EC2104 Lecture 4 - Leibniz Integration (extra)

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We illustrate how to use Mean Value Theorem, some tricks in integration and the definition of differentiation to prove Leibniz Integration. This is optional content and is for your interest only.

Mean Value Theorem

If a function f is continuous over the closed interval $[a, b]$ and differentiable on the open interval (a, b).

Then there exists a point c in the interval (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

- Meaning, if function is continuous and differentiable over an closed interval $[a, b]$
- Then there is a point (let's call it c) where if you take the differentiation at that point
- The differentiation will equals to the secant line joining a and b

Proving Leibniz Integration

WTS:

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)dx = \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(x,t)dx + f(b(t),t)\frac{db(t)}{dt} - f(a(t),t)\frac{da(t)}{dt}$$

1. Let $I(x,t) = \int_{a(t)}^{b(t)} f(x,t)dx$
2. Applying definition of differentiation

$$\frac{\partial}{\partial t} I(x,t) = \lim_{\Delta t \to 0} \frac{I(x, t+\Delta t) - I(x,t)}{\Delta t}$$

3. Substitute the definition of $I(x,t)$

$$\lim_{\Delta t \to 0} \frac{I(x, t+\Delta t) - I(x,t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(\int_{a+\Delta a}^{b+\Delta b} f(x, t+\Delta t)dx - \int_a^b f(x,t)dx\right)$$

4. Applying property of integration: $\int_{-b}^{b} f(x)dx = \int_{-b}^{0} f(x)dx + \int_0^b f(x)dx$

$$\begin{aligned}
&\lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(\int_{a+\Delta a}^{b+\Delta b} f(x, t+\Delta t)dx - \int_a^b f(x,t)dx\right) \\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(\int_{a+\Delta a}^{a} f(x, t+\Delta t)dx + \int_a^b f(x, t+\Delta t)dx + \int_b^{b+\Delta b} f(x, t+\Delta t)dx - \int_a^b f(x,t)dx\right)
\end{aligned}$$

5. Applying property of integration: $\int_a^b f(x)dx = -\int_b^a f(x)dx$

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(\int_{a+\Delta a}^{a} f(x,t+\Delta t)dx + \int_a^b f(x,t+\Delta t)dx + \int_b^{b+\Delta b} f(x,t+\Delta t)dx - \int_a^b f(x,t)dx\right) =$$
$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(\int_a^b f(x,t+\Delta t) - f(x,t)dx + \int_b^{b+\Delta b} f(x,t+\Delta t)dx - \int_a^{a+\Delta a} f(x,t+\Delta t)dx\right)$$

6. Using Mean Value Theorem: $f'(c) = \frac{f(b)-f(a)}{b-a} \Rightarrow f(b)-f(a) = f'(c)(b-a)$

$$\begin{aligned}\int_b^{b+\Delta b} f(x,t+\Delta t)dx &= F(b+\Delta b, t+\Delta t) - F(b, t+\Delta t)\\ &= f(c,t+\Delta t)(b+\Delta b - b)\\ &= f(c,t+\Delta t)\Delta b\end{aligned}$$

   - Note: $F'(x) = f(x)$

7. Further using implications from Mean Value Theorem, as $\Delta t \to 0 \Rightarrow b+\Delta b \to b \Rightarrow c = b$ as $\Delta t \to 0$
   - Layman: as the changes in d gets smaller, the changes in b will get smaller (since $b(t)$ is a function of t).
   - By Mean Value Theorem, there must exist a c between the bounds such that $f(b) - f(a) = f'(c)(b-a)$
   - Since the bound $[b, b+\Delta b]$ gets increasingly smaller with t getting smaller, eventually the bound will become $[b, b]$
   - Implying that $c = b$ (the only possible value in [b, b])
   - Formally:
$$\lim_{\Delta t\to 0} f(c, t+\Delta t) = f(b,t)$$

8. Distribute the terms

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(\int_a^b f(x,t+\Delta t) - f(x,t)dx + \int_b^{b+\Delta b} f(x,t+\Delta t)dx - \int_a^{a+\Delta a} f(x,t+\Delta t)dx\right)$$
$$= \left[\int_a^b \lim_{\Delta t\to 0}\frac{f(x,t+\Delta t)-f(x,t)}{\Delta t}dx\right] + \left[\lim_{\Delta t\to 0}\frac{f(c_b,t+\Delta t)\Delta b(t)}{\Delta t}\right] - \left[\lim_{\Delta t\to 0}\frac{f(c_a,t+\Delta t)\Delta a(t)}{\Delta t}\right]$$

9. Applying results from 6, 7 and definition of differentiation $f'(t) = \lim_{\Delta t\to 0}\frac{f(t+\Delta t)-f(t)}{\Delta t} = \lim_{\Delta t\to 0}\frac{\Delta f(t)}{\Delta t}$

$$\begin{aligned}\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)dx &= \left[\int_a^b \lim_{\Delta t\to 0}\frac{f(x,t+\Delta t)-f(x,t)}{\Delta t}dx\right] + \left[\lim_{\Delta t\to 0}\frac{f(c_b,t+\Delta t)\Delta b(t)}{\Delta t}\right] - \left[\lim_{\Delta t\to 0}\frac{f(c_a,t+\Delta t)\Delta a(t)}{\Delta t}\right]\\ &= \int_a^b \frac{\partial}{\partial t}f(x,t)dx + f(b(t),t)\frac{db(t)}{dt} - f(a(t),t)\frac{da(t)}{dt} \text{ (shown)}\end{aligned}$$