5.5 The Substitution Rule

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Definitions & Theorems:

1. The Substitution Rule:

If $u = g(x)$ is a differentiable function whose rang is an interval I and f is continuous on I, then

$$\int f(\mathrm{g}(x))g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u$$

Proofs or Explanations:

1. Rule1:

$$\frac{\mathrm{d}}{\mathrm{d}x}F(\mathrm{g}(x)) = F'(g(x))g'(x)$$

Let $u = g(x)$

$$\int F'(g(x))g'(x)\,\mathrm{d}x = F(\mathrm{g}(x)) + C = F(u) + C = \int F'(u)\,\mathrm{d}u$$

$F' = f$

$$\int f(\mathrm{g}(x))g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u$$

Examples:

1. $\int \mathrm{e}^{4y}\,\mathrm{d}y$

Method1: $\int \mathrm{e}^{4y}\,\mathrm{d}y = \frac{1}{4}e^{4y} + C$

Method2: Let $u = 4y \to du = 4dy \to \int \mathrm{e}^{4y}\,\mathrm{d}y = \int \mathrm{e}^{\mathrm{u}}\frac{du}{4} = \frac{1}{4}\int \mathrm{e}^{\mathrm{u}}du = \frac{1}{4}e^{u} + C = \frac{1}{4}e^{4y} + C$

2. $\int 2x\sqrt{9 + x^2}\,\mathrm{d}x$

Let $u = 9 + x^2 \to du = 2xdx$

$$\int 2x\sqrt{9 + x^2}\,\mathrm{d}x = \int \sqrt{9 + x^2}\,2x\,\mathrm{d}x = \int \sqrt{u}\,\mathrm{d}u = \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}(9 + x^2)^{\frac{3}{2}} + C$$

3. $\int 3(8y - 1)\mathrm{e}^{4y^2 - \mathrm{y}}\,\mathrm{d}y$

Let $u = 4y^2 - y \to du = (8y - 1)dx$

$$\int 3(8y - 1)\mathrm{e}^{4y^2 - \mathrm{y}}\,\mathrm{d}y = 3\int \mathrm{e}^{\mathrm{u}}du = 3e^{u} + C = 3e^{4y^2 - y} + C$$

4. $\int x^2(3 - 10x^3)^4\,\mathrm{d}x$

Let $u = 3 - 10x^3 \to du = -30x^2dx$

$$\int x^2(3 - 10x^3)^4\,\mathrm{d}x = \int u^4\frac{\mathrm{d}u}{-30} = \frac{u^5}{5}\left(-\frac{1}{30}\right) + C = -\frac{1}{150}(3 - 10x^3)^5 + C$$

5. $\int \left(1 - \frac{1}{\theta}\right)\cos(\theta - \ln\theta)\,\mathrm{d}\theta$

Let $u = \theta - \ln\theta \to du = \left(1 - \frac{1}{\theta}\right)d\theta$

$$\int\left(1-\frac{1}{\theta}\right)\cos(\theta-\ln\theta)\,\mathrm{d}\theta=\int\cos u\,\mathrm{d}u=\sin u+C=\sin(\theta-\ln\theta)+C$$

6. $\displaystyle\int\frac{\mathrm{d}x}{x^2+9}$

Let $u=\frac{x}{3}\rightarrow du=\frac{1}{3}dx$

$$\int\frac{\mathrm{d}x}{x^2+9}=\int\frac{\mathrm{d}x}{9(\frac{x^2}{9}+1)}=\frac{1}{9}\int\frac{\mathrm{d}x}{\left(\frac{x}{3}\right)^2+1}=\frac{1}{9}\int\frac{3du}{u^2+1}=\frac{1}{9}(3\tan^{-1}u)+C=\frac{1}{3}\tan^{-1}\frac{x}{3}+C$$

7. $\displaystyle\int\tan\theta\,\mathrm{d}\theta$

$$\int\tan\theta\,\mathrm{d}\theta=\int\frac{\sin\theta}{\cos\theta}\mathrm{d}\theta$$

$\begin{cases}u=\sin\theta\\u=\cos\theta\end{cases}$

but $u=\sin\theta$ does not work. (why?)

Let $u=\cos\theta\rightarrow du=-\sin\theta d\theta$

$$\int\tan\theta\,\mathrm{d}\theta=\int\frac{\sin\theta}{\cos\theta}\mathrm{d}\theta=\int\frac{-\,\mathrm{d}u}{u}=-\ln|u|+C=-\ln|\cos\theta|+C$$

8. $\displaystyle\int\sec\theta\,\mathrm{d}\theta$

$$\int\sec\theta\,\mathrm{d}\theta=\int\frac{1}{\cos\theta}\mathrm{d}\theta=\int\frac{1}{\cos\theta}\left(\frac{\sec\theta\tan\theta+\sec^2\theta}{\sec\theta\tan\theta+\sec^2\theta}\right)\mathrm{d}\theta=\int\frac{\sec\theta\tan\theta+\sec^2\theta}{\tan\theta+\sec\theta}\mathrm{d}\theta$$

Let $u=\tan\theta+\sec\theta\rightarrow du=(\sec^2\theta+\sec\theta\tan\theta)d\theta$

$$\int\sec\theta\,\mathrm{d}\theta=\int\frac{\sec\theta\tan\theta+\sec^2\theta}{\tan\theta+\sec\theta}d\theta=\int\frac{du}{u}=\ln|\tan\theta+\sec\theta|+C$$

9. $\displaystyle\int\csc\theta\,\mathrm{d}\theta$

$$\int\csc\theta\,\mathrm{d}\theta=\int\frac{1}{\sin\theta}d\theta=\int\frac{1}{\sin\theta}\left(\frac{\csc\theta\cot\theta+\csc^2\theta}{\csc\theta\cot\theta+\csc^2\theta}\right)\mathrm{d}\theta=\int\frac{\csc\theta\cot\theta+\csc^2\theta}{\cot\theta+\csc\theta}d\theta$$

Let $u=\cot\theta+\csc\theta\rightarrow du=\left(-\csc^2\theta-\csc\theta\cot\theta\right)d\theta$

$$\int\csc\theta\,\mathrm{d}\theta=\int\frac{\csc\theta\cot\theta+\csc^2\theta}{\cot\theta+\csc\theta}\mathrm{d}\theta=\int\frac{-\mathrm{d}u}{u}=-\ln|u|+C=-\ln|\cot\theta+\csc\theta|+C$$

10. $\displaystyle\int\cot\theta\,\mathrm{d}\theta$

$$\int\cot\theta\,\mathrm{d}\theta=\int\frac{\cos\theta}{\sin\theta}d\theta$$

Let $u=\sin\theta\rightarrow du=\cos\theta\,d\theta$

$$\int\cot\theta\,\mathrm{d}\theta=\int\frac{\cos\theta}{\sin\theta}\mathrm{d}\theta=\int\frac{du}{u}=\ln|u|+C=\ln|\sin\theta|+C$$

! 11. $\displaystyle\int\frac{7x+1}{x^2+4}\mathrm{d}x$

$$\int \frac{7x+1}{x^2+4}\mathrm{d}x = \int \frac{7x}{x^2+4}\mathrm{d}x + \int \frac{1}{x^2+4}\mathrm{d}x$$

Let $t = x^2+4$, $s = \frac{x}{2}$

$$\int \frac{7x+1}{x^2+4}\mathrm{d}x = \int \frac{7x}{x^2+4}\mathrm{d}x + \int \frac{1}{x^2+4}\mathrm{d}x = \frac{7}{2}\ln\left(x^2+4\right) + \frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

12. $\displaystyle\int \frac{\cos\sqrt{x}}{\sqrt{x}}\mathrm{d}x$

Let $u = \sqrt{x} \to du = \frac{dx}{2\sqrt{x}}$

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}\mathrm{d}x = \int \cos u\, 2du = 2\sin u + C = 2\sin\sqrt{x} + C$$

13. $\displaystyle\int \mathrm{e}^{t+\mathrm{e}^t}\,\mathrm{d}t$

$$\int \mathrm{e}^{t+\mathrm{e}^t}\,\mathrm{d}t = \int \mathrm{e}^t\mathrm{e}^{\mathrm{e}^t}\,\mathrm{d}t$$

Let $u = e^t \to du = e^t dt$

$$\int \mathrm{e}^{t+\mathrm{e}^t}\,\mathrm{d}t = \int e^u\,\mathrm{d}u = e^u + C = e^{e^t} + C$$

14. $\displaystyle\int x\sqrt{1-x}\,\mathrm{d}x$

Let $u = 1-x \to x = 1-u, du = -dx$

$$\int x\sqrt{1-x}\,\mathrm{d}x = \int (1-u)\sqrt{\mathrm{u}}(-\,\mathrm{d}u) = \int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)du = \frac{u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{(1-x)^{\frac{5}{2}}}{\frac{5}{2}} - \frac{(1-x)^{\frac{3}{2}}}{\frac{3}{2}} + C$$

15. $\displaystyle\int 2x^3\sqrt{x^2+1}\,\mathrm{d}x$

Let $u = x^2+1 \to x^2 = u-1, du = 2xdx$

$$\int 2x^3\sqrt{x^2+1}\,\mathrm{d}x = \int (\mathrm{u}-1)\sqrt{u}\,\mathrm{d}u$$

16. $\displaystyle\int_0^4 2x\sqrt{9+x^2}\,\mathrm{d}x$

Let $u = 9+x^2 \to du = 2xdx$

$$\int 2x\sqrt{9+x^2}\,\mathrm{d}x = \int \sqrt{9+x^2}2x\,\mathrm{d}x = \int \sqrt{u}\,\mathrm{d}u = \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}(9+x^2)^{\frac{3}{2}} + C$$

Method1: $\int_0^4 2x\sqrt{9+x^2}\,\mathrm{d}x = \frac{2}{3}(9+x^2)^{\frac{3}{2}}\Big]_0^4$

Method2: $\int_0^4 2x\sqrt{9+x^2}\,\mathrm{d}x = \int_9^{25}\sqrt{u}\,\mathrm{d}u = \frac{2}{3}u^{\frac{3}{2}}\Big]_9^{25} = \frac{196}{3}$

17. $\displaystyle\int_0^{\pi} \sin(t)\sin^9(\cos(t))\,dt$

Let $u = \cos t \to du = -\sin t\,dt$

$$\int_0^{\pi} \sin(t)\sin^9(\cos(t))\,dt = \int_1^{-1}\sin^9 u\,(-du) = \int_{-1}^{1}\sin^9 u\,du = 0$$

18. $\displaystyle\int_{e^3}^{e^6} \frac{(\ln t)^4}{t}\mathrm{d}t$

Let $u = \ln t \rightarrow du = \frac{1}{t}dt$

$$\int_{e^3}^{e^6} \frac{(\ln t)^4}{t}\,\mathrm{d}t = \int_3^6 u^4 du = \frac{u^5}{5}\bigg]_3^6 = \frac{1}{5}(6^5 - 3^5)$$

19. Let f be a continuous even function, show that $F(x) = \int_0^x f(t)\,\mathrm{d}t$ is odd.

Let $s = -t$, $ds = -dt$

$$F(-x) = \int_0^{-x} f(t)\,\mathrm{d}t = \int_0^x f(-s)\,(-\,\mathrm{d}s) = \int_0^x f(s)\,(-\,\mathrm{d}s) = -\int_0^x f(s)\,(\mathrm{d}s) = -F(x)$$