Lecture 9 - Matrix algebra

ling

Section 1

Determinants

Determinants

Result: an invertible matrix $\mathbf{A}$ has $det(\mathbf{A}) \neq 0$

Laplace Expansion

$$det(\mathbf{A}) = \sum_{j=1}^{n} (-1)^{i+j} a_{i,j} M_{i,j}$$

Procedure:

1. Find sub-matrices $\mathbf{A}_{ij}$
   - $\mathbf{A}_{ij}$ sub-matrix is obtained with deleting row i and column j
2. Find Minor $\mathbf{M_{ij}}$
   - $M_{ij} = det(\mathbf{A}_{ij})$
3. Find Cofactors C_{ij}
   - $C_{ij} = (-1)^{i+j} M_{ij}$
4. Find $C_{ij} a_{ij}$ in any column and row
   - Tip: find the column/row with most $a_{ij} = 0$

Section 2

Inverse

Inverse

Inverse

Square matrix $\mathbf{A}$ has an inverse $\mathbf{A}^{-1}$ if and only if $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$

Note:

- The concept of fraction $\frac{1}{a}$ does not exist in linear algebra.
- Inverse only exist for square matrices
- Since the definition is ‘if and only if’, meaning it is a two way relationship.
  1. If we know $\mathbf{A}$ is invertible, we know that $\mathbf{A}^{-1}$ exist.
  2. If we can find a matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$ then matrix $\mathbf{A}$ is invertible with inverse $\mathbf{A}^{-1} = \mathbf{B}$

Subsection 1

Finding inverse

Finding inverse

1. Using Gauss-Jordon elimination
    1. Form augmented matrix $[\mathbf{A}|\mathbf{I}]$
    2. Find RREF $[\mathbf{I}|\mathbf{A}^{-1}]$
2. Using adjoint matrices
    1. Find adjoint matrix $adj(\mathbf{A}) = \mathbf{C}^T$
    2. Find $\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} adj(\mathbf{A})$

Section 3

Solving system of linear equations

Solving system of linear equations

To solve: $\mathbf{Ax} = \mathbf{b}$

1. Using inverse
   1. G-J: $[\mathbf{A}|\mathbf{b}] \Rightarrow [\mathbf{I}|\mathbf{A}^{-1}\mathbf{b}]$
   2. Pre-multiply inverse: $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$
2. Cramer's Rule
   - $x_i = \frac{|\mathbf{A}_i|}{|\mathbf{A}|}$
   - where $\mathbf{A_i}$ is the matrix $\mathbf{A}$ with ith column vector replaced by $\mathbf{b}$

Section 4

Eigenvalues and Eigenvector

Eigenvalues and Eigenvector

Eigenvalues and Eigenvector

λ is called eigenvalues of $\mathbf{A}$ when the following relationship holds for $\mathbf{v} \neq \mathbf{0}$

$$\mathbf{Av} = \lambda \mathbf{v}$$

- Multiplying $\mathbf{A}$ to $\mathbf{v}$ only scale $\mathbf{v}$ by a factor of λ
- To find $\lambda, \mathbf{v}$
  1. To find λ: solve $det(\mathbf{A} - \lambda \mathbf{I}) = 0$
  2. To find $\mathbf{v}$: solve $(\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = \mathbf{0}$
     - Use RREF to find $[\mathbf{A} - \lambda \mathbf{I} | \mathbf{0}]$
     - Eigenvector is not unique (infinite solutions)

Eigenvalues properties (among a long list)

1. Product of $\lambda = det(\mathbf{A})$
2. If the hessian matrix has all non-negative λ, the multi-variable function is convex