W2D2, T1, T2

- the eigenvalue / eigenvector determines the behavior of the dynamical system

- discrete map: $x_{t+1} = Ax_t$

$$x_k = A^k x_0 \qquad A = Q\Lambda Q^{-1}$$

$$\underbrace{(A \cdot A \cdot A \cdot \ldots A)}_{k \text{ terms}} = Q\Lambda Q^{-1} Q\Lambda Q^{-1} \ldots Q\Lambda Q^{-1}$$

$$= Q\Lambda^k Q^{-1}$$

$$= Q \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_N^k \end{bmatrix} Q^{-1}$$

$$\lambda_n = r_n e^{i\theta} \longrightarrow \lambda_n^k = r^k e^{ik\theta}$$

$$\begin{cases} r > 1 & r^k \to +\infty \\ r = 1 & r^k = 1 \\ r < 1 & r^k \to 0 \end{cases}$$

- example: markov model stationary distribution is associated with eigvec with $r = 1$

• continuous time dynamics

$$\frac{dx}{dt} = Ax \qquad \phi(t, x_0) = \underbrace{\exp\{At\}}_{\text{matrix exponential}} \vec{x}_0$$

$$= \sum_{n=0}^{\infty} \frac{(At)^n}{n!} \cdot \vec{x}_0$$

$$A = Q \Lambda Q^{-1} = Q \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} Q^{-1}$$

$$= \underbrace{\sum_{n=0}^{\infty} Q \frac{\begin{bmatrix} \lambda_1 t & \\ & \lambda_2 t \end{bmatrix}^n}{n!} Q^{-1} \cdot x_0}_{\text{sum of series}}$$

$$\Downarrow$$

$$Q \begin{bmatrix} e^{\lambda_1 t} & \\ & e^{\lambda_2 t} \end{bmatrix} Q^{-1} \cdot x_0$$

initial state

$$\Rightarrow e^{\lambda_1 t} = e^{(a+ib)t} = e^{at} e^{ibt}$$

$$= e^{at} (\cos bt + i \sin bt)$$

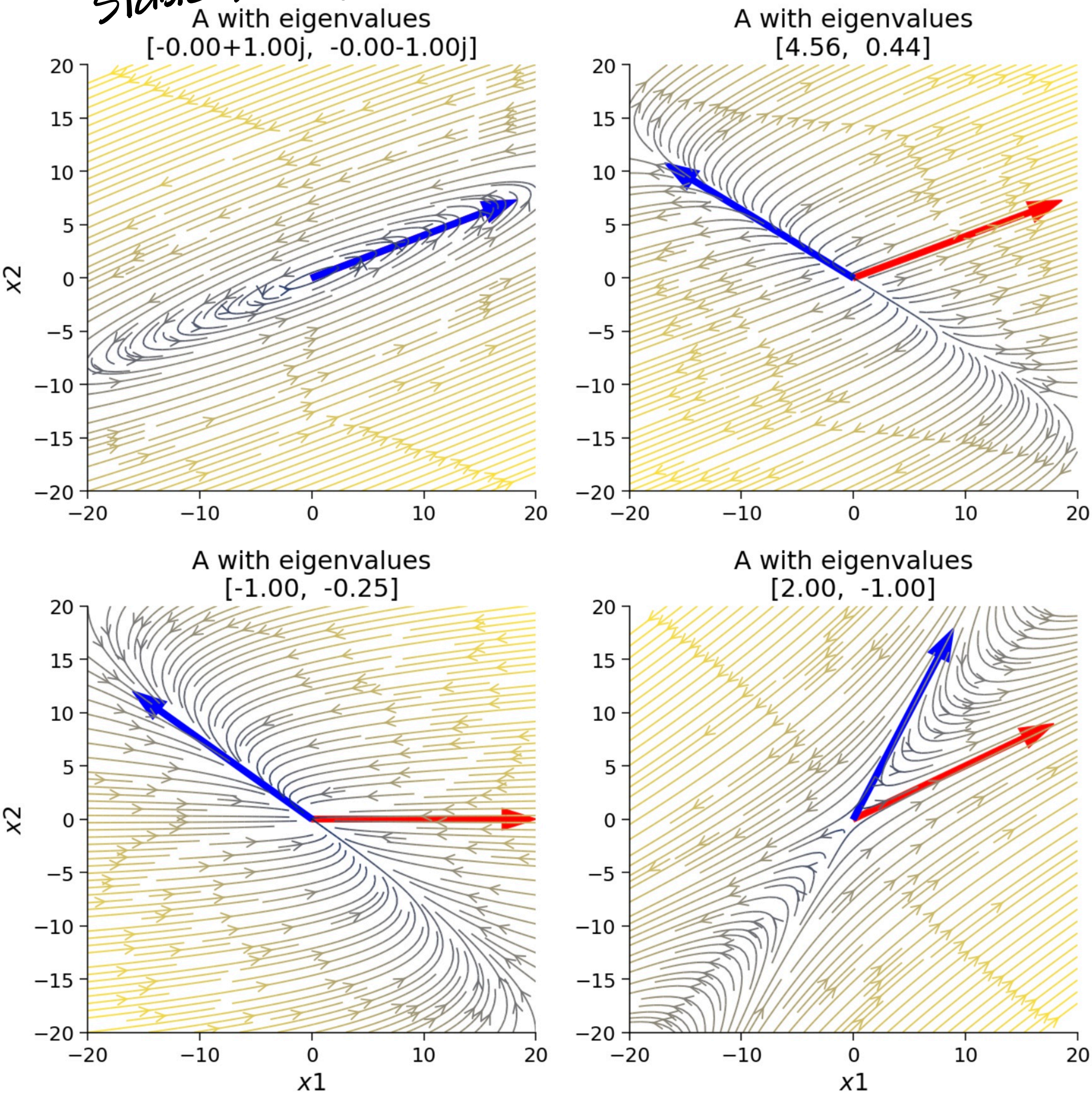
→ a determines convergence

→ b determines oscillation and its frequency along eigenvector associated with λ_i

$a>0$ divergence, $a=0$ stable, $a<0$ convergence towards 0

- use streamplot to understand the dynamics

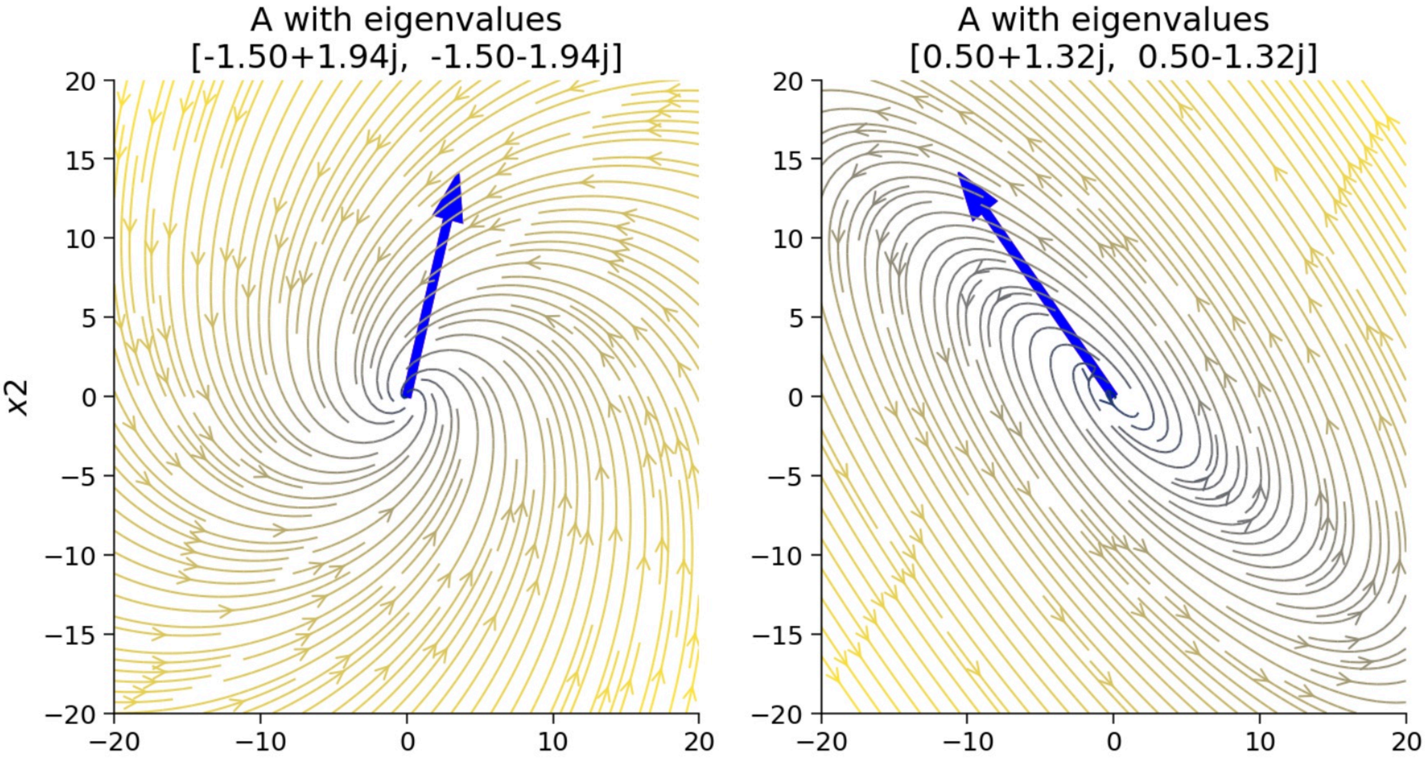
stable with oscillation — divergence



convergence → 0

divergence along one vect along another

convergence along ...



Analysis of non-linear system with fixed point by linearization of dynamics around fixed point *

example: perturbation around fixed point

Similar to what we did in the linear system above, in order to determine the stability of a fixed point r^* of the excitatory population dynamics, we perturb Equation (1) around r^* by ϵ, i.e. $r = r^* + \epsilon$. We can plug in Equation (1) and obtain the equation determining the time evolution of the perturbation $\epsilon(t)$:

$$\tau \frac{d\epsilon}{dt} \approx -\epsilon + wF'(w \cdot r^* + I_{\text{ext}}; a, \theta)\epsilon$$

where $F'(\cdot)$ is the derivative of the transfer function $F(\cdot)$. We can rewrite the above equation as:

$$\frac{d\epsilon}{dt} \approx \frac{\epsilon}{\tau}[-1 + wF'(w \cdot r^* + I_{\text{ext}}; a, \theta)]$$

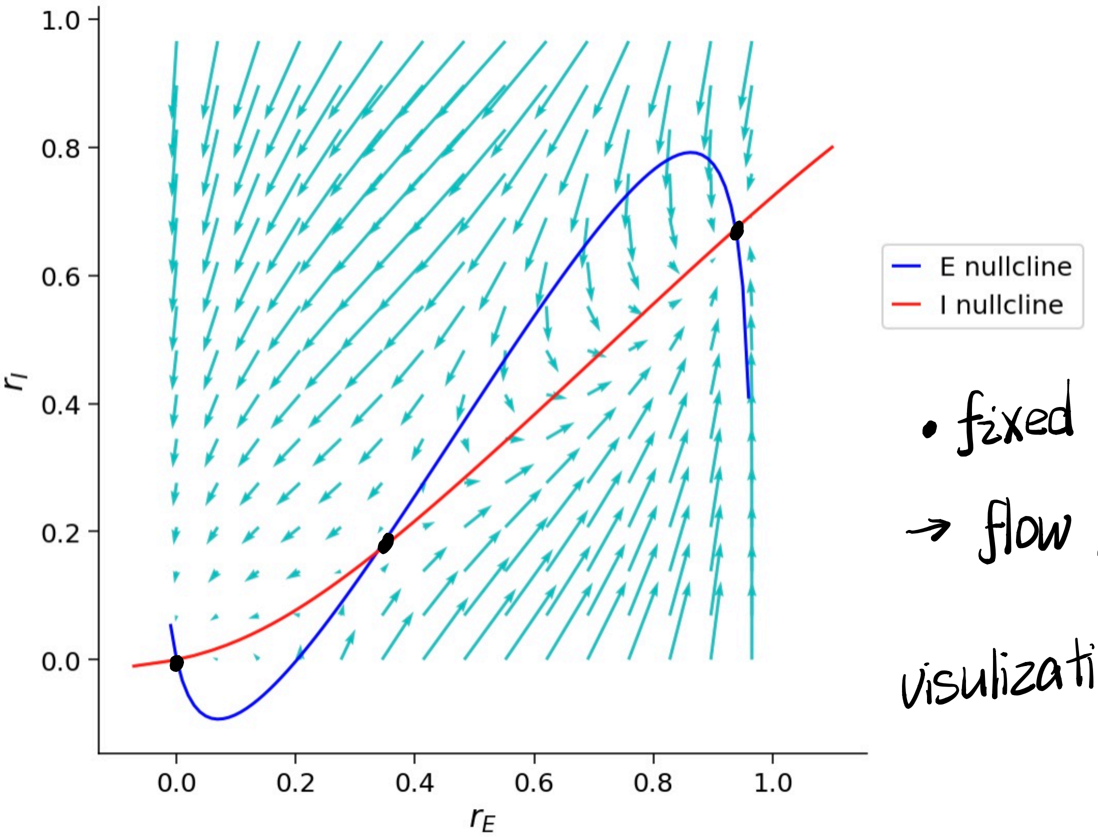
That is, as in the linear system above, the value of

$$\lambda = [-1 + wF'(w \cdot r^* + I_{\text{ext}}; a, \theta)]/\tau \qquad (4)$$

determines whether the perturbation will grow or decay to zero, i.e., λ defines the stability of the fixed point. This value is called the **eigenvalue** of the dynamical system.

• W3D2

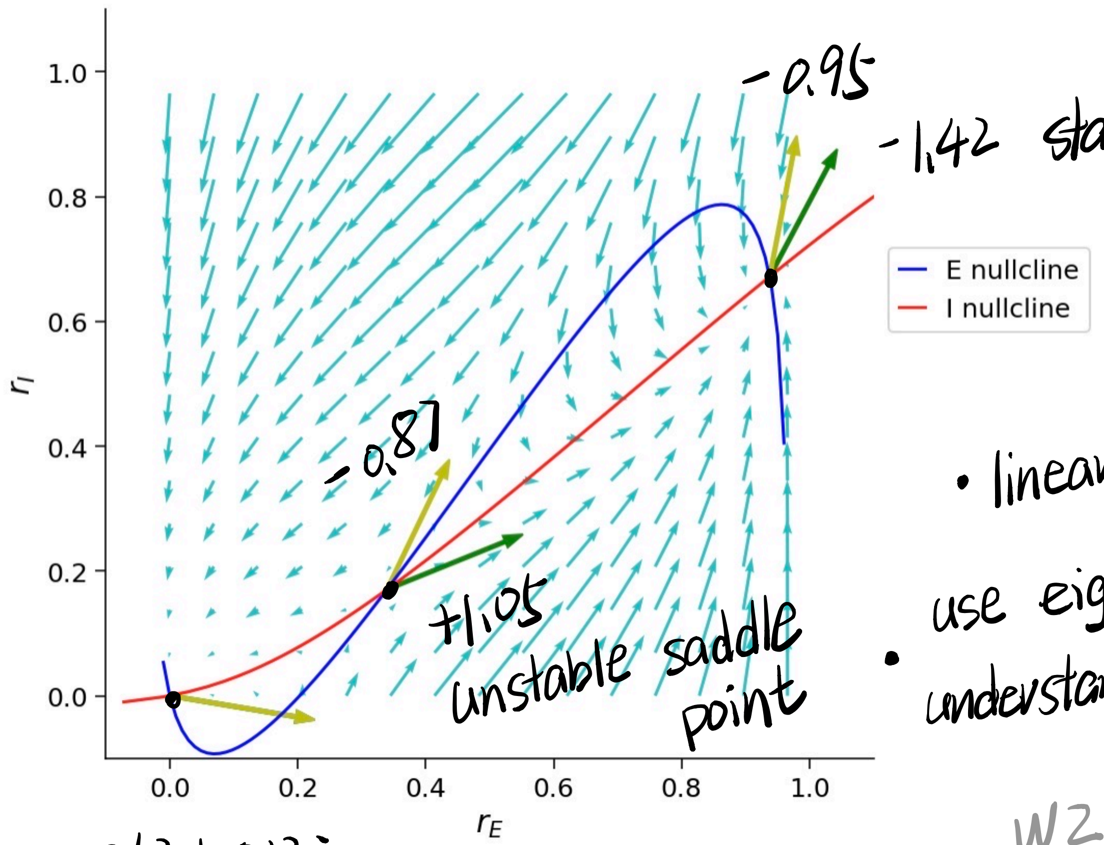
e.g., 2D system with 3 fixed point



• fixed point

→ flow field

visulization of dynamics



−1.42 stable fixed point

• linearization

• use eigval/vec for understanding the stability

−0.62 ± 0.13j
stable fixed point

W2D2, linear dynamical system

fun example of oscillations & limited cycle