

Optimal budget-pacing for Real-time Bidding

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Abstract

We consider an algorithm buying online inventory via Real-time Bidding (RTB) auctions on a given segment. We consider the problem of how to smooth the spent throughout the day in an optimal fashion (maximize the number of impressions – under an assumption of homogeneity). We introduce an optimization framework based on variational calculus which permits to obtain closed formulas for the budget-pacing problem in RTB. Our main result is that the optimal spending is not linear but proportional to the number of auction-requests we observe. We also show that the same framework can be applied in the presence of frequency-capping.

Keywords: Real-time Bidding, budget pacing, frequency capping, programmatic advertising, variational calculus

1. Introduction

Real-time Bidding (RTB in short) is a growing technology used in display-advertising enabling advertisers to target users in a per-access basis in order to turn website visits into final conversions; this by taking into account information
5 about the user, website, time of the day among others [5, 7, 8]. Media-trading desks and DSP² work in behalf of advertisers to accomplish their marketing and return-on-investment goals. On a nutshell, these intermediaries connect

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²Demand-side platforms: intermediaries serving as interface connecting advertisers with ad-exchanges and offering customized strategies.

with virtual marketplaces (known as ad-exchanges) where inventory (the right to publish a banner) is purchased. These ad-exchanges send auction-requests to which DSP and media-trading desks send back a bid and the higher bid wins the auction (usually of a Vickrey type). These auctions happens thousands of time every minute for each advertising campaign. More information about the per-auction optimization is given in [2] (see [1, 3, 4, 6, 9, 10] for other different approaches).

Different problems are faced by algorithms optimizing the bidding process, these can be the choice of the optimal bid for the different segments (e.g. users, websites etc.), the optimal allocation of budget across different segments or how to optimally smooth the budget spent throughout the trading session once the segments are already defined (also known as *budget-pacing*). In this paper we will tackle the last problem by proposing a new approach using variational calculus techniques and focusing in the pacing problem on segments which are homogeneous in quality. This in contrast with the literature [3] where the problem is usually approached by online-learning procedures and trying to accomplish several tasks (e.g. optimal allocation and bidding) at the same time.

Concretely, our goal is to solve the problem of optimally smoothing a budget during a time window (in which the likelihood of conversion during the window is uniform), taking as input the number of bid requests, the average price we paid per impression and the win-rate, this, in order to maximize the number of impressions. Our model is essentially continuous, this is not a strong assumption due to the large number of auctions we participate during the day . Usually, practitioners try to smooth linearly the spent during the time window, we will prove mathematically that this approach can be improved by spending in proportion to the number of bid-requests.

In a second time, we study the optimization by including the effects of *frequency-capping*; i.e. do not displaying impressions to users whom have already be exposed to a given number of impressions. In this case, part of the bid requests are censored if the user already saw a maximum of impressions, thus, in order to keep a given spending pace, the algorithm should increase its win-

rate (hence, the level of bid) hence, creating a feedback effect on the optimal
40 strategy. We show that the same variational-calculus approach can be used and
formulas for the optimal strategy are obtained.

Throughout the article we address some practical considerations when im-
plementing that kind of strategies.

2. Model setup

45 Let us consider an algorithm participating in a sequence of Real-Time Bid-
ding auctions (e.g. a media trading-desk or a DSP in behalf of an advertiser)
wanting to spent a given budget $S \in \mathbb{R}_+$ over a given period of time $[0, T]$
(this can represent few hours) on a given segment (e.g. users, domains, time
of the day etc.). Because of the large number of auctions algorithms face in
50 practice, we consider a continuous function $\mu : t \mapsto \mu(t)$ representing the num-
ber of auction-requests per unit of time the algorithm receives. Thus, the total
auction-requests the algorithm receives, is given by $\int_0^T \mu(t) dt$.

At each instant $t \in [0, T]$, the algorithm controls the bid-level $b(t)$ it posts
in the auctions. We make the assumption that the segment has a homogeneous
55 quality in every dimension, so the only goal of the algorithm is to control this
bid-level in order to maximize the inventory purchased.

At this point, two quantities are introduced:

- The *winrate*: A function $w : b \mapsto w(b)$, given the ratio of auctions won
when the posted bid is equal to b .
- 60 • The *CPM (or cost-per-mille)*: A function $p : b \mapsto p(b)$, representing the
price paid per one thousand units of inventory when the bid posted is
equal to b . We use the name CPM as it is the standard in advertising,
however, throughout the calculations we will use the cost-per-unit in order
to avoid carrying the *per-mille* factor.

65 It is an obvious fact, in practice, that those two functions are increasing in
 b . In a first time we will suppose they are strictly increasing and differentiable

(this is a strong hypothesis as we will see in a discussion about floor-prices on section IV).

With this in mind, for a bidding strategy $(b(t))_{t \in [0, T]}$, we can define now the evolution of the *running purchased-inventory* and the *running spent* as follows.

- Running purchased-inventory $I : t \mapsto I(t)$, $I(0) = 0$.

$$I'(t) = \underbrace{\mu(t)}_{\text{nb. of requests}} \times \underbrace{w(b(t))}_{\text{winrate}}.$$

- Running spent $S : t \mapsto S(t)$, $S(0) = 0$.

$$S'(t) = \underbrace{I'(t)}_{\text{new inventory}} \times \underbrace{p(b(t))}_{\text{current cost}}.$$

2.1. A change of time

Let us introduce the following change of time by defining the process

$$\tau(t) = \int_0^t \mu(s) ds$$

i.e. the total number of auction-requests until time t .

Hence, we obtain

$$I'(t) = \tilde{I}'(\tau) \mu(t)$$

and

$$S'(t) = \tilde{S}'(\tau) \mu(t).$$

Leading to the following dynamic equations:

$$\begin{aligned} \tilde{I}'(\tau) &= w(b(\tau)) \\ \tilde{S}'(\tau) &= \tilde{I}'(\tau) p(b(\tau)). \end{aligned}$$

Where the $\tilde{\cdot}$ symbol denotes that the underlying variable is τ

(i.e. $I(t) = \tilde{I}(\tau(t))$ – the same for S).

Because we supposed w and p strictly increasing and differentiable (so their inverse), we have the following relation

$$\tilde{S}'(\tau) = \tilde{I}'(\tau) (p \circ w^{-1})(\tilde{I}'(\tau)).$$

Otherwise said, $\tilde{S}'(\tau)$ is totally driven by $\tilde{I}'(\tau)$.

For simplicity we will work without the symbol \sim as we will stick with the variable τ as measure of time.

2.2. Optimization problem

80 The goal of the algorithm is to spend the total budget S in a way that maximizes the purchased inventory. An equivalent problem is to minimize the total spent by fixing the number of impressions. From a dynamical standpoint, those problems are the same as they only differs in their boundary conditions.

From a mathematical point of view, using the relation

$$S'(\tau) = I'(\tau)(p \circ w^{-1})(I'(\tau))$$

it is easier to tackle the problem of spent-minimization (rather than inventory-
85 maximization)

Formally, we want to solve

$$\min S_{\tau(T)} = \int_0^{\tau(T)} S'(\tau) d\tau$$

by fixing the total purchased-inventory $I(\tau(T))$.

We will see that this is easily solved by variational calculus techniques (i.e. the Euler-Lagrange equation).

3. Solution

The problem defined above can be solved by means of the Euler-Lagrange equation, which in this case reads³:

$$\left(\frac{\partial}{\partial I} - \frac{d}{d\tau} \frac{\partial}{\partial \dot{I}} \right) [\dot{I}(p \circ w^{-1})(\dot{I})] = 0.$$

In this case, this means, it exists C , constant, such as

$$\frac{\partial}{\partial \dot{I}} [\dot{I}(p \circ w^{-1})(\dot{I})] = C$$

³In variational calculus, \dot{I} is the standard notation for $I'(\tau)$.

90 Which in particular implies that the quantity $I'(\tau)$ is constant throughout the entire period (this, because there is no dependency on other quantities in the last equation).

Moreover, as the evolution of the budget only depends on $I'(\tau)$, we state our result in the following way:

The optimal budget-pacing strategy satisfies

$$S'(t) \propto \mu(t).$$

95 4. Practical considerations

4.1. Choice of the optimal bidding-strategy

Our result states that the optimal budget-pacing strategy is such as we should spent proportional to the number of auction-requests we observe (and not linearly in time).

100 With that in mind, there is no need in practice to define the optimal bid in a explicit way as it can be controlled by a on-line feedback-control system trying to follow, for example, an optimal budget-pacing curve.

Moreover, using for example the observed daily patterns of the number of auction-requests (e.g. assuming that is stable across the days), the optimal
105 budget-pacing curve to follow could be defined beforehand.

4.2. The effect of floors prices

Along our reasoning we use the hypothesis the winrate is strictly increasing, and differentiable, as a function of the control (i.e. the level of bid at a given instant). This is not always the case as, in practice, some ad-exchanges set
110 (a priori, unknown) floor-prices creating discontinuities on the winrate function (until some level of bid the winrate is zero and, suddenly, a infinitesimal increase can make our algorithm purchase a considerable amount of inventory).

This phenomena does not impact the main result of the previous section (i.e. $S'(t) \propto \mu(t)$), however, it creates challenges for media-trading desks and DSP
115 when controlling the bid through a feedback-control system.

4.3. Time-homogeneity

The underlying hypothesis of this work is that buying inventory at two different moments $t, s \in [0, T]$ is equivalent in terms of likelihood of conversions (i.e. we consider segments and time-context as being homogeneous in quality).

120 In practice, we can expect that throughout the day, on a given campaign, there can be some interest to diffuse in some periods of the day more than in others. This is not a problem as we can always split the day in different phases such as inside each phase the *homogeneity hypothesis* is applicable.

5. Dealing with frequency-capping

125 In the set-up of advertising campaigns, practitioners like to define a frequency-capping factor, limiting the exposure (i.e. impressions) each user can see. We assume this parameter is global, that is, not changing from user to user.

From a statistical standpoint, the effects on capping are similar than in a urn model with a given quantity of balls (potential users) where, in the beginning, all balls are labeled by 0 (number of exposures) and, as they are drawn randomly
130 from the urn, they are replaced by a ball with a label with a number one unit higher (i.e. $n \mapsto n + 1$). If the maximum capping is $K \in \mathbb{N}$, then that means all the balls labeled with a number larger or equal than K go back into the urn without being replaced, and the more the process is repeated, the larger the
135 ratio of balls already labeled by K .

5.1. Effect on the dynamics

In our model, we will represent the capping effect as a function $\theta : \mathbb{R}_+ \mapsto [0, 1]$, depending on the number of running inventory-purchased (indeed, from the urn analogy, it is more natural think that the capping depends on the
140 number of impressions more than time) such as, $\theta(I(\tau))$ represents the number of users are still available to be exposed to advertising, after we already showed $I(\tau)$ impressions.

The new equations for I and S are:

$$\begin{aligned} I'(\tau) &= w(b(\tau))\theta(I(\tau)) \\ S'(\tau) &= I'(\tau)p(b(\tau)). \end{aligned}$$

Thus,

$$S'(\tau) = I'(\tau)(p \circ w^{-1}) \left(\frac{I'(\tau)}{\theta(I(\tau))} \right)$$

5.2. A (new) change of time

Proceeding similarly as in section II, we define a change of time by defining:

$$\nu = \int_0^\tau \theta(I(u)) du$$

Hence, we obtain

$$I'(\tau) = \tilde{I}'(\nu)\theta(I(\tau))$$

and

$$S'(\tau) = \tilde{S}'(\nu)\theta(I(\tau)).$$

Leading to the dynamics

$$\tilde{S}'(\nu) = \tilde{I}'(\nu)(p \circ w^{-1})(\tilde{I}'(\nu))$$

145 Where the symbol $\tilde{\cdot}$ denotes the fact that the underlying time is ν .

5.3. Solution

By applying the same variational approach than in section III, we obtain that at the optimum

$$S'(\tau) \propto \theta(I(\tau)),$$

or equivalently:

$$S'(t) \propto \theta(I(t))\mu(t).$$

Otherwise said, the optimal solution is spending proportional to the number of auction-requests that are not ‘banned’ yet by the capping.

In practice this is slightly more difficult as it demands to track users in order
150 to approximate the effects of the frequency-capping. These can also be obtained
by simulating the effects of the capping via urn-models.

6. Conclusion

In this article we solved the problem of optimal budget-pacing in real-time bidding by proposing a variational calculus approach. Our main result is that the optimal way to pace the budget is by spending proportional to the number of auction-requests we observe, and not linearly as practitioners do. We shown that this approach can be also applied in the case advertising campaigns have, in their set-up, frequency-capping factors limiting the exposure to advertising. Future directions of this research is how to include information about periods of the day where it is better to display advertising, the feedback-control system to choose the right bid in order to follow the optimal spent (this is specially challenging in presence of hard-floor prices) and, another possible direction, is to tackle this same problem by considering spent and inventory-purchased as stochastic process (that in this work are treated through their fluid-limits).

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