EC3102 Macroeconomics II

Lingjie, November 22, 2020

Infinite-Period Framework

inflation (real) $\pi_{t+1} = \frac{P_{t+1}-P_t}{P_t}$

interest rate (nominal) $1+i_t = \frac{S_{t+1}+D_{t+1}}{S_t}$

Subjective Discount Factor

β, lower means more impatient

Utility function

$$\nu(c_t) = \sum_{t\in[1,\infty]} \beta^{t-1}u(c_t)$$

Budget Constraint (stocks)

consumption + changes in bonds = income + dividend

$$\{P_t c_t + S_t(a_t - a_{t-1}) = Y_t + D_t a_{t-1}\}_{t=1}^{\infty}$$

P_t : nominal price of goods
c_t : period t real consumption
S_t : nominal price of stock
a_t : quantity of stock
Y_t : period t nominal income
D_t : nominal dividend

Equilibrium condition

solve

$$\max_{c_t,a_t,\lambda} \sum_{t\in[1,\infty]} [\beta^{t-1}u(c_t) - \beta^{t-1}\lambda_t(P_t c_t + S_t(a_t - a_{t-1}) - Y_t - D_t a_{t-1})]$$

$$\begin{aligned} \frac{\partial}{\partial c_t} &= u'(c_t) - \lambda_t P_t &= 0\\ \frac{\partial}{\partial a_t} &= -S_t\lambda_t + \beta\lambda_{t+1}[S_{t+1}+D_{t+1}] &= 0\\ \frac{\partial}{\partial c_{t+1}} &= \beta u'(c_{t+1}) - \beta\lambda_{t+1}P_{t+1} &= 0 \end{aligned}$$

Asset Pricing (Basic)

$$\begin{aligned} S_t &= \left(\frac{\beta\lambda_{t+1}}{\lambda_t}\right)(S_{t+1}+D_{t+1})\\ &= \left(\frac{\beta u'(c_{t+1})}{u'(c_t)}\right)\left(\frac{P_t}{P_{t+1}}\right)(S_{t+1}+D_{t+1})\\ &= \left(\frac{\beta u'(c_{t+1})}{u'(c_t)}\right)\left(\frac{1}{1+\pi_{t+1}}\right)(S_{t+1}+D_{t+1}) \end{aligned}$$

Macro events affect Asset Pricing

c_t, c_{t+1}, π_{t+1} and any factors that will affect inflation and GDP

Consumer Optimization

$$\begin{aligned} \frac{u'(c_t)}{\beta u'(c_{t+1})} &= \left(\frac{1}{1+\pi_{t+1}}\right)\frac{(S_{t+1}+D_{t+1})}{S_t}\\ &= \frac{1+i_t}{1+\pi_{t+1}} = 1+r_t \end{aligned}$$

Long Run theory of Macro

Steady state: condition in which all real variables settle down to constant value, nominal values might change
$\Rightarrow c_t = c_{t+1} = c, r_t = r_{t+1} = r$

Real Interest Rate

$$\begin{aligned} \frac{u'(c_t)}{\beta u'(c_{t+1})} &= 1+r_t\\ \Rightarrow \frac{1}{\beta} &= 1+r, \text{ in long run} \end{aligned}$$

modern view is that $\beta < 1 \Rightarrow r > 0$

Government and Fiscal Policy in Consumption-Saving Model

Government adopt discretionary fiscal policy:
1. Purchase of new goods and services
2. Tax rate changes

Government saving: changes in wealth during the period

Budget Constraint (Tax)

Consumer

$$c_1 + \frac{c_2}{1+r} = y_1 - t_1 + \frac{y_2 - t_2}{1+r} + (1+r)a_0$$

Government

$$g_1 + \frac{g_2}{1+r} = t_1 + \frac{t_2}{1+r} + (1+r)b_0$$

Economy-wide Resource Frontier

assume $a_0 = 0, b_0 = 0$

$$c_1 + \frac{c_2}{1+r} = y_1 - g_1 + \frac{y_2 - g_2}{1+r}$$

observe how tax is not directly reflected in resource frontier

Consumer Optimization

solve $\max_{c_1,c_2} U(c_1,c_2) - \lambda\left[c_1 + \frac{c_2}{1+r} - (y_1 - g_1) - \frac{y_2-g_2}{1+r}\right]$

$$\begin{aligned} \frac{\partial}{\partial c_1} &= U_1(c_1,c_2) - \lambda &= 0\\ \frac{\partial}{\partial c_2} &= U_2(c_1,c_2) - \frac{\lambda}{1+r} &= 0\\ \text{MRS} &= \frac{U_1(c_1,c_2)}{U_2(c_1,c_2)} &= \frac{1+r}{1} \end{aligned}$$

National Savings

$$\begin{aligned} s^{private} &= y_1 - t_1 - c_1\\ s^{govern} &= t_1 - g_1\\ s^{nation} &= y_1 - g_1 - c_1 \end{aligned}$$

Effects of Tax policy

Objective: does tax affect savings
1. assume g_1, g_2 unchanged
$\Rightarrow g_1 + \frac{g_2}{1+r} = t_1 + \frac{t_2}{1+r} - \Delta t_1 + \frac{\Delta t_2}{1+r} \Rightarrow -\Delta t_1 + \frac{\Delta t_2}{1+r} = 0$
2. tax does not change utility function
3. tax does not change consumer BC, $\because -\Delta t_1 + \frac{\Delta t_2}{1+r} = 0$

Under the 3 conclusions, tax does not affect savings

Ricardian Equivalence Theorem

For a given PDV of government spending, neither consumption nor national saving is affected by the precise timing of lump-sum taxes

Lump-Sum Tax

A tax whose total incidence does not depend in any way on any choices an individual make, no real world example, *yet*
$c_1 + \frac{c_2}{1+r} = y_1 - t_1 + \frac{y_2 - t_2}{1+r}$

Proportional (distortionary) Tax

Tax whose total incidence depends on choices individual makes
$(1+\tau_1)c_1 + \frac{(1+\tau_2)c_2}{1+r} = y_1 + \frac{y_2}{1+r} \Rightarrow$ slope $-\frac{1+\tau_1}{1+\tau_2}(1+r)$
changes in tax changes consumer budget constraint $\Rightarrow$ savings change

Money and Bonds

The Roles of Money (liquidity)

1. Medium of exchange
2. Unit of account
3. Store of value

Bond Markets

Bond is a risk free assets
Common types of bonds: short/long term t-bills, coupon bonds (coupon payment every period), zero-coupon bonds (only pay back at maturity)
This mod: short term zero-coupon bonds, with face value 1
$P_t^b = \frac{\text{Face Value}_{t+1}}{1+i_t} = \frac{1}{1+i_t} \Rightarrow i_t = \frac{1}{P_t^b} - 1$

Money Markets and Bond Markets

Since $i_t = \frac{1}{P_t^b} - 1$, Fed open-market operations conducted via short-term bond markets and changing money supply

Money In Utility (MIU) Function

$$u\left(c_t, \frac{M_t^D}{P_t}\right)$$

In equilibrium, $\frac{M_t}{P_t} = \frac{M_t^D}{P_t} = \frac{M_t^S}{P_t}$

Budget Constraint (Money)

let face value of bond = 1
consumption + bond purchase + changes in money + changes in saving = income + bond payout + dividend

$$\begin{aligned} &P_t c_t + P_t^b B_t + (M_t - M_{t-1}) + (S_t a_t - S_t a_{t-1}) \\ &= Y_t + B_{t-1} + D_t a_{t-1} \end{aligned}$$

Equilibrium condition

solve

$$\begin{aligned} \max_{c_t, a_t, B_t, M_t \lambda} \sum_{t \in [1,\infty]} [\beta^{t-1} u\left(c_t, \frac{M_t}{P_t}\right) \\ - \beta^{t-1}\lambda_t (P_t c_t + P_t^b B_t + (M_t - M_{t-1}) \\ + S_t(a_t - a_{t-1}) - Y_t - B_{t-1} - D_t a_{t-1})] \end{aligned}$$

$$\begin{aligned} \tfrac{\partial}{\partial c_t} &= u_1\left(c_t, \tfrac{M_t}{P_t}\right) - \lambda_t P_t &= 0 \\ \tfrac{\partial}{\partial a_t} &= -S_t \lambda_t + \beta \lambda_{t+1}[S_{t+1} + D_{t+1}] &= 0 \\ \tfrac{\partial}{\partial B_t} &= -P_t^b \lambda_t + \beta \lambda_{t+1} &= 0 \\ \tfrac{\partial}{\partial M_t} &= \tfrac{\partial\left(c_t, \frac{M_t}{P_t}\right)}{\partial M_t} - \lambda_t + \beta \lambda_{t+1} &= 0 \end{aligned}$$

note: $\frac{\partial\left(c_t, \frac{M_t}{P_t}\right)}{\partial M_t} = \frac{\partial\left(c_t, \frac{M_t}{P_t}\right)}{\partial\left(\frac{M_t}{P_t}\right)} \frac{\partial\left(\frac{M_t}{P_t}\right)}{\partial M_t} = u_2\left(c_t, \frac{M_t}{P_t}\right)\frac{1}{P_t}$

Asset Pricing Revisited

$$\begin{aligned} S_t &= \left(\frac{\beta \lambda_{t+1}}{\lambda_t}\right)(S_{t+1} + D_{t+1}) \\ &= \left(P_t^b\right)(S_{t+1} + D_{t+1}) \\ &= \left(\frac{1}{1+i_t}\right)(S_{t+1} + D_{t+1}) \end{aligned}$$

Consumption-Money Optimality Condition

$$\frac{u_2\left(c_t, \frac{M_t}{P_t}\right)}{u_1\left(c_t, \frac{M_t}{P_t}\right)} = \frac{i_t}{1+i_t}$$

Money Demand

Suppose $\left(c_t, \frac{M_t}{P_t}\right) = \ln c_t + \ln\left(\frac{M_t}{P_t}\right)$
$\frac{u_2\left(c_t, \frac{M_t}{P_t}\right)}{u_1\left(c_t, \frac{M_t}{P_t}\right)} = \frac{P_t c_t}{M_t} = \frac{i_t}{1+i_t} \Rightarrow \frac{M_t}{P_t} = c_t\left(\frac{1+i_t}{i_t}\right)$

Short Run effects of Monetary Policy

Money is neutral if changes in the money supply have no effect on the real economy

Neutral: RBC

Price is not sticky in the short run $\Rightarrow P_t$ changes, c_t unchanged

Non-neutral: Keynesian

Sticky price in the short run $\Rightarrow P_t$ unchanged, c_t change

Long Run effects of Monetary Policy

Question: what determines inflation in the long run

Money and Inflation in LR

Since $\left(c_t, \frac{M_t}{P_t}\right) = \ln c_t + \ln\left(\frac{M_t}{P_t}\right)$, $\frac{M_t}{P_t} = c_t\left(\frac{1+i_t}{i_t}\right)$
$\frac{M_t/P_t}{M_{t-1}/P_{t-1}} = \frac{c_t}{c_{t-1}}\left(\frac{1+i_t}{i_t}\right)\left(\frac{i_{t-1}}{1+i_{t-1}}\right)$
Let inflation $\pi_t = \frac{P_t}{P_{t-1}} - 1$, money growth rate $\mu_t = \frac{M_t}{M_{t-1}} - 1$
$\Rightarrow \frac{1+\mu_t}{1+\pi_t} = \frac{c_t}{c_{t-1}}\left(\frac{1+i_t}{i_t}\right)\left(\frac{i_{t-1}}{1+i_{t-1}}\right)$, at steady stage: $\frac{1+\mu_t}{1+\pi_t} = 1$

$$\mu = \pi$$

Monetarism: In steady state, inflation determined solely by how quickly central bank expands the nominal money supply. In the long run, money supply only affects inflation.

Fiscal and Monetary Interactions

Main idea: budget constraint/balance sheet of one policy authority affect the other policy authority

Fiscal Authority

Congress/Treasury: government spending + bond repayment = tax + bond selling + profit from central bank

$$P_t g_t + B_{t-1}^T = T_t + P_t^b B_t^T + RCB_t$$

Monetary Authority

Central bank: buy bond + pass profit to congress = repayment from bond + changes in money supply

$$P_t^b B_t^M + RCB_t = B_{t-1}^M + M_t - M_{t-1}$$

Consolidated Government Budget

Government: government spending + private sector bond repayment = tax + private sector bond sales + changes in money supply

$$\begin{aligned} P_t g_t + B_{t-1}^T - B_{t-1}^M &= T_t + P_t^b(B_t^T - B_t^M) + M_t - M_{t-1} \\ P_t g_t + B_{t-1} &= T_t + P_t^b B_t + M_t - M_{t-1} \end{aligned}$$

Active vs Passive Policy

Both monetary or fiscal policy can be active/passive

Active policy: every instrument at its disposal can be completely freely chosen, without any concern for the consolidated government budget constraint

Passive policy: not every instrument at its disposal can be completely freely chosen, without any concern for the GBC

PV policy interaction

since $P_t g_t + B_{t-1} = T_t + P_t^b B_t + M_t - M_{t-1}$

$$g_t + \frac{B_{t-1}}{P_t} = \frac{T_t}{P_t} + \frac{P_t^b B_t}{P_t} + \frac{M_t - M_{t-1}}{P_t}$$

Seignorage Revenue

Real quantity of resources the government raises for itself through money creation

$$sr_t = \frac{M_t - M_{t-1}}{P_t}$$

Lifetime Consolidated GBC

real debt = sum of discounted s.r. + fiscal surplus

$$\begin{aligned} \frac{B_{t-1}}{P_t} &= sr_t + \sum_{s \in [1,\infty)} \frac{sr_{t+s}}{\prod_{x \in [1,s]}(1 + r_{t+x-1})} \\ &+ (t_t - g_t) + \sum_{s \in [1,\infty)} \frac{t_{t+s} - g_{t+s}}{\prod_{x \in [1,s]}(1 + r_{t+x-1})} \end{aligned}$$

Ricardian vs Non-Ricardian Policy

Ricardian: fiscal authority set planned sequence of tax and spending to ensure PVGBC is balanced

non-Ricardian: without regard for whether PVGBC is balanced

That is, if Ricardian, authority planned the sequence of t, g, sr but can change the precise timing of collection

Fiscal Theory of Inflation (FTI)

If fiscal authority is non-Ricardian, monetary authority can choose to adjust sr to balance PVGBC
$\Rightarrow \Delta M^S$ which leads to inflation (possibly long and sustained)

Fiscal Theory of Price Level (FTPL)

If fiscal authority is non-Ricardian, and monetary authority do not adjust sr to balance PVGBC
only P_t can be changed now $\Rightarrow$ one time change in price level

Fiscal Theory of Exchange Rates

An international macroeconomic application of the fiscal theory of the price level (FTPL)
Note: x^* represent the foreign currency, not the equilibrium for this chapter
Exchange Rate definitions

(Nominal) ER, E_t:	A nominal exchange rate is the price of one currency in terms of another currency
(Real) ER, e_t:	Price of one country's consumption basket in terms of another country's consumption basket.
Appreciation:	Currency becomes stronger $\rightarrow$ one unit of currency can exchange for more of another currency
Depreciation:	Currency becomes weaker
Foreign reserves, B_t^G:	Foreign reserves are a central bank's holding of foreign currencies for the purpose of government international transactions
Foreign reserves change:	A country's foreign reserves change during a given time period measures by how much its foreign reserves changed during that time period. A flow variable
Fixed exchange rate:	Exchange rate determined through government intervention in exchange markets in order to fix the rate at some value
Floating exchange rate:	Exchange rate determined solely by the forces of market supply and demand

Monetary policy and Fiscal policy interaction affecting ER

Auxiliary Assumptions

1. Constant consumption in every period, $c = \bar{c}$
2. No foreign inflation, $P_t^* = 1$ for every period
3. Foreign real interest rate never changes, $r_t^* = r^*$ for every period

4 equations

Money Demand Function:	$\frac{M_t}{P_t} = \phi(\bar{c}, i_t)$
Purchasing Power Parity (PPP):	$E_t = P_t$
Interest-Rate Parity (IRP):	$1 + i_t = \frac{(1+r^*)E_{t+1}^e}{E_t}$
Consolidated Flow Government Budget Constraint:	$B_t^G - B_{t-1}^G = \frac{M_t - M_{t-1}}{P_t} - DEF_t$

Real Exchange Rates

$e_t = \frac{E_t P_t^*}{P_t}$

Money Demand Function

Derived from MIU model
$\frac{M_t}{P_t} = \phi(c_t, i_t)$

Purchasing Power Parity

A statement about how real exchange rates behave over time.
In steady state, $e = 1$
Assume PPP holds in every period $\rightarrow e_t = 1 \Rightarrow E_t P_t^* = P_t$
Furthermore assume $P_t^* = 1 \Rightarrow E_t = P_t$, a fixed exchange rate eliminates domestic inflation.

Interest Rate Parity (IRP)

$1 + i_t = (1 + i_t^*)(\frac{E_{t+1}}{E_t})$
When no arbitrage exist, interest rates are equalised after adjusting into common currencies.
Assume zero foreign inflation rate, $i_t^* = r_t^* + \pi_t^* = r^*$ for every period $\Rightarrow 1 + i_t = (1 + r^*)\frac{E_{t+1}^e}{E_t}$

Consolidated Flow Government Budget Constraint

fiscal-monetary budget constraint
$P_t G_t = T_t + (P_t^b B_t - B_{t-1}) + M_t - M_{t-1}$, $B_t :=$ borrowing

In developing countries, borrowings are held as foreign-dominated bonds. Then
$P_t G_t = T_t + (M_t - M_{t-1}) + E_t r^* B_{t-1}^G + E_t(B_{t-1}^G - B_t^G)$, or
$P_t G_t + E_t(B_t^G - B_{t-1}^G) = T_t + (M_t - M_{t-1}) + E_t r^* B_{t-1}^G$,
where $B_t^G :=$ foreign reserve holdings at end of period t, units in foreign currency

Since assume $E_t = P_t$
$B_t^G - B_{t-1}^G = \frac{M_t - M_{t-1}}{P_t} - DEF_t$
Δ in foreign reserve $= \Delta$ in monetary policy + fiscal policy

How BOP crisis unfolds

Case 1: Fixed ER + no expected change

Key results

1. $i_t = r^*$, domestic nominal ir = foreign ir
2. $sr_t = 0$, seignorage revenue = 0
3. if $DEF_t > 0$, foreign reserve level falls

Case 2: Unanticipated, one-time devaluation

Key results

1. $i_t = r^*$, domestic ir did not change
2. $sr_t > 0$, positive seignorage revenue in the period of devaluation

Case 3: Fixed ER + fiscal deficit $\rightarrow$ BOP crisis

Key results

1. $i_{t-1} > r^*$, domestic nominal interest rate rises when devaluation is imminent (about to happen)
2. $M_{t-1} < M_{t-2}$, domestic nominal money supply falls when collapse is imminent
3. $sr_{t-1} < 0$, seignorage revenue is negative when collapse is imminent
4. currency run occurs, self-fulfilling prophecies

Growth model

Explaining the growth in output (Y)

Production model

5 equations	Production function:	$Y = F(K, L)$
	Hiring capital:	$MPK = r$
	Hiring labour:	$MPL = w$
	Labour supply:	$L = \bar{L}$
	Capital supply:	$K = \bar{K}$

5 variables	Y:	output	K:	capital
	L:	labour	w:	wage
	r:	rent		

2 parameters $L^* = \bar{L}$ $K^* = \bar{K}$

Solution

$$\begin{aligned} Y^* &= \bar{A}\bar{K}^{1/3}\bar{L}^{2/3} \\ r^* &= \frac{1}{3}\bar{A}\frac{\bar{L}}{\bar{K}}^{2/3} \\ w^* &= \frac{2}{3}\bar{A}\frac{\bar{K}}{\bar{L}}^{1/3} \\ Y^* &= r^*K^* + w^*L^* \end{aligned}$$

Solow model

adds dynamics of capital accumulation ($K^* \neq \bar{K}$)

5 equations

Production function:	$Y_t = \bar{A}K_t^{1/3}L_t^{2/3}$
Capital accumulation:	$\Delta K_{t+1} = I_t - \bar{d}K_t$
Labour force:	$L_t = \bar{L}$
Resource constraint:	$C_t + I_t = Y_t$
Allocation of resources:	$I_t = \bar{s}Y_t$

5 variables

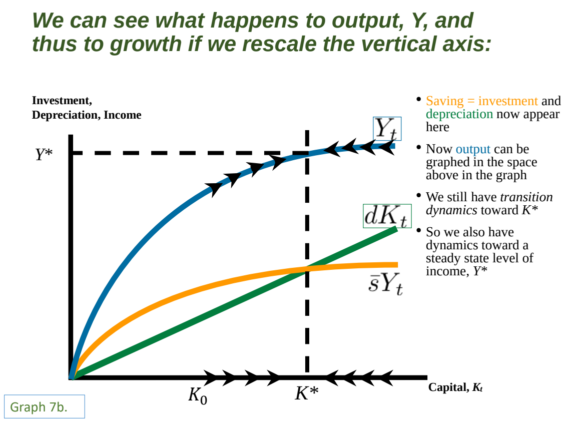
Y_t:	output at time t	K_t:	capital at time t
L_t:	labour at time t	C_t:	consumption at time t
I_t:	investment at time t		

5 parameters

$\bar{A}$:	Fixed total factor productivity (TFP)
$\bar{s}$:	Saving rate
$\bar{d}$:	Depreciation rate
$\bar{L}$:	Labour force size
$\bar{K}_0$:	Initial capital

Solution

1. output curve $F(K_t, \bar{L})$
2. saving curve $sF(K_t, \bar{L})$
3. depreciation curve (or break even curve) δK_t



Solving for steady state

At steady state, investment = depreciation

$$\begin{aligned} \bar{s}Y^* &= \bar{d}K^* \\ &= \bar{s}\bar{A}K^{*\alpha}\bar{L}^{1-\alpha} \\ \Rightarrow K^* &= \bar{L}\left(\frac{\bar{s}\bar{A}}{\bar{d}}\right)^{\frac{1}{1-\alpha}} \\ \Rightarrow Y^* &= \bar{A}K^{*\alpha}\bar{L}^{1-\alpha} \\ &= \bar{A}^{\frac{1}{1-\alpha}}\left(\frac{\bar{s}}{\bar{d}}\right)^{\frac{\alpha}{1-\alpha}}\bar{L} \end{aligned}$$

Note

population ($\bar{L}$) growth can increase aggregate output (Y^*) but not output per capita (y^*)

$g_y = g_Y - g_L = 0$

Transition Dynamics

Explains the different growth rate:
below steady state → grow
above steady state → negative growth

Strength and weakness

strength:
1. explains steady state
2. has principal of transition dynamics

weakness:
1. focus only on investment can capital (TFP unexplained)
2. no explanation for different investment and productivity rates
3. no explanation for sustained long-run economic growth

Romer model

sustained economic growth due to the growth of new ideas (A_t)
Increasing then constant Return to Scale

$Y = \frac{X-\bar{F}}{10} \Rightarrow \frac{Y}{X} = \frac{1-\frac{\bar{F}}{X}}{10}$

where $X :=$ production, $\bar{F} :=$ fixed cost

4 equations

Output production function:	$Y_t = A_t L_{yt}$
Idea production function:	$\Delta A_{t+1} = \bar{z}A_t L_{at}$
Resource constraint:	$L_{yt} + L_{at} = \bar{L}$
Allocation of labour:	$L_{at} = \bar{l}\bar{L}$

4 variables

Y_t:	output at time t	A_t:	ideas at time t
L_{yt}:	labour for output, time t	L_{at}:	researchers at time t

4 parameters

$\bar{z}$:	research productivity
$\bar{L}$:	total labour force
$\bar{l}$:	proportion of research labour
$\bar{A}_0$:	initial pool of ideas

Solution

$$\begin{aligned} y_t := \frac{Y_t}{\bar{L}} &= A_t(1-\bar{l}) \\ g_A = \frac{\Delta A_{t+1}}{A_t} &= \bar{z}L_{at} = \bar{z}\bar{l}\bar{L} \\ A_t = &\ \bar{A}_0(1+\bar{g}_A)^t \text{ (stock of knowledge)} \\ y_t = &\ \bar{A}_0(1-\bar{l})(1+\bar{g}_A)^t \end{aligned}$$

If diminishing return to idea

constant return to idea: $\Delta A_{t+1} = \bar{z}A_t L_{at}$
diminishing return to idea: $\Delta A_{t+1} = \bar{z}A_t^{\alpha} L_{at}$

there will still be sustained growth, but growth effect are eliminated due to diminishing returns

Combining Solow and Romer models

Explains the different growth rate (instead of constant growth rate as proposed in Romer model)
Exhibits transition dynamics if economy is not on its balanced growth path

short period: countries have different growth rate
long period: countries grow at the same rate

Growth accounting

determine the sources of growth in an economy and how they changes over time

$Y_t = A_t K_t^{1/3} L_{yt}^{2/3} \Rightarrow g_{Y_t} = g_{A_t} + \frac{1}{3}g_{K_t} + \frac{2}{3}g_{L_{yt}}$

adjust for labour hours

$g_{Y_t} - g_{L_t} = \frac{1}{3}(g_{K_t} - g_{L_t}) + \frac{2}{3}(g_{L_{yt}} - g_{L_t}) + g_{A_t}$

Combining Solow and Romer Algebraically

5 equations

Output production function:	$Y_t = A_t K_t^{1/3} L_{yt}^{2/3}$
Capital production function:	$\Delta K_{t+1} = \bar{s}Y_t - \bar{d}K_t$
Idea production function:	$\Delta A_{t+1} = \bar{z}A_t L_{at}$
Resource constraint:	$L_{yt} + L_{at} = \bar{L}$
Allocation of labour:	$L_{at} = \bar{l}\bar{L}$

4 variables

Y_t:	output at time t	A_t:	ideas at time t
L_{yt}:	labour for output at time t	L_{at}:	researchers at time t

5 parameters

$\bar{z}$:	research productivity
$\bar{L}$:	total labour force
$\bar{l}$:	proportion of research labour
$\bar{s}$:	saving rate
$\bar{d}$:	depreciation rate

Solution

Production function now is:
constant return to scale in objects
increasing return to scale in ideas and objects together
At balanced growth path, g_Y^*, g_K^*, g_A^* are constant

$$\begin{aligned} g_{Y_t} &= g_{A_t} + \tfrac{1}{3}g_{K_t} + \tfrac{2}{3}g_{L_{yt}} \\ g_{K_t} &= \frac{\Delta K_{t+1}}{K_t} = \bar{s}\frac{Y_t}{K_t} - \bar{d} = \text{constant} \Rightarrow g_K^* = g_Y^* \\ g_{A_t} &= \bar{z}\bar{l}\bar{L} \\ g_{L_{yt}} &= 0 \end{aligned}$$

$\Rightarrow g_Y^* = \bar{g}_A + \frac{1}{3}g_Y^* \Rightarrow g_Y^* = \frac{3}{2}\bar{g} = \frac{3}{2}\bar{z}\bar{l}\bar{L}$

Higher growth rate than in Romer model due to growth in ideas

per capita growth:

since $\frac{K_t^*}{Y_t^*} = \frac{\bar{s}}{g_y^* + \bar{d}} \Rightarrow y_t^* = \left(\frac{\bar{s}}{g_y^* + \bar{d}}\right)^{1/2} A_t^{*3/2}(1-\bar{l})$