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Maximally representative allocations for guaranteed delivery advertising campaigns

R. Preston McAfee · Kishore Papineni · Sergei Vassilvitskii

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Abstract There are around 400 advertising networks that match opportunities for "display" advertising, which include banner ads, video ads and indeed all ads other than text-based ads, on web pages and candidate advertisements. This is about a $25 billion business annually. The present study derives a method of pricing such advertisements based on their relative scarcity while ensuring that all campaigns obtain a reasonably representative sample of the relevant opportunities. The mechanism is well-behaved under supply uncertainty. A method based on the mechanism described in this paper was implemented by Yahoo! Inc.

Keywords Advertising · Representativeness · Mechanism design · Market design · Bidding · Exchange

JEL Classification D02 · M37

There are around 400 advertising networks, as well as individual publishers, that match opportunities for "display" advertising, which include banner ads, video ads and indeed all ads other than text-based ads, on web pages. This is about a $25 billion business annually. The present study derives a method of pricing such advertisements, using randomized bidding, based on their relative scarcity, while ensuring that all campaigns obtain a reasonably representative sample of the relevant opportunities. The mechanism is well-behaved under supply uncertainty. A method based on the mechanism described in this paper was implemented by Yahoo! Inc.

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R. P. McAfee (✉) · K. Papineni · S. Vassilvitskii
Google, Inc, 1600 Amphitheatre Parkway, Mountain View, CA 94043, USA
e-mail: randolph@mcafee.cc

The economic problem addressed by the present study arises when a seller—in our application a publisher or website—with some degree of price-setting ability or market power, has an extremely large variety of goods for sale and cannot control exactly the product mix available. The key problem is the large variety of distinct goods for sale. With a moderate to small number of goods, a seller would simply use the generalized inverse-elasticity rule to set prices.[1] With today's technologies, something on the order of a million prices could be set this way, especially if the cross-price elasticity matrix is block-diagonal or otherwise sparse. In the advertising setting, however, there are trillions of distinct products; indeed modern advertising auctions now price each ad seen by each individual on each page separately. Consequently, the pricing problem requires simplification. This paper derives a practical mechanism that comes from the solution to a reasonable optimization problem, and has useful properties, such as handling supply uncertainty and implementation via an advertising exchange.

To describe the specific problem in more detail, we consider an online publisher, such as Yahoo!, the Wall Street Journal, or the Huffington Post, who sells advertising space on their pages. Ads and the space they occupy are both, confusingly, known as impressions. Advertisers such as Proctor and Gamble or General Motors buy guaranteed advertising campaigns from the publisher. These fix the number of impressions, the time interval in which the impressions run, targeting criteria and price. For example, Proctor and Gamble might buy a campaign aimed at women aged 30-45 in suburban locations, while a General Motors campaign for minivans targets middle-income individuals likely to have children. Besides demographic targeting criteria, these campaigns may come with additional requirements on the specific sites (e.g. prohibiting email or comment pages) and other user characteristics (e.g. allowing only users who visited an autos page in the past week).

When an advertiser purchases a guaranteed contract, often there will be many kinds of content or inventory, the pages on which the ads may run, that can be used to satisfy the contract. For example, a buyer of ten million impressions on autos.com may get high-income males in Los Angeles or blind senior citizens in Fargo, ND. Automobile advertisers are going to be unhappy if they mostly get the latter. On the other hand, guaranteeing the specific quantities of each type of customer provides little leeway in the event that demand or supply change, and supply varies substantially over time. A key feature of the problem we address is that the seller cannot control the available set of inventory at the time the campaign is run, but must instead allocate the inventory that arises among the campaigns. Guaranteed advertising contracts should simultaneously respect advertisers' wishes to avoid poor quality opportunities, while providing the seller with adequate flexibility to adjust to changing circumstances.[2]

A strategy for ensuring overall quality would be to provide representativeness across all supply types. This is the strategy used by major websites historically; guaranteed campaigns served first and thus received a relatively representative bundle of impressions. Thus, a guaranteed display advertiser might be provided with a mix of locations, with flexibility in the mix given to the network subject to some kind of budget. This

[1] The cleanest treatment of the inverse elasticity rule is Varian (1985).

[2] The problem of display advertising is discussed in Babaioff et al. (2008), Boutilier et al. (2008), Contantin et al. (2009) and Ghosh et al. (2009).

solution sounds good in principle but encounters practical problems almost instantly. Advertisers increasingly specify target audience attributes. These attributes include standard demographic variables like age, gender, income, and geography. However, many other targeting attributes are used, for example "interested in autos" as judged by visits to auto-related web pages. Advertisers have created advertising campaigns to show their advertisements only in cities where the sun is shining, and only when the Dow Jones Industrial Average is higher than its previous close. Assigning a random mix fails to use inventory strategically to meet a set of campaigns, because it does not account for which inventory types are in short supply.

Detailed targeting creates "orphan categories", in which no one is specifically interested; these categories come about because they have distinct groups of customers. For example, if one advertiser buys ads based on "the sun is shining" and another based on "the stock market is up", four categories of inventory, each with distinct demand, are created (e.g. stock market down and sun shining is one category). There are trillions of such categories just based on demographics, geographic selection and interest-based "behavioral" targeting created by a hundred campaigns, and large advertising networks handle hundreds of campaigns. Ostensibly all of these trillions of categories must be priced to carry out a delivery mix based on pricing.[3] Unfortunately, it is not practical to actually compute the prices of trillions of categories, nor is it sensible to price categories in which no one is directly interested. Our mechanism is the only existing methodology for pricing campaigns—sets of advertisements like "males in California" or "people who live in cities in which the sun is currently shining"—that avoids the need to price orphan categories and simultaneously ensures that advertisers obtain as representative an allocation of inventory as is technically feasible.

For a concrete motivating example, consider a publisher that has 6M impressions to sell, equally divided across those within and outside the US. The demand comes from two advertisers, the first looking to buy 2M impressions in the US only, the second looking to buy 3M impressions spread equally across all of the supply. How should one price the two different supply types?

This paper starts by deriving "maximally representative allocations". This produces a parameterized family of demands. These demands aggregate simply and naturally and lead to market prices by equating supply and demand. Moreover the pricing system based on supply and demand is very well-behaved, and accommodates supply uncertainty effectively.

1 Equilibrium pricing and targets

Consider an agency, publisher or network that has a set of supply pools of inventory. Examples of supply pools could include "young men on auto pages" or "unknown gender on a computer related news article".

Index campaigns by superscripts. Campaign j specifies a quantity Y^j and a set of eligible supply pools. A supply pool i is eligible for campaign j if $s_i^j = 1$, and $s_i^j = 0$

[3] The problem of orphan categories arises in the literature on expressive bidding, exemplified by Boutilier et al. (2008) and Agarwal et al. (2007).

otherwise. (Subscripts are used to denote supply pools.) The variable s_i^j also captures the possibility that one might sell males, but deliver unknowns *weighting them by the likelihood of obtaining males.* Thus, if 55 % of the pool "unknown gender" are males, one can deliver males by providing "unknown genders" but only at the rate $s_i^j = 55\,\%$. The following additional notation is convenient:

x_i is the volume of supply pool i
r_i is the minimum price for supply pool i,[4]
y_i^j is the amount of supply pool i that will go toward contract j, $Y^j = \sum_i s_i^j y_i^j$, and
V^j is the priority or value attributed to campaign j.

The objective we posit is to minimize the weighted squared distance of the within-supply pool market shares from representative market shares. By representative shares, we mean the values of the background population. For campaign j, the representative share of inventory type i is $\frac{s_i^j x_i}{\sum_k s_k^j x_k}$; this is the proportion of the total feasible inventory represented by the ith type.

The squared distance is chosen primarily for tractability and we will also investigate the *KL* divergence (maximum entropy) as an alternate objective in a subsequent section. The squared distance has useful properties in that the statistics of mean squared error are well understood. Moreover, finding solutions to minimizing quadratics are possible using fast algorithms. It is desirable to weight the squared deviations by $\frac{\sum_k s_k^j x_k}{s_i^j x_i}$, to make the solution independent of supply pool division,[5] and by V^j to reflect potentially differential campaign weighting. (In practice the weights depend on the total spend of the advertiser, the premium the advertiser offers over base-priced inventory, the length of the contract and the quality of the advertisements as viewed by users.) The campaign weights are $V^j Y^j$ to reflect that a campaign of twice the size should be twice as important, or that merging two identical campaigns should not influence the solution. Thus the objective function is

$$\min \frac{1}{2} \sum_j V^j Y^j \sum_i \frac{\sum\limits_k s_k^j x_k}{s_i^j x_i} \left(\frac{s_i^j x_i}{\sum\limits_k s_k^j x_k} - \frac{s_i^j y_i^j}{Y^j} \right)^2$$

$$= \min \frac{1}{2} \sum_j V^j Y^j \sum_i s_i^j \frac{\sum_k s_k^j x_k}{x_i} \left(\frac{x_i}{\sum\limits_k s_k^j x_k} - \frac{y_i^j}{Y^j} \right)^2 \tag{1}$$

4 The role of r_i is as a floor or reserve price. It is the minimum that should ever be accepted, and can be thought of as the price that obtains by auctioning the impression in an exchange or the value of running a "house ad" (an ad for the website's own content) or a public service advertisement.

5 Otherwise merging two similar supply pools changes the objective function. Consider merging two supply pools, with the same relative shares. The change in the objective function, using the proportional weights, is $\frac{1}{x_1}\left(\frac{x_1}{X} - \frac{y_1}{Y}\right)^2 + \frac{1}{x_2}\left(\frac{x_2}{X} - \frac{y_2}{Y}\right)^2 - \frac{1}{x_1+x_2}\left(\frac{x_1+x_2}{X} - \frac{y_1+y_2}{Y}\right)^2 = \frac{x_1 x_2}{Y^2(x_1+x_2)}\left(\frac{y_2}{x_2} - \frac{y_1}{x_1}\right)^2 = 0.$

There are three kinds of constraints. The allocation must be feasible, meaning there is adequate inventory in each supply pool. Mathematically this requires

$$x_i \geq \sum_j y_i^j, \quad \text{for every supply pool } i. \tag{2}$$

Second, each campaign must meet its guarantee, requiring

$$Y^j = \sum_i s_i^j y_i^j \quad \text{for all campaigns } j. \tag{3}$$

In addition, quantities are non-negative, creating a constraint

$$y_i^j \geq 0, \quad \text{for all } i \text{ and } j. \tag{4}$$

Let p_i be the Lagrangian multiplier (shadow value) on the inventory constraint (2) and α^j be the multiplier on the campaign guarantee constraint (3). We will handle the non-negativity constraint manually. The Lagrangian becomes:

$$\min \frac{1}{2} \sum_j V^j Y^j \sum_i s_i^j \frac{\sum_k s_k^j x_k}{x_i} \left(\frac{x_i}{\sum_k s_k^j x_k} - \frac{y_i^j}{Y^j} \right)^2$$
$$+ \sum_i p_i \left(\sum_j y_i^j - x_i \right) + \sum_j \alpha^j \left(Y^j - \sum_i s_i^j y_i^j \right)$$

We refer to the solution that minimizes (1) subject to (2)–(4) as the L^2 maximally representative allocation.

Theorem 1 *There are campaign shadow values p^{*j} and inventory prices p_i such that the maximally representative allocation is given by:*

$$\frac{s_i^j y_i^j}{Y^j} = Max \left\{ 0, \ \frac{1}{V^j} \left(\frac{x_i}{\sum_k s_k^j x_k} \right) \left(s_i^j p^{*j} - p_i \right) \right\} \tag{5}$$

Theorem 1 shows that the demand system under maximally representative allocations is linear. Moreover, given the values p^{*j}, there is a closed form for the inventory-specific prices. In particular, summing (5) over j,

$$x_i \geq \sum_j y_i^j = \sum_j Max \left\{ 0, \ \frac{Y^j}{V^j} \left(\frac{x_i}{\sum_k s_k^j x_k} \right) \left(p^{*j} - p_i \ / \ s_i^j \right) \right\}. \tag{6}$$

If $x_i > \sum_j Max\left\{0,\ \frac{Y^j}{V^j}\left(\frac{x_i}{\sum_k s_k^j x_k}\right)\left(p^{*j} - r_i \ / \ s_i^j\right)\right\}$, then $p_i = r_i$, and otherwise p_i is determined by

$$1 = \sum_j Max\left\{0,\ \frac{1}{V^j}\left(\frac{Y^j}{\sum_k s_k^j x_k}\right)\left(p^{*j} - p_i \ / \ s_i^j\right)\right\}$$

This equation solves for prices, given the list of p^{*j}. The equation is piecewise linear with the number of pieces equal to the number of campaigns. Thus, a problem with potentially trillions of dimensions (pricing each supply pool) is reduced to a problem with hundreds of dimensions in practice by maximally representative allocations.

The shadow prices are constrained to exceed an outside option r_i. In essence, this solution "sells" some of the capacity in an outside market if the shadow value p_i would otherwise fall below r_i. The shadow prices are denominated in dollars and thus are interpretable as actual prices to charge the buyer for that bucket of inventory.

Note that V^j allows weighting campaign j differently from other campaigns. The value V^j measures how important a representative mix is for campaign j. For higher values of V^j, the importance of a representative mix is higher. The premium is beyond the minimum prices r_i, that is, the customer paying higher values of $\sum_i (p_i - r_i)y_i^j$ motivates a higher value of V^j.

The price of a given supply pool enters in the form p_i/s_i^j and thus "penalizes" supply pool i when the campaign can't readily substitute; low values of s_i^j entail high effective prices. For example, a lipstick campaign that seeks only females might obtain inventory from a mixed gender group, but only count impressions that reach females. This counts showing impressions to the wrong group as having zero value, which is reasonable if the wrong group does not find the ads offensive.

2 Practical implementation of the L^2 solution

The campaign pricing described above controls the allocation of each campaign with a single variable, p^{*j}. Prices for inventory are determined by a single piecewise linear equation, either r_i or $1 = \sum_j Max\left\{0,\ \frac{1}{V^j}\left(\frac{Y^j}{\sum_k s_k^j x_k}\right)\left(p^{*j} - p_i \ / \ s_i^j\right)\right\}$, a piecewise linear expression decreasing in p_i. The entire allocation process is controlled by a vector equal to the number of campaigns. This is generally a much smaller number than the number of inventory types, hundreds versus trillions.

Finding the solution to the maximal representative allocation problem is thus a much lower dimensional problem and in fact is solvable in practice (see the work by Bharadwaj et al. 2012) describing a simple combinatorial algorithm). In a subsequent paper published earlier, Ghosh et al. (2009) show that the maximally representative allocations can be implemented using randomized bidding in an exchange environment.[6] The basis for that solution is provided by manipulating Eq. (5), which can be

[6] Beck and Milgrom (2012) also use an alternative randomized allocation, based on the gap between first and second prices, to improve efficiency of the allocation.

rewritten to look like a conditional bid. First note that the probability of obtaining inventory is $\frac{y_i^j}{x_i}$, which by Eq. (5) can be expressed as:

$$\frac{y_i^j}{x_i} = Max\left\{0,\quad \frac{1}{V^j}\left(\frac{p^{*j}Y^j}{\sum\limits_k s_k^j x_k}\right)\left(1 - p_i \;/\; s_i^j p^{*j}\right)\right\}.$$

Let k minimize $p_i \;/\; s_i^j p^{*j}$ over i such that $s_i^j > 0$. The minimized value $p_k/s_k^j p^{*j} < 1$ for otherwise the firm would obtain no inventory. By construction, $\frac{1-p_i\,/\,s_i^j p^{*j}}{1-p_k\,/\,s_k^j p^{*j}} \leq 1$. Thus, we can rewrite (5) to be:

$$\frac{y_i^j}{x_i} = \frac{1}{V^j}\left(\frac{p^{*j}Y^j}{\sum\limits_k s_k^j x_k}\right)\left(1 - p_k \;/\; s_k^j p^{*j}\right)\; Max\left\{0,\quad \frac{1 - p_i \;/\; s_i^j p^{*j}}{1 - p_k \;/\; s_k^j p^{*j}}\right\} \tag{5'}$$

Note as well that $1 \geq \frac{y_k^j}{x_k} = \frac{1}{V^j}\left(\frac{p^{*j}Y^j}{\sum_k s_k^j x_k}\right)\left(1 - p_k \;/\; s_k^j p^{*j}\right)$. Thus, $\frac{y_i^j}{x_i}$ is written as the product of two numbers, one of which is independent of i, and the other which depends on p_i. Both of these numbers are less than or equal to one and hence can be interpreted to be probabilities.

By submitting a bid which is uniform on $\left[s_i^j p_k \;/\; s_k^j, s_i^j p^{*j}\right]$, a firm wins with probability $Max\left\{0,\quad \frac{s_i^j p^{*j} - p_i}{s_i^j p^{*j} - s_i^j p_k\,/\,s_k^j}\right\} = Max\left\{0,\quad \frac{1-p_i\,/\,s_i^j p^{*j}}{1-p_k\,/\,s_k^j p^{*j}}\right\}$. Therefore, an implementation of a campaign exists where with probability $\frac{1}{V^j}\left(\frac{p^{*j}Y^j}{\sum_k s_k^j x_k}\right)\left(1 - p_k \;/\; s_k^j p^{*j}\right)$, a bid is drawn randomly from $U\left[s_i^j p_k \;/\; s_k^j, s_i^j p^{*j}\right]$. This implementation requires knowing the eligibility s_i^j of inventory in question but does not otherwise depend on the inventory type and in particular does not depend on either x_i or p_i. Thus, when the values p_i are generated by an exchange (exogenous to an advertiser), there is an implementation of maximally representative allocations using a probability of submitting a randomized bid into the exchange according to a uniform distribution.[7]

The demand system developed here has the *gross substitutes* property: an increase in the price of item i decreases the demand y_i^j for item i and increases the demand y_k^j for other goods k. Adjustment mechanisms, known as *Walrasian* mechanisms, which change the price proportionally to the excess demand $\sum_j y_i^j - x_i$, are known to converge to competitive equilibrium (excess demand = 0), and moreover that equilibrium

[7] The contribution of Ghosh et al. (2009) is to show how to rationalize a set of such campaigns that might conflict with each other; in particular it could be necessary to divert inventory that one campaign loses to another campaign. This in particular arises whenever $1 > \sum_j \frac{1}{V^j}\left(\frac{p^{*j}Y^j}{\sum_k s_k^j x_k}\right)\left(1 - p_k \;/\; s_k^j p^{*j}\right)$.

is unique. See Arrow et al. (1959) for the proof. Thus the system has a unique solution and is well-behaved and readily computable.

3 A 2 × 2 example

Recall the motivating example. Stated formally, there are two supply pools, 1 and 2, each with 3M units of supply. Either supply pool sells for \$1 per unit in the wholesale market (r_i). There are two buyers. Buyer 1 wants 2M from supply pool 1. Buyer 2 wants 3M units spread across both. Set $V_1 = V_2 = 1$.

It is possible to serve the entire demand at the reserve price, with buyer 1 getting 2M from supply pool 1, and buyer 2 getting 1 M from supply pool 1 and 2M from supply pool 2.

By a routine calculation, the price of supply pool 1 is \$1.67. The price is used to shift the second buyer away from supply pool 1, so as to accommodate the focused demand of buyer 1. Thus, the ability of buyer 1 to purchase an entire pool is limited by the buyer paying for the distortion in allocation to buyer 2, and such price changes are significant.

4 KL Divergence

The Kullback–Leibler (KL) divergence is a measure of deviation from an ideal distribution, and is closely related to maximum entropy (Csiszar 1991). Mathematically, the KL divergence is

$$\sum_j V^j \sum_i \left(s_i^j y_i^j \log\left(y_i^j \,/\, x_i \right) + s_i^j x_i - s_i^j y_i^j \right) \tag{7}$$

This objective function faces the same constraints (2)–(4). There is an important property of the *KL* minimization:

$$\text{If } s_i^j > 0, \quad \text{then } y_i^j > 0. \tag{8}$$

This property arises because $\lim_{z\to 0} \ (z \log z)' = -\infty$, so that it is always desirable to allocate a small bit of feasible inventory to each contract. Thus KL minimization is always an interior solution, in contrast to the L^2 case where zeros will arise whenever a contract value p^{*j} is lower than the effective price of the inventory $p_i \,/\, s_i^j$. As a result, the Lagrangian method produces a nearly closed form solution for the *KL* minimization problem. Let

$$L = \sum_j V^j \sum_i \left(s_i^j y_i^j \log\left(y_i^j \,/\, x_i \right) + s_i^j x_i - s_i^j y_i^j \right) + \sum_i p_i \left(\sum_j y_i^j - x_i \right) + \sum_j \alpha^j \left(Y^j - \sum_i s_i^j y_i^j \right)$$

The solution for y_i^j is

$$y_i^j = x_i e^{\frac{-1}{V^j}\left(\frac{p_i}{s_i^j} - \alpha^j\right)}.$$

Here p_i is the price of inventory of type i, and α^j is the shadow value of campaign j. These must sum to the overall campaign objective, which determines α^j, so

$$\frac{y_i^j}{Y^j} = \frac{x_i e^{\frac{-p_i}{V^j s_i^j}}}{\sum_k s_k^j x_k e^{\frac{-p_k}{V^j s_k^j}}}$$

The KL objective function is convex. While it is generally not strictly convex, it has a unique optimal allocation that is readily found through gradient descent. Consequently, both the *KL* divergence solution and the L^2 objective are readily solved in practice.

5 Supply uncertainty

With supply uncertainty, the means by which the allocations are chosen become relevant. We consider the situation where selection is fractional, that is, the publisher dictates that a fraction of the inventory will be applied to various advertisers, and then the supply is realized and the fractions executed. There are several reasons for focusing on fractions. First, rotational schemes are not uncommon in practice. A rotational scheme comes in the form "first impression to A, second impression to B, …" and implements a fractional allocation. Second, advertisers recognize a product sometimes known as "share of voice", which contractually guarantees a certain fraction of the impressions. Third, fractional allocation skirts the priority problem; otherwise a publisher must determine which of the various advertisers get which parts of the available inventory. Thus, fractional allocations have both mathematical simplicity and real world relevance.

Let f_i^j be the fraction of inventory i assigned to campaign j; in the previous notation, $f_i^j = y_i^j \ / \ x_i$. We can rewrite the *KL* objective function to obtain

$$\begin{aligned}
&\sum_j V^j \sum_i \left(s_i^j y_i^j \log\left(y_i^j \ / \ x_i\right) + s_i^j x_i - s_i^j y_i^j\right) \\
&\quad = \sum_j V^j \sum_i \left(s_i^j f_i^j x_i \log\left(f_i^j\right) + s_i^j x_i - s_i^j f_i^j x_i\right)
\end{aligned}$$

This objective function is linear in supply variables x_i. Thus the *KL* divergence has the same solution under uncertainty that it has under the expected value of the supply.

In practice, publishers may experience an asymmetric loss from failure to meet the campaign quantity. This can be modeled by imposing the cost of not meeting the campaign quantity objective only when the allocation is too low. A penalty for over-delivery already exists through the price of the inventory. This penalty consideration induces a rewrite of the Lagrangian as:

$$L = \sum_j V^j \sum_{i|S_i^j>0} \left(s_i^j f_i^j x_i \log \left(f_i^j \right) + s_i^j x_i - s_i^j f_i^j x_i \right) + \sum_i p_i \left(\sum_j f_i^j x_i - x_i \right) + \sum_j \alpha^j \left(Y^j - \sum_i s_i^j f_i^j x_i \right)^+$$

In this case, the solution comes in the form:

$$f_i^j = e^{\frac{-1}{V^j}\left(\frac{p_i}{s_i^j} - \gamma^j\right)},$$

where γ^j is the probability of not meeting the campaign objective times the penalty α^j.

6 Conclusion

This paper develops a methodology for pricing supply pools of advertising opportunities and allocating available inventory to campaigns. It has the distinct advantages of straightforward computation and a pricing rule which reflects both scarcity and the desire for a representative allocation. Representativeness is a useful property for much the same reason that mutual funds are good investment vehicles; it reduces risk and mitigates the problem of being assigned the worst possible inventory consistent with the contract. This mitigation has the important advantage of encouraging advertisers to provide flexibility to the network in choosing the locations of their advertisements. Such flexibility is critical to actually being able to implement a set of advertising campaigns.

A second advantage to the system is that the problem of pricing inventory and the problem of inventory allocation to campaigns are considered as a single optimization problem. While this integration is not special to maximally representative allocations, and indeed could be operated in any pricing system, it is nevertheless extremely valuable. Integrating the allocations of inventory with the pricing of inventory permits the pricing of campaigns to be consistent with the pricing of inventory, in other words, the price charged to an advertiser can be a markup of the value of the inventory allocated. Rather than using a "price first, allocate second" methodology common in the industry, maximum representative allocations ensure that the pricing of campaigns reflects not only the expected cost of the inventory but also a cost for the distortion of inventory provided to other campaigns, by integrating the pricing and the serving plan.

A third advantage is that the implementation can be achieved by using a randomized bidding mechanism. This means that the allocation can be achieved without *ex ante* knowledge of the price of a particular piece of inventory, because the same bidding distribution works for all eligible inventory. Moreover, the distribution takes a particularly simple form, uniform for L^2 and exponential for KL. This ability to execute the guaranteed plan through bidding in an exchange is advantageous because the publishers are increasingly integrating their guaranteed campaigns and sales of leftover impressions via auctions.

Yahoo! implemented a version of this system in its display advertising serving system, based on the analysis of this and subsequent papers.

Appendix

Proof of Theorem 1 The first order conditions, when $s_i^j > 0$, come in the form

$$0 = -s_i^j V^j \left(\frac{x_i}{\sum_k s_k^j x_k} - \frac{y_i^j}{Y^j} \right) \frac{\sum_k s_k^j x_k}{x_i} + p_i - s_i^j \alpha^j, \text{ or } y_i^j = 0 \quad \text{and } \frac{\partial L}{\partial y_i^j} \geq 0.$$

Thus,

$$\frac{s_i^j y_i^j}{Y^j} = \left(\frac{s_i^j x_i}{\sum_k s_k^j x_k} \right) \left(1 - \frac{p_i / s_i^j - \alpha^j}{V^j} \right) \text{ or } \quad y_i^j = 0$$

and

$$0 \geq \left(\frac{s_i^j x_i}{\sum_k s_k^j x_k} \right) \left(1 - \frac{p_i / s_i^j - \alpha^j}{V^j} \right). \tag{9}$$

Let $A^j = \{i | y_i^j > 0\}$. Summing (9) over $i \in A^j$, and solving for α^j we obtain

$$\alpha^j = \frac{V^j \sum_k s_k^j x_k + \sum_{i \in A^j} x_i p_i}{\sum_{i \in A^j} s_i^j x_i} - V^j. \tag{10}$$

Substituting the value of α^j into (9), we obtain a solution for y_i^j:

$$\begin{aligned} \frac{s_i^j y_i^j}{Y^j} &= \left(\frac{s_i^j x_i}{\sum_k s_k^j x_k} \right) \left(1 - \frac{p_i / s_i^j - \alpha^j}{V^j} \right) \\ &= \frac{1}{V^j} \left(\frac{s_i^j x_i}{\sum_k s_k^j x_k} \right) \left(\frac{V^j \sum_k s_k^j x_k + \sum_{i \in A^j} x_i p_i}{\sum_{i \in A^j} s_i^j x_i} - p_i / s_i^j \right) \end{aligned} \tag{11}$$

Define

$$p^{*j} = \frac{V^j \sum_k s_k^j x_k + \sum_{i \in A^j} x_i p_i}{\sum_{i \in A^j} s_i^j x_i} \tag{12}$$

Now, note that $y_i^j = 0$ if and only if $p_i \leq s_i^j p^{*j}$. Thus we have:

$$\frac{s_i^j y_i^j}{Y^j} = \left(\frac{s_i^j x_i}{\sum_k s_k^j x_k} \right) \frac{Max[0, \quad p^{*j} - p_i/s_i^j]}{V^j}, \text{ and} \tag{13}$$

$$A^j = \{i | y_i^j > 0\} = \{i | p_i < p^{*j}\}. \tag{14}$$

This solution delivers quantities as a function of prices. The prices are endogenized as follows. Recall that there is an exogenous price r_i at which the guaranteed inventory can be sold to the outside. Then the prices are given by complementary slackness—either

$$p_i = r_i \quad \text{and} \quad x_i \geq \sum_j y_i^j \quad \text{or} \quad p_i > r_i \quad \text{and} \quad x_i = \sum_j y_i^j. \tag{15}$$

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