Block structured random matrices

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Motivation of Project

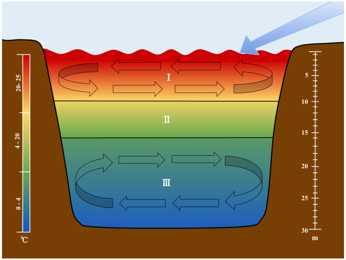


Figure: Lakes are stratified into three separate sections

- Normally, we have a system of nonlinear differential equations modelling an interacting ecosystem

$$\frac{dX_i(t)}{dt} = f_i(\boldsymbol{X}(t))$$

where $\boldsymbol{X}(t) = (X_1(t), \cdots, X_n(t))$, each represents population density of species i.

Motivation of Project

- By linearization, we determine the local stability around equilibrium point X^* using Jacobian matrix M

$$M_{ij} = \frac{\partial f_i}{\partial X_j}(\boldsymbol{X}(t))|_{\boldsymbol{X}^*}$$

by finding the rightmost eigenvalues of M. $(\Re(\lambda_1) < 0)$

Motivation of Project

- Problem: Different set of equations for different systems and biological system is usually large (i.e huge number of parameters)

Solution

- How to proceed?

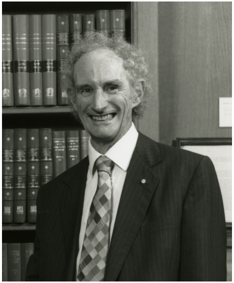


Figure: Robert May

- Idea: Take advantage of **large** ecosystem (S) and apply random matrix theory.

Detour: Random Matrix Theory

Definition (ESD)

Let M be a $S \times S$ non-Hermitian random matrix. Then, the Empirical Spectral Distribution (ESD) is defined as

$$\mu(\lambda) = \frac{1}{S} \sum_{i=1}^{S} \delta(\lambda - \Re\lambda_i)\delta(\lambda - \Im\lambda_i)$$

This function counts the number of eigenvalues in a given region.

Detour: Random Matrix Theory

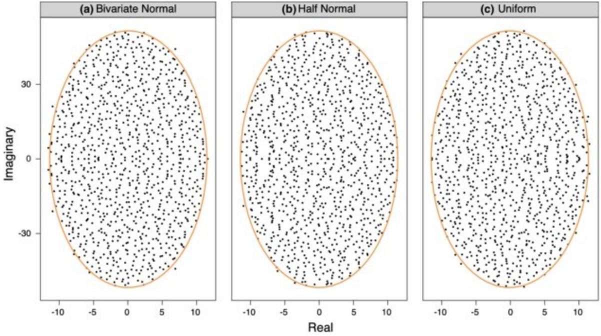
Theorem (Elliptic law)

Let M be a $S \times S$ nonsymmetric matrix, where the pairs of coefficients (M_{ij}, M_{ji}) are sampled independently from a bivariate distribution with mean $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and covariance matrix $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$. Then the ESD of $\frac{M}{\sqrt{S}}$ converges to the elliptic law as $S \to \infty$, i.e

$$\mu(\lambda) = \begin{cases} \frac{1}{\pi(1-\rho^2)} & \text{if } \frac{(\Re(\lambda))^2}{(1+\rho)^2} + \frac{(\Im(\lambda))^2}{(1-\rho)^2} \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Important tools to locate bulk of eigenvalues.

Detour: Random Matrix Theory



- Universality - not affected by the choice of distribution given same **mean and variance.**
- Self-averaging - position of eigenvalues converge to averaged location.
- Can study local stability regardless of system.

Modelling ecosystem using community matrix

$$M = \begin{bmatrix} \frac{\partial f_1}{\partial X_1} & \frac{\partial f_1}{\partial X_2} & \cdots & \frac{\partial f_1}{\partial X_S} \\ \frac{\partial f_2}{\partial X_1} & \cdots & \cdots & \frac{\partial f_2}{\partial X_S} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_S}{\partial X_1} & \cdots & \cdots & \frac{\partial f_S}{\partial X_S} \end{bmatrix}$$

Biology	Mathematics
Ecosystem not balanced	Non-zero mean for off-diagonal entries
Species affect each other	Symmetric entries not necessarily independent.
Species divides into groups	**Modular structure**

Example

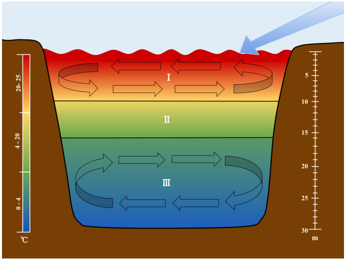


Figure: Lakes are stratified into three separate sections

S total number of species.
N number of groups of the ecosystem.
γ_1 - Group 1
γ_2 - Group 2
$\vdots$
γ_N - Group N

Modelling modular ecosystem

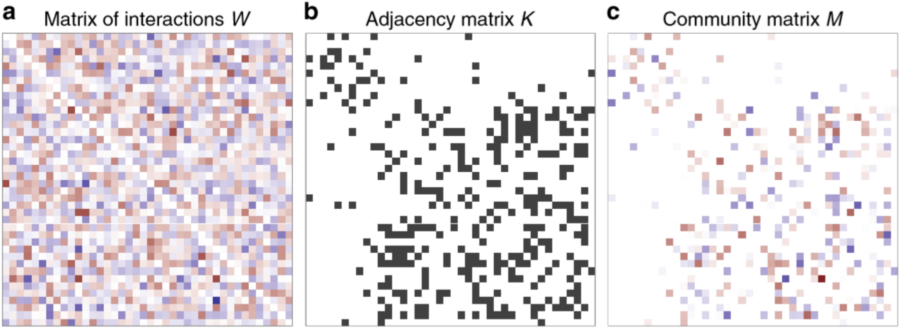
Interpretation (Deterministic model)

Build M with $M = W \circ K$ (W interaction strength matrix, K adjacency block-structured matrix), where

$$(W_{ij}, W_{ji}) \sim \mathcal{Z}\left(\begin{pmatrix}\mu \\ \mu\end{pmatrix}, \sigma^2 \begin{pmatrix}1 & \rho \\ \rho & 1\end{pmatrix}\right)$$

$$K_{ij} = \begin{cases} 1 & \text{if there is interaction between species } i \text{ and } j \\ 0 & \text{if there is no interaction between species } i \text{ and } j \end{cases}$$

Modelling modular ecosystem



Modelling modular ecosystem

Interpretation (Deterministic model)

K is a block-structured random matrices with N subsystems:

- *S species are partitioned into N groups given by $\gamma_1, \cdots, \gamma_N$ each of size $\alpha_1 S, \cdots, \alpha_N S$.*
- *Number of observed interactions within same(different) subsystem is denoted $L_w(L_b)$*
- *Species within same (different) subsystems interact with probability $C_w(C_b)$.*

The degree of modularity is controlled by Q

$$Q = \frac{L_w - \mathbb{E}(L_w)}{L_w + L_b}$$

The overall connectance C is the porpotion of non-zero entries of matrix K, i.e

$$C = \frac{(L_w + L_b)}{S(S-1)/2}$$

Modelling modular ecosystem

Interpretation (Deterministic model)

$$C_w = C\left(1 + \frac{Q}{\sum_{i=1}^{N} \alpha_i^2}\right)$$

$$C_b = \frac{C(1 - Q - \sum_{i=1}^{N} \alpha_i^2)}{2\sum_{i<j, i\neq j} \alpha_i \alpha_j}$$

in terms of Q, C *and* $[\alpha_1, \alpha_2, \cdots, \alpha_N]$

In short, matrix M is controlled only by $\mu, \sigma^2, \rho, [\alpha_1, \cdots, \alpha_N], C$ and Q that encodes the biological information.

Modelling modular ecosystem

Interpretation (Stochastic model)

Build M directly with $M_{ii} = 0$ and

$$(M_{ij}, M_{ji}) \sim \begin{cases} \mathcal{Z}_w\left(\begin{pmatrix}\mu_w \\ \mu_w\end{pmatrix}, \sigma_w^2\begin{pmatrix}1 & \rho_w \\ \rho_w & 1\end{pmatrix}\right) & \text{if } \gamma_i = \gamma_j \\ \mathcal{Z}_b\left(\begin{pmatrix}\mu_b \\ \mu_b\end{pmatrix}, \sigma_b^2\begin{pmatrix}1 & \rho_b \\ \rho_b & 1\end{pmatrix}\right) & \text{if } \gamma_i \neq \gamma_j \end{cases}$$

Modelling modular ecosystem

Why we need different interpretations ?
Connecting two different interpretations, we have following set of equations

$$
\begin{aligned}
\mu_w &= C_w \mu \\
\mu_b &= C_b \mu \\
\sigma_w^2 &= C_w(\sigma^2 + (1 - C_w)\mu^2) \\
\sigma_b^2 &= C_b(\sigma^2 + (1 - C_b)\mu^2) \\
\rho_w &= \frac{\rho\sigma^2 + (1 - C_w)\mu^2}{\sigma^2 + (1 - C_w)\mu^2} \\
\rho_b &= \frac{\rho\sigma^2 + (1 - C_b)\mu^2}{\sigma^2 + (1 - C_b)\mu^2}
\end{aligned}
$$

LHS: Useful and easier for derivations.
RHS: Control the variation of matrices using important parameters, namely $\mu, \sigma^2.\rho, [\alpha_1, \cdots, \alpha_N], C$ and Q.

Project

Problem Statement: How to determine the rightmost eigenvalues for a **large, block-structured** random matrices ?

1. Easiest method: Find it using computer by brute force.
2. Find it using properties of random matrix and a method from statistical physics.

Find rightmost eigenvalues of M

(O'Rourke & Renfrew) Let $M = A + B$ be a $S \times S$ random matrix. If A has low rank (few nonzero eigenvalues) relative to S and B is a large random matrix with eigenvalues following elliptic law, then

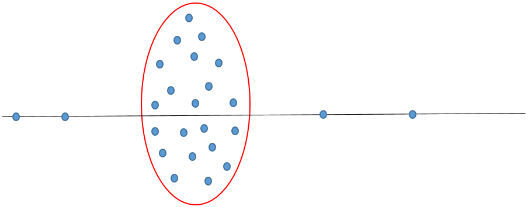


Figure: Location of eigenvalues of M

Find rightmost eigenvalues of M

Idea: Decompose community matrix M as $A + B$, where A is a block structure with

$$A_{ij} = \begin{cases} \mu_w & \text{if } \gamma_i = \gamma_j \\ \mu_b & \text{if } \gamma_i \neq \gamma_j \end{cases}$$

and $B = M - A$. With this and previous theorem, it suffices to :

1. show A has low rank and determine its eigenvalue.
2. show B follows elliptic law and determine its eigenvalue.

Finding eigenvalues of A

Result

Let A be the block structure matrix decomposed from M. Then there are $S - N$ zero eigenvalues and the N nonzero eigenvalues is the eigenvalues of the $N \times N$ matrix

$$A' = \begin{bmatrix} \alpha_1 S\mu_w & \alpha_2 S\mu_b & \alpha_3 S\mu_b & \cdots & \alpha_N S\mu_b \\ \alpha_1 S\mu_b & \alpha_2 S\mu_w & \alpha_3 S\mu_b & \cdots & \alpha_N S\mu_b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_1 S\mu_b & \alpha_2 S\mu_b & \alpha_3 S\mu_b & \cdots & \alpha_N S\mu_w \end{bmatrix}$$

Detour: Cavity Method

Definition

Let B be a $S \times S$ non-Hermitian random matrix. Then, the Empirical Spectral Distribution (ESD) is defined as

$$p_B(z) = \frac{1}{S} \sum_{i=1}^{S} \delta(z - \Re z_i)\delta(z - \Im z_i)$$

Reformulate the definition using cavity method.

Detour: Cavity Method

- **Step 1**: rewrite ESD as

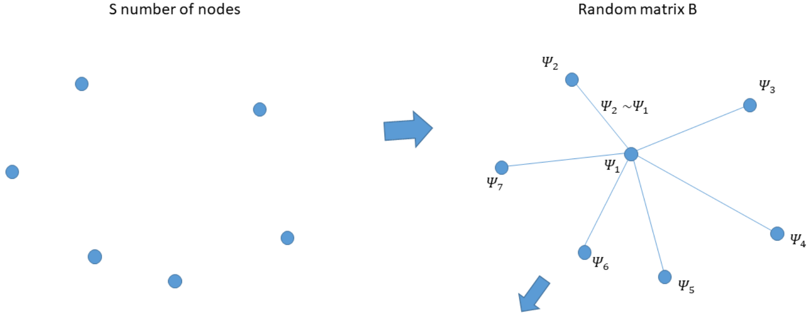
$$p_B(z,\bar{z}) = \frac{1}{\pi S} \lim_{\epsilon\to 0} \partial_{\bar{z}} \partial_z \ln \det H'$$

where H' is a $2S \times 2S$ Hermitian matrix. (Hermitization) given by

$$H' \equiv H'(z,\bar{z},\epsilon) = \begin{pmatrix} \epsilon 1_S & B - z1_S \\ B^\dagger - \bar{z}1_S & \epsilon 1_S \end{pmatrix}$$

Detour: Cavity Method

- **Step 2**: Translate the matrix B into a graph G_B with entries B_{ij} represents the interaction strength between particle ψ_i (node i) and ψ_j (node j), ∂_i is set of neighbors of i, k_i is number of neighbors of i.



$$P(\psi) = \frac{1}{\mathcal{Z}} e^{-\mathcal{H}(\psi, z, \bar{z}, \epsilon)}$$

Detour: Cavity Method

- Can write ESD as

$$p_B(z,\bar{z}) = \lim_{\epsilon\to 0} \frac{1}{\pi S} \sum_{i=1}^{S} -i\partial_{\bar{z}} \left\langle \psi_i^\dagger \sigma^+ \psi_i \right\rangle_P$$

where $\sigma^+ = \frac{\sigma_x + i\sigma_y}{2}$ and $\langle \ldots \rangle_P$ is the average over the probability distribution

$$P(\psi) = \frac{1}{\mathcal{Z}} e^{-\mathcal{H}(\psi,z,\bar{z},\epsilon)}$$

Remark:

$$\mathcal{H}(\psi,z,\bar{z},\epsilon) = \sum_{i}^{S} \psi_i^\dagger [\epsilon 1_S + i(x\sigma_x - y\sigma_y)]\psi_i - i \sum_{i,j=1}^{S} \psi_i^\dagger (B_{ij}^h \sigma_x - B_{ij}^s \sigma_y)\psi_j$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

and $B = B^h + iB^s$, B^h Hermitian, B^s skew-Hermitian.

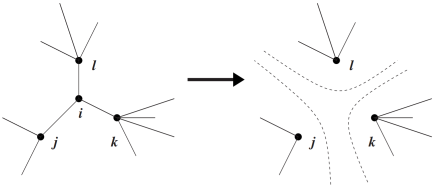
Detour: Cavity Method

- **Step 3**: Find marginal distribution $P_i(\psi_i)$ to evaluate $\left\langle \psi_i^\dagger \sigma^+ \psi_i \right\rangle_P$.

$$P_i(\psi_i) = \frac{e^{-\mathcal{H}_i}}{\mathcal{Z}_i} \int \mathcal{D}(\boldsymbol{\psi}_{\partial i}) e^{-\sum_{l \in \partial i} \mathcal{H}_{il}} P^{(i)}\left(\{\psi_l\}_{l \in \partial i}\right)$$

- If B is treelike, then

$$P^{(i)}(\{\psi_l\}_{l \in \partial i}) = \prod_{l \in \partial i} P_l^{(i)}(\psi_l)$$



Detour: Cavity Method

- Then, we have

$$P_i(\psi_i) = \frac{e^{-\mathcal{H}_i}}{\mathcal{Z}_i} \int \mathcal{D}(\boldsymbol{\psi}_{\partial i}) e^{-\sum_{l \in \partial i} \mathcal{H}_{il}} \prod_{l \in \partial i} P_l^{(i)}(\psi_l)$$

  Suffices to find each $P_l^{(i)}(\psi_l)$. Iterate these steps until it reaches the leaves of the graph.

- **Step 4**: This equation can be self-consistently solved by Gaussian

$$P_l^{(i)}(\psi_l) = \frac{1}{\mathcal{Z}_l^{(i)}} exp\left(-\psi_l^{\dagger}[C_l^{(i)}]^{-1}\psi_l\right)$$

$$P_i(\psi_i) = \frac{1}{\mathcal{Z}_i} exp(-\psi_i^{\dagger}[C_i]^{-1}\psi_i)$$

Detour: Cavity Method

- After solving, we have a set of recursive equations

$$C_i^{(j)} = [\epsilon 1_2 + i(x\sigma_x - y\sigma_y) + F(C_{\partial i \setminus j}^{(i)})]^{-1}$$

where

$$F(C_{\partial i \setminus j}^{(i)}) = \sum_{l \in \partial i \setminus j} \left(B_{il}^h \sigma_x - B_{il}^s \sigma_y \right) C_l^{(i)} \left(B_{li}^h \sigma_x - B_{li}^s \sigma_y \right)$$

- Similarly, we have

$$C_i = [\epsilon 1_2 + i(x\sigma_x - y\sigma_y) + F(C_{\partial i}^{(i)})]^{-1}$$

- Write

$$C_i \equiv \begin{pmatrix} a_i & i\bar{b}_i \\ ib_i & d_i \end{pmatrix}$$

where $a_i, d_i \in \mathbb{R}^+$, $b_i \in \mathbb{C}$

Detour: Cavity Method

Result (ESD using cavity method)

Let B be a treelike, $S \times S$ *non-Hermitian random matrix.*

$$p_B(z, \bar{z}) = -\frac{1}{\pi S} \lim_{\epsilon \to 0} \sum_{i=1}^{S} \partial_{\bar{z}} b_i(z, \bar{z}, \epsilon)$$

Finding eigenvalues of B

Definition

Let B be a $S \times S$ non-Hermitian random matrix. Then, the Empirical Spectral Distribution (ESD) is defined as

$$\mu(\lambda) = \frac{1}{S} \sum_{i=1}^{S} \delta(\lambda - \Re\lambda_i)\delta(\lambda - \Im\lambda_i)$$

Definition (Resolvent)

The resolvent is a function

$$\mathcal{G}(\boldsymbol{q}) = \frac{1}{S} \sum_{i} (z_i - \boldsymbol{q})^{-1}$$

where $\boldsymbol{q}$ is a quaternion, i.e

$$\boldsymbol{q} = z + wj = \begin{pmatrix} z & w \\ \bar{w} & \bar{z} \end{pmatrix}$$

Finding eigenvalues of B

Result (ESD using resolvent function)

$$p_B(x,y) = -\frac{1}{\pi} \lim_{\epsilon \to 0^+} \Re \left(\frac{\partial}{\partial \bar{z}} \mathcal{G}(z + \epsilon j) \right) |_{z=x+iy}$$

Can also write

$$\mathcal{G}(\boldsymbol{q}) = \frac{1}{S} \sum_i \boldsymbol{G}_{ii}(\boldsymbol{q})$$

where $\boldsymbol{G}_{ii}(\boldsymbol{q}) = (\boldsymbol{B}_{ii} - \boldsymbol{q})^{-1}$ and $\boldsymbol{B}_{ij} = \begin{pmatrix} B_{ij} & 0 \\ 0 & B_{ji} \end{pmatrix}$ because $tr(B) = \sum_i z_i$. Then we have

$$p_B(x,y) = -\frac{1}{\pi S} \lim_{\epsilon \to 0^+} \Re \left(\frac{\partial}{\partial \bar{z}} \sum_i \boldsymbol{G}_{ii}(\boldsymbol{z} + \epsilon j) \right) |_{z=x+iy}$$

Finding eigenvalues of B

Combining

$$p_B(x,y) = -\frac{1}{\pi S}\lim_{\epsilon\to 0^+}\Re\left(\frac{\partial}{\partial\bar{z}}\sum_{i=1}^{S}\boldsymbol{G}_{ii}(z+\epsilon j)\right)|_{z=x+iy}$$

$$p_B(z,\bar{z}) = -\frac{1}{\pi S}\lim_{\epsilon\to 0}\sum_{i=1}^{S}\partial_{\bar{z}}b_i(z,\bar{z},\epsilon)$$

$$C_i \equiv \begin{pmatrix} a_i & i\bar{b}_i \\ ib_i & d_i \end{pmatrix}$$

We have the relation

$$\boldsymbol{G}_{ii} = -\boldsymbol{j}C_i\boldsymbol{i}$$

Finding eigenvalues of B

Using

$$C_i = [\epsilon 1_2 + i(x\sigma_x - y\sigma_y) + F(C^{(i)}_{\partial i})]^{-1}$$

we obtain

$$\boldsymbol{G}_{ii}(\boldsymbol{q}) = -\left(\boldsymbol{q} + \sum_{j\neq i} \boldsymbol{B}_{ij}\boldsymbol{G}^{(i)}_{jj}\boldsymbol{B}_{ji}\right)^{-1}$$

where $\boldsymbol{G}^{(i)}$ is the resolvent of the matrix obtained by removing row and column i from B. (removing nodes i).

Finding eigenvalues of B

- Putting all species i (node) into groups $\gamma = (\gamma_1, \cdots, \gamma_N)$ and letting the sum in RHS converges to average behaviour by law of large number, we have

$$\boldsymbol{G}_{\gamma_i}(\boldsymbol{q}) = -\left(\boldsymbol{q} + \mathbb{E}\left(\sum_j \boldsymbol{B}_{ij}\boldsymbol{G}_{\gamma_j}\boldsymbol{B}_{ji}\right)\right)^{-1}$$

for $\gamma_i \in (1, \cdots, N)$.

Finding eigenvalues of B

By some algebraic manipulation,

Result

$$\boldsymbol{G_m}(\boldsymbol{q}) = -\left(\boldsymbol{q} + \alpha_m \Sigma_{\boldsymbol{w}} \circ \boldsymbol{G_m} + \sum_{i \neq m} \alpha_i \Sigma_b \circ \boldsymbol{G}_i\right)^{-1} \tag{1}$$

where $m \in (1, \cdots, N)$, $\Sigma_{\boldsymbol{w}} = S\sigma_w^2 \begin{pmatrix} \rho_w & 1 \\ 1 & \rho_w \end{pmatrix}$ *and* $\Sigma_{\boldsymbol{b}} = S\sigma_b^2 \begin{pmatrix} \rho_b & 1 \\ 1 & \rho_b \end{pmatrix}$.

Furthermore,

$$\mathcal{G}(\boldsymbol{q}) = \sum_{m=1}^{N} \alpha_m \boldsymbol{G_m}(\boldsymbol{q})$$

Finding the eigenvalues of B

Result

$$p_B(z, \bar{z}, \epsilon) = -\frac{1}{\pi} \lim_{\epsilon \to 0} \Re \left(\frac{\partial}{\partial \bar{z}} \sum_{m=1}^{N} \alpha_m \boldsymbol{G_m}(\boldsymbol{q}) \right)$$

If we let $\boldsymbol{G_m}(\boldsymbol{q}) = r_m + \beta_m j$, *then as* $\epsilon \to 0$, *we have the ESD given by*

$$p_B(z, \bar{z}) = -\frac{1}{\pi} \sum_{m=1}^{N} \alpha_m \Re \frac{\partial r_m}{\partial \bar{z}} \tag{2}$$

Equation (1) and (2) is the core result to determine location of eigenvalues of B.

Equal-sized subsystem

Result

If $\alpha_1, \alpha_2, \cdots, \alpha_N = \frac{1}{N}$, *we have*

$$p(z, \bar{z}) = \begin{cases} \frac{1}{\pi S \tilde{\sigma}^2 (1-\tilde{\rho})^2} & \text{if } \frac{x^2}{S(1+\tilde{\rho})^2 \tilde{\sigma}^2} + \frac{y^2}{S(1-\tilde{\rho})^2 \tilde{\sigma}^2} < 1 \\ 0 & \text{otherwise} \end{cases}$$

where

$$\tilde{\rho} = \frac{\rho A \sigma^2 + B\mu^2}{A\sigma^2 + B\mu^2}$$

$$\tilde{\sigma}^2 = -\frac{C}{N}(A\sigma^2 + B\mu^2)$$

$$A = -4Q + N(2Q - 1)$$

$$B = 4Q(2C - 1) + N(2Q - 1)(2CQ - C + 1)$$

Verification

Example

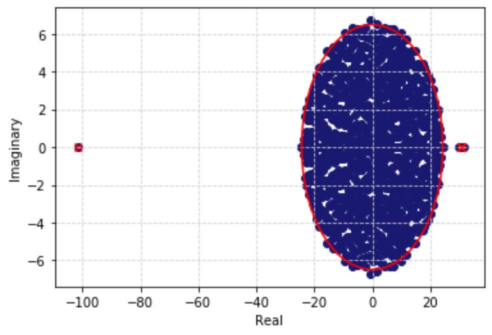


Figure: Five equal-sized subsystem community matrix M sampled from normal bivariate distribution with $S = 1000$, $C = 0.2$, $\mu = -0.5$, $\sigma = 1$, $\rho = 0.5$, $Q = -0.2$, $[\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5] = [0.2, 0.2, 0.2, 0.2, 0.2]$

.

Advantage of this approach

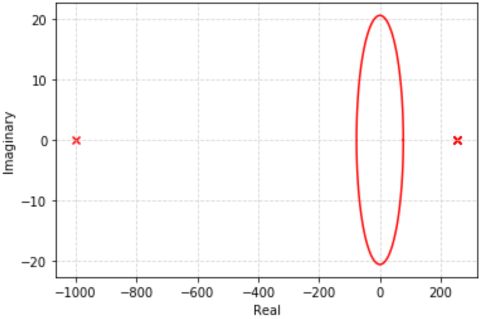


Figure: Five equal-sized subsystem community matrix M sampled from normal bivariate distribution with $S = 10000$, $C = 0.2$, $\mu = -0.5$, $\sigma = 1$, $\rho = 0.5$, $Q = -0.2$, $[\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5] = [0.2, 0.2, 0.2, 0.2, 0.2]$. The rightmost eigenvalue is predicted to be 255.5

Future study

How about unequal-sized subsystem ?

$$\boldsymbol{G_m}(\boldsymbol{q}) = -\left(\boldsymbol{q} + \alpha_m \Sigma_{\boldsymbol{w}} \circ \boldsymbol{G_m} + \sum_{i \neq m} \alpha_i \Sigma_b \circ \boldsymbol{G}_i\right)^{-1}$$

$$p(z, \bar{z}) = -\frac{1}{\pi} \sum_{m=1}^{N} \alpha_m \Re \frac{\partial r_m}{\partial \bar{z}}$$

Possible approach:

- Approximate unequal-sized subsystem with equal-sized subsystem using a loss function.