EC2104 Lecture 3 - Single Variable Optimization

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Section 1

Implicit Differentiation

Implicit Differentiation

Solving $\frac{dy}{dx}$ without an explicit function

Steps:

1. Take derivatives of both side with respect to x
2. Solve for $\frac{dy}{dx}$

E.g. Solve $\frac{dy}{dx}$ for $y^2 + xy = 5$

1. $2y\frac{dy}{dx} + y + x\frac{dy}{dx} = 0$
2. $\frac{dy}{dx} = \frac{-y}{2y+x}$

Subsection 1

Differentiating Inverse Function

Differentiating Inverse Function

Common trick used in probability to retrieve transformed random variable

Differentiating Inverse Function

If x_0 is an interior point of interval I and $f'(x_0) \neq 0$, then g is differentiable at $y_0 = f(x_0)$ and

$$g'(y_0) = \frac{1}{f'(x_0)}$$

Section 2

Taylor Approximation

Taylor Approximation

nth-order Taylor polynomial

Approximate the function $f(x)$ around $x = a$

$$f(x) \approx \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

- Building block in many numerical approximation for solving optimization problems.
  - Besides economics, machine learning and operational research field heavily uses this approximation in their algorithms.
- Note that we are approximating $f(x)$ at a point a
  - When a change, the approximated value $\tilde{y}$ changes
- Error of approximation $(\tilde{y} - y)$ increase as x moves further away from a
- Note:
  - $0! = 1, n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$
  - $f^{(k)}(a)$ is the kth derivatives evaluated at a

Section 3

Convexity

Convexity

Key building blocks used in many techniques, including optimization

Convexity

A function is said to be convex if for any a, b and for any $t \in [0, 1]$

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b)$$

- Strictly convex if $f(ta + (1-t)b) > tf(a) + (1-t)f(b)$
- Concave function when $f(ta + (1-t)b) \geq tf(a) + (1-t)f(b)$
  - Strictly concave when $f(ta + (1-t)b) < tf(a) + (1-t)f(b)$
- Note that a concave function is the negative of convex function
  - If $f(x)$ is concave, then $-f(x)$ is convex
- Why study convex problem?
  - For a convex minimization problem, a local minimizer is also a global minimizer (which is in general not true for other type of problem!)

Subsection 1

Testing Convexity with second derivative

Testing Convexity with second derivative

Second derivative test for convexity

Suppose that f is continuous in the interval I and twice differentiable is in the interior of I. Then:

- $f''(x) > 0 \Rightarrow f(x)$ is strictly convex in I
- $f''(x) \geq 0 \Rightarrow$ convex only
- $f''(x) < 0 \Rightarrow f(x)$ is strictly concave in I
- $f''(x) \leq 0 \Rightarrow$ concave only

Section 4

Single variable optimization

Subsection 1

Extreme Value Theorem

Extreme Value Theorem

Useful to know if extrema exist (note: but it does not tell you if extrema does not exist)

Extreme Value Theorem

Suppose that f is continuous function over a closed and bounded interval $[a, b]$. Then there exist point a minimum point d and a maximum point c

- Must be continuous
- Must be closed (skipping formal definition)
  - Intuitively: point exist at the boundary
  - Closed: $[1, 2]$
  - Open: $(1, 2), (1, 2], [1, 2)$
- Must be bounded (skipping formal definition)
  - Intuitively: you can find a positive number M that is larger than all the values in the set of values
  - points must not infinity

Subsection 2

Stationary/ critical/ turning points

Stationary/ critical/ turning points

Stationary point or critical point

Let $f : X \to R$, where X is an open subset of R. An interior point $x^* \in X$ is called a stationary point of f if $f'(x^*) = 0$

Types of stationary points:

1. Maximum point
2. Minimum point
3. Inflection point (saddle point)

Q: How do we know what is the type of stationary points?

- First derivative test
- Second derivative test

Subsection 3

Finding extrema using turning points

Finding extrema using turning points

First Order Necessary Condition

If x^* is a local minimizer of f and f is continuously differentiable in an open neighbourhood of x^*, then

$$\frac{d}{dx}f(x^*) = 0$$

- Note: FOC are only necessary conditions for extrema ($P \Rightarrow Q$)
  - If x^* is a local minimizer/maximizer of f on the domain,then x^* is a stationary point ($\Rightarrow f'(x) = 0$).
  - Provided the function is differentiable at the point
- Given a turning point, it is uncertain if it's min/max point
  - $P \Leftarrow Q$ not clear, so is x^* a max? min? or inflection?
- Other points that are extrema but might not satisfy FOC
  - Extreme points
  - Non-differentiable points

Subsection 4

Testing identify of turning points using First Derivative Test

Testing identify of turning points using First Derivative Test

Condition for First Derivative Test

Assume 1. Interior point 2. Function is differentiable within the neighbourhood (a, b) of the stationary point at c

$f'(a)$	$f'(c)$	$f'(b)$	result
≥ 0	$= 0$	≤ 0	local max
≤ 0	$= 0$	≥ 0	local min
> 0	$= 0$	> 0	not a extrema
< 0	$= 0$	< 0	not a extrema

Intuitively: gradient is the slope, draw the different cases

Subsection 5

Testing identify of turning points using Second Derivative Test

Testing identify of turning points using Second Derivative Test

Condition for Second Derivative Test

Assume 1. Interior point 2. Function is twice differentiable at $x = c$ where $f'(c) = 0$ (stationary point)

$f''(c)$	result
≤ 0	local max
< 0	strict local max
≥ 0	local min
> 0	strict local min
$= 0$	no results

Note:

- Recall the property of convex function $f''(x) \geq 0 \forall x$
- Therefore, a stationary point in the convex function is always the min

Subsection 6

Determine point of inflection

Determine point of inflection

Recall: second derivative test say nothing about $f''(c) = 0$

Definition for Inflection points

If function f is twice differentiable, the point c is called an inflection point for f if there exists an interval (a, b) about c such that

- $f''(x) \geq 0$ in (a, c) and $f''(x) \leq 0$ in (c, b)
- $f''(x) \leq 0$ in (a, c) and $f''(x) \geq 0$ in (c, b)

Test for Inflection Points

Let f be a function with a continuous second derivative in an interval I, and let c be an interior point of I.

- If c is an inflection point for f, then $f''(c) = 0$
- If $f''(c) = 0$ and f'' changse sign at c, then c is an inflection point for f

Subsection 7

Note on Global vs Local Extrema

Note on Global vs Local Extrema

Extreme points are points where functions reach their highest/lowest values

Extrema

If $f(x)$ has domain D,

- $c \in D$ is maximum point for $f \Leftrightarrow f(x) \leq f(c), \forall x \in D$
- $d \in D$ is maximum point for $f \Leftrightarrow f(x) \geq f(c), \forall x \in D$

- Local extrema when the point is max/min point in a neighbourhood (within a interval of values)
- Global extrema when the point is max/min point across all the domain
- Generally, a function will have multiple local extrema
    - Only for a special subset of functions, the local extrema = global extreme (when functions are strictly convex/concave)
    - Determining global extrema remain an unsolved problem outside of convex functions. Best way is to compare the functional values

Subsection 8

Summary on finding the extrema of functions

Summary on finding the extrema of functions

Setup

Find maximum/minimum values of a differentiable function f defined on a closed, bounded interval $[a, b]$

Then the General step for finding the extrema:

1. Consider interior point
   - Using First Order Condition (FOC)
2. Consider boundary (end points)
   - In the case of the setup, $f(a), f(b)$
3. Consider non-differentiable points

Compare the functional values to determine global extrema

Determine extrema of unbounded functions

Value of extreme must be found through methods of limits