CS5321 Numerical Optimization Homework 3

Due 1/6/2023

1. (20%) In the trust region method (unit 3), we need to solve the model problem m_k

$$\min_{\vec{p}} m_k(\vec{p}) = f_k + \vec{g}_k^T \vec{p} + \frac{1}{2}\vec{p}^{\,T} B_k \vec{p}.$$
$$\text{s.t.} \|\vec{p}\| \leq \Delta$$

Show that $\vec{p}*$ is the optimal solution if and only if it satisfies

$$(B_k + \lambda I)\vec{p}* = -\vec{g}$$
$$\lambda(\Delta - \|\vec{p}*\|) = 0$$

where $B_k + \lambda I$ is positive definite. (Hint: using KKT conditions.)

2. (15%) Prove that for the matrix $\begin{bmatrix} G & A^T \\ A & 0 \end{bmatrix}$, if A has full row-rank and the reduced Hessian $Z^T G Z$ is positive definite, where span$\{Z\}$ is the null space of span$\{A^T\}$ then the matrix is nonsingular. (You may reference Lemma 16.1 in the textbook.)

3. (30%) Consider the problem

$$\begin{array}{ll} \min\limits_{x_1, x_2} & (x_1 - 3)^2 + 10x_2^2 \\ \text{s.t.} & x_1^2 + x_2^2 - 1 \leq 0 \end{array} \tag{1}$$

   (a) Write down the KKT conditions for (1).
   (b) Solve the KKT conditions and find the optimal solutions, including the Lagrangian parameters.
   (c) Compute the reduced Hessian and check the second order conditions for the solution.

4. (20%) Consider the following constrained optimization problem

$$\begin{array}{ll} \min\limits_{x_1, x_2} & x_1^3 + 2x_2^2 \\ \text{s.t.} & x_1^2 + x_2^2 - 1 = 0 \end{array}$$

   (a) What is the optimal solution and the optimal Lagrangian multiplier?
   (b) Formulate this problem to the equation of augmented Lagrangian method, and derive the gradient of Lagrangian.
   (c) Let $\rho_0 = -1, \mu_0 = 1$. What is x_1 if it is solved by the augmented Lagrangian problem.

(d) To make the solution of augmented Lagrangian method exact, what is the minimum ρ should be?

5. (15%) Consider the following constrained optimization problem

$$
\begin{aligned}
\min_{x_1, x_2} \quad & -3x_1 + x_2 \\
\text{s.t.} \quad & 2x_1 + x_2 \leq 20 \\
& x_1 + 2x_2 \leq 16 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

Formulate this problem to the equation of the interior point method, and derive the gradient and Jacobian.