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Option valuation with long-run and short-run volatility components ☆

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ABSTRACT

This paper presents a new model for the valuation of European options, in which the volatility of returns consists of two components. One is a long-run component and can be modeled as fully persistent. The other is short-run and has a zero mean. Our model can be viewed as an affine version of Engle and Lee [1999. A permanent and transitory component model of stock return volatility. In: Engle, R., White, H. (Eds.), Cointegration, Causality, and Forecasting: A Festschrift in Honor of Clive W.J. Granger. Oxford University Press, New York, pp. 475–497], allowing for easy valuation of European options. The model substantially outperforms a benchmark single-component volatility model that is well established in the literature, and it fits options better than a model that combines conditional heteroskedasticity and Poisson–normal jumps. The component model's superior performance is partly due to its improved ability to model the smirk and the path of spot volatility, but its most distinctive feature is its ability to model the volatility term structure. This feature enables the component model to jointly model long-maturity and short-maturity options.

1. Introduction

A consensus exists in the equity options literature that combining time-variation in the conditional variance of asset returns (Engle, 1982; Bollerslev, 1986) with a leverage effect (Black, 1976) constitutes a potential solution to well-known biases associated with the Black and Scholes (1973) model, such as the implied volatility smirk. These asymmetric dynamic volatility models generate negative skewness in the distribution of asset returns, which in turn generates higher prices for out-of-the-money put options as compared with the Black and Scholes formula. In the continuous-time option valuation literature, the Heston (1993) model addresses some of these biases. This model contains a leverage effect as well as stochastic volatility. In the discrete-time literature, the NGARCH(1,1) option valuation model proposed by Duan

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(1995) contains time-variation in the conditional variance as well as a leverage effect. The model by Heston and Nandi (2000) is closely related to Duan's model.

Many existing empirical studies confirm the importance of time-varying volatility, the leverage effect and negative skewness in continuous-time and discrete-time setups, using parametric (see Chernov and Ghysels, 2000; Jones, 2003) as well as nonparametric (see Ait-Sahalia and Lo, 1998) techniques. However, while these models help explain the biases of the Black and Scholes model in a qualitative sense, they come up short in a quantitative sense. Using parameters estimated from returns or options data, these models reduce the biases of the Black and Scholes model, but the magnitude of the effects is insufficient to completely resolve the biases. The resulting pricing errors have the same sign as the Black and Scholes pricing errors but are smaller in magnitude. We therefore need models that possess the same qualitative features as the models in Heston (1993) and Duan (1995) as well as stronger quantitative effects. These models need to generate more flexible skewness and volatility of volatility dynamics to fit observed option prices. Existing studies attempt to address this by combining stochastic volatility specifications with jump processes (see, for example, Bakshi, Cao, and Chen, 1997; Bates, 2000; Broadie, Chernov, and Johannes, 2007) or by using non-normal innovations in heteroskedastic models as in Christoffersen, Heston, and Jacobs (2006).

The shortcomings of existing models in modeling the moneyness dimension are compounded by their shortcomings in modeling the term structure of volatility and the path of spot volatility. It has been observed using a variety of diagnostics that fitting the dynamics of return volatility is difficult using a benchmark model such as a GARCH(1,1). A similar observation applies to stochastic volatility models such as Heston (1993). The main problem is that volatility autocorrelations are too high at longer lags to be explained by a GARCH(1,1), unless the process is extremely persistent. This extreme persistence could impact negatively on other aspects of option valuation, such as the valuation of short-maturity options.

In fact, it has been observed in the literature that volatility could be better modeled using a fractionally integrated process, as in Baillie, Bollerslev, and Mikkelsen (1996), rather than an exponential GARCH process. Andersen, Bollerslev, Diebold, and Labys (2003) confirm this finding using realized volatility. Bollerslev and Mikkelsen (1996, 1999) and Comte, Coutin, and Renault (2001) investigate and discuss some of the implications of long memory for option valuation. Using fractional integration models for option valuation is somewhat cumbersome. Optimization is time-intensive and certain ad hoc choices have to be made regarding implementation.

This paper attempts to remedy remaining option biases by modeling richer volatility dynamics. We use a model that is relatively easy to implement and that captures the stylized facts addressed by long-memory models at horizons relevant for option valuation. The model builds on Heston and Nandi (2000) and Engle and Lee (1999). The volatility of returns consists of two components. One of these components is a long-run component, and it can be modeled as (fully) persistent. The other component is short-run and mean zero. We study two models: one in which the long-run component is constrained to be fully persistent and another in which it is not. We refer to these models as the persistent component model and the component model, respectively. These models are able to generate autocorrelations that are richer than those of a GARCH(1,1) model while using just a few additional parameters.

Unobserved component or factor models are popular in the finance literature. See Fama and French (1988), Poterba and Summers (1988), and Summers (1986) for applications to stock prices. In the option pricing literature, Bates (2000) and Taylor and Xu (1994) investigate two-factor stochastic volatility models. Duffie, Pan, and Singleton (2000) provide a general continuous-time framework for the valuation of contingent claims using multifactor affine models. Eraker (2004) suggests the usefulness of a multifactor approach based on his empirical results. Alizadeh, Brandt, and Diebold (2002) uncover two factors in stochastic volatility models of exchange rates using range-based estimation. Bollerslev and Zhou (2002), Brandt and Jones (2006), Chacko and Viceira (2003), Chernov, Gallant, Ghysels, and Tauchen (2003), and Maheu (2005) also find that two-factor stochastic volatility models outperform single factor models when modeling daily asset return volatility. Adrian and Rosenberg (2008) investigate the relevance of a two-component volatility model for pricing the cross section of stock returns. Unobserved component models are also popular in the term structure literature, although in this literature the models are more commonly referred to as multifactor models. See for example Dai and Singleton (2000), Duffee (1999), Duffie and Singleton (1999), and Pearson and Sun (1994). Interesting parallels exist between our approach and results and stylized facts in the term structure literature. In the term structure literature, short-run fluctuations are customarily modeled around a time-varying long-run mean of the short rate. In our framework we model short-run fluctuations around a time-varying long-run volatility.

Dynamic factor and component models can be implemented in continuous time (see Duffie, Pan and Singleton, 2000) or discrete time. We choose a discrete-time approach because of the ease of implementation. In particular, our model is related to the GARCH class of processes, and therefore volatility filtering and forecasting are relatively straightforward, which is critically important for option valuation. French, Schwert, and Stambaugh (1987) provide one of the first applications of GARCH to stock returns. Because the filtering problem is extremely simple in the GARCH framework, we are able to analyze an extensive option sample. An additional advantage of our model is parsimony: The most general model we investigate has seven parameters. We speculate that parsimony could help our model's out-of-sample performance.

The component model is a generalization of the GARCH(1,1) model, and because its implementation uses similar techniques, the GARCH(1,1) is a natural benchmark. Moreover, Heston and Nandi (2000) find that the GARCH(1,1) slightly outperforms the ad hoc implied volatility benchmark model in Dumas, Fleming, and Whaley (1998). As there is evidence that Poisson–normal jump processes can alleviate some of the biases associated with the Heston (1993) model and the GARCH(1,1) model, we also include a GARCH(1,1) model augmented with Poisson–normal jumps in our analysis.

We provide two different analyses of the component model. We first estimate the physical model parameters by maximum likelihood estimation (MLE) on historical Standard & Poor's (S&P) 500 returns for 1962 through 2001. We compare the component model and the persistent component model with the GARCH(1,1) benchmark as well as with the more general GARCH(1,1)-Jump model. Based on the log-likelihood criterion, the GARCH (1,1)-Jump model performs the best, followed by the component model, the persistent component model, and the GARCH(1,1) model. However, when we compare the models based on option root mean squared error (RMSE) fit using MLE parameters, the best fit is obtained using the component and persistent component models, followed by the GARCH(1,1)-Jump model. The GARCH(1,1) model is again the worst performer. The RMSE in the component and persistent component models is about 23.7% below that of the GARCH(1,1).

We also use the MLE parameters to emphasize differences in important model features, such as the conditional volatility of variance, the correlation between returns and conditional variance, the term structure of conditional skewness and kurtosis, the volatility smirk, and the volatility term structure. The improvement in the model's performance is primarily due to its richer dynamics, which result in different modeling of the term structure and enable the component model to capture patterns in long-maturity as well as short-maturity options.

In a second empirical investigation, we estimate the models using options data, while filtering the latent volatility from the underlying returns data. The performance of the component model is impressive when compared with a benchmark GARCH(1,1) model. When using option data in estimation, the RMSE of the component model is between 10.3% and 22.7% lower than that of the benchmark GARCH model in-sample and between 20.8% and 23.5% lower out-of-sample. Our out-of-sample results strongly suggest that the in-sample results are not simply due to spurious overfitting. The persistent component model performs better than the benchmark GARCH(1,1) model. But, in contrast to the results obtained using MLE parameters, it is clearly inferior to the component model both in- and out-of-sample.

The paper proceeds as follows. Section 2 introduces the model. Section 3 discusses the volatility term structure, and Section 4 discusses option valuation. Sections 5 and 6 present the empirical results, and Section 7 concludes.

2. Return dynamics with volatility components

In this section we first present the Heston and Nandi GARCH(1,1) model, which serves as the benchmark model throughout the paper. We then construct the component model as a natural extension of the GARCH(1,1) model. We also present the persistent component model as a special case of the component model.

2.1. The Heston and Nandi GARCH(1,1) model

Heston and Nandi (2000) propose a class of GARCH models that allow for a closed-form solution for the price of a European call option. They present an empirical analysis of the GARCH(1,1) version of this model, which is given by

$$R_{t+1} \equiv \ln(S_{t+1}/S_t) = r + \lambda h_{t+1} + \sqrt{h_{t+1}} z_{t+1} \quad \text{and}$$
$$h_{t+1} = w + bh_t + a(z_t - c\sqrt{h_t})^2, \tag{1}$$

where S_{t+1} denotes the underlying asset price; r, the risk free rate; λ, the price of risk; and h_{t+1}, the daily variance on day $t+1$, which is known at the end of day t. The z_{t+1} shock is assumed to be independently and identically distributed (i.i.d.) $N(0,1)$. The Heston and Nandi model captures time-variation in the conditional variance as in Engle (1982) and Bollerslev (1986), and the parameter c captures the so-called leverage effect. The leverage effect captures the negative relation between shocks to returns and volatility (Black, 1976), which results in a negatively skewed distribution of returns. Its importance for option valuation has been emphasized, among others, by Christoffersen and Jacobs (2004) and Nandi (1998). The GARCH(1,1) dynamic in Eq. (1) is slightly different from the more conventional NGARCH model used by Engle and Ng (1993) and Hentschel (1995), which is used for option valuation in Duan (1995). The reason is that the dynamic in Eq. (1) is engineered to yield a closed-form solution for option valuation, whereas a closed-form solution does not obtain for the more conventional GARCH dynamic. Hsieh and Ritchken (2005) provide evidence that the more traditional GARCH model could slightly dominate the fit of Eq. (1). Our main point can be demonstrated using either dynamic. Because of the convenience of the closed-form solution provided by dynamics such as Eq. (1), we use this as a benchmark in our empirical analysis and we model the richer component structure within the Heston and Nandi framework. We refer to Engle and Mustafa (1992) and Hsieh and Ritchken (2005) for other empirical studies of European option valuation using GARCH dynamics. Ritchken and Trevor (1999) discuss the pricing of American options with GARCH processes.

To better appreciate the workings of the component models presented below, by using the expression for the unconditional variance

$$E[h_{t+1}] \equiv \sigma^2 = \frac{w+a}{1-b-ac^2} \tag{2}$$

to substitute out w, the variance process can be rewritten as

$$h_{t+1} = \sigma^2 + b(h_t - \sigma^2) + a((z_t - c\sqrt{h_t})^2 - (1 + c^2\sigma^2)). \tag{3}$$

2.2. Building a component volatility model

The expression for the GARCH(1,1) variance process in Eq. (3) highlights the role of the parameter σ^2 as the constant unconditional mean of the conditional variance process. A natural generalization is then to specify σ^2 as time-varying. Denoting this time-varying component by q_{t+1}, the expression for the variance in Eq. (3) can be generalized to

$$h_{t+1} = q_{t+1} + \beta(h_t - q_t) + \alpha((z_t - \gamma_1\sqrt{h_t})^2 - (1 + \gamma_1^2 q_t)). \tag{4}$$

This model is similar in spirit to the component model of Engle and Lee (1999). The difference between our model and Engle and Lee (1999) is that the functional form of the GARCH dynamic in Eq. (4) allows for a closed-form solution for European option prices. This is similar to the difference between the Heston and Nandi (2000) GARCH(1,1) dynamic and the more traditional NGARCH (1,1) dynamic. In Eq. (4), the conditional volatility h_{t+1} can most usefully be thought of as having two components. Following Engle and Lee (1999), we refer to the component q_{t+1} as the long-run component and to $h_{t+1} - q_{t+1}$ as the short-run component. By construction, the unconditional mean of the short-run component $h_{t+1} - q_{t+1}$ is zero.

The model can also be written as

$$\begin{aligned} h_{t+1} &= q_{t+1} + (\alpha\gamma_1^2 + \beta)(h_t - q_t) + \alpha((z_t - \gamma_1\sqrt{h_t})^2 - (1 + \gamma_1^2 h_t)) \\ &= q_{t+1} + \tilde{\beta}(h_t - q_t) + \alpha((z_t - \gamma_1\sqrt{h_t})^2 - (1 + \gamma_1^2 h_t)), \end{aligned} \tag{5}$$

where $\tilde{\beta} = \alpha\gamma_1^2 + \beta$. This representation is useful because we can think of

$$\begin{aligned} v_{1,t} &\equiv (z_t - \gamma_1\sqrt{h_t})^2 - (1 + \gamma_1^2 h_t) \\ &= (z_t^2 - 1) - 2\gamma_1\sqrt{h_t}z_t \end{aligned} \tag{6}$$

as a mean-zero innovation.

The model is completed by specifying the functional form of the long-run volatility component. In a first step, we assume that q_{t+1} follows the process

$$q_{t+1} = \omega + \rho q_t + \varphi((z_t^2 - 1) - 2\gamma_2\sqrt{h_t}z_t). \tag{7}$$

If $\rho < 1$ then we have that $\mathrm{E}[q_{t+1}] = \mathrm{E}[h_{t+1}] = \sigma^2 = \omega/(1-\rho)$. We can therefore write the component volatility model as

$$\begin{aligned} h_{t+1} &= q_{t+1} + \tilde{\beta}(h_t - q_t) + \alpha v_{1,t} \\ q_{t+1} &= \omega + \rho q_t + \varphi v_{2,t} \\ &= \sigma^2 + \rho(q_t - \sigma^2) + \varphi v_{2,t} \end{aligned} \tag{8}$$

with

$$v_{i,t} = (z_t^2 - 1) - 2\gamma_i\sqrt{h_t}z_t \quad \text{for } i = 1,2 \tag{9}$$

and $\mathrm{E}_{t-1}[v_{i,t}] = 0$, $i = 1,2$. In addition to the price of risk, λ, the model contains seven parameters: α, $\tilde{\beta}$, γ_1, γ_2, ω, ρ, and φ. In comparison, the GARCH(1,1) model has four variance parameters.

2.3. A fully persistent special case

In our empirical work, we also investigate a special case of the model in Eq. (8). In Eq. (8) the long-run component of volatility is a mean-reverting process for $\rho < 1$. We also estimate a version of the model which imposes $\rho = 1$. The resulting process is

$$\begin{aligned} h_{t+1} &= q_{t+1} + \tilde{\beta}(h_t - q_t) + \alpha v_{1,t} \quad \text{and} \\ q_{t+1} &= \omega + q_t + \varphi v_{2,t} \end{aligned} \tag{10}$$

and $v_{i,t}$, $i = 1,2$ are defined as in Eq. (9). In addition to the price of risk, λ, the model now contains six parameters: $\alpha, \tilde{\beta}, \gamma_1, \gamma_2, \omega$, and φ.

In this case the process for long-run volatility contains a unit root and shocks to the long-run volatility never die out: They have a permanent effect. Following Engle and Lee (1999) in Eq. (8) we refer to q_{t+1} as the long-run component and to $h_{t+1} - q_{t+1}$ as the short-run component. In the special case of Eq. (10) we can also refer to q_{t+1} as the permanent component, because innovations to q_{t+1} are truly permanent and do not die out. It is then customary to refer to $h_{t+1} - q_{t+1}$ as the transitory component, which reverts to zero. This permanent-effects version of the model is most closely related to models that have been studied more extensively in the finance and economics literature, not the more general model in Eq. (8). We refer to this model as the persistent component model.

Eq. (10) is clearly nested by Eq. (8). It is therefore to be expected that the in-sample fit of Eq. (8) is superior. However, out-of-sample this might not necessarily be the case. It is often the case that more parsimonious models perform better out-of-sample if the restriction imposed by the model is a sufficiently adequate representation of reality. The persistent component model could also be better able to capture structural breaks in volatility out-of-sample, because a unit root in the process allows it to adjust to a structural break, which is not possible for a mean-reverting process. It therefore is of interest to verify how close ρ is to one when estimating the more general model in Eq. (8).

3. Variance term structures

To intuitively understand the shortcomings of existing models such as the GARCH(1,1) model in Eq. (1) and the improvements provided by our model in Eq. (8), it is instructive to graphically show some of the models' statistical properties that are key for option valuation. In this section we therefore illustrate the models' variance term structures and impulse response functions.

3.1. The variance term structure for the GARCH(1,1) model

Following the logic used for the component model in Eq. (8), we can rewrite the GARCH(1,1) variance dynamic in Eq. (3):

$$h_{t+1} = \sigma^2 + \tilde{b}(h_t - \sigma^2) + a((z_t^2 - 1) - 2c\sqrt{h_t}z_t), \tag{11}$$

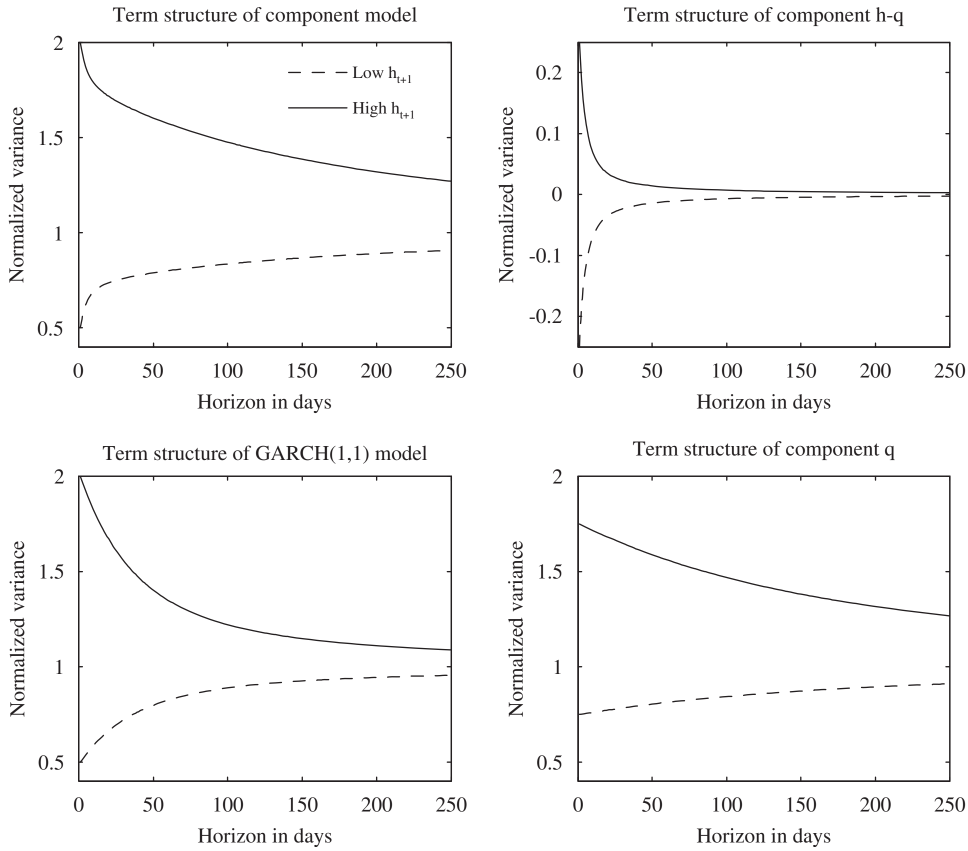

Fig. 1. Term structure of variance with high and low initial variance, component model and GARCH(1,1), normalized by unconditional variance. In the left-side panels, we plot the variance term structure implied by the component GARCH and GARCH(1,1) models for 1 through 250 days. In the right-side panels, we plot the term structure of the short-run $(h-q)$, and long-run (q) component. The parameter values are obtained from maximum likelihood estimation on returns in Table 1. For the low initial variance (dashed lines), the initial value of q_{t+1} is set to $0.75\sigma^2$, the initial value of h_{t+1} is set to $0.5\sigma^2$, and the initial value for h_{t+1} in the GARCH(1,1) is set to $0.5\sigma^2$. For the high initial variance (solid lines), the initial value of q_{t+1} is set to $1.75\sigma^2$, the initial value of h_{t+1} is set to $2\sigma^2$, and the initial value for h_{t+1} in the GARCH(1,1) is set to $2\sigma^2$. All values are normalized by the unconditional variance σ^2.

where $\tilde{b} = b + ac^2$ and the innovation term has a zero conditional mean. From Eq. (11) the multi-step forecast of the conditional variance is

$$\mathrm{E}_t[h_{t+k}] = \sigma^2 + \tilde{b}^{k-1}(h_{t+1} - \sigma^2), \tag{12}$$

where the conditional expectation is taken at the end of day t. The parameter $\tilde{b}$ is directly interpretable as the variance persistence in this representation of the model.

We can now define a convenient measure of the variance term structure for maturity T as

$$\begin{aligned} h_{t+1:t+T} &\equiv \frac{1}{T}\sum_{k=1}^{T} \mathrm{E}_t[h_{t+k}] = \frac{1}{T}\sum_{k=1}^{T} \sigma^2 + \tilde{b}^{k-1}(h_{t+1} - \sigma^2) \\ &= \sigma^2 + \frac{1-\tilde{b}^T}{1-\tilde{b}} \frac{(h_{t+1} - \sigma^2)}{T}. \end{aligned} \tag{13}$$

This variance term structure measure succinctly captures important information about the model's potential to explain the variation of option values across maturities.[1]

[1] Notice that due to the price of risk term in the conditional mean of returns, the term structure of variance as defined here is not exactly equal to the conditional variance of cumulative returns over T days.

To compare different models, it is convenient to set the current variance, h_{t+1}, to a simple m multiple of the long-run variance. In this case the variance term structure relative to the unconditional variance is given by

$$h_{t+1:t+T}/\sigma^2 \equiv 1 + \frac{1-\tilde{b}^T}{1-\tilde{b}} \frac{(m-1)}{T}. \tag{14}$$

The bottom-left panel in Fig. 1 shows the term structure of variance for the GARCH(1,1) model for low (dashed line) and high (solid line) initial conditional variances. We use parameter values estimated via MLE on daily S&P 500 returns (see Table 1). We set $m = \frac{1}{2}$ to investigate low initial variance and $m = 2$ to investigate high initial variance. The figures present the variance term structure for up to 250 days, which corresponds approximately to the number of trading days in a year and therefore captures the empirically relevant term structure for option valuation. It can be clearly seen that, for the GARCH(1,1) model, the conditional variance converges quickly to the long-run variance.

We can also learn about the dynamics of the variance term structure through impulse response functions. For

Table 1
Maximum likelihood estimates and properties using an estimation sample of daily returns, 1962–2001

We use daily total returns from July 1, 1962 to December 31, 2001 on the Standard and Poor's 500 index to estimate the three models using maximum likelihood. Robust standard errors are calculated from the outer product of the gradient at the optimum parameter values. Persistence refers to the persistence of the conditional variance as defined in the text. Average annual volatility refers to the average annualized standard deviation during 1990–1995. Average volatility of variance refers to the average standard deviation of the conditional variance during 1990–1995. Average correlation refers to the average correlation between the return and the conditional variance during 1990–1995. Ln likelihood refers to the logarithm of the likelihood at the optimal parameter values. Option RMSE (root mean squared error) refers to the fit of the models on option prices observed during 1990–1995.

Parameter	GARCH(1,1)		Parameter	Component GARCH		Parameter	Persistent component	
	Estimate	Standard error		Estimate	Standard error		Estimate	Standard error
λ	2.231E+00	1.123E+00	λ	2.092E+00	7.729E−01	λ	2.017E−07	4.316E−01
w	2.101E−17	1.120E−07	$\tilde{\beta}$	6.437E−01	2.759E−02	$\tilde{\beta}$	8.822E−01	9.931E−03
b	9.012E−01	4.678E−03	α	1.580E−06	2.430E−07	α	2.057E−06	1.539E−07
a	3.317E−06	1.380E−07	γ_1	4.151E+02	6.341E+01	γ_1	2.516E+02	2.237E+01
c	1.276E+02	8.347E+00	γ_2	6.324E+01	5.300E+00	γ_2	1.187E+02	1.126E+01
			ω	8.208E−07	7.620E−08	ω	1.187E−07	1.393E−08
			φ	2.480E−06	1.160E−07	φ	7.966E−07	4.599E−08
			ρ	9.896E−01	9.630E−04	ρ	1.000E+00	
Ln likelihood	33,955			34,102			34,005	
Persistence	0.9553			0.9963			1.0000	
Average annual volatility	0.1206			0.1174			0.1239	
Average volatility of variance	7.997E−06			1.341E−05			1.044E−05	
Average correlation	−0.7940			−0.8849			−0.9014	
Option RMSE	2.236			1.706			1.705	
Normalized	1.000			0.763			0.763	

the GARCH(1,1) model, the effect of a shock at time t, z_t, on the expected k-day ahead variance is

$$\partial(\mathrm{E}_t[h_{t+k}])/\partial z_t^2 = \tilde{b}^{k-1} a(1 - c\sqrt{h_t}/z_t) \tag{15}$$

and thus the effect on the variance term structure is

$$\partial \mathrm{E}_t[h_{t+1:t+T}]/\partial z_t^2 = \frac{1-\tilde{b}^T}{1-\tilde{b}}\frac{a}{T}(1 - c\sqrt{h_t}/z_t). \tag{16}$$

The bottom-left panel of Fig. 2 plots the impulse responses to the term structure of variance for $h_t = \sigma^2$ and $z_t = 2$ (dashed line) and $z_t = -2$ (solid line), respectively, again using the parameter estimates from Table 1.[2] The impulse responses are normalized by the unconditional variance. The effect of a shock dies out rather quickly for the GARCH(1,1) model. Fig. 2 also illustrates the asymmetric response of the variance term structure from a positive versus negative shock to returns. Due to the presence of a positive c, a positive shock has less impact than a negative shock along the entire term structure of variance.

3.2. The variance term structure for the component model

In the component model,

$$h_{t+1} = q_{t+1} + \tilde{\beta}(h_t - q_t) + \alpha v_{1,t} \quad \text{and}$$
$$q_{t+1} = \sigma^2 + \rho(q_t - \sigma^2) + \varphi v_{2,t}. \tag{17}$$

The multi-day forecast of the two components are

$$\mathrm{E}_t[h_{t+k} - q_{t+k}] = \tilde{\beta}^{k-1}(h_{t+1} - q_{t+1}) \quad \text{and}$$
$$\mathrm{E}_t[q_{t+k}] = \sigma^2 + \rho^{k-1}(q_{t+1} - \sigma^2). \tag{18}$$

The simplicity of these multi-day forecasts is a key advantage of the component model. The multi-day variance forecast is a simple sum of the forecasts for the two variance components. The parameters $\tilde{\beta}$ and ρ correspond directly to the persistence of the short-run and long-run components, respectively.

We can now calculate the variance term structure in the component model for maturity T as

$$\begin{aligned} h_{t+1:t+T} &\equiv \frac{1}{T}\sum_{k=1}^{T} \mathrm{E}_t[q_{t+k}] + \mathrm{E}_t[h_{t+k} - q_{t+k}] \\ &= \frac{1}{T}\sum_{k=1}^{T} \sigma^2 + \rho^{k-1}(q_{t+1} - \sigma^2) + \tilde{\beta}^{k-1}(h_{t+1} - q_{t+1}) \\ &= \sigma^2 + \frac{1-\rho^T}{1-\rho}\frac{q_{t+1}-\sigma^2}{T} + \frac{1-\tilde{\beta}^T}{1-\tilde{\beta}}\frac{h_{t+1}-q_{t+1}}{T}. \end{aligned} \tag{19}$$

If we set q_{t+1} and h_{t+1} equal to m_1 and m_2 multiples of the long-run variance, respectively, then we get the variance term structure relative to the unconditional variance simply as

$$h_{t+1:t+T}/\sigma^2 = 1 + \frac{1-\rho^T}{1-\rho}\frac{m_1 - 1}{T} + \frac{1-\tilde{\beta}^T}{1-\tilde{\beta}}\frac{m_2 - m_1}{T}. \tag{20}$$

The top-left panel in Fig. 1 shows the term structure of variance for the component model using parameters

[2] Engle and Patton (2001) refer to the effect of a shock on the expected k-day ahead variance and on the variance term structure as the forward persistence and the cumulative forward persistence, respectively.

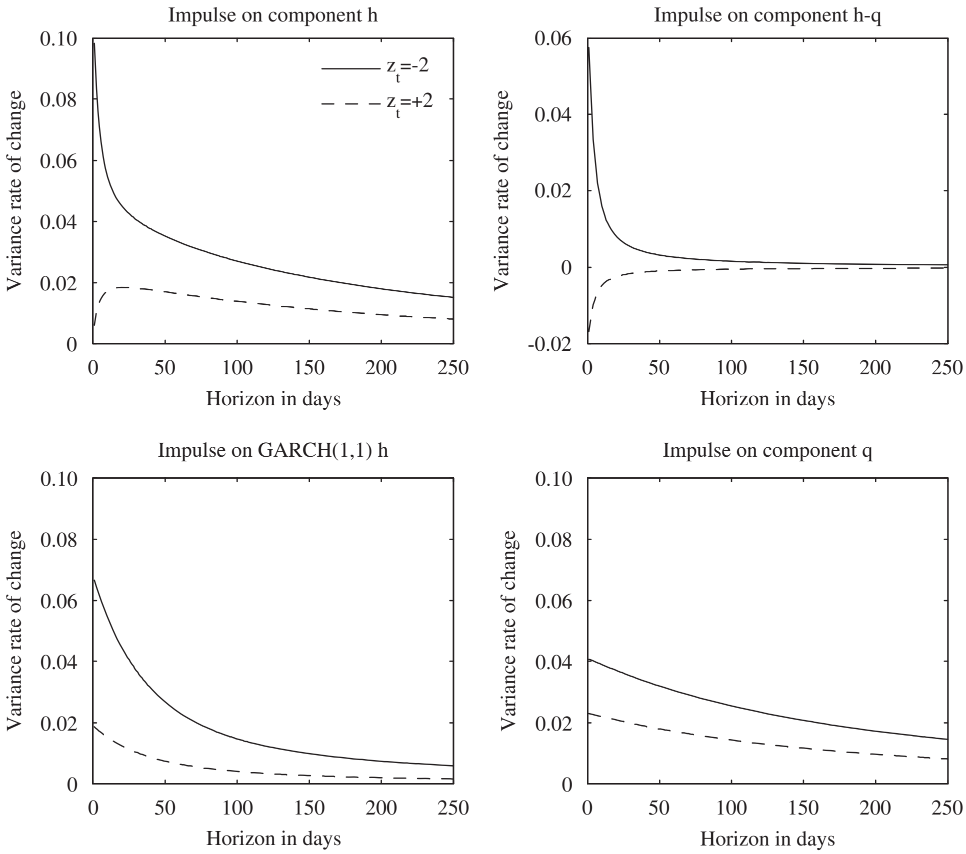

Fig. 2. Variance term structure impulse response to positive and negative return shocks, component model versus GARCH(1,1), normalized by unconditional variance. We plot the variance term structure responses to two shocks: $z_t = 2$ (dashed lines) and $z_t = -2$ (solid lines). In the left-hand panels, we plot the variance (h) responses to shocks to returns in the component and GARCH(1,1) models. For the component model, the right-hand panels show the response of the short-run ($h - q$) and long run (q) components. The parameter values are obtained from the maximum likelihood estimation on returns in Table 1. The current variance is set equal to the unconditional value. All values are normalized by the unconditional variance.

estimated via MLE on daily S&P 500 returns from Table 1. We set $m_1 = \frac{3}{4}$, $m_2 = \frac{1}{2}$ to investigate low initial variance (dashed line) and $m_1 = \frac{7}{4}$, $m_2 = 2$ to investigate high initial variance (solid line). By picking m_2 equal to the m used for the GARCH(1,1) model, we ensure comparability across models within each figure because the spot variances relative to their long-run variances are identical.[3] The main conclusion from Fig. 1 is that, compared with GARCH(1,1), the conditional variance converges more slowly to the unconditional variance in the component model. This is particularly so on days with a high spot variance. The right-side panels show the contribution to the total variance from each component. The strong persistence in the long-run component is clear.

We can also calculate impulse response functions in the component model. The effects of a shock at time t, z_t on the expected k-day ahead variance components are

$$\partial \mathrm{E}_t[q_{t+k}]/\partial z_t^2 = \rho^{k-1}\varphi(1 - \gamma_2\sqrt{h_t}/z_t),$$
$$\partial \mathrm{E}_t[h_{t+k} - q_{t+k}]/\partial z_t^2 = \tilde{\beta}^{k-1}\alpha(1 - \gamma_1\sqrt{h_t}/z_t), \quad \text{and}$$
$$\partial \mathrm{E}_t[h_{t+k}]/\partial z_t^2 = \tilde{\beta}^{k-1}\alpha(1 - \gamma_1\sqrt{h_t}/z_t) + \rho^{k-1}\varphi(1 - \gamma_2\sqrt{h_t}/z_t). \tag{21}$$

Notice again the simplicity due to the component structure. The impulse response on the term structure of variance is then

$$\partial \mathrm{E}_t[h_{t+1:t+T}]/\partial z_t^2 = \frac{1 - \tilde{\beta}^T}{1 - \tilde{\beta}}\frac{\alpha}{T}(1 - \gamma_1\sqrt{h_t}/z_t) + \frac{1 - \rho^T}{1 - \rho}\frac{\varphi}{T}(1 - \gamma_2\sqrt{h_t}/z_t). \tag{22}$$

The top-left panel of Fig. 2 plots the impulse responses to the term structure of variance for $h_t = \sigma^2$ and $z_t = 2$ (dashed line) and $z_t = -2$ (solid line), respectively. This figure reinforces the message from Fig. 1 that, when using parameters estimated from the data, the component model is different from the GARCH(1,1) model. The effects

[3] We need $m_1 \neq m_2$ in this numerical experiment to generate a short-term effect in Eq. (20). Changing m_1 changes the picture, but the main conclusions stay the same.

of shocks are much longer-lasting in the component model using estimated parameter values because of the parameterization of the long-run component. Fig. 2 also indicates that the term structure of the volatility asymmetry is more flexible in the component model. As a result, current shocks and the current state of the economy potentially have a much more profound impact on the pricing of options across maturities in the component model.

Bollerslev and Mikkelsen (1996, 1999), Baillie, Bollerslev, and Mikkelsen (1996), and Ding, Granger, and Engle (1993) have argued that the hyperbolic rate of decay displayed by long-memory processes could be a more adequate representation for the conditional variance of returns. We do not disagree with these findings. Instead, we argue that Figs. 1 and 2 demonstrate that in the component model the combination of two variance components with exponential decay gives rise to a slower decay pattern that adequately captures the hyperbolic decay pattern of long-memory processes for the horizons relevant for option valuation. This is of interest because, although the long-memory model could be a more adequate representation of the data, it is harder to implement.

4. Option valuation

We now turn to the ultimate purpose of this paper, namely, the valuation of derivatives on an underlying asset with dynamic variance components. For the purpose of option valuation we first derive the conditional moment generating function (MGF) for the return process and then present the risk-neutral return dynamics. For the underlying theory on risk-neutral distributions in discrete-time option valuation, we refer to Rubinstein (1976), Brennan (1979), Amin and Ng (1993), Duan (1995), Camara (2003), and Schroder (2004).

4.1. The moment generating function

For the return dynamics in this paper we can characterize the MGF of the log stock price with a set of difference equations, using the techniques in Heston and Nandi (2000). Appendix A demonstrates that, for the component GARCH model, the MGF defined by

$$f(t,T;\phi) \equiv \mathrm{E}_t[\exp(\phi \ln(S_T))] \tag{23}$$

can be written as

$$f(t,T;\phi) = S_t^{\phi} \exp[A_t + B_{1,t}(h_{t+1} - q_{t+1}) + B_{2,t} q_{t+1}] \tag{24}$$

with coefficients

$$\begin{aligned} A_t &= A_{t+1} + r\phi - (\alpha B_{1,t+1} + \varphi B_{2,t+1}) - 1/2 \ln(1 - 2\alpha B_{1,t+1} \\ &\quad - 2\varphi B_{2,t+1}) + B_{2,t+1}\omega, \\ B_{1,t} &= B_{1,t+1}\tilde{\beta} + \lambda\phi + 2\frac{(\alpha\gamma_1 B_{1,t+1} + \varphi\gamma_2 B_{2,t+1} - 0.5\phi)^2}{1 - 2\alpha B_{1,t+1} - 2\varphi B_{2,t+1}}, \quad \text{and} \\ B_{2,t} &= B_{2,t+1}\rho + \lambda\phi + 2\frac{(\alpha\gamma_1 B_{1,t+1} + \varphi\gamma_2 B_{2,t+1} - 0.5\phi)^2}{1 - 2\alpha B_{1,t+1} - 2\varphi B_{2,t+1}} \end{aligned} \tag{25}$$

and terminal conditions

$$A_T = B_{1,T} = B_{2,T} = 0. \tag{26}$$

For the MGF in the GARCH(1,1) case we refer to Heston and Nandi (2000).

4.2. The risk-neutral GARCH(1,1) dynamic

The risk-neutral dynamics for the GARCH(1,1) model are given in Heston and Nandi (2000) as

$$\begin{aligned} R_{t+1} &= r - \tfrac{1}{2}h_{t+1} + \sqrt{h_{t+1}}z_{t+1}^* \quad \text{and} \\ h_{t+1} &= w + bh_t + a(z_t^* - c^*\sqrt{h_t})^2, \end{aligned} \tag{27}$$

with $c^* = c + \lambda + 0.5$ and $z_t^* \sim \mathrm{N}(0,1)$.

4.3. The risk-neutral component GARCH dynamic

Appendix B demonstrates that the risk-neutral component GARCH dynamic is given by

$$\begin{aligned} h_{t+1} &= q_{t+1} + \tilde{\beta}^*(h_t - q_t) + \alpha((z_t^* - \gamma_1^*\sqrt{h_t})^2 - (1 + \gamma_1^{*2}h_t)) \quad \text{and} \\ q_{t+1} &= \omega + \rho^* q_t + \varphi((z_t^* - \gamma_2^*\sqrt{h_t})^2 - (1 + \gamma_2^{*2}h_t)), \end{aligned} \tag{28}$$

where the risk-neutral parameters are defined as

$$\begin{aligned} \tilde{\beta}^* &= \tilde{\beta} + \alpha(\gamma_1^{*2} - \gamma_1^2) + \varphi(\gamma_2^{*2} - \gamma_2^2), \\ \rho^* &= \rho + \alpha(\gamma_1^{*2} - \gamma_1^2) + \varphi(\gamma_2^{*2} - \gamma_2^2), \quad \text{and} \\ \gamma_i^* &= \gamma_i + \lambda + 0.5, \quad i = 1,2. \end{aligned} \tag{29}$$

The MGF for the risk-neutral component GARCH process is therefore equal to the one for the physical component GARCH process, setting $\lambda = -0.5$ and using the risk-neutral parameters $\gamma_1^*, \gamma_2^*, \rho^*$, and $\tilde{\beta}^*$ as well as ω, α, and φ.

4.4. The option valuation formula

Given the MGF and the risk-neutral dynamics, option valuation is relatively straightforward. We use the result of Heston and Nandi (2000) that, at time t, a European call option with strike price K that expires at time T is worth

$$\begin{aligned} \text{Call Price} &= e^{-r(T-t)}\mathrm{E}_t^*[\mathrm{Max}(S_T - K, 0)] \\ &= S_t\left(\frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}\left[\frac{K^{-i\phi}f^*(t,T;i\phi+1)}{i\phi S_t e^{r(T-t)}}\right]d\phi\right) \\ &\quad - Ke^{-r(T-t)}\left(\frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}\left[\frac{K^{-i\phi}f^*(t,T;i\phi)}{i\phi}\right]d\phi\right), \end{aligned} \tag{30}$$

where $f^*(t,T;i\phi)$ is the conditional characteristic function of the logarithm of the spot price under the risk-neutral measure.

5. Empirical results from return estimation

This section presents the first set of empirical results. We study the models estimated on a long time series of S&P 500 index returns and emphasize a number of important stylized facts. We also report the differences between the models when using these parameters for option valuation and compare the models with a GARCH(1,1)-Jump model that allows for jumps in returns and variance. Finally, we analyze the differences in option valuation along various dimensions.

5.1. Properties of the physical return process

Table 1 presents maximum likelihood estimates of the physical model parameters obtained using returns data for 1962–2001. We use a long sample of returns on the S&P 500 because it is well known that it is difficult to estimate GARCH parameters precisely using relatively short samples of return data. We compare the models using goodness-of-fit statistics, and we discuss differences in model properties. We present results for three models: the GARCH(1,1) model in Eq. (1), the component model in Eq. (8) and the persistent component model in Eq. (10). Almost all parameters are estimated significantly different from zero at conventional significance levels.[4] The price of risk, λ, is marginally significant in the case of the GARCH(1,1) and not significant in the persistent component model. The log likelihood values indicate that the fit of the component model is much better than that of the persistent component model, which in turn fits much better than the GARCH(1,1) model.

The dynamic variance models can be compared by assessing their persistence properties. The variance persistence in the GARCH(1,1) model is defined by $\tilde{b} = b + ac^2$ from Eq. (11). In the component model, the total variance persistence is a confluence of the persistence in the two factors. If we substitute out q_{t+1} and q_t from the h_{t+1} equation in Eq. (8), then persistence can be computed as the sum of the coefficients on h_t and h_{t-1}. In this way, the component persistence formula can be derived to be $\rho + \tilde{\beta}(1-\rho)$.

The improvement in fit for the component GARCH model over the persistent component GARCH model is perhaps somewhat surprising when inspecting the persistence of the component GARCH model. The persistence is equal to 0.9963. It therefore would appear that equating this persistence to 1, as is done in the persistent component model, is an interesting hypothesis, but apparently modeling these small differences from 1 is important. While the persistence of the long-run component (ρ) is 0.9896 for the component model as opposed to 1 for the persistent component model, the persistence of the short-run component ($\tilde{\beta}$) is 0.6437 versus 0.8822, which could account for the differences in likelihood. The persistence of the GARCH(1,1) model is estimated at 0.9553, which is consistent with earlier literature. It is slightly lower than the estimate in Christoffersen, Heston, and Jacobs (2006) and a bit higher than the average of the estimates in Heston and Nandi (2000).

In Fig. 1, we illustrate differences across models in terms of variance term structures that are key for option valuation. Following Das and Sundaram (1999) and Bates (2000), we can use Eq. (24) to further investigate the conditional term structure of higher moments. The conditional skewness and excess kurtosis for maturity T are easily defined from the conditional cumulant generating function, $\ln(f(t,T;\phi))$, as

$$Skewness(t,T) = \frac{\partial^3 \ln f(t,T;\phi)/\partial\phi^3|_{\phi=0}}{(\partial^2 \ln f(t,T;\phi)/\partial\phi^2|_{\phi=0})^{3/2}} \quad \text{and}$$

$$Kurtosis(t,T) = \frac{\partial^4 \ln f(t,T;\phi)/\partial\phi^4|_{\phi=0}}{(\partial^2 \ln f(t,T;\phi)/\partial\phi^2|_{\phi=0})^2}. \tag{31}$$

We compute these moments by taking numerical derivatives of $\ln(f(t,T;\phi))$ in Eq. (24).

In Fig. 3 we plot the term structure of skewness and kurtosis in the three GARCH models. The initial volatility is set to its long-run value in the GARCH(1,1) and component GARCH models. In the persistent component model the initial volatility is set to the unconditional volatility from the component model. The parameter estimates are again taken from Table 1.

Fig. 3 reveals important differences between the term structures of these moments for the GARCH(1,1) model, the component model, and the persistent component model. While the term structures of skewness and kurtosis are hump-shaped for the GARCH(1,1) model over the maturities relevant for pricing the options in our sample, they are downward sloping for skewness and upward sloping for kurtosis in the persistent component model. For the component model, the minimum and the maximum, respectively, for the conditional skewness and kurtosis occur for options with approximately a six-month maturity, but the skewness and kurtosis for longer maturities are very close to these extrema. These fundamental differences in higher moment term structures could have important implications for the option valuation properties across models.

Fig. 4 further analyzes the component model's improvement in performance over the benchmark GARCH(1,1) model. It presents the 1990–1995 sample path for the spot variance in the GARCH(1,1) model and the component model, as well as the sample path for volatility components for the component models. We plot results for the 1990–1995 subsample here because this is the sample that we subsequently use for option valuation. Fig. 4 indicates that the mean zero short-run component in the top-right panel adds high-frequency noise to the long-run component in the bottom-right panel. This results in a volatility dynamic for the component model in the top-left panel that is more noisy than the volatility dynamic for the GARCH(1,1) model in the bottom-left panel. This more noisy sample path suggests a higher value for the volatility of variance in the component model. An analysis of the persistent component model (not reported) supports similar conclusions, even though the sample paths are somewhat different from those in Fig. 4.

We now investigate in more detail differences between the models in the modeling of the volatility (standard deviation) of the conditional variance, as well as differences in the modeling of the covariance and correlation between returns and variance. For option valuation, the

[4] One exception is the parameter w. The reason for this is probably that the point estimate for w is very small. This in turn is due to the enforcement of the constraint that $w > 0$, which is a sufficient condition for positive conditional variances. All standard errors are computed using the outer product of the gradient at the optimal parameter values.

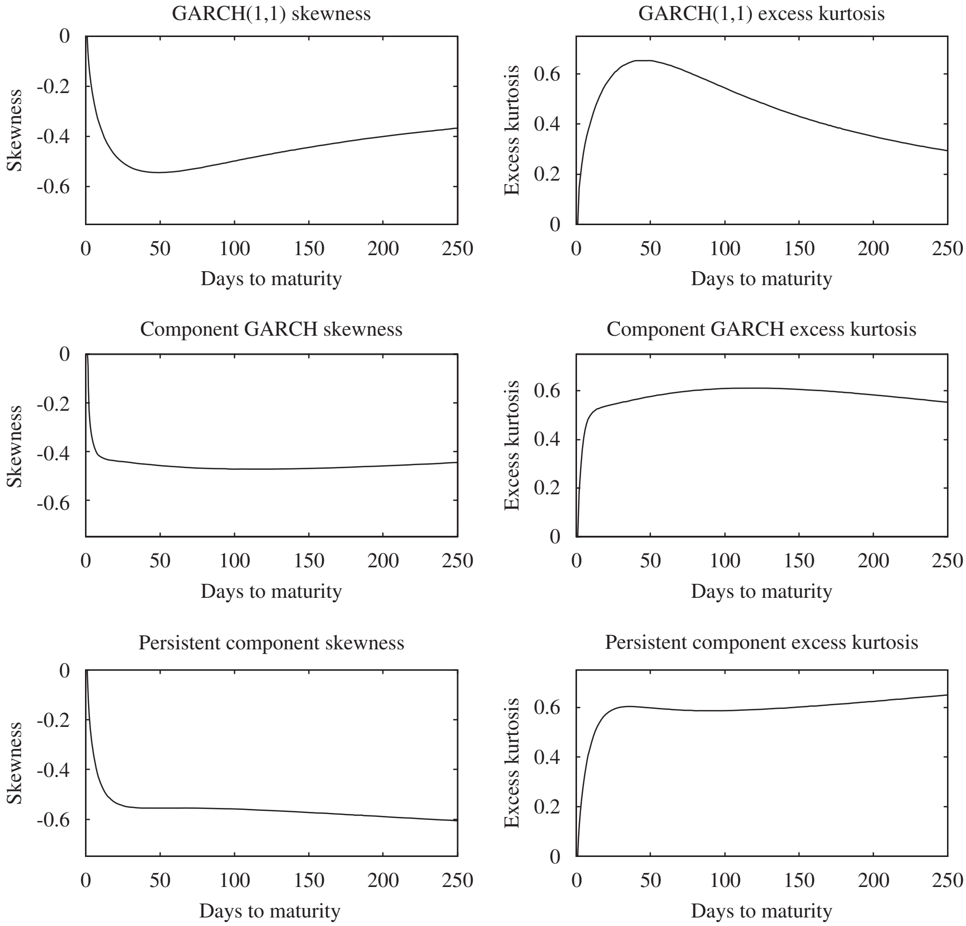

Fig. 3. Term structure of skewness and kurtosis. We use the numerical derivatives of the log moment generating function to compute the term structure of conditional skewness and kurtosis in the three GARCH models. The initial volatility is set to its long-run value in the GARCH(1,1) and component GARCH models. In the persistent component model, the initial volatility is set to the unconditional volatility from the component model. The parameter values are obtained from the maximum likelihood estimates on returns in Table 1.

conditional versions of these quantities and their variation through time are just as important as the unconditional versions.

For the GARCH(1,1) model, the conditional variance of variance is

$$\begin{aligned} Var_t(h_{t+2}) &= \mathrm{E}_t[h_{t+2} - \mathrm{E}_t[h_{t+2}]]^2 \\ &= 2a^2 + 4a^2c^2h_{t+1}, \end{aligned} \tag{32}$$

and for the component and persistent component models, the conditional variance of variance is

$$Var_t(h_{t+2}) = 2(\alpha + \varphi)^2 + 4(\gamma_1\alpha + \gamma_2\varphi)^2 h_{t+1}. \tag{33}$$

The three left panels in Fig. 5 use the parameters from Table 1 to plot the standard deviation of variance in the three GARCH models. The standard deviation of variance in the component model is in general much higher than in the GARCH(1,1) model, and it is also more volatile. The average level of the conditional standard deviation of variance in the persistent component model is in between that of the other two models. Table 1 reports the average volatility of variance during 1990–1995.

If we think of the option price as being a function of the spot variance, then we can view variation in the option prices as being driven by the volatility of variance. The volatility of variance is also related to kurtosis. Fig. 5 thus shows that the component model is able to generate richer time-varying kurtosis dynamics than the GARCH(1,1) model and thus potentially richer option price dynamics.

The conditional covariance between return and variance in the GARCH(1,1) model is given by

$$\begin{aligned} Cov_t(R_{t+1}, h_{t+2}) &= \mathrm{E}_t[(R_{t+1} - \mathrm{E}_t[R_{t+1}])(h_{t+2} - \mathrm{E}_t[h_{t+2}])] \\ &= \mathrm{E}_t[\sqrt{h_{t+1}}z_{t+1}(az_{t+1}^2 - 2acz_{t+1}\sqrt{h_{t+1}} - a)] \\ &= -2ach_{t+1}. \end{aligned} \tag{34}$$

Conditional correlation is easier to interpret than conditional covariance. The conditional correlation in the GARCH(1,1) model is

$$Corr_t(R_{t+1}, h_{t+2}) = \frac{-2c\sqrt{h_{t+1}}}{\sqrt{2 + 4c^2h_{t+1}}}, \tag{35}$$

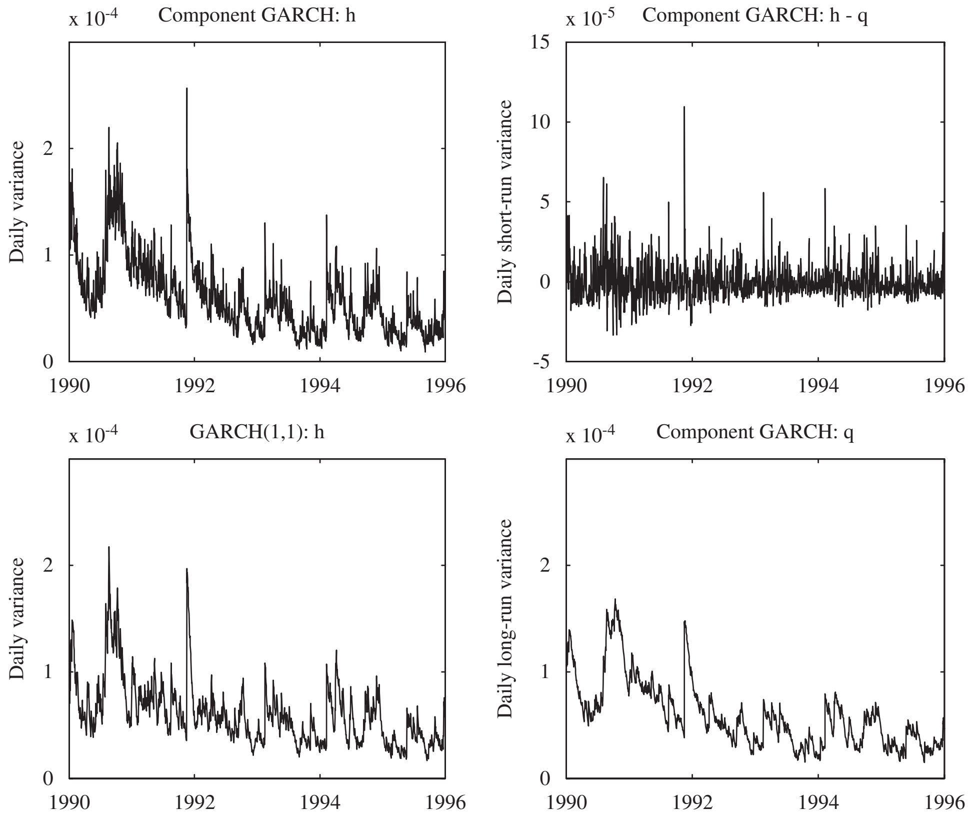

Fig. 4. Spot variance of component GARCH versus GARCH(1,1). The left-hand panels plot the variance paths from the component and GARCH(1,1) models. The right-hand panels plot the individual components. The parameter values are obtained from maximum likelihood estimation on returns in Table 1.

where we use the conditional variance of variance from Eq. (32).

The conditional covariance in the component model is

$$Cov_t(R_{t+1}, h_{t+2}) = -2(\gamma_1\alpha + \gamma_2\varphi)h_{t+1}, \tag{36}$$

and the conditional correlation in the component model is thus given by

$$Corr_t(R_{t+1}, h_{t+2}) = \frac{-2(\gamma_1\alpha + \gamma_2\varphi)\sqrt{h_{t+1}}}{\sqrt{2(\alpha+\varphi)^2 + 4(\gamma_1\alpha + \gamma_2\varphi)^2 h_{t+1}}}. \tag{37}$$

The three right panels in Fig. 5 plot the conditional correlation for the three models. The conditional correlation is clearly more negative in the component models than in the GARCH(1,1) model, and, furthermore, the component correlation paths are much more volatile. Table 1 gives the average correlations during 1990–1995 as −79.40% (GARCH(1,1)), −88.49% (component), and −90.14% (persistent component).

For the component and persistent component models, we can also compute the conditional correlations between the return and each volatility component separately:

$$Corr_t(R_{t+1}, h_{t+2} - q_{t+2}) = \frac{-2\gamma_1\sqrt{h_{t+1}}}{\sqrt{2 + 4\gamma_1^2 h_{t+1}}} \quad \text{and}$$

$$Corr_t(R_{t+1}, q_{t+2}) = \frac{-2\gamma_2\sqrt{h_{t+1}}}{\sqrt{2 + 4\gamma_2^2 h_{t+1}}}. \tag{38}$$

We find that the conditional correlation of the return with the short-run variance component is on average more negative than the conditional correlation between the return and the long-run variance component (not reported). This difference can be traced to Table 1, where $\gamma_1 > \gamma_2$ in both models. The correlations with the long-run factor are relatively more negative in the persistent component model, whereas the correlations with the short-run factor are relatively more negative in the component model. This can also be traced back to Table 1, where γ_1 is larger in the component model than in the persistent component model, whereas γ_2 is larger in the persistent component model.

Fig. 6 shows the correlation between returns and variances from a different perspective. We plot the correlations from Eqs. (35) and (37) against levels of the annualized conditional volatility. For all three models, the relation between the level of volatility and the correlation is negative. This is shown by Jones (2003) to be a desirable feature for option valuation, and it is a feature missing in the standard Heston (1993) stochastic volatility model where the correlation is constant. The Heston and Nandi (2000) GARCH(1,1) model does have this negative relation as Fig. 6 shows. Fig. 6 also shows that, when fitted on the more general component model, the return data wants a correlation that is more negative than the simple GARCH(1,1) model for all levels of volatility. The differences in

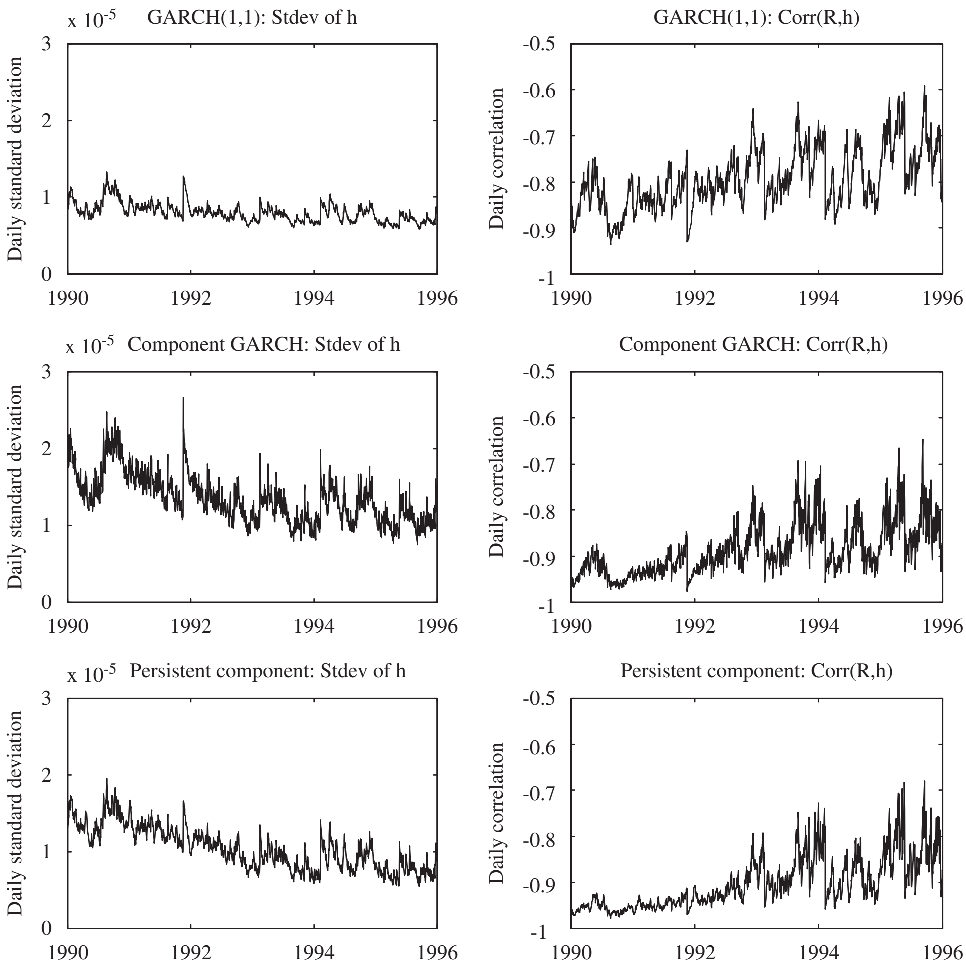

Fig. 5. Conditional standard deviation of variance and conditional correlation. In the left panels, we plot the conditional standard deviation of conditional variance as implied by the GARCH models. In the right panels, we plot the conditional correlation between return and variance. The scales are identical across the rows of panels to facilitate comparison across models. The parameter values are obtained from maximum likelihood estimation on returns in Table 1.

correlation are large for the most common levels of volatility.

Figs. 1–3 show that the component models are able to generate more flexible variance, skewness, and kurtosis term structures than the simple GARCH(1,1) model. Fig. 4 shows that the component models generate richer spot variance dynamics. We conclude from Fig. 5 that the more flexible component models are capable of generating more flexible dynamics for the conditional correlation between returns and variance and the conditional variance of variance. These dynamics are critically important for skewness and kurtosis dynamics, which in turn are key for explaining the variation in index options prices.

5.2. Option valuation results

We use a rich sample of S&P 500 call options from the period 1990–1995. Following Bakshi, Cao, and Chen (1997), we apply standard filters to the data. We only use Wednesday options data. Wednesday is the day of the week least likely to be a holiday. It is also less likely than other days such as Monday and Friday to be affected by day-of-the-week effects. For those weeks in which Wednesday is a holiday, we use the next trading day. The decision to pick one day every week is to some extent motivated by computational constraints. Using only Wednesday data allows us to study a fairly long time series, which is useful considering the highly persistent volatility processes. An additional motivation for using Wednesday data is that, following the work of Dumas, Fleming, and Whaley (1998), several studies including Heston and Nandi (2000) have used this setup.

Table 2 presents descriptive statistics for the options data for 1990–1995 by moneyness and maturity. Panels A and B indicate that the data are standard. We can observe the implied volatility smirk from Panel C, and the slope of the smirk clearly differs across maturities. Descriptive statistics for different subperiods (not reported) demonstrate that the slope changes over time, but that the smirk

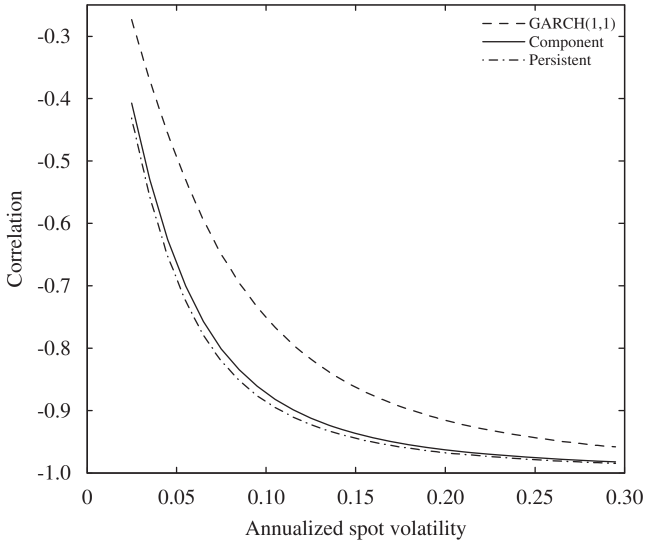

Fig. 6. Correlation between return and variance as a function of volatility level. We plot the conditional correlation between the return on the underlying index and the daily variance. This conditional correlation is plotted against the level of annualized volatility. The dashed line corresponds to the GARCH(1,1), the solid line to the component model, and the dash-dots to the persistent component model. The parameter estimates are from Table 1.

Table 2
S&P 500 index call option data, 1990–1995

We use European call options on the Standard and Poor's 500 index. The prices are taken from quotes within 30 minutes from closing on each Wednesday during the January 1, 1990 to December 31, 1995 period. The moneyness and maturity filters used by Bakshi, Cao, and Chen, 1997 are applied here as well. The implied volatilities are calculated using the Black and Scholes formula. DTM refers to the number of days to maturity and S/K refers to moneyness defined as the underlying index level divided by the option strike price.

Moneyness	DTM < 20	20 < DTM < 80	80 < DTM < 180	DTM > 180	All
Panel A. Number of call option contracts					
S/K < 0.975	101	1,884	1,931	1,769	5,685
0.975 < S/K < 1.00	283	1,272	706	477	2,738
1.00 < S/K < 1.025	300	1,212	726	526	2,764
1.025 < S/K < 1.05	261	1,167	654	409	2,491
1.05 < S/K < 1.075	245	1,039	582	390	2,256
1.075 < S/K	549	2,345	1,679	1,245	5,818
All	1,739	8,919	6,278	4,816	21,752
Panel B. Average call price					
S/K < 0.975	0.88	2.30	6.25	11.94	6.62
0.975 < S/K < 1.00	2.29	6.83	15.19	27.50	12.12
1.00 < S/K < 1.025	8.35	13.60	22.48	34.41	19.32
1.025 < S/K < 1.05	17.57	22.00	30.11	42.14	26.97
1.05 < S/K < 1.075	27.11	30.84	38.14	48.83	35.43
1.075 < S/K	50.67	52.79	58.99	68.34	57.70
All	24.32	23.66	28.68	36.07	27.91
Panel C. Average implied volatility from call options					
S/K < 0.975	0.1625	0.1269	0.1350	0.1394	0.1342
0.975 < S/K < 1.00	0.1308	0.1296	0.1449	0.1562	0.1383
1.00 < S/K < 1.025	0.1527	0.1459	0.1558	0.1606	0.1520
1.025 < S/K < 1.05	0.1915	0.1647	0.1665	0.1656	0.1681
1.05 < S/K < 1.075	0.2433	0.1828	0.1775	0.1739	0.1865
1.075 < S/K	0.3897	0.2356	0.1961	0.1868	0.2283
All	0.2434	0.1703	0.1622	0.1607	0.1717

is present throughout the sample. There is substantial clustering in implied volatilities, and volatility is higher in the early part of the sample.

The last two rows of Table 1 compare the performance of the four models for option valuation. We use the MLE parameters to compute root mean squared errors (RMSEs)

Table 3
1990–1995 root mean squared error (RMSE) and ratio RMSE by moneyness and maturity with parameters estimated from daily returns, 1962–2001

We use the maximum likelihood estimates from Table 1 to compute the root mean squared option valuation error (RMSE) for various moneyness and maturity bins during 1990–1995. Panel A shows the RMSEs for the GARCH(1,1) model. Panels B and C show the ratio of the RMSEs of the component and persistent component models, respectively, over the RMSE of the GARCH(1,1) model. DTM refers to the number of days to maturity and S/K refers to moneyness defined as the ratio of the index level divided by the strike price.

Moneyness	DTM < 20	20 < DTM < 80	80 < DTM < 180	DTM > 180	All
Panel A. GARCH(1,1) RMSE					
S/K < 0.975	0.454	1.778	3.032	4.155	3.090
0.975 < S/K < 1.00	0.671	2.116	3.087	3.548	2.603
1.00 < S/K < 1.025	0.638	1.650	2.574	2.955	2.154
1.025 < S/K < 1.05	0.549	1.204	2.099	2.487	1.700
1.05 < S/K < 1.075	0.735	1.013	1.879	2.227	1.516
1.075 < S/K	0.759	1.024	1.424	1.917	1.360
All	0.683	1.503	2.448	3.228	2.236
Panel B. Ratio of component GARCH to GARCH(1,1) RMSE					
S/K < 0.975	0.782	0.684	0.712	0.782	0.749
0.975 < S/K < 1.00	0.788	0.631	0.657	0.739	0.678
1.00 < S/K < 1.025	0.870	0.655	0.669	0.733	0.691
1.025 < S/K < 1.05	0.968	0.832	0.744	0.755	0.773
1.05 < S/K < 1.075	1.043	1.000	0.849	0.800	0.870
1.075 < S/K	1.000	1.037	0.974	0.907	0.962
All	0.949	0.750	0.735	0.784	0.763
Panel C. Ratio of persistent component to GARCH(1,1) RMSE					
S/K < 0.975	0.985	0.726	0.774	0.842	0.808
0.975 < S/K < 1.00	0.978	0.640	0.635	0.722	0.669
1.00 < S/K < 1.025	0.982	0.672	0.600	0.681	0.653
1.025 < S/K < 1.05	0.947	0.773	0.613	0.660	0.675
1.05 < S/K < 1.075	1.010	0.909	0.663	0.718	0.750
1.075 < S/K	1.002	0.981	0.836	0.784	0.856
All	0.990	0.746	0.719	0.796	0.763

for the 1990–1995 option sample.[5] As was the case in the log likelihood comparison, the GARCH(1,1) model is the worst performer based on the option RMSE. Whereas the component model had a considerably better log likelihood than the persistent component model, the persistent component model now achieves a marginally better RMSE than the component model. Both component models have much better RMSEs than the GARCH(1,1) model.

Table 3 provides additional evidence on the option fit of the three models. We report option RMSE by moneyness and maturity. The top panel reports the RMSE for the GARCH(1,1) model, while the two other panels report the ratio of the RMSE for the two other models to that of the GARCH(1,1). The improvements of the component models over the GARCH(1,1) model are fairly robust across maturity and moneyness. Importantly, the component models are never much worse than the GARCH(1,1) model, and they fail to improve on the GARCH(1,1) model only for short-term deep-in-the-money call options. This finding leads us to consider jumps in returns, which by way of adding non-normality to the conditional density could lead to improvements in the valuation of short-term options.

5.3. Comparison with a GARCH(1,1)-Jump model

The Heston and Nandi GARCH(1,1) model is a useful first benchmark, but it has well-known empirical biases. These biases are similar to those displayed by the Heston (1993) model. The continuous-time literature has attempted to improve the performance of the Heston (1993) model by adding to it (potentially correlated) jumps in returns and volatility, and this strategy has been partly successful. Poisson–normal jumps in returns and volatility lead to improved modeling of the historical time series of returns. When model parameters are estimated using the cross section of option prices, Poisson–normal jumps usually do not lead to improved model fit, but Broadie, Chernov, and Johannes (2007) find evidence of the importance of jumps for option pricing when imposing consistency between the physical and risk-neutral parameters. For evidence on the importance of Poisson–normal jumps, see, for example, Andersen, Benzoni, and Lund (2002), Bakshi, Cao, and Chen (1997), Bates (1996, 2006), Chernov, Gallant, Ghysels, and Tauchen (2003), Eraker, Johannes, and Polson (2003), Eraker (2004), and Pan (2002).

[5] As the price of risk parameter, λ, is poorly estimated in Table 1, and to keep the persistence at unity under both measures in the persistent model, we simply set $\lambda = -0.5$ across models when computing option prices.

In the discrete-time literature, some studies attempt to address these model biases by combining conditional heteroskedasticity with non-normal innovations. This strategy could seem different from including jumps in the return process, but both approaches essentially introduce conditional non-normalities in the daily return distribution. However, Christoffersen, Heston, and Jacobs (2006) find that inverse Gaussian innovations do not improve out-of-sample model fit. To provide a more challenging benchmark for the component model, we therefore combine the GARCH(1,1) dynamic with a Poisson jump process similar to the one used in the continuous-time option valuation literature. We refer to the resulting model as the GARCH(1,1)-Jump model. Consider the return and variance dynamics

$$R_{t+1} = r + \lambda_z h_{t+1} + \lambda_y \chi + \sqrt{h_{t+1}} z_{t+1} + y_{t+1} - \mu\chi \quad \text{and}$$
$$h_{t+1} = w + bh_t + a\left(z_t + \frac{y_t}{\sqrt{h_t}} - c\sqrt{h_t}\right)^2. \tag{39}$$

The innovation to returns consists of a diffusive component z_t and a jump component y_t. The return dynamic has been designed to ensure that the expected log excess return equals the sum of the diffusive and jump risk premium. The diffusive innovation z_t is assumed to be i.i.d. $N(0,1)$. We consider a simple jump structure, where y_t is a compound Poisson process defined as

$$y_t = \sum_{j=1}^{N_t} X_t^{(j)} \tag{40}$$

with

$$X_t^{(j)} \sim N(\mu, \tau^2) \quad \text{for } j = 1, 2, \ldots, N_t \tag{41}$$

and N_t is a Poisson random variable with constant intensity χ. We therefore have

$$y_t \sim \text{CPoisson}(\chi, \mu, \tau^2). \tag{42}$$

For the purpose of comparison with the Heston and Nandi GARCH(1,1) model and our component model, we allow for time-varying variance only in the diffusive term, and we specify an affine variance dynamic h_{t+1}. The model nests the Heston and Nandi GARCH(1,1) model as a special case for $\chi = 0$. Maheu and McCurdy (2004) and Duan, Ritchken, and Sun (2005, 2006) provide alternative specifications of the jump process in a discrete-time setup.

It is relatively straightforward to estimate this model by maximum likelihood. Using the Esscher transform, it can be shown that the resulting risk-neutral dynamic has the same functional form with parameterization

$$R_{t+1} = r - \tfrac{1}{2}h_{t+1} + (\lambda_y^* - \mu^*)\chi^* + \sqrt{h_{t+1}} z_{t+1}^* + y_{t+1}^* \quad \text{and}$$
$$h_{t+1} = w + bh_t + a\left(z_t^* + \frac{y_t^*}{\sqrt{h_t}} - c^*\sqrt{h_t}\right)^2, \tag{43}$$

where $z_t^* \sim N(0,1)$ and $y_t^* \sim \text{CPoisson}(\chi^*, \mu^*, \tau^2)$, with $c^* = c + \frac{1}{2} + \lambda_z$, $\lambda_y^* = (\lambda_y - \mu)\chi/\chi^* + \mu^*$, $\mu^* = \mu + \pi_y \tau^2$, $\chi^* = \chi \exp(\pi_y^2 \tau^2/2 + \pi_y \mu)$, and where π_y is the market price of jump risk, which can be solved for numerically from the physical maximum likelihood estimates.[6] The term $(\lambda_y^* - \mu^*)\chi^*$ makes the discounted price process a martingale. No closed-form solution exists for the GARCH(1,1)-Jump model so that option prices must be computed by Monte Carlo simulation.

The structure of the jump component in the GARCH(1,1)-Jump model in Eq. (43) is similar to that used in several papers in the continuous-time option valuation literature. In this literature a closely related model is known as the stochastic volatility with correlated jumps (SVCJ) process with perfect correlation. See, for example, Broadie, Chernov, and Johannes (2007), Eraker (2004), and Eraker, Johannes, and Polson (2003). Our setup also allows us to estimate a process with one extra parameter that captures less than perfect correlation, but this parameter is often estimated imprecisely in the continuous-time literature, and so we do not estimate it here.

Table 4 reports the empirical results for the GARCH(1,1)-Jump model. Panel A reports the parameter estimates from maximum likelihood estimation on the sample of daily S&P 500 returns used in Table 1. Most parameters are precisely estimated, except for λ_z and especially λ_y. To prevent these imprecise estimates from influencing the comparison with other models when valuing options, we compute option prices using values for λ_z and λ_y, which yield zero risk premia associated with the normal and the jump components, respectively. This approach ensures comparability with the GARCH models, where we let $\lambda = -0.5$, which makes the risk-neutral parameters equal to the physical ones. In the GARCH(1,1)-Jump model this is accomplished by using $\lambda_z = -0.5$ and $\lambda_y = \mu - (e^{(1/2)\tau^2 + \mu} - 1) = -4.56 \times 10^{-4}$.

Our parameter estimates are generally in line with the literature. The average jump size is -1.2% and the jump intensity is equal to 0.0098, which can be annualized by multiplying by 252, indicating approximately 2.5 jumps per year.[7] The standard deviation of the jump τ is almost identical to the estimate in Eraker, Johannes, and Polson (2003). The estimated mean jump size μ is similar to the estimate in Chernov, Gallant, Ghysels, and Tauchen (2003), but about half of the estimate in Eraker, Johannes, and Polson (2003) and Eraker (2004). Our estimated jump intensity χ is higher than the estimates in those papers. Presumably our smaller mean jump and higher jump intensity estimates are due to the fact that compared with these papers we use a more extended sample period and that consequently the 1987 crash is relatively less important in determining the parameters.

The log likelihood value for the GARCH(1,1)-Jump model is considerably larger than for the three models in Table 1. The GARCH(1,1)-Jump model thus provides an excellent description of the conditional density for daily S&P 500 returns. The option RMSE for the GARCH(1,1)-Jump model is \$1.939, which is much better than the \$2.236 for the GARCH(1,1) in Table 1, but somewhat worse

[6] See Huang and Wu (2004) and Carr and Wu (2004) for applications of the Esscher transform to option valuation. See Dai, Le, and Singleton (2006) for applications to fixed income pricing.

[7] See Table 1 in Broadie, Chernov, and Johannes (2007) for a good overview of estimates available in the literature.

Table 4
GARCH(1,1)-Jump model parameters estimated from daily returns, 1962–2001

In Panel A, we use daily total returns from July 1, 1962 to December 31, 2001 on the Standard and Poor's 500 index to estimate the GARCH(1,1)-Jump model using maximum likelihood. Robust standard errors are calculated from the outer product of the gradient at the optimum parameter values. In Panel B, we compute the ratio of the option root mean squared error (RMSE) from the GARCH(1,1)-Jump model to the RMSE of the GARCH(1,1) in Panel A of Table 3. DTM refers to the number of days to maturity and S/K refers to moneyness defined as the underlying index level divided by the option strike price.

Panel A. GARCH(1,1)-Jump MLE estimates

Parameters	Estimate	Standard error
λ_z	2.793E + 00	1.864E + 00
λ_y	−3.419E − 03	1.153E − 02
w	−1.186E − 06	9.457E − 08
b	9.551E − 01	3.979E − 03
a	2.191E − 06	1.306E − 07
c	1.088E + 02	8.983E + 00
χ	9.845E − 03	2.100E − 03
μ	−1.174E − 02	4.879E − 03
τ	2.799E − 02	2.048E − 03
Ln likelihood	34,260	
Option RMSE	1.939	

Panel B. Ratio of GARCH(1,1)-Jump to GARCH(1,1) RMSE

Moneyness	DTM < 20	20 < DTM < 80	80 < DTM < 180	DTM > 180	All
S/K < 0.975	1.265	0.538	0.654	0.631	0.629
0.975 < S/K < 1.00	1.166	0.655	0.758	0.809	0.749
1.00 < S/K < 1.025	1.578	1.027	0.989	0.908	0.978
1.025 < S/K < 1.05	1.405	1.467	1.231	1.016	1.224
1.05 < S/K < 1.075	0.989	1.556	1.377	1.201	1.346
1.075 < S/K	1.177	1.277	1.348	1.223	1.274
All	1.247	0.942	0.902	0.800	0.867

than the \$1.706 and \$1.705 for the component model and persistent model, respectively.

Panel B in Table 4 shows the ratio of the RMSE of the GARCH(1,1)-Jump to the GARCH(1,1) model. The GARCH(1,1)-Jump model in general performs better than the GARCH(1,1), and its performance is especially strong for deep out-of-the money and for long-maturity options. Somewhat surprisingly, the GARCH(1,1)-Jump model does not perform well for short-maturity options. Given that the component model also does not perform well in this dimension, the component structure and this particular jump structure might not be complementary in addressing the biases of the GARCH(1,1) model. The GARCH(1,1)-Jump model could perform better for short-maturity options if it was estimated using option data as the jump parameters may be difficult to estimate precisely from returns only. The strong performance by the jump model for long-maturity options suggests that the jump-in-volatility feature incorporated in this model is useful. Jumps in volatility, which in turn is persistent, yield non-normal behavior in the conditional distribution at long horizons. This long-horizon non-normality helps fit an option smile that persists at longer horizons.

The performance of the GARCH(1,1)-Jump model is encouraging. We also find that alternative specifications of the jump risk premium can lead to improvements in performance, but it is not clear how to calibrate this risk premium and how to ensure a meaningful comparison with the component models. Different specifications for the jump process could provide an even better option pricing fit, but such an investigation is beyond the scope of this paper. In summary, we conclude that, while the GARCH(1,1)-Jump model significantly outperforms the GARCH(1,1) model for the purpose of option valuation, the component model compares favorably with the GARCH(1,1)-Jump model. It is important to keep in mind that the component model does so with eight parameters, and the persistent component model with seven, whereas the GARCH(1,1)-Jump model contains nine parameters. Nevertheless, because the GARCH(1,1)-Jump model provides a more challenging benchmark for the component model, we analyze it in some more detail.

5.4. Analyzing option valuation performance

The component models' performance is remarkable and to some extent surprising. First, the GARCH(1,1) model is a good benchmark, which itself has a solid empirical performance (see Heston and Nandi, 2000). The model captures important stylized facts about option prices such as volatility clustering and the leverage effect (or equivalently negative skewness). When estimating models from option prices, Christoffersen and Jacobs (2004) find that GARCH models with richer news-impact parametrization do not improve the model fit out-of-sample. Christoffersen, Heston, and Jacobs (2006) find that a GARCH model with non-normal innovations

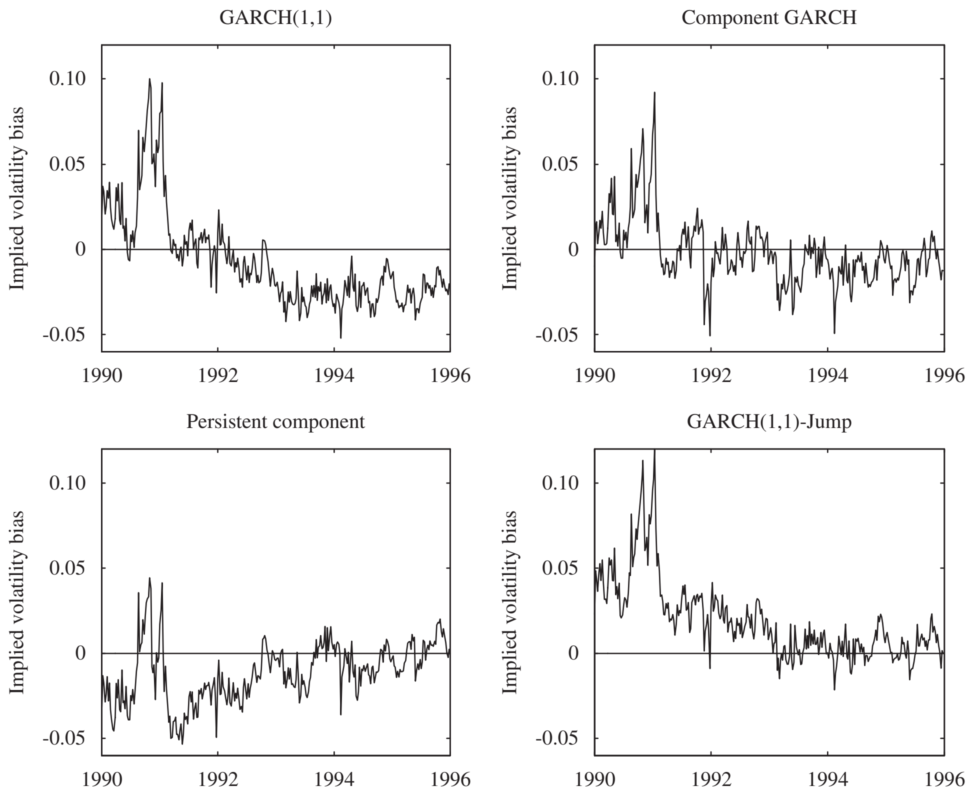

Fig. 7. Weekly implied volatility bias for at-the-money options. On each Wednesday we compute the Black and Scholes implied volatility for the at-the-money option contracts. Options with moneyness (index value over strike price) between 0.975 and 1.025 are considered at-the-money. The implied volatility is computed both for the market price and for each model price. We plot the weekly average difference between the market and model implied volatility. The top-left panel shows the GARCH(1,1) model, the top-right panel shows the component model, the bottom-left panel shows the persistent component model, and the bottom-right panel shows the GARCH(1,1)-Jump model. The maximum likelihood estimates from Tables 1 and 4 are used.

improves the model's fit in-sample and for short out-of-sample horizons, but not for long out-of-sample horizons. Furthermore, the analysis in Table 4 demonstrates that the component model provides a better option fit than the GARCH(1,1) model augmented with Poisson–normal jumps.

We now provide some more insight into the improved performance of the component models by analyzing the differences across models along three critical dimensions: the (spot) volatility level, the volatility term structure, and the modeling of the smirk. First, component models could better match the volatility patterns over time. We investigate this by comparing the differences in the time paths between implied volatilities from the data and the models. Second, it could be the case that the component models more adequately capture the term structure of volatility than the GARCH(1,1) model. We investigate this by comparing the models' term structures of implied volatility for at-the-money options. Third, it could be the case that the component models better capture the implied volatility smirk at various maturities. We study the differences between the models in this dimension in different volatility regimes.

Fig. 7 presents the average weekly at-the-money implied volatility bias (average observed market implied volatility less average model implied volatility) over the 1990–1995 option sample, using the estimates from Table 1. Here and below we define at-the-money to be options with strike price within 2.5% of the underlying index. Clearly the component models outperform the GARCH (1,1) model in this dimension: The GARCH(1,1) model shows significant underpricing (positive bias) during the high volatility episode in 1990–1991 and extended periods of overpricing (negative bias) during the low volatility period in 1993–1995. In comparison, the component model has smaller (positive) bias in 1990–1991 and also smaller (negative) bias in 1993–1995, suggesting that it is much better able to capture the dynamics of market volatility. The persistent model has the smallest (positive) bias in late 1990 but instead has significant (negative) bias in early 1990 and in late 1991. The GARCH(1,1)-Jump model performs the best during the 1993–1995 low volatility period. Somewhat surprisingly, however, it has a rather large bias during the high volatility 1990–1991 episode.

Broadie, Johannes, and Chernov (2007) report that the implied volatility from at-the-money options on average is higher than the realized volatility from returns. It is therefore interesting to ask whether this feature of the data is captured by the GARCH option pricing models. We compute the average implied volatility for at-the-money options during 1990–1995 to be 12.53%. The average at-the-money implied volatility in the GARCH(1,1) model is 13.11% compared with 12.68% for the component and 13.49% for the persistent component model. The component model thus comes close to matching the average implied volatility in the at-the-money options. Compare these implied volatility numbers with the average annual volatility in Table 1, which were 12.06% for the GARCH (1,1), 11.74% for the component model, and 12.39% for the persistent component model. All three models thus produce average implied at-the-money volatility that is roughly 1% higher than the average return volatility during the 1990–1995 period.

Fig. 8 studies the implied volatility term structure for at-the-money options in the four models. We again use the parameters from Table 1 to compute option prices. To study the importance of different volatility regimes, we study at-the-money options in three different sample periods. The first sample period is from September 1, 1995 until December 31, 1995. The average value of the option-implied volatility index (VIX) over this period is 12.70%, and we therefore think of it as a low volatility period. The second sample period is from August 1, 1991 until November 31, 1991, with an average VIX value of 16.66%, and it represents a medium volatility period. The third sample period is from August 1, 1990 until November 31, 1990, which is a high volatility period with an average VIX value of 27.97%. The main conclusion is that the differences between the models are sometimes very pronounced, but that these differences depend on the volatility regime. Whereas the fit of the GARCH(1,1) model and the component GARCH model do not differ much in the medium volatility regime, the component GARCH model significantly outperforms the GARCH(1,1) model in the low and high volatility periods. The GARCH-Jump

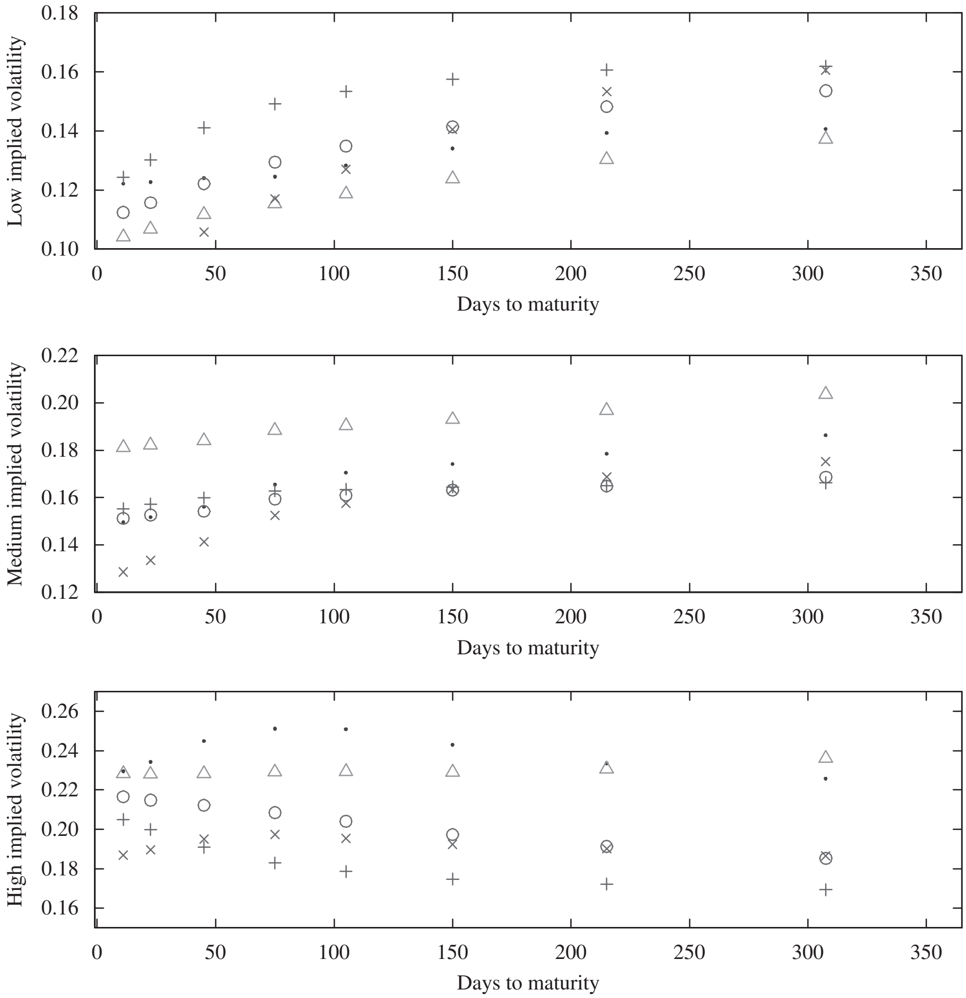

Fig. 8. Implied volatility term structures for at-the-money options. We compute average implied annualized Black and Scholes volatilities based on the data and the models for three different periods: a high volatility period (bottom panel), a medium volatility period (middle panel), and a low volatility period (top panel). We report on the three GARCH models and the GARCH-Jump model for options with different maturity. A dot represents the implied volatility of the data, a "+" represents the GARCH(1,1) model, a circle represents the component GARCH model, a triangle represents the persistent component model, and an "×" represents the GARCH(1,1)-Jump model. The maturity is on the horizontal axis. The maximum likelihood estimates from Tables 1 and 4 are used.

model captures some of the same biases as the component model but perhaps does a bit poorer in the high volatility period. The persistent component model clearly performs well in the high volatility regime.

These results can be explained as follows: In the component model, the initial volatility is much more important for the valuation of longer maturity options than in the GARCH(1,1) model, and even more so in the persistent component model. Put differently, in the GARCH(1,1) model, today's level of volatility has virtually no impact on the implied volatility for one-year maturity options. For the component model, the initial volatility has an effect on the implied volatility for one-year maturity options, and in the fully persistent model the effect of initial volatility is as large at the one-year maturity as it is at short maturities. The more persistent component therefore allows the component model to capture the variation in longer-maturity implied

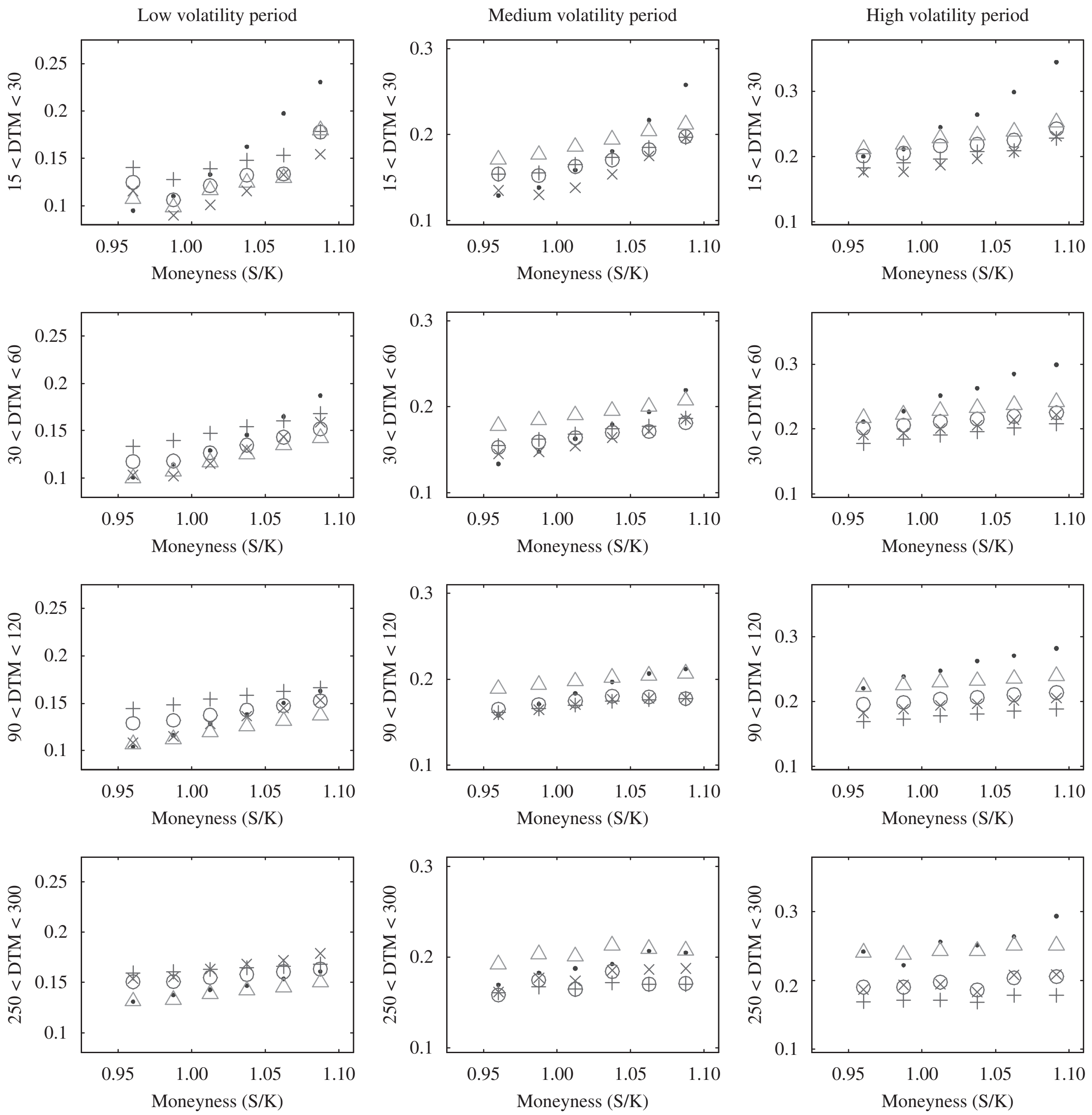

Fig. 9. Implied volatility smirks. We compute average implied annualized Black and Scholes volatilities based on the data and the models for three different periods: a high volatility period, a medium volatility period, and a low volatility period. We report on the three normal GARCH models and the GARCH(1,1)-Jump model for various levels of moneyness defined as index value divided by strike price (S/K) and days-to-maturity (DTM). A dot represents the implied volatility of the data, a "+" represents the GARCH(1,1) model, a circle represents the component GARCH model, a triangle represents the persistent component model, and an "×" represents the GARCH(1,1)-Jump model. The maximum likelihood estimates from Tables 1 and 4 are used.

volatilities. Moreover, the transitory component allows the component model to adequately capture short-maturity implied volatilities in different volatility regimes.

Fig. 9 builds on these findings and also analyzes a third source of differences in fit between the models. For each of the models, we plot the implied volatility smirks across moneyness for four different maturity buckets: 15–30, 30–60, 90–120, and 250–300 days to maturity. Following the exposition in Fig. 8, we repeat the analysis for three different volatility periods. The main conclusion from Fig. 9 is that, whereas the models greatly differ with respect to the level of the smirk, the slope of the smirk does not seem to differ much between models, although it is steeper for the GARCH(1,1)-Jump model in some cases. While the GARCH(1,1)-Jump model generates a steeper smirk in some cases, it underperforms the other models with respect to matching the level of implied volatilities. These two aspects of the model's performance explain the variations in performance across moneyness and maturity in Table 4, Panel B. All models tend to undervalue in-the-money call options across volatility regimes for the shortest maturity (top row panels) and for most maturities in the high volatility regimes (right-side panels). Even richer volatility and jump specifications could thus be needed.

We conclude that important differences exist between the GARCH(1,1), GARCH(1,1)-Jump, component, and persistent component models in terms of the path of spot volatility, the term structure of volatility, and the level of the volatility smirk. Importantly, the GARCH(1,1)-Jump, component, and persistent component models outperform the benchmark GARCH(1,1) model in virtually every dimension. The GARCH(1,1)-Jump model outperforms the component model in some dimensions, such as the fit of the volatility path in the less volatile part of the sample periods, but it underperforms the component model in the more volatile sample period. The performance of the component models for short-maturity and long-maturity options in high volatility regimes seems to be the main reason for their success. The model is able to successfully address both these dimensions thanks to the different time-series properties of the transitory and persistent components.

6. Estimation using option price information

So far we have used the option price information only to evaluate the different models. However, it stands to reason that the observed option prices should be helpful in estimating the models as well. In this section we therefore implement the GARCH(1,1), component, and persistent component models by minimizing the mean squared option valuation error instead of maximizing the log likelihood on returns as we did in Table 1.

We obtain parameters by minimizing the dollar mean squared error

$$\$MSE = \frac{1}{N^T}\sum_{t,i}(C^D_{i,t} - C^M_{i,t})^2, \tag{44}$$

where $C^D_{i,t}$ is the market price of option i at time t, $C^M_{i,t}$ is the model price, and $N^T = \sum_{t=1}^{T} N_t$. T is the total number of days included in the sample and N_t is the number of options included in the sample at date t. The variance dynamic is used to update the variance from one Wednesday to the next using daily returns, and the option valuation formula in Eq. (30) is used to compute the model prices on each Wednesday. The volatility updating rule is applied to the five hundred days predating the first Wednesday used in the estimation exercise, and it is initialized at the model's unconditional variance.

This nonlinear least squares (NLS) estimation technique is much more computationally intensive than the simple MLE on returns in Table 1. For each function evaluation performed by the numerical optimizer, the model price must be calculated for every option in the sample. The optimizer performs many function evaluations for each parameter update and consequently it is crucial to be able to compute option prices quickly and reliably. The pricing formula in Eq. (30) makes this estimation technique feasible. As we unfortunately do not have a closed-form pricing formula for the GARCH (1,1)-Jump model, we do not consider that model in this section.

Table 5 presents parameter estimates obtained using the 1990–1992 options data and in-sample RMSEs for the 1990–1992 data, as well as out-of-sample RMSEs using the 1993 data. The shortest maturity is seven days because options with very short maturities were filtered out. Table 6 presents parameter estimates obtained using options data for 1992–1994, as well as 1992–1994 in-sample and 1995 out-of-sample option RMSEs. Table 7 presents RMSE results by moneyness and maturity for the two in-sample and two out-of-sample periods.[8]

In Table 5 we present results for the 1990–1992 period (in-sample) and the 1993 period (out-of-sample). The standard errors indicate that almost all parameters are estimated significantly different from zero. There are some differences with the parameters estimated from returns in Table 1, but the parameters are mostly of the same order of magnitude. This is also true for critical determinants of the models' performance, such as average annual volatility, average volatility of variance, and average return correlation. Also, the persistence of the short-run components and the long-run components is not dramatically different from Table 1. The persistence of the GARCH(1,1) process is higher than in Table 1, though. In fact, the persistence of the GARCH(1,1) model and the component GARCH model is close to one. This motivates the use of the persistent component model, where the persistence is restricted to be one. Also, the average correlation between return and volatility is now close to minus one in all three models.

Table 5 contains two sets of RMSEs. The RMSEs for NLS are obtained using the parameter values in the table. We also report RMSEs based on the parameter values

[8] The risk-neutral dynamics in Eqs. (27) and (28) show that the parameter λ is not separately identified using option prices. We therefore simply set $\lambda = -0.5$, and we do not report λ in Tables 5 and 6.

Table 5

Nonlinear least squares (NLS) estimates and properties using two sets of data: 1990–1992 (in-sample) and 1993 (out-of-sample)

We use Wednesday option prices from January 1, 1990 to December 31, 1992 on the Standard and Poor's 500 index to estimate the three GARCH models using nonlinear least squares on the valuation errors. Robust standard errors are calculated from the outer product of the gradient at the optimum parameter values. RMSE refers to the square root of the mean-squared valuation errors. RMSE (90–92) refers to the 1990–1992 in-sample period, and RMSE (93) refers to the 1993 out-of-sample period. NLS refers to the model estimated using option data, and MLE (maximum likelihood estimation) refers to the model estimated using returns only. Normalized values are divided by the respective RMSE from GARCH(1,1).

Parameter	GARCH(1,1)		Parameter	Component GARCH		Parameter	Persistent component	
	Estimate	Standard error		Estimate	Standard error		Estimate	Standard error
w	3.891E − 14	3.560E − 12	$\tilde{\beta}$	7.050E − 01	2.565E − 01	$\tilde{\beta}$	7.201E − 01	1.021E − 01
b	6.801E − 01	3.211E − 03	α	1.770E − 06	3.444E − 07	α	1.597E − 06	2.279E − 06
a	2.666E − 07	6.110E − 09	γ_1	5.617E + 02	1.494E + 02	γ_1	7.481E + 02	8.974E + 01
c	1.090E + 03	5.432E + 01	γ_2	5.638E + 02	1.555E + 02	γ_2	4.767E + 02	1.246E + 02
			ω	2.424E − 07	1.212E − 07	ω	5.343E − 08	1.345E − 08
			φ	5.249E − 07	3.525E − 07	φ	5.123E − 07	1.285E − 07
			ρ	9.981E − 01	3.519E − 03	ρ	1.000E + 00	

	GARCH(1,1)	Component GARCH	Persistent component
Persistence	0.9970	0.9994	1.000
Average annual volatility	0.1347	0.1405	0.1431
Average volatility of variance	4.283E − 06	1.962E − 05	2.197E − 05
Average correlation	−0.9967	−0.9876	−0.9914

Option fitting performance	Estimation method		Estimation method		Estimation method	
	NLS	MLE	NLS	MLE	NLS	MLE
RMSE (90–92)	1.038	1.896	0.931	1.609	0.984	2.143
Normalized	1.000	1.000	0.897	0.849	0.948	1.130
RMSE (93)	1.284	2.229	0.983	1.584	1.198	1.260
Normalized	1.000	1.000	0.765	0.710	0.933	0.565

Table 6

Nonlinear least squares (NLS) estimates and properties using two sets of data: 1992–1994 (in-sample) and 1995 (out-of-sample)

We use Wednesday option prices from January 1, 1992 to December 31, 1994 on the Standard and Poor's 500 index to estimate the three GARCH models using nonlinear least squares on the valuation errors. Robust standard errors are calculated from the outer product of the gradient at the optimum parameter values. RMSE refers to the square root of the mean-squared valuation errors. RMSE (92–94) refers to the 1992–1994 in-sample period, and RMSE (95) refers to the 1995 out-of-sample period. NLS refers to the model estimated using option data, and MLE (maximum likelihood estimation) refers to the model estimated using returns only. Normalized values are divided by the respective RMSE from GARCH(1,1).

Parameter	GARCH(1,1)		Parameter	Component GARCH		Parameter	Persistent component	
	Estimate	Standard error		Estimate	Standard error		Estimate	Standard error
w	7.521E − 16	3.498E − 09	$\tilde{\beta}$	9.297E − 01	3.346E − 02	$\tilde{\beta}$	9.587E − 01	3.821E − 05
b	4.694E − 01	1.251E − 01	α	1.808E − 06	1.320E − 07	α	1.943E − 06	1.614E − 06
a	1.936E − 06	3.986E − 07	γ_1	5.854E + 02	2.362E + 02	γ_1	2.589E + 02	8.383E + 01
c	5.061E + 02	1.041E + 02	γ_2	5.749E + 02	4.025E + 02	γ_2	2.254E + 02	5.063E + 02
			ω	2.204E − 07	3.470E − 08	ω	6.927E − 08	1.262E − 08
			φ	2.835E − 07	1.586E − 07	φ	6.971E − 07	1.253E − 08
			ρ	9.966E − 01	1.277E − 03	ρ	1.000E + 00	

	GARCH(1,1)	Component GARCH	Persistent component
Persistence	0.9654	0.9998	1.0000
Average annual volatility	0.1074	0.1129	0.1082
Average volatility of variance	1.423E − 05	1.838E − 05	1.085E − 05
Average correlation	−0.9701	−0.9781	−0.9095

Option fitting performance	Estimation method		Estimation method		Estimation method	
	NLS	MLE	NLS	MLE	NLS	MLE
RMSE (92–94)	1.107	2.000	0.855	1.524	1.058	1.402
Normalized	1.000	1.000	0.773	0.762	0.956	0.701
RMSE (95)	1.227	2.775	0.972	1.920	1.174	1.249
Normalized	1.000	1.000	0.792	0.692	0.957	0.450

Table 7

Root mean squared errors (RMSE) and ratio RMSE by moneyness and maturity

We use the nonlinear least squares estimates from Table 5 and 6 to compute the root mean squared option valuation error (RMSE) for various moneyness and maturity bins. The RMSE is reported in levels for the GARCH(1,1) model. For the component and persistent component models, we report the RMSE ratio with respect to the GARCH(1,1) model. DTM refers to the number of days to maturity and S/K refers to moneyness defined as the underlying index level divided by the option strike price.

Option features	Panel A. 1990–1992 in sample			Panel B. 1993 out-of-sample		
	GARCH(1,1)	Component	Persistent	GARCH(1,1)	Component	Persistent
	RMSE	RMSE ratio	RMSE ratio	RMSE	RMSE ratio	RMSE ratio
Moneyness						
S/K < 0.975	1.078	0.873	0.987	1.461	0.556	1.028
0.975 < S/K < 1.00	1.054	0.927	0.977	1.631	0.644	0.671
1.00 < S/K < 1.025	0.956	0.947	1.003	1.356	0.749	0.763
1.025 < S/K < 1.05	0.919	0.936	0.941	1.008	0.883	0.876
1.05 < S/K < 1.075	1.032	0.902	0.891	0.860	0.902	1.071
1.075 < S/K	1.079	0.883	0.861	1.166	0.977	1.068
All	1.038	0.897	0.948	1.284	0.765	0.933
Maturity						
DTM < 20	0.610	0.923	0.903	0.813	0.994	0.994
20 < DTM < 80	0.976	0.865	0.859	1.124	0.828	0.782
80 < DTM < 180	1.124	0.883	0.891	1.258	0.778	0.914
DTM > 180	1.151	0.960	1.121	1.822	0.652	1.079
All	1.038	0.897	0.948	1.284	0.765	0.933
	Panel C. 1992–1994 in sample			Panel D. 1995 out-of-sample		
	GARCH(1,1)	Component	Persistent	GARCH(1,1)	Component	Persistent
	RMSE	RMSE ratio	RMSE ratio	RMSE	RMSE ratio	RMSE ratio
Moneyness						
S/K < 0.975	1.122	0.604	1.010	1.771	0.757	0.781
0.975 < S/K < 1.00	1.275	0.747	0.964	1.546	0.783	0.985
1.00 < S/K < 1.025	1.228	0.760	0.918	1.389	0.708	0.950
1.025 < S/K < 1.05	1.002	0.790	0.900	1.110	0.705	1.021
1.05 < S/K < 1.075	0.991	0.785	0.856	0.896	0.796	1.092
1.075 < S/K	1.032	0.919	0.969	0.681	1.026	1.286
All	1.107	0.773	0.956	1.227	0.792	0.957
Maturity						
DTM < 20	0.857	0.909	0.899	0.744	0.978	0.952
20 < DTM < 80	1.009	0.808	0.927	0.846	0.929	1.129
80 < DTM < 180	1.045	0.717	1.000	1.187	0.847	1.016
DTM > 180	1.422	0.744	0.959	1.848	0.670	0.848
All	1.107	0.773	0.956	1.227	0.792	0.957

obtained from MLE in Table 1. First consider the RMSEs obtained using NLS. In the in-sample 1990–1992 period, the RMSE of the component model is 89.7% of that of the benchmark GARCH(1,1) model. For the out-of-sample 1993 period, the ratio of the RMSEs is 76.5%. For the persistent component model, the ratios are 94.8% and 93.3%, respectively. Using the MLEs, the relative RMSEs are similar for the component model: 84.9% in 1990–1992 and 71.0% in 1993. Using the maximum likelihood estimates, the persistent component model performs relatively worse in 1990–1992 with 113.0%, but better in 1993 with 56.5% of the RMSE in the GARCH(1,1). When comparing across maximum likelihood and NLS estimates, the RMSEs from NLS are typically much smaller than those from MLE. The information in option prices is clearly valuable for estimating the models. The only example in which the RMSE from MLE is close to that of the NLS counterpart is for the persistent component model in the 1993 out-of-sample period.

Table 6 presents the results for the 1992–1994 period (in-sample) and the 1995 period (out-of-sample). The results largely confirm those obtained in Table 5. The most important difference is that the in-sample and out-of-sample performance of the component model is even better relative to the benchmark, as compared with the results in Table 5. When using NLS estimates, the component model's RMSE is 77.3% of that of the GARCH(1,1) model for the 1992–1994 in-sample period, and for the 1995 out-of-sample period the ratio is 79.2%. For the persistent component model the ratios are 95.6% and 95.7%, respectively. When using maximum likelihood estimates the component model is 76.2% of the GARCH (1,1) in 1992–1994 and 69.2% in 1995. The persistent model performs very well relative to the GARCH(1,1) MLE generating a 70.1% relative RMSE for 1992–1994 and 45.0% in 1995.

Comparing RMSEs across NLS and MLE parameters, we again find that the option prices add important information and drive the NLS RMSEs down from their MLE levels. Again, the only case in which the RMSE from MLE comes close to that from NLS is for the out-of-sample persistent model. Interesting differences with Table 5 are that the persistence of the short-run component is higher and that the persistence of the GARCH(1,1) process in Table 6 is lower than in Table 5 but in line with the maximum likelihood estimate in Table 1.

Table 7 provides a more detailed analysis of the valuation errors by presenting RMSE results by moneyness and maturity, using the parameter estimates from Tables 5 and 6. The table contains the RMSE in dollars for the GARCH(1,1) model. To facilitate the interpretation of the table, the RMSEs for the component and persistent component models are normalized by the corresponding RMSE for the GARCH(1,1) model. An overall RMSE, which is not too different across the three models in Tables 5 and 6, can mask large differences in the models' performance for a given moneyness or maturity. Inspection of the out-of-sample results in Panels B and D is especially instructive. We conclude that the improved out-of-sample performance of the component models is due to the improved valuation of long-maturity options. This is an interesting affirmation of the intuition obtained previously using return-based estimates in Figs. 1–2 and Figs. 8–9. The richer volatility dynamic in the component model enables more accurate modeling of variations in long-maturity option prices.

Overall, we conclude that, based on the parameter values obtained using NLS, the performance of the component GARCH model is impressive. Its RMSE is between 76.5% and 89.7% of the RMSE of the benchmark GARCH(1,1) model. The performance of the persistent component model is less impressive, both in-sample and out-of-sample. However the persistent component model performs relatively well in the out-of-sample experiments when MLE parameters are used. This suggests that the persistent component model could be valuable for option valuation when no option price information is available for parameter estimation.

7. Conclusion and directions for future work

This paper presents a new option valuation model based on the work by Engle and Lee (1999) and Heston and Nandi (2000). The empirical performance of the new variance component model is significantly better than that of the benchmark GARCH(1,1) model, in-sample as well as out-of-sample, and regardless of the information used in estimation. This is an important finding because the literature has demonstrated that it is difficult to find empirical models that improve on the GARCH(1,1) model or the Heston (1993) model. We also compare the component model with a GARCH(1,1)-Jump model, which combines conditional heteroskedasticity with Poisson–normal jumps. The GARCH(1,1)-Jump model achieves a better statistical fit to daily returns than the component model, but the component model performs better when using the parameter estimates to fit options. Overall, the GARCH(1,1)-Jump model and the variance component model remedy some of the same biases left unaddressed by the GARCH(1,1) model.

An important aspect of the model's improved performance is that its richer parameterization allows for improved joint modeling of long-maturity and short-maturity options. The model captures the stylized fact that shocks to current conditional volatility impact on the conditional variance forecast up to a year in the future, which results in a very different implied volatility term structure for at-the-money options. The component model also results in a different path for spot volatility compared with the GARCH(1,1) model.

Because the estimated persistence of the model is close to one, we also investigate a special case of our model in which shocks to the variance never die out. When estimating model parameters by maximum likelihood using a historical time series of returns, the persistent component model is somewhat inferior to the component model when judged by the likelihood criterion. When the MLE parameters are used to price options, the persistent component model performs similarly to the component model in terms of overall fit. This is partly due to its ability to model long-maturity options, especially in high volatility periods.

When model parameters are estimated from option prices, the component model significantly outperforms the other models both in- and out-of-sample. We also find that for a given model the parameters obtained from historical return data always lead to higher RMSEs than the parameters directly estimated from option data.

Given the success of the proposed component models, a number of further extensions to this work are warranted. First, the empirical performance of the model should be validated using other data sets. In particular, it would be interesting to test the model using long-term equity anticipation security (LEAPS) data, because the model could excel at modeling long-maturity LEAPS options. In this regard a direct comparison between component and fractionally integrated volatility models could be interesting. It could also be useful to combine the stylized features of the model with other modeling components that improve option valuation. One interesting experiment could be to replace the Poisson jump innovations considered in this paper with another non-Gaussian distribution. Combining the model in this paper with the inverse Gaussian shock model in Christoffersen, Heston, and Jacobs (2006) could be a viable approach. In this paper we propose a component model that gives a closed-form solution using results from Heston and Nandi (2000) who rely on an affine GARCH model. We believe that this is a logical first step, but the affine structure of the model could be restrictive in ways that are not immediately apparent. It could therefore prove worthwhile to investigate nonaffine variance component models. A final remark is that the performance of the GARCH(1,1)-Jump model, which we use here as a benchmark for the component models, is encouraging. It could prove worthwhile to further analyze its performance, for instance, by estimating the model using option data.

Appendix A

This appendix derives the MGF for the component GARCH process. The component GARCH process is given by

$$\begin{aligned}\ln(S_{t+1}) &= \ln(S_t) + r + \lambda h_{t+1} + \sqrt{h_{t+1}}z_{t+1},\\ h_{t+1} &= q_{t+1} + \tilde{\beta}(h_t - q_t) + \alpha((z_t - \gamma_1\sqrt{h_t})^2 - (1+\gamma_1^2 h_t)), \quad \text{and}\\ q_{t+1} &= \omega + \rho q_t + \varphi((z_t - \gamma_2\sqrt{h_t})^2 - (1+\gamma_2^2 h_t)). \end{aligned} \tag{45}$$

Let $x_t = \ln(S_t)$. For convenience we denote the time t conditional generating function of S_T [or equivalently the conditional MGF of x_T] by f_t instead of the more cumbersome $f(t;T,\phi)$. By definition

$$f_t = \mathrm{E}_t[\exp(\phi x_T)]. \tag{46}$$

We guess that the MGF has the log-linear form. We use the more parsimonious notation A_t to indicate $A(t;T,\phi)$ and similarly for $B_{1,t}$ and $B_{2,t}$ and write

$$f_t = \exp(\phi x_t + A_t + B_{1,t}(h_{t+1} - q_{t+1}) + B_{2,t}q_{t+1}). \tag{47}$$

We have the terminal condition $A_T = B_{1,T} = B_{2,T} = 0$. Applying the law of iterated expectations to f_t in Eq. (46) we get

$$f_t = \mathrm{E}_t[f_{t+1}] = \mathrm{E}_t \exp(\phi x_{t+1} + A_{t+1} + B_{1,t+1}(h_{t+2} - q_{t+2}) + B_{2,t+1}q_{t+2}). \tag{48}$$

Substituting the dynamics of x_{t+1} gives

$$\begin{aligned} f_t &= \mathrm{E}_t \exp\begin{pmatrix} \phi(x_t + r) + \phi\lambda h_{t+1} + \phi\sqrt{h_{t+1}}z_{t+1} + A_{t+1}\\ +B_{1,t+1}(h_{t+2} - q_{t+2}) + B_{2,t+1}q_{t+2}\end{pmatrix}\\ &= \mathrm{E}_t \exp\begin{pmatrix} \phi(x_t + r) + \phi\lambda h_{t+1} + \phi\sqrt{h_{t+1}}z_{t+1} + A_{t+1}\\ +B_{1,t+1}(\tilde{\beta}(h_{t+1} - q_{t+1}) + \alpha((z_{t+1} - \gamma_1\sqrt{h_{t+1}})^2\\ -(1+\gamma_1^2 h_{t+1})))\\ +B_{2,t+1}(\omega + \rho q_{t+1} + \varphi((z_{t+1} - \gamma_2\sqrt{h_{t+1}})^2\\ -(1+\gamma_2^2 h_{t+1})))\end{pmatrix}\\ &= \mathrm{E}_t \exp\begin{pmatrix} \phi(x_t + r) + \phi\lambda h_{t+1} + A_{t+1} + B_{1,t+1}\tilde{\beta}(h_{t+1} - q_{t+1})\\ +B_{2,t+1}(\omega + \rho q_{t+1}) - (\alpha B_{1,t+1} + \varphi B_{2,t+1})\\ +(\alpha B_{1,t+1} + \varphi B_{2,t+1})\\ \times\left(z_{t+1} - \dfrac{\alpha\gamma_1 B_{1,t+1} + \varphi\gamma_2 B_{2,t+1} - 0.5\phi}{(\alpha B_{1,t+1} + \varphi B_{2,t+1})}\sqrt{h_{t+1}}\right)^2\\ -\dfrac{(\alpha\gamma_1 B_{1,t+1} + \varphi\gamma_2 B_{2,t+1} - 0.5\phi)^2}{(\alpha B_{1,t+1} + \varphi B_{2,t+1})}h_{t+1}\end{pmatrix}. \end{aligned} \tag{49}$$

Using the following result for a standard normal variable, z,

$$\mathrm{E}[\exp(x(z+y)^2)] = \exp(-\tfrac{1}{2}\ln(1-2x) + xy^2/(1-2x)), \tag{50}$$

we get

$$f_t = \mathrm{E}_t \exp\begin{pmatrix} \phi(x_t + r) + A_{t+1} - (\alpha B_{1,t+1} + \varphi B_{2,t+1})\\ -1/2\ln(1 - 2\alpha B_{1,t+1} - 2\varphi B_{2,t+1}) + B_{2,t+1}\omega\\ +B_{1,t+1}\tilde{\beta}(h_{t+1} - q_{t+1}) + B_{2,t+1}\rho q_{t+1}\\ +\left(\lambda\phi + 2\dfrac{(\alpha\gamma_1 B_{1,t+1} + \varphi\gamma_2 B_{2,t+1} - 0.5\phi)^2}{1 - 2\alpha B_{1,t+1} - 2\varphi B_{2,t+1}}\right)h_{t+1}\end{pmatrix}. \tag{51}$$

Matching terms in Eq. (51) and Eq. (47) gives

$$\begin{aligned} A_t &= A_{t+1} + r\phi - (\alpha B_{1,t+1} + \varphi B_{2,t+1}) - 1/2\ln(1 - 2\alpha B_{1,t+1} - 2\varphi B_{2,t+1}) + B_{2,t+1}\omega,\\ B_{1,t} &= B_{1,t+1}\tilde{\beta} + \lambda\phi + 2\frac{(\alpha\gamma_1 B_{1,t+1} + \varphi\gamma_2 B_{2,t+1} - 0.5\phi)^2}{1 - 2\alpha B_{1,t+1} - 2\varphi B_{2,t+1}},\\ B_{2,t} &= B_{2,t+1}\rho + \lambda\phi + 2\frac{(\alpha\gamma_1 B_{1,t+1} + \varphi\gamma_2 B_{2,t+1} - 0.5\phi)^2}{1 - 2\alpha B_{1,t+1} - 2\varphi B_{2,t+1}}. \end{aligned} \tag{52}$$

Appendix B

The physical component GARCH dynamic is given by

$$\begin{aligned}\ln(S_{t+1}) &= \ln(S_t) + r + \lambda h_{t+1} + \sqrt{h_{t+1}}z_{t+1},\\ h_{t+1} &= q_{t+1} + \tilde{\beta}(h_t - q_t) + \alpha((z_t - \gamma_1\sqrt{h_t})^2 - (1+\gamma_1^2 h_t)), \quad \text{and}\end{aligned} \tag{53}$$

$$q_{t+1} = \omega + \rho q_t + \varphi((z_t - \gamma_2\sqrt{h_t})^2 - (1+\gamma_2^2 h_t)). \tag{54}$$

Under the risk-neutral measure, we need $\mathrm{E}^*[S_{t+1}/S_t] = \exp(r)$, which requires that

$$\ln(S_{t+1}) = \ln(S_t) + r - 0.5h_{t+1} + \sqrt{h_{t+1}}z^*_{t+1}. \tag{55}$$

This implies in turn that

$$z^*_{t+1} = z_{t+1} + (\lambda + 0.5)\sqrt{h_{t+1}}. \tag{56}$$

We also want to ensure that the conditional variances are the same under the two measures

$$Var_t(R_{t+1}) = Var^*_t(R_{t+1}). \tag{57}$$

We therefore need to have the same variance innovations under the two measures. Thus we need

$$(z_t - \gamma_i\sqrt{h_t})^2 = (z^*_t - \gamma^*_i\sqrt{h_t})^2, \quad i = 1, 2, \tag{58}$$

which can be achieved by defining new risk-neutral parameters $\gamma^*_i = \gamma_i + \lambda + 0.5$, $i = 1, 2$.

Consider the following candidate for the risk-neutral component GARCH dynamic:

$$\begin{aligned} h_{t+1} &= q_{t+1} + \tilde{\beta}^*(h_t - q_t) \\ &\quad + \alpha((z^*_t - \gamma^*_1\sqrt{h_t})^2 - (1 + \gamma^{*2}_1 h_t)) \quad \text{and} \end{aligned} \tag{59}$$

$$q_{t+1} = \omega + \rho^* q_t + \varphi((z^*_t - \gamma^*_2\sqrt{h_t})^2 - (1 + \gamma^{*2}_2 h_t)), \tag{60}$$

where $z^*_t \sim N(0, 1)$ and the risk-neutral parameters are defined as

$$\begin{aligned} \tilde{\beta}^* &= \tilde{\beta} + \alpha(\gamma^{*2}_1 - \gamma^2_1) + \varphi(\gamma^{*2}_2 - \gamma^2_2) \quad \text{and} \\ \rho^* &= \rho + \alpha(\gamma^{*2}_1 - \gamma^2_1) + \varphi(\gamma^{*2}_2 - \gamma^2_2). \end{aligned} \tag{61}$$

For this candidate risk-neutral dynamic to be valid, we have to verify that it is consistent with Eqs. (53) and (54). Using Eqs. (56), (61), and (60) in Eq. (59) we get

$$\begin{aligned} h_{t+1} &= \omega + \rho q_t + \varphi((z_t - \gamma_2\sqrt{h_t})^2 - (1 + \gamma^2_2 h_t)) + \tilde{\beta}(h_t - q_t) \\ &\quad + \alpha((z_t - \gamma_1\sqrt{h_t})^2 - (1 + \gamma^2_1 h_t)), \end{aligned} \tag{62}$$

which is identical to what we get using the physical dynamic in Eqs. (53) and (54).

References

Adrian, T., Rosenberg, J., 2008. Stock returns and volatility: pricing the long-run and short-run components of market risk. Journal of Finance, forthcoming.

Ait-Sahalia, Y., Lo, A., 1998. Nonparametric estimation of state-price densities implicit in financial asset prices. Journal of Finance 53, 499–547.

Alizadeh, S., Brandt, M., Diebold, F., 2002. Range-based estimation of stochastic volatility models. Journal of Finance 57, 1047–1091.

Amin, K., Ng, V., 1993. ARCH processes and option valuation. Unpublished working paper. University of Michigan, Ann Arbor, MI.

Andersen, T., Benzoni, L., Lund, J., 2002. An empirical investigation of continuous-time equity return models. Journal of Finance 57, 1239–1284.

Andersen, T., Bollerslev, T., Diebold, F., Labys, P., 2003. Modeling and forecasting realized volatility. Econometrica 71, 529–626.

Baillie, R., Bollerslev, T., Mikkelsen, H., 1996. Fractionally integrated generalized autoregressive conditional heteroskedasticity. Journal of Econometrics 74, 3–30.

Bakshi, C., Cao, C., Chen, Z., 1997. Empirical performance of alternative option pricing models. Journal of Finance 52, 2003–2049.

Bates, D., 1996. Jumps and stochastic volatility: exchange rate processes implicit in deutsche mark options. Review of Financial Studies 9, 69–107.

Bates, D., 2000. Post-87 crash fears in S&P 500 futures options. Journal of Econometrics 94, 181–238.

Bates, D., 2006. Maximum likelihood estimation of latent affine processes. Review of Financial Studies 19, 909–965.

Black, F., 1976. Studies of stock price volatility changes. In: Proceedings of the 1976 Meetings of the Business and Economic Statistics Section. American Statistical Association, pp. 177–181.

Black, F., Scholes, M., 1973. The pricing of options and corporate liabilities. Journal of Political Economy 81, 637–659.

Bollerslev, T., 1986. Generalized autoregressive conditional heteroskedasticity. Journal of Econometrics 31, 307–327.

Bollerslev, T., Mikkelsen, H., 1996. Modeling and pricing long memory in stock market volatility. Journal of Econometrics 73, 151–184.

Bollerslev, T., Mikkelsen, H., 1999. Long-term equity anticipation securities and stock market volatility dynamics. Journal of Econometrics 92, 75–99.

Bollerslev, T., Zhou, H., 2002. Estimating stochastic volatility using conditional moments of integrated volatility. Journal of Econometrics 109, 33–65.

Brandt, M., Jones, C., 2006. Forecasting volatility with range-based EGARCH models. Journal of Business and Economic Statistics 24, 470–486.

Brennan, M., 1979. The pricing of contingent claims in discrete-time models. Journal of Finance 34, 53–68.

Broadie, M., Chernov, M., Johannes, M., 2007. Model specification and risk premiums: evidence from futures options. Journal of Finance 62, 1453–1490.

Camara, A., 2003. A generalization of the Brennan-Rubinstein approach for the pricing of derivatives. Journal of Finance 58, 805–819.

Carr, P., Wu, L., 2004. Time-changed Lévy processes and option pricing. Journal of Financial Economics 17, 113–141.

Chacko, G., Viceira, L., 2003. Spectral GMM estimation of continuous-time processes. Journal of Econometrics 116, 259–292.

Chernov, M., Gallant, R., Ghysels, E., Tauchen, G., 2003. Alternative models for stock price dynamics. Journal of Econometrics 116, 225–257.

Chernov, M., Ghysels, E., 2000. A study towards a unified approach to the joint estimation of objective and risk-neutral measures for the purpose of option valuation. Journal of Financial Economics 56, 407–458.

Christoffersen, P., Heston, S., Jacobs, K., 2006. Option valuation with conditional skewness. Journal of Econometrics 131, 253–284.

Christoffersen, P., Jacobs, K., 2004. Which GARCH model for option valuation? Management Science 50, 1204–1221.

Comte, F., Coutin, L., Renault, E., 2001. Affine fractional stochastic volatility models. Unpublished working paper, University of Montreal, Montreal, Canada.

Dai, Q., Le, A., Singleton, K., 2006. Discrete-time dynamic term structure models with generalized market prices of risk. Unpublished working paper. Stanford University, Stanford, CA.

Dai, Q., Singleton, K., 2000. Specification analysis of affine term structure models. Journal of Finance 55, 1943–1978.

Das, S., Sundaram, R., 1999. Of smiles and smirks: a term structure perspective. Journal of Financial and Quantitative Analysis 34, 211–239.

Ding, Z., Granger, C., Engle, R., 1993. A long memory property of stock market returns and a new model. Journal of Empirical Finance 1, 83–106.

Duan, J.-C., 1995. The GARCH option pricing model. Mathematical Finance 5, 13–32.

Duan, J.-C., Ritchken, P., Sun, Z., 2005. Jump starting GARCH: pricing and hedging options with jumps in returns and volatilities. Unpublished working paper. University of Toronto, Rotman School, Toronto, Canada.

Duan, J.-C., Ritchken, P., Sun, Z., 2006. Approximating GARCH-jump models, jump-diffusion processes, and option pricing. Mathematical Finance 16, 21–52.

Duffee, G., 1999. Estimating the price of default risk. Review of Financial Studies 12, 197–226.

Duffie, D., Pan, J., Singleton, K., 2000. Transform analysis and asset pricing for affine jump-diffusions. Econometrica 68, 1343–1376.

Duffie, D., Singleton, K., 1999. Modeling term structures of defaultable bonds. Review of Financial Studies 12, 687–720.

Dumas, B., Fleming, J., Whaley, R., 1998. Implied volatility functions: empirical tests. Journal of Finance 53, 2059–2106.

Engle, R., 1982. Autoregressive conditional heteroskedasticity with estimates of the variance of UK inflation. Econometrica 50, 987–1008.

Engle, R., Lee, G., 1999. A permanent and transitory component model of stock return volatility. In: Engle, R., White, H. (Eds.), Cointegration, Causality, and Forecasting: A Festschrift in Honor of Clive W.J. Granger. Oxford University Press, New York, pp. 475–497.

Engle, R., Mustafa, C., 1992. Implied ARCH models from options prices. Journal of Econometrics 52, 289–311.

Engle, R., Ng, V., 1993. Measuring and testing the impact of news on volatility. Journal of Finance 48, 1749–1778.

Engle, R., Patton, A., 2001. What good is a volatility model? Quantitative Finance 1, 237–245.

Eraker, B., 2004. Do stock prices and volatility jump? Reconciling evidence from spot and option prices. Journal of Finance 59, 1367–1403.

Eraker, B., Johannes, M., Polson, N., 2003. The impact of jumps in volatility and returns. Journal of Finance 58, 1269–1300.

Fama, E., French, K., 1988. Permanent and temporary components of stock prices. Journal of Political Economy 96, 246–273.

French, K., Schwert, G.W., Stambaugh, R., 1987. Expected stock returns and volatility. Journal of Financial Economics 19, 3–30.

Hentschel, L., 1995. All in the family: nesting symmetric and asymmetric GARCH models. Journal of Financial Economics 39, 71–104.

Heston, S., 1993. A closed-form solution for options with stochastic volatility with applications to bond and currency options. Review of Financial Studies 6, 327–343.

Heston, S., Nandi, S., 2000. A closed-form GARCH option pricing model. Review of Financial Studies 13, 585–626.

Hsieh, K., Ritchken, P., 2005. An empirical comparison of GARCH option pricing models. Review of Derivatives Research 8, 129–150.

Huang, J.-Z., Wu, L., 2004. Specification analysis of option pricing models based on time-changed Lévy processes. Journal of Finance 59, 1405–1439.

Jones, C., 2003. The dynamics of stochastic volatility: evidence from underlying and options markets. Journal of Econometrics 116, 181–224.

Maheu, J., 2005. Can GARCH models capture long-range dependence? Studies in Nonlinear Dynamics and Econometrics 9, Article 1.

Maheu, J., McCurdy, T., 2004. News arrival, jump dynamics, and volatility components for individual stock returns. Journal of Finance 59, 755–793.

Nandi, S., 1998. How important is the correlation between returns and volatility in a stochastic volatility model? Empirical evidence from pricing and hedging in the S&P 500 index options market. Journal of Banking and Finance 22, 589–610.

Pan, J., 2002. The jump-risk premia implicit in options: evidence from an integrated time-series study. Journal of Financial Economics 63, 3–50.

Pearson, N., Sun, T., 1994. Exploiting the conditional density in estimating the term structure: an application to the Cox, Ingersoll, and Ross model. Journal of Finance 49, 1279–1304.

Poterba, J., Summers, L., 1988. Mean reversion in stock returns: evidence and implications. Journal of Financial Economics 22, 27–60.

Ritchken, P., Trevor, R., 1999. Pricing options under generalized GARCH and stochastic volatility processes. Journal of Finance 54, 377–402.

Rubinstein, M., 1976. The valuation of uncertain income streams and the pricing of options. Bell Journal of Economics 7, 407–425.

Schroder, M., 2004. Risk-neutral parameter shifts and derivative pricing in discrete time. Journal of Finance 59, 2375–2401.

Summers, L., 1986. Does the stock market rationally reflect fundamental values? Journal of Finance 41, 591–600.

Taylor, S., Xu, X., 1994. The term structure of volatility implied by foreign exchange options. Journal of Financial and Quantitative Analysis 29, 57–74.