limit of Bayes estimators* If T has constant risk and $\liminf_j r_j \ge R_T$, then T is minimax.
[Simul est] Simultaneous estimate vector-valued $\mathcal{V}$ with e.g. squared loss $L(\theta,a) = \|a - \theta\|^2 = \sum_{i=1}^p (a_i - \theta_i)^2$
[Asymptotic] (Posterior Consistency) $X \sim P_{\theta_0}$ and $\Pi(U|X_n) \xrightarrow{P_{\theta_0}} 1$ for all open U containing θ_0. (Wald type consistency) Assume $p_\theta(x)$ is continuous, measurable, θ_* is unique maximizer then MLE converge to true parameter θ^* P_* a.s. Furthermore, if θ^* is in the support of the prior, then posterior converges to θ^* in probability. (Posterior Robustness) all priors that lead to consistent posteriors are equivalent.
[BM] Bernstein-von Mises: assume regularity conditions, posterior $T_n = \sqrt{n}(\tilde{\theta_n} - \hat{\theta_n}) \sim \mathcal{N}(\hat{\theta}_n, V^*/n)$ asymptotically. (Well-specified) $V^* = E_*\left[-\nabla^2_\theta \log p_{\theta^*}(Y)\right]^{-1}$ (same as MLE, with θ^* as true parameter, CI = CR) (Mis-specified) $V^* = \mathbb{E}_*\left[-\nabla^2_\theta \log p_{\theta_*}(Y)\right]^{-1} = \mathbb{E}_*\left[-\nabla^2_\theta \log p_{\theta^*}(Y)\right]^{-1} \mathrm{Var}_*\left(\nabla \log p_{\theta^*}(Y)\right) \mathbb{E}_*\left[-\nabla^2_\theta \log p_{\theta^*}(Y)\right]^{-1}$ (differ from MLE, with θ_* the projection of P_* to parameter space) (Result) $\sqrt{n}\left(\hat{\theta}_n - E_\theta[\theta|X_1, \cdots, X_n]\right) \xrightarrow{P} 0$ (If MLE has asym normality, so is posterior mean)

Linear Model

[Linear Model] $X = Z\beta + \epsilon$ (or $X_i = Z_i^T\beta + \epsilon_i$) Estimate with $b = \min_b \|X - Zb\|^2 = \|X - Z\hat{\beta}\|^2$, (solution = normal equation) $Z^Z b = Z^T X$ (Full rank): $\hat{\beta} = (Z^TZ)^{-1}Z^TX$ (Non-full rank): $\hat{\beta} = (Z^TZ)^- Z^TX$ (A1 Gaussian noise) $\epsilon \sim N_n(0, \sigma^2 I_n)$ (A2 homoscedastic noise) $E(\epsilon) = 0$, $Var(\epsilon) = \sigma^2 I_n$ (A3 general noise) $E(\epsilon) = 0$, $Var(\epsilon) = \Sigma$
[Inference] Estimate linear combination of coefficient (General) Necce and Suff condition: $\ell\ inR(Z) = R(Z^TZ)$ (A3) LSE $\ell^T\hat{\beta}$ is unique and unbiased (A1) if $\ell \notin R(Z)$, $\ell^T\beta$ not estimable

[Properties] Require $\ell \in R(Z) = R(Z^TZ)$ (A1) (i) LSE $\ell^T\hat{\beta}$ is UMVUE of $\ell^T\beta$, (ii) UMVUE of $\hat{\sigma}^2 = (n-r)^{-1}\|X - Z\hat{\beta}\|^2$, r is rank of Z (iii) $\ell\hat{\beta}$ and $\hat{\sigma}^2$ are independent, $\ell^T\hat{\beta} \sim N(\ell^T\beta, \sigma^2\ell^T(Z^TZ)^-\ell)$, $(n-r)\hat{\sigma}/\sigma^2 \sim \chi^2_{n-r}$ (A2) LSE $\ell^T\hat{\beta}$ is BLUE (Best Linear Unbiased Estimator, best as in min var) [A3] Following are equivalent: (a) $\ell^T\hat{\beta}$ is BLUE for $\ell^T\beta$ (also UMVUE), (b) $E[\ell^T\hat{\eta}^TX] = 0]$, any η is s.t. $E[\eta^TX] = 0$ (c) $Z^Tvar(\epsilon)U = 0$, for U s.t. $Z^TU = 0$, $R(U^T) + R(Z^T) = R^n$ (d) $Var(\epsilon) = Z\Lambda_1Z^T + U\Lambda_2U^T$, for some Λ_1, Λ_2, U s.t. $Z^TU = 0$, $R(U^T) + R(Z^T) = R^n$ (e) $Z(Z^TZ)^-Z^TVar(\epsilon)$ is symmetric

[Asymptotic] $\lambda_+[A]$ is the largest eigenvalue of $A_n = (Z^TZ)^-$. (Consistency) Suppose $\sup_n \lambda_+[Var\epsilon)] < \infty$ and $\lim_{n\to\infty}\lambda_+[A_n] = 0$, $\ell^T\hat{\beta}$ is consistent in MSE. (Asym Normality) $\ell^T(\hat{\beta} - \beta)/\sqrt{Var(\ell^T\hat{\beta})} \to_d N(0,1)$ suff cond: $\lambda_+[A_n] \to 0$, $Z_n^TA_nZ_n \to 0$ as $n \to \infty$, and there exist $\{a_n\}$ s.t. $a_n \to \infty$, $a_n/a_{n+1} \to 1$, Z^TZ/a_n converge to positive definite matrix.

[Testing] Under A1, $\ell \in R(Z)$, θ_0 fixed constant, (Hypothesis testing) (simple) $\ell \in R(Z)$, $H_0 : \ell^T\beta \le \theta_0$, $H_1 : \ell^T\beta > \theta_0$, or $H_0 : \ell^T\beta = \theta_0$, $H_1 : \ell^T\beta \ne \theta_0$, $t(X) = \frac{\ell^T\hat{\beta} - \theta_0}{\sqrt{\ell^T(Z^TZ)^-\ell}\sqrt{SSR/(n-r)}} \sim t_{n-r}$ under H_0, UMPU reject $t(X) > t_{n-r,\alpha}$ or $|t(X)| > t_{n-r,\alpha/2}$ (multiple) $L_{s\times p}$, $s \le r$ and all rows $= \ell_j \in R(Z)$ $H_0 : L\beta = 0$, $H_1 : L\beta \ne 0$ $W = \frac{(\|X - Z\hat{\beta}_0\|^2 - \|X - Z\hat{\beta}\|^2)/s}{\|X - Z\hat{\beta}\|^2/(n-r)} \sim F_{s,n-r}$ with non-central param $\sigma^{-2}\|Z\beta - \Pi_0Z\beta\|^2$, reject $W > F_{s,n-r,1-\alpha}$ (Confidence set) Pivotal qty: $\mathcal{R}(X,\beta) = \frac{(\hat{\beta}-\beta)^TZ^TZ(\hat{\beta}-\beta)/p}{\|X - Z\hat{\beta}\|^2/(n-p)} \sim F_{p,n-p}$, $\hat{\beta}$ is LSE of β, $C(X) = \{\beta : \mathcal{R}(X,\beta) \le F_{p,n-p,1-\alpha}\}$

Sufficiency [Factorization] $T(X)$ is sufficient for $\theta \Leftrightarrow \exists\, h(x), g_P(t)$ s.t. $f(x|\theta) = g_P(T(x))h(x)$

[Min. Sufficient] T is min sufficient $\Leftrightarrow$ for any other stat S, $T = \psi(S)$. Min suff is unique and usually exist.

[Method 1] (A) If $P_0 \subset P$ and P_0 a.s. implies P a.s., if T is suff for P and min suff for P_0, then T is min suff for P. (B1) $T(X) = \{f_i(x)/f_\infty(x)\}$ is min suff for P, where $f_\infty(x) = \sum_{i=0}^\infty c_if_i(x)$, $c_i > 0, \sum_{i=0}^\infty c_i = 1$ (B2) $T(X) = \{f_i(x)/f_0(x)\}$ is min suff for P, if $\{x : f_i(x) > 0\} \subset \{x : f_0(x) > 0\}$

[Method 2] (C) $T(X)$ is suff, $\exists\ \phi$ s.t. $f_P(x) = f_P(y)\phi(x,y) \implies T(x) = T(y)$ Then $T(X)$ is min suff.

[Exp Fam] T is suff, $\exists\Theta_0 \subset \Theta$ s.t. $\eta_i = \eta(\theta_i) - \eta(\theta_0), i = 1, \cdots, p$ are linear indep, then T is min suff. e.g. Θ Full rank.

Completeness T is complete for θ if $E_\theta[g(T)] = 0 \implies f(T) = 0$ a.s. Suff + bounded complete $\implies$ min suff.

[Exp Fam] if η is full-rank in NEF, then T is complete and suff.

[Varying Support] $\int_0^\theta g(x)x^{n-1}dx = 0 \implies g(\theta)\theta^{n-1} = 0, \implies g(\theta) = g(X_{(n)}) = 0$ and thus $X_{(n)})$ is complete.

[Basu] If V is ancillary and T is boundedly complete and sufficnet, then V and T are indep.

Estimation [MoM] $\mu_j = E_\theta X^j = h_j(\theta), \implies \hat{\theta} = h_j^{-1}(\hat{\mu}_j)$. Provided h_j^{-1} exists and $\hat{\mu}_j = \frac{1}{n}\sum_{i=1}^n X_i^j$.

[MLE] $\hat{\theta} = \arg\max_\theta L(\theta)$. Consider (a) boundary opint (b) $\partial L(\theta)/\partial\theta = 0$ and $\partial^2L(\theta)/\partial\theta^2 < 0$, MLE may not exist (Asym Normality of RLEs) If $I(\theta)$ positive definite at θ, then $\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{D} N(0, I(\theta)^{-1})$

Decision Rule (Loss) $L : P \times \mathcal{A} \to [0,\infty]$, (Risk) $R_T(P) = E_P[L(P, T(X)]$. T_2 dominated by T_1 if $R_{T_1}(P) < R_{T_2}(P)$ (Bayes Risk) $r_T(\Pi) = \int_P R_T(P)d\Pi(P)$, find via $\min E[L(\theta, T)|X]$

[Decision] (Optimal) T is optimal if $\forall T'$, $R_T(P) \le R_{T'}(P)$, or as good as any other rule. (Admissibility) T is admissible if no T' s.t. $R_T(P) > R_{T'}(P)\forall P$, or not dominated by any other rule. (MiniMax) T is mini-max if $\sup_{\theta\in\Theta} R_T(P) \le \sup_{\theta\in\Theta} R_{T'}(P)$ for any T'. (Bayes Rule) $r_T(\Pi) \le r'_T(\Pi)$ for any T'.

[Rao-Blackwell] Consider rule S_0 and $S_1 = E[S_0(X)|T]$. If $L(P,a)$ convex in a then $R_{S_1}(P) \le R_{S_0}(P)$. If L is strictly convex, and S_0 is not function of T, then S_0 is inadmissible and dominated by S_1

[UMVUE] T is unbiased and $Var(T) \le Var(S)$ for any unbiased S.

[Method 1] (Lehmann-Scheffe) if T is suff and complete, UNVUE is in form $h(T)$ and is unique. $\implies$ UMVUE $\theta = E[h(T)]$

[Method 2] Find unbiased estimator U, find suff and complete T, UNVUE is $E[U|T]$.

[Method 3] When no complete and suff stat, let T be unbiased estimator and $\mathcal{U}$ be unbiased estimator of 0. If S is UMVUE $\Leftrightarrow E[SU] = 0$ for any U and P, if $S = h(T)$ then $E[SU(T)] = 0$, T is suff stat. Useful to find UMVUE, check if S is UMVUE, show non-existence of UMVUE. e.g. find UMVUE of $X \sim Unif(0,\theta), \Theta = [1,\infty)$

[Fisher Info] $I(\theta) = E\left(\frac{\partial}{\partial\theta}\log f_\theta(X)\right)^2$, provided $\frac{\partial f_\theta(x)}{\partial\theta}$ exists. Note if $\theta = \psi(\eta)$, $\tilde{I}(\eta) = \psi'(\eta)^2I(\psi(\eta))$. Suppose f_θ is twice differentiable, and $\int\frac{\partial^2}{\partial\theta^2}f_\theta(x)d\nu = 0$, then $I(\theta) = -E[\frac{\partial^2}{\partial\theta^2}\log f_\theta(X)]$ Suppose $\int\frac{\partial}{\partial\theta}f_\theta(x)d\nu = 0$, $I_{X+Y}(\theta) = I_X(\theta) + I_Y(\theta)$

[Cramer-Rao LB] LB $Var(T) \ge \frac{g'(\theta)^2}{I(\theta)}$, where T is unbiased estimator of $g(\theta)$, s.t. $g'(\theta) = \frac{\partial}{\partial\theta}\int Tf_\theta(x)d\nu = \int T(x)\frac{\partial}{\partial\theta}f_\theta(x)d\nu$. Require f_θ differentiable and $0 = \frac{\partial}{\partial\theta}\int f_\theta(x)d\nu = \int\frac{\partial}{\partial\theta}f_\theta(x)d\nu$

[Convergence] (a.s.) $P(\lim_{n\to\infty}X_n = X) = 1$. ($L^p$) $\lim_{n\to\infty}E|X_n - X|^p = 0$. (Prob) $\forall\epsilon > 0, \lim_{n\to\infty}P(|X_n - X| > \epsilon) = 0$. (Dist) $\lim_{n\to\infty}F_n(x) = F(x)$.

[Showing a.s.] (1st Borel-Cantelli) If $\sum_{n=1}^\infty P(A_n) < \infty$, then $P(\limsup A_n) = 0$. (or A_n occurs finitely often) (2nd BC) If A_n are pairwise indep and $\sum_{n=1}^\infty P(A_n) = \infty$, then $P(\limsup A_n) = 1$. (Thm) If $\sum_{n=1}^\infty P(A_n(\epsilon)) < \infty\ \forall\epsilon > 0$, then $X_n \xrightarrow{a.s.} X$

[Dist with chf] (Levy continuity) $X_n \xrightarrow{D} X \Leftrightarrow \phi_{X_n}(t) \to \psi_X(t)$ where $\phi_X(t)$ is chf.

[Continuous mapping] If $X_n \xrightarrow{a.s.} X$, then $g(X_n) \xrightarrow{a.s.} g(X)$. If $X_n \xrightarrow{P} X$, then $g(X_n) \xrightarrow{P} g(X)$. If $X_n \xrightarrow{D} X$, then $g(X_n) \xrightarrow{D} g(X)$.

[Slutsky's thm]

If $X_n \xrightarrow{D} X$ and $Y_n \xrightarrow{P} c$, then $X_n + Y_n \xrightarrow{D} X + c$, $X_nY_n \xrightarrow{D} cX$, $X_n/Y_n \xrightarrow{D} X/c$ if $c \ne 0$.

[δ-method] Suppose $a_n(X_n - c) \xrightarrow{D} Y$, then $a_n[g(X_n) - g(c) \xrightarrow{D} g'(c)Y$. $a_n^m[g(X_n) - g(c)] \xrightarrow{D} \frac{1}{m!}g^{(m)}(c)Y^m$

[SLLN] (iid) If $E|X| < \infty$, then $\frac{1}{n}\sum_{i=1}^n X_i \xrightarrow{a.s.} \mu$. (non ident) if $\exists p \in [1,2]$ s.t. $\sum_{i=1}^\infty\frac{E|X_i|^p}{i^p} < \infty$, then $\frac{1}{n}\sum_{i=1}^n(X_i - EX_i) \xrightarrow{a.s.} 0$

[WLLN] (iid) If $nP(|X_1| > n) \to 0$, then $\frac{1}{n}\sum_{i=1}^n X_i - E[X_1I_{|X|\le n}] \xrightarrow{P} 0$ (non ident) If $\exists p \in [1,2]$ s.t. $\lim_{n\to\infty}\frac{1}{n^p}\sum_{i=1}^n E|X_i|^p = 0$, then $\frac{1}{n}\sum_{i=1}^n(X_i - EX_i) \xrightarrow{P} 0$

[CLT] (iid) If $\Sigma = Var(X_1) < \infty$, then $\frac{\sum_{i=1}^n(X_i - EX_i)}{\sqrt{n}} \xrightarrow{D} N(0,\Sigma)$ (non ident - Lindeberg's CLT) For each n, let $\{X_{nj}\}$ with $j = 1, \cdots, k_n$ Suppose $k_n \to \infty$ as $n \to \infty$ and $0 < \sigma_n^2 = Var\left(\sum_{j=1}^{k_n}X_{nj}\right) < \infty$. If for any $\epsilon > 0$, $\frac{1}{\sigma_n^2}\sum_{j=1}^{k_n}E\left\{(X_{nj} - EX_{nj})^2I_{\{|X_{nj} - EX_{nj}| > \epsilon\sigma_n\}}\right\} \to 0$, then $\frac{1}{\sigma_n}\sum_{j=1}^{k_n}(X_{nj} - EX_{nj}) \xrightarrow{D} N(0,1)$

[Consistency] (Consistent) $T_n(X) \xrightarrow{P} \theta$ (Strongly Consistent) $T_n(X) \xrightarrow{a.s.} \theta$ (a_n-consistent) $a_n(T_n(X) - \theta) = O_p(1)$ (L_r-consistent) $T_n(X) \xrightarrow{L^r} \theta$ (Proving consistency) LLN + CLT + Slustsky's thm + continuous mapping thm + δ-method

[Asymptotic] (Approx unbiased) $b_{T_n}(P) := ET_n(X) - \theta \to 0$ as $n \to \infty$ (Asym Expectation) $a_n \xi_n \xrightarrow{D} \xi$, $E|\xi| < \infty$, then $E\xi/a_n$ is asym. expect of ξ_n [Asym Bias] Asym. expect $\tilde{b}_{T_n} = T_n - \theta$ (Asym unbiase) if $\lim_{n\to\infty} \tilde{b}_{T_n}(P) = 0$ (Asym MSE) Suppose $a_n(T_n - \theta) \xrightarrow{D} Y$, amse is EY^2/a_n^2 (Asym var) $\sigma^2_{T_n}(P) = Var(Y)/a_n^2$ (Asym relative efficiency) $e_{T_n' T_n}(P) = amse_{T_n}/amse_{T_n'}$ (note the order)
[Hypo Test] (UMP) Satisfy (1) pre-set size $\alpha = E_{H_0}(T)$ (2) max power $\beta_T(P) = E_{H_1}(T)$ (Neyman-Pearson) $T(X) = I(f_1(X) > cf_0(X)) + \gamma I(f_1(X) = cf_0(X))$ (unique up to randomised test)
[MLR] (Monotone Likelihood ratio in $Y(X)$) for any $\theta_1 < \theta_2$, $f_{\theta_2}(x)/f_{\theta_1}(x)$ nondecreasing in $Y(x)$. [MLR for one-param exp fam] $\eta(\theta)$ nondecreasing in θ. [Simply NP test] $T(X) = I(Y(X) > c) + \gamma I(Y(X) = c)$ (increasing MLR, $H_0 : \theta \le \theta_0$, $H_1 : \theta > \theta_0$)
[Generalised NP] (Want to) $\max_\phi \int \phi f_{m+1} d\nu$ s.t. $\int \phi f_1 d\nu \le t_1$, $\int \phi f_2 d\nu \le t_2$, $\cdots$ $\int \phi f_m d\nu \le t_m$, (condition) If $\exists\ c_1, \cdots, c_m$ s.t. $\phi_*(x) = I(f_{m+1}(x) > c_1 f_1(x) + \cdots + c_m f_m(x))$, then ϕ_* maximises objective function with equality constraint. If $c_i \ge 0$ then ϕ_* maximises with inequality constraint.
[UMPU] (UMPU) UMP amongst unbiased test of size α. (NEF) U is sufficient and complete when θ is known. [$H_0 : \theta \le \theta_0$, $H_1 : \theta > \theta_0$] $T(Y,U) = I(Y > c(U))$ and $E_{\theta_0}(T|U = u) = \alpha$ [$H_0 : \theta \notin (\theta_1, \theta_2)$, $H_1 : \theta \in (\theta_1, \theta_2)$] $T(Y,U) = I(c_1(U) < Y c_2(U))$ and $E_{\theta_1}[T|U = u] = E_{\theta_2}[T|U = u] = \alpha$ [$H_0 : \theta_1 \le \theta \le \theta_2$, $H_1 : \theta < \theta_1$ or $\theta > \theta_2$] $T(Y,U) = I(Y < c_1(U)$ or $Y > c_2(U))$ and $E_{\theta_1}(T|U = u) = E_{\theta_2}(T|U = u) = \alpha$ [$H_0 : \theta = \theta_0$, $H_1 : \theta \ne \theta_0$] $T(Y,U) = I(Y < c_1(U)$ or $Y > c_2(U))$ and $E_{\theta_0}(T|U = u) = \alpha E_{\theta_0}(Y|U = u)$
[UMPU, Normal] $V(Y,U)$ independent of U when $\theta = \theta_j$ (in H_0), increasing in y for each u. Usually can be shown via Basu. [$H_0 : \theta = \theta_0$, $H_1 : \theta \ne \theta_0$] If $V(y,u) = a(u)y + b(u)$, $a(u) > 0$, then $T(V) = I(V < c_1$ or $V > c_2)$ and $E_{\theta_0}(T) = \alpha$, and $E_{\theta_0}(TV) = \alpha E_{\theta_0}(V)$ [$H_0 : \theta \le \theta_1$ or $\theta \ge \theta_0$, $H_1 : \theta_1 < \theta < \theta_2$] $T(V) = I(c_1 < V < c_2)$, and $E_{\theta_1}(T) = E_{\theta_2}(T) = \alpha$ [$H_0 : \theta_1 \le \theta \le \theta_2$, $H_1 : \theta < \theta_1$ or $\theta > \theta_2$] $T(V) = I(V < c_1$ or $V > c_2)$ and $E_{\theta_1}(T) = E_{\theta_2}(T) = \alpha$ [$H_0 : \theta \le \theta_0$, $H_1 : \theta > \theta_0$] $T(V) = I(V > c)$ and $E_{\theta_0}(T) = \alpha$
[LRT] (Likelihood Ratio Test) Reject H_0 if $\lambda(X) < c$, $c \in [0,1]$ $\lambda(X) = \frac{\sup_{\theta\in\Theta_0} \ell(\theta)}{\sup_{\theta\in\Theta} \ell(\theta)}$. [Asym Dist] Under H_0 $-2\log\lambda(X) \to \chi^2_r$ (r is dim of Θ) (Rejection region) $\lambda(X) < exp(-\chi^2_{r,1-\alpha}/2)$, same for other asym test
[Wald's test] $H_0 : R(\theta) = 0$, reject large $W_n = R(\hat\theta)^T [C(\hat\theta)^T I_n^{-1}(\hat\theta) C(\hat\theta)]^{-1} R(\hat\theta)$. $C(\theta) = \partial R(\theta)/\partial\theta$, $I_n(\theta)$ fisher info, $\hat\theta$ is MLE.
[Rao's score test] $H_0 : R(\theta) = 0$, reject large $Q_n = s_n(\tilde\theta)^T I_n^{-1}(\tilde\theta) s_n(\tilde\theta)$. $s_n(\theta) = \partial \log \ell(\theta)/\partial\theta$, $\tilde\theta$ is MLE in $H_0 : R(\theta) = 0$
[Confidence Set] $C(X)$ confidence set for θ (Pivotal quantity) $\mathcal{R}(X,\theta)$ does not depend on P (Invert accept region) accept region of $H_0 : \theta = \theta_0$ (Bonferroni's method) Simultaneous test simply α/r
[Shortest CI] (Pivot $(T - \theta)/U$) $f(x)$ unimodal at x_0, $f(a_*) = f(b_*) > 0$, $C = \{[T - bU, T - aU] : \int_a^b f(x)dx = 1 - \alpha\}$ (Pivotal T/θ) $x^2 f(x)$ unimodal at x_0, $a_*^2 f(a_*) = b_*^2 f(b_*) > 0$, $C = \{[T/b, T/a] : \int_a^b f(x)dx = 1 - \alpha\}$ (General) f unimodal at x_0, $\min b - a$ s.t. $\int_a^b f(x)dx = A$ at $a_* \le x_0 \le b_*$ and $f(a_*) = f(b_*) > 0$ and $\int_{a_*}^{b_*} f(x)dx = A$
[Bayes action] (Bayes Action) $\min_a E[L(\theta,a)|X = x]$ (Generalised) $\min_a \int_\Theta L(\theta,a) f_\theta(x) d\Pi$ (improper prior)
[Admissibility] $\delta(X)$ is a Bayes rule with prior Π, δ is admissible if (1) if δ is unique (2) If Θ is countable, $\Pi(\theta) > 0\ \forall\Theta$. Note, not true for generalised Bayes rules unless limit is Bayes rule.
[Minimaxity] If T is Bayes estimator and $R_T(\theta) = \sup_{\theta'} R_T(\theta')$, then T is minimax. If T is unique, it is unique minimax.
[Bernstein-von Mises] (asymp normality) $\tilde\theta_n$ posterior $\hat\theta_n$ MLE, $T_n = \sqrt{n}(\tilde\theta_n - \hat\theta_n) \xrightarrow{D} N(\hat\theta_n, V^*/n)$, θ^* is true value. (assume MLE is asym normal) (well-specified) $V^* = E_*[-\nabla^2_\theta \log p_{\theta^*}(Y)]^{-1}$ (mis-specified) $V^* = E_*[-\nabla^2_\theta \log p_{\theta^*}(Y)]^{-1} Var_*(\nabla \log p_{\theta^*}(Y)) E_*[-\nabla^2_\theta \log_{p_{\theta^*}}(Y)]^{-1}$
Linear models (Normal equation) $Z^T Z b = Z^T X$ (LSE) $\hat\beta = (Z^T Z)^- Z^T X$ (Generalised inverse) Moore-Penrose inverse $A^+ A A^+ = A^+$, $A = (Z^T Z)$ (Projection matrix) $P_Z = Z(Z^T Z)^- Z^T$, $P_Z^2 = P_Z$, $P_Z Z = Z$, $rank(P_Z) = tr(P_Z) = r$
[Assumptions] (A1 Gaussian noise) $\epsilon \sim N_n(0, \sigma^2 I_n)$ (A2 homoscedastic noise) $E(\epsilon) = 0$, $Var(\epsilon) = \sigma^2 I_n$ (A3 general noise) $E(\epsilon) = 0$, $Var(\epsilon) = \Sigma$
[Estimable] Estimate $\nu = \ell^T \beta$ for some $\ell \in \mathcal{R}^p$. (Necc Suff) $\ell \in \mathcal{R}(\mathcal{Z}) = \mathcal{R}(Z^T Z)$ (linear subspace) (A3 + above) LSE $\ell^T \hat\beta$ unique and unbiased (A1 + not cond) $\ell^T \beta$ not estimable
[asym] (Asym Norm) $\ell^T(\hat\beta - \beta)/\sqrt{Var(\ell^T\hat\beta} \xrightarrow{d} N(0,1)$
[Hypo test - one] ($H_0 : \ell^T\beta \le \theta_0$, $H_1 : \ell^T\beta > \theta_0$) ($H_0 : \ell^T\beta = \theta_0$, $H_1 : \ell^T\beta \ne \theta_0$) $t(X) = \frac{\ell^T\hat\beta - \theta_0}{\sqrt{\ell^T(Z^T Z)^- \ell}\sqrt{SSR/(n-r)}} \sim t_{n-r}$ under H_0
[Hypo test - multi] ($H_0 : L\beta = 0$, $H_1 : L\beta \ne 0$) $W = \frac{(||X - Z\hat\beta_0||^2 - ||X - Z\hat\beta||^2)/s}{||X - Z\hat\beta||^2/(n-r)} \sim F_{s,n-r}$ under H_0