Outline

January 18, 2017 13:02

Department of Mathematics & Statistics
Concordia University

MAST 218 (MATH 264)
Multivariate Calculus I
Winter 2017

Instructor: Dr. P. Gora, Office: LB 901-17 (SGW), Phone: 514-848-2424, Ext. 3257
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Class Schedule: Wednesday, 18:00-20:15 in H 620, SGW Campus.

Office Hours: TBA

Prerequisites: Math 205 or an equivalent Calculus II course.

Text: *Multivariable Calculus*, 8th Edition by J. Stewart, Cengage Learning, 2015.

Assignments: Assignments are ***very important*** as they indicate the level of difficulty of the problems that the students are expected to solve. Therefore, every effort should be made to **do and understand the assignment problems.** The assignments will be corrected and graded.

Web Resources: Many excellent animated illustrations to the text of the book are collected at the site **www.stewartcalculus.com**, see TEC (Tools for Enriching Calculus) for the edition 6. Regular use of this resource is much recommended.

Use of Computer Algebra System: It is optional but much recommended to install and use Maple or Mathematica. These computer tools can be used to verify and illustrate any analytical results you get while doing your assignment problems.

Calculators: Electronic communication devices (including cell phones) are not allowed in the examination rooms. Only "Faculty Approved Calculators" **SHARP EL-531** or **CASIO FX-300MS**) are allowed in the examination rooms during the midterm exam and the final exam.

Test: Midterm exam covering the first six weeks will be given in week 8.

Final Grade: The highest of the following:
- 90% final exam, 10% assignments.
- 30% midterm, 10% assignments, and 60% final exam.

Departmental website → www.concordia.ca/artsci/math-stats.html

Week	Sections	Topics
1	10.1, 10.2	Parametric equations of curves.
2	10.3, 10.4, 10.5	Areas and lengths in polar coordinates. Conic sections.
3	10.6, 11.10, 12.1	Conic sections in polar coordinates. Taylor series: review. Three-dimensional coordinate systems.
4	12.2, 12.3, 12.4	Vectors. Dot product. Cross product.
5	12.5, 12.6	Equations of lines and planes. Cylinders and quadric surfaces.
6	13.1, 13.2	Vector functions and space curves. Derivatives and integrals of vector functions.
7	13.3, 13.4	Arc length and curvature of space curve. Velocity and acceleration.
8	14.1, 14.2	Functions of several variables, their limits and continuity.
9	14.3, 14.4	Partial derivatives. Tangent planes and linear approximation.
10	14.5, 14.6	Chain rule. Directional derivatives and gradient vector.
11	14.7	Maximum and minimum values.
12	14.8	Lagrange multipliers.
13		**Review**

10.1 Curves Defined by Parametric Equations (P712)

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Definitions & Theorems:

1. Parametric Equations

   Suppose that x and y are both given as functions of a third variable t (called a parameter) by the equations
   $$x = f(t) \qquad y = g(t)$$
   These equations are called parametric equations.

2. Parametric Curve

   Each value of t determines a point (x, y), which we can plot in a coordinate plane. As t varies, the point $(x, y) = (f(t), g(t))$ varies and traces out a curve C, which we call a parametric curve.

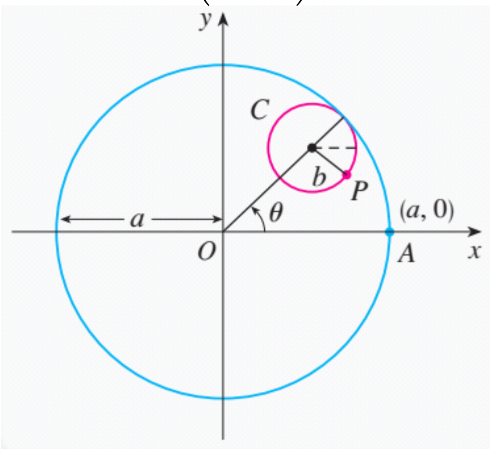
   In general, the curve with parametric equations
   $$x = f(t) \qquad y = g(t) \qquad a \le t \le b$$
   has ***initial point*** $(f(a), g(a))$ and ***terminal point*** $(f(b), g(b))$.

3. The Cycloid

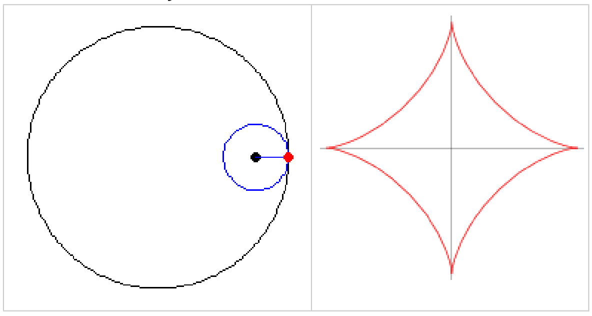
   See Example 4

4. Astroid

   a) Hypocycloid: A hypocycloid is a curve traced out by a fixed point P on a circle C of radius b as C rolls on the inside of a circle with center O and radius a. Show that if the initial position of P is $(a, 0)$ and the parameter θ is chosen as in the figure, then parametric equations of the hypocycloid are
   $$x = (a-b)\cos\theta + b\cos\left(\frac{a-b}{b}\theta\right), \qquad y = (a-b)\sin\theta - b\sin\left(\frac{a-b}{b}\theta\right)$$



   b) Astroid: $a = 4, b = 1$
   $$x = 4\cos^3\theta, \qquad y = 4\sin^3\theta$$



http://mathworld.wolfram.com/images/gifs/astroid.gif

Proofs or Explanations:

1. Why parametric equations?

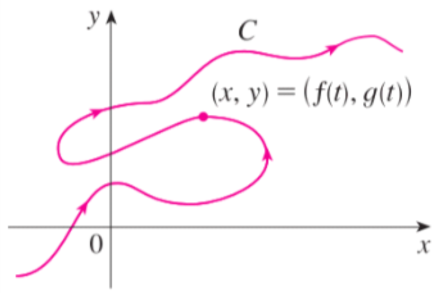


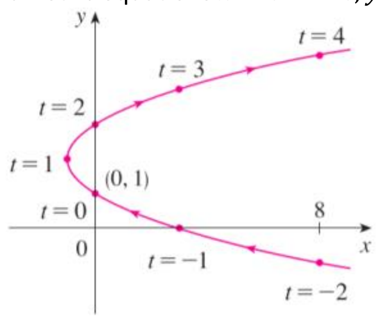
FIGURE 1

Imagine that a particle moves along the curve C shown in Figure 1.
It is impossible to describe C by an equation of the form $y = f(x)$ because C fails the ***Vertical Line Test***.
But the $x -$ and $y -$coordinates of the particle are functions of time and so we can write $x = f(t)$ and $y = g(t)$.
Such a pair of equations is often a convenient way of describing a curve.

Examples:

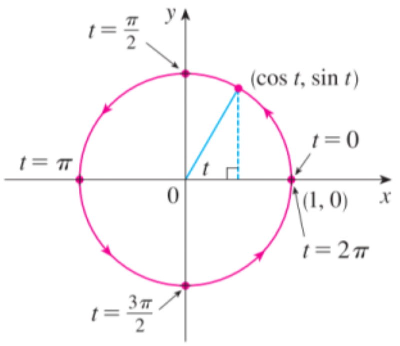
1. Sketch and identify the curve defined by the parametric equations $x = t^2 - 2t, y = t + 1$

t	x	y
-2	8	-1
-1	3	0
0	0	1
1	-1	2
2	0	3
3	3	4
4	8	5

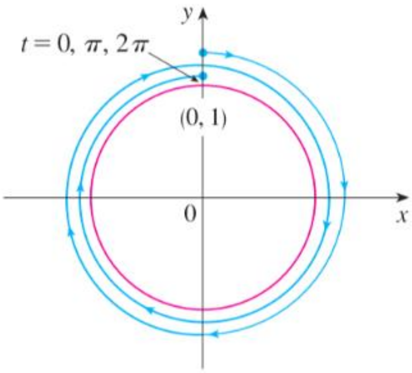


$y = t + 1 \Rightarrow t = y - 1$
$x = t^2 - 2t \Rightarrow x = (y - 1)^2 - 2(y - 1) = y^2 - 4y + 3$
$\Rightarrow$ The curve represented by the given parametric equations is the parabola $x = y^2 - 4y + 3$.

2. What curve is represented by the following parametric equations?
   a) $x = \cos t, y = \sin t, 0 \le t \le 2\pi$
      $x^2 + y^2 = \cos^2 t + \sin^2 t = 1 \Rightarrow$ The point (x, y) moves on the unit circle $x^2 + y^2 = 1$.

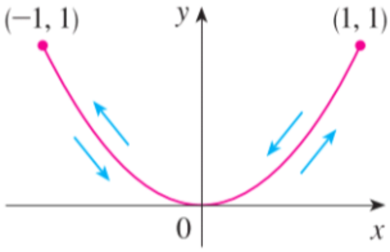


   b) $x = \sin 2t, y = \cos 2t, 0 \le t \le 2\pi$
      $x^2 + y^2 = \cos^2 2t + \sin^2 2t = 1 \Rightarrow$ The point (x, y) moves on the unit circle $x^2 + y^2 = 1$.
      But as t increases from 0 to 2π, the point $(x, y) = (\sin 2t, \cos 2t)$ starts at $(0,1)$ and moves twice around the circle in the clockwise direction.

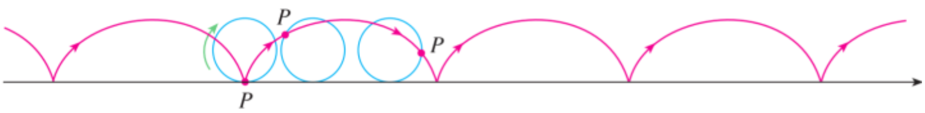


- a) and b) show that different *sets* of parametric equations can represent the *same* curve. Thus we distinguish between a *curve*, which is a set of points, and a *parametric curve*, in which the points are traced in a particular way.

3. Sketch the curve with parametric equations $x = \sin t, y = \sin^2 t$.
   $y = \sin^2 t = x^2 \Rightarrow$ The point (x, y) moves on the parabola $y = x^2$.
   But $-1 \le \sin t \le 1 \Rightarrow -1 \le x \le 1$
   Since $\sin t$ is periodic, the point $(x, y) = (\sin t, \sin^2 t)$ moves back and forth infinitely often along the parabola from $(-1,1)$ to $(1,1)$.



4. The curve traced out by a point P on the circumference of a circle as the circle rolls along a straight line is called a cycloid. If the circle has radius r and rolls along the x-axis and if one position P is the origin, find parametric equations of the cycloid.



We choose as parameter the angle of rotation θ of the circle ($\theta = 0$) when P is at the origin). Suppose the circle has rotated through θ radians. Because the circle has been in contact with the line, we see that the distance it has rolled from the origin if

$$|OT| = arc\ PT = r\theta$$

Therefore the center of the circle is $C(r\theta, r)$. Let the coordinates of P be (x, y). The we see that

$$x = |OT| - |PQ| = r\theta - r\sin\theta = r(\theta - \sin\theta)$$
$$y = |TC| - |QC| = r - r\cos\theta = r(1 - \cos\theta)$$

Therefore parametric equations of the cycloid are

$$x = r(\theta - \sin\theta), y = r(1 - \cos\theta), \qquad \theta \in \mathbb{R}$$

10.2 Calculus with Parametric Curves (P721)

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Definitions & Theorems:

1. Tangents
$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y/\,\mathrm{d}t}{\mathrm{d}x/\,\mathrm{d}t}, \qquad \frac{\mathrm{d}x}{\mathrm{d}t}\neq 0$$
   - the curve has a ***horizontal*** tangent when
$$\frac{\mathrm{d}y}{\mathrm{d}t}=0, \qquad \frac{\mathrm{d}x}{\mathrm{d}t}\neq 0$$
   - the curve has a ***vertical*** tangent when
$$\frac{\mathrm{d}x}{\mathrm{d}t}=0, \qquad \frac{\mathrm{d}y}{\mathrm{d}t}\neq 0$$
   - $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=\frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{\frac{\mathrm{d}x}{\mathrm{d}t}}$
2. Areas
$$A=\int_a^b y\,dx=\int_\alpha^\beta g(t)f'(t)\,dt\,, \qquad or \qquad \left[\int_\beta^\alpha g(t)f'(t)\,dt\right]$$
3. Theorem: Arc Length (P725)
   If a curve C is described by the parametric equations $x=f(t), y=g(t), \alpha\le x\le\beta$, where f' and g' are continuous on $[\alpha,\beta]$ and C is traversed exactly once as t increases from α to β, then the length of C is
$$L=\int_\alpha^\beta\sqrt{\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2}\,dt$$
4. Surface Area
$$S=\int_\alpha^\beta 2\pi y\sqrt{\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2}\,dt\,, \qquad t\in[\alpha,\beta]$$

Proofs or Explanations:

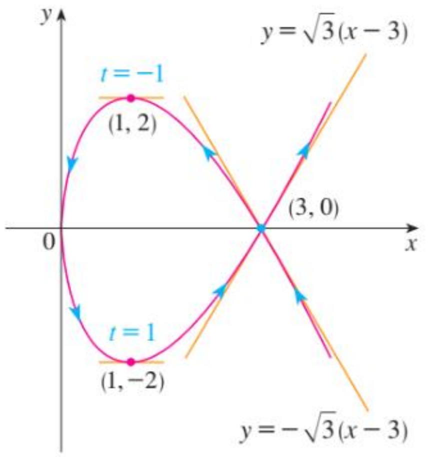
1. Tangents
$$\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{\mathrm{d}y}{\mathrm{d}x}\cdot\frac{\mathrm{d}x}{\mathrm{d}t}$$
$$\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}, \qquad \frac{\mathrm{d}x}{\mathrm{d}t}\neq 0$$
2. Area
   When $x=a$, t is either α or β. When $x=b$, t is the remaining value.
   $y=g(t)$
   $dx=d\big(f(t)\big)=f'(t)\,dt$
3. Arc Length
   See Formula 8.1.3
$$L=\int_a^b\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,dx=\int_\alpha^\beta\sqrt{1+\left(\frac{\mathrm{d}y/\,\mathrm{d}t}{\mathrm{d}x/\,\mathrm{d}t}\right)^2}\,\frac{\mathrm{d}x}{\mathrm{d}t}dt=\int_\alpha^\beta\sqrt{\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2}\,dt\,, \qquad t\in[\alpha,\beta]$$

Examples:

1. A curve C is defined by the parametric equations $x=t^2, y=t^3-3t$.
   a) Show that C has two tangents at the point $(3,0)$ and find their equations.
   b) Find the points on C where the tangent is horizontal or vertical.
   c) Determine where the curve is concave upward or downward.
   d) Sketch the curve.

   Solution:
   a) $y=t^3-3t=0\Rightarrow t=0, t=\pm\sqrt{3}$
      $\Rightarrow$ The point (3,0) on C arises from two values of the parameter, $t=\sqrt{3}$ and $t=-\sqrt{3}$.
      $\Rightarrow$ C crosses itself at (3,0).
      $$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y/\,\mathrm{d}t}{\mathrm{d}x/\,\mathrm{d}t}=\frac{3t^2-3}{2t}=\frac{3}{2}\left(t-\frac{1}{t}\right)\Rightarrow when\ t=\pm\sqrt{3}, \frac{\mathrm{d}y}{\mathrm{d}x}=\pm\sqrt{3}$$
      $\Rightarrow$ The equations of the tangetns at (3,0) are
      $$y=\sqrt{3}(x-3), \qquad y=-\sqrt{3}(x-3)$$
   b) $\frac{\mathrm{d}y}{\mathrm{d}t}=0, \frac{\mathrm{d}x}{\mathrm{d}t}\neq 0\Rightarrow t=\pm 1\Rightarrow horizontal\ tangent\ at\ (1,-2)\ and\ (1,2)$
      $\frac{\mathrm{d}x}{\mathrm{d}t}=0, \frac{\mathrm{d}y}{\mathrm{d}t}\neq 0\Rightarrow t=0\Rightarrow vertical\ tangent\ at\ (0,0)$
   c) $$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=\frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{\frac{\mathrm{d}x}{\mathrm{d}t}}=\frac{\frac{3}{2}\left(1+\frac{1}{t^2}\right)}{2t}=\frac{3(t^2+1)}{4t^3}$$

⇒The curve is concave upward when $t > 0$ and concave downward when $t < 0$.

d) 

10.3 Polar Coordinates (P730)

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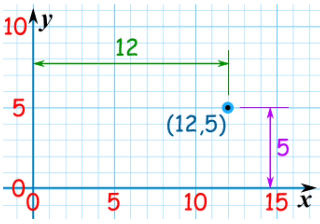
Definitions & Theorems:

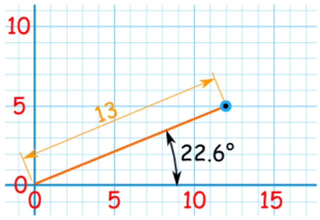
1. Cartesian coordinates

   Using Cartesian Coordinates we mark a point by **how far along** and **how far up** it is. In the Cartesian coordinate system every point has only one representation.

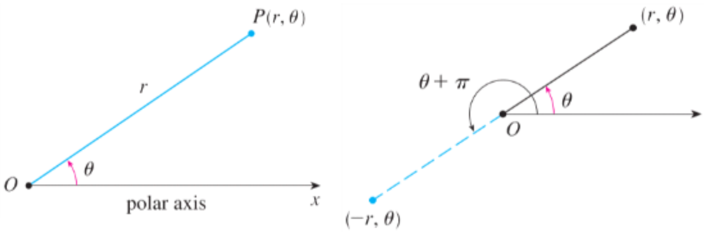


2. Polar coordinates

   Using Polar Coordinates we mark a point by **how far away**, and **what angle** it is. In the polar coordinate system each point has many representations (Example 2).

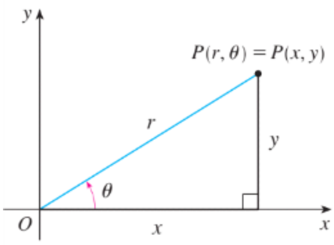


3. Polar coordinate system



- Pole: the origin, is labeled O
- Polar axis: is drawn horizontally to the right and corresponds to the positive x-axis in Cartesian coordinates.
- r: the distance from O to P
- θ: the angle (usually measured in radians) between the polar axis and the line OP
- Polar coordinates: the ordered pair (r, θ)
- $(0, \theta)$ represents the pole for any value of θ
- an angle is positive if measured in the counter clockwise direction from the polar axis and negative in the clockwise direction.
- Negative point: the points (r, θ) and$(-r, \theta)$ lie on the same line through O and at the same distance $|r|$ from O, but on opposite sides of O.

4. The connection between polar and Cartesian coordinates



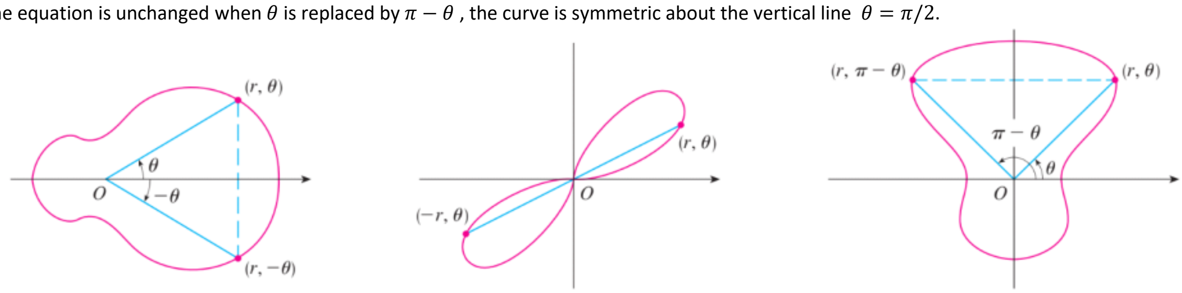
$$\cos\theta = \frac{x}{r} \qquad x = r\cos\theta \qquad r^2 = x^2 + y^2$$

$$\sin\theta = \frac{y}{r} \qquad y = r\sin\theta \qquad \tan\theta = \frac{y}{x}$$

5. Polar Curves

   The graph of a polar equation $r = f(\theta)$, or more generally $F(r, \theta) = 0$, consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

6. Symmetry
   1) If a polar equation is unchanged when θ is replaced by $-\theta$, the curve is symmetric about the polar axis.
   2) If the equation is unchanged when r is replaced by $2r$, or when θ is replaced by $\theta + \pi$, the curve is symmetric about the pole. (This means that the curve remains unchanged if we rotate it through 180° about the origin.)
   3) If the equation is unchanged when θ is replaced by $\pi - \theta$, the curve is symmetric about the vertical line $\theta = \pi/2$.



(a) (b) (c)

7. Tangents to Polar Curves

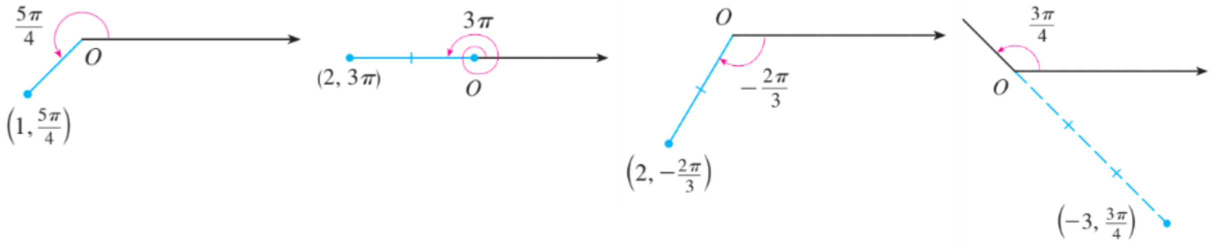
   $r = f(\theta) \Rightarrow x = r\cos\theta = f(\theta)\cos\theta \,, y = r\sin\theta = f(\theta)\sin\theta$

$$\frac{dy}{dx} = \frac{dy/\,d\theta}{dx/\,d\theta} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

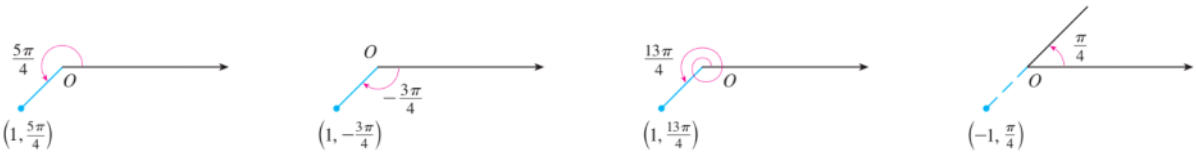
- the curve has a ***horizontal*** tangent when
$$\frac{dy}{d\theta} = 0, \qquad \frac{dx}{d\theta} \neq 0$$
- the curve has a ***vertical*** tangent when
$$\frac{dx}{d\theta} = 0, \qquad \frac{dy}{d\theta} \neq 0$$
- If we are looking for tangent lines at the pole, then $r = 0$
$$\frac{dy}{dx} = \tan\theta\,, \qquad \frac{dr}{d\theta} \neq 0$$

Examples:

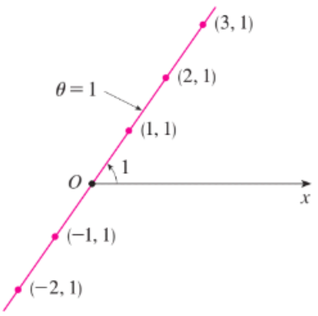
1. Plot the points whose polar coordinates are given.
$$\left(1, \frac{5\pi}{4}\right), (2, 3\pi), \left(2, -\frac{2\pi}{3}\right), \left(-\frac{3{,}3\pi}{4}\right)$$



2. In the polar coordinate system each point has many representations.

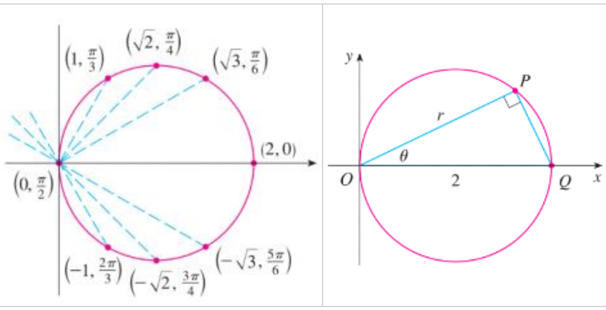


3. Sketch the polar curve $\theta = 1$.



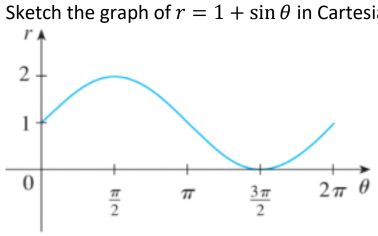
4. Sketch the curve with polar equation $r = 2\cos\theta$, find a Cartesian equation for this curve.

θ	$r = 2\cos\theta$
0	2
$\pi/6$	$\sqrt{3}$
$\pi/4$	$\sqrt{2}$
$\pi/3$	1
$\pi/2$	0
$2\pi/3$	-1
$3\pi/4$	$-\sqrt{2}$
$5\pi/6$	$-\sqrt{3}$
π	-2



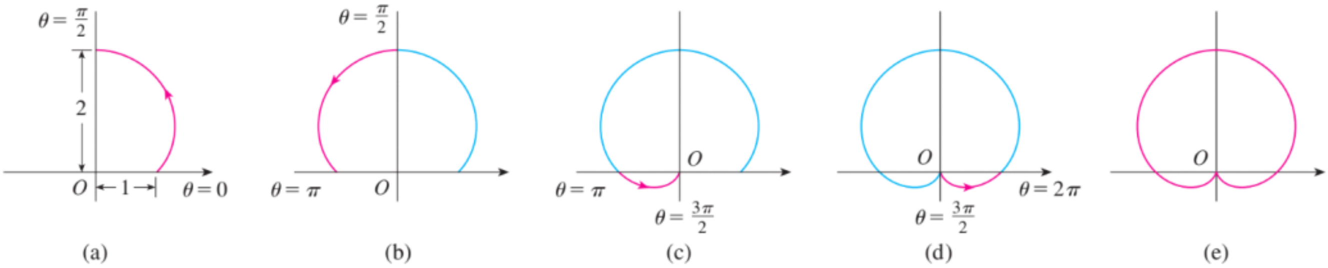
$r = 2\cos\theta \Rightarrow r^2 = 2r\cos\theta \Rightarrow x^2 + y^2 = 2x \Rightarrow (x-1)^2 + y^2 = 0$ which is an equation of a circle with center (1,0) and radius 1.

5. (Cardioid, P734) Sketch the curve $r = 1 + \sin\theta$; find the slope of the tangent line when $\theta = \pi/3$; find the points on the cardioid where the tangent line is horizontal or vertical.
   1) Sketch the curve
      i. Sketch the graph of $r = 1 + \sin\theta$ in Cartesian coordinates.



      ii. As increases from 0 to $\pi/2$, r (the distance from O) increases from 1 to 2, so we sketch the corresponding part of the polar curve in (a).
      iii. As increases from $\pi/2$ to π, Figure 10 shows that r decreases from 2 to 1, so we sketch the next part of the curve as in (b).
      iv. As increases from π to $3\pi/2$, r decreases from 1 to 0 as shown in (c).
      v. Finally, as increases from $3\pi/2$ to 2π, r increases from 0 to 1 as shown in (d).

vi. Putting together all the parts of the curve, we sketch the complete curve in (e).



2) Find the slope of the tangent line when $\theta = \frac{\pi}{3}$

$$\frac{dy}{dx} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\cos\theta\,\sin\theta + (1 + \sin\theta)\cos\theta}{\cos\theta\,\cos\theta - (1 + \sin\theta)\sin\theta}$$

$$= \frac{\cos\theta\,(1 + 2\sin\theta)}{1 - 2\sin^2\theta - \sin\theta} = \frac{\cos\theta\,(1 + 2\sin\theta)}{(1 + \sin\theta)(1 - 2\sin\theta)}$$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/3} = \frac{\cos(\pi/3)(1 + 2\sin(\pi/3))}{(1 + \sin(\pi/3))(1 - 2\sin(\pi/3))} = \frac{\frac{1}{2}(1 + \sqrt{3})}{(1 + \sqrt{3}/2)(1 - \sqrt{3})}$$

$$= \frac{1 + \sqrt{3}}{(2 + \sqrt{3})(1 - \sqrt{3})} = \frac{1 + \sqrt{3}}{-1 - \sqrt{3}} = -1$$

3) Find the points on the cardioid where the tangent line is horizontal or vertical

$$\frac{dy}{d\theta} = \cos\theta\,(1 + 2\sin\theta) = 0 \qquad \text{when } \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$\frac{dx}{d\theta} = (1 + \sin\theta)(1 - 2\sin\theta) = 0 \qquad \text{when } \theta = \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$$

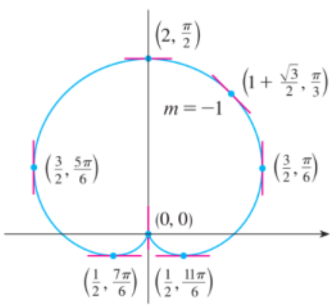
Therefore there are horizontal tangents at the points $(2, \pi/2)$, $(\frac{1}{2}, 7\pi/6)$, $(\frac{1}{2}, 11\pi/6)$ and vertical tangents at $(\frac{3}{2}, \pi/6)$ and $(\frac{3}{2}, 5\pi/6)$. When $\theta = 3\pi/2$, both $dy/d\theta$ and $dx/d\theta$ are 0, so we must be careful. Using l'Hospital's Rule, we have

$$\lim_{\theta\to(3\pi/2)^-}\frac{dy}{dx} = \left(\lim_{\theta\to(3\pi/2)^-}\frac{1 + 2\sin\theta}{1 - 2\sin\theta}\right)\left(\lim_{\theta\to(3\pi/2)^-}\frac{\cos\theta}{1 + \sin\theta}\right)$$

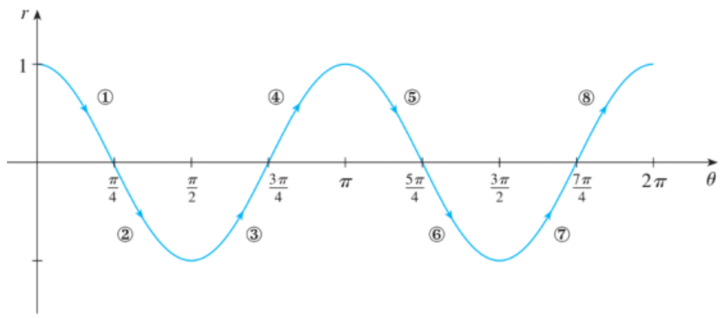
$$= -\frac{1}{3}\lim_{\theta\to(3\pi/2)^-}\frac{\cos\theta}{1 + \sin\theta} = -\frac{1}{3}\lim_{\theta\to(3\pi/2)^-}\frac{-\sin\theta}{\cos\theta} = \infty$$

By symmetry, $$\lim_{\theta\to(3\pi/2)^+}\frac{dy}{dx} = -\infty$$

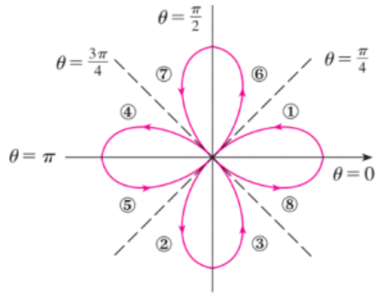
Thus there is a vertical tangent line at the pole (see Figure 15).



6. (Four-leaved rose, P 734) Sketch the curve $r = \cos 2\theta$.



r
1
①
②
③
④
⑤
⑥
⑦
⑧
$\frac{\pi}{4}$
$\frac{\pi}{2}$
$\frac{3\pi}{4}$
π
$\frac{5\pi}{4}$
$\frac{3\pi}{2}$
$\frac{7\pi}{4}$
2π
θ



$\theta = \frac{\pi}{2}$
$\theta = \frac{3\pi}{4}$
$\theta = \frac{\pi}{4}$
$\theta = \pi$
$\theta = 0$
①
②
③
④
⑤
⑥
⑦
⑧

10.4 Areas and Lengths in Polar Coordinates (P741)

January 18, 2017 23:36

Definitions & Theorems:

1. Area

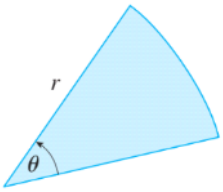
$$A=\int_a^b \frac{1}{2}r^2\,\mathrm{d}\theta=\frac{1}{2}\int_a^b r^2\,\mathrm{d}\theta$$

2. Arc Length

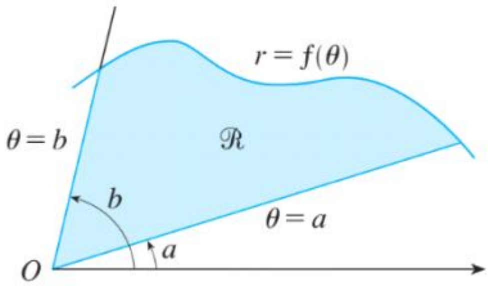
$$L=\int_a^b \sqrt{r^2+\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$$

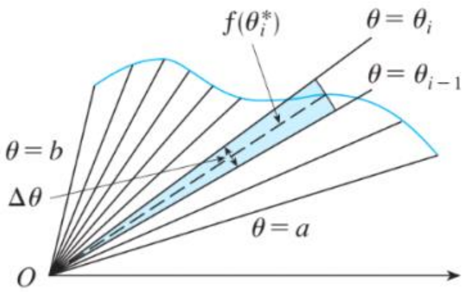
Proofs or Explanations:

1. Area



$$A=\frac{1}{2}r^2\theta$$



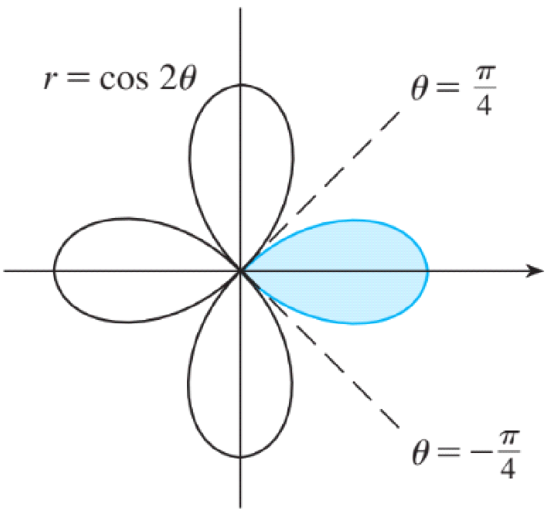


$$A\approx\sum_{i=1}^{n}\frac{1}{2}\left[f(\theta_i^*)\right]^2\Delta\theta$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{2}\left[f(\theta_i^*)\right]^2\Delta\theta=\int_a^b\frac{1}{2}\left[f(\theta)\right]^2\,\mathrm{d}\theta=\int_a^b\frac{1}{2}r^2\,\mathrm{d}\theta$$

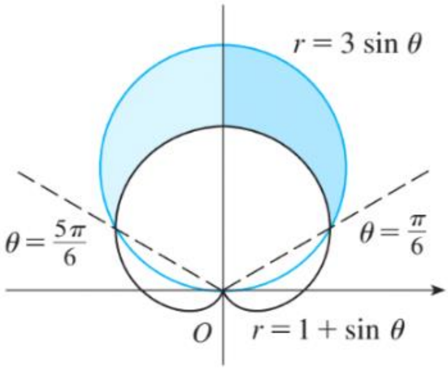
Examples:

1. Find the area enclosed by one leaf of the four-leaved rose $r=\cos 2\theta$.



$$A=\int_a^b\frac{1}{2}r^2\,\mathrm{d}\theta=\int_{-\pi/4}^{\pi/4}\frac{1}{2}(\cos 2\theta)^2\mathrm{d}\theta=\int_0^{\pi/4}\cos^2 2\theta\,\mathrm{d}\theta=\frac{\pi}{8}$$

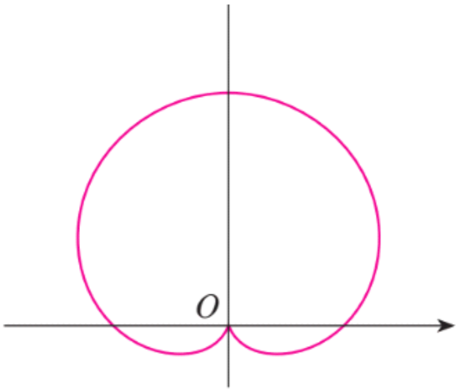
2. Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.



$$A = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (3\sin\theta)^2 d\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1+\sin\theta)^2\, d\theta$$

$$A = 2\left[\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (3\sin\theta)^2\, d\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1+\sin\theta)^2\, d\theta\right] = \pi$$

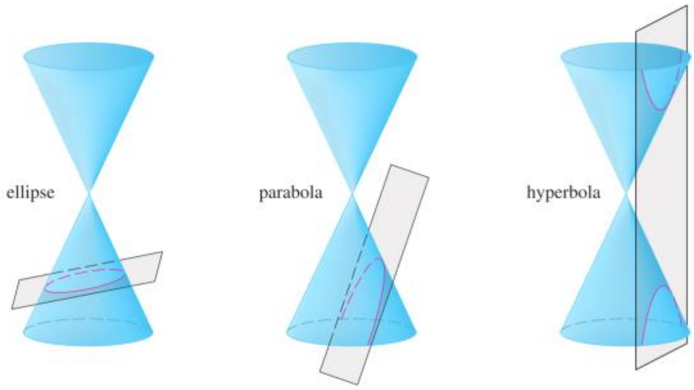
3. Find the length of the cardioid $r = 1 + \sin\theta$.



$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = L = \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta}\, d\theta = 8$$

10.5 Conic Sections (P746)

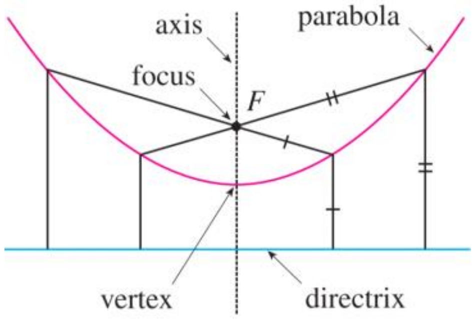
January 18, 2017 23:36



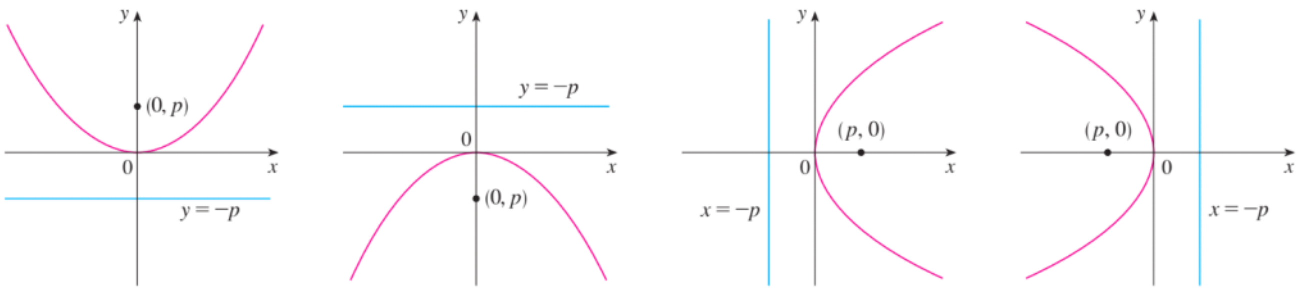
Definitions & Theorems:

1. Parabola

   A parabola is the set of points in a plane that are ***equidistant*** from a fixed point F (called the ***focus***) and a fixed line (called the ***directrix***). The point halfway between the focus and the directrix lies on the parabola is called the ***vertex***. The line through the focus perpendicular to the directrix is called the ***axis*** of the parabola.



   1) An equation of the parabola with focus $(0, p)$, and directrix $y = -p$ is
      $x^2 = 4py$

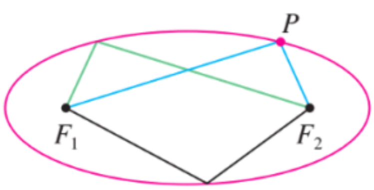


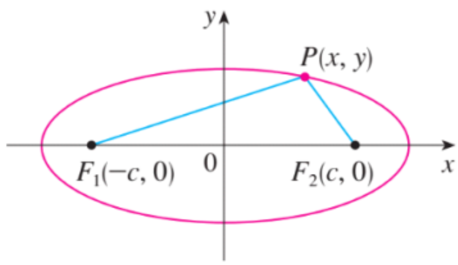
(a) $x^2 = 4py,\ p > 0$ (b) $x^2 = 4py,\ p < 0$ (c) $y^2 = 4px,\ p > 0$ (d) $y^2 = 4px,\ p < 0$

   2) If we interchange x and y, we obtain $y^2 = 4px$

2. Ellipse

   An ellipse is the set of points in a plane the ***sum*** of whose distances from two fixed points F_1 and F_2 is a constant. These two fixed points are called the foci (plural of focus).

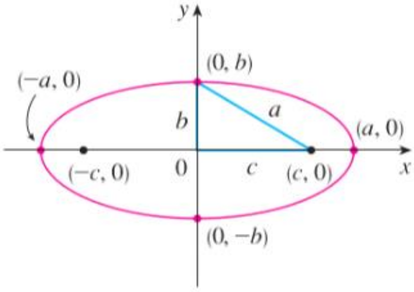




   1) $(a, 0)$ and $(-a, 0)$ are called the ***vertices*** of the ellipse
   2) The line segment joining the vertices is called the major axis
   3) The line segment joining $(0, b)$ and $(0, -b)$ is the minor axis.
   4) The ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \qquad a \ge b > 0$$

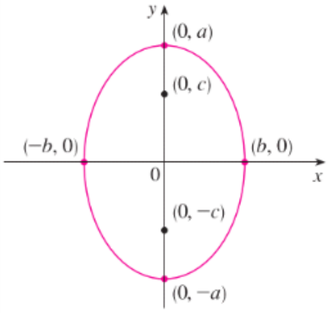
has foci $(\pm c, 0)$, where $c^2 = a^2 - b^2$, and vertices $(\pm a, 0)$.



5) The ellipse

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1, \qquad a \ge b > 0$$

has foci $(0, \pm c)$, where $c^2 = a^2 - b^2$, and vertices $(\pm a, 0)$.



3. Hyperbolas

A hyperbola is the set of all points in a plane the ***difference*** of whose distances from two fixed points F_1 and F_2 (the ***foci***) is a constant.

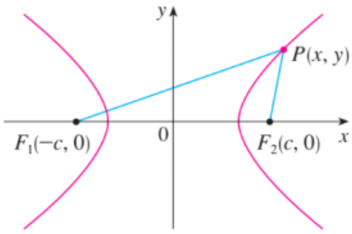
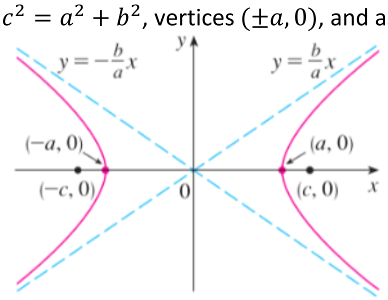


FIGURE 11
P is on the hyperbola when
$|PF_1| - |PF_2| = \pm 2a$.

1) The hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

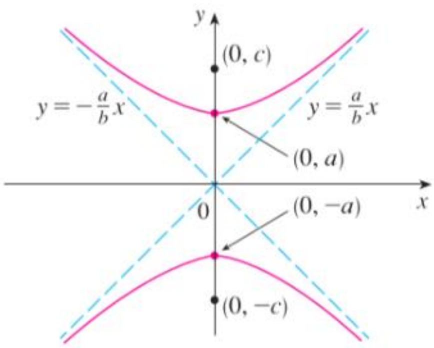
has foci $(\pm c, 0)$, where $c^2 = a^2 + b^2$, vertices $(\pm a, 0)$, and asymptotes $y = \pm(b/a)x$



2) The hyperbola

$$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$$

has foci $(0, \pm c)$, where $c^2 = a^2 + b^2$, vertices $(0, \pm a)$, and asymptotes $y = \pm(a/b)x$



4. Shifted Conics
   See Examples 6 and 7

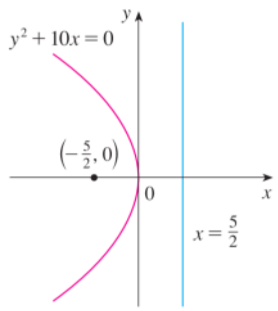
Proofs or Explanations:

1.

Examples:

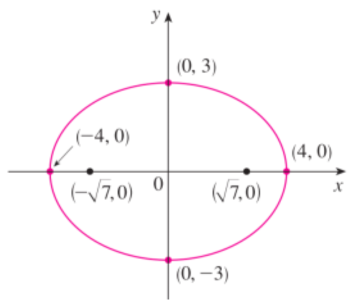
1. Find the focus and directrix of the parabola $y^2 + 10x = 0$ and sketch the graph.

   $y^2 + 10x = 0 \Rightarrow y^2 = -10x$
   $\Rightarrow 4p = -10 \Rightarrow p = -\frac{5}{2}$
   $\Rightarrow (p, 0)$ is $(-\frac{5}{2}, 0)$
   $\Rightarrow$ the directrix is $x = \frac{5}{2}$



2. Sketch the graph of $9x^2 + 16y^2 = 144$ and locate the foci.

   $9x^2 + 16y^2 = 144 \Rightarrow \frac{x^2}{16} + \frac{y^2}{9} = 1$
   $\Rightarrow a^2 = 16, b^2 = 9 \Rightarrow c^2 = a^2 - b^2 = 7$
   $\Rightarrow$ the x-intercepts are ± 4
   $\Rightarrow$ the y-intercepts are ± 3
   $\Rightarrow$ the foci are $(\pm\sqrt{7}, 0)$

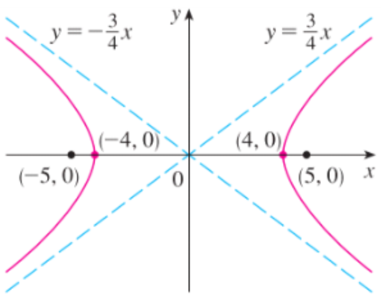


3. Find an equation of the ellipse with foci $(0, \pm 2)$ and vertices $(0, \pm 3)$.

   $c = 2, a = 3 \Rightarrow b^2 = a^2 - c^2 = 5$
   $\Rightarrow \frac{x^2}{5} + \frac{y^2}{9} = 1$

4. Find the foci and asymptotes of the hyperbola $9x^2 - 16y^2 = 144$ and sketch its graph.

   $9x^2 - 16y^2 = 144 \Rightarrow \frac{x^2}{16} - \frac{y^2}{9} = 1 \Rightarrow a = 4, b = 3 \Rightarrow c^2 = 16 + 9 = 25$
   $\Rightarrow$ the foci are (± 5); the asymptotes are the line $y = \frac{3}{4}x$ and $y = -\frac{3}{4}x$.



5. Find the foci and equation of the hyperbola with vertices $(0, \pm 1)$ and asymptote $y = 2x$.

   $a = 1, \frac{a}{b} = 2 \Rightarrow b = \frac{1}{2}, c^2 = a^2 + b^2 = \frac{5}{4}$

$\Rightarrow$ the foci are $(0, \pm\frac{\sqrt{5}}{2})$

$$\Rightarrow \frac{y^2}{1^2} - \frac{x^2}{\left(\frac{1}{2}\right)^2} = 1 \Rightarrow y^2 - 4x^2 = 1$$

6. Find an equation of the ellipse with foci $(2,-2),(4,-2)$ and vertices $(1,-2),(5,-2)$.

   The major axis is the line segment that joins the vertices $(1,-2),(5,-2)$ and has length 4, so $a = 2$.
   The distance between the foci is 2, so $c = 1$.
   $\Rightarrow b^2 = a^2 - c^2 = 3$
   $\Rightarrow$ the center of the ellipse is $(3,-2)$.

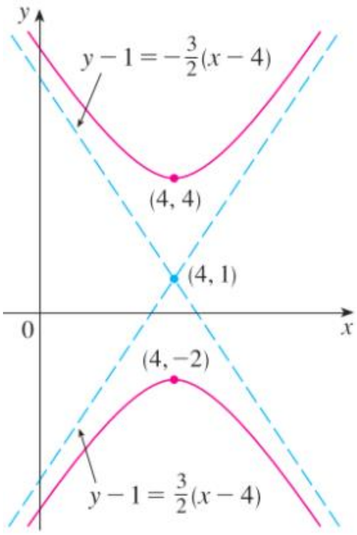
   $$\Rightarrow \frac{(x-3)^2}{4} + \frac{(y+2)^2}{3} = 1$$

7. Sketch the conic $9x^2 - 4y^2 - 72x + 8y + 176 = 0$ and find its foci.

   $$9x^2 - 4y^2 - 72x + 8y + 176 = 0 \Rightarrow \frac{(y-1)^2}{9} - \frac{(x-4)^2}{4} = 1$$

   $\Rightarrow a^2 = 9, b^2 = 4 \Rightarrow c^2 = 13$

   The hyperbola is shifted four units to the right and one unit upward $\Rightarrow$ foci are $(4, 1 \pm \sqrt{13})$ and the vertices are $(4,4)$ and $(4,-2)$; the asymptotes are $y - 1 = \pm\frac{3}{2}(x-4)$.



10.6 Conic Sections in Polar Coordinates (P754)

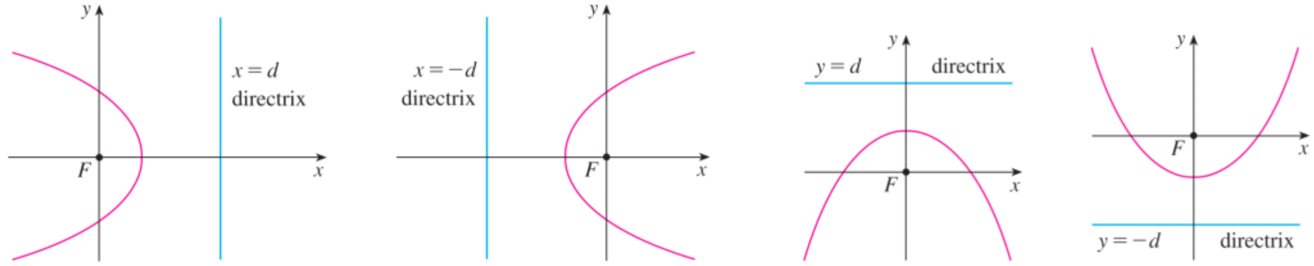
January 18, 2017 17:28

Definitions & Theorems:

1. Theorem

   Let F be a fixed point (called the ***focus***) and l be a fixed line (called the ***directrix***) in a plane. Let e be a fixed positive number (called the ***eccentricity***). The set of all points P in the plane such that
   $$\frac{|PF|}{|Pl|} = e$$
   (that is, the ratio of the distance from F to the distance from l is the constant e) is a conic section. The conic is
   a) an ellipse if $e < 1$
   b) a parabola if $e = 1$
   c) a hyperbola if $e > 1$
2. Theorem

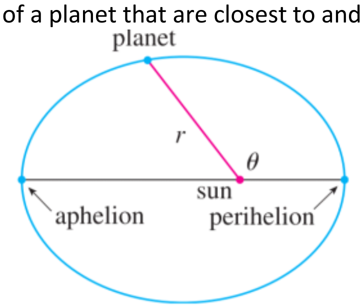
   A polar equation of the form
   $$r = \frac{ed}{1 \pm e\cos\theta} \quad or \quad r = \frac{ed}{1 \pm e\sin\theta}, \qquad e = \frac{c}{a}$$
   represents a conic section with eccentricity e. The conic is an ellipse if $e < 1$, a parabola if $e = 1$, or a hyperbola if $e > 1$.



(a) $r = \dfrac{ed}{1 + e\cos\theta}$ (b) $r = \dfrac{ed}{1 - e\cos\theta}$ (c) $r = \dfrac{ed}{1 + e\sin\theta}$ (d) $r = \dfrac{ed}{1 - e\sin\theta}$

3. The polar equation of an ellipse with focus at the origin, semimajor axis a, eccentricity e, and directrix $x = d$ can be written in the form
   $$r = \frac{a(1 - e^2)}{1 + e\cos\theta}$$
4. Kepler's Law
   1) A planet revolves around the sun in an elliptical orbit with the sun at one focus.
   2) The line joining the sun to a planet sweeps out equal areas in equal times.
   3) The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.
5. Perihelion and aphelion

   The positions of a planet that are closest to and farthest from the sun are called its perihelion and aphelion, respectively.



   The perihelion distance from a planet to the sun is $a(1 - e)$ and the aphelion distance is $a(1 + e)$.

Proofs or Explanations:

1.

Examples:

1. Find a polar equation for a parabola that has its focus at the origin and whose directrix is the line $y = -6$.

   a parabola $\Rightarrow e = 1$

   $y = -6 \Rightarrow d = 6$
   $$r = \frac{ed}{1 - e\sin\theta} \Rightarrow r = \frac{6}{1 - \sin\theta}$$
2. A conic is given by the polar equation
   $$r = \frac{10}{3 - 2\cos\theta}$$

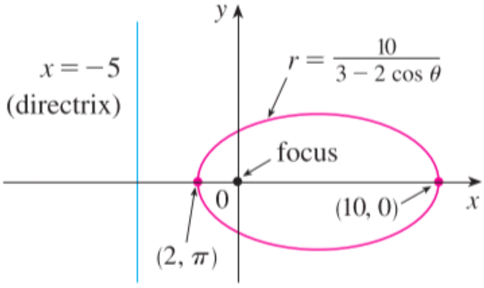
Find the eccentricity, identify the conic, locate the directrix, and sketch the conic.

$$r = \frac{10}{3 - 2\cos\theta} = \frac{10/3}{1 - 2/3\cos\theta}$$

$$\Rightarrow e = \frac{2}{3}, ed = \frac{10}{3} \Rightarrow d = 5$$

$\Rightarrow$ the directrix has Cartesian equation $x = -5$.

$\theta = 0, r = 10$ *and* $\theta = \pi, r = 2 \Rightarrow$ the vertices have polar coordinates $(10,0)$ and $(2,\pi)$



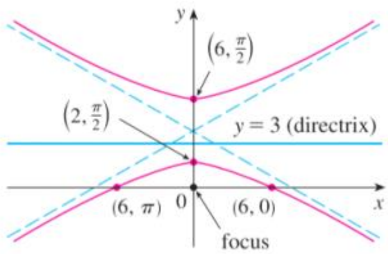
3. Sketch the conic $r = \frac{12}{2+4sin\theta}$

$$r = \frac{12}{2 + 4sin\theta} = \frac{6}{1 + 2\sin\theta}$$

$\Rightarrow e = 2, d = 3$

$\Rightarrow$ the directrix has Cartesian equation $y = 3$.

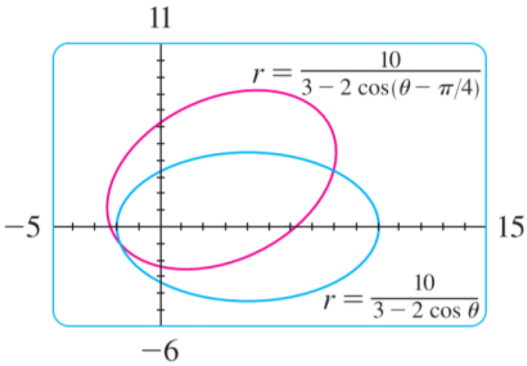
we see that the eccentricity is $e = 2$ and the equation therefore represents a hyperbola. Since $ed = 6$, $d = 3$ and the directrix has equation $y = 3$. The vertices occur when $\theta = \pi/2$ and $3\pi/2$, so they are $(2, \pi/2)$ and $(-6, 3\pi/2) = (6, \pi/2)$. It is also useful to plot the x-intercepts. These occur when $\theta = 0, \pi$; in both cases $r = 6$. For additional accuracy we could draw the asymptotes. Note that $r \to \pm\infty$ when $1 + 2\sin\theta \to 0^+$ or 0^- and $1 + 2\sin\theta = 0$ when $\sin\theta = -\frac{1}{2}$. Thus the asymptotes are parallel to the rays $\theta = 7\pi/6$ and $\theta = 11\pi/6$. The hyperbola is sketched in Figure 4.



4. If the ellipse of Example 2 is rotated through an angle $\pi/4$ about the origin, find a polar equation and graph the resulting ellipse.

replacing θ with $\theta - \pi/4$

$$r = \frac{10}{3 - 2\cos\theta} \Rightarrow r = \frac{10}{3 - 2\cos(\theta - \pi/4)}$$



5. (a) Find an approximate polar equation for the elliptical orbit of the earth around the sun (at one focus) given that the eccentricity is about 0.017 and the length of the major axis is about 2.99×10^8 km. (b) Find the distance from the earth to the sun at perihelion and at aphelion.

SOLUTION

(a) The length of the major axis is $2a = 2.99 \times 10^8$, so $a = 1.495 \times 10^8$. We are given that $e = 0.017$ and so, from Equation 7, an equation of the earth's orbit around the sun is

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta} = \frac{(1.495 \times 10^8)[1 - (0.017)^2]}{1 + 0.017\cos\theta}$$

or, approximately,

$$r = \frac{1.49 \times 10^8}{1 + 0.017\cos\theta}$$

(b) From (8), the perihelion distance from the earth to the sun is

$$a(1 - e) \approx (1.495 \times 10^8)(1 - 0.017) \approx 1.47 \times 10^8 \text{ km}$$

and the aphelion distance is

$$a(1 + e) \approx (1.495 \times 10^8)(1 + 0.017) \approx 1.52 \times 10^8 \text{ km}$$

11.10 Taylor and Maclaurin Series (P831)

January 18, 2017 23:36

Definitions & Theorems:

1. Radius of convergence

$$f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n$$

Ratio test: $\left|\frac{a_{n+1}}{a_n}\right|$

$$R=\frac{1}{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|}$$

convergent for $|x-a|<R$

$x\in(a-R,a+R)$

2. Theorem

If f has a power series representation (expansion) at a, that is, if

$$f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n, \qquad |x-a|<R$$

then its coefficients are given by the formula

$$c_n=\frac{f^{(n)}(a)}{n!}$$

3. Taylor series of the function f at a (or about a or centered at a)

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$
$$=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\cdots$$

4. Maclaurin series

When $a=0$, the Taylor series becomes

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots$$

5. Theorem

Power series can be differentiated and integrated term by term in its interval convergence.

$$\sin x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$$

$$\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$$

6. Theorem

If $f(x)=T_n(x)+R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a and

$$\lim_{n\to\infty}R_n(x)=0$$

for $|x-a|<R$, then f is equal to the sum of its Taylor series on the interval $|x-a|<R$.

7. Theorem: Alternating series estimation theorem

If $s=\sum(-1)^{n-1}b_n$ is the sum of alternating series that satisifies

(i) $0\le b_{n+1}\le b_n$

(ii) $\lim_{n\to\infty}b_n=0$

then $|R_n|=|s-s_n|\le b_{n+1}$

8. Equation

$$\lim_{n\to\infty}\frac{x^n}{n!}=0, \qquad for\ every\ real\ number\ x$$

9. Important Maclaurin Series and Their Radii of Convergence

$$1)\ \frac{1}{1-x}=\sum_{n=0}^{\infty}x^n=1+x^1+x^2+x^3+x^4+\cdots, \qquad R=1$$

2) $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots, \qquad R = \infty$

$$c_n = \frac{1}{n!} \Rightarrow \frac{c_{n+1}}{c_n} = \frac{1}{n+1} \Rightarrow R = \infty$$

3) $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \qquad R = \infty$

$\sin 0 = 0$
$(\sin x)' = \cos x, \qquad (\sin 0)' = \cos 0 = 1$
$(\sin x)'' = -\sin x, \qquad (\sin 0)'' = -\sin 0 = 0$
$(\sin x)''' = -\cos x, \qquad (\sin 0)''' = -\cos 0 = -1$
$(\sin x)'''' = \sin x, \qquad (\sin 0)'''' = \sin 0 = 0$

4) $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots, \qquad R = \infty$

5) $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \frac{x^1}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \qquad R = 1$

6) $\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \frac{x^1}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \qquad R = 1$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$$

$$\ln(1+x) = \int \frac{dx}{1+x} = \frac{x^1}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots$$

7) $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{x} = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \cdots, \qquad R = 1$

Examples:

1. Expand in Taylor Series at $a = 0$. Find the Radiis of convergence.

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{-2x^2} = 1 + \frac{(-2x^2)^1}{1!} + \frac{(-2x^2)^2}{2!} + \frac{(-2x^2)^3}{3!} + \frac{(-2x^2)^4}{4!} + \cdots$$

$R = +\infty$
$(-\infty, +\infty)$

2. Expand in Taylor Series at $a = 5$. Find the Radiis of convergence.

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{5x} = e^{5(x-5)+25} = e^{25} e^{5(x-5)} = e^{25} \left[1 + \frac{(5(x-5))^1}{1!} + \frac{(5(x-5))^2}{2!} + \frac{(5(x-5))^3}{3!} + \frac{(5(x-5))^4}{4!} + \cdots \right]$$

$R = +\infty$
$(-\infty, +\infty)$

3. Expand in Taylor Series at $a = 0$. Find the Radiis of convergence.

$$\frac{1}{1-5x} = 1 + 5x + (5x)^2 + (5x)^3 + (5x)^4 + \cdots$$

since $\frac{1}{1-y}$ has $R = 1$

$$\Rightarrow |5x| < 1 \Rightarrow |x| < \frac{1}{5} \Rightarrow R = \frac{1}{5}$$

4. Expand in Taylor Series at $a = 5$. Find the Radiis of convergence.

$$\frac{1}{1-5x}=\frac{1}{1-5(x-5)-25}=-\frac{1}{24+5(x-5)}=-\frac{1}{24}\times\frac{1}{1+\frac{5}{24}(x-5)}=-\frac{1}{24}\times\frac{1}{1-\left(-\frac{5}{24}(x-5)\right)}$$

$$=-\frac{1}{24}\left[1+\left(-\frac{5}{24}(x-5)\right)+\left(-\frac{5}{24}(x-5)\right)^2+\left(-\frac{5}{24}(x-5)\right)^3+\left(-\frac{5}{24}(x-5)\right)^4+\cdots\right]$$

$$\left|\frac{5}{24}(x-5)\right|<1\Rightarrow|x-5|<\frac{24}{5}=R$$

$$\left(5-\frac{24}{5},5+\frac{24}{5}\right)\Rightarrow\left(\frac{1}{5},\frac{49}{5}\right)$$

5. Find Taylor approximation of order 7, and find $x=0.75$

$$\int_0^x \sin(2t^2)\,dt$$

$$\sin(2t^2)=2t^2-\frac{2^3t^6}{3!}+\frac{2^5t^{10}}{5!}-\frac{2^7t^{14}}{7!}+\cdots$$

$$\int_0^x \sin(2t^2)\,dt=\left[\frac{2t^3}{3}-\frac{2^3t^7}{3!\times 7}+\frac{2^5t^{11}}{5!\times 11}-\frac{2^7t^{15}}{7!\times 15}+\cdots\right]_0^x=\frac{2x^3}{3}-\frac{2^3x^7}{3!\times 7}+\frac{2^5x^{11}}{5!\times 11}-\frac{2^7x^{15}}{7!\times 15}+\cdots\approx\frac{2x^3}{3}-\frac{2^3x^7}{3!\times 7}$$

$$F(0.75)=\frac{2(0.75)^3}{3}-\frac{2^3(0.75)^7}{3!\times 7}+\frac{2^5(0.75)^{11}}{5!\times 11}-\frac{2^7(0.75)^{15}}{7!\times 15}+\cdots\approx\frac{2(0.75)^3}{3}-\frac{2^3(0.75)^7}{3!\times 7}$$

$$error<\frac{2^5(0.75)^{11}}{5!\times 11}$$

Theorem: Alternating series estimation theorem

If $s=\sum(-1)^{n-1}b_n$ is the sum of alternating series that satisifies

(i) $0\le b_{n+1}\le b_n$

(ii) $\lim_{n\to\infty} b_n=0$

then $|R_n|=|s-s_n|\le b_{n+1}$

6. limits

$$\lim_{x\to 0}\frac{e^{x^4}-1}{2x^4}=\lim_{x\to 0}\frac{\left(1+\frac{x^4}{1!}+\frac{x^8}{2!}+\frac{x^{12}}{3!}+\frac{x^{16}}{4!}+\cdots\right)-1}{2x^4}=\lim_{x\to 0}\frac{x^4+x^8(\ldots)}{2x^4}=\frac{1}{2}\ (red\ part\ is\ convergent)$$

$$\lim_{x\to 0}\frac{\sin x^2-x^2}{\cos x^3-1}=\lim_{x\to 0}\frac{\left(x^2-\frac{x^6}{3!}+\frac{x^{10}}{5!}-\frac{x^{14}}{7!}+\cdots\right)-x^2}{\left(1-\frac{x^6}{2!}+\frac{x^{12}}{4!}-\frac{x^{18}}{6!}+\cdots\right)-1}=\lim_{x\to 0}\frac{-\frac{x^6}{3!}+\frac{x^{10}}{5!}-\frac{x^{14}}{7!}+\cdots}{-\frac{x^6}{2!}+\frac{x^{12}}{4!}-\frac{x^{18}}{6!}+\cdots}=\frac{1}{3}$$

7. Find $f^{1000}(0)$ of $f(x)=\cos 3x^2$

$$\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}=\frac{x^0}{0!}-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$$

$$c_n=\frac{f^{(n)}(a)}{n!}$$

$$\cos 3x^2=\sum_{n=0}^{\infty}(-1)^n\frac{(3x^2)^{2n}}{(2n)!},n=\frac{1000}{4}=250$$

$$\Rightarrow(-1)^n\frac{(3x^2)^{2n}}{(2n)!}=\frac{3^{500}x^{1000}}{500!}$$

$$c_n=\frac{f^{(n)}(a)}{n!}\Rightarrow c_{1000}=\frac{f^{(1000)}(0)}{1000!}=\frac{3^{500}}{500!}$$

$$\Rightarrow f^{1000}(0)=\frac{1000!\,3^{500}}{500!}$$

$$\frac{1}{1-4t^2}=1+4t^2+4^2t^4+4^4t^8$$

What is R: $|4t^2|<1 \Rightarrow t^2<\frac{1}{4} \Rightarrow |t|<\frac{1}{2}$

$R=\frac{1}{2}$ $\quad (-\frac{1}{2}, \frac{1}{2})$

12.1 Three-Dimensional Coordinate Systems (P864)

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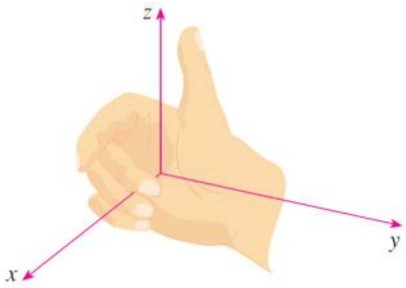
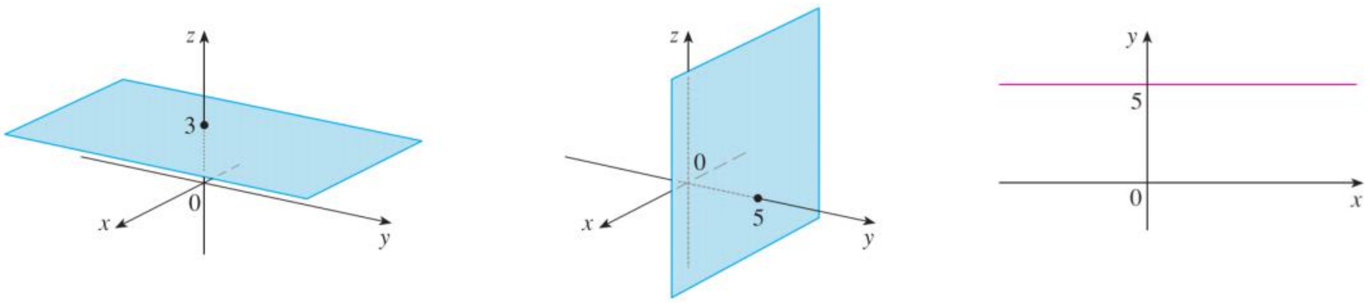


FIGURE 2
Right-hand rule

EXAMPLE 1 What surfaces in $\mathbb{R}^3$ are represented by the following equations?
(a) $z = 3$ (b) $y = 5$

SOLUTION
(a) The equation $z = 3$ represents the set $\{(x, y, z) \mid z = 3\}$, which is the set of all points in $\mathbb{R}^3$ whose z-coordinate is 3 (x and y can each be any value). This is the horizontal plane that is parallel to the xy-plane and three units above it as in Figure 7(a).



(a) $z = 3$, a plane in $\mathbb{R}^3$ (b) $y = 5$, a plane in $\mathbb{R}^3$ (c) $y = 5$, a line in $\mathbb{R}^2$

EXAMPLE 2

(a) Which points (x, y, z) satisfy the equations

$$x^2 + y^2 = 1 \qquad \text{and} \qquad z = 3$$

(b) What does the equation $x^2 + y^2 = 1$ represent as a surface in $\mathbb{R}^3$?

SOLUTION

(a) Because $z = 3$, the points lie in the horizontal plane $z = 3$ from Example 1(a). Because $x^2 + y^2 = 1$, the points lie on the circle with radius 1 and center on the z-axis. See Figure 8.

(b) Given that $x^2 + y^2 = 1$, with no restrictions on z, we see that the point (x, y, z) could lie on a circle in any horizontal plane $z = k$. So the surface $x^2 + y^2 = 1$ in $\mathbb{R}^3$ consists of all possible horizontal circles $x^2 + y^2 = 1$, $z = k$, and is therefore the circular cylinder with radius 1 whose axis is the z-axis. See Figure 9.

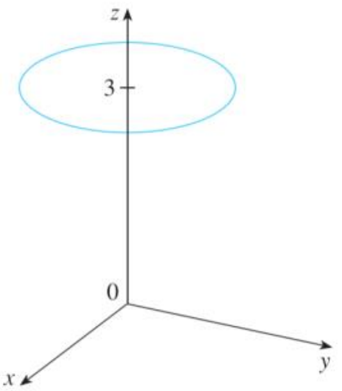


FIGURE 8
The circle $x^2 + y^2 = 1$, $z = 3$

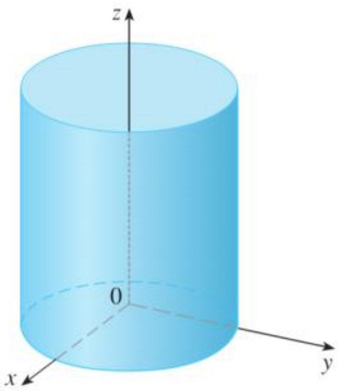


FIGURE 9
The cylinder $x^2 + y^2 = 1$ ■

EXAMPLE 3 Describe and sketch the surface in $\mathbb{R}^3$ represented by the equation $y = x$.

SOLUTION The equation represents the set of all points in $\mathbb{R}^3$ whose x- and y-coordinates are equal, that is, $\{(x, x, z) \mid x \in \mathbb{R}, z \in \mathbb{R}\}$. This is a vertical plane that intersects the xy-plane in the line $y = x$, $z = 0$. The portion of this plane that lies in the first octant is sketched in Figure 10. ■

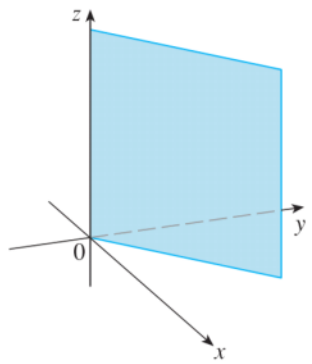
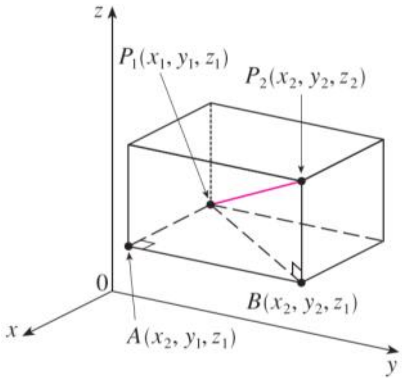


FIGURE 10
The plane $y = x$

Distance Formula in Three Dimensions The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

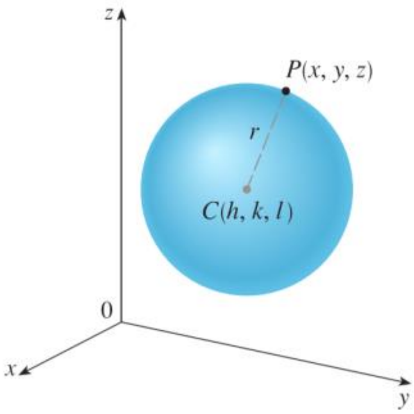


Equation of a Sphere An equation of a sphere with center $C(h, k, l)$ and radius r is

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

In particular, if the center is the origin O, then an equation of the sphere is

$$x^2 + y^2 + z^2 = r^2$$



EXAMPLE 6 Show that $x^2 + y^2 + z^2 + 4x - 6y + 2z + 6 = 0$ is the equation of a sphere, and find its center and radius.

SOLUTION We can rewrite the given equation in the form of an equation of a sphere if we complete squares:

$$(x^2 + 4x + 4) + (y^2 - 6y + 9) + (z^2 + 2z + 1) = -6 + 4 + 9 + 1$$

$$(x + 2)^2 + (y - 3)^2 + (z + 1)^2 = 8$$

Comparing this equation with the standard form, we see that it is the equation of a sphere with center $(-2, 3, -1)$ and radius $\sqrt{8} = 2\sqrt{2}$. ■

EXAMPLE 7 What region in $\mathbb{R}^3$ is represented by the following inequalities?

$$1 \leqslant x^2 + y^2 + z^2 \leqslant 4 \qquad z \leqslant 0$$

SOLUTION The inequalities

$$1 \leqslant x^2 + y^2 + z^2 \leqslant 4$$

can be rewritten as

$$1 \leqslant \sqrt{x^2 + y^2 + z^2} \leqslant 2$$

so they represent the points (x, y, z) whose distance from the origin is at least 1 and at most 2. But we are also given that $z \leqslant 0$, so the points lie on or below the xy-plane. Thus the given inequalities represent the region that lies between (or on) the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ and beneath (or on) the xy-plane. It is sketched in Figure 13. ■

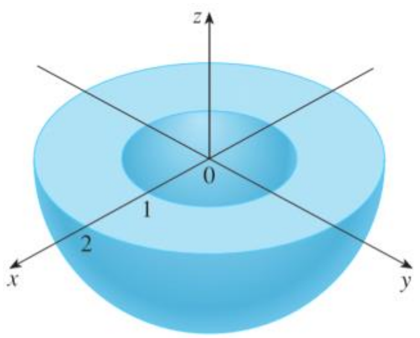


FIGURE 13

12.2 Vectors (P870)

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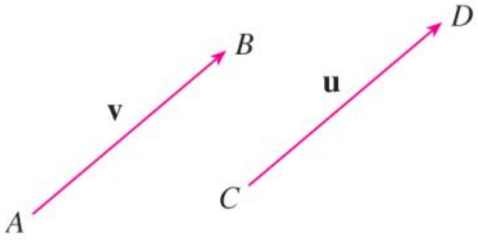


FIGURE 1
Equivalent vectors

Definition of Vector Addition If **u** and **v** are vectors positioned so the initial point of **v** is at the terminal point of **u**, then the **sum u + v** is the vector from the initial point of **u** to the terminal point of **v**.

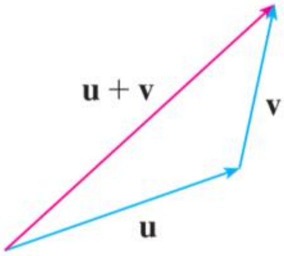


FIGURE 3
The Triangle Law

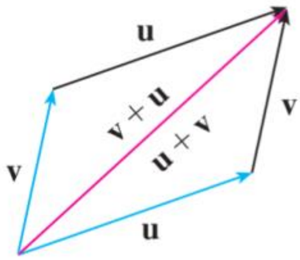
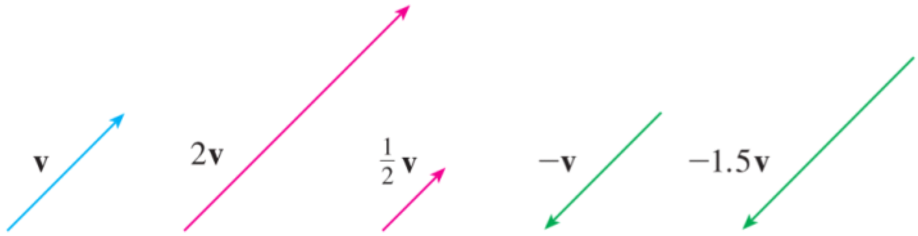


FIGURE 4
The Parallelogram Law

Definition of Scalar Multiplication If c is a scalar and **v** is a vector, then the **scalar multiple** $c\mathbf{v}$ is the vector whose length is $|c|$ times the length of **v** and whose direction is the same as **v** if $c > 0$ and is opposite to **v** if $c < 0$. If $c = 0$ or $\mathbf{v} = \mathbf{0}$, then $c\mathbf{v} = \mathbf{0}$.



1 Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector $\mathbf{a}$ with representation $\overrightarrow{AB}$ is

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

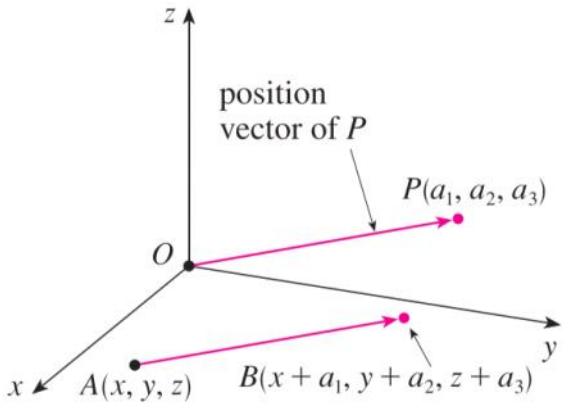


FIGURE 13
Representations of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

The length of the two-dimensional vector $\mathbf{a} = \langle a_1, a_2 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$

The length of the three-dimensional vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle \qquad \mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$$

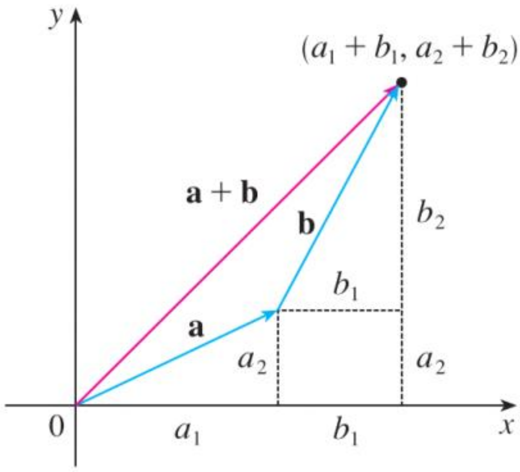
$$c\mathbf{a} = \langle ca_1, ca_2 \rangle$$

Similarly, for three-dimensional vectors,

$$\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

$$\langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$

$$c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$$



Properties of Vectors If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in V_n and c and d are scalars, then

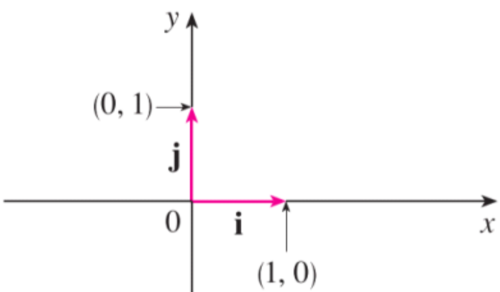
1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ **2.** $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$

3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$ **4.** $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$

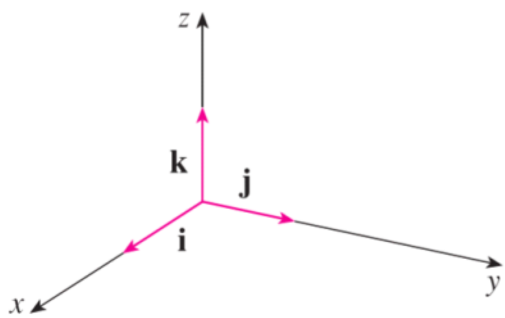
5. $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$ **6.** $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$

7. $(cd)\mathbf{a} = c(d\mathbf{a})$ **8.** $1\mathbf{a} = \mathbf{a}$

$$\mathbf{i} = \langle 1, 0, 0 \rangle \qquad \mathbf{j} = \langle 0, 1, 0 \rangle \qquad \mathbf{k} = \langle 0, 0, 1 \rangle$$



(a)



(b)

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, then we can write

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle = \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle$$

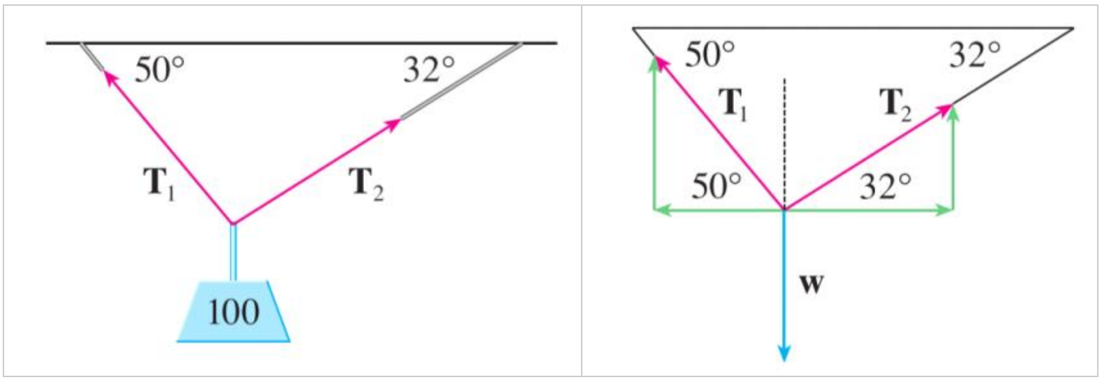
$$= a_1\langle 1, 0, 0 \rangle + a_2\langle 0, 1, 0 \rangle + a_3\langle 0, 0, 1 \rangle$$

2 $$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

A **unit vector** is a vector whose length is 1. For instance, **i**, **j**, and **k** are all unit vectors. In general, if $\mathbf{a} \neq \mathbf{0}$, then the unit vector that has the same direction as **a** is

4
$$\mathbf{u} = \frac{1}{|\mathbf{a}|}\mathbf{a} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

❖ Example 7: A 100-lb weight hangs from two wires. Find the tensions (forces) $\boldsymbol{T_1}$ and $\boldsymbol{T_2}$ in both wires and the magnitudes of the tension.



$$\begin{cases} \boldsymbol{T_1} = -|\boldsymbol{T_1}|\cos 50°\,\boldsymbol{i} + |\boldsymbol{T_1}|\sin 50°\,\boldsymbol{j} \\ \boldsymbol{T_2} = -|\boldsymbol{T_2}|\cos 32°\,\boldsymbol{i} + |\boldsymbol{T_2}|\sin 32°\,\boldsymbol{j} \end{cases}$$

$$\boldsymbol{T_1} + \boldsymbol{T_2} = -\boldsymbol{w} = 100\boldsymbol{j}$$

$$\Rightarrow \left(-|\boldsymbol{T_1}|\cos 50° + |\boldsymbol{T_2}|\cos 32°\right)\boldsymbol{i} + \left(|\boldsymbol{T_1}|\sin 50° + |\boldsymbol{T_2}|\sin 32°\right)\boldsymbol{j} = 100\boldsymbol{j}$$

$$\Rightarrow \begin{cases} -|\boldsymbol{T_1}|\cos 50° + |\boldsymbol{T_2}|\cos 32° = 0 \\ |\boldsymbol{T_1}|\sin 50° + |\boldsymbol{T_2}|\sin 32° = 100 \end{cases}$$

$$\Rightarrow \begin{cases} \boldsymbol{T_1} \approx -55.05\boldsymbol{i} + 65.60\boldsymbol{j} \\ \boldsymbol{T_2} \approx 55.05\boldsymbol{i} + 34.40\boldsymbol{j} \end{cases}$$

12.3 The Dot Product (P879)

January 18, 2017 23:37

1 Definition If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **dot product** of $\mathbf{a}$ and $\mathbf{b}$ is the number $\mathbf{a} \cdot \mathbf{b}$ given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

2 Properties of the Dot Product If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in V_3 and c is a scalar, then

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$

2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

4. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$

5. $\mathbf{0} \cdot \mathbf{a} = 0$

3 Theorem If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

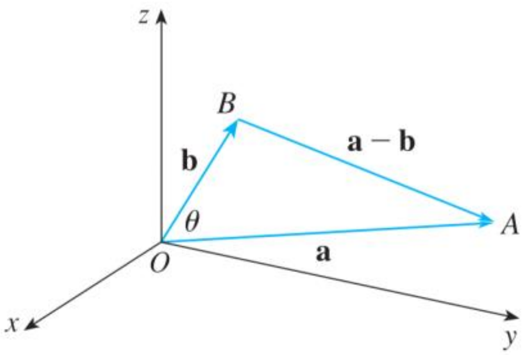


FIGURE 1

PROOF If we apply the Law of Cosines to triangle OAB in Figure 1, we get

$$\boxed{4} \qquad |AB|^2 = |OA|^2 + |OB|^2 - 2|OA||OB|\cos\theta$$

(Observe that the Law of Cosines still applies in the limiting cases when $\theta = 0$ or π, or $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$.) But $|OA| = |\mathbf{a}|$, $|OB| = |\mathbf{b}|$, and $|AB| = |\mathbf{a} - \mathbf{b}|$, so Equation 4 becomes

$$\boxed{5} \qquad |\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta$$

Using Properties 1, 2, and 3 of the dot product, we can rewrite the left side of this equation as follows:

$$\begin{aligned} |\mathbf{a} - \mathbf{b}|^2 &= (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \\ &= \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} \\ &= |\mathbf{a}|^2 - 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2 \end{aligned}$$

Therefore Equation 5 gives

$$|\mathbf{a}|^2 - 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta$$

Thus $$-2\mathbf{a} \cdot \mathbf{b} = -2|\mathbf{a}||\mathbf{b}|\cos\theta$$

or $$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$ ■

6 Corollary If θ is the angle between the nonzero vectors **a** and **b**, then

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

EXAMPLE 3 Find the angle between the vectors $\mathbf{a} = \langle 2, 2, -1 \rangle$ and $\mathbf{b} = \langle 5, -3, 2 \rangle$.

SOLUTION Since

$$|\mathbf{a}| = \sqrt{2^2 + 2^2 + (-1)^2} = 3 \qquad \text{and} \qquad |\mathbf{b}| = \sqrt{5^2 + (-3)^2 + 2^2} = \sqrt{38}$$

and since

$$\mathbf{a} \cdot \mathbf{b} = 2(5) + 2(-3) + (-1)(2) = 2$$

we have, from Corollary 6,

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{2}{3\sqrt{38}}$$

So the angle between **a** and **b** is

$$\theta = \cos^{-1}\left(\frac{2}{3\sqrt{38}}\right) \approx 1.46 \quad (\text{or } 84°)$$

7 Two vectors **a** and **b** are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

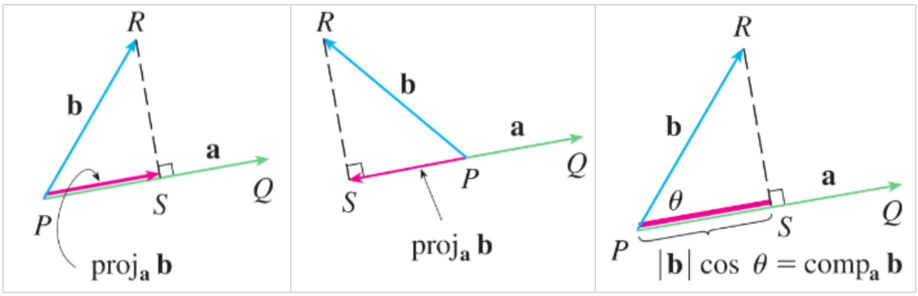
Scalar projection of **b** onto **a**:

$$|\mathbf{b}|\cos\theta = \text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$$

Scalar projection of **b** onto **a**: $\quad \text{comp}_{\mathbf{a}}\,\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$

Vector projection of **b** onto **a**: $\quad \text{proj}_{\mathbf{a}}\,\mathbf{b} = \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\,\mathbf{a}$

Notice that the vector projection is the scalar projection times the unit vector in the direction of **a**.



EXAMPLE 6 Find the scalar projection and vector projection of $\mathbf{b} = \langle 1, 1, 2\rangle$ onto $\mathbf{a} = \langle -2, 3, 1\rangle$.

SOLUTION Since $|\mathbf{a}| = \sqrt{(-2)^2 + 3^2 + 1^2} = \sqrt{14}$, the scalar projection of **b** onto **a** is

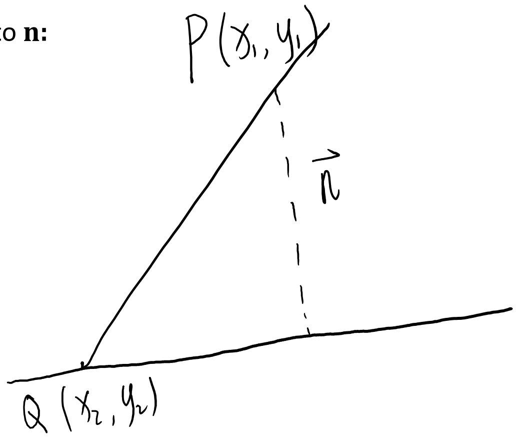
$$\text{comp}_{\mathbf{a}}\,\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \frac{(-2)(1) + 3(1) + 1(2)}{\sqrt{14}} = \frac{3}{\sqrt{14}}$$

The vector projection is this scalar projection times the unit vector in the direction of **a**:

$$\text{proj}_{\mathbf{a}}\,\mathbf{b} = \frac{3}{\sqrt{14}}\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{3}{14}\mathbf{a} = \left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14}\right\rangle \qquad \blacksquare$$

Example: The distance between $P(x_1, y_1)$ to the line $ax + by + c = 0$.

1. Normal Vector: $\mathbf{n} = < a, b >$
2. Let $Q(x_2, y_2)$ be a point on the line $ax + by + c = 0$.
   $\Rightarrow ax_2 + by_2 + c = 0$
3. The distance is the absolute value of Scalar projection of $\overrightarrow{QP}$ onto **n**:



$$\left|\overrightarrow{QP}\right|\cos\theta = \text{comp}_{\mathbf{n}}\overrightarrow{QP} = \frac{\mathbf{n}\cdot\overrightarrow{QP}}{|\mathbf{n}|}$$
$$= \frac{< a, b > \cdot < x_1 - x_2, y_1 - y_2 >}{\sqrt{a^2 + b^2}}$$
$$= \frac{a(x_1 - x_2) + b(y_1 - y_2)}{\sqrt{a^2 + b^2}}$$
$$= \frac{(ax_1 + by_1 + c) - (ax_2 + by_2 + c)}{\sqrt{a^2 + b^2}}$$
$$= \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}$$
$$d = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$

12.4 The Cross Product (P886)

January 18, 2017 23:37

4 Definition If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of $\mathbf{a}$ and $\mathbf{b}$ is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \qquad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

8 Theorem The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

PROOF In order to show that $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{a}$, we compute their dot product as follows:

$$\begin{aligned} (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3 \\ &= a_1(a_2 b_3 - a_3 b_2) - a_2(a_1 b_3 - a_3 b_1) + a_3(a_1 b_2 - a_2 b_1) \\ &= a_1 a_2 b_3 - a_1 b_2 a_3 - a_1 a_2 b_3 + b_1 a_2 a_3 + a_1 b_2 a_3 - b_1 a_2 a_3 \\ &= 0 \end{aligned}$$

A similar computation shows that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$. Therefore $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$. ■

$$\vec{a} \cdot \left(\vec{a} \times \vec{b}\right) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

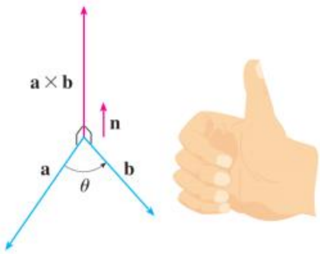


FIGURE 1
The right-hand rule gives the direction of $\mathbf{a} \times \mathbf{b}$.

9 Theorem If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ (so $0 \leqslant \theta \leqslant \pi$), then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$$

PROOF From the definitions of the cross product and length of a vector, we have

$$\begin{aligned} |\mathbf{a} \times \mathbf{b}|^2 &= (a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2 \\ &= a_2^2 b_3^2 - 2a_2 a_3 b_2 b_3 + a_3^2 b_2^2 + a_3^2 b_1^2 - 2a_1 a_3 b_1 b_3 + a_1^2 b_3^2 \\ &\qquad + a_1^2 b_2^2 - 2a_1 a_2 b_1 b_2 + a_2^2 b_1^2 \\ &= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2 \\ &= |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \\ &= |\mathbf{a}|^2 |\mathbf{b}|^2 - |\mathbf{a}|^2 |\mathbf{b}|^2 \cos^2\theta \qquad \text{(by Theorem 12.3.3)} \\ &= |\mathbf{a}|^2 |\mathbf{b}|^2 (1 - \cos^2\theta) \\ &= |\mathbf{a}|^2 |\mathbf{b}|^2 \sin^2\theta \end{aligned}$$

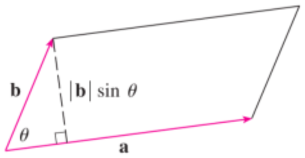
Taking square roots and observing that $\sqrt{\sin^2\theta} = \sin\theta$ because $\sin\theta \geqslant 0$ when $0 \leqslant \theta \leqslant \pi$, we have

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$$

10 Corollary Two nonzero vectors **a** and **b** are parallel if and only if

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}$$

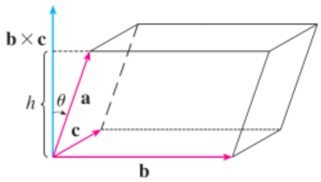
The length of the cross product $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by **a** and **b**.



$$A = |\boldsymbol{a}|(|\boldsymbol{b}| \sin\theta) = |\boldsymbol{a} \times \boldsymbol{b}|$$

14 The volume of the parallelepiped determined by the vectors **a**, **b**, and **c** is the magnitude of their scalar triple product:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$



$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$V = Ah = |\mathbf{b} \times \mathbf{c}||\mathbf{a}||\cos\theta| = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

Distance from a point P to the plane crossing QRS is

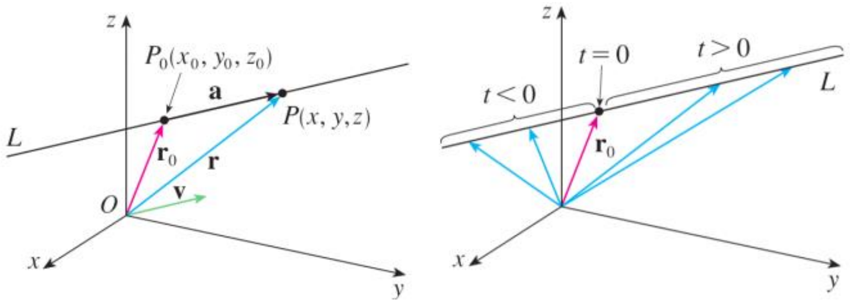
$$d = \frac{\left|\vec{a} \cdot \left(\vec{b} \times \vec{c}\right)\right|}{|\vec{b} \times \vec{c}|}$$

12.5 Equations of Lines and Planes (P895)

January 18, 2017 23:37

1. Vector Equation of L

1 $$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$



2. Parametric Equation of L

2 Parametric equations for a line through the point (x_0, y_0, z_0) and parallel to the direction vector $\langle a, b, c\rangle$ are

$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct$$

3. Symmetric Equation of L

3 $$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

If one of a, b, or c is 0, we can still eliminate t. For instance, if $a = 0$, then $x = x_0, y = y_0 + bt, z = z_0 + ct$, we could write the equations of L as

$$x = x_0 \qquad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

EXAMPLE 2
(a) Find parametric equations and symmetric equations of the line that passes through the points $A(2, 4, -3)$ and $B(3, -1, 1)$.
(b) At what point does this line intersect the xy-plane?

SOLUTION

(a) We are not explicitly given a vector parallel to the line, but observe that the vector $\mathbf{v}$ with representation $\overrightarrow{AB}$ is parallel to the line and

$$\mathbf{v} = \langle 3 - 2, -1 - 4, 1 - (-3)\rangle = \langle 1, -5, 4\rangle$$

Thus direction numbers are $a = 1$, $b = -5$, and $c = 4$. Taking the point $(2, 4, -3)$ as P_0, we see that parametric equations (2) are

$$x = 2 + t \qquad y = 4 - 5t \qquad z = -3 + 4t$$

and symmetric equations (3) are

$$\frac{x - 2}{1} = \frac{y - 4}{-5} = \frac{z + 3}{4}$$

(b) The line intersects the xy-plane when $z = 0$, so we put $z = 0$ in the symmetric equations and obtain

$$\frac{x - 2}{1} = \frac{y - 4}{-5} = \frac{3}{4}$$

This gives $x = \frac{11}{4}$ and $y = \frac{1}{4}$, so the line intersects the xy-plane at the point $\left(\frac{11}{4}, \frac{1}{4}, 0\right)$. ■

EXAMPLE 3 Show that the lines L_1 and L_2 with parametric equations

$$L_1: \quad x = 1 + t \qquad y = -2 + 3t \qquad z = 4 - t$$
$$L_2: \quad x = 2s \qquad y = 3 + s \qquad z = -3 + 4s$$

are **skew lines**; that is, they do not intersect and are not parallel (and therefore do not lie in the same plane).

SOLUTION The lines are not parallel because the corresponding direction vectors $\langle 1, 3, -1 \rangle$ and $\langle 2, 1, 4 \rangle$ are not parallel. (Their components are not proportional.) If L_1 and L_2 had a point of intersection, there would be values of t and s such that

$$\begin{aligned} 1 + t &= 2s \\ -2 + 3t &= 3 + s \\ 4 - t &= -3 + 4s \end{aligned}$$

But if we solve the first two equations, we get $t = \frac{11}{5}$ and $s = \frac{8}{5}$, and these values don't satisfy the third equation. Therefore there are no values of t and s that satisfy the three equations, so L_1 and L_2 do not intersect. Thus L_1 and L_2 are skew lines. ■

The lines L_1 and L_2 in Example 3, shown in Figure 5, are skew lines.

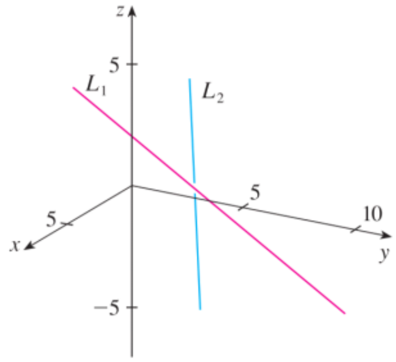


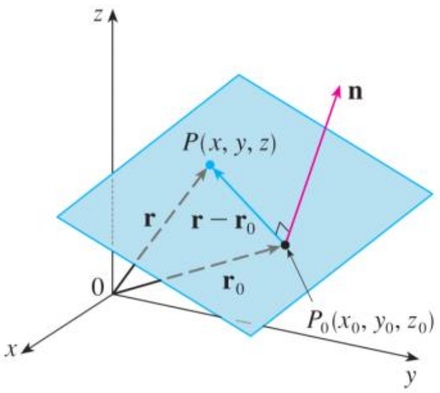
FIGURE 5

4. Vector Equation of the plane

5 $$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

which can be rewritten as

6 $$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0$$



A plane in space is determined by a point (x_0, y_0, z_0) in the plane and a vector n that is orthogonal to the plane.

This orthogonal vector n is called a ***normal vector***.

5. Scalar Equation of a plane

7 A **scalar equation of the plane** through point $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle a, b, c \rangle$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

6. Linear Equation in x, y and z

8

$$ax + by + cz + d = 0$$

where $d = -(ax_0 + by_0 + cz_0)$

EXAMPLE 5 Find an equation of the plane that passes through the points $P(1, 3, 2)$, $Q(3, -1, 6)$, and $R(5, 2, 0)$.

SOLUTION The vectors $\mathbf{a}$ and $\mathbf{b}$ corresponding to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are

$$\mathbf{a} = \langle 2, -4, 4 \rangle \qquad \mathbf{b} = \langle 4, -1, -2 \rangle$$

Since both $\mathbf{a}$ and $\mathbf{b}$ lie in the plane, their cross product $\mathbf{a} \times \mathbf{b}$ is orthogonal to the plane and can be taken as the normal vector. Thus

$$\mathbf{n} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = 12\,\mathbf{i} + 20\,\mathbf{j} + 14\,\mathbf{k}$$

With the point $P(1, 3, 2)$ and the normal vector $\mathbf{n}$, an equation of the plane is

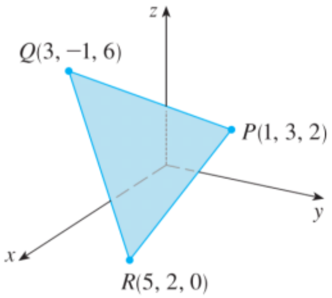
$$12(x - 1) + 20(y - 3) + 14(z - 2) = 0$$

or

$$6x + 10y + 7z = 50$$

■

Figure 8 shows the portion of the plane in Example 5 that is enclosed by triangle PQR.



EXAMPLE 6 Find the point at which the line with parametric equations $x = 2 + 3t$, $y = -4t$, $z = 5 + t$ intersects the plane $4x + 5y - 2z = 18$.

SOLUTION We substitute the expressions for x, y, and z from the parametric equations into the equation of the plane:

$$4(2 + 3t) + 5(-4t) - 2(5 + t) = 18$$

This simplifies to $-10t = 20$, so $t = -2$. Therefore the point of intersection occurs when the parameter value is $t = -2$. Then $x = 2 + 3(-2) = -4$, $y = -4(-2) = 8$, $z = 5 - 2 = 3$ and so the point of intersection is $(-4, 8, 3)$. ■

EXAMPLE 7

(a) Find the angle between the planes $x + y + z = 1$ and $x - 2y + 3z = 1$.
(b) Find symmetric equations for the line of intersection L of these two planes.

SOLUTION

(a) The normal vectors of these planes are

$$\mathbf{n}_1 = \langle 1, 1, 1 \rangle \qquad \mathbf{n}_2 = \langle 1, -2, 3 \rangle$$

and so, if θ is the angle between the planes, Corollary 12.3.6 gives

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{1(1) + 1(-2) + 1(3)}{\sqrt{1+1+1}\sqrt{1+4+9}} = \frac{2}{\sqrt{42}}$$

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{42}}\right) \approx 72^\circ$$

(b) We first need to find a point on L. For instance, we can find the point where the line intersects the xy-plane by setting $z = 0$ in the equations of both planes. This gives the equations $x + y = 1$ and $x - 2y = 1$, whose solution is $x = 1$, $y = 0$. So the point $(1, 0, 0)$ lies on L.

Now we observe that, since L lies in both planes, it is perpendicular to both of the normal vectors. Thus a vector $\mathbf{v}$ parallel to L is given by the cross product

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 5\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$$

and so the symmetric equations of L can be written as

$$\frac{x-1}{5} = \frac{y}{-2} = \frac{z}{-3}$$ ■

7. Distance D from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$

9
$$\boxed{D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}}$$

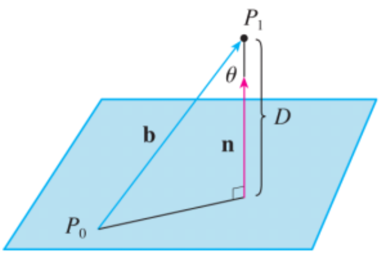


FIGURE 12

From Figure 12 you can see that the distance D from P_1 to the plane is equal to the absolute value of the scalar projection of $\mathbf{b}$ onto the normal vector $\mathbf{n} = \langle a, b, c \rangle$. (See Section 12.3.) Thus

$$D = |\text{comp}_\mathbf{n}\mathbf{b}| = \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|}$$

$$= \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|(ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}}$$

EXAMPLE 9 Find the distance between the parallel planes $10x + 2y - 2z = 5$ and $5x + y - z = 1$.

SOLUTION First we note that the planes are parallel because their normal vectors $\langle 10, 2, -2 \rangle$ and $\langle 5, 1, -1 \rangle$ are parallel. To find the distance D between the planes, we choose any point on one plane and calculate its distance to the other plane. In particular, if we put $y = z = 0$ in the equation of the first plane, we get $10x = 5$ and so $\left(\frac{1}{2}, 0, 0\right)$ is a point in this plane. By Formula 9, the distance between $\left(\frac{1}{2}, 0, 0\right)$ and the plane $5x + y - z - 1 = 0$ is

$$D = \frac{|5(\frac{1}{2}) + 1(0) - 1(0) - 1|}{\sqrt{5^2 + 1^2 + (-1)^2}} = \frac{\frac{3}{2}}{3\sqrt{3}} = \frac{\sqrt{3}}{6}$$

So the distance between the planes is $\sqrt{3}/6$. ■

EXAMPLE 10 In Example 3 we showed that the lines

$$L_1: \quad x = 1 + t \qquad y = -2 + 3t \qquad z = 4 - t$$

$$L_2: \quad x = 2s \qquad y = 3 + s \qquad z = -3 + 4s$$

are skew. Find the distance between them.

SOLUTION Since the two lines L_1 and L_2 are skew, they can be viewed as lying on two parallel planes P_1 and P_2. The distance between L_1 and L_2 is the same as the distance between P_1 and P_2, which can be computed as in Example 9. The common normal vector to both planes must be orthogonal to both $\mathbf{v}_1 = \langle 1, 3, -1 \rangle$ (the direction of L_1) and $\mathbf{v}_2 = \langle 2, 1, 4 \rangle$ (the direction of L_2). So a normal vector is

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -1 \\ 2 & 1 & 4 \end{vmatrix} = 13\mathbf{i} - 6\mathbf{j} - 5\mathbf{k}$$

If we put $s = 0$ in the equations of L_2, we get the point $(0, 3, -3)$ on L_2 and so an equation for P_2 is

$$13(x - 0) - 6(y - 3) - 5(z + 3) = 0 \qquad \text{or} \qquad 13x - 6y - 5z + 3 = 0$$

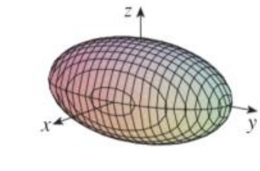
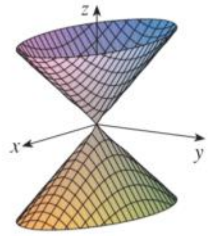
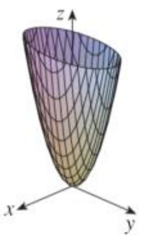
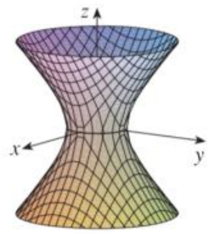
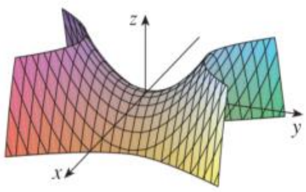
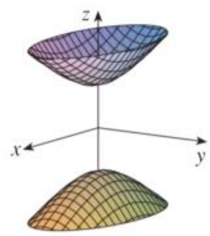
If we now set $t = 0$ in the equations for L_1, we get the point $(1, -2, 4)$ on P_1. So the distance between L_1 and L_2 is the same as the distance from $(1, -2, 4)$ to $13x - 6y - 5z + 3 = 0$. By Formula 9, this distance is

$$D = \frac{|13(1) - 6(-2) - 5(4) + 3|}{\sqrt{13^2 + (-6)^2 + (-5)^2}} = \frac{8}{\sqrt{230}} \approx 0.53$$ ■

12.6 Cylinders and Quadric Surfaces (P906)

January 18, 2017 23:37

Table 1 Graphs of Quadric Surfaces

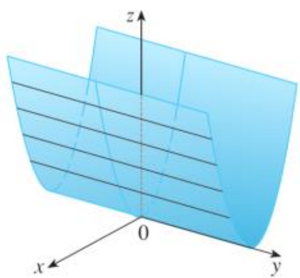
Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ All traces are ellipses. If $a=b=c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x=k$ and $y=k$ are hyperbolas if $k\neq 0$ but are pairs of lines if $k=0$.
Elliptic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c<0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ Horizontal traces in $z=k$ are ellipses if $k>c$ or $k<-c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

In order to sketch the graph of a surface, it is useful to determine the curves of intersection of the surface with planes parallel to the coordinate planes. These curves are called traces (or cross-sections) of the surface.

1. Cylinder
   A cylinder is a surface that consists of all lines (called rulings) that are parallel to a given line and pass through a given plane curve.

2. Quadric Surfaces
   A quadric surface is the graph of a second-degree equation in three variables x, y, and z. The most general such equation is
   $$Ax^2+By^2+Cz^2+Dxy+Eyz+Fxz+Gx+Hy+Iz+J=0$$

EXAMPLE 1 Sketch the graph of the surface $z=x^2$.



EXAMPLE 2 Identify and sketch the surfaces.

(a) $x^2 + y^2 = 1$ (b) $y^2 + z^2 = 1$

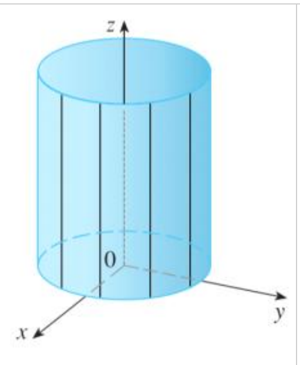


FIGURE 2
$x^2 + y^2 = 1$

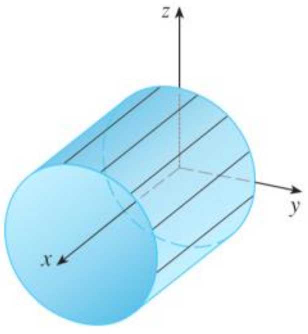


FIGURE 3
$y^2 + z^2 = 1$

EXAMPLE 6 Sketch the surface $\frac{x^2}{4} + y^2 - \frac{z^2}{4} = 1$.

SOLUTION The trace in any horizontal plane $z = k$ is the ellipse

$$\frac{x^2}{4} + y^2 = 1 + \frac{k^2}{4} \qquad z = k$$

but the traces in the xz- and yz-planes are the hyperbolas

$$\frac{x^2}{4} - \frac{z^2}{4} = 1 \qquad y = 0 \qquad \text{and} \qquad y^2 - \frac{z^2}{4} = 1 \qquad x = 0$$

This surface is called a **hyperboloid of one sheet** and is sketched in Figure 9. ■

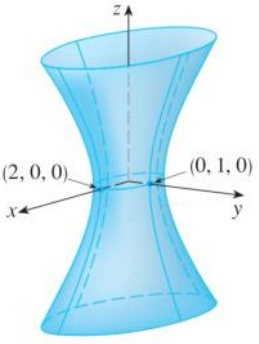


FIGURE 9

EXAMPLE 3 Use traces to sketch the quadric surface with equation

$$x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$$

SOLUTION By substituting $z = 0$, we find that the trace in the xy-plane is $x^2 + y^2/9 = 1$, which we recognize as an equation of an ellipse. In general, the horizontal trace in the plane $z = k$ is

$$x^2 + \frac{y^2}{9} = 1 - \frac{k^2}{4} \qquad z = k$$

which is an ellipse, provided that $k^2 < 4$, that is, $-2 < k < 2$.

Similarly, vertical traces parallel to the yz- and xz-planes are also ellipses:

$$\frac{y^2}{9} + \frac{z^2}{4} = 1 - k^2 \qquad x = k \qquad (\text{if } -1 < k < 1)$$

$$x^2 + \frac{z^2}{4} = 1 - \frac{k^2}{9} \qquad y = k \qquad (\text{if } -3 < k < 3)$$

Figure 4 shows how drawing some traces indicates the shape of the surface. It's called an **ellipsoid** because all of its traces are ellipses. Notice that it is symmetric with respect to each coordinate plane; this is a reflection of the fact that its equation involves only even powers of x, y, and z. ■

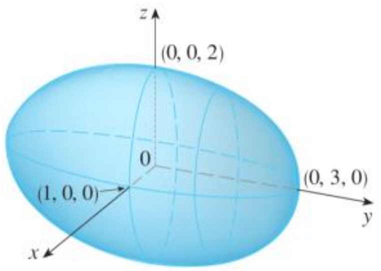


FIGURE 4
The ellipsoid $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$

EXAMPLE 4 Use traces to sketch the surface $z = 4x^2 + y^2$.

SOLUTION If we put $x = 0$, we get $z = y^2$, so the yz-plane intersects the surface in a parabola. If we put $x = k$ (a constant), we get $z = y^2 + 4k^2$. This means that if we slice the graph with any plane parallel to the yz-plane, we obtain a parabola that opens upward. Similarly, if $y = k$, the trace is $z = 4x^2 + k^2$, which is again a parabola that opens upward. If we put $z = k$, we get the horizontal traces $4x^2 + y^2 = k$, which we recognize as a family of ellipses. Knowing the shapes of the traces, we can sketch the graph in Figure 5. Because of the elliptical and parabolic traces, the quadric surface $z = 4x^2 + y^2$ is called an **elliptic paraboloid**. ■

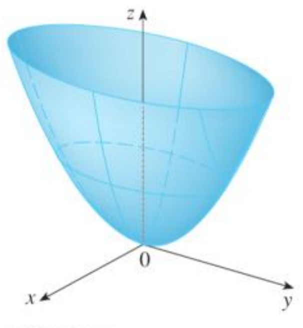
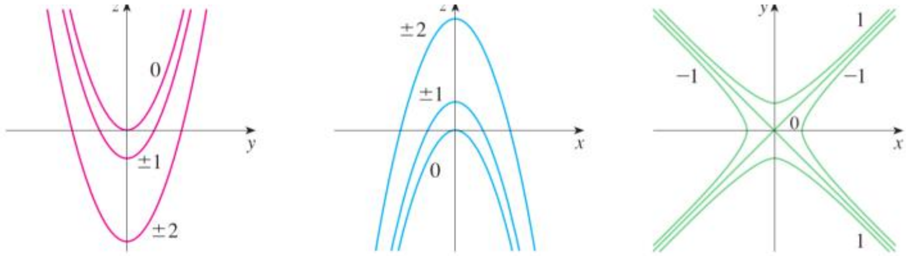


FIGURE 5
The surface $z = 4x^2 + y^2$ is an elliptic paraboloid. Horizontal traces are ellipses; vertical traces are parabolas.

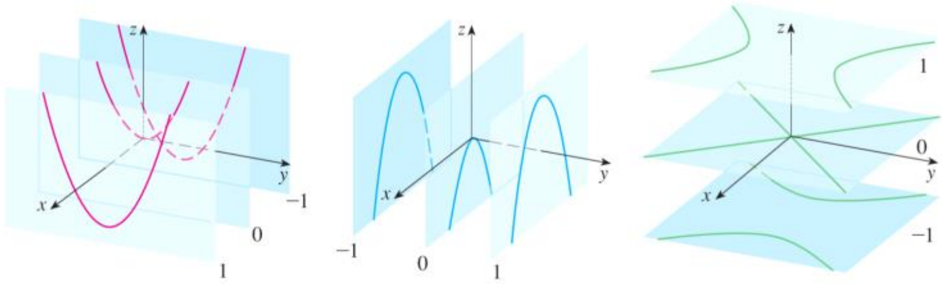
EXAMPLE 5 Sketch the surface $z = y^2 - x^2$.

SOLUTION The traces in the vertical planes $x = k$ are the parabolas $z = y^2 - k^2$, which open upward. The traces in $y = k$ are the parabolas $z = -x^2 + k^2$, which open downward. The horizontal traces are $y^2 - x^2 = k$, a family of hyperbolas. We draw the families of traces in Figure 6, and we show how the traces appear when placed in their correct planes in Figure 7.



Traces in $x = k$ are $z = y^2 - k^2$.　Traces in $y = k$ are $z = -x^2 + k^2$.　Traces in $z = k$ are $y^2 - x^2 = k$.

FIGURE 6
Vertical traces are parabolas; horizontal traces are hyperbolas. All traces are labeled with the value of k.



Traces in $x = k$　Traces in $y = k$　Traces in $z = k$

FIGURE 7
Traces moved to their correct planes

In Figure 8 we fit together the traces from Figure 7 to form the surface $z = y^2 - x^2$, a **hyperbolic paraboloid**. Notice that the shape of the surface near the origin resembles that of a saddle. This surface will be investigated further in Section 14.7 when we discuss saddle points.

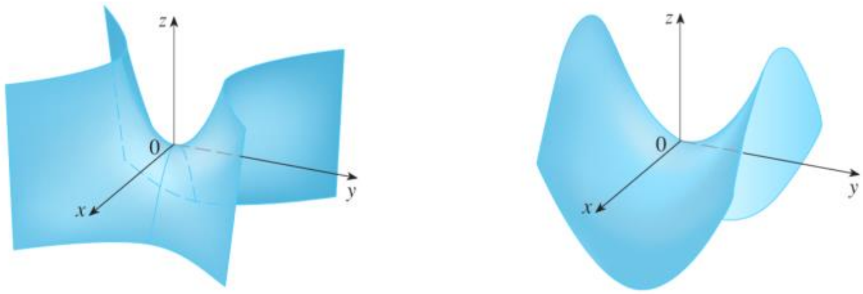


FIGURE 8
Two views of the surface $z = y^2 - x^2$, a hyperbolic paraboloid

■

13.1 Vector Functions and Space Curves (P920)

January 18, 2017 23:37

1. Vector-valued Functions

   A vector-valued function, or vector function, is a function whose domain is a set of real numbers and whose range is a set of vectors.

   $$\boldsymbol{r}: t \to \boldsymbol{r}(t)$$

2. Component Functions

   If $g(t), g(t), h(t)$ are the components of the vector $\boldsymbol{r}(t)$, then f, g, and h are real-valued functions called the component functions of r and we can write

   $$\boldsymbol{r}(t) = < f(t), g(t), h(t) > = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$$

EXAMPLE 6 Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.

SOLUTION Figure 5 shows how the plane and the cylinder intersect, and Figure 6 shows the curve of intersection C, which is an ellipse.

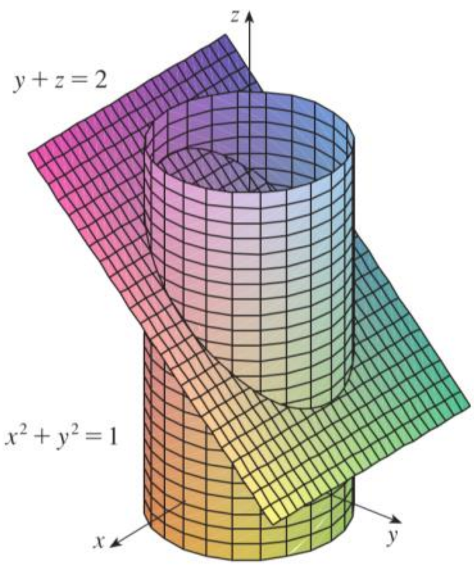


FIGURE 5

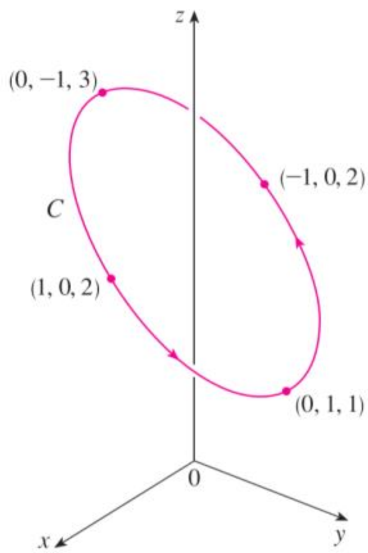


FIGURE 6

The projection of C onto the xy-plane is the circle $x^2 + y^2 = 1$, $z = 0$. So we know from Example 10.1.2 that we can write

$$x = \cos t \qquad y = \sin t \qquad 0 \leqslant t \leqslant 2\pi$$

From the equation of the plane, we have

$$z = 2 - y = 2 - \sin t$$

So we can write parametric equations for C as

$$x = \cos t \qquad y = \sin t \qquad z = 2 - \sin t \qquad 0 \leqslant t \leqslant 2\pi$$

The corresponding vector equation is

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + (2 - \sin t)\,\mathbf{k} \qquad 0 \leqslant t \leqslant 2\pi$$

3. Limits and Continuity

1 If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t \to a} \mathbf{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$$

provided the limits of the component functions exist.

A vector function $\boldsymbol{r}$ is continuous at a if

$$\lim_{t \to a} \boldsymbol{r}(t) = \boldsymbol{r}(a)$$

EXAMPLE 2 Find $\lim_{t \to 0} \mathbf{r}(t)$, where $\mathbf{r}(t) = (1 + t^3)\,\mathbf{i} + te^{-t}\,\mathbf{j} + \dfrac{\sin t}{t}\,\mathbf{k}$.

SOLUTION According to Definition 1, the limit of $\mathbf{r}$ is the vector whose components are the limits of the component functions of $\mathbf{r}$:

$$\lim_{t \to 0} \mathbf{r}(t) = \left[\lim_{t \to 0} (1 + t^3)\right]\mathbf{i} + \left[\lim_{t \to 0} te^{-t}\right]\mathbf{j} + \left[\lim_{t \to 0} \frac{\sin t}{t}\right]\mathbf{k}$$

$$= \mathbf{i} + \mathbf{k} \qquad \text{(by Equation 2.4.2)}$$

4. Space Curves

The set C of all points (x, y, z) in space, where

$$x = f(t), y = g(t), z = h(t)$$

and t varies throughout the interval I, is called a space curve.

13.2 Derivatives and Integrals of Vector Functions (P927)

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1. Derivatives

1 $$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

2 Theorem If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, where f, g, and h are differentiable functions, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}$$

$\boldsymbol{r}'(t)$ is the tangent vector to the curve $\boldsymbol{r}(t)$

2. Unit Tangent Vector

$$\boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|}$$

EXAMPLE 1

(a) Find the derivative of $\mathbf{r}(t) = (1 + t^3)\,\mathbf{i} + te^{-t}\,\mathbf{j} + \sin 2t\,\mathbf{k}$.

(b) Find the unit tangent vector at the point where $t = 0$.

SOLUTION

(a) According to Theorem 2, we differentiate each component of $\mathbf{r}$:

$$\mathbf{r}'(t) = 3t^2\,\mathbf{i} + (1 - t)e^{-t}\,\mathbf{j} + 2\cos 2t\,\mathbf{k}$$

(b) Since $\mathbf{r}(0) = \mathbf{i}$ and $\mathbf{r}'(0) = \mathbf{j} + 2\mathbf{k}$, the unit tangent vector at the point $(1, 0, 0)$ is

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{\mathbf{j} + 2\mathbf{k}}{\sqrt{1+4}} = \frac{1}{\sqrt{5}}\,\mathbf{j} + \frac{2}{\sqrt{5}}\,\mathbf{k}$$

EXAMPLE 3 Find parametric equations for the tangent line to the helix with parametric equations

$$x = 2\cos t \qquad y = \sin t \qquad z = t$$

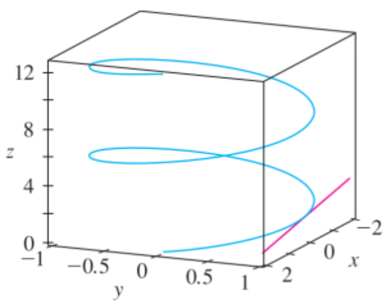
at the point $(0, 1, \pi/2)$.

SOLUTION The vector equation of the helix is $\mathbf{r}(t) = \langle 2\cos t, \sin t, t \rangle$, so

$$\mathbf{r}'(t) = \langle -2\sin t, \cos t, 1 \rangle$$

The parameter value corresponding to the point $(0, 1, \pi/2)$ is $t = \pi/2$, so the tangent vector there is $\mathbf{r}'(\pi/2) = \langle -2, 0, 1 \rangle$. The tangent line is the line through $(0, 1, \pi/2)$ parallel to the vector $\langle -2, 0, 1 \rangle$, so by Equations 12.5.2 its parametric equations are

$$x = -2t \qquad y = 1 \qquad z = \frac{\pi}{2} + t$$



3. Differentiation Rules

3 Theorem Suppose **u** and **v** are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\dfrac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$

2. $\dfrac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

3. $\dfrac{d}{dt}[f(t)\,\mathbf{u}(t)] = f'(t)\,\mathbf{u}(t) + f(t)\,\mathbf{u}'(t)$

4. $\dfrac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$

5. $\dfrac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$

6. $\dfrac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$ (Chain Rule)

EXAMPLE 4 Show that if $|\mathbf{r}(t)| = c$ (a constant), then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t.

SOLUTION Since

$$\mathbf{r}(t)\cdot\mathbf{r}(t) = |\mathbf{r}(t)|^2 = c^2$$

and c^2 is a constant, Formula 4 of Theorem 3 gives

$$0 = \frac{d}{dt}[\mathbf{r}(t)\cdot\mathbf{r}(t)] = \mathbf{r}'(t)\cdot\mathbf{r}(t) + \mathbf{r}(t)\cdot\mathbf{r}'(t) = 2\mathbf{r}'(t)\cdot\mathbf{r}(t)$$

Thus $\mathbf{r}'(t)\cdot\mathbf{r}(t) = 0$, which says that $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$.

Geometrically, this result says that if a curve lies on a sphere with center the origin, then the tangent vector $\mathbf{r}'(t)$ is always perpendicular to the position vector $\mathbf{r}(t)$. (See Figure 4.)

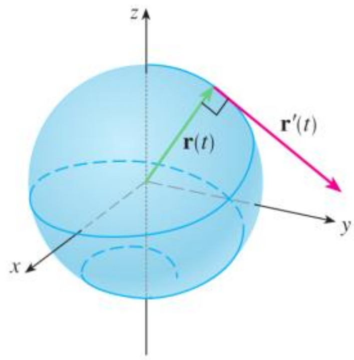


FIGURE 4

4. Integrals

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\mathbf{i} + \left(\int_a^b g(t)\,dt\right)\mathbf{j} + \left(\int_a^b h(t)\,dt\right)\mathbf{k}$$

Example:

Find $\boldsymbol{r}(t)$, where

$$\boldsymbol{r}'(t) =< \tan t, \frac{1}{1+t^2}, \frac{1}{1-t^2} >, \boldsymbol{r}(0) =< 1,1,1 >$$

$$\boldsymbol{r}(t) = \int \boldsymbol{r}'(t)$$

$$=< \int \tan t, \int \frac{1}{1+t^2}, \int \frac{1}{1-t^2} >$$

$$= < -\ln(\cos t) + c_1, \arctan t + c_2, \frac{1}{2}\left(-\ln(1-t) + \ln(1+t)\right) >$$

$$= < -\ln(\cos t) + c_1, \arctan t + c_2, \frac{1}{2}\left(\ln\frac{1+t}{1-t}\right) + c_3 >$$

$$\boldsymbol{r}(0) =< 1,1,1 > = < c_1, c_2, c_3 >$$

$$\boldsymbol{r}(t) =< -\ln(\cos t) + 1, \arctan t + 1, \frac{1}{2}\left(\ln\frac{1+t}{1-t}\right) + 1 >$$

14.3 Partial Derivatives (P983)

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4 If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

Notations for Partial Derivatives If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Rule for Finding Partial Derivatives of $z = f(x, y)$

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

Clairaut's Theorem Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

$$f(x,y) = x^2 + y^4 + x \sin y + e^{xy} + \tan(x^2 + y^2)$$
$$\frac{\partial f}{\partial x} = 2x + 0 + \sin y + ye^{xy} + \frac{1}{\cos^2(x^2 + y^2)} \cdot 2x$$
$$\frac{\partial f}{\partial x} = 0 + 4y^3 + x \cos y + xe^{xy} + \frac{1}{\cos^2(x^2 + y^2)} \cdot 2y$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial^2 x}$$
$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial y\partial x}$$

$$f(x,y)=\sin(xy)+x^2y^2$$
$$\frac{\partial f}{\partial x}=y\cos(xy)+2xy^2\Rightarrow\frac{\partial^2 f}{\partial y\partial x}=\cos(xy)-xy\sin(xy)+4xy$$
$$\frac{\partial f}{\partial y}=x\cos(xy)+2yx^2\Rightarrow\frac{\partial^2 f}{\partial x\partial y}=\cos(xy)-xy\sin(xy)+4xy$$

$$f(x,y)=x^3y^2$$
$$f_x=\frac{\partial f}{\partial x}=3x^2y^2$$
$$f_y=\frac{\partial f}{\partial y}=2x^3y\Rightarrow f_{yx}=6x^2y,f_{yy}=2x^3$$
$$\Rightarrow f_{yxy}=\frac{\partial^3 f}{\partial y\partial x\partial y}=6x^2,f_{yyx}=\frac{\partial^3 f}{\partial x\partial y\partial y}=6x^2$$

Implicit differentiation

$$x^3+y^3+z^3+6xyz=1$$
$$z=f(x,y)$$

in x:

$$3x^2+0+3z^2\cdot\frac{\partial z}{\partial x}+6yz+6xy\cdot\frac{\partial z}{\partial x}=0$$
$$\frac{\partial z}{\partial x}(3z^2+6xy)=-3x^2-6yz$$
$$\frac{\partial z}{\partial x}=\frac{-3x^2-6yz}{3z^2+6xy}$$

in y:

$$0+3y^2+3z^2\cdot\frac{\partial z}{\partial y}+6xy\cdot\frac{\partial z}{\partial y}=0$$
$$\frac{\partial z}{\partial y}(3z^2+6xy)=-3y^2-6xz$$
$$\frac{\partial z}{\partial y}=\frac{-3y^2-6xz}{3z^2+6xy}$$

13.3 Arc Length and Curvature (P933)

January 18, 2017 23:37

1. Length of a Curve

$\boxed{1}$ $$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$\boxed{2}$ $$\boxed{\begin{aligned} L &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt \\ &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \end{aligned}}$$

$\boxed{3}$ $$\boxed{L = \int_a^b |\mathbf{r}'(t)|\, dt}$$

Example: Find the length of the curve $\boldsymbol{r}(t) =< \sqrt{2}t, e^t, e^{-t} >, t \in [0,6]$

$\boldsymbol{r}'(t) =< \sqrt{2}, e^t, -e^{-t} >$

$|\boldsymbol{r}'(t)| = \sqrt{2 + e^{2t} + e^{-2t}} = e^t + e^{-t}$

$\int_0^6 (e^t + e^{-t})dt = [e^t - e^{-t}]_0^6 = e^6 - e^{-6}$

2. Parametrization of a Curve

A single curve C can be represented by more than one vector function. For instance, the twisted cubic

$\boxed{4}$ $$\mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle \qquad 1 \leqslant t \leqslant 2$$

could also be represented by the function

$\boxed{5}$ $$\mathbf{r}_2(u) = \langle e^u, e^{2u}, e^{3u} \rangle \qquad 0 \leqslant u \leqslant \ln 2 \ \ (t = e^u)$$

We say that Equations 4 and 5 are parametrizations of the curve C. If we were to use Equation 3 to compute the length of C, using Equations 4 and 5 we would get the same answer.

$r_1'(t) =< 1, 2t, 3t^2 >, r_2'(u) =< e^u, 2e^{2u}, 3e^{3u} >$

$t = \frac{3}{2} \Rightarrow u = \ln\frac{3}{2}$

$\Rightarrow r_1'\left(\frac{3}{2}\right) =< 1, 3, \left(\frac{27}{4}\right) >, r_2'\left(\ln\frac{3}{2}\right) =< \frac{3}{2}, \left(\frac{9}{2}\right), \left(\frac{81}{8}\right) >$

$\Rightarrow r_2'\left(\ln\frac{3}{2}\right) = \frac{3}{2} r_1'\left(\frac{3}{2}\right)$

3. The Arc Length Function

$\boxed{6}$ $$s(t) = \int_a^t |\mathbf{r}'(u)|\, du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du$$

It is often useful to **parametrize a curve with respect to arc length** because arc length arises naturally from the shape of the curve and does not depend on a particular coordinate system. If a curve $\boldsymbol{r}(t)$ is already given in terms of a parameter t and $s(t)$ is the arc length function given by Equation 6, then we may be able to solve for t as a function of s: $t = t(s)$. Then the curve can be reparametrized in terms of s by substituting for t: $\boldsymbol{r} = \boldsymbol{r}\big(t(s)\big)$. Thus, if $s = 3$ for instance, $\boldsymbol{r}\big(t(3)\big)$ is the position vector of the point 3 units of length along the curve from its starting point.

Example: Curve $\boldsymbol{r}(t) =< \sqrt{2}t, e^t, e^{-t} >$

$\boldsymbol{r}'(t) =< \sqrt{2}, e^t, -e^{-t} >$

$|\boldsymbol{r}'(t)| = \sqrt{2 + e^{2t} + e^{-2t}} = e^t + e^{-t}$

$$s(t) = \int (e^t + e^{-t})dt = e^t - e^{-t}$$

$$s(t)e^t = e^{2t} - 1 \Rightarrow e^t = \frac{s + \sqrt{s^2+4}}{2} \Rightarrow t = \ln \frac{s + \sqrt{s^2+4}}{2}$$

$$\Rightarrow \boldsymbol{r}(s) = < \sqrt{2}\ln \frac{s+\sqrt{s^2+4}}{2}, \frac{s+\sqrt{s^2+4}}{2}, \frac{2}{s+\sqrt{s^2+4}} >$$

4. Curvature

8 Definition The **curvature** of a curve is

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right|$$

where $\mathbf{T}$ is the unit tangent vector.

9
$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$

10 Theorem The curvature of the curve given by the vector function $\mathbf{r}$ is

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

PROOF Since $\mathbf{T} = \mathbf{r}'/|\mathbf{r}'|$ and $|\mathbf{r}'| = ds/dt$, we have

$$\mathbf{r}' = |\mathbf{r}'|\mathbf{T} = \frac{ds}{dt}\mathbf{T}$$

so the Product Rule (Theorem 13.2.3, Formula 3) gives

$$\mathbf{r}'' = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\mathbf{T}'$$

Using the fact that $\mathbf{T} \times \mathbf{T} = \mathbf{0}$ (see Example 12.4.2), we have

$$\mathbf{r}' \times \mathbf{r}'' = \left(\frac{ds}{dt}\right)^2 (\mathbf{T} \times \mathbf{T}')$$

Now $|\mathbf{T}(t)| = 1$ for all t, so $\mathbf{T}$ and $\mathbf{T}'$ are orthogonal by Example 13.2.4. Therefore, by Theorem 12.4.9,

$$|\mathbf{r}' \times \mathbf{r}''| = \left(\frac{ds}{dt}\right)^2 |\mathbf{T} \times \mathbf{T}'| = \left(\frac{ds}{dt}\right)^2 |\mathbf{T}||\mathbf{T}'| = \left(\frac{ds}{dt}\right)^2 |\mathbf{T}'|$$

Thus
$$|\mathbf{T}'| = \frac{|\mathbf{r}' \times \mathbf{r}''|}{(ds/dt)^2} = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^2}$$

and
$$\kappa = \frac{|\mathbf{T}'|}{|\mathbf{r}'|} = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3} \qquad ■$$

EXAMPLE 3 Show that the curvature of a circle of radius a is $1/a$.

SOLUTION We can take the circle to have center the origin, and then a parametrization is

$$\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j}$$

Therefore $\quad \mathbf{r}'(t) = -a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j} \quad$ and $\quad |\mathbf{r}'(t)| = a$

so $$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$$

and $$\mathbf{T}'(t) = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j}$$

This gives $|\mathbf{T}'(t)| = 1$, so using Formula 9, we have

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{1}{a}$$ ■

For the special case of a plane curve with equation $y = f(x)$, we choose x as the parameter and write $\mathbf{r}(x) = x\,\mathbf{i} + f(x)\,\mathbf{j}$. Then $\mathbf{r}'(x) = \mathbf{i} + f'(x)\,\mathbf{j}$ and $\mathbf{r}''(x) = f''(x)\,\mathbf{j}$. Since $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ and $\mathbf{j} \times \mathbf{j} = \mathbf{0}$, it follows that $\mathbf{r}'(x) \times \mathbf{r}''(x) = f''(x)\,\mathbf{k}$. We also have $|\mathbf{r}'(x)| = \sqrt{1 + [f'(x)]^2}$ and so, by Theorem 10,

11 $$\boxed{\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}}$$

Note: $\boldsymbol{r}(x) =< x, f(x) >$
or $\boldsymbol{r}(x) =< x, f(x), 0 >$

EXAMPLE 5 Find the curvature of the parabola $y = x^2$ at the points (0, 0), (1, 1), and (2, 4).

SOLUTION Since $y' = 2x$ and $y'' = 2$, Formula 11 gives

$$\kappa(x) = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{2}{(1 + 4x^2)^{3/2}}$$

The curvature at (0, 0) is $\kappa(0) = 2$. At (1, 1) it is $\kappa(1) = 2/5^{3/2} \approx 0.18$. At (2, 4) it is $\kappa(2) = 2/17^{3/2} \approx 0.03$. Observe from the expression for $\kappa(x)$ or the graph of κ in Figure 5 that $\kappa(x) \to 0$ as $x \to \pm\infty$. This corresponds to the fact that the parabola appears to become flatter as $x \to \pm\infty$.

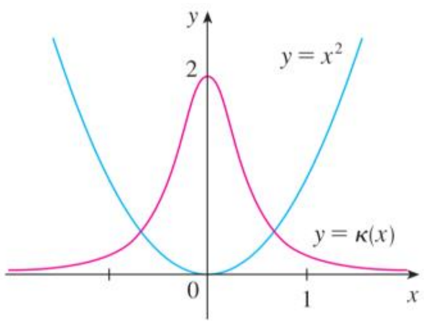

FIGURE 5
The parabola $y = x^2$ and its curvature function

5. The Normal and Binormal Vectors

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \qquad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \qquad \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

EXAMPLE 6 Find the unit normal and binormal vectors for the circular helix

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$$

SOLUTION We first compute the ingredients needed for the unit normal vector:

$$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k} \qquad |\mathbf{r}'(t)| = \sqrt{2}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{2}}(-\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k})$$

$$\mathbf{T}'(t) = \frac{1}{\sqrt{2}}(-\cos t\,\mathbf{i} - \sin t\,\mathbf{j}) \qquad |\mathbf{T}'(t)| = \frac{1}{\sqrt{2}}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j} = \langle -\cos t, -\sin t, 0\rangle$$

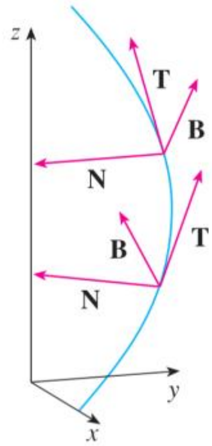

FIGURE 7

This shows that the normal vector at any point on the helix is horizontal and points toward the z-axis. The binormal vector is

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\langle \sin t, -\cos t, 1\rangle$$

6. Normal plane and Osculating plane
    - Normal plane: The plane determined by the normal and binormal vectors $\boldsymbol{N}$ and $\boldsymbol{B}$ at a point P on a curve C is called the normal plane of C at P. It consists of all lines that are orthogonal to the tangent vector T.
    - Osculating plane: The plane determined by the vectors T and N is called the osculating plane of C at P.
    - Osculating circle: The circle that lies in the osculating plane of C at P, has the same tangent as C at P, lies on the concav e side of C (toward which N points), and has radius $\rho = 1/\kappa$ (the reciprocal of the curvature) is called the osculating circle (or the circle of curvature) of C at P.

13.4 Motion in Space: Velocity and Acceleration

January 18, 2017 23:37

1. Velocity Vector

2
$$\mathbf{v}(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \mathbf{r}'(t)$$

2. Speed
$$v(t) = |\boldsymbol{v}(t)|$$

3. Acceleration
$$\boldsymbol{a}(t) = \boldsymbol{v}'(t) = \boldsymbol{r}''(t)$$

Example: A projectile is fired with the angle of elevation α and initial velocity $\boldsymbol{v}_0$. Assuming the only external force is gravity, find the trajectory $\boldsymbol{r}(t)$ of the projectile.

$$v_0 = |\boldsymbol{v}_0|$$
$$\boldsymbol{v}_0 =< v_0 \cos\alpha , 0, v_0 \sin\alpha >$$
$$\boldsymbol{F} = m \cdot \boldsymbol{a} = m \cdot < 0,0,-g >$$
$$\boldsymbol{a} = \boldsymbol{v}' =< 0,0,-g >$$
$$\boldsymbol{v} = \int < 0,0,-g > dt =< C_1, C_2, -gt + C_3 >$$
$$\boldsymbol{v}(0) =< v_0 \cos\alpha , 0, v_0 \sin\alpha > = < C_1, C_2, -gt + C_3 >$$
$$\boldsymbol{v}(t) =< v_0 \cos\alpha , 0, -gt + v_0 \sin\alpha >$$
$$\boldsymbol{r}(t) = \int \boldsymbol{v}(t)\, dt =$$
$$< v_0 \cos\alpha\, t + C_1, C_2, -\frac{1}{2} g t^2 + v_0 \sin\alpha\, t + C_3 >$$
$$\boldsymbol{r}(0) =< 0,0,0 >$$
$$\boldsymbol{r}(t) =< v_0 \cos\alpha\, t, 0, -\frac{1}{2} g t^2 + v_0 \sin\alpha\, t >$$

How long will the projectile be in the air?

$$-\frac{1}{2} g t^2 + v_0 \sin\alpha\, t = 0$$
$$t = 0 \; or \; t = \frac{v_0 \sin\alpha}{\frac{1}{2} g}$$

How far will it travel?

$$d = v_0 \cos\alpha \left(\frac{v_0 \sin\alpha}{\frac{1}{2} g} \right) = \frac{v_0^2 \sin 2\alpha}{g}$$
$$d_{max} = d\left(\alpha = \frac{\pi}{2}\right)$$

4. Tangential and Normal Components of Acceleration

If we write $v = |\boldsymbol{v}|$ for the speed of the particle, then

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{\mathbf{v}}{v}$$

and so
$$\mathbf{v} = v\mathbf{T}$$

If we differentiate both sides of this equation with respect to t, we get

5
$$\mathbf{a} = \mathbf{v}' = v'\mathbf{T} + v\mathbf{T}'$$

If we use the expression for the curvature given by Equation 13.3.9, then we have

6
$$\kappa = \frac{|\mathbf{T}'|}{|\mathbf{r}'|} = \frac{|\mathbf{T}'|}{v} \quad \text{so} \quad |\mathbf{T}'| = \kappa v$$

The unit normal vector was defined in the preceding section as $\mathbf{N} = \mathbf{T}'/|\mathbf{T}'|$, so (6) gives

$$\mathbf{T}' = |\mathbf{T}'|\mathbf{N} = \kappa v \mathbf{N}$$

and Equation 5 becomes

7
$$\mathbf{a} = v'\mathbf{T} + \kappa v^2 \mathbf{N}$$

Writing a_T and a_N for the tangential and normal components of acceleration, we have

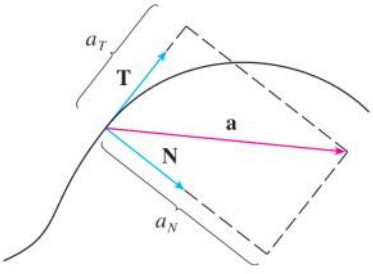

FIGURE 7

$\boxed{7}$ $$\mathbf{a} = v'\mathbf{T} + \kappa v^2\mathbf{N}$$

Writing a_T and a_N for the tangential and normal components of acceleration, we have

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$$

where

$\boxed{8}$ $$a_T = v' \qquad \text{and} \qquad a_N = \kappa v^2$$

$$\begin{aligned}\mathbf{v}\cdot\mathbf{a} &= v\mathbf{T}\cdot(v'\mathbf{T} + \kappa v^2\mathbf{N})\\ &= vv'\mathbf{T}\cdot\mathbf{T} + \kappa v^3\mathbf{T}\cdot\mathbf{N}\\ &= vv' \qquad (\text{since } \mathbf{T}\cdot\mathbf{T} = 1 \text{ and } \mathbf{T}\cdot\mathbf{N} = 0)\end{aligned}$$

Therefore

$\boxed{9}$ $$a_T = v' = \frac{\mathbf{v}\cdot\mathbf{a}}{v} = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|}$$

Using the formula for curvature given by Theorem 13.3.10, we have

$\boxed{10}$ $$a_N = \kappa v^2 = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}|\mathbf{r}'(t)|^2 = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|}$$

EXAMPLE 7 A particle moves with position function $\mathbf{r}(t) = \langle t^2, t^2, t^3\rangle$. Find the tangential and normal components of acceleration.

SOLUTION

$$\begin{aligned}\mathbf{r}(t) &= t^2\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}\\ \mathbf{r}'(t) &= 2t\,\mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k}\\ \mathbf{r}''(t) &= 2\,\mathbf{i} + 2\,\mathbf{j} + 6t\,\mathbf{k}\\ |\mathbf{r}'(t)| &= \sqrt{8t^2 + 9t^4}\end{aligned}$$

Therefore Equation 9 gives the tangential component as

$$a_T = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \frac{8t + 18t^3}{\sqrt{8t^2 + 9t^4}}$$

Since $$\mathbf{r}'(t)\times\mathbf{r}''(t) = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2t & 2t & 3t^2\\ 2 & 2 & 6t\end{vmatrix} = 6t^2\,\mathbf{i} - 6t^2\,\mathbf{j}$$

Equation 10 gives the normal component as

$$a_N = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \frac{6\sqrt{2}\,t^2}{\sqrt{8t^2 + 9t^4}}$$

14.1 Functions of Several Variables (P960)

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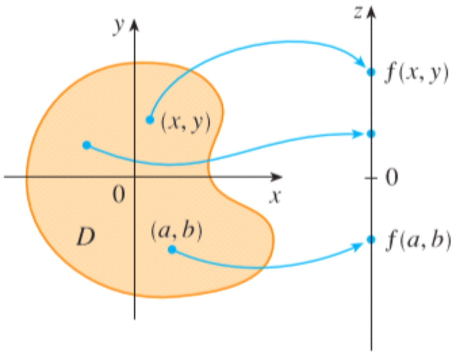
1. Functions of Two Variables

Definition A **function f of two variables** is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the **domain** of f and its **range** is the set of values that f takes on, that is, $\{f(x, y) \mid (x, y) \in D\}$.

x and y are independent variables
z is the dependent variable



EXAMPLE 1 For each of the following functions, evaluate $f(3, 2)$ and find and sketch the domain.

(a) $f(x, y) = \dfrac{\sqrt{x + y + 1}}{x - 1}$ (b) $f(x, y) = x \ln(y^2 - x)$

SOLUTION

(a) $f(3, 2) = \dfrac{\sqrt{3 + 2 + 1}}{3 - 1} = \dfrac{\sqrt{6}}{2}$

The expression for f makes sense if the denominator is not 0 and the quantity under the square root sign is nonnegative. So the domain of f is

$$D = \{(x, y) \mid x + y + 1 \geqslant 0,\ x \neq 1\}$$

(b) $$f(3, 2) = 3 \ln(2^2 - 3) = 3 \ln 1 = 0$$

Since $\ln(y^2 - x)$ is defined only when $y^2 - x > 0$, that is, $x < y^2$, the domain of f is $D = \{(x, y) \mid x < y^2\}$. This is the set of points to the left of the parabola $x = y^2$.

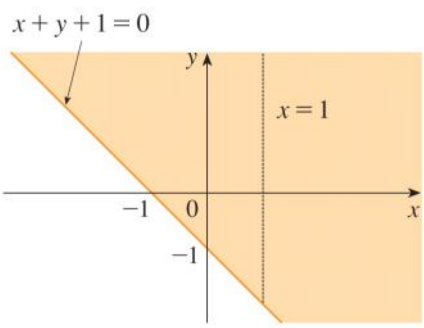


FIGURE 2

Domain of $f(x, y) = \dfrac{\sqrt{x + y + 1}}{x - 1}$

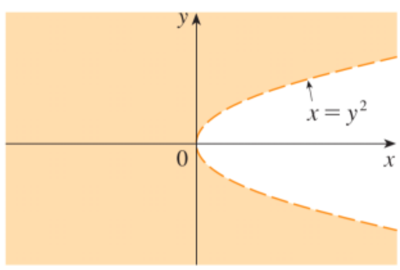


FIGURE 3

Domain of $f(x, y) = x \ln(y^2 - x)$

EXAMPLE 4 Find the domain and range of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

SOLUTION The domain of g is

$$D = \{(x, y) \mid 9 - x^2 - y^2 \geqslant 0\} = \{(x, y) \mid x^2 + y^2 \leqslant 9\}$$

which is the disk with center $(0, 0)$ and radius 3. (See Figure 4.) The range of g is

$$\left\{z \mid z = \sqrt{9 - x^2 - y^2},\ (x, y) \in D\right\}$$

Since z is a positive square root, $z \geqslant 0$. Also, because $9 - x^2 - y^2 \leqslant 9$, we have

$$\sqrt{9 - x^2 - y^2} \leqslant 3$$

So the range is

$$\{z \mid 0 \leqslant z \leqslant 3\} = [0, 3]$$

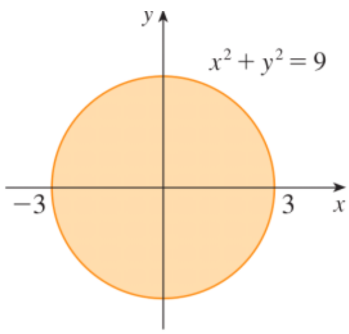
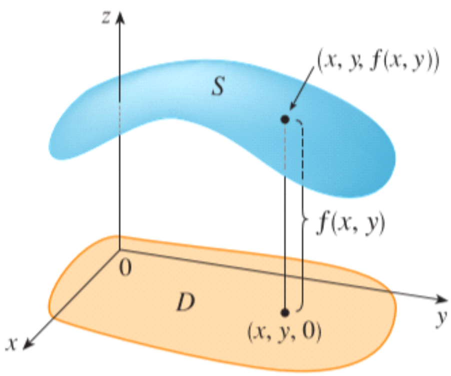


FIGURE 4
Domain of $g(x, y) = \sqrt{9 - x^2 - y^2}$

2. Graphs

> **Definition** If f is a function of two variables with domain D, then the **graph** of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.



EXAMPLE 6 Sketch the graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

SOLUTION The graph has equation $z = \sqrt{9 - x^2 - y^2}$. We square both sides of this equation to obtain $z^2 = 9 - x^2 - y^2$, or $x^2 + y^2 + z^2 = 9$, which we recognize as an equation of the sphere with center the origin and radius 3. But, since $z \geqslant 0$, the graph of g is just the top half of this sphere (see Figure 7).

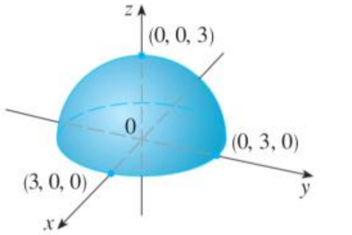


FIGURE 7
Graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$

3. Level Curves

> **Definition** The **level curves** of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).

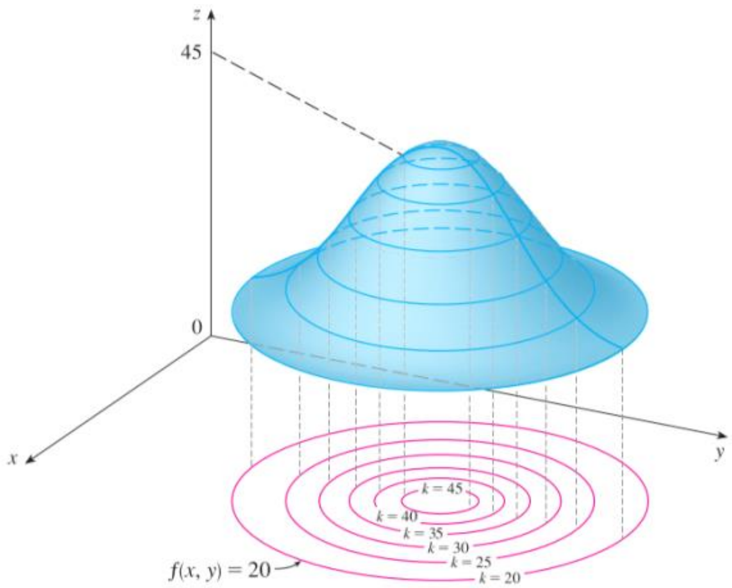


FIGURE 11

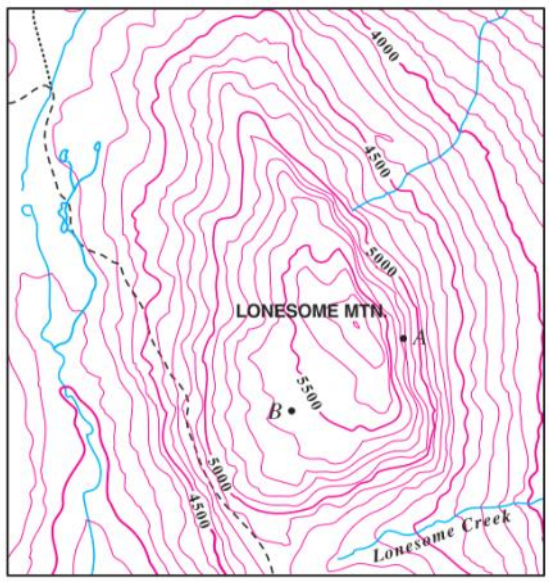


FIGURE 12

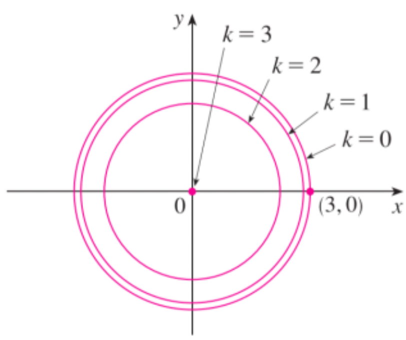
EXAMPLE 11 Sketch the level curves of the function

$$g(x, y) = \sqrt{9 - x^2 - y^2} \qquad \text{for} \quad k = 0, 1, 2, 3$$

SOLUTION The level curves are

$$\sqrt{9 - x^2 - y^2} = k \qquad \text{or} \qquad x^2 + y^2 = 9 - k^2$$

This is a family of concentric circles with center $(0, 0)$ and radius $\sqrt{9 - k^2}$. The cases $k = 0, 1, 2, 3$ are shown in Figure 17. Try to visualize these level curves lifted up to form a surface and compare with the graph of g (a hemisphere) in Figure 7. (See TEC Visual 14.1A.)



4. Functions of Three or More Variables

A function of three variables, f, is a rule that assigns to each ordered triple (x, y, z) in a domain $D \subset \mathbb{R}^3$ a unique real number denoted by $f(x, y, z)$. For instance, the temperature T at a point on the surface of the earth depends on the longitude x and latitude y of the point and on the time t, so we could write $T = f(x, y, t)$.

A function of n variables is a rule that assigns a number $z = f(x_1, x_2, \dots, x_n)$ to an n-tuple $(x_1, x_2, \dots, x_n)$ of real numbers. We denote by $\mathbb{R}^n$ the set of all such n-tuples. For example, if a company uses n different ingredients in making a fodd product, c_i is the cost per unit of th ith ingredient, and x_i units of the ith ingredients are used, then the total cost C of the ingredients is a function of the n variables $x_1, x_2, \dots, x_n$: $C = f\left(x_1, x_2, \dots, x_n\right) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$

If $\boldsymbol{X} =< x_1, x_2, \dots, x_n >, \boldsymbol{c} =< c_1, c_2, \dots, c_n > \Rightarrow f(\boldsymbol{X}) = \boldsymbol{c} \cdot \boldsymbol{X}$

In view of the one-to-one correspondence between points $(x_1, x_2, \dots, x_n)$ in $\mathbb{R}^n$ and their position vectors $\boldsymbol{X} =< x_1, x_2, \dots, x_n >$ in V_n we have three ways of looking at a function f defined on a subsset of $\mathbb{R}^n$:

1) As a function of n real variables $x_1, x_2, \dots, x_n$
2) As a function of a single point variable $\left(x_1, x_2, \dots, x_n\right)$
3) As a function of a single vector variable $\boldsymbol{X} =< x_1, x_2, \dots, x_n >$

5. Exercises

a)

14.2 Limits and Continuity (P975)

January 18, 2017 23:37

1. Limits

1 Definition Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the **limit of $f(x, y)$ as (x, y) approaches (a, b)** is L and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

if $(x, y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x, y) - L| < \varepsilon$

If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

2 $$\lim_{(x,y)\to(a,b)} x = a \qquad \lim_{(x,y)\to(a,b)} y = b \qquad \lim_{(x,y)\to(a,b)} c = c$$

The Squeeze Theorem also holds.

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$$

Let $t = x^2 + y^2$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{t\to(0,0)} \frac{\sin t}{t} = 1$$

EXAMPLE 1 Show that $\lim_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2}$ does not exist.

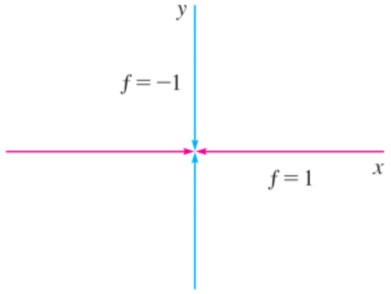
SOLUTION Let $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$. First let's approach $(0, 0)$ along the x-axis. Then $y = 0$ gives $f(x, 0) = x^2/x^2 = 1$ for all $x \neq 0$, so

$$f(x, y) \to 1 \quad \text{as} \quad (x, y) \to (0, 0) \text{ along the } x\text{-axis}$$

We now approach along the y-axis by putting $x = 0$. Then $f(0, y) = \dfrac{-y^2}{y^2} = -1$ for all $y \neq 0$, so

$$f(x, y) \to -1 \quad \text{as} \quad (x, y) \to (0, 0) \text{ along the } y\text{-axis}$$

Since f has two different limits along two different lines, the given limit does not exist.



EXAMPLE 2 If $f(x, y) = xy/(x^2 + y^2)$, does $\lim_{(x,y)\to(0,0)} f(x, y)$ exist?

SOLUTION If $y = 0$, then $f(x, 0) = 0/x^2 = 0$. Therefore

$$f(x, y) \to 0 \qquad \text{as} \qquad (x, y) \to (0, 0) \text{ along the } x\text{-axis}$$

If $x = 0$, then $f(0, y) = 0/y^2 = 0$, so

$$f(x, y) \to 0 \qquad \text{as} \qquad (x, y) \to (0, 0) \text{ along the } y\text{-axis}$$

Although we have obtained identical limits along the axes, that does not show that the given limit is 0. Let's now approach (0, 0) along another line, say $y = x$. For all $x \neq 0$,

$$f(x, x) = \frac{x^2}{x^2 + x^2} = \frac{1}{2}$$

Therefore $\qquad f(x, y) \to \frac{1}{2} \qquad$ as $\qquad (x, y) \to (0, 0)$ along $y = x$

(See Figure 5.) Since we have obtained different limits along different paths, the given limit does not exist. ■

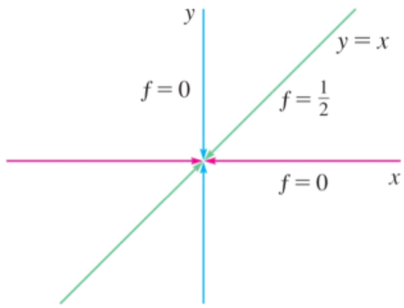


FIGURE 5

EXAMPLE 3 If $f(x, y) = \dfrac{xy^2}{x^2 + y^4}$, does $\lim_{(x,y)\to(0,0)} f(x, y)$ exist?

SOLUTION With the solution of Example 2 in mind, let's try to save time by letting $(x, y) \to (0, 0)$ along any line through the origin. If the line is not the y-axis, then $y = mx$, where m is the slope, and

$$f(x, y) = f(x, mx) = \frac{x(mx)^2}{x^2 + (mx)^4} = \frac{m^2x^3}{x^2 + m^4x^4} = \frac{m^2x}{1 + m^4x^2}$$

So $\qquad f(x, y) \to 0 \qquad$ as $\qquad (x, y) \to (0, 0)$ along $y = mx$

We get the same result as $(x, y) \to (0, 0)$ along the line $x = 0$. Thus f has the same limiting value along every line through the origin. But that does not show that the given limit is 0, for if we now let $(x, y) \to (0, 0)$ along the parabola $x = y^2$, we have

$$f(x, y) = f(y^2, y) = \frac{y^2 \cdot y^2}{(y^2)^2 + y^4} = \frac{y^4}{2y^4} = \frac{1}{2}$$

so $\qquad f(x, y) \to \frac{1}{2} \qquad$ as $\qquad (x, y) \to (0, 0)$ along $x = y^2$

Since different paths lead to different limiting values, the given limit does not exist.

EXAMPLE 4 Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2 + y^2}$ if it exists.

Solution:

$$\frac{x^2}{x^2+y^2} \le 1$$
$$0 \le \left|3y \cdot \frac{x^2}{x^2+y^2}\right| \le 3|y|$$
$$\lim_{(x,y)\to(0,0)} 3|y| = 0$$
$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$$

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = \lim_{r\to 0} \frac{(r\cos\alpha)^3+(r\sin\alpha)^3}{(r\cos\alpha)^2+(r\sin\alpha)^2}$$
$$= \lim_{r\to 0} \frac{(r\cos\alpha)^3+(r\sin\alpha)^3}{r^2}$$
$$= \lim_{r\to 0} r\left(\cos^3\alpha + \sin^3\alpha\right) = 0$$

$$\lim_{(x,y)\to(0,0)} \frac{\left(x^2+y^2\right)\sin(x^2+y^2)}{x^4+y^4} = \lim_{r\to 0} \frac{r^2 \sin r^2}{r^4\cos^4\theta + r^4\sin^4\theta}$$
$$= \lim_{r\to 0} \frac{\sin r^2}{r^2} \cdot \frac{1}{\cos^4\theta + \sin^4\theta}$$
$$\lim_{r\to 0} \frac{\sin r^2}{r^2} = 1$$
$$\lim_{r\to 0} \frac{1}{\cos^4\theta + \sin^4\theta} =?$$
$$\theta = 0 \Rightarrow \lim_{r\to 0} \frac{1}{\cos^4\theta + \sin^4\theta} = 1$$
$$\theta = \frac{\pi}{4} \Rightarrow \lim_{r\to 0} \frac{1}{\cos^4\theta + \sin^4\theta} = 2$$
$$\lim_{r\to 0} \frac{1}{\cos^4\theta + \sin^4\theta} \ \textit{does not exist}$$

2. Continuity

4 Definition A function f of two variables is called **continuous at** (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

We say f is **continuous on** D if f is continuous at every point (a, b) in D.

3. Functions of Three or More Variables

5 If f is defined on a subset D of $\mathbb{R}^n$, then $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$ means that for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if} \quad \mathbf{x} \in D \quad \text{and} \quad 0 < |\mathbf{x} - \mathbf{a}| < \delta \quad \text{then} \quad |f(\mathbf{x}) - L| < \varepsilon$$

14.4 Tangent Planes and Linear Approximations (P999)

January 18, 2017 23:38

1. Tangent Planes

2 Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

EXAMPLE 1 Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

SOLUTION Let $f(x, y) = 2x^2 + y^2$. Then

$$f_x(x, y) = 4x \qquad f_y(x, y) = 2y$$
$$f_x(1, 1) = 4 \qquad f_y(1, 1) = 2$$

Then (2) gives the equation of the tangent plane at $(1, 1, 3)$ as

$$z - 3 = 4(x - 1) + 2(y - 1)$$

or

$$z = 4x + 2y - 3$$

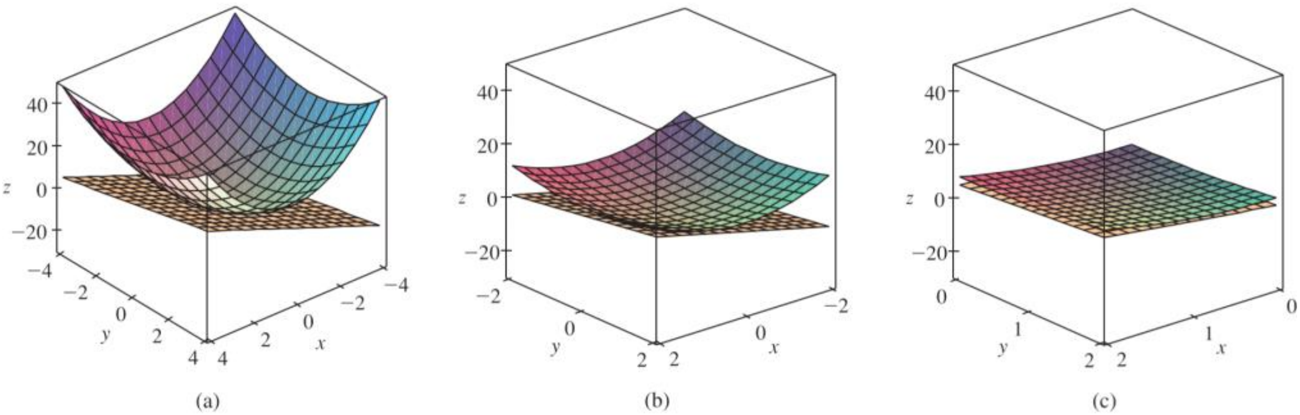


FIGURE 2 The elliptic paraboloid $z = 2x^2 + y^2$ appears to coincide with its tangent plane as we zoom in toward $(1, 1, 3)$.

2. Linear Approximations

In general, we know from (2) that an equation of the tangent plane to the graph of a function f of two variables at the point $(a, b, f(a, b))$ is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

The linear function whose graph is this tangent plane, namely

3 $$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linearization** of f at (a, b) and the approximation

4 $$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linear approximation** or the **tangent plane approximation** of f at (a, b).

7 Definition If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \ \Delta y) \to (0, 0)$.

8 Theorem If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

Find a tangent plane to the graph of $f(x, y) = \sin xy$ at the point $(1,1,\sin 1)$

$$\frac{\partial f}{\partial x} = y \cos xy$$

$$\frac{\partial f}{\partial y} = x \cos xy$$

both partial derivatives are continuous at $(1,1) \Rightarrow f(x,y)$ is differentiable at $(1,1)$

$$\Rightarrow z - \sin 1 = \cos 1\,(x-1) + \cos 1\,(y-1)$$

10

$$dz = f_x(x, y)\,dx + f_y(x, y)\,dy = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy$$

This gives approximation of f provided by the tangent plane.

EXAMPLE 4

(a) If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz.
(b) If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz.

SOLUTION

(a) Definition 10 gives

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy = (2x + 3y)\,dx + (3x - 2y)\,dy$$

(b) Putting $x = 2$, $dx = \Delta x = 0.05$, $y = 3$, and $dy = \Delta y = -0.04$, we get

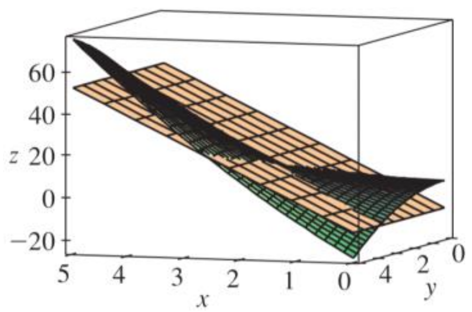
$$dz = [2(2) + 3(3)]0.05 + [3(2) - 2(3)](-0.04) = 0.65$$

The increment of z is

$$\begin{aligned}\Delta z &= f(2.05, 2.96) - f(2, 3)\\ &= [(2.05)^2 + 3(2.05)(2.96) - (2.96)^2] - [2^2 + 3(2)(3) - 3^2]\\ &= 0.6449\end{aligned}$$

Notice that $\Delta z \approx dz$ but dz is easier to compute.

In Example 4, dz is close to Δz because the tangent plane is a good approximation to the surface $z = x^2 + 3xy - y^2$ near $(2, 3, 13)$. (See Figure 8.)



14.5 The Chain Rule (P1009)

January 18, 2017 23:38

1. The Chain Rule

2 The Chain Rule (Case 1) Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

EXAMPLE 1 If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find dz/dt when $t = 0$.

SOLUTION The Chain Rule gives

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\\ &= (2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t)\end{aligned}$$

It's not necessary to substitute the expressions for x and y in terms of t. We simply observe that when $t = 0$, we have $x = \sin 0 = 0$ and $y = \cos 0 = 1$. Therefore

$$\left.\frac{dz}{dt}\right|_{t=0} = (0 + 3)(2\cos 0) + (0 + 0)(-\sin 0) = 6$$

3 The Chain Rule (Case 2) Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

EXAMPLE 3 If $z = e^x \sin y$, where $x = st^2$ and $y = s^2t$, find $\partial z/\partial s$ and $\partial z/\partial t$.

SOLUTION Applying Case 2 of the Chain Rule, we get

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (e^x \sin y)(t^2) + (e^x \cos y)(2st)\\ &= t^2e^{st^2}\sin(s^2t) + 2ste^{st^2}\cos(s^2t)\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (e^x \sin y)(2st) + (e^x \cos y)(s^2)\\ &= 2ste^{st^2}\sin(s^2t) + s^2e^{st^2}\cos(s^2t)\end{aligned}$$

4 **The Chain Rule (General Version)** Suppose that u is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$. Then u is a function of $t_1, t_2, \ldots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for each $i = 1, 2, \ldots, m$.

2. Implicit Differentiation

$$F(x,y) = 0, \frac{dy}{dx}?$$

By Chain Rule:

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{\partial F}{\partial x}\frac{dx}{dx} = \frac{\partial F}{\partial x} = -\frac{\partial F}{\partial y}\frac{dy}{dx}$$

$$\frac{\partial F}{\partial y} \neq 0$$

6

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

$$F(x,y,z) = 0, \frac{\partial z}{\partial x}?$$

By Chain Rule:

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} + \frac{\partial F}{\partial z}\frac{dz}{dx} = 0$$

$$\frac{dx}{dx} = 1$$

$$\frac{dy}{dx} = 0 \ (z = f(x,y) \Rightarrow \ x, y \ are \ independent \ variables)$$

$$\Rightarrow \frac{\partial F}{\partial x} = -\frac{\partial F}{\partial z}\frac{dz}{dx}$$

7

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

EXAMPLE 9 Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.

SOLUTION Let $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1$. Then, from Equations 7, we have

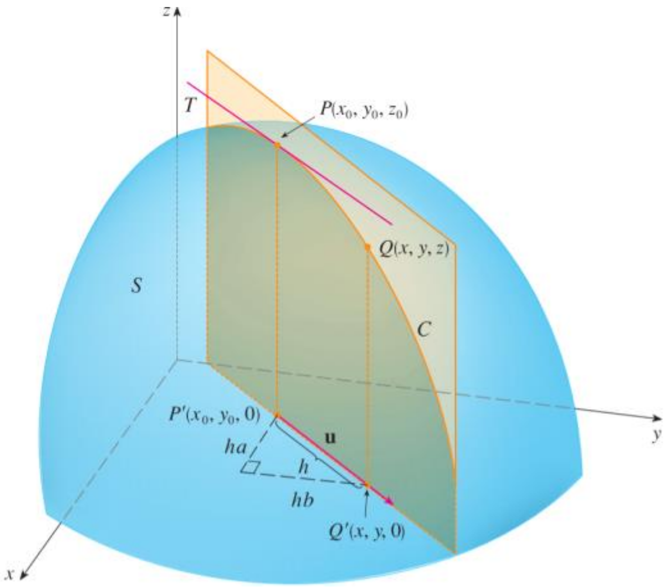
$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + 6yz}{3z^2 + 6xy} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3y^2 + 6xz}{3z^2 + 6xy} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$

14.6 Directional Derivatives and the Gradient Vector (P1018)

January 18, 2017 23:38

1. Directional Derivatives



2 Definition The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

3 Theorem If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\, a + f_y(x, y)\, b$$

If the unit vector $\mathbf{u}$ makes an angle θ with the positive x-axis (as in Figure 2), then we can write $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$ and the formula in Theorem 3 becomes

6 $$D_{\mathbf{u}} f(x, y) = f_x(x, y) \cos\theta + f_y(x, y) \sin\theta$$

EXAMPLE 2 Find the directional derivative $D_{\mathbf{u}} f(x, y)$ if

$$f(x, y) = x^3 - 3xy + 4y^2$$

and $\mathbf{u}$ is the unit vector given by angle $\theta = \pi/6$. What is $D_{\mathbf{u}} f(1, 2)$?

SOLUTION Formula 6 gives

$$D_{\mathbf{u}} f(x, y) = f_x(x, y) \cos\frac{\pi}{6} + f_y(x, y) \sin\frac{\pi}{6}$$

$$= (3x^2 - 3y)\frac{\sqrt{3}}{2} + (-3x + 8y)\tfrac{1}{2}$$

EXAMPLE 2 Find the directional derivative $D_{\mathbf{u}} f(x, y)$ if

$$f(x, y) = x^3 - 3xy + 4y^2$$

and $\mathbf{u}$ is the unit vector given by angle $\theta = \pi/6$. What is $D_{\mathbf{u}} f(1, 2)$?

SOLUTION Formula 6 gives

$$\begin{aligned} D_{\mathbf{u}} f(x, y) &= f_x(x, y) \cos \frac{\pi}{6} + f_y(x, y) \sin \frac{\pi}{6} \\ &= (3x^2 - 3y) \frac{\sqrt{3}}{2} + (-3x + 8y)\tfrac{1}{2} \\ &= \tfrac{1}{2}\left[3\sqrt{3}\,x^2 - 3x + \left(8 - 3\sqrt{3}\right)y\right] \end{aligned}$$

Therefore

$$D_{\mathbf{u}} f(1, 2) = \tfrac{1}{2}\left[3\sqrt{3}(1)^2 - 3(1) + \left(8 - 3\sqrt{3}\right)(2)\right] = \frac{13 - 3\sqrt{3}}{2}$$

2. The Gradient Vector

8 Definition If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

9

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

3. Maximizing the Directional Derivative

15 Theorem Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}} f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

PROOF From Equation 9 or 14 we have

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}| \cos \theta = |\nabla f| \cos \theta$$

where θ is the angle between ∇f and $\mathbf{u}$. The maximum value of $\cos \theta$ is 1 and this occurs when $\theta = 0$. Therefore the maximum value of $D_{\mathbf{u}} f$ is $|\nabla f|$ and it occurs when $\theta = 0$, that is, when $\mathbf{u}$ has the same direction as ∇f. ■

EXAMPLE 6

(a) If $f(x, y) = xe^y$, find the rate of change of f at the point $P(2, 0)$ in the direction from P to $Q(\frac{1}{2}, 2)$.

(b) In what direction does f have the maximum rate of change? What is this maximum rate of change?

SOLUTION

(a) We first compute the gradient vector:

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \langle e^y, xe^y \rangle$$

$$\nabla f(2, 0) = \langle 1, 2 \rangle$$

The unit vector in the direction of $\overrightarrow{PQ} = \langle -\frac{3}{2}, 2 \rangle$ is $\mathbf{u} = \langle -\frac{3}{5}, \frac{4}{5} \rangle$, so the rate of change of f in the direction from P to Q is

$$D_{\mathbf{u}} f(2, 0) = \nabla f(2, 0) \cdot \mathbf{u} = \langle 1, 2 \rangle \cdot \langle -\tfrac{3}{5}, \tfrac{4}{5} \rangle$$

$$= 1\left(-\tfrac{3}{5}\right) + 2\left(\tfrac{4}{5}\right) = 1$$

(b) According to Theorem 15, f increases fastest in the direction of the gradient vector $\nabla f(2, 0) = \langle 1, 2 \rangle$. The maximum rate of change is

$$|\nabla f(2, 0)| = |\langle 1, 2 \rangle| = \sqrt{5}$$ ■

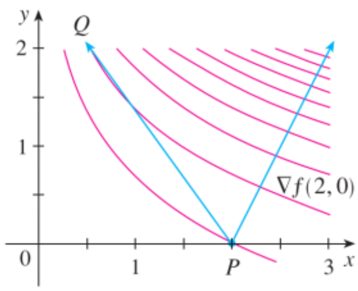


FIGURE 7

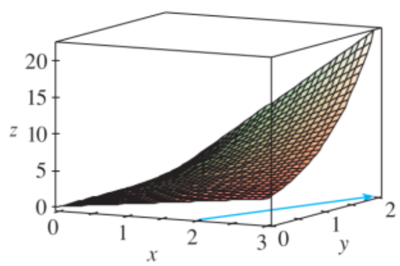
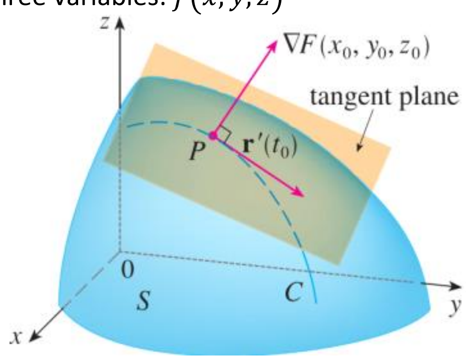


FIGURE 8

4. Geometric applications of Gradient
   a. Function of two variables: $f(x, y)$
      ∇f is the diretion of maximal growth, so it perpendicular to level curves.
      Thus the euqation of tangent line to a level curve at (x_0, y_0) is
      $$\nabla f \cdot < x - x_0, y - y_0 > = 0$$

   b. Function of three variables: $f(x, y, z)$



∇f is the diretion of maximal growth, so it normal to level surfaces.
Thus the euqation of tangent plane to a level surface at (x_0, y_0, z_0) is
$$\nabla f \cdot < x - x_0, y - y_0, z - z_0 > = 0$$

$$\boxed{19} \quad F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

The normal line

$$\boxed{20} \quad \frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

EXAMPLE 8 Find the equations of the tangent plane and normal line at the point $(-2, 1, -3)$ to the ellipsoid

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$$

SOLUTION The ellipsoid is the level surface (with $k = 3$) of the function

$$F(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$$

Therefore we have

$$F_x(x, y, z) = \frac{x}{2} \qquad F_y(x, y, z) = 2y \qquad F_z(x, y, z) = \frac{2z}{9}$$

$$F_x(-2, 1, -3) = -1 \qquad F_y(-2, 1, -3) = 2 \qquad F_z(-2, 1, -3) = -\tfrac{2}{3}$$

Then Equation 19 gives the equation of the tangent plane at $(-2, 1, -3)$ as

$$-1(x + 2) + 2(y - 1) - \tfrac{2}{3}(z + 3) = 0$$

which simplifies to $3x - 6y + 2z + 18 = 0$.

By Equation 20, symmetric equations of the normal line are

$$\frac{x + 2}{-1} = \frac{y - 1}{2} = \frac{z + 3}{-\frac{2}{3}}$$

Figure 10 shows the ellipsoid, tangent plane, and normal line in Example 8.

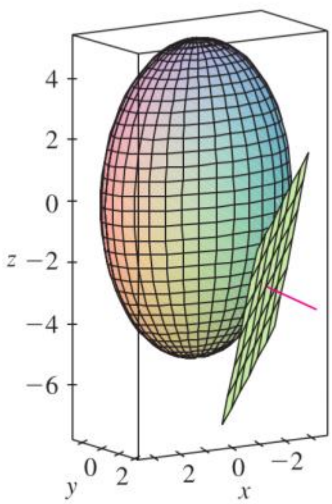
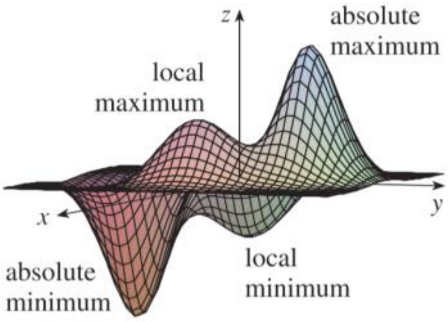


FIGURE 10

14.7 Maximum and Minimum Values (P1031)

January 18, 2017 23:38



1. Local Maximum and Minimum Values

1 Definition A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leqslant f(a, b)$ when (x, y) is near (a, b). [This means that $f(x, y) \leqslant f(a, b)$ for all points (x, y) in some disk with center (a, b).] The number $f(a, b)$ is called a **local maximum value**. If $f(x, y) \geqslant f(a, b)$ when (x, y) is near (a, b), then f has a **local minimum** at (a, b) and $f(a, b)$ is a **local minimum value**.

2 Theorem If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Notice that the conclusion of Theorem 2 can be stated in the notation of gradient vectors as $\nabla f(a, b) = \mathbf{0}$.

A point (a, b) is called a critical point (or stationary point) of f if
$f_x(a, b) = 0$ and
$f_y(a, b) = 0$, or if one of these partial derivatives does not exist.

EXAMPLE 1 Let $f(x, y) = x^2 + y^2 - 2x - 6y + 14$. Then

$$f_x(x, y) = 2x - 2 \qquad f_y(x, y) = 2y - 6$$

These partial derivatives are equal to 0 when $x = 1$ and $y = 3$, so the only critical point is $(1, 3)$. By completing the square, we find that

$$f(x, y) = 4 + (x - 1)^2 + (y - 3)^2$$

Since $(x - 1)^2 \geqslant 0$ and $(y - 3)^2 \geqslant 0$, we have $f(x, y) \geqslant 4$ for all values of x and y. Therefore $f(1, 3) = 4$ is a local minimum, and in fact it is the absolute minimum of f.

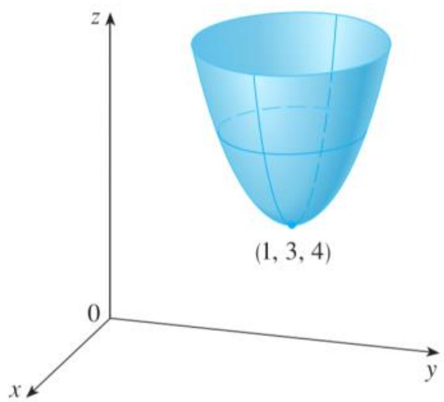


FIGURE 2
$z = x^2 + y^2 - 2x - 6y + 14$

3 Second Derivatives Test Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
(c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

NOTE 1 In case (c) the point (a, b) is called a **saddle point** of f and the graph of f crosses its tangent plane at (a, b).

NOTE 2 If $D = 0$, the test gives no information: f could have a local maximum or local minimum at (a, b), or (a, b) could be a saddle point of f.

NOTE 3 To remember the formula for D, it's helpful to write it as a determinant:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2$$

EXAMPLE 3 Find the local maximum and minimum values and saddle points of $f(x, y) = x^4 + y^4 - 4xy + 1$.

SOLUTION We first locate the critical points:

$$f_x = 4x^3 - 4y \qquad f_y = 4y^3 - 4x$$

Setting these partial derivatives equal to 0, we obtain the equations

$$x^3 - y = 0 \qquad \text{and} \qquad y^3 - x = 0$$

To solve these equations we substitute $y = x^3$ from the first equation into the second one. This gives

$$0 = x^9 - x = x(x^8 - 1) = x(x^4 - 1)(x^4 + 1) = x(x^2 - 1)(x^2 + 1)(x^4 + 1)$$

so there are three real roots: $x = 0, 1, -1$. The three critical points are $(0, 0)$, $(1, 1)$, and $(-1, -1)$.

Next we calculate the second partial derivatives and $D(x, y)$:

$$f_{xx} = 12x^2 \qquad f_{xy} = -4 \qquad f_{yy} = 12y^2$$

$$D(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = 144x^2y^2 - 16$$

Since $D(0, 0) = -16 < 0$, it follows from case (c) of the Second Derivatives Test that the origin is a saddle point; that is, f has no local maximum or minimum at $(0, 0)$. Since $D(1, 1) = 128 > 0$ and $f_{xx}(1, 1) = 12 > 0$, we see from case (a) of the test that $f(1, 1) = -1$ is a local minimum. Similarly, we have $D(-1, -1) = 128 > 0$ and $f_{xx}(-1, -1) = 12 > 0$, so $f(-1, -1) = -1$ is also a local minimum.

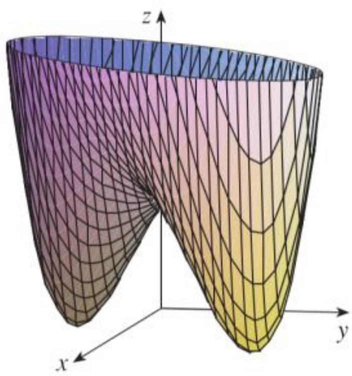


FIGURE 4
$z = x^4 + y^4 - 4xy + 1$

EXAMPLE 4 Find and classify the critical points of the function

$$f(x, y) = 10x^2y - 5x^2 - 4y^2 - x^4 - 2y^4$$

Also find the highest point on the graph of f.

SOLUTION The first-order partial derivatives are

$$f_x = 20xy - 10x - 4x^3 \qquad f_y = 10x^2 - 8y - 8y^3$$

So to find the critical points we need to solve the equations

$$\boxed{4} \qquad 2x(10y - 5 - 2x^2) = 0$$

$$\boxed{5} \qquad 5x^2 - 4y - 4y^3 = 0$$

From Equation 4 we see that either

$$x = 0 \qquad \text{or} \qquad 10y - 5 - 2x^2 = 0$$

In the first case ($x = 0$), Equation 5 becomes $-4y(1 + y^2) = 0$, so $y = 0$ and we have the critical point $(0, 0)$.

EXAMPLE 6 A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

SOLUTION Let the length, width, and height of the box (in meters) be x, y, and z, as shown in Figure 10. Then the volume of the box is

$$V = xyz$$

We can express V as a function of just two variables x and y by using the fact that the area of the four sides and the bottom of the box is

$$2xz + 2yz + xy = 12$$

Solving this equation for z, we get $z = (12 - xy)/[2(x + y)]$, so the expression for V becomes

$$V = xy\,\frac{12 - xy}{2(x + y)} = \frac{12xy - x^2y^2}{2(x + y)}$$

We compute the partial derivatives:

$$\frac{\partial V}{\partial x} = \frac{y^2(12 - 2xy - x^2)}{2(x + y)^2} \qquad \frac{\partial V}{\partial y} = \frac{x^2(12 - 2xy - y^2)}{2(x + y)^2}$$

If V is a maximum, then $\partial V/\partial x = \partial V/\partial y = 0$, but $x = 0$ or $y = 0$ gives $V = 0$, so we must solve the equations

$$12 - 2xy - x^2 = 0 \qquad 12 - 2xy - y^2 = 0$$

These imply that $x^2 = y^2$ and so $x = y$. (Note that x and y must both be positive in this problem.) If we put $x = y$ in either equation we get $12 - 3x^2 = 0$, which gives $x = 2$, $y = 2$, and $z = (12 - 2 \cdot 2)/[2(2 + 2)] = 1$.

We could use the Second Derivatives Test to show that this gives a local maximum of V, or we could simply argue from the physical nature of this problem that there must be an absolute maximum volume, which has to occur at a critical point of V, so it must occur when $x = 2$, $y = 2$, $z = 1$. Then $V = 2 \cdot 2 \cdot 1 = 4$, so the maximum volume of the box is 4 m^3. ■

2. Absolute Maximum and Minimum Values

8 Extreme Value Theorem for Functions of Two Variables If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

9 To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

EXAMPLE 7 Find the absolute maximum and minimum values of the function $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x, y) \mid 0 \leqslant x \leqslant 3, 0 \leqslant y \leqslant 2\}$.

SOLUTION Since f is a polynomial, it is continuous on the closed, bounded rectangle D, so Theorem 8 tells us there is both an absolute maximum and an absolute minimum. According to step 1 in (9), we first find the critical points. These occur when

$$f_x = 2x - 2y = 0 \qquad f_y = -2x + 2 = 0$$

so the only critical point is $(1, 1)$, and the value of f there is $f(1, 1) = 1$.

In step 2 we look at the values of f on the boundary of D, which consists of the four line segments L_1, L_2, L_3, L_4 shown in Figure 12. On L_1 we have $y = 0$ and

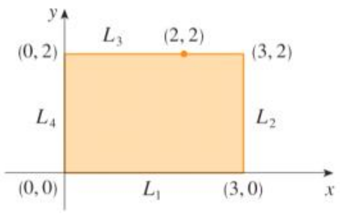


FIGURE 12

$$f(x, 0) = x^2 \qquad 0 \leqslant x \leqslant 3$$

This is an increasing function of x, so its minimum value is $f(0, 0) = 0$ and its maximum value is $f(3, 0) = 9$. On L_2 we have $x = 3$ and

$$f(3, y) = 9 - 4y \qquad 0 \leqslant y \leqslant 2$$

This is a decreasing function of y, so its maximum value is $f(3, 0) = 9$ and its minimum value is $f(3, 2) = 1$. On L_3 we have $y = 2$ and

$$f(x, 2) = x^2 - 4x + 4 \qquad 0 \leqslant x \leqslant 3$$

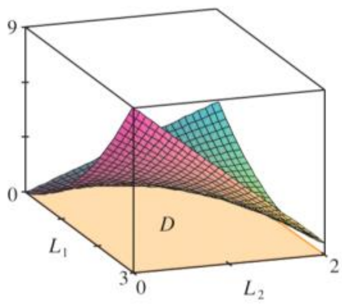


FIGURE 13
$f(x, y) = x^2 - 2xy + 2y$

By the methods of Chapter 3, or simply by observing that $f(x, 2) = (x - 2)^2$, we see that the minimum value of this function is $f(2, 2) = 0$ and the maximum value is $f(0, 2) = 4$. Finally, on L_4 we have $x = 0$ and

$$f(0, y) = 2y \qquad 0 \leqslant y \leqslant 2$$

with maximum value $f(0, 2) = 4$ and minimum value $f(0, 0) = 0$. Thus, on the boundary, the minimum value of f is 0 and the maximum is 9.

In step 3 we compare these values with the value $f(1, 1) = 1$ at the critical point and conclude that the absolute maximum value of f on D is $f(3, 0) = 9$ and the absolute minimum value is $f(0, 0) = f(2, 2) = 0$. Figure 13 shows the graph of f. ■

Find the absolute maximum and minimum values of $f(x,y) = xy^2$ on the $D = \{(x,y): x \geq 0, y \geq 0, x^2 + y^2 \leq 3\}$

1) Critical points

   $f_x = y^2 = 0, f_y = 2xy = 0$
   $\Rightarrow P_t = (t, 0)$

2) Sides points

   a) $y = 0$
      $\Rightarrow f(x,y) = 0$

   b) $x = 0$
      $\Rightarrow f(x,y) = 0$

   c) $x^2 + y^2 = 3$
      $\Rightarrow f(x) = x(3 - x^2) = 3x - x^3$
      $= f'(x) = 3 - 3x^2 \Rightarrow x = 1$
      $P_1 = (1, \sqrt{2})$

3) Vertices (extreme values) $V_1 = (0,0), V_2 = \left(0, \sqrt{3}\right), V_3 = \left(\sqrt{3}, 0\right)$

4) Comparing the values

   $f(P_t) = 0$
   $f(P_1) = 2$
   $f(V_1) = 0$
   $f(V_2) = 0$
   $f(V_3) = 0$

14.8 Lagrange Multipliers (P1043)

January 18, 2017 23:38

Method of Lagrange Multipliers To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$]:

(a) Find all values of x, y, z, and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and

$$g(x, y, z) = k$$

(b) Evaluate f at all the points (x, y, z) that result from step (a). The largest of these values is the maximum value of f; the smallest is the minimum value of f.

EXAMPLE 1 A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

SOLUTION As in Example 14.7.6, we let x, y, and z be the length, width, and height, respectively, of the box in meters. Then we wish to maximize

$$V = xyz$$

subject to the constraint

$$g(x, y, z) = 2xz + 2yz + xy = 12$$

Using the method of Lagrange multipliers, we look for values of x, y, z, and λ such that $\nabla V = \lambda \nabla g$ and $g(x, y, z) = 12$. This gives the equations

$$V_x = \lambda g_x$$

$$V_y = \lambda g_y$$

$$V_z = \lambda g_z$$

$$2xz + 2yz + xy = 12$$

which become

$$yz = \lambda(2z + y) \tag{2}$$

$$xz = \lambda(2z + x) \tag{3}$$

$$xy = \lambda(2x + 2y) \tag{4}$$

$$2xz + 2yz + xy = 12 \tag{5}$$

There are no general rules for solving systems of equations. Sometimes some ingenuity is required. In the present example you might notice that if we multiply (2) by x, (3) by y, and (4) by z, then the left sides of these equations will be identical. Doing this, we have

$$xyz = \lambda(2xz + xy) \tag{6}$$

$$xyz = \lambda(2yz + xy) \tag{7}$$

$$xyz = \lambda(2xz + 2yz) \tag{8}$$

We observe that $\lambda \neq 0$ because $\lambda = 0$ would imply $yz = xz = xy = 0$ from (2), (3), and (4) and this would contradict (5). Therefore, from (6) and (7), we have

$$2xz + xy = 2yz + xy$$

which gives $xz = yz$. But $z \neq 0$ (since $z = 0$ would give $V = 0$), so $x = y$. From (7) and (8) we have

$$2yz + xy = 2xz + 2yz$$

which gives $2xz = xy$ and so (since $x \neq 0$) $y = 2z$. If we now put $x = y = 2z$ in (5), we get

$$4z^2 + 4z^2 + 4z^2 = 12$$

Since x, y, and z are all positive, we therefore have $z = 1$ and so $x = 2$ and $y = 2$. This agrees with our answer in Section 14.7. ■

EXAMPLE 2 Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

SOLUTION We are asked for the extreme values of f subject to the constraint $g(x, y) = x^2 + y^2 = 1$. Using Lagrange multipliers, we solve the equations $\nabla f = \lambda \nabla g$ and $g(x, y) = 1$, which can be written as

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad g(x, y) = 1$$

or as

$$2x = 2x\lambda \tag{9}$$

$$4y = 2y\lambda \tag{10}$$

$$x^2 + y^2 = 1 \tag{11}$$

From (9) we have $x = 0$ or $\lambda = 1$. If $x = 0$, then (11) gives $y = \pm 1$. If $\lambda = 1$, then $y = 0$ from (10), so then (11) gives $x = \pm 1$. Therefore f has possible extreme values at the points $(0, 1)$, $(0, -1)$, $(1, 0)$, and $(-1, 0)$. Evaluating f at these four points, we find that

$$f(0, 1) = 2 \qquad f(0, -1) = 2 \qquad f(1, 0) = 1 \qquad f(-1, 0) = 1$$

Therefore the maximum value of f on the circle $x^2 + y^2 = 1$ is $f(0, \pm 1) = 2$ and the minimum value is $f(\pm 1, 0) = 1$. In geometric terms, these correspond to the highest and lowest points on the curve C in Figure 2, where C consists of those points on the paraboloid $z = x^2 + 2y^2$ that are directly above the constraint circle $x^2 + y^2 = 1$. ■

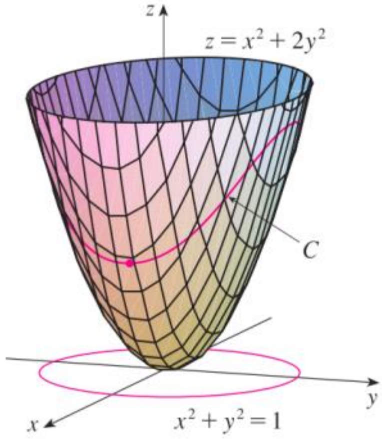


FIGURE 2

EXAMPLE 4 Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farthest from the point $(3, 1, -1)$.

SOLUTION The distance from a point (x, y, z) to the point $(3, 1, -1)$ is

$$d = \sqrt{(x-3)^2 + (y-1)^2 + (z+1)^2}$$

but the algebra is simpler if we instead maximize and minimize the square of the distance:

$$d^2 = f(x, y, z) = (x-3)^2 + (y-1)^2 + (z+1)^2$$

The constraint is that the point (x, y, z) lies on the sphere, that is,

$$g(x, y, z) = x^2 + y^2 + z^2 = 4$$

According to the method of Lagrange multipliers, we solve $\nabla f = \lambda \nabla g$, $g = 4$. This gives

$$\boxed{12} \qquad 2(x-3) = 2x\lambda$$

$$\boxed{13} \qquad 2(y-1) = 2y\lambda$$

$$\boxed{14} \qquad 2(z+1) = 2z\lambda$$

$$\boxed{15} \qquad x^2 + y^2 + z^2 = 4$$

The simplest way to solve these equations is to solve for x, y, and z in terms of λ from (12), (13), and (14), and then substitute these values into (15). From (12) we have

$$x - 3 = x\lambda \qquad \text{or} \qquad x(1-\lambda) = 3 \qquad \text{or} \qquad x = \frac{3}{1-\lambda}$$

[Note that $1 - \lambda \neq 0$ because $\lambda = 1$ is impossible from (12).] Similarly, (13) and (14) give

$$y = \frac{1}{1-\lambda} \qquad z = -\frac{1}{1-\lambda}$$

Therefore, from (15), we have

$$\frac{3^2}{(1-\lambda)^2} + \frac{1^2}{(1-\lambda)^2} + \frac{(-1)^2}{(1-\lambda)^2} = 4$$

which gives $(1-\lambda)^2 = \frac{11}{4}$, $1 - \lambda = \pm\sqrt{11}/2$, so

$$\lambda = 1 \pm \frac{\sqrt{11}}{2}$$

These values of λ then give the corresponding points (x, y, z):

$$\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right) \qquad \text{and} \qquad \left(-\frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$$

It's easy to see that f has a smaller value at the first of these points, so the closest point is $\left(6/\sqrt{11}, 2/\sqrt{11}, -2/\sqrt{11}\right)$ and the farthest is $\left(-6/\sqrt{11}, -2/\sqrt{11}, 2/\sqrt{11}\right)$. ■

Midterm-Review

March 1, 2017 19:10

1.

$$x(t) = \frac{3t}{1+t^3}, y(t) = \frac{3t^2}{1+t^3}, t \geq 0$$

Find vertical and horizontal tangents.

$$x'(t) = \frac{3(1-3t^3)}{(1+t^3)^2}$$

$$y'(t) = \frac{3t(2-t^3)}{(1+t^3)^2}$$

Horizontal:

$$\frac{dy}{dt} = 0 \Rightarrow 3t(2-t^3) = 0 \Rightarrow t = 0, \sqrt[3]{2}$$

Vertical:

$$\frac{dx}{dt} = 0 \Rightarrow 3(1-3t^3) = 0 \Rightarrow t = \sqrt[3]{1/3}$$

2. Show the line L_1, L_2 are skew.

$$\frac{x-1}{2} = \frac{y-3}{3} = \frac{z-1}{2}$$

$$\frac{x-2}{4} = \frac{y-1}{3} = \frac{z-2}{2}$$

Not parallel:

$$\boldsymbol{v}_1 =< 2,3,2 >$$

$$\boldsymbol{v}_2 =< 4,3,2 >$$

Not intersect:

$$< 1+2t, 3+3t, 1+2t >$$

$$< 2+4s, 1+3s, 2+2s >$$

$$1+2t = 2+4s$$

$$3+3t = 1+3s$$

$$1+2t = 2+2s$$

Common normal vector:

$$\boldsymbol{n} = \boldsymbol{v}_1 \times \boldsymbol{v}_2 =< 0,4,-6 >$$

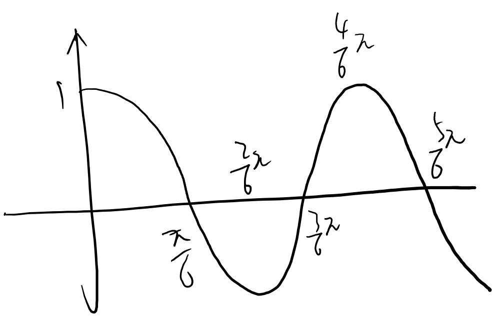
$$P_1 \in L_1: (1,3,1)$$

$$P_2 \in L_2: (2,1,2)$$

$$\boldsymbol{w} = P_1P_2 =< 1,-2,1 >$$

$$Comp_{\boldsymbol{n}}\boldsymbol{w} = \frac{\boldsymbol{w} \cdot \boldsymbol{n}}{|\boldsymbol{n}|}$$

$$d = |Comp_{\boldsymbol{n}}\boldsymbol{w}| = \frac{14}{\sqrt{52}}$$



3. Area

$$r = 3 + \cos 3\theta$$

$$A = \int_0^{2\pi} \frac{1}{2} r^2 d\theta = 6 \int_0^{\pi/3} \frac{1}{2} r^2 d\theta$$

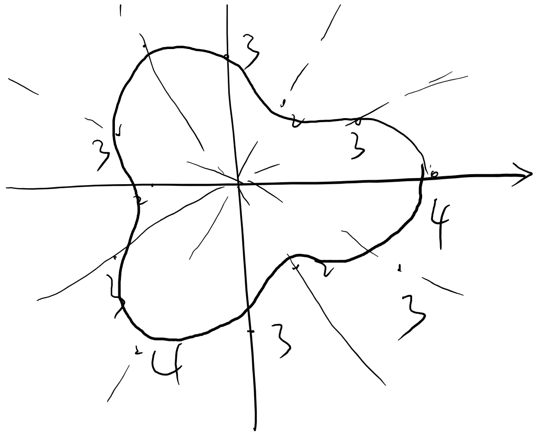
$$= 3 \int_0^{\pi/3} (9 + 2\cos 3\theta + \cos^2 3\theta) d\theta$$

$$= 3\left[9 \cdot \frac{\pi}{3} + 0\right] + 3 \int_0^{\pi/3} \left(\frac{1}{2} + \frac{1}{2}\cos 6\theta\right) d\theta$$

$$= 3\left[9 \cdot \frac{\pi}{3} + 0\right] + 3\int_0^{\pi/3} \left(\frac{1}{2} + \frac{1}{2}\cos 6\theta\right) d\theta$$

$$= 3\left[9 \cdot \frac{\pi}{3} + 0\right] + 3\left(\frac{1}{2} \cdot \frac{\pi}{3} + 0\right)$$

$$= 9\pi + \frac{\pi}{2}$$



Final Review

April 24, 2017 18:15

Series
Chain Rule
Local mini, max, second derivation test
Global mini, max
Lagrange multiplier
Cross product
Conic
Curvature
Tangent, normal, binormal
limit