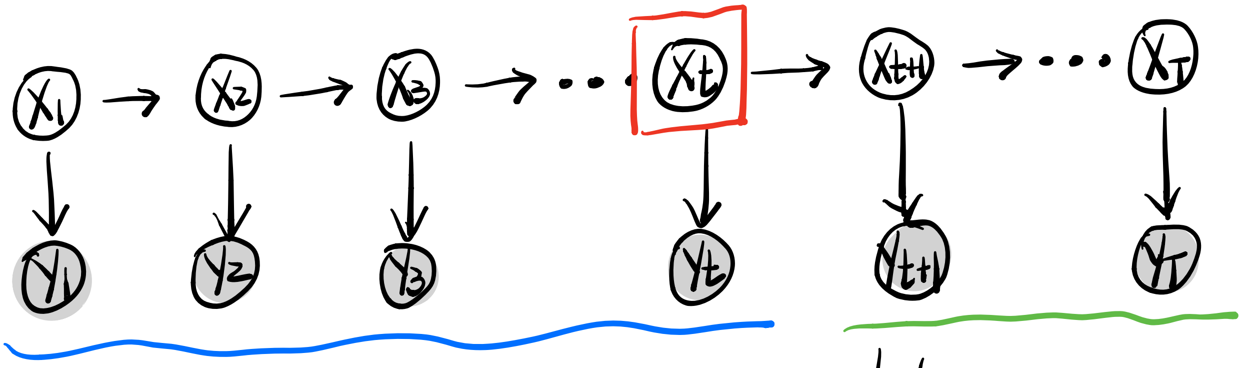

HMM

inference in hidden markov model:

forward - backward algorithm

* → full posterior $p(X_t \mid Y_{1:T})$

W2D3, T2, T3

forward pass $\quad a(t) = \underline{P(Y_{1:t}, X_t) \propto P(X_t \mid Y_{1:t})}$

- recursive formula

$$a(1) = P(Y_1 \mid X_1)\, P(X_1)$$

$$a(t) = \sum_{X_{t-1}} \underbrace{P(Y_t \mid X_t)}_{\text{likelihood}}\ \underbrace{P(X_t \mid X_{t-1})}_{\text{dynamics}}\ \underbrace{P(X_{t-1}, Y_{1:t-1})}_{\substack{a(t-1) \\ \text{"old posterior"}}}$$

proof:

$$
\begin{aligned}
P(Y_{1:t}, X_t) &= P(Y_{1:t} \mid X_t)\, P(X_t) && \text{chain rule} \\
&= P(Y_t \mid X_t)\, P(Y_{1:t-1} \mid X_t)\, P(X_t) && \text{cond. independent} \\
&= P(Y_t \mid X_t) \sum_{X_{t-1}} P(Y_{1:t-1}, X_t, X_{t-1}) \\
&= P(Y_t \mid X_t) \sum_{X_{t-1}} P(X_t \mid X_{t-1})\, P(X_{t-1}, Y_{1:t-1})
\end{aligned}
$$

backward pass

$$b(t) = P(Y_{t+1:T} \mid X_t)$$

- recursive formula

$$b(T) = 1$$

for $t < T$

$$b(t) = \sum_{S_{t+1}} \underbrace{P(y_{t+1} \mid S_{t+1}) \overbrace{P(y_{t+2:T} \mid S_{t+1})}^{b(t+1)} P(S_{t+1} \mid S_t)}$$

↓ cond. independent

proof: $\sum_{S_{t+1}} P(y_{t+1:T} \mid S_{t+1}) P(S_{t+1} \mid S_t)$

$$\downarrow$$

$$P(Y_{t+1:T} \mid S_t)$$

---

posterior

$$P(X_t \mid Y_{1:T}) \propto \underbrace{P(Y_{1:t}, X_t)}_{a(t)} \underbrace{P(Y_{t+1:T} \mid X_t)}_{b(t)}$$

$$= P(Y_{1:t}, X_t) P(Y_{t+1:T} \mid X_t, Y_{1:t})$$

$$= P(Y_{1:T}, X_t)$$

normalized posterior

$$P(X_t \mid Y_{1:T}) = a(t)\, b(t) / P(Y_{1:T})$$

$$= a(t)\, b(t) / \sum_{X_T} \underbrace{P(Y_{1:T}, X_T)}_{a(T)}$$