Outline

September 6, 2017 10:48

Department of Mathematics and Statistics
Concordia University

MATH 251
Linear Algebra I
Fall 2017

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Office hours: Mondays: 16:15-17:45, Fridays: 13:30-15:00.

Text: *Linear Algebra*, 4th Edition, by S. Friedberg, A. Insel, L. Spence, (Prentice Hall).

Assignments: You will be required to hand in weekly assignments. They reflect the content of the course. No late assignments will be accepted. Solutions will be posted at the Digital Store (LB-115).
Assignments for each week must be handed in at the beginning of the next week second class.

Class Test: There will be one class test in the seventh week of classes, covering the first five weeks of the course. <u>There will be no make-up test</u>.

Final Grade: The final examination will be three hours long. It will cover material from the entire course.

Grading: Your final grade is the maximum of the final examination grade counted as 100%, and a grade computed by adding 60% of your mark on the final examination to your class test 30%, and your assignments 10%.

Calculators: Only calculators approved by the Department are permitted in the class test(s) and final examination. The calculators are the **Sharp EL 531** and the **Casio FX 300MS**, available at the Concordia Bookstore.

Departmental website → http://www.mathstat.concordia.ca

Week	Section	Topic	Problems
1	1.2, 1.3	Vector Spaces, Subspaces	1.2: 19, 20 1.3: 10, 12, 17
2	1.4, 1.5	Linear Combinations, Systems of Equations Linear Dependence and Independence	1.4: 5(d,f,h) , 6, 12 1.5: 2(b,d,f), 8a, 10
3	1.6	Basis and Dimension	1.6: 3(b,d), 8, 14, 16, 30
4	2.1	Linear Transformations, Null Spaces, Ranges	2.1: 3, 6, 9b, 11, 14
5	2.2	Matrix Representation of Linear Transformation	2.2: 2(b,e), 4, 5(a,d,f), 10
6	2.3	Composition of Linear Transformations, Matrix Multiplication	2.3: 3(a,b), 9, 11, 12c, 13, 15
7		**CLASS TEST**	
8	2.4 2.5	Invertibility and Isomorphisms Change of Coordinate Matrix	2.4: 6, 9, 15, 16, 17 2.5: 2(b,d), 3f, 6(b,d)
9	3.1, 3.2, 3.3	Elementary Matrices, Rank of Matrices, Matrix Inverses, Systems of Equations	3.2: 2f, 4b, 5h, 6(d,f), 20a 3.3: 2d, 3d
10	3.4	Systems of Equations	3.4: 2j, 6*, 8, 10, 12 (*In question 6: Determine A if the first, third and FIFTH columns…)
11	4.4 5.1	Summary about Determinants, Eigenvalues and Eigenvectors	4.4: 3h, 4h 5.1: 2d, 3(b,d), 4(c,d,g), 15(a,b)
12	5.2	Diagonalizability	5.2: 2(b,d,f), 3(b,f), 7, 8, 9
13		**REVIEW**	

1.2 Vector Spaces

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Definitions & Theorems:

1. **Definition field F:** A filed F is a set on which two operations (called addition and scalar multiplication, respectively) are defined so that for each pair of elements a, b in F there are unique elements $a + b$ and ab in F, such that the following conditions hold for all elements a, b, c in F.

$$F \times F \to F$$
$$(a, b) \to a + b \in F$$
$$(a, b) \to a \cdot b \in F$$

   1) $a + b = b + a$ and $a \cdot b = b \cdot a$ (commutativity of addition and multiplication)
   2) $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (associativity of addition and multiplication)
   3) There exists distinct elements 0 and 1 such that $0 + a = a$ and $1 \cdot a = a$ (existence of identify elements for addition and multiplication).
   4) For each element a in F and each nonzero element b in F, there exists elements c and d in F such that $a + c = 0$ and $b \cdot d = 1$ (existence of inverses for addition and multiplication).
   5) $a \cdot (b + c) = a \cdot b + a \cdot c$ (distribution of multiplication over addition).

2. **Definition Vector Space V:** A vector space (or linear space) V over a filed F consists of a set on which two operations (called addition and scalar multiplication, respectively) are defined so that for each pair of elements x, y in V there is a unique element $x + y$ in V and for each element a in F and each element x in V there is a unique element ax in V, such that the following conditions hold.

$$V \times V \to V$$
$$(x, y) \to x + y \in V$$
$$F \times V \to V$$
$$(c, x) \to cx \in V$$

   1) For all x, y in V, $x + y = y + x$ (commutativity of addition)
   2) For all x, y, z in V, $(x + y) + z = x + (y + z)$ (associativity of addition)
   3) There exists an element in V denoted by 0_V such that $x + 0_V = x$ for each x in V (existence of identify element).
   4) For each element x in V there exists an element y in V such that $x + y = 0_V$ (additive inverse).
   5) For each element x in V, $1x = x$.
   6) For each pair of elements a, b in F and each element x in V, $(ab)x = a(bx)$.
   7) For each element a in F and each pair of elements x, y in V, $a(x + y) = ax + ay$.
   8) For each pair of elements a, b in F and each element x in V, $(a + b)x = ax + bx$.

   The elements of the filed F are called scalars and the elements of the vector space V are called vectors.

3. Properties of vector space:
   1) If $x, y, z \in V$ such that $x + z = y + z$, then $x = y$ (Cancellation Law for Vector Addition).
   2) The vector 0_V is unique.
   3) The additive inverse is unique.
   4) $0 \cdot x = 0_V$
   5) $c \cdot 0_V = 0_V$
   6) $(-c) \cdot x = c \cdot (-x) = -cx$

4. n-tuple: An object of the form $(a_1, a_2, \dots, a_n)$, where the entries $a_1, a_2, \dots, a_n$ are elements of a field F, is called an $n - tuple$ with elements from F. The elements $a_1, a_2, \dots, a_n$ are caled the entries or componets of the $n - tuple$.
   Two n-tuple $(a_1, a_2, \dots, b_n)$ and $(b_1, b_2, \dots, b_n)$ with entries from a field F are called equal if $a_i = b_i$ for $i = 1,2, \dots, n$.
5. F^n: The set of all n-tuples with entries from a filed F is denoted by F^n.
6. Diagonal Entries: An $m \times n$ matrix with entries from a field F where each entry a_{ij} $(1 \le i \le m, 1 \le j \le n)$ is an element of F. The entries a_{ij} with $i = j$ are the diagonal entries of the matrix.

Proofs or Explanations:

1.

Examples:

1. F^n is a vector space over F with the operations of coordinatewise addition and scalar multiplication; that is, if $u = (a_1, a_2, \dots, a_n) \in F^n$, $v = (b_1, b_2, \dots, b_n) \in F^n$, and $c \in F$, then

$$u + v = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$$
$$cu = (ca_1, ca_2, \dots, ca_n)$$

   $\Rightarrow C^2$ is a vector space over C, R^3 is a vector space over R.

2. The set of all $m \times n$ matrics with entries from a filed F is a vector space, denoted by $M_{m \times n}(F)$, with the following operations of matrix addition and scalar multiplication: For $A, B \in M_{m \times n}(F)$ and $c \in F$,

$$(A + B)_{ij} = A_{ij} + B_{ij}$$
$$(cA)_{ij} = cA_{ij}$$

   for $1 \le i \le m$ and $1 \le j \le n$.

   For instance,

$$\begin{pmatrix} 2 & 0 & -1 \\ 1 & -3 & 4 \end{pmatrix} + \begin{pmatrix} -5 & -2 & 6 \\ 3 & 4 & -1 \end{pmatrix} = \begin{pmatrix} -3 & -2 & 5 \\ 4 & 1 & 3 \end{pmatrix}$$

and

$$-3\begin{pmatrix} 2 & 0 & -1 \\ 1 & -3 & 4 \end{pmatrix} = \begin{pmatrix} -6 & 0 & 3 \\ -3 & 9 & -12 \end{pmatrix}$$

3. Let $P(F)$ be a set of all polynomials with coefficients in a field F; that is,

   $$P(F) = \{f = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n, n\ is\ positive\ integer.\}$$

   Define:

   $f, g \in P(F) \Rightarrow f + g = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots (a_n + b_n)x^n \in P(F), \qquad a_i, b_i \in F$

   $c \in F \Rightarrow c \cdot f = ca_0 + ca_1 x + ca_2 x^2 + \cdots + ca_n x^n \in P(F)$

   $\Rightarrow P(F)$ is a vector space over F.

   Zero polynomial:

   $P(F) = 0$, that is, $a_n = a_{n-1} = \cdots = a_0 = 0$

   For convenience, its degree is defined to be -1, that is, $n = -1$.

   Recall: constant polynomial: $P(F) = C$ has a degree of 0.

4. Let S be a set and F be a field. Denote $\mathcal{F}(S, F)$ as the set of all functions from S to F.

   $\mathcal{F}(S, F) = \{f: S \to F\}$

   $x \in S \to f(x) \in F$

   $f, g \in \mathcal{F}(S, F) \Rightarrow f + g: s \to F \Rightarrow (f + g)(x) = f(x) + g(x)$

   $c \in F, f \in \mathcal{F} \Rightarrow c \cdot f: S \to F \Rightarrow (c \cdot f)(x) = c \cdot f(x)$

   Zero vector:

   $f(x) = 0$ constant function for any $x \in S$

   $\Rightarrow \mathcal{F}(S, F)$ is a vector space over F.

1.3 Subspaces

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Definitions & Theorems:

1. **Definition Subspace**: A subset W of a vector space V over a filed F is called a *subspace* of V if W is a vector space over F with the operations of addition and scalar multiplication defined on V.
   In any vector space V, note that V and $\{0\}$ are subspaces, and they are called trivial subspaces.
2. **Theorem 1.3**: Let V be a vector space and W a subset of V. Then W is a subspace of V if and only if the following three conditions hold for the operations defined in V.
   1) $0 \in W$.
   2) $x \in W$ and $y \in W \Rightarrow x + y \in W$. ($W$ is closed under addition.)
   3) $c \in F$ and $x \in W \Rightarrow cx \in W$. ($W$ is closed under scalar multiplication.)
3. Transpose A^t of matrix A
   The transpose A^t of an $m \times n$ matrix A is the $n \times m$ matrix obtained from A by interchanging the rows with the columns; that is, $(A^t)_{ij} = A_{ji}$.
   For example,
   $$\begin{pmatrix} 1 & -2 & 3 \\ 0 & 5 & -1 \end{pmatrix}^t = \begin{pmatrix} 1 & 0 \\ -2 & 5 \\ 3 & -1 \end{pmatrix}$$
4. Symmetric matrix
   A symmetric matrix is a matrix A such that $A^t = A$.
   For example,
   $$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}^t = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$$
   The set W of all symmetric matrices in $M_{n\times n}(F)$ is a subspace of $M_{n\times n}(F)$ by Theorem 1.3.
5. Diagonal matrix
   An $M_{n\times n}$ is called a diagonal matrix if $M_{ij} = 0$ whenever $i \neq j$, that is, if all its nondiagonal entries are zero.
   The set of diagonal matrices is a subspace of $M_{n\times n}(F)$ by Theorem 1.3.
6. **Theorem 1.4**: Any intersection S of subspaces $(W_1, W_2, \dots, W_n)$ of a vector space V is a subspace of V.
   1) $0_V \in S$
   2) $x, y \in S \Rightarrow x + y \in S$
   3) $c \in F, x \in S \Rightarrow cx \in S$

   But in general, the union of subspaces is not a subspace.
   1) $0_V \in \cup W$
   2) ?
      $W_1 = \{(x, 0), x \in R\} \subseteq R^2, W_2 = \{(0, y), y \in R\}$
      $W_1 \cap W_2 = \{(0,0)\} \subseteq R^2$
      $W_1 \cup W_2: (1,0) + (0,1) = (1,1) \notin W_1 \cup W_2$
      Note: $W_1 \cup W_2 \Rightarrow W_1 \subseteq W_2 \ or \ W_2 \subseteq W_1$
      $W_1 \subseteq W_2 \ or \ W_2 \subseteq W_1 \Rightarrow W_1 \cup W_2$
   3) $c \in F, x \in \cup W \Rightarrow cx \in \cup W$
7. **Definition**: If S_1 and S_2 are nonempty subsets of a vector space V, then the sum of S_1 and S_2:
   $S_1 + S_2 = \{x + y, x \in S_1, y \in S_2\}$.
8. **Definition** $\oplus$: A vector space V is called the the direct sum $\oplus$ of W_1 and W_2; that is, $V = W_1 \oplus W_2$; if W_1 and W_2 are subspace of V such that
   1) $V = W_1 + W_2$
   2) $W_1 \cap W_2 = \{0_V\}$
9. Proposition: Let W_1, W_2 be subspaces of V over F.
   1) $W_1 + W_2$ is a subapce of V that contains both W_1 and W_2.
      i. $0_V \in W_1 + W_2 \ (0_V = 0_V + 0_V)$
      ii. $x, y \in W_1 + W_2 \Rightarrow x + y \in W_2 + W_2$
         $x = x_1 + x_2, x_1 \in W_1, x_2 \in W_2$
         $y = y_1 + y_2, y_1 \in W_1, y_2 \in W_2$
         $\Rightarrow x + y = (x_1 + y_1) + (x_2 + y_2)$
      iii. $c \in F, x \in W_1 + W_2 \Rightarrow cx \in W_1 + W_2$
         $x = x_1 + x_2, x_1 \in W_1, x_2 \in W_2$
         $cx = cx_1 + cx_2$
   2) If W is a sbuspace of V that contains both W_1 and W_2, then W contains $W_1 + W_2$.
      Note: $A \subseteq B$: for any $x \in A$, we need to show $x \in B$.
      Show $W_1 \subseteq W_1 + W_2$
      $x \in W_1 \Rightarrow x = x + 0_V, x \in W_1, 0_V \in W_2$

Show $W_2 \subseteq W_1 + W_2$

$x \in W_2 \Rightarrow x = 0_V + x, 0_V \in W_1, x \in W_2$

i. $x = x_1 + x_2, x_1 \in W_1, x_2 \in W_2 \Rightarrow x_1 \in W, x_2 \in W$

W is a subspace $\Rightarrow$close on addition $\Rightarrow x \in W$

ii. $x = cy$

10. Trace of a matrix

The trace of an $n \times n$ matrix M, denoted $tr(M)$, is the sum of the diagonal entries of M; that is,

$tr(M) = M_{11} + M_{22} + \cdots + M_{nn}$

Proofs or Explanations:

1. Theorem 1.3
   1) If W is a subspace of V

      W is a vector space over F $\Rightarrow$ 2,3 are true.

      $x + 0_V = x, x \in V \Rightarrow x + 0_V = x, x \in W$

      W is a vector space $\Rightarrow$ there exsits 0_W

      $\Rightarrow 0_V = 0_W \Rightarrow 1$ is true.
   2) If 1,2,3 are true

      $-x \in W$

      $-x = (-1) \cdot x \in W$ ($cx \in W$ whenever $c \in F$ and $x \in W$.)

Examples:

1. $W = \{x = (x_1, x_2, x_3, x_4) \in R^4 : x_1 - 2x_2 + 3x_3 = 0\} \subseteq R^4$

   W is subsapce of R^4/R?
   1) $0_{R^4} = (0,0,0,0): 0 - 2 \cdot 0 + 3 \cdot 0 = 0 \Rightarrow 0_{R^4} \in W$
   2) $x, y \in W \Rightarrow x + y \in W$?

      $x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4)$

      $(x_1 + y_1) - 2(x_2 + y_2) + 3(x_3 + y_3) = 0$?

      $= (x_1 - 2x_2 + 3x_3) + (y_1 - 2y_2 + y_3) = 0$

      $\Rightarrow x + y \in W$

      Because $x \in W, y \in W$
   3) If $c \in R, x \in W \Rightarrow cx \in W$?

      $cx = (cx_1, cx_2, cx_3, cx_4)$

      $(cx_1) - 2(cx_2) + (cx_3) = 0$?

      $c(x_1 - 2x_2 + 3x_3) = 0$

      $\Rightarrow cx \in W$
2. $W = \{(x_1, x_2, x_3) : x_1 - 2x_2 + 4x_3 = 1\} \subseteq R^3$

   $0_{R^3} \notin W \Rightarrow$ not subspace
3. $W = \{A \in M_{n\times n}(R) : tr(A) = 0\}$

   $tr(A) = A_{11} + A_{22} + A_{33} + \cdots A_{nn}$

   W is a subspace?
   1) $0_{n,n} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$

      $tr(0_{n,n}) = 0$
   2) $A, B \in W \Rightarrow A + B \in W$?

      $A + B = \begin{pmatrix} A_{11} + B_{11} & \cdots & A_{1n} + B_{1n} \\ A_{21} + B_{21} & \ddots & \vdots \\ \cdots & \cdots & A_{nn} + B_{nn} \end{pmatrix}$

      $tr(A + B) = (A_{11} + B_{11}) + (A_{22} + B_{22}) + \cdots + (A_{nn} + B_{nn}) = (A_{11} + \cdots + A_{nn}) + (B_{11} + \cdots + B_{nn})$

      $= tr(A) + tr(B) = 0$
   3) $c \in R, A \in W \Rightarrow cA \in W$?

      $cA = \begin{pmatrix} cA_{11} & \cdots & cA_{1n} \\ \vdots & \ddots & \vdots \\ cA_{n1} & \cdots & cA_{nn} \end{pmatrix}$

      $tr(cA) = cA_{11} + cA_{22} + \cdots + cA_{nn} = c(A_{11} + A_{22} + \cdots + A_{nn}) = c \cdot tr(A) = 0$
4. Let S be a set and F be a field. Denote $C(R,R) = \{f: R \to R, f\ continous\}, D(R,R) = \{f: R \to R, f\ differentiable\}$
   1) $f(x) = 0, x \in R \to continous$
   2) $f, g \in C(R,R) \Rightarrow f + g \in C(R,R)$
   3) $c \in R, f \in C(R,R) \Rightarrow cf \in C(R,R)$
5. In $M_{m\times n}(F)$ define $W_1 = \{A \in M_{m\times n}(F): A_{ij} = 0\ whenever\ i > j\}$ and $W_2 = \{A \in M_{m\times n}(F): A_{ij}\ whenever\ i \leq j\}$. Show that

$M_{m\times n}(F) = W_1 \oplus W_2$

1.4 Linear Combinations and Systems of Linear Equations

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Definitions & Theorems:

1. **Definition linear combination**: Let V be a vector space over F and S a nonempty **subset** of V. A vector $v \in V$ is called a **linear combination** of vectros of S if there exist
   a) a finite number of vectors $u_1, u_2, \dots, u_n$ in S and
   b) scalars $a_1, a_2, \dots, a_n$ in F such that
      $v = a_1u_1 + a_2u_2 + \cdots + a_nu_n$

   In this case we also say that v is a linear combination of $u_1, u_2, \dots, u_n$ and call $a_1, a_2, \dots, a_n$ the coefficients of the linear combination. In any vector space V, $\mathbf{0} = 0v$ for each $v \in V$. Thus the zero vector is a linear combination of any nonempty subset of V.
2. **Definition span**: Let S be a nonempty **subset** of a vector space V. The **span** of S, denoted $span(S)$, is the set consisting of all linear combinations of the vectors in S. For convenience, we define $span(\emptyset) = \{0\}$.
3. **Theorem 1.5**: The span of any subset S of a vector space V is a subspace of V. Moreover, any subspace of V that contains S must also contain the span of S.
4. Proposition: $span(S)$ is the smallest subspace containing S.
5. Let V be a vector space over F.
   a. $S_1 \subseteq S_2 \Rightarrow span(S_1) \subseteq span(S_2)$
      $S_1 \subseteq S_2, S_2 \subseteq span(S_2) \Rightarrow S_1 \subseteq span(S_2) \Rightarrow span(S_1) \subseteq span(S_2)$(smallest subspce)
   b. $span(S_1 \cap S_2) \subseteq span(S_1) \cap span(S_2)$
      $S_1 \cap S_2 \subseteq S_1 \Rightarrow span(S_1 \cap S_2) \subseteq span(S_1)$
      $S_1 \cap S_2 \subseteq S_2 \Rightarrow span(S_1 \cap S_2) \subseteq span(S_2)$
      $\Rightarrow span(S_1 \cap S_2) \subseteq span(S_1) \cap span(S_2)$
   c. $span(S_1 \cup S_2) = span(S_1) + span(S_2)$
      i. $span(S_1 \cup S_2) \subseteq span(S_1) + span(S_2)$
         $x \in span(S_1 \cup S_2)$
         $x = a_1u_1 + a_2u_2 + \cdots + a_mu_m + b_1v_1 + b_2v_2 + \cdots + b_mv_n$
         $u_1, u_2, \dots, u_n \in S_1, v_1, v_2, \dots, v_n \in S_2, v_1, v_2, \dots, v_n \notin S_1$
         $\Rightarrow a_1u_1 + a_2u_2 + \cdots + a_mu_m \in span(S_1), b_1v_1 + b_2v_2 + \cdots + b_mv_n \in span(S_2)$
      ii. $span(S_1) + span(S_2) \subseteq span(S_1 \cup S_2)$
         Let $W_1 = span(S_1), W_2 = span(S_2) \Rightarrow W_1, W_2$ are subspaces$\Rightarrow$ $W_1 + W_2$ is a subspace
         Because $W_1 + W_2$ is the smallest subspace containing W_1 and W_2, we need to prove $W_1 \subseteq span(S_1 \cup S_2)$ and $W_2 \subseteq span(S_1 \cup S_2)$, so $W_1 + W_2 \subseteq span(S_1 \cup S_2)$
6. S is a subspace of $V \Leftrightarrow S = span(S)$.
7. **Definition**: A subset S of a vector space V generates (or spans) V if $span(S) = V$. In this case, we also say that the vectors of S generate (or span) V.

Proofs or Explanations:

1. Theorem 1.5
   a. If $S = \emptyset \Rightarrow span(S) = \{0\}$
   b. If $S \neq \emptyset \Rightarrow S$ constains a vector z
      i. $0z = 0_S$
      ii. Let $x, y \in span(S), u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n \in S$
         $x = a_1u_1 + a_2u_2 + \cdots + a_mu_m$
         $y = b_1v_1 + b_2v_2 + \cdots + b_mv_n$
         $x + y = a_1u_1 + a_2u_2 + \cdots + a_mu_m + b_1v_1 + b_2v_2 + \cdots + b_mv_n \in span(S)$
      iii. $c \in F \Rightarrow cx = (ca_1)u_1 + (ca_2)u_2 + \cdots + (ca_m)u_m \in span(S)$
2. 4Proposition
   We need to show: if W is a subspace of V, S is a subset of $W \Rightarrow span(S) \subseteq W$
   Let $x = a_1u_1 + a_2u_2 + \cdots + a_mu_m \in span(S), u_1, u_2, \dots, u_n \in S$
   $\Rightarrow u_1, u_2, \dots, u_n \in S \in W \Rightarrow a_1u_1 \in W, a_2u_2 \in W, \dots, a_mu_m \in W$ (close under scalar mutiplication)
   $\Rightarrow a_1u_1 + a_2u_2 + \cdots + a_mu_m \in W$ (close under addition)
   $\Rightarrow x \in W$
3. 7Definition, explanation
   a. $V = R^n$
      $S_i = (0,0, \dots, i, \dots, 0), i = 1$
      $span(S_i) = R^n$

b. $V = M_{m\times n}(R)$
$S = \{E_{ij} = 1\}$?

c. $V = P_n(R) = \{f - polynomial, \deg(f) \leq n\}$
$S = \{1, x, x^2, \dots, x^n\}: span(S) = P_n(R)$

d. $V = P(R) = \{f - polynomial\}$
$S = \{1, x, x^2, \dots, x^n, \dots\}: span(S) = P(R)$

Examples:

1. $f_1 = x^3 - 3x + 5, f_2 = x^3 + 2x^2 - x + 1, f_3 = x^3 + 3x^2 - 1$
$f_1 = a \cdot f_2 + b \cdot f_3$
$x^3 - 3x + 5 = a(x^3 + 2x^2 - x + 1) + b(x^3 + 3x^2 - 1)$
$x^3 - 3x + 5 = (a + b)x^3 + (2a + 3b)x^2 - ax + (a - b)$
$\Rightarrow a + b = 1, 2a + 3b = 0, -a = -3, a - b = 5 \Rightarrow a = 3, b = -2$
2. 5(g)
$S = \left\{\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\right\}$
$\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix} =? \, span(S)$
Show $\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + c\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ has at least one solution.

1.5 Linear Dependence and Linear Independence

September 6, 2017 11:05

Definitions & Theorems:

1. **Definition linearly dependent**: A **subset** S of a vector space V is called linearly dependent if there exist a finite number of distinct vectors $u_1, u_2, \dots, u_n$ in S and scalars $a_1, a_2, \dots, a_n$, not all zero, such that $a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0$. In this case we also say that the vectors of S are linearly dependent.
   For any vectors $u_1, u_2, \dots, u_n$, we have $a_1u_2 + a_2u_2 + \cdots a_nu_n = 0$ if $a_1 = a_2 = \cdots = a_n = 0$. We call this the trivial representation of 0 as a linear combination of $u_1, u_2, \dots, u_n$.
   Thus, for a set to be linearly dependent, there must **exist a nontrivial** representation of 0 as a linear combination of vectors in the set.
   Consequently, any subset of a vector space that contains the zero vector is linearly dependent, because $0 = 1 \cdot 0$ is a nontrivial representation of 0 as a linear combination of vectors in the set.
2. **Definition**: A subset S of a vector space that is not linearly dependent is called linearly independent.
   a. $\emptyset$ is linearly independent, for linearly dependent sets must be nonempty.
   b. $S = \{u\}, u \neq \emptyset \Rightarrow S$ is linearly independent. For if $\{u\}$ is linearly dependent, then $au = 0$ for some nonzero scalar a. Thus $u = a^{-1}(au) = 1 \cdot 0 = 0$.
   c. $0_V \in S \Rightarrow S$ is linearly dependent.
   d. $S = \{u, v\} \Leftrightarrow u$ or v is a multiplication of the other.
   e. A set is linearly independent $\Leftrightarrow$ representations of 0 as linearly combinations of its vectors are trivial representations.
3. **Theorem 1.6**: Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_1 is linearly **dependent**, then S_2 is linearly **dependent**.
4. **Corollary**: Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_2 is linearly **independent**, then S_1 is linearly **independent**.
5. **Theorem 1.7**: Let S be a linearly independent subset of a vector space V, and let v be a vector in V that is not in S, that is $v \in V - S$. Then
   a. $S \cup \{v\}$ is linearly dependent $\Leftrightarrow v \in span(S)$.
      Exists $a_1u_1 + a_2u_2 + \cdots a_nu_n + bv = 0_V$
      $a_1, a_2, \dots, a_n$ not all zero
      $a_1u_1 + a_2u_2 + \cdots a_nu_n \neq 0_V$
      $\Rightarrow b \neq 0$
   b. $S \cup \{v\}$ is linearly independent $\Leftrightarrow v \notin span(S)$

Proofs or Explanations:

1.

Examples:

1. 2c: $\{x^3 + 2x^2, -x^2 + 3x + 1, x^3 - x^2 + 2x - 1\}$ in $P_3(R)$
   $a(x^3 + 2x^2) + b(-x^2 + 3x + 1) + c(x^3 - x^2 + 2x - 1) = 0_V$
   $\Rightarrow a + c = 0, 2a - b - c = 0, 3b + 2c = 0, b - c = 0$
2. $\left\{\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -2 & -3 \\ 7 & 1 \end{pmatrix}\right\}$
   $\begin{pmatrix} a - 2c & -b - 3c \\ -2a + b + 7c & a + b + c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
   $a - 2c = 0, -b - 3c = 0$
   $-2a + b + 7c = 0, a + b + c = 0$

1.6 Bases and Dimension

September 6, 2017 11:05

Definitions & Theorems:

1. **Definition Basis**: A basis β for a vector space V is a linearly independent subset of V that generates V. If β is a basis for V, we also say that the vectors of β form a basis for V.
   β is a basis $\Rightarrow$ β is linearly independent and $span(\beta) = V$.
2. **Theorem 1.8**: Let V be a vector space and $\beta = \{u_1, u_2, \dots, u_n\}$ be a subset of V. Then
   β is a basis for $V \Leftrightarrow$ each $v \in V$ can be **uniquely** expressed as a linear combination of vectors of β.
   That is, can be express in the form $v = a_1u_1 + a_2u_2 + \cdots + a_nu_n$ for unique scalars $a_1, a_2, \dots, a_n$.
3. **Theorem 1.9**: If a vector space V is generated by a finite set S, then some subsets of S is a basis for V. Hence V has a finite basis.
4. **Theorem 1.10** (Replacement Theorem): Let V be a vector space that is generated by a set G containing exactly n vectors, and let L be a linearly independent subset of V containing exactly m vectors. Then $m \le n$ and there exsits a subset H of G containing exactly $n - m$ vectors such that $L \cup H$ generates V.
5. **Corollary 1**: Let V be a vector space having a finite basis. Then every basis for V contains the same number of vectors.
6. **Definition Dimension**: A vector space is called finite-dimensional if it has a basis consisting of a finite number of vectors. The unique number of vectors in each basis for V is called the dimension of V and is denoted by $\dim(V)$. A vector space that is not finite-dimensional is called infinite-dimensional.
7. **Corollary 2**: Let V be a vector space with dimension n.
   a. Any finite **generating** set for V contains **at least** n vectors, and a generating set for V that contains exactly n vectors is a basis for V; that is, it is also linearly independent.
   b. Any **linearly independent** subset of V contains **at most** n vectors; and that contains exactly n vectors is a basis for V.
   c. Every linearly independent subset of V can be extended to a basis for V.
      S is a basis $\Leftrightarrow$ S has n vectors: S is linearly independent, or generating set.
   d. Each generating set for V contains $\ge n$ vectors and can be reduced to a basis for V.
      a) If $S = \emptyset \Rightarrow V = span(\emptyset) = \{0_V\}, basis\ is\ \emptyset$
      b) If $S = \{0_V\} \Rightarrow V = span(\{0_V\}) = \{0_V\}, basis\ is\ \emptyset$
      c) If $S \ne \emptyset, S \ne \{0_V\} \Rightarrow$ there exists at least one non-zero vector in S, say $u_1 \ne 0_V, u_1 \in S$
8. **Theorem 1.11**: Let W be a **subspace** of a finite-dimensional vector space V. Then W is finite-dimensional and $\dim(W) \le \dim(\mathrm{V})$. Moreover, if $\dim(W) = \dim(V)$, then $V = W$.
9. Remark: In general, if W and V are **vector space** that $\dim(W) = \dim(V)$, then W is not necessarily equal to V.
   $V = P_3(R)$: $\dim(V) = 4$
   $W = M_{2\times2}(R)$: $\dim(W) = 4$
10. Corollary: If W is a **subspace** of a finite-dimensional vector space V, then any basis for W can be extended to a basis for V.
    a. W_1, W_2 are subspaces of vector space $V \Rightarrow W_1 \cap W_2$ is subspace of V.
       $\dim(W_1 \cap W_2) \le \dim(\mathrm{W}_1)$
       $\dim(W_1 \cap W_2) \le \dim(\mathrm{W}_2)$
       $\{0_V\}$: $basis\ is\ \emptyset, \dim(0_V) = 0$
    b. W_1, W_2 are subspaces of vector space $V \Rightarrow W_1 + W_2$ is subspace of V.
       $\dim(W_1 + W_2) = \dim(\mathrm{W}_1) + \dim(\mathrm{W}_2) - \dim(W_1 \cap W_2)$
       $\dim(W_1) = m, \dim(W_2) = n.$
       Find a basis for $W_1 + W_2$
          Method 1:
             $\beta = \{u_1, u_2, \dots, u_k\}$ is basis for $W_1 \cap W_2$
             Then
                extends β to a basis for W_1: $\{u_1, u_2, \dots, u_k, v_1, v_2, \dots, v_{m-k}\}$
                extends β to a basis for W_2: $\{u_1, u_2, \dots, u_k, v_1, v_2, \dots, v_{n-k}\}$
                $\Rightarrow$basis for $W_1 + W_2$: $\{u_1, u_2, \dots, u_k, v_1, v_2, \dots, v_{m-k}, v_1, v_2, \dots, v_{n-k}\}$
                $\Rightarrow \dim(W_1 + W_2) = k + (m - k) + (n - k) = m + n - k = \dim(\mathrm{W}_1) + \dim(\mathrm{W}_2) - \dim(W_1 \cap W_2)$
          Method 2: $span(S_1 \cup S_2) = span(S_1) + span(S_2)$
             S_1 is a basis for $W_1 \Rightarrow span(S_1) = W_1$
             S_2 is a basis for $W_2 \Rightarrow span(S_2) = W_2$
             $\Rightarrow span(S_1 \cup S_2) = W_1 + W_2$
             $\Rightarrow S_1 \cup S_2$ generates $W_1 + W_2$
             $\Rightarrow S_1 \cup S_2$ is reduced to a bsis for $W_1 + W_2$
11. The Lagrange Polynomials
    Let $c_0, c_1, \dots, c_n$ be distinct scalars in an infinite field F. The polynomials $f_0(x), f_1(x), \dots, f_n(x)$ defined by
    $$f_i(x) = \frac{(x - c_0)\dots(x - c_{i-1})(x - c_{i+1})\dots(x - c_n)}{(c_i - c_0)\dots(c_i - c_{i-1})(c_i - c_{i+1})\dots(c_i - c_n)} = \prod_{\substack{k=0 \\ k\ne i}}^{n} \frac{x - c_k}{c_i - c_k}$$
    Note: $f_i(x)$ is a polynomial of degree n and hence is in $P_n(F)$.
    $$f_i(c_j) = \begin{cases} 0, & if\ i \ne j \\ 1, & if\ i = j \end{cases}$$
    $\beta = \{f_0, f_1, f_2, \dots, f_n\}$ is a linealry independent subset of $P_n(F)$.

Let $x = c_0 \Rightarrow a_0 = 0$
Let $x = c_1 \Rightarrow a_1 = 0$
Let $x = c_n \Rightarrow a_n = 0$

12. The Lagrange Interpolations Formula

    $\beta = \{f_0, f_1, f_2, \dots, f_n\}$ is a basis for $P_n(F)$.

    Every polynomial function g in $P_n(F)$ is a linear combination of polynomial functions of β, say,

    $$g = \sum_{i=0}^{n} b_i f_i$$

    It follows that

    $$g(c_j) = \sum_{i=0}^{n} b_i f_i(c_j) = b_j$$

    So

    $$g = \sum_{i=0}^{n} g(c_i) f_i$$

    is the unique representation of g as a linear combination of elements of β.

    This representations is called the Lagrange interpolation formula.

13. An important consequence of the Lagrange interpolations formula:

    If $f \in P_n(F)$ and $f(c_i) = 0$ for $n + 1$ distinct scalars $c_0, c_1, \dots, c_n$ in F, then f is the zero function.

    $g(x) \in P_n, g(c_0) = g(c_1) = \cdots = g(c_n) = 0 \Rightarrow g(x) = 0$

Proofs or Explanations:

1. Theorem 1.8

    Let β be a basis for V.

    If $v \in V$, then $v \in span(\beta)$ because $span(\beta) = V$. Thus v is a linear combination of the vectors of β.

    Let $v = a_1 u_1 + a_2 u_2 + \cdots + a_n u_n$ and $v = b_1 u_1 + b_2 u_2 + \cdots + b_n u_n$

    $\Rightarrow v = (a_1 - b_1)u_1 + (a_2 - b_2)u_2 + \cdots + (a_n - b_n)u_n = 0_V$

    β is linearly independent $\Rightarrow a_1 - b_1 = a_2 - b_2 = \cdots = a_n - b_n = 0$

    $\Rightarrow a_1 = b_1, a_2 = b_2, \dots, a_n = b_n$

Examples:

1. Example 1: $span(\emptyset) = \{0_V\} \Rightarrow \emptyset$ is a basis for the zero vector space.
2. Example 2: In R^n, let $e_1 = (1,0,0,\dots,0), e_2 = (0,1,0,\dots,0), \dots, e_n = (0,0,0,\dots,1)$; $\{e_1, e_2, \dots, e_n\}$ is a basis for R^n and is called the standard basis for R^n. $\dim(R^n) = n$
3. Example 3: In $M_{m\times n}(F)$, let E^{ij} denote the matrix whose only nonzero entry is a 1 in the ith row and jth column. Then $\{E^{ij}: 1 \le i \le m, 1 \le j \le n\}$ is a basis for $M_{m\times n}(F)$. $\dim(M_{m\times n}) = m \times n$
4. Example 4: In $P_n(F)$ the set $\{1, x, x^2, \dots, x^n\}$ is a basis for $P_n(F)$, and it is the standard basis for $P_n(F)$. $\dim(P_n) = n + 1$
5. Example 5: In $P(F)$ the set $\{1, x, x^2, \dots\}$ is a basis.

    No basis for $P(F)$ can be finite.
6. P54(7) Generating set reduce to basis

    $u_1 \ne 0_{R^3} \Rightarrow u_1$ is linearly independent

    $\{u_1, u_2\}$ is linearly independent. (not a multiple of the other)

    $\{u_1, u_2, u_3\}$ is linearly dependent, dicard u_3. ($u_3 = -4u_1$)

    $\{u_1, u_2, u_4\}$ is linearly dependent, dicard u_4. ($3cu_1 - 7cu_2 + cu_4 = 0_{R^3}$)

    $\{u_1, u_2, u_5\}$ is linearly independent.

    $\{u_1, u_2, u_5\}$ is maximal linearly independent subset of $G \Rightarrow \{u_1, u_2, u_5\}$ is a basis.
7. Page20(8+9): W_1, W_2, W_3 are subspaces. Find dimensions for $W_1 \cap W_3, W_1 \cap W_4, W_3 \cap W_4$

    $W_1 = \{(a_1, a_2, a_3): a_1 = 3a_2, a_3 = -a_2\}$

    $\Rightarrow W_1 = \{(3t, t, -t), t \in \mathbb{R}\} \Rightarrow W_1 = \{t(3,1,-1), t \in \mathbb{R}\}$

    $\Rightarrow (3,1,-1)$ generates W_1, it is also linearly independent

    $\Rightarrow (3,1,-1)$ is a basis, $\dim(W_1) = 1$

    $W_3 = \{(a_1, a_2, a_3): 2a_1 - 7a_2 + a_3 = 0\}$

    $\Rightarrow W_3 = \{(a_1, a_2, -2a_1 + 7a_2): a_1, a_2 \in \mathbb{R}\}$

    $(a_1, a_2, -2a_1 + 7a_2) = (a_1, 0, -2a_1) + (0, a_2, 7a_2) = a_1(1,0,-2) + a_2(0,1,7)$

    $\Rightarrow S = \{(1,0,-2), (0,1,7)\}$ generates W_3

    Linearly independent $\Rightarrow S$ is a basis, $\dim(W_3) = 2$

    $W_4 = \{(a_1, a_2, a_3): a_1 - 4a_2 - a_3 = 0\}$

    $\Rightarrow W_4 = \{(4a_2 + a_3, a_2, a_3): a_2, a_3 \in \mathbb{R}\}$

    $W_1 \cap W_3$:

    Let $a \in W_1 \cap W_3 \Rightarrow a \in W_1, a \in W_3$

    $\Rightarrow$

    $a_1 = 3a_2, a_3 = -2a_2, 2a_1 - 7a_2 + a_3 = 0$

$\Rightarrow a_1 = a_2 = a_3$

$\Rightarrow a = (0,0,0) = 0_{R^3}$

$\Rightarrow W_1 \cap W_3 = \{0_{R^3}\}, \dim(W_1 \cap W_3) = 0$

8. Prove W_1, W_2 are subspace, find β and $\dim()$ for $W_1, W_2, W_1 \cap W_2, W_1 + W_2$

$$W_1 = \left\{ \begin{pmatrix} a & b & b \\ c & d & d \\ e & 0 & e \end{pmatrix}, a,b,c,d,e \in \mathbb{R} \right\}, W_2 = \left\{ \begin{pmatrix} f & 0 & g \\ f & h & l \\ 0 & 0 & q \end{pmatrix}, f,g,h,l,q \in \mathbb{R} \right\}$$

1) W_1 is subspace
   a) Let $a = b = c = d = e = 0 \Rightarrow 0_V \in W_1$
   b) Let $A_1, A_2 \in W_1 \Rightarrow$

$$A_1 + A_2 = \begin{pmatrix} a_1 & b_1 & b_1 \\ c_1 & d_1 & d_1 \\ e_1 & 0 & e_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 & b_2 \\ c_2 & d_2 & d_2 \\ e_2 & 0 & e_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 & d_1 + d_2 \\ e_1 + e_2 & 0 & e_1 + e_2 \end{pmatrix} \in W_1$$

   c) Let $c \in F \Rightarrow$

$$cA_1 = c \begin{pmatrix} a_1 & b_1 & b_1 \\ c_1 & d_1 & d_1 \\ e_1 & 0 & e_1 \end{pmatrix} = \begin{pmatrix} ca_1 & cb_1 & cb_1 \\ cc_1 & cd_1 & cd_1 \\ ce_1 & 0 & ce_1 \end{pmatrix} \in W_1$$

   $\Rightarrow W_1$ is subspace

2) W_2 is subspace
   a) Let $g = g = h = l = q = 0 \Rightarrow 0_V \in W_2$
   b) Let $B_1, B_2 \in W_2 \Rightarrow$

$$B_1 + B_2 = \begin{pmatrix} f_1 & 0 & g_1 \\ f_1 & h_1 & l_1 \\ 0 & 0 & q_1 \end{pmatrix} + \begin{pmatrix} f_2 & 0 & g_2 \\ f_2 & h_2 & l_2 \\ 0 & 0 & q_2 \end{pmatrix} = \begin{pmatrix} f_1 + f_2 & 0 & g_1 + g_2 \\ f_1 + f_2 & h_1 + h_2 & l_1 + l_2 \\ 0 & 0 & q_1 + q_2 \end{pmatrix} \in W_2$$

   c) Let $c \in F \Rightarrow$

$$cB_1 = c \begin{pmatrix} f_1 & 0 & g_1 \\ f_1 & h_1 & l_1 \\ 0 & 0 & q_1 \end{pmatrix} = \begin{pmatrix} cf_1 & 0 & cg_1 \\ cf_1 & ch_1 & cl_1 \\ 0 & 0 & cq_1 \end{pmatrix} \in W_2$$

   $\Rightarrow W_2$ is subspace

3) W_1
   a)

$$\begin{pmatrix} a & b & b \\ c & d & d \\ e & 0 & e \end{pmatrix} = a\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} + e\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= aA_1 + bA_2 + cA_3 + dA_4 + eA_5$$

   $\Rightarrow \{A_1, A_2, A_3, A_4, A_5\}$ is a generating set.
   b) Check $\{A_1, A_2, A_3, A_4, A_5\}$ linearly independence.
      Let $aA_1 + bA_2 + cA_3 + dA_4 + eA_5 = 0_V \Rightarrow a = b = c = d = e = 0$
   $\Rightarrow \{A_1, A_2, A_3, A_4, A_5\}$ is linearly independent.

$\Rightarrow \{A_1, A_2, A_3, A_4, A_5\}$ is a basis for W_1, $\dim(W_1) = 5$

4) W_2
   a)

$$\begin{pmatrix} f & 0 & g \\ f & h & l \\ 0 & 0 & q \end{pmatrix} = f\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + g\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + h\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + l\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + q\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= fB_1 + gB_2 + hB_3 + lB_4 + qB_5$$

   $\Rightarrow \{B_1, B_2, B_3, B_4, B_5\}$ is a generating set.
   b) Check $\{B_1, B_2, B_3, B_4, B_5\}$ linearly independence.
      Let $aB_1 + bB_2 + cB_3 + dB_4 + eB_5 = 0_V \Rightarrow a = b = c = d = e = 0$
   $\Rightarrow \{B_1, B_2, B_3, B_4, B_5\}$ is linearly independent.

$\Rightarrow \{B_1, B_2, B_3, B_4, B_5\}$ is a basis for W_2, $\dim(W_2) = 5$

5) $W_1 \cap W_2$

Let $M = \begin{pmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix} \in (W_1 \cap W_2) \Rightarrow M \in W_1, M \in W_2 \Rightarrow$

$$M \in W_1 \Rightarrow \begin{cases} y = z \\ y_1 = z_1 \\ x_2 = z_2 \\ y_2 = 0 \end{cases}$$

$$M \in W_2 \Rightarrow \begin{cases} x = x_1 \\ y = 0 \\ x_2 = 0 \\ y_2 = 0 \end{cases}$$

$$\Rightarrow M = \begin{pmatrix} x & 0 & 0 \\ x & y_1 & y_1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow W_1 \cap W_2 = \left\{ \begin{pmatrix} x & 0 & 0 \\ x & y & y \\ 0 & 0 & 0 \end{pmatrix}, x, y \in \mathbb{R} \right\}$$

a) $\begin{pmatrix} x & 0 & 0 \\ x & y & y \\ 0 & 0 & 0 \end{pmatrix} = x \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + y \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = xM_1 + yM_2$

$\Rightarrow \{M_1, M_2\}$ is a generating set.

b) Check $\{M_1, M_2\}$ linearly independence.

Let $aM_1 + bM_2 = 0_V \Rightarrow a = b = 0$

$\Rightarrow \{M_1, M_2\}$ is linearly independent.

$\Rightarrow \{M_1, M_2\}$ is a basis for $W_1 \cap W_2$, $\dim(W_1 \cap W_2) = 2$

6) $W_1 + W_2$

$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) = 8$

If $\dim(W_1 + W_2) = 9 = \dim(M_{3\times3})$, W_1 and W_2 are subspaces of $M_{3\times3}$

$\Rightarrow (W_1 + W_2) = M_{3\times3}$, so we can choose any basis from $M_{3\times3}$ as a basis for $W_1 + W_2$

Find a basis for $(W_1 + W_2)$:

Method 1:

Extend $\beta_{W_1\cap W_2}$ to $\beta_{W_1+W_2}$

(1) $\beta_{W_1\cap W_2}$ to β_{W_1}

$$\beta_{W_1\cap W_2} = \{M_1, M_2\} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

For W_1:

$$\{A_1, A_2, A_3, A_4, A_5\} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \right\}$$

$A_4 = M_2, A_1 + A_3 = M_1$

$\Rightarrow$ remove A_4, remove A_1 or A_3

$$\Rightarrow \beta_{W_1} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \right\}$$

(2) $\beta_{W_1\cap W_2}$ to β_{W_2}

$$\beta_{W_1\cap W_2} = \{M_1, M_2\} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

For W_2:

$$\{B_1, B_2, B_3, B_4, B_5\} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

$B_1 = M_1, B_3 + B_4 = M_2$

$\Rightarrow$ remove B_1, remove B_3 or B_4

$$\Rightarrow \beta_{W_2} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

$\beta_{W_1+W_2} = \beta_{W_1} + \beta_{W_1}$

$$= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

Method 2:

Reduce a generating set to a basis

$W_1 + W_2 = span\left(\beta_{W_1}\right) + span\left(\beta_{W_2}\right) = span\left(\beta_{W_1} + \beta_{W_2}\right)$

$\Rightarrow \beta_{W_1} + \beta_{W_2}$ generates $W_1 + W_2$

$\beta_{W_1} + \beta_{W_2} = \{A_1, A_2, A_3, A_4, A_5, B_1, B_2, B_3, B_4, B_5\}$

$A_1 + A_3 = B_1, B_3 + B_4 = A_4$

$\Rightarrow$ remove A_1 or A_3, remove B_3 or B_4

$$\Rightarrow \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} 0, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

9. P56,Q10

Construct the polynomial of smallest degree whose graph contains the following points: $(-2,-6), (-1,5), (1,3) \in graph(g)$.

$g(-2) = -6$

$g(-1) = 5$

$g(1) = 3$

$$g(x) = g(c_0)f_0(x) + g(c_1)f_1(x) + g(c_2)f_2(x) = -6 \cdot f_0(x) + 5 \cdot f_1(x) + 3 \cdot f_2(x)$$

$$f_0(x) = \frac{(x-c_1)(x-c_2)}{(c_0-c_1)(c_0-c_2)} = \frac{(x+1)(x-1)}{(-2+1)(-2-1)} = \frac{x^2-1}{3}$$

$$f_1(x) = \frac{(x-c_0)(x-c_2)}{(c_1-c_0)(c_1-c_2)} = \frac{(x+1)(x-1)}{(-2+1)(-2-1)} = \frac{(x+2)(x-1)}{-2}$$

$$f_2(x) = \frac{(x-c_0)(x-c_1)}{(c_2-c_0)(c_2-c_1)} = \frac{(x+1)(x-1)}{(-2+1)(-2-1)} = \frac{(x+2)(x+1)}{6}$$

2.1 Linear Transformations, Null Spaces, and Ranges

September 6, 2017 11:05

Definitions & Theorems:

1. Definition: Let V and W be vector spaces over F. We call a function $T: V \to W$ a linear transformatoin from V to W (or simply call T linear) if, for all $x, y \in W$ and $c \in F$, we have
   1) $T(x+y) = T(x) + T(y)$
   2) $T(cx) = cT(x)$
2. Properties
   1) If T is linear, then $T(\mathbf{0}) = \mathbf{0}$.
   2) T is linear $\Leftrightarrow T(cx+y) = cT(x) + T(y)$ for all $x, y \in V$ and $c \in F$.
   3) If T is linear, then $T(x-y) = T(x) - T(y)$ for all $x, y \in V$.
   4) T is linear $\Leftrightarrow$ for $x_1, x_2, \dots, x_n \in V$ and $a_1, a_2, \dots, a_n \in F$, we have
      $$T(a_1x_1 + a_2x_2 + \cdots a_nx_n) = a_1T(x_1) + a_2T(x_2) + \cdots + a_nT(x_n)$$
3. Zero transformation:
   $T_0: V \to W, T_0(x) = 0_W, for\ all\ x \in V$
   T_0 is linear, T_0 is homogeneous.
   $N(T_0) = V \Rightarrow T_0$ is not one-to-one.
   $R(T_0) = \{0_W\} \Rightarrow T_0$ is not onto.
4. Identity transformation:
   $I_V: V \to V, I_V(x) = x, for\ all\ x \in V$
   I_V is linear.
   $N(I_V) = \{0_V\} \Rightarrow$ one-to-one.
   $R(I_V) = V \Rightarrow$ onto.
5. Definition: Let V and W be vector spaces, and let $T: V \to W$ be linear. We define the ***null space*** (or kernel) $N(T)$ of T to be the set of all vectors x in V such that $T(x) = 0$; that is, $N(T) = \{x \in V: T(x) = 0\}$.
6. Review: $f: V \to W$
   1) One-to-one (Injective): $\forall a, b \in V, f(a) = f(b) \Rightarrow a = b$
   2) Onto (Surjective): $\forall y \in W, \exists x \in V, f(x) = y$, that is, $f(V) = range(f) = W$
   3) One-to-One & Onto $\Rightarrow$ bijective
7. Definition: The ***range*** (or image) $R(T)$ of T is the subset of W consisting of all images (under T) of vectors in V; that is, $R(T) = \{T(x), x \in V\}$.
8. Theorem 2.1: Let V and W be vector spaces and $T: V \to W$ be linear. Then $N(T)$ and $R(T)$ are subspaces of V and W, repectively.
   $T(cx) = c \cdot T(x) \Rightarrow T(x) = 0_V \Rightarrow c \cdot T(x) = 0_V$
9. Theorem 2.2: Let V and W be vector spaces and $T: V \to W$ be linear. If $\beta = \{v_1, v_2, \dots, v_n\}$ is a basis for V, then
   $$R(T) = span\left(T(\beta)\right) = span(\{T(v_1), T(v_2), \dots, T(v_n)\})$$
10. Definition: Let V and W be vector spaces and $T: V \to W$ be linear. If $N(T)$ and $R(T)$ are finite-dimensional, then we define the ***nullity*** of T, denoted by $nullity(T)$, and the ***rank*** of T, denoted by $rank(T)$, to be the dimensions of $N(T)$ and $R(T)$, respectively.
    $nullity(T) = \dim(N(T))$
    $rank(T) = \dim(R(T))$
11. Theorem 2.3 (Dimension Theorem): Let V and W be vector spaces and $T: V \to W$ be linear. If V is finite-dimensional, then
    $$\boldsymbol{nullity(T) + rank(T) = \dim(V)}$$
12. Theorem 2.4: Let V and W be vector spaces and $T: V \to W$ be linear. Then T is one-to-one $\Leftrightarrow N(T) = \{0_V\}$.
13. Theorem: Let V and W be vector spaces and $T: V \to W$ be linear. Then T is onto $\Leftrightarrow rank(T) = \dim(W)$.
14. Theorem 2.5: Let V and W be vector spaces of equal (finite) dimension, and let $T: V \to W$ be linear. Then the following are equivalent:
    1) T is one-to-one.
    2) T is onto.
    3) $rank(T) = \dim(V)$.

    Proof:
    T is one-to-one $\Leftrightarrow$
    $$N(T) = \{0_V\} \Leftrightarrow \dim(N(T)) = 0 \Leftrightarrow \dim(V) = nullity(T) + rank(T) \Leftrightarrow \dim(V) = rank(T) \Leftrightarrow \dim(W) = rank(T)$$
    $\Leftrightarrow T$ is onto.
15. Theorem 2.6: Let V and W be vector spaces over F, and suppose that $\{v_1, v_2, \dots, v_n\}$ is a basis for V. For $w_1, w_2, \dots, w_n$ in W, there exists exactly one linear transformation $T: V \to W$ such that $T(v_i) = w_i\ for\ i = 1,2, \dots, n$.
16. Corollary: Let V and W be vector spaces, and suppose that V has a finite basis $\{v_1, v_2, \dots, v_n\}$. If $U, T: V \to W$ are linear and $U(v_i) = T(v_i)$ for $i = 1,2,..,n$, then $U = T$.

Proofs or Explanations:

1. Theorem 2.2
   1) $\subseteq$
      Let $v \in R(T) \Rightarrow v = T(x)$ for some $x \in V$
      $x \in V \Rightarrow x = a_1v_1 + a_2v_2 + \cdots + a_nv_n$ for some scalar $a_1, a_2, \dots, a_n \in F$
      $$v = T(x) = T(a_1v_1 + a_2v_2 + \cdots + a_nv_n) = a_1T(v_1) + a_2T(v_2) + \cdots + a_nT(v_n) = span(\{T(v_1), T(v_2), \dots, T(v_n)\})$$
   2) $\supseteq$

$\{T(v_1), T(v_2), \dots, T(v_n)\} \subseteq R(T)$
$span(\{T(v_1), T(v_2), \dots, T(v_n)\}) \subseteq R(T)$
Because $span(\{T(v_1), T(v_2), \dots, T(v_n)\})$ is the smallest subspace contains all $\{T(v_1), T(v_2), \dots, T(v_n)\}$

2. Theorem 2.4
   1) T is one-to-one $\Rightarrow N(T) = \{0_V\}$
      Let $x \in N(T) \Rightarrow T(x) = 0_W$
      $T(0_V) = 0_W$
      $\Rightarrow T(x) = T(0_W)$
      One-to-one $\Rightarrow x = 0_V \Rightarrow N(T) \subseteq \{0_V\}$
      $\{0_V\} \subseteq N(T) \Rightarrow N(T) = \{0_V\}$
      $\dim(N(T)) = 0$
   2) $N(T) = \{0_V\} \Rightarrow T$ is one-to-one
      Let x, y such that $T(x) = T(y)$
      $T(x) = T(y) \Rightarrow T(x) - T(y) = 0_W = T(x - y) = T(0_W) \Rightarrow x - y \in N(T)$
      $N(T) = \{0_V\} \Rightarrow x - y = 0_V \Rightarrow x = y$
3. Theorem 2.6
   1) Let $x \in V \Rightarrow x = a_1 v_1 + a_2 v_2 + \cdots a_n v_n$, for some unique $a_1, a_2, \dots, a_n \in F$.
      Define $T: V \to W$ as $T(x) = a_1 w_1 + a_2 w_2 + \cdots + a_n w_n$
      $v_1 = 1 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_n \Rightarrow T(v_1) = 1 \cdot w_1 + 0 \cdot w_2 + \cdots + 0 \cdot w_n = w_1$
      $v_2 = w_2$
      …
      $v_n = w_n$
      $\Rightarrow T(v_i) = w_i \; for \; i = 1,2, \dots, n$
   2) T is linear $\Rightarrow T(cx + y) = cT(x) + T(y)$
      $T(cx + y) = T\left(c(a_1 v_1 + a_2 v_2 + \cdots a_n v_n) + (b_1 v_1 + b_2 v_2 + \cdots b_n v_n)\right)$
      $= T\left((ca_1 + b_1)v_1 + (ca_2 + b_2)v_2 + \cdots (ca_n + b_n)v_n\right)$
      $= (ca_1 + b_1)w_1 + (ca_2 + b_2)w_2 + \cdots (ca_n + b_n)w_n$
      $= c(a_1 w_1 + a_2 w_2 + \cdots a_n w_n) + (b_1 w_1 + b_2 w_2 + \cdots b_n w_n) = cT(x) + T(y)$
   3) Unique
      Assume that there exists another linear $U: V \to W$, such that $U(v_i) = w_i \; for \; i = 1,2, \dots, n$.
      Let $x =\in V \Rightarrow x = a_1 v_1 + a_2 v_2 + \cdots a_n v_n$, for some unique $a_1, a_2, \dots, a_n \in F$.
      $U(x) = U(a_1 v_1 + a_2 v_2 + \cdots a_n v_n) = a_1 U(v_1) + a_2 U(v_2) + \cdots + a_n U(v_n) = a_1 w_1 + a_2 w_2 + \cdots a_n w_n = T(x) \Rightarrow U = T$

Examples:

1. $T: R^3 \to R^2, T(a_1, a_2, a_3) = (3a_1, a_2 - a_3)$
   1) T is linear
      Method1:
      i. Additive:
         $T(a + b) = T(a_1 + b_1, a_2 + b_2, a_3 + b_3) = \left(3(a_1 + b_1), ((a_2 + b_2) - (a_3 + b_3))\right)$
         $T(a) + T(b) = T(a_1, a_2, a_3) + T(b_1, b_2, b_3) = \left(3(a_1 + b_1), ((a_2 + b_2) - (a_3 + b_3))\right)$
      ii. $T(c \cdot a) = c \cdot T(a)$
         $T(ca) = T(ca_1, ca_2, ca_3)$

      Method2:
         $T(ca + b) = cT(a) + T(b)$
   2) $N(T), \beta_{N(T)}, nullity(T)$, one-to-one
      Let $a \in N(T) \Rightarrow T(a) = 0_{R^2} \Rightarrow (3a_1, a_2 - a_3) = (0,0) \Rightarrow a_1 = 0, a_2 = a_3$
      $\Rightarrow N(T) = \{a = (0, a_2, a_2) = a_2(0,1,1), a_2 \in R\}$
         (0,1,1) is a generating set
         (0,1,1) is linearly independent
         $\Rightarrow$ $\{(0,1,1)\}$ is a basis for $N(T) \Rightarrow nullity(T) = 1$
      Not one-to-one, because $nullity(T) \neq 0$ or $N(T) \neq \{0_V\}$
   3) $R(T), \beta_{R(T)}, rank(T)$, onto
      Let $\beta_{R^3} = \{e_1, e_2, e_3\}$ be a basis for R^3
      $\Rightarrow R(T) = span\left(T(\beta)\right) = span(\{T(e_1), T(e_2), T(e_3)\})$
      $T(e_1) = T(1,0,0) = (3,0)$
      $T(e_2) = T(0,1,0) = (0,1)$
      $T(e_3) = T(0,0,1) = (0,-1)$
      $\Rightarrow R(T) = span(\{(3,0), (0,1), (0,-1)\})$
      $\beta_{R(T)} = \{(3,0), (0,1)\} = \{(1,0), (0,1)\}$
      $rank(T) = 2 = \dim(R^2) \Rightarrow$ onto

      $nullity(T) + rank(T) = \dim(R^3)$
      $1 + 2 = 3$

4) Bijective

2. Prove that there exists a linear transformation $T: R^2 \to R^3$ such that $T(1,2) = (0,2,4)$ and $T(-1,3) = (1,-2,3)$. What is $T(2,9)$?
   1) $v_1 = (1,2), v_2 = (-1,3), w_1 = (0,2,4), w_2 = (1,-2,3), T(v_1) = w_1, T(v_2) = w_2$
   2) Show $\beta = \{v_1, v_2\}$ is a basis for R^2.

$(x,y) = a_1 v_1 + a_2 v_2$

$x = a_1 - a_2$

$y = 2a_1 + 3a_2$

$\Rightarrow a_1 = \frac{1}{5}(3x+y), a_2 = \frac{1}{5}(-2x+y)$

$\Rightarrow (x,y) = \frac{1}{5}(3x+y) \cdot (1,2) + \frac{1}{5}(-2x+y) \cdot (-1,3)$

$\Rightarrow$ there exists $T: R^2 \to R^3$ such that $T(v_1) = w_1$ and $T(v_2) = w_2$; that is,

$T(x,y) = \frac{1}{5}(3x+y) \cdot w_1 + \frac{1}{5}(-2x+y) \cdot w_2 = \frac{1}{5}(3x+y) \cdot (0,2,4) + \frac{1}{5}(-2x+y) \cdot (1,-2,3)$

$T(2,9) \Rightarrow x = 2, y = 9 \Rightarrow T(2,9) = \frac{1}{5}(3 \cdot 2 + 9) \cdot (0,2,4) + \frac{1}{5}(-2 \cdot 2 + 9) \cdot (1,-2,3) = 3 \cdot (0,2,4) + 1 \cdot (1,-2,3) = (1,4,15)$

3. Prove that there exists a linear transformation $T: P_2(R) \to P_3(R)$.

$T(f(x)) = 4 \cdot f'(x) + \int_0^x 3f(t)dt$

   1) T is lienar.

$T(c \cdot f(x) + g(x)) = cT(f(x)) + T(g(x))$

$T(c \cdot f(x) + g(x)) = 4 \cdot (c \cdot f(x) + g(x))' + \int_0^x 3(c \cdot f(x) + g(x))dt = 4cf'(x) + 4g'(x) + c\int_0^x 3f(x)dt + \int_0^x 3g(x)dt$

$= c\left(4f'(x) + \int_0^x 3f(x)dt\right) + \left(4g'(x) + \int_0^x 3g(x)dt\right) = cT(f(x)) + T(g(x))$

   2) $N(T), nullity(T), T$ is one-to-one

Let $f(x) \in N(T) \Rightarrow T(f(x)) = 0(0\, polinomial)$

$f(x) = a + bx + cx^2 \Rightarrow T(f(x)) = 4 \cdot f'(x) + \int_0^x 3f(t)dt = 4 \cdot (b + 2cx) + \int_0^x 3(a + bt + ct^2)dt$

$= 4b + 8cx + 3at\Big|_0^x + \frac{3bt^2}{2}\Big|_0^x + ct^3\Big|_0^x = 4b + 8cx + 3ax + \frac{3bx^2}{2} + cx^3 = 0$

$\Rightarrow b = 0, 8c + 3a = 0, \frac{3b}{2} = 0, c = 0$

$\Rightarrow f(x) = 0 \Rightarrow N(T) = \{0_{P_2(R)}\} \Rightarrow nullity(T) = 0 \Rightarrow T$ is one-to-one

   3) $R(T), rank(T), T$ is onto

$R(T) = span\left(T(\beta)\right), \beta$ is a basis for $P_2(R)$

Let $\beta = \{1, x, x^2\} \Rightarrow R(T) = span(\{T(1), T(x), T(x^2)\})$

$T(1) = 4 \cdot 1' + \int_0^x 3 \cdot 1dt = 3x$

$f(1) = 1$

$T(x) = 4 \cdot x'1 + \int_0^x 3t\, dt = 4 + \frac{3t^2}{2}\Big|_0^x = 4 + \frac{3x^2}{2}$

$f(x) = x$

$T(x^2) = 4 \cdot (x^2)' + \int_0^x 3f(x)dt = 8x + t^3\Big|_0^x = 8x + x^3$

$f(x) = x^2$

$\Rightarrow R(T) = span\left\{3x, 4 + \frac{3x^2}{2}, 8x + x^3\right\}$

According 4) $rank(T) = 3 \leq 4$ is not onto.

   4) Dimension theorem

$\dim(P_2(R)) = 3 = nullity(T) + rank(T) \Rightarrow rank(T) = 3$

2.2 The Matrix Representation of a Linear Transformation

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Definitions & Theorems:

1. Definition: Let V be a finite-dimensional vector spaces. An ordered basis for V is a basis for V endowed (inˈdou,enˈdou) with a specific order; that is, an ordered basis for V is a finite sequence of linearly independent vectors in V that generates V.
2. Standard ordered basis:
   $F^n: \{e_1, e_2, \dots, e_n\}$
   $P_n(F): \{1, x, \dots, x^n\}$
3. Definition: Let $\beta = \{u_1, u_2, \dots, u_n\}$ be an ordered basis for a finite dimensional vector space V. For $x \in V$, let $a_1, a_n, \dots, a_n$ be the unique scalars such that
   $$x = \sum_{i=1}^{n} a_i u_i$$
   We define the coordinate vector of x relative to β, denoted $[x]_\beta$, by
   $$[x]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$
4. Definition: Suppose that V and W are finite-dimensional vector spaces with ordered bases $\beta = \{v_1, v_2, \dots, v_n\}$ and $\gamma = \{w_1, w_2, \dots, w_m\}$, respectively. Let $T: V \to W$ be linear. Then for each $j, 1 \le j \le n$, there exists unqiue scalars $a_{ij} \in F, 1 \le j \le m$, such that
   $$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i \quad for\ 1 \le j \le n$$
   We call the $m \times n$ matrix A defined by $A_{ij} = a_{ij}$ the matrix representation of T in the ordered bases β and γ and write $A = [T]_\beta^\gamma$.
   $T: V \to W$ is linear.
   $\dim(V) = n, \beta = \{v_1, v_2, \dots, v_n\}$ ordered basis
   $\dim(W) = m, \gamma = \{w_1, w_2, \dots, w_m\}$ ordered basis
   $$T(v_1) \in W \Rightarrow T(v_1) = a_{11}w_1 + a_{21}w_2 + \cdots + a_{m1}w_m \Rightarrow [T(v_1)]_\gamma = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}$$
   $$T(v_2) \in W \Rightarrow T(v_2) = a_{12}w_1 + a_{22}w_2 + \cdots + a_{m2}w_m \Rightarrow [T(v_2)]_\gamma = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}$$
   $$T(v_n) \in W \Rightarrow T(v_n) = a_{1n}w_1 + a_{2n}w_2 + \cdots + a_{mn}w_m \Rightarrow [T(v_n)]_\gamma = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$
   The matrix representation of T with respect to the ordered bases β and γ is:
   $$[T]_\beta^\gamma = \left([T(v_1)]_\gamma \quad [T(v_2)]_\gamma \quad \cdots \quad [T(v_n)]_\gamma\right) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{pmatrix} \in M_{m\times n}(F)$$
5. If $V = W$ and $\beta = \gamma, T: V \to W$
   Then $[T]_\beta^\gamma = [T]_\beta^\beta = [T]_\beta$
6. Property:
   If $T, U: V \to W$ is liear, and $[T]_\beta^\gamma = [U]_\beta^\gamma \Rightarrow T = U$.
7. Definition: Let $T, U: V \to W$ be arbitrary functions, where V and W are vector spaces over F, and let $a \in F$. We define
   a) $T + U: V \to W$ by $(T + U)(x) = T(x) + U(x)$ for all $x \in V$, and
   b) $aT: V \to W$ by $(aT)(x) = aT(x)$ for all $x \in V$.
8. Theorem 2.7: Let V and W be vector spaces over a filed F, and let $T, U: V \to W$ be lienar.
   a. For all $a \in F, aT + U$ is linear.
   b. Using the operations of addition and scalar multiplication in the preceding definition, the collection of all linear transformation from V to W is a vector space over F.
9. Definition: Let V and W be vector spaces over F. We denote the vector space of all linear transformations from V into W by $\mathcal{L}(V, W)$. In the case that $V = W$, we write $\mathcal{L}(V)$ instead of $\mathcal{L}(V, W)$.
   $V = W \Rightarrow \mathcal{L}(V, W) = \mathcal{L}(V) = \{T: V \to V, T\ is\ linear\}$ (linear operator)
10. $\mathcal{L}(V, W)$ is a vector space over F with the operations of addition and scalar mutilication.
    Proof:
    Check closure properties, that is,
    T, U is linear $\Rightarrow T + U$ is linear.

T is linear $\Rightarrow k \cdot T$ is linear.

Step1:

$T + U \text{ is linear} \Leftrightarrow (T+U)(cx+y) = c\cdot(T+U)(x) + (T+U)(y)$

$(T+U)(cx+y) = T(cx+y) + U(cx+y) = cT(x) + T(y) + cU(x) + U(y)$

$= c(T(x)+U(x)) + (T(y)+U(y)) = c(T+U)(x) + (T+U)(y)$

Step2:

$k\cdot T \text{ is linear} \Leftrightarrow (kT)(cx+y) = c\cdot(k\cdot T)(x) + (kT)(y)$

$(kT)(cx+y) = k\cdot T(cx+y) = k\cdot(cT(x)+T(y)) = k\cdot(cT(x)) + k\cdot T(y)$

$= k\cdot c\cdot T(x) + k\cdot T(y) = c\cdot k\cdot T(x) + k\cdot T(y) = c\cdot(kT)(x) + (kT)(y)$

11. Theorem 2.8: Let V and W be finite-dimensional vector spaces with ordered bases β and γ, respectively, and let $U, T: V \to W$ be linear transformations, then
    a. $[T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$
    b. $[aT]_\beta^\gamma = a[T]_\beta^\gamma =$ for all scalars a.

Proofs or Explanations:

1. $V = R^3$, ordered bases $\beta = \{e_1, e_2, e_3\}, \gamma = \{e_2, e_1, e_3\}$

$v = (1,-4,3) = 1\cdot e_1 + (-4)\cdot e_2 + 3\cdot e_3$

$\Rightarrow [v]_\beta = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}, [v]_\gamma = \begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix}$

2. $f(x) = 1 - 2x + 4x^2 \in P_2(R)$, ordered bases $\beta = \{1, x, x^2\}, \gamma = \{x^2, 1, x\}$

$\Rightarrow [f(x)]_\beta = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}, [f(x)]_\gamma = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$

3. $T: R^3 \to R^2, T(a_1, a_2, a_3) = (a_1 + a_2, a_2 - a_3)$ is linear.

$\beta = \{e_1, e_2, e_3\}$ ordered basis for R^3

$\gamma = \{a_1, e_2\}$ ordered basis for R^2

$T(e_1) = T(1,0,0) = (1,0) = 1\cdot e_1 + 0\cdot e_2 \Rightarrow [T(e_1)]_\gamma = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$T(e_2) = T(0,1,0) = (1,1) = 1\cdot e_1 + 1\cdot e_2 \Rightarrow [T(e_2)]_\gamma = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$T(e_3) = T(0,0,1) = (0,-1) = 0\cdot e_1 - 1\cdot e_2 \Rightarrow [T(e_3)]_\gamma = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$[T]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \in M_{2\times3}(R)$

4. $T: M_{2\times2}(R) \to P_2(R), T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = b + c + (3d)x + ax^2$ is linear.

$\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ ordered basis for $M_{2\times2}(R)$

$\gamma = \{1, x, x^2\}$ ordered basis for $P_2(R)$

$[T]_\beta^\gamma \to 3\times4$

$T(E_{11}) = T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0 + 0 + 0x + 1x^2 = 0 + 0x + 1x^2 \Rightarrow [T(E_{11})]_\gamma = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$T(E_{12}) = T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 1 + 0 + 0x + 0x^2 = 1 + 0x + 0x^2 \Rightarrow [T(E_{12})]_\gamma = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$T(E_{21}) = T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 1 + 0 + 0x + 0x^2 = 1 + 0x + 0x^2 \Rightarrow [T(E_{21})]_\gamma = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$T(E_{22}) = T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 1 + 0 + 0x + 0x^2 = 1 + 0x + 0x^2 \Rightarrow [T(E_{22})]_\gamma = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$

$\Rightarrow [T]_\beta^\gamma = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 3 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

5. $T: M_{2\times2}(R) \to M_{2\times2}(R), T(A) = A + A^T$ is linear.

$\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ ordered basis for $M_{2\times2}(R)$

$[T]_\beta^\gamma \to 4\times4$

$$T(E_{11}) = T\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} = \begin{pmatrix}2 & 0\\0 & 0\end{pmatrix} \Rightarrow 2 \cdot E_{11} + 0 \cdot E_{12} + 0 \cdot E_{21} + 0 \cdot E_{21} \Rightarrow [T(E_{11})]_\gamma = \begin{pmatrix}2\\0\\0\\0\end{pmatrix}$$

$$T(E_{12}) = T\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \Rightarrow 0 \cdot E_{11} + 1 \cdot E_{12} + 1 \cdot E_{21} + 0 \cdot E_{21} \Rightarrow [T(E_{12})]_\gamma = \begin{pmatrix}0\\1\\1\\0\end{pmatrix}$$

$$T(E_{21}) = T\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} = \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} + \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \Rightarrow 0 \cdot E_{11} + 1 \cdot E_{12} + 1 \cdot E_{21} + 0 \cdot E_{21} \Rightarrow [T(E_{21})]_\gamma = \begin{pmatrix}0\\1\\1\\0\end{pmatrix}$$

$$T(E_{22}) = T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} + \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 2\end{pmatrix} \Rightarrow 0 \cdot E_{11} + 0 \cdot E_{12} + 0 \cdot E_{21} + 2 \cdot E_{21} \Rightarrow [T(E_{22})]_\gamma = \begin{pmatrix}0\\0\\0\\2\end{pmatrix}$$

$$\Rightarrow [T]_\beta^\gamma = \begin{pmatrix}2 & 0 & 0 & 0\\0 & 1 & 1 & 0\\0 & 1 & 1 & 0\\0 & 0 & 0 & 2\end{pmatrix}$$

Examples:

1. P84,Q3:

   $T: R^2 \to R^3, T(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$ be linear. $\beta = \{e_1, e_2\}, \gamma = \{(1,1,0), (0,1,1), (2,2,3)\}, \alpha = \{(1,2), (2,3)\}$
   $U: R^2 \to R^3, U(a_1, a_2) = (a_1 + a_2, a_2, -3a_1 + a_2)$ be linear. $[T]_\beta^\gamma, [T + U]_\beta^\gamma$

   a. $T(a_1, a_2)$

      i. $T(e_1) = (1,1,2) = -\frac{1}{3}(1,1,0) + 0 \cdot (0,1,1) + \frac{2}{3}(2,2,3) \Rightarrow [T(e_1)]_\gamma = \begin{pmatrix}-1/3\\0\\2/3\end{pmatrix}$

      ii. $T(e_2) = (-1,0,1) = -1 \cdot (1,1,0) + 1 \cdot (0,1,1) + 0 \cdot (2,2,3) \Rightarrow [T(e_2)]_\gamma = \begin{pmatrix}-1\\1\\0\end{pmatrix}$

      iii. $[T]_\beta^\gamma = \begin{pmatrix}-1/3 & -1\\0 & 1\\2/3 & 0\end{pmatrix}$

   b. $[T + U]_\beta^\gamma$

      Method1:

      $(T + U)(a_1, a_2) = T(a_1, a_2) + U(a_1, a_2) = (2a_1, a_1 + a_2, -a_1 + 2a_2)$

      i. $(T + U)(e_1) = (T + U)(1,0) = (2,1,-1) \Rightarrow [(T + U)(e_1)]_\gamma = \begin{pmatrix}2\\-1\\0\end{pmatrix}$

      ii. $(T + U)(e_2) = (T + U)(0,1) = (0,1,2) \Rightarrow [(T + U)(e_2)]_\gamma = \begin{pmatrix}-2/3\\1\\1/3\end{pmatrix}$

      iii. $[T + U]_\beta^\gamma = \begin{pmatrix}2 & -2/3\\-1 & 1\\0 & 1/3\end{pmatrix}$

      Method2:

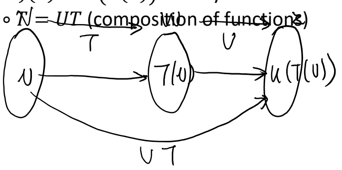
      $[T + U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$

2.3 Composition of Linear Transformation and Matrix Multiplication

September 6, 2017 11:05

Definitions & Theorems:

1. Review:
   Matrix: $m \times n$, m rows, n columns
   $(m \times n)(n \times p) = (m \times p)$
   $AB = C = (C_{ij}), C_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + A_{i3}B_{3j} + \cdots + A_{in}B_{nj}$
   Properties:
   $A \in M_{m\times n}(F), B, C \in M_{n\times p}(F), D, E \in M_{p\times m}(F)$
   1) $A(B + C) = AB + AC, (D + E)A = DA + EA$
   2) $a(AB) = (aA)B = A(aB)$
   3) $AI_n = A = I_m A$
   4) $I_V: V \to V$ is the identity transformation, $\dim(V) = n$.
      $I_V(x) = x \; for \; any \; x$
      β is a basis for $V \Rightarrow [I_V]_\beta = I_n$
   Power of A:
   $A^1 = A, A^2 = AA, A^n = A^{n-1}A, A \in M_{n\times n}(F)$
   Convention:
   $A^0 = I_n$
   Property:
   $A \in M_{m\times n}(F), B \in M_{n\times p}(F) \Rightarrow (AB)^t = B^t \cdot A^t$
2. Composition of linear transformation
   Let $T: V \to W$ and $U: W \to Z$ be linear.
   Then
   $UT: V \to Z$
   $(UT)(v) = U(T(v))$ for any $v \in V$
   $U \circ T = UT$ (composition of functions)



3. Theorem 2.9: Let V, W and Z be vector spaces over the same field F, and let $T: V \to W$ and $U: W \to Z$ be linear. Then $UT: V \to Z$ is linear.
   Proof:
   Let $x, y \in V$ and $a \in F$. Then
   $UT(ax + y) = U\left(T(ax + y)\right) = U(aT(x) + T(y)) = aU(T(x)) + U\left(T(y)\right) = a(UT)(x) + UT(y).$
4. Nullity:
   $N(T) \subseteq N(UT)$
   Proof:
   $N(T) = \{v \in V: T(v) = 0_W\}$
   $N(UT) = \{v \in V: (UT)(v) = 0_Z\}$
   Let $v \in N(T)? \, v \in N(UT)$
   $T(v) = 0_W \Rightarrow U(T(v)) = U(0_W) = 0_Z \Rightarrow (UT)(v) = 0_Z \Rightarrow v \in N(UT).$
5. Property: If UT is one-to-one, then T is one-to-one.
   Proof:
   UT is one-to-one and linear $\Rightarrow N(UT) = \{0_V\}$
   But $N(T) \subseteq N(UT) \Rightarrow N(T) \subseteq \{0_V\}$
   $\{0_V\} \subseteq N(T) \Rightarrow N(T) = \{0_V\} \Rightarrow T$ is one-to-one.
6. Range:
   U is linear $\Rightarrow R(U)$ is a subspace of $Z \Rightarrow R(UT) \subseteq R(U)$
   UT is linear $\Rightarrow R(UT)$ is a subspace of Z.
   Proof:
   Let $z \in R(UT) \Rightarrow z = (UT)(v)$ for some $v \in V \Rightarrow z = U(T(v))$ for some $v \in V$.
   $T(v) = w \in W \Rightarrow z = U(w)$ for some $w \in W \Rightarrow z \in R(U)$
7. Property: If UT is onto, then U is onto.
   Proof:
   UT is linear and onto$\Rightarrow R(UT) = Z$
   But $R(UT) \subseteq R(U) \Rightarrow Z \subseteq R(U)$
   Also, $R(U) \subseteq Z \Rightarrow R(U) = Z \Rightarrow U$ is onto.
8. Property: $R(UT) = U(R(T))$
   Proof:
   Let $z \in R(UT) \Rightarrow z = (UT)(v)$ for some $v \in V \Rightarrow z = U(T(v))$
   $T(v) = w \in R(T) \Rightarrow z = U(w), w \in R(T) \Rightarrow z \in U(R(T))$
9. Theorem 2.10: Let V be a vector space. Let $T, U_1, U_2 \in \mathcal{L}(V)$. Then
   (a) $T(U_1 + U_2) = TU_1 + TU_2$
   (b) $(U_1 + U_2)T = U_1T + U_2T$
   (c) $T(U_1U_2) = (TU_1)U_2$
   (d) $TI = IT = T$
   (e) $a(U_1U_2) = (aU_1)U_2 = U_1(aU_2)$ for all scalars a.
10. Definition: Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. We define the product of A and B, denoted AB, to be the $m \times p$ matrix such that
   $$(AB)_{ij} = \sum_{k=1}^{m} A_{ik}B_{kj} \quad for \; 1 \le i \le m, 1 \le j \le p$$
11. Theorem 2.11: Let V, W and Z be finite-dimensional vector spaces with ordered bases α, β, and γ, respectively. Let $T: V \to W$ and $U: W \to Z$ be linear transformations,

dim$(V) = n$, dim$(W) = m$, dim$(Z) = p$. Then

$[T]_\alpha^\beta \in M_{m\times n}(F), [U]_\beta^\gamma \in M_{p\times m}(F), [UT]_\alpha^\gamma \in M_{p\times n}(F)$

$[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$

If $V = W = Z, \alpha = \beta = \gamma \Rightarrow [UT]_\alpha = [U]_\alpha [T]_\alpha$

12. Corollary: Let V be a finite-dimensional vector space with and ordered basis β. Let $T, U \in \mathcal{L}(V)$. Then $[UT]_\beta = [U]_\beta [T]_\beta$
13. Definition: We define the *Kronecker delta* δ_{ij} by $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Then $n \times n$ identity matrix I_n is defined by $(I_n)_{ij} = \delta_{ij}$. Thus, for example.

$$I_1 = (1)$$
$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

14. Theorem 2.12: Let A be an $m \times m$ matrix, B and C be $n \times p$ matrices, and D and E be $q \times m$ matrices. Then
    (a) $A(B + C) = AB + AC$ *and* $(D + E)A = DA + EA$.
    (b) $a(AB) = (aA)B = A(aB)$ for any scalar a.
    (c) $I_m A = A = AI_n$.
    (d) If V is an n-dimensional vector space with an ordered basis β, then $[I_V]_\beta = I_n$.
15. Corollary: Let A be an $m \times n$ matrix, $B_1, B_2, \ldots, B_k$ be $n \times p$ matrices, $C_1, C_2, \ldots, C_k$ be $q \times m$ matrices, and $a_1, a_2, \ldots, a_k$ be scalars. Then

$$A\left(\sum_{i=1}^{k} a_i B_i\right) = \sum_{i=1}^{k} a_i AB_i$$
$$\left(\sum_{i=1}^{k} a_i C_i\right) A = \sum_{i=1}^{k} a_i C_i A$$

16. Theorem 2.13: Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. For each $j (1 \le j \le p)$ let u_j and v_j denote the jth columns of AB and B, respectively. Then
    (a) $u_j = Av_j$
    (b) $v_j = Be_j$, where e_j is the jth standard vector of F_p.
    The column j of AB is the linear combination of columns of A, with the coefficients being the entries of column j of B
    The row i of AB is the linear combination of rows of B, with the coefficients being the entries of row i of A
17. Theorem 2.14: Let V and W be finite-diemensional vector spaces having ordered bases β and γ, respectively, and let $T: V \to W$ be linear. Then, for each $u \in V$, we have

$$[T(u)]_\gamma = [T]_\beta^\gamma [u]_\beta$$

    Proof:

    Fix $u \in V$, and define the linear transformations $f: F \to V$ by $f(a) = au$ and $g: F \to W$ by $g(a) = aT(u)$ for all $a \in F$. Let $\alpha = \{1\}$ be the standard ordered basis for F. Notice that $g = Tf$. Identifying column vectors as matrices and using Theorem 2.11, we obtain

$$[T(u)]_\gamma = [g(1)]_\gamma = [g]_\alpha^\gamma = [Tf]_\alpha^\gamma = [T]_\beta^\gamma [f]_\alpha^\beta = [T]_\beta^\gamma [f(1)]_\beta = [T]_\beta^\gamma [u]_\beta$$

18. Definition: Let $A \in M_{m\times n}$ with entries from a filed F. We denote by L_A the mapping $L_A: F^n \to F^m$ defined by $L_A(x) = Ax$ (the matrix product of A and x) for each column vector $x \in F^n$. We call L_A a left-multiplication transformation.

$$F^n = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{pmatrix} \right\}, F^m = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_m \end{pmatrix} \right\}$$

    e.g.:

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 3 \end{pmatrix} \Rightarrow m = 2, n = 3 \Rightarrow L_A: R^3 \to R^2$$
$$L_A(x) = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

19. Theorem 2.15: Let A be an $m \times n$ matrix with entries from F. Then the left-multiplication transformation $L_A: F^n \to F^m$ is linear. Furthermore, if B is any other $m \times n$ matrix (with entries from F) and β and γ are the standard ordered bases for F^n and F^m, respectively, then we have the following properties.
    1) L_A is linear.
    2) $L_A = L_B \Leftrightarrow A = B$.
    3) $L_{A+B} = L_A + L_B$ and $L_{aA} = aL_A$ for all $a \in F$.
    4) If E is an $n \times p$ matrix, then $L_{AE} = L_A \circ L_E$.
    5) If $m = n, A = I_n \Rightarrow L_{I_n}: F^n \to F^n \Rightarrow L_{I_n}(x) = I_n \cdot x = x = I_{F_n}(x) \Rightarrow L_{I_n} = I_{F_n}$.
    6) $[L_A]_\beta^\gamma = A$.
    7) If $T: F^n \to F^m$ is linear, then there exists a unique $m \times n$ matrix C such that $T = L_C$. In fact, $C = [T]_\beta^\gamma$.

    Proof:
    1) $L_A: F^n \to F^m, L_A(x) = A \cdot x$
       L_A is linear $\Leftrightarrow L_A(cx + y) = c \cdot L_A(x) + L_A(y)$
       $L_A(cx + y) = A \cdot (cx + y) = A \cdot cx + Ay = c \cdot Ax + Ay = c \cdot L_A(x) + L_A(y)$
    2) $L_A = L_B \Rightarrow L_A(x) = L_B(x)$ for any $x \in F^n \Rightarrow Ax = Bx$ for any $x \in F^n$.
       Let $x = e_1 \Rightarrow C_1^A = C_1^B$ (first column of A and first column of B)
       Let $x = e_2 \Rightarrow C_2^A = C_2^B$
       Let $x = e_n \Rightarrow C_n^A = C_n^B$
       $\Rightarrow A = B$
    3) $L_{A+B}(x) = (A + B) \cdot x = A \cdot x + B \cdot x = L_A(x) + L_B(x) = (L_A + L_B)(x)$
       $L_{aA}(x) = (aA) \cdot x = a \cdot (A \cdot x) = a \cdot L_A(x) = (aL_A)(x)$
    4) $L_{AB}(x) = (AB)x$
       $(L_A \circ L_B)(x) = L_A(L_B(x)) = A \cdot (Bx) = (AB)x$
    6) $\beta = \{e_1, e_2, \ldots, e_n\}, \gamma = \{e_1, e_2, \ldots, e_m\}$
       $L_A(e_1) = Ae_1 = C_1^A = A_{11}e_1 + A_{21}e_1 + \cdots + A_{n1}e_1$
    7) $[L_C]_\beta^\gamma = C \Rightarrow \left[L_{[L_C]_\beta^\gamma}\right]_\beta^\gamma = C \Rightarrow \left[L_{[T]_\beta^\gamma}\right]_\beta^\gamma = [T]_\beta^\gamma \Rightarrow L_{[T]_\beta^\gamma} = T$
20. Theorem 2.16: Let A, B and C be matrices such that $A(BC)$ is defined. Then $(AB)C$ is also defined and $A(BC) = (AB)C$; that is, matrix multiplication is associative.

Proofs or Explanations:

1.

Examples:

1. $T: P_3(R) \to P_2(R), T\big(f(x)\big) = 2 \cdot f'(x);$

   $U: P_2 \to R^2, U\big(a + bx + cx^2\big) = (a + b, a + c);$

   U, T is linear.

   $\alpha = \{1, x, x^2, x^3\}, \beta = \{1 + x, 1 + x^2, x + x^2\}, \gamma = \{e_1, e_2\}$

   (a) $[T]_\alpha^\beta, [U]_\beta^\gamma, [UT]_\alpha^\gamma$

   Method1:

   $[T]_\alpha^\beta$

   $f(x) = a + bx + cx^2 = a_1(1 + x) + a_2(1 + x^2) + a_3(x + x^2)$

   $$\Rightarrow \begin{cases} a_1 + a_2 = a \\ a_1 + a_3 = b \\ a_2 + a_3 = c \end{cases} \Rightarrow \begin{cases} a_1 = \dfrac{a + b - c}{2} \\ a_2 = \dfrac{a - b + c}{2} \\ a_3 = \dfrac{-a + b + c}{2} \end{cases}$$

   $$\Rightarrow a + bx + cx^2 = \left(\frac{a + b - c}{2}\right)(1 + x) + \left(\frac{a - b + c}{2}\right)(1 + x^2) + \left(\frac{-a + b + c}{2}\right)(x + x^2)$$

   $$T(1) = 2 \cdot 1' = 0 \Rightarrow a = b = c = 0 \Rightarrow [T(1)]_\beta = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

   $$T(x) = 2 \cdot x' = 2 \Rightarrow a = 2, b = c = 0 \Rightarrow [T(x)]_\beta = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

   $$T(x^2) = 2 \cdot (x^2)' = 4x \Rightarrow a = 0, b = 4, c = 0 \Rightarrow [T(x^2)]_\beta = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$$

   $$T(x^3) = 2 \cdot (x^3)' = 6x^2 \Rightarrow a = b = 0, c = 6 \Rightarrow [T(x^3)]_\beta = \begin{pmatrix} -3 \\ 3 \\ 3 \end{pmatrix}$$

   $$[T]_\alpha^\beta = \begin{pmatrix} 0 & 1 & 2 & -3 \\ 0 & 1 & -2 & 3 \\ 0 & -1 & 2 & 3 \end{pmatrix}$$

   $[U]_\beta^\gamma$

   $1 + x \Rightarrow a = 1, b = 1, c = 0$

   $U(1 + x) = (2,1)$

   $U(1 + x^2) = (1,2)$

   $U(x + x^2) = (1,1)$

   $$[U]_\beta^\gamma = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

   $[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$

   Method2:

   $(UT)\big(f(x)\big) = U\Big(T\big(f(x)\big)\Big) = U\Big(T\big(a + bx + cx^2 + dx^3\big)\Big) = U\Big(2 \cdot \big(b + 2cx + 3dx^2\big)\Big) = U\big(2b + 4cx + 6dx^2\big) = (2b + 4c, 2b + 6d)$

   $\Rightarrow (UT)\big(a + bx + cx^2 + dx^3\big) = (2b + 4c, 2b + 6d)$

   $(UT)(1) = (0,0)$

   $(UT)(x) = (2,3)$

   $(UT)(x^2) = (4,0)$

   $(UT)(x^3) = (0,6)$

   $$[UT]_\alpha^\gamma = \begin{pmatrix} 0 & 2 & 4 & 0 \\ 0 & 3 & 0 & 6 \end{pmatrix}$$

   (b) $P(x) = 1 - 2x + 3x^2 - x^3 \to check\ \big[T\big(P(x)\big)\big]_\beta = [T]_\alpha^\beta [P(x)]_\alpha$

   $T\big(1 - 2x + 3x^2 - x^3\big) = -4 + 12x - 6x^2$

   $$a + bx + cx^2 = \left(\frac{a + b - c}{2}\right)(1 + x) + \left(\frac{a - b + c}{2}\right)(1 + x^2) + \left(\frac{-a + b + c}{2}\right)(x + x^2)$$

   $\Rightarrow 7(1 + x) + 5(1 + x^2) + 5(x + x^2)$

2.4 Invertibility and Isomorphisms

September 6, 2017 11:05

Definitions & Theorems:

1. Definition: Let $T: V \to W$ be linear. A function $U: W \to V$ is said to be an inverse of T if $TU = I_W$ and $UT = I_V$.
   $TU = I_W \Leftrightarrow (TU)(y) = y$
   $UT = I_V \Leftrightarrow (UT)(x) = x$
   If T has an inverse, then T is said to be invertible.
2. Properties:
   1) If T is invertible, then T has a unique inverse, denoted by $T^{-1} \Rightarrow (TT^{-1})(x) = x, x \in V$ and $(T^{-1}T)(y) = y, y \in W$
   2) If T is invertible, then T^{-1} is also invertible and $(T^{-1})^{-1} = T$
   3) T is invertible $\Leftrightarrow$ T is one-to-one and onto.
   4) $T: V \to W, U: W \to Z$. If T, U are invetible, then $UT: V \to Z$ is invetible and $(UT)^{-1} = T^{-1} \circ U^{-1}$
3. Recall: $T: V \to W$ is linear, $\dim(V) = \dim(W) \Rightarrow T$ is one-to-one$\Leftrightarrow$ T is onto$\Leftrightarrow$ $rank(T) = \dim(V)$
   T is invertible $\Leftrightarrow rank(T) = \dim(V)$
   T is invertible $\Leftrightarrow$ for any $y \in W$ there exists unique $x \in V$ such that $T(x) = y$, and therefore, $T^{-1}(y) = x$
4. Theorem 2.17: $T: V \to W$ be linear and invertible. Then its inverse $T^{-1}: W \to V$ is also linear.
   Proof:
   T^{-1} is linear $\Rightarrow T^{-1}(cy_1 + y_2) = cT^{-1}(y_1) + T^{-1}(y_2), y_1, y_2 \in W$
   $\Rightarrow y_1 = T(x_1)$ for some $x_1 \in V \Rightarrow x_1 = T^{-1}(y_1)$
   $\Rightarrow y_2 = T(x_2)$ for some $x_2 \in V \Rightarrow x_2 = T^{-1}(y_2)$
   $T^{-1}(cy_1 + y_2) = T^{-1}\left(c \cdot T(x_1) + T(x_2)\right) = T^{-1}\left(T(cx_1 + x_2)\right)$
   $= (TT^{-1})(cx_1 + x_2) = I_v(cx_1 + x_2) = cx_1 + x_2$
5. Lemma: Let $T: V \to W$ be linear and invertible. Then, V is finite dimentional $\Leftrightarrow$ W is finite dimentional, moreover, $\dim(V) = \dim(W)$
   Proof:
   V is finite dimentional$\Rightarrow$ there exists a finite basis $\beta = \{v_1, v_2, \dots, v_n\}$ of V
   T is linear $\Rightarrow R(T) = span\left(T(\beta)\right) = span(\{T(v_1), T(v_2), \dots, T(v_n)\})$
   $\Rightarrow$ there exists a basis γ in $\{T(v_1), T(v_2), \dots, T(v_n)\}$ for $R(T) \Rightarrow \gamma$ is finite
   T is onto $\Rightarrow R(T) = W \Rightarrow \gamma$ is a finite basis for $W \Rightarrow W$ is finite dimentional.
   T is invertible$\Rightarrow T^{-1}: W \to V$
   $\dim(V) = nullity(T) + rank(T) = 0 + rank(T) = \dim(W)$
6. Theorem 2.18: Let V and W be linear and finite dimentional with ordered bases β and γ. Then T is invertible $\Leftrightarrow [T]_\beta^\gamma$ is invertible.
   Furthermore, $[T^{-1}]_\gamma^\beta = \left([T]_\beta^\gamma\right)^{-1}$.
7. Corollary 1: $V = W, T: V \to W$ and $\beta = \gamma$. Then T is invertible $\Leftrightarrow [T]_\beta$ is invetible and $[T^{-1}]_\beta = \left([T]_\beta\right)^{-1}$.
8. Corollary 2: Let $A \in M_{n \times n}$. Then A is invertible $\Leftrightarrow L_A$ is inveritble. Furthermore, $(L_A)^{-1} = L_{A^{-1}}$.
   Proof: $L_A: F^n \to F^n, L_A(x) = A \cdot x$
   By theorem, L_A is invertible $\Leftrightarrow [L_A]_\beta^\gamma = A$ is invertible, β, γ be standard bases.
   $L_A \circ L_{A^{-1}} = L_{AA^{-1}} = L_{I_n} = I_{F^n}$
   $L_{A^{-1}} \circ L_A = L_{A^{-1}A} = L_{I_n} = I_{F^n}$

   e.g.:
   1) $(AB)C = A(BC)$
   2) $A \in M_{m \times n}(F), A$ is invertible$\Rightarrow m = n$
   3) A, B are invertible $\Rightarrow AB$ is invertible

   Proof:
   1) $L_{(AB)C} = L_{AB} \circ L_C = (L_A \circ L_B) \circ L_C = L_A \circ (L_B \circ L_C) = L_A \circ L_{BC} = L_{A(BC)}$
   2) A is invertible, $A \in M_{m \times n}(F) \Rightarrow L_A: F^n \to F^m$ is invertible. And L_A is linear$\Rightarrow \dim(F^n) = \dim(F^m) \Rightarrow n = m$
   3) A is inveritble $\Rightarrow L_A$ is inveritble. Same $\Rightarrow L_B$ is invertible.$\Rightarrow L_A \circ L_B$ is invertible$\Rightarrow L_{AB}$ is invertible$\Rightarrow AB$ is invertible.
      If AB is inveritble $\Rightarrow L_{AB}$ is invertible$\Rightarrow L_A \circ L_B$ is invertible$\Rightarrow L_B$ is one-to-one, L_A is onto.? But dim() is the same, so one-to-one = onto.
9. Definition: Let V and W be vector spaces over the same field F. We say that V is isomorphic to W if there exists a linear transformation $T: V \to W$ that is linear and invertible. Such a linear transformation is called an isomorphism from V to W.
10. Properties:
    1) V is isomorphic to V. (reflexivity)
       $I_V: V \to V, I_V(x) = x$
    2) V is isomorphic to W, then W is isomorphic to V. That is, V and W are isomorphic. (symmetric)
    3) If V is isomporphic to W, and W is isomorphic to U, then V is isomorphic to U. (transitivity)
11. Remark: If V and W are isomorphic, then either both V and W are finite-dimetional, or both V and W are infinite-dimentional.
12. Theorem 2.19: Let V and W be finite dimentional over the same field F. Then V is isomorphic to $W \Leftrightarrow \dim(V) = \dim(W)$.
13. Corollary: Let V be a vector space over F. Then V is isomorphic to F^n if and only if $\dim(V) = n$.
    e.g.:

V is finite-dimentional, V and F^n are isomorphic $\Leftrightarrow \dim(V) = n$

14. Theorem 2.20: Let V and W be finite-dimentional vector spaces over F of dimensions n and m, respectivly, and let β and γ be ordered bases for V and W, respectively. Then the function $\Phi: \mathcal{L}(V, W) \to M_{m\times n}(F)$, define by $\Phi(T) = [T]_\beta^\gamma$ for $T \in \mathcal{L}(V, W)$, is an isomorphism. And therefore, $\dim(\mathcal{L}(V, W)) = \dim(M_{m\times n}(F)) = mn$.
    Recall: $\mathcal{L}(V, W) = \{T: V \to W, T\ is\ linear\}$

Proofs or Explanations:

1.

Examples:

1. R^n is isomorphic to $P_{n-1}(R)$
    $\dim(R^n) = n$
    $\dim(P_{n-1}(R)) = n$
    $T(a_0, a_1, \dots, a_{n-1}) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}$ is linear and invertible.
2.

2.5 The Change of Coordinate Matrix

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Definitions & Theorems:

1. Recall

   $$[T(v)]_\gamma = [T]_\beta^\gamma [v]_\beta$$

   Special case: $T = I_V$

   $$[I(v)]_\gamma = [I_V]_\beta^\gamma [v]_\beta \Rightarrow [v]_\gamma = [I_V]_\beta^\gamma [v]_\beta$$

2. Theorem 2.22 (Change of coordinate matrix): Let β and β' be two ordered bases for a finite-dimetional vector space V, and let $Q = [I_V]_{\beta'}^{\beta}$. Then
   a) Q is invertible. Moreover, Q^{-1} changes β coordiantes into β' coordinates.

   $$\left([I_V]_{\beta'}^{\beta}\right)^{-1} = [I_V^{-1}]_\beta^{\beta'} = [I_V]_\beta^{\beta'}$$

   b) For any $v \in V$, $[v]_\beta = Q[v]_{\beta'}$. We say Q changes β'-coordinates into β-coordinates.

   $$[v]_\beta = [I_V(v)]_\beta = [I_V]_{\beta'}^{\beta}[v]_{\beta'} = Q[v]_{\beta'}$$

   Example 1:

   In R^2, let $\beta = \{(1,1),(1,-1)\}$ and $\beta' = \{(2,4),(3,1)\}$.

   $$(2,4) = 3(1,1) - 1(1,-1)$$
   $$(3,1) = 2(1,1) + 1(1,-1)$$

   The matrix that changes β'-coordiantes into β-coordiante is

   $$Q = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$$

   Thus, for instance,

   $$[(2,4)]_\beta = Q[(2,4)]_{\beta'} = Q\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

3. Linear Operation: Linear transformations that map a vector space V into itself. Such a linear transformation is called a linear operator on V.
4. Theorem 2.23: Let T be a linear operator on a finite-dimenstional vector space V, and let β and β' be ordered bases for V. Suppose the Q is the change of coordinate matrix that changes β'-coordinates into β-coordinates. Then

   $$[T]_{\beta'} = Q^{-1}[T]_\beta Q$$

   Example 2:

   Let T be a linear operator on R^2 defined by

   $$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3a - b \\ a + 3b \end{pmatrix}$$

   and let $\beta = \{(1,1),(1,-1)\}$ and $\beta' = \{(2,4),(3,1)\}$ be the ordered bases.

   $$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
   $$T\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
   $$\Rightarrow [T]_\beta = \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}, Q = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix} (\text{see example 1})$$
   $$\Rightarrow Q^{-1} = \frac{1}{5}\begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$$
   $$\Rightarrow [T]_{\beta'} = Q^{-1}[T]_\beta Q = \begin{pmatrix} 4 & 1 \\ -2 & 2 \end{pmatrix}$$

5. Corollary: Let $A \in M_{n\times n}(F)$, and let γ be an ordered basis for F^n. Then $[L_A]_\gamma = Q^{-1}AQ$, where Q is the $n \times n$ matrix whose jth column is the jth vector of γ.

   Proof:

   $$A \in M_{n\times n}(F): L_A: F^n \to F^n, L_A(x) = x, x = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$$

   $$\underset{\gamma}{\mathrm{F^n}} \xrightarrow{I_{F^n}} \underset{\beta}{\mathrm{F^n}} \xrightarrow{L_A} \underset{\beta}{\mathrm{F^n}} \xrightarrow{I_{F^n}} \underset{\gamma}{\mathrm{F^n}}$$

   $$\Rightarrow L_A = I_{F^n} \circ L_A \circ I_{F^n}$$
   $$\Rightarrow [L_A]_\gamma = [I_{F^n}]_\beta^\gamma [L_A]_\beta [I_{F^n}]_\gamma^\beta$$

   Special case: $\beta = \{e_1, e_2, \dots, e_n\} \Rightarrow [L_A]_\beta = A$

   $$\Rightarrow [L_A]_\gamma = [I_{F^n}]_\beta^\gamma [L_A]_\beta [I_{F^n}]_\gamma^\beta = Q^{-1}AQ$$

   Q changes γ into β

   That is, the jth column of Q is the jth vector of γ

6. Definition: Let A and B be matrices in $M_{n\times n}(F)$. We say that B is similar to A if there exists an invertible matrix Q such that $B = Q^{-1}AQ$.
7. Generalization of Theorem 2.23 (P117): Let $T: V \to W$ be a linear transformation from a finite-dimensional vector space V to a finite-dimensional vector space W. Let β and β' be ordered bases for V, and let γ and γ' be ordered bases for W. Then

$$[T]_{\beta'}^{\gamma'} = P^{-1}[T]_\beta^\gamma Q = [I_W]_\gamma^{\gamma'}[T]_\beta^\gamma[I_V]_{\beta'}^\beta$$

where $Q = [I_V]_{\beta'}^\beta$ is the matrix that changes β'-coordiantes into β-coordinates and

$P = [I_W]_{\gamma'}^\gamma$ is the matrix that changes γ'-coordinates into γ-coordinates.

Proof:

$$\underset{\beta'}{V} \xrightarrow{I_V} \underset{\beta}{V} \xrightarrow{T} \underset{\gamma}{W} \xrightarrow{I_W} \underset{\gamma'}{W}$$

$T: V \to W$

a) β, γ for V, W respectively
b) β', γ' for V, W respectively

$T = I_W \circ T \circ I_V$

$\Rightarrow [T]_{\beta'}^{\gamma'} = [I_W]_\gamma^{\gamma'}[T]_\beta^\gamma[I_V]_{\beta'}^\beta = P^{-1}[T]_\beta^\gamma Q$

Special case: $V = W, \beta = \gamma, \beta' = \gamma'$

$T = I_V \circ T \circ I_V$

$[T]_{\beta'} = [I_V]_\beta^{\beta'}[T]_\beta[I_V]_{\beta'}^\beta = Q^{-1}[T]_\beta Q$

Proofs or Explanations:

1.

Examples:

1. P116(2c) $V = R^2, \beta = \{(2,5), (-1,-3)\}, \beta' = \{(1,0), (0,1)\}$

$Q = [I_{R^2}]_{\beta'}^\beta$

$(x, y) = a \cdot (2,5) + b \cdot (-1,-3) = (2a - b, 5a - 3b) \Rightarrow a = 3x - y, b = 5x - 2y$

$\Rightarrow (x, y) = (3x - y) \cdot (2,5) + (5x - 2y) \cdot (-1,-3)$

$I_{R^2}(1,0) = (1,0) = 3 \cdot (2,5) + 5 \cdot (-1,-3)$

$I_{R^2}(0,1) = (0,1) = -1 \cdot (2,5) - 2 \cdot (-1,-3)$

$\Rightarrow [I_{R^2}]_{\beta'}^\beta = \begin{pmatrix} 3 & -1 \\ 5 & -2 \end{pmatrix} = Q$

$I_{R^2}(2,5) = (2,5) = 2 \cdot (1,0) + 5 \cdot (0,1)$

$I_{R^2}(2,5) = (-1,-3) = (-1) \cdot (1,0) + (-3) \cdot (0,1)$

$\Rightarrow [I_{R^2}]_\beta^{\beta'} = \begin{pmatrix} 2 & -1 \\ 5 & -3 \end{pmatrix} = Q^{-1}$

2. P116(3e) $V = P_2(R), \beta = \{x^2 - x, x^2 + 1, x - 1\}, \beta' = \{5x^2 - 2x - 3, -2x^2 + 5x + 5, 2x^2 - x - 3\}$

$Q = \left[I_{P_2(R)}\right]_{\beta'}^\beta$

$I_{P_2(R)}(5x^2 - 2x - 3) = 5x^2 - 2x - 3 = 5(x^2 - x) + 0(x^2 - 1) + 3(x - 1)$

$I_{P_2(R)}(-2x^2 + 5x + 5) = -2x^2 + 5x + 5 = (-6)(x^2 - x) + 4(x^2 - 1) + (-1)(x - 1)$

$I_{P_2(R)}(2x^2 - x - 3) = 2x^2 - x - 3 = 3(x^2 - x) + (-1)(x^2 - 1) + 2(x - 1)$

$ax^2 + bx + c = r \cdot (x^2 - x) + s \cdot (x^2 - 1) + t \cdot (x - 1) \Rightarrow r = \frac{1}{2}(a - b - c), s = \frac{1}{2}(a + b + c), t = \frac{1}{2}(a + b - c)$

$\Rightarrow ax^2 + bx + c = \frac{1}{2}(a - b - c) \cdot (x^2 - x) + \frac{1}{2}(a + b + c) \cdot (x^2 - 1) + \frac{1}{2}(a + b - c) \cdot (x - 1)$

$$\Rightarrow Q = [I_{R^2}]_{\beta'}^\beta = \begin{pmatrix} 5 & -6 & 3 \\ 0 & 4 & -1 \\ 3 & -1 & 2 \end{pmatrix}$$

3. $T: R^4 \to R^3$

$$\underset{\beta'}{\mathrm{R}^4} \xrightarrow{I_{R^4}} \underset{\beta}{\mathrm{R}^4} \xrightarrow{T} \underset{\gamma}{\mathrm{R}^3} \xrightarrow{I_{R^3}} \underset{\gamma'}{\mathrm{R}^3}$$

$\beta = \{e_1, e_2, e_3, e_4\}, \gamma = \{e_1, e_2, e_3\}$,

$\beta' = \{(1,1,0,0), (0,1,-1,0), (0,0,1,-1), (1,0,0,1)\}, \gamma' = \{(1,1,0), (0,1,1,), (1,0,1)\}$

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 \end{pmatrix}, [T]_{\beta'}^{\gamma'}?$$

$T = I_{R^3} \circ T \circ I_{R^4}$

$$[T]_{\beta'}^{\gamma'} = [I_{R^3}]_\gamma^{\gamma'} [T]_\beta^\gamma [I_{R^4}]_{\beta'}^\beta = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 \end{pmatrix}^{-1} ????$$

Find two invertible matrices P, Q such that $[T]_{\beta'}^{\gamma'} = P[T]_\beta^\gamma Q$

$I_{R^3}(1,0,0) = (1,0,0)$
$I_{R^3}(0,1,0) = (0,1,0)$
$I_{R^3}(0,0,1) = (0,0,1)$
$(x, y, z) = a(1,1,0) + b(0,1,1) + c(1,0,1)$

$$= \frac{1}{2}(x + y - z)(1,1,0) + \frac{1}{2}(-x + y + z)(0,1,1) + \frac{1}{2}(x - y + z)(1,0,1)$$

$$P = [I_{R^3}]_\gamma^{\gamma'} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$I_{R^4}(1,1,0,0) = (1,1,0,0) = e_1 + e_2$

$$Q = [I_{R^4}]_{\beta'}^\beta = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

$$[T]_{\beta'}^{\gamma'} = [I_{R^3}]_\gamma^{\gamma'} [T]_\beta^\gamma [I_{R^4}]_{\beta'}^\beta = P[T]_\beta^\gamma Q$$

4. $T: R^4 \to R^3$

   $[T]_\beta^\gamma \to [T]_{\beta'}^\gamma$

   $T = T \circ I_{R^4}$

   $[T]_{\beta'}^\gamma = [T]_\beta^\gamma [I_{R^4}]_{\beta'}^\beta = [T]_\beta^\gamma Q$

   Q is invertible and changes β' to β

5. $T: R^4 \to R^3$

   $[T]_\beta^\gamma \to [T]_\beta^{\gamma'}$

   $T = I_{R^3} \circ T$

   $[T]_\beta^{\gamma'} = [I_{R^3}]_\gamma^{\gamma'} [T]_\beta^\gamma = Q[T]_\beta^\gamma$, where Q is invertible and changes γ to γ'.

6. P117(6a) $[L_A]_\beta = Q^{-1}AQ$

   $A = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}, \beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$

   $L_A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 6 \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

   $L_A \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix} = 11 \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 4 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

   $[L_A]_\beta = \begin{pmatrix} 6 & 11 \\ -2 & -4 \end{pmatrix}$

   $Q = [I_{R^2}]_\beta^\gamma$ changes β into the standard basis γ.

   $Q = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

3.1 Elementary Matrix Operations and Elementary Matrices

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Definitions & Theorems:

1. Definition: Let A be an $m \times n$ matrix. Any one of the following three operations on the rows [columns] of A is called an elementary row [column] operations.
   (1) Type 1: Interchanging any two rows [columns] of A;
   (2) Type 2: Multiplying any row [column] of A by a nonzero scalar;
   (3) Type 3: Adding any scalar multiple of a row [column] of A to another row [column].
2. Definition: An $n \times n$ elementary matrix is a matrix obtained by performing an elementary operation on I_n. The elementary matrix is said to be of type 1, 2, or 3 according to whether the elementary operation performed on I_n is a type 1, 2, or 3 operation, respectively.
3. Theorem 3.1: Let $A \in M_{m\times n}(F)$, and suppose that
   B is obtained from A by performing an elementary row [column] operation.
   $\Rightarrow$ there exists an $m \times m$ $[n \times n]$ elementary matrix E such that $B = EA$ $[B = AE]$.
   In fact, E is obtained from I_m $[I_n]$ by performing the same elementary row [column] operation as that which was performed on A to obtain B.
   Conversely, if E is an elementary $m \times m$ $[n \times n]$ matrix, then EA $[AE]$ is the matrix obtained from A by performing the same elementary row [column] operation as that which produces E from I_m $[I_n]$.
4. Theorem 3.2 : Elementary matrices are invertible, and the inverse of an elementary matrix is an elementary matrix of the same type.

Proofs or Explanations:

1.

Examples:

1. $A = \begin{pmatrix} 1 & -1 & 0 & 1 \\ -1 & 2 & 1 & 1 \\ 0 & -1 & 1 & 2 \end{pmatrix}$

   Type 1: $R_1 \Leftrightarrow R_2$

   $$B = \begin{pmatrix} -1 & 2 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & -1 & 1 & 2 \end{pmatrix}$$

   $$B = EA, E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   Type 2: $2 \cdot C_2$

   $$B = \begin{pmatrix} 1 & -2 & 0 & 1 \\ -1 & 4 & 1 & 1 \\ 0 & -2 & 1 & 2 \end{pmatrix}$$

   $$B = AE, E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

   $B: R_2 \Leftrightarrow R_3$

   $$C = \begin{pmatrix} 1 & -2 & 0 & 1 \\ 0 & -2 & 1 & 2 \\ -1 & 4 & 1 & 1 \end{pmatrix} = E'B, E' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

   $\Rightarrow C = E'AE$

3.2 The Rank of a Matrix and Matrix Inverses

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Definitions & Theorems:

1. Definition: If $A \in M_{m\times n}(F)$, we define the rank of A, denoted $rank(A)$, to be the rank of the linear transformation $L_A: F^n \to F^m \Rightarrow rank(L_A) = \dim\left(R(L_A)\right)$
2. Theorem 3.3: Let $T: V \to W$ be a linear transformation between finite-dimentional vector spaces, and let β and γ be ordered bases for V and W, respectively. Then $rank(T) = rank([T]_\beta^\gamma)$.
3. Theorem 3.4: Let $A \in M_{m\times n}(F)$. If P and Q are invertible $m \times m$ and $n \times n$ matrices, respectively, then
   (a) $rank(AQ) = rank(A)$
   (b) $rank(PA) = rank(A)$
   (c) $rank(PAQ) = rank(A)$
4. Corollary: Elementary row and column operations on a matrix are rank-preserving.
5. Theorem 3.5: The rank of any matrix equals the maximum number of its linearly independent columns; that is, the rank of a matrix is the dimension of the subspace generated by its columns.

   e.g.:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \Rightarrow rank(A) = \dim\left(span\left(\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right\}\right)\right) = 2$$

6. Theorem 3.6: Let A be an $m \times n$ matrix of rank r. Then $r \le m, r \le n$, and, by means of a finite number of elementary row and column operations, A can be transformed into the matrix

$$D = \begin{pmatrix} I_r & 0_1 \\ 0_2 & 0_3 \end{pmatrix}$$

   where $0_1, 0_2, 0_3$ are zero matrices. Thus $D_{ii} = 1$ for $i \le r$ and $D_{ij} = 0$ otherwise.
7. Corollary 1: Let $A \in M_{m\times n}(F), rank(A) = r$. Then there exist invetible matrices B and C of sizes $m \times n$ and $n \times n$ such that $D = BAC$, where

$$D = \begin{pmatrix} I_r & 0_1 \\ 0_2 & 0_3 \end{pmatrix}$$

   is the $m \times n$ matrix in which $0_1, 0_2, 0_3$ are zero matrices.
8. Corollary 2: Let $A \in M_{m\times n}(F)$, then
   (a) $rank(A^t) = rank(A)$
   (b) The rank of any matrix equals the maximum number of its linear independent rows; that is, the rank

Proofs or Explanations:

1. Theorem 3.5

$$rank(A) = rank(L_A) = \dim\left(R(L_A)\right)$$
$$R(T) = span\left(T(\beta)\right) \Rightarrow$$
$$R(L_A) = span\left(L_A(\beta)\right), \beta = \{e_1, e_2, \dots, e_n\}$$
$$R(L_A) = span(\{L_A(e_1), L_A(e_2), \dots, L_A(e_n)\}) = span(\{Ae_1, Ae_2, \dots, Ae_n\}) = span(\{C_1^A, C_2^A, \dots, C_n^A\})$$

Examples:

1. 5(d), P166

$$A = \begin{pmatrix} 0 & -2 & 4 \\ 1 & 1 & -1 \\ 2 & 4 & 5 \end{pmatrix}$$

$$A|I_3 = \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & -2 & 4 & 1 & 0 & 0 \\ 2 & 4 & 5 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & -2 & -1/2 & 0 & 0 \\ 2 & 4 & 5 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & -2 & -1/2 & 0 & 0 \\ 0 & 4 & 5 & 0 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & -2 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right)$$

3.3 Systems of Linear Equations — Theoretical Aspects

September 6, 2017 11:05

Definitions & Theorems:

1. The system of equations, where a_{ij} and b_i are scalars in a field F and $x_1, x_2, \dots, x_n$ are n variables taking values in F, is called a system of m linear equations in n unkonws over the filed F.
   $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$
   $a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_m$
   …
   $a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$
2. The $m \times n$ matrix A is called the coefficient matrix of the system.
   $$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$
   $$x = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}, b = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix}$$
   $Ax = b, A \in M_{m \times n} \Leftrightarrow L_A(x) = b, L_A: F^n \to F^m$
3. A solution to the system is an n-tuple such that $As = b$.
   $$s = \begin{pmatrix} s_1 \\ s_2 \\ \dots \\ s_n \end{pmatrix} \in F^n$$
4. The set of all solutions to the system is called the solution set of the system. System is called consistent if its solution is nonempty; otherwise it is called non-consistent.
5. Definition: A system $Ax = b$ of m linear equation in n unknows is said to be homogeneous if $b = 0$. Otherwsie the system is said to be nonhomogeneous.
6. Remark: Any homogeneous system has at least one solution, namely, the zero vector $S = (0,0,\dots,0)^T$.
7. Theorem 3.8: Let $Ax = 0$ be a homogeneous system of m linear equations in n unknows over a filed F. Let S_H denote the set of all solutions to $Ax = 0$. Then $S_H = N(L_A)$; hence
   S_H is a subspace of F^n and therefore,
   $\dim(S_H) = \dim\left(N(L_A)\right) = n - rank(L_A) = n - rank(A)$.
   Proof: $S_H = \{S \in F^n \text{ such that } As = 0\} = \{s \in F^n \text{ such that } L_A(s) = 0\} = N(L_A)$
8. Corollary: If $m < n$, then the system $Ax = 0$ has a nonzero solution.
   Proof:
   1) $Ax = 0 \Leftrightarrow L_A(x) = 0$
   2) $L_A: F^n \to F^m$
   3) if $n > m$
   $\Rightarrow L_A$ is not one-to-one$\Rightarrow N(L_A) \neq \{0_V\}$
   $\Rightarrow S_H \neq \{0_V\}$
9. Theorem 3.9: Let S_N be the solution set of the non-homogeneous system $Ax = b, S_H$ be the solution set of the corresponding homogeneous system $Ax = 0$. Then
   if s is a particular solution of the non-homogeneous system, then
   $S_N = s + S_H = \{s + k : k \in S_H\}$
10. Theorem 3.10: Let $Ax = b$ be a system of n linear equations in n unknows.
    The system has q unique solution $\Leftrightarrow$ A is invertible
    Moreover, the solution is $x = A^{-1}b$.
    Proof: The system has a unique solution
    $\Leftrightarrow$ the equation $Ax = b$ has a unique solution
    $\Leftrightarrow$ they equation $L_A(x) = b$ has a unique solution.
11. Theorem 3.11: Let $Ax = b$ be a system of linear equations. Then
    the system is consistent $\Leftrightarrow rank(A) = rank(A|b)$
    Proof: the system is consistent
    $\Leftrightarrow$ the solution is non-empty
    $\Leftrightarrow$ there exists at least one solution to the system
    $\Leftrightarrow$ there exists $s \in F^n$ such that $As = b$
    $\Leftrightarrow$ there exists $s \in F^n$ such that $L_A(s) = b$
    $\Leftrightarrow b \in R(L_A) = span(\{C_1^A, C_2^A, \dots, C_n^A\}) = span(\{C_1^A, C_2^A, \dots, C_n^A, b\})$
    $\Leftrightarrow rank(A) = rankd(A|b)$

Proofs or Explanations:

1.

Examples:

1. One solution

$$\begin{cases} x_1 + x_2 = 3 \\ x_1 - x_2 = 1 \end{cases} \Rightarrow \begin{cases} x_1 = 2 \\ x_2 = 1 \end{cases} \Rightarrow s = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

2. Many solution

$$\begin{cases} 2x_1 + 3x_2 + x_3 = 3 \\ x_1 - x_2 + 2x_3 = 6 \end{cases} \Rightarrow \begin{cases} x_1 = \\ x_2 = \\ x_3 = \end{cases} \Rightarrow s = \begin{pmatrix} -6 \\ 2 \\ 7 \end{pmatrix} \; and \; S = \begin{pmatrix} 8 \\ -4 \\ -3 \end{pmatrix}$$

3. No solution

$$\begin{cases} -x_1 + x_2 = 0 \\ x_1 - x_2 = 1 \end{cases}$$

4. S

$$\begin{cases} x_1 + 2x_2 + x_3 = 0 \\ x_1 - x_2 - x_3 = 0 \end{cases} \Rightarrow A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & -1 \end{pmatrix} \Rightarrow rank(A) = 2, \dim(S_H) = 3 - 2 = 1$$

$\Rightarrow$ any nonzero solution constitutes a basis for K. For example,

$$s = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

$\Rightarrow \left\{ \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \right\}$ is a solution to the given system.

$$\Rightarrow S_H = t \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, t \in \mathbb{R}$$

5. $\begin{cases} x_1 + 2x_2 + x_3 = 7 \\ x_1 - x_2 - x_3 = -4 \end{cases} \Rightarrow s = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} \Rightarrow S_N = \left\{ \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, t \in \mathbb{R} \right\}$

3.4 Systems of Linear Equations — Computational Aspects

September 6, 2017 11:05

Definitions & Theorems:

1. Definition: Two systems of linear equations are called equivalent if they have the same solution set.
2. Theorem 3.13: Let $Ax = b$ be a system of m linear equations in n unknows, and let C be an invertible $m \times m$ matrix.
   Then the system $(CA)x = Cb$ is equivalent to $Ax = b$.
3. Corollary: Let $Ax = b$ be a system of m linear equations in n unknows. If $(A'|b')$ is obtained from $(A|b)$ by a finite number of elementary row operations, then tye system $A'x = b'$ is equivalent to the original system.
4. Computational aspects
   1) Transform the augmented matrix $(A|b)$ to its RREF, that is $(A'|b')$
   2) If there is a row in the RREF in which the only non-zero element lies in the last column, then the system is inconsistent; otherwise the system is consistent and discard any zero rows of $(A'|b')$. Then write the corresponding equations.
   3) Divide the variable into 2 sets. Set 1 contains leftmost variable, Set 2 has the remaining variables. Then the general solution of the system is given by the following theorem.
5. Definition: A matrix is said to be in reduced row echelon form if the following three conditions are satisfied.
   (a) Any row containing a nonzero entry precedes any row in which all the entries are zero (if any).
   (b) The first nonzero entry in each row is the only nonzero entry in its column.
   (c) The first nonzero entry is each row is 1 and it occurs in a column to the right of the first nonzero entry in the preceding row.
6. Theorem 3.14: Gaussian elimination (P183-185) transforms any matrix into its reduced row echelon form.
   (a) In the forward pass, the augmented matrix is transformed into an upper triangular matrix in which the first nonzero entry of each row is 1, and it occurs in a column to the right of the first nonzero entry of each preceding row.
   (b) In the backward pass or back-substitution, the upper triangular matrix is transformed into reduced echelon form by making the first nonzero entry of each row the only nonzero entry of its column.

   e.g.:

$$\begin{cases} 2x_1 + 3x_2 + x_3 + 4x_4 - 9x_5 = 17 \\ x_1 + x_2 + x_3 + x_4 - 3x_5 = 6 \\ x_1 + x_2 + x_3 + 2x_4 - 5x_5 = 8 \\ 2x_1 + 2x_2 + 2x_3 + 3x_4 - 8x_5 = 14 \end{cases}$$

$$\left(\begin{array}{ccccc|c} 2 & 3 & 1 & 4 & -9 & 17 \\ 1 & 1 & 1 & 1 & -3 & 6 \\ 1 & 1 & 1 & 2 & -5 & 8 \\ 2 & 2 & 2 & 3 & -8 & 14 \end{array}\right) \to \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -3 & 6 \\ 2 & 3 & 1 & 4 & -9 & 17 \\ 1 & 1 & 1 & 2 & -5 & 8 \\ 2 & 2 & 2 & 3 & -8 & 14 \end{array}\right) \to \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -3 & 6 \\ 0 & 1 & -1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 1 & -2 & 2 \end{array}\right)$$

$$\to \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -3 & 6 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & -4 & 5 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -2 & 3 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \begin{cases} x_1 + 2x_3 - 2x_5 = 3 \\ x_2 - x_3 + x_5 = 1 \\ x_4 - 2x_5 = 2 \end{cases} \Rightarrow \begin{cases} x_1 = -2x_3 + 2x_5 + 3 \\ x_2 = x_3 - x_5 + 1 \\ x_4 = 2x_5 + 2 \end{cases}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2x_3 + 2x_5 + 3 \\ x_3 - x_5 + 1 \\ x_3 \\ 2x_5 + 2 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 2 \\ -1 \\ 0 \\ 2 \\ 1 \end{pmatrix}$$

$\Rightarrow \left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \\ 2 \\ 1 \end{pmatrix} \right\}$ is a basis for the solution set the corresponding homogeneous system of equations.

$\begin{pmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}$ is a particular solution to the original system.

7. Theorem 3.15: Assume that the system $Ax = b$ is consistent
   $\Leftrightarrow rank(A) = rank(A|b) = r =$ the number of non-zero rows in its RREF
   Then, the general solution can be written as
   $$s = s_0 + t_1 u_1 + t_2 u_2 + \cdots + t_{n-r} u_{n-r}$$
   where
   s_0 is a particular to the non-homogeneous $Ax = b$,
   the variable in Set 1 are $x_1, x_2, \ldots, x_r$ and
   the variables in Set 2 are $x_{r+1}, x_{r+2}, \ldots, x_n$ $(x_{r+1} = t_1, x_{r+2} = t_2, \ldots, x_n = t_{n-r})$
   $\{u_1, u_2, \ldots, u_{n-r}\}$ is a basis for the solution set of the corresponding homogeneous system.
8. Theorem 3.16: Let A be an $m \times n$ matrix of rank r, where $r > 0$, and let B be the reduced row echelon form of A. Then
   (a) The number of nonzero rows in B is r.
   (b) For each $i = 1,2, \ldots, r$, there is a column b_{j_i} of B such that $b_{j_i} = e_i$, more precisely, there exists $b_{j_1} = e_1, b_{j_2} = e_2, \ldots, b_{j_r} = e_r$,

(c) The columns of A numbered $j_1, j_2, \dots, j_r$ are linearly independent.

(d) For each $k = 1,2,\dots,n$, if column k of B is $d_1e_1 + d_2e_2 + \cdots + d_re_r$, then column k if A is $d_1a_{j1} + d_2a_{j2} + \cdots + d_ra_{jr}$.

$C_k^B = d_1e_1 + d_2e_2 + \cdots + d_re_r$

$C_k^A = d_1C_{j_2}^A + d_1C_{j_2}^A + \cdots + d_rC_{j_r}^A$

Notice that their coefficients are same.

9. Corollary: The reduced row echelon form of a matrix is unique.

Example 2:

$$A = \begin{pmatrix} 2 & 4 & 6 & 2 & 4 \\ 1 & 2 & 3 & 1 & 1 \\ 2 & 4 & 8 & 0 & 0 \\ 3 & 6 & 7 & 5 & 9 \end{pmatrix}$$

$\Rightarrow$ the reduced row echelon form of A is

$$B = \begin{pmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ B has three nonzero rows $\Rightarrow rank(A) = 3$.

$\{C_1^B, C_3^B, C_5^B\}$ are e_1, e_2 and $e_3 \Rightarrow \{C_1^A, C_3^A, C_5^A\}$ are linearly independent.

Proofs or Explanations:

1.

Examples:

Applications:

1.

$$B = \text{RREF of } A = \begin{pmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Find A such that the columns 1, 3 and 5 are $\begin{pmatrix} 2 \\ 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 6 \\ 3 \\ 8 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 0 \\ 9 \end{pmatrix}$

$rank(A) = 3 = r$

By RREF, $\{C_1^A, C_3^A, C_5^A\}$ are linearly independent.

$C_2^B = 2e_1 = 2C_1^B \Rightarrow C_2^A = 2C_1^A$

$C_4^B = 4e_1 - e_2 = 4C_1^B - C_2^B \Rightarrow C_4^A = 4C_1^A - C_2^A$

$$\Rightarrow A = \begin{pmatrix} 2 & 4 & 6 & 2 & 4 \\ 1 & 2 & 3 & 1 & 1 \\ 2 & 4 & 8 & 0 & 0 \\ 3 & 6 & 4 & 5 & 9 \end{pmatrix}$$

2. Reduce any finite generating set to a basis (Find a maximal linearly independent subset)

$S = \{2 + x + 2x^2 + 3x^3, 4 + 2x + 4x^2 + 6x^3, 6 + 3x + 8x^2 + 7x^3, 2 + x + 5x^3, 4 + x + 9x^3\}$ generates a subspace V of $P_3(R)$.

Reduce S to a basis for V.

$$S' = \left\{ [P_1(x)]_\gamma = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 3 \end{pmatrix}, [P_2(x)]_\gamma = \begin{pmatrix} 4 \\ 2 \\ 4 \\ 6 \end{pmatrix}, [P_3(x)]_\gamma = \begin{pmatrix} 6 \\ 3 \\ 8 \\ 7 \end{pmatrix}, [P_4(x)]_\gamma = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 5 \end{pmatrix}, [P_5(x)]_\gamma = \begin{pmatrix} 4 \\ 1 \\ 0 \\ 9 \end{pmatrix} \right\}$$

$$A = \begin{pmatrix} 2 & 4 & 6 & 2 & 4 \\ 1 & 2 & 3 & 1 & 1 \\ 2 & 4 & 8 & 0 & 0 \\ 3 & 6 & 7 & 5 & 9 \end{pmatrix} \xrightarrow{\frac{1}{2}R_1, \frac{1}{2}R_3} \begin{pmatrix} 1 & 2 & 3 & 1 & 2 \\ 1 & 2 & 3 & 1 & 1 \\ 1 & 2 & 4 & 0 & 0 \\ 3 & 6 & 7 & 5 & 9 \end{pmatrix}$$

$$\xrightarrow{R_2 - R_1, R_3 - R_1, R_4 - 3R_1} \begin{pmatrix} 1 & 2 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & -2 & 2 & 3 \end{pmatrix} \xrightarrow{2R_3 + R_4} \begin{pmatrix} 1 & 2 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$\Rightarrow \{C_1^A, C_3^A, C_5^A\}$ is linearly independent$\Rightarrow \{P_1(x), P_3(x), P_5(x)\}$ is linear independent.

3. Extend a linear independent subset S to a basis for V.

Step 1. Find an explicit basis β for V.

Step 2. Consider the ordered set S' consisting of the vectors in S followed by the vectors in β. Then S' generates V.

Proof: $\beta \subseteq S' \Rightarrow span(\beta) \subseteq span(S')$

$V \subseteq span(S), but\ span(S') \subseteq V \Rightarrow span(S') = V$

Step 3. Reduce S' to a basis.

e.g.:

$V = \{(x_1, x_2, x_3, x_4, x_5) \in R^5 : 2x_1 - x_3 + x_4 - 3x_5 = 0\}$ is a vector space over R.

a) Show that $S = \{(1,0,0,-2,0), (0,1,0,3,1)\}$ is a linearly indepdendent set in V.

v_1 and v_2 are not multiples of each other $\Rightarrow S$ is linearly independent.

$v_1 \in V, v_2 \in V$

$\Rightarrow S$ is a linearly independent set in V.

b) Extend S to a full basis for V.

Step 1:

$$V = \left\{\left(x_1, x_2, x_3, x_4, \frac{2}{3}x_1 - \frac{1}{3}x_3 + \frac{1}{3}x_4\right), x_1, x_2, x_3, x_4 \in R\right\}$$

$$= \left\{x_1\left(1,0,0,0,\frac{2}{3}\right) + x_2(0,1,0,0,0) + x_3\left(0,0,1,0,-\frac{1}{3}\right) + x_4\left(0,0,0,1,\frac{1}{3}\right)\right\}$$

$$= span\left\{\left(1,0,0,0,\frac{2}{3}\right), (0,1,0,0,0), \left(0,0,1,0,-\frac{1}{3}\right), \left(0,0,0,1,\frac{1}{3}\right)\right\} = span\{w_1, w_2, w_3, w_4\}$$

Check linearly independent:

$$x_1\left(1,0,0,0,\frac{2}{3}\right) + x_2(0,1,0,0,0) + x_3\left(0,0,1,0,-\frac{1}{3}\right) + x_4\left(0,0,0,1,\frac{1}{3}\right) = 0 \Rightarrow x_1 = x_2 = x_3 = x_4 = 0$$

Step 2:

$S' = \{v_1, v_2, w_1, w_2, w_3, w_4\}$ generates V.

Step 3:

S' can be reduced to a basis for V.

$\dim(V) = 4$

$[v_1]_\gamma, [v_2]_\gamma, [w_1]_\gamma, [w_2]_\gamma, [w_3]_\gamma, [w_4]_\gamma, \gamma = \{e_1, e_2, e_3, e_4, e_5\}$

$$\Rightarrow A = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \\ 0 & 1 & \frac{2}{3} & 0 & -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & \frac{3}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow basis = \{v_1, v_2, w_1, w_3\}$

4.

4.4 Summary — Important Facts about Determinants

September 6, 2017 11:05

Definitions & Theorems:

1. Definition: Let $A \in M_{n\times n}(F), n > 2$. Then the determinant of A: $\det(A)$ can be evaluated by expanding along the ith row or jth column.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

   a) expanding along the ith row.

$$\det(A) = a_{i1}(-1)^{i+1}\det(\tilde{a}_{i1}) + a_{i2}(-1)^{i+2}\det(\tilde{a}_{i2}) + \cdots + a_{in}(-1)^{i+n}\det(\tilde{a}_{in})$$

   b) expanding along the jth column.

$$\det(A) = a_{1j}(-1)^{1+j}\det\left(\tilde{a}_{1j}\right) + a_{2j}(-1)^{2+j}\det(\tilde{a}_{2j}) + \cdots + a_{nj}(-1)^{n+j}\det(\tilde{a}_{nj})$$

   $\tilde{a}_{ij}$ is a $(n-1)\times(n-1)$ matrix obtained from A by removing row i and column j

   $(-1)^{i+j}\det\left(\tilde{a}_{ij}\right)$ is the cofactor of A.

2. Determinants:
   a) $n = 1 \Rightarrow A = A_{11} \Rightarrow \det(A) = A_{11}$
   b) $n = 2 \Rightarrow \det(A) = A_{11}A_{22} - A_{12}A_{21}$
   c) $n = 3 \Rightarrow \det(A) = A_{i1}(-1)^{i+1}\det(\tilde{A}_{i1}) + A_{i2}(-1)^{i+2}\det(\tilde{A}_{i2}) + A_{i3}(-1)^{i+3}\det(\tilde{A}_{i3})$
3. Recall: We say that A and B are similar if there exists an invertible matrix Q such that
   $B = Q^{-1}AQ \Leftrightarrow QBQ^{-1} = A$
4. Properties of the Determinant
   a) $\det(A) = \det(A^t)$
   b) If B is a matrix obtained by interchanging any two rows or interchanging any two columns of an $n\times n$ matrix A, then $\det(B) = -\det(A)$
   c) If B is a matrix obtained by multiplying each entry of some row or column of an $n\times n$ matrix A by a scalar k, then $\det(B) = k\cdot\det(A)$

$$\det\begin{pmatrix} 2 & 4 & -2 \\ 2 & 6 & 10 \\ -2 & 8 & 2 \end{pmatrix} = 2\cdot\det\begin{pmatrix} 1 & 2 & -1 \\ 2 & 6 & 10 \\ -2 & 8 & 2 \end{pmatrix} = 2\cdot 2\cdot\det\begin{pmatrix} 1 & 2 & -1 \\ 1 & 3 & 5 \\ -2 & 8 & 2 \end{pmatrix}$$

   d) If B is a matrix obtained from an $n\times n$ matrix A by adding a multiple of row i to row j ro a multiple of comumn i to column j for $i \neq j$, then $\det(B) = \det(A)$.
   e) A has 2 identical rows or 2 identical columns, then $\det(A) = 0$.
   f) The determinant of an upper triangular matrix is the product of its diagonal entries. In particular, $\det(I) = 1$.
   g) For any $n\times n$ matrices A and B, $\det(AB) = \det(A)\cdot\det(B)$.
   h) An $n\times n$ matrix A is invertible $\Leftrightarrow \det(A) \neq 0 \Leftrightarrow rank(A) = n$.

$$A\ is\ invertible\ \Rightarrow \det\left(A^{-1}\right) = \frac{1}{\det(A)}$$

   i) If A and B are similar matrices, then $\det(A) = \det(B)$.
5. Computing A^{-1}, method2
   a) Theorem: if A is invertible, then $\det(A) \neq 0$ and $A^{-1} = \frac{1}{\det(A)}\cdot\tilde{A}$.

   $\tilde{A}$ is a classical adjoint matrix and is the transpose of the matrix of cofactors of A. That is, $\tilde{A} = C^T$, where $C = \left(C_{ij}\right)_{ij}, C_{ij} = (-1)^{i+j}\cdot\det(\tilde{A}_{ij})$

Proofs or Explanations:

1.

Examples:

1. ?

$$\det\begin{pmatrix} 1 & 0 & 0 & 0 \\ a & b-a & c-a & d-a \\ a^2 & b^2-a^2 & c^2-a^2 & d^2-a^2 \\ a^3 & b^3-a^3 & c^3-a^3 & d^3-a^3 \end{pmatrix} = 1\cdot(-1)^{1+1}\cdot\det\begin{pmatrix} b-a & c-a & d-a \\ b^2-a^2 & c^2-a^2 & d^2-a^2 \\ b^3-a^3 & c^3-a^3 & d^3-a^3 \end{pmatrix}$$

$$= (b-a)(c-a)(d-a) \cdot \det\begin{pmatrix} 1 & 1 & 1 \\ b-a & c-a & d-a \\ d^2+da+a^2 & c^2+ca+a^2 & d^2+da+a^2 \end{pmatrix} =$$

$$= (b-a)(c-a)(d-a)(c-b)(\square)$$

2. ?

$$\det\begin{pmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{pmatrix} = (b-a)(c-a)(d-a)(c-b)(d-b)(d-c)$$

5.1 Eigenvalues and Eigenvectors

September 6, 2017 11:05

Definitions & Theorems:

1. Definition: Let V be a finite-dimensional vector space over filed F and let $T: V \to V$ be linear. We say that T is diagonalizable if there exists an ordered basis β such that $[T]_\beta = diagonal(P246)$
2. Definition: Let $A \in M_{n\times n}(F)$. Then A is diagonalizable if $L_A: F^n \to F^n$ is diagonalizable. That is, there exists an ordered basis β such that $[L_A]_\beta = diagonal$.
3. Definition: $T: V \to V$ be linear. A non-zero vector v in V is an eigenvector of T if there exists λ in F such that $T(v) = \lambda v$ and $\lambda = eigenvalue\ corresponding\ to\ the\ eigenvector$
4. Definition: Let $A \in M_{n\times n}(F)$. A non-zero vector v in F^n is an eigenvector of A if there exists λ in F such that $L_A(v) = \lambda \cdot v \Rightarrow L_A(v) = Av = \lambda v$.
5. Summary:
   a) $T: V \to V$ is linear; v = eigenvector if $v \neq 0$ and $T(v) = \lambda v$
   b) $A \in M_{n\times n}$; v = eigenvector of A if $v \neq 0$ and $L_A(v) = Av = \lambda v$
6. Remark:
   a) A vector v is an eigenvector of $A \Leftrightarrow v$ is eigenvector of L_A
   b) λ is an eigenvalue of $A \Leftrightarrow \lambda$ is an eigenvalue of L_A
7. Theorem 5.2: Let $A \in M_{n\times n}(F)$. Then
   a scalar λ is an eigenvalue of $A \Leftrightarrow \det(A - \lambda I_n) = 0$
8. Definition: Let $A \in M_{n\times n}(F)$. The polynomial $f(t) = \det(A - tI_n)$ is called the characteristic polynomial of A.
9. Definition: Let T be a linear operator on an n-dimensional vector space V with ordered basis β. We define the characteristic polynomial $f(t)$ of T to be the characteristic polynomial of $A = [T]_\beta$. That is, $f(t) = \det(A - tI_n)$.
10. Theorem 5.4
11. v is eigenvecctor of $A \Leftrightarrow L_A(v) = Av = \lambda v \Leftrightarrow Av - \lambda v = 0 \Leftrightarrow (Av - \lambda I_n)(v) = 0 \Leftrightarrow (A - \lambda I_n)(v) = 0 \Leftrightarrow L_{A-\lambda I_n}(v) = 0 \Rightarrow v \in N\left(L_{A-\lambda I_n}\right)$
12. Theorem: Let $T: V \to V$ be linear and finite-dimensional. Then,
   T is diagonalizable$\Leftrightarrow$ there exists an ordered basis β for V consisting of $eigenvectors$ of T.
   If the matrix $[T]_\beta$ is a diagonal matrix, where each entry is eigenvalue corresponding to the eigenvector v_i.

Proofs or Explanations:

1.

Examples:

1.

5.2 Diagonalizability

September 6, 2017 11:05

Definitions & Theorems:

1. Theorem 5.5: Let T be a linear operator on a vector space V, and let $\lambda_1, \lambda_2, \dots, \lambda_k$ be distinct eigenvalues of T. If $v_1, v_2, \dots, v_k$ are eigenvectors of T such that λ_i corresponding to v_i, then $\{v_1, v_2, \dots, v_k\}$ is linearly independent.
   Proof (Final exam)
2. $A \in M_{n\times n}(F)$ with n distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$ and the corresponding eigenvectors $v_1, v_2, \dots, v_k$. Then $\{v_1, v_2, \dots, v_k\}$ is linear independent.
   Proof:
   1) by Induction on k
   $k = 1: \{v_1\}, v_1 \neq 0 \Rightarrow linear\ independent$
   $k = 2: \{v_1, v_2\}$
   $a_1 v_1 + a_2 v_2 = 0_V$
   $T(a_1 v_1 + a_2 v_2) = T(0_V) \Rightarrow a_1 T(v_1) + a_2 T(v_2) \Rightarrow a_1 \lambda_1 v_1 + a_2 \lambda_2 v_2 = 0_V$
   $-a_1 \lambda_2 v_1 + a_1 \lambda_1 v_1 = 0 \Rightarrow a_1(\lambda_1 - \lambda_2) v_1 = 0 \Rightarrow a_1 = 0$
3. Corollary:
   a) Let T be a linear operator on an n-dimensional vector space V. If T has n distinct eigenvalues, then T is diagonalizable.
   b) Let $A \in M_{n\times n}(F)$. If A has n distinct eigenvalues, then $A(L_A)$ is diagonalizable.
   Remark: The converse statement is not always true.
   Example:
   $T: I_V, V \to V; \dim(V) = n$
   $[I_v]_\beta = I_n \Rightarrow I_V$ is diagonalizable.
   $f(\lambda) = \det\left([I_v]_\beta - \lambda I_n\right) = \det(I_n - \lambda I_n) = \det((1-\lambda) I_n) = (1-\lambda)^n = 0 \Rightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_n = 1$
4. Definition: A polynomial $f(t)$ in $P(F)$ splits over F if there are scalars $c, a_1, a_2, \dots, a_n$ (not necessarily distinct) in F such that
   $f(t) = c(t - a_1)(t - a_1) \dots (t - a_n)$
   e.g.:
   $f(t) = t^2 - 4 = (t-2)(t+2)$ splits over R
   $f(t) = (t-1)(t^2+4) = (t-1)(t+2i)(t-2i)$ splits over C but not R
5. Theorem 5.6: The characteristic polynomial of any diagonalizable linear operator splits.
6. Definition: Let λ be an eigenvalue of a linear operator or matrix with characteristic polynomial $f(t)$. The (algebraic) multiplicity of λ is the largest positive integer k for which $(t-\lambda)^k$ is a factor of $f(t)$.
   e.g.:
   $f(\lambda) = (\lambda-1)^3(\lambda+1)^2(\lambda-3), k_1 = 3, k_2 = 2, k_3 = 1$
7. Definition: Let T be a linear operator on a vector space V, and let λ be an eigenvalue of T. Define $E_\lambda = \{v \in V: T(v) = \lambda v\} = \{v \in V: (T - \lambda I_V)v = 0_V\} = N(T - \lambda I_V)$. The set E_λ is called the eigenspace of T corresponding to the egiengvalue λ.
   Analogously, we define the eigenspace of a square matrix A to be the eigenspace of L_A.
   Let $A \in M_{n\times n}(F)$, and let λ be an eigenvalue of A. Define $E_\lambda = \{v \in F^n: Av = \lambda v\} = \{v \in F^n: (A - \lambda I_V)v = 0_V\} = N(L_{A-\lambda I_V})$, which is a subspace of F^n. The set E_λ is called the eigenspace of A corresponding to the egiengvalue λ.
8. Theorem 5.7: Let T be a linear operator on a finite-dimensional vector space V, and let λ be an eigenvalue of T having multiplicity m. Then $1 \leq \dim(E_\lambda) \leq m$.
9. Theorem 5.8: Let T be a linear operator on a vector space V, and let $\lambda_1, \lambda_2, \dots, \lambda_k$ be distinct eigenvalues of T. For each $i = 1,2,\dots,k$, let S_i be a finite linearly independent subset of the eigenspace E_λ. Then $S = S_1 \cup S_2 \cup \cdots \cup S_k$ is linear independent subset of V.
10. Theorem 5.9: Let T be a linear operator on a finite-dimensional vector space V such that the characteristic polynomial of T splits. Let $\lambda_1, \lambda_2, \dots, \lambda_k$ be the distinct eigenvalues of T. Then
    a) T is diagonalizable if and only if the multiplicity of λ_i is equal to $\dim(E_{\lambda_i})$ for all i.
    b) If T is diagonalizable and β_i is an ordered basis for E_{λ_i} for each i, then $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ is an ordered bais for V consisting of eigenvectors of T.
11. Test for Diagonalization: Let T be a linear operator on an n-dimensional vector space V. Then T is diagonalizable if and only if both of the following conditions hold.
    a) The characteristic polynomial of T splits.
    b) For each eigenvalue λ of T, the multiplicity of λ equals $n - rank(T - \lambda I) = \dim(E_\lambda) = \dim\left(N(T - \lambda I_V)\right)$

Proofs or Explanations:

1.

Examples:

1.

Midterm-2014

October 18, 2017 11:50

1. $span(A \cap B) \subseteq span(A) \cap span(B)$

   $span(A \cup B) \nsubseteq span(A) \cup span(B)$

   $A = \{(1,0)\} \Rightarrow span(A) = \{(a,0), a \in \mathbb{R}\}$

   $B = \{(0,1)\} \Rightarrow span(B) = \{(0,b), b \in \mathbb{R}\}$

   $span(A \cup B) = \{a(1,0) + b(0,1)\} = \{(a,b), a,b \in \mathbb{R}\} = \mathbb{R}^2$
2. $W = \{f(x) \in P_2(\mathbb{R}) \big| f(5) = 0\}$
   a. Subspace
      i. $0_P \in P_2(\mathbb{R})$
      ii. $f(x), g(x) \in W$

         $(f+g)(x) = f(x) + g(x) = 0$
      iii. $f(x) \in W, c \in \mathbb{R}$

         $(cf)(x) = c \cdot f(x) = 0$
   b. basis

      $f(x) = a + bx + cx^2$

      $f(5) = a + 5b + 25c = 0$

      $\Rightarrow W = \{f(x) = -5b - 25c + bx + cx^2, b,c \in \mathbb{R} | f(5) = 0\}$

      $W = \{b(x-5) + c(x^2 - 25), b,c \in \mathbb{R}\} = span\left((x-5), (x^2-25)\right)$

   If:

   $W = \{f(x) \in P_4(\mathbb{R}) \big| f(1) = 0, f(2) = 0\}$

   $f(x) = a + bx + cx^2 + dx^3 + ex^4$

   $f(1) = a + b + c + d + e = 0$

   $f(2) = a + 2b + 4c + 8d + 16e = 0$
3. $T: P_2(\mathbb{R}) \to \mathbb{R}$
   a. $N(T)$

      $f(x) \in N(T) \Rightarrow T(f(x)) = 0$

      $f(x) = a + bx + cx^2$

      $T(f(x)) \Rightarrow \int_0^1 f(x)\,dx = \int_0^1 (a + bx + cx^2)\,dx = a + \frac{b}{2} + \frac{c}{3}$

      $\Rightarrow f(x) = -\frac{b}{2} - \frac{c}{3} + bx + cx^2$

      $\Rightarrow N(T) = \left\{f(x) = -\frac{b}{2} - \frac{c}{3} + bx + cx^2 = b\left(x - \frac{1}{2}\right) + c\left(x^2 - \frac{1}{3}\right), b,c \in \mathbb{R}\right\}$

      $= span\left(x - \frac{1}{2}, x^2 - \frac{1}{3}\right)$

      Check linear independence.

      $\Rightarrow \left\{x - \frac{1}{2}, x^2 - \frac{1}{3}\right\}$ *basis*

      $\Rightarrow \dim(N(T)) = 2$
   b. $rank(T)$

      $\dim(P_2(\mathbb{R})) = \dim(N(T)) + rank(T) \Rightarrow 3 = 2 + rank(T)$

      $\Rightarrow rank(T) = 1 = \dim(\mathbb{R}) \Rightarrow \text{onto}$

      If not onto

      $R(T) = span\left(T(\beta)\right)$
4. $T: \mathbb{R}^2 \to \mathbb{R}^2$
   a. $\{(1,1), (1,-1)\} \Rightarrow (x,y) = a(1,1) + b(1,-1) \Rightarrow a = \left(\frac{x+y}{2}\right), b = \left(\frac{x-y}{2}\right)$

      $\Rightarrow (x,y) = \left(\frac{x+y}{2}\right)(1,1) + \left(\frac{x-y}{2}\right)(1,-1)$

5. $\{u, v, w\}$ is linearly independent.

$$a(u+v)+b(v+w)+c(w+u)=0_V$$
$$(a+c)u+(a+b)v+(b+c)w=0_V$$
$$\Rightarrow \begin{cases} a+c=0 \\ a+b=0 \\ b+c=0 \end{cases}$$

Midterm-2015

October 18, 2017 12:11

1. 4

   $T: V \to W$
   $\dim(V) < \dim(W) \Rightarrow T\ not\ onot$
   $\dim(V) > \dim(W) \Rightarrow T\ not\ one - to - one$

   $rank(T) = \dim(V) - nullity(T) \leq \dim(V) < \dim(W)$
   $nullity(T) = \dim(V) - rank(T) \geq \dim(V) - \dim(W) > 0$
2. 6
   a. $T\big(p(x) + g(x)\big) = T\big(p(x)\big) + T\big(g(x)\big)$
   b. $T\big(c \cdot p(x)\big) = c \cdot T\big(p(x)\big)$
3. 7

   $R(T + U) \subseteq R(T) + R(U)$
   Let $w \in R(T + U) \Rightarrow w = (T + U)(v) for\ some\ v \in V$
   $\Rightarrow w = T(v) + U(v)$
   $T(v) \in R(T), U(v) \in R(U)$
   $\Rightarrow w \in R(T) + R(U)$

   1.2 or 1.3
   $e^{rt}, e^{st}, r \neq s \Rightarrow \{e^{rt}, e^{st}\}\ is\ linearly\ independent$
4. 3

   $ae^{x} + be^{3x} + cx(x - 1) = 0$
   $x = 0 \Rightarrow a + b = 0$
   $x = 1 \Rightarrow ae + be^{3} = 0$
   $x = 2 \Rightarrow ae^{2} + be^{6} + 2c = 0$

Midterm-2017

Wednesday, November 29, 2017 6:03 PM

2. Let V and W be fiite-dimensional vectors spaces over a field F.
   a) Let $T: V \to W$ be linear and $\dim(V) = \dim(W)$. Show that T is onto if and only if T is one-to-one.

      $\Rightarrow$

      T is onto $\Rightarrow R(T) = W \Rightarrow \dim(R(T)) = rank(T) = \dim(W)$
      $nullity(T) + rank(T) = \dim(V) \Rightarrow nullity(T) + \dim(W) = \dim(W)$
      $\Rightarrow nullity(T) = 0 \Rightarrow T$ is one-to-one
      Remark:
      $nullity(T) + rank(T) = dim(V)$

      $\Leftarrow$

      T is one-to-one $\Rightarrow nullity(T) = 0$
      $nullity(T) + rank(T) = \dim(V) \Rightarrow rank(T) = \dim(W) \Rightarrow T$ is onto.
   b) Give an example of a linear transformation T such that T is one-to-one but not onto.
      $T: R^2 \to R^3$
      $T(a_1, a_2) = (a_1, a_2, 0)$
3. $w_1 = \{(a_1, a_2, a_3) \in R^3 : a_1 - 5a_2 - 3a_3 = 0\}, w_2 = \{(a_1, a_2, a_3) \in R^3 : 4a_3 - a_1 = 0 \; and \; 5a_2 - a_3 = 0\}$
   a) w_1, w_2 are subspaces of $\mathbb{R}^3$? Find bases and dimensions.
      i. $0_{\mathbb{R}^3} \in w_1$
      ii. Let $a, b \in w_1 \Rightarrow a + b \in w_1$
      iii. Let $a \in w_1, c \in \mathbb{R} \Rightarrow ca \in w_1$
      $\Rightarrow w_1$ is a subspace of $\mathbb{R}^3$
      $w_1 = \{(a_1, a_2, a_3) \in R^3 : a_1 - 5a_2 - 3a_3 = 0\}$
      $= \{(5a_2 + 3a_3, a_2, a_3) : a_2, a_3 \in \mathbb{R}\} = \{a_2(5,1,0) + a_3(3,0,1,) : a_2, a_3 \in \mathbb{R}\}$
      $\Rightarrow w_1 = span(\{(5,1,0), (3,0,1,)\})$
      $\{(5,1,0), (3,0,1,)\}$ is linear independent $\Rightarrow \{(5,1,0), (3,0,1,)\}$ is a basis for w_1 and $\dim(w_1) = 2$

      i. $0_{\mathbb{R}^3} \in w_2$
      ii. Let $a, b \in w_2 \Rightarrow a + b \in w_2$
      iii. Let $a \in w_2, c \in \mathbb{R} \Rightarrow ca \in w_2$
      $\Rightarrow w_2$ is a subspace of $\mathbb{R}^3$
      $w_2 = \{(a_1, a_2, a_3) \in R^3 : 4a_3 - a_1 = 0 \; and \; 5a_2 - a_3 = 0\}$
      $= \left\{\left(4a_3, \frac{1}{5}a_3, a_3\right) : a_3 \in \mathbb{R}\right\} = \left\{a_3\left(4, \frac{1}{5}, 1\right) : a_3 \in \mathbb{R}\right\}$
      $\Rightarrow w_2 = span\left(\left\{\left(4, \frac{1}{5}, 1\right)\right\}\right)$
      $\left\{\left(4, \frac{1}{5}, 1\right)\right\}$ is linear independent since it is non-zero vector $\Rightarrow \left\{\left(4, \frac{1}{5}, 1\right)\right\}$ is a basis for w_2 and $\dim(w_2) = 1$
   b) Find bases and dimensions of $w_1 \cap w_2, w_1 + w_2$
      i. Let $a \in w_1 \cap w_2 \Rightarrow a \in w_1 \; and \; a \in w_2$
      $$\Rightarrow \begin{cases} a_1 - 5a_2 - 3a_3 = 0 \\ 4a_3 - a_1 = 0 \; and \; 5a_2 - a_3 = 0 \end{cases} \Rightarrow$$
      $$4a_3 - a_1 = 0 \; and \; 5a_2 - a_3 = 0 \Rightarrow \begin{cases} a_1 = 4a_3 \\ a_2 = \frac{a_3}{5} \\ a_3 = a_3 \end{cases}$$
      Substitute back to the first equation:
      $a_1 - 5a_2 - 3a_3 = 4a_3 - 5\left(\frac{a_3}{5}\right) - 3a_3 = 0 \Rightarrow w_2 \subseteq w_1 \Rightarrow w_1 \cap w_2 = w_2$
      $\Rightarrow \left\{\left(4, \frac{1}{5}, 1\right)\right\}$ is a basis for $w_1 \cap w_2$ and $\dim(w_1 \cap w_2) = 1$
      ii. $w_1 + w_2 = w_1$
      $\Rightarrow \{(5,1,0), (3,0,1,)\}$ is a basis for $w_1 + w_2$ and $\dim(w_1 + w_2) = 2$
   c) Is $w_1 \cup w_2$ subspace ?
      Since $w_2 \subseteq w_1 \Rightarrow w_1 \cup w_2 = w_1$, which is a subspace of $\mathbb{R}^3$.
4. Let $T: P_2(\mathbb{R}) \to \mathbb{R}^3$ be linear, $T(a + bx + cx^2) = (a + 2b, b + 2c, a + b + c), \beta = \{1, x, x^2\}$ be a basis for $P_2(\mathbb{R})$ and $\gamma = \{(1,-1,1), (0,1,2), (1,0,1)\}$ be a basis for $\mathbb{R}^3$, compute $[T]_\beta^\gamma$.
   $T(1) = (1,0,1) = 0 \cdot (1,-1,1) + 0 \cdot (0,1,2) + 1 \cdot (1,0,1)$

$T(x) = (2,1,1) = -\frac{3}{2} \cdot (1,-1,1) - \frac{1}{2} \cdot (0,1,2) + \frac{7}{2} \cdot (1,0,1)$

$T(x^2) = (0,2,1) = -\frac{3}{2} \cdot (1,-1,1) + \frac{1}{2} \cdot (0,1,2) + \frac{3}{2} \cdot (1,0,1)$

$\Rightarrow [T]_\beta^\gamma = \begin{pmatrix} 0 & -\frac{3}{2} & -\frac{3}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 1 & \frac{7}{2} & \frac{3}{2} \end{pmatrix}$

5. Let $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be linear,

   $T(f(x)) = 2xf'(x) + \frac{3}{4}x^2 \cdot \int_0^4 f(t)\,dt$

   a) Find a basis for $N(T)$ and $nullity(T)$. Is T one-to-one ?

      Let $f(x) \in N(T)$, then $T(f(x)) = T(a + bx + cx^2) = 0$

      $\Rightarrow 2x(b + 2cx) + \frac{3}{4}x^2 \int_0^4 (a + bt + ct^2)\,dt = 0$

      $\Rightarrow 2bx + 4cx^2 + \frac{3}{4}x^2 \left(4a + \frac{4^2}{2}b + \frac{4^3}{3}c\right) = 0$

      $\Rightarrow 2bx + (4c + 3a + 6b + 16c)x^2 = 0$

      $\Rightarrow \begin{cases} 2b = 0 \\ 4c + 3a + 6b + 16c = 0 \end{cases} \Rightarrow \begin{cases} b = 0 \\ c = -\frac{3}{20}a \end{cases}$

      $\Rightarrow N(T) = \left\{a - \frac{3a}{20}x^2 : a \in \mathbb{R}\right\} = \left\{\frac{a}{20}(20 - 3x^2) : a \in \mathbb{R}\right\} = span(\{20 - 3x^2\})$

      $\{20 - 3x^2\}$ is linear independent

      $\Rightarrow \{20 - 3x^2\}$ is a basis for $N(T) \Rightarrow nullity(T) = 1 \Rightarrow T$ is not one-to-one.

   b) Find a basis for $R(T)$ and $rank(T)$. Is T onto ?

      Let $\beta = \{1, x, x^2\}$ be the standard basis for $P_2(\mathbb{R})$

      $\Rightarrow R(T) = span(\{T(1), T(x), T(x^2)\}) = span(\{3x^2, 2x + 6x^2, 20x^2\})$

      $\Rightarrow R(T) = span(\{x^2, 2x + 6x^2\})$, where $\{x^2, 2x + 6x^2\}$ is linear independent.

      $\Rightarrow \{x^2, 2x + 6x^2\}$ is a basis for $R(T), rank(T) = 2 \Rightarrow T$ is not onto since $\dim(P_2(\mathbb{R})) = 3$.

6. $U: M_{2\times2}(\mathbb{R}) \to \mathbb{R}^2, U(A) = (tr(A), 4 \cdot tr(A))$

   a) Show U is linear.

      Let $A, B \in M_{2\times2}(R)$ and $c \in \mathbb{R}$

      $\Rightarrow U(cA + B) = cU(A) + U(B)$

   b) $T: M_{2\times2}(\mathbb{R}) \to \mathbb{R}^2$ is linear. $U(E_{11}) = T(E_{11}), U(E_{12}) = T(E_{12}), U(E_{21}) = T(E_{21}), U(E_{22}) = T(E_{22})$. Find T and $[T(A)]_\alpha$, where $\alpha = \{(1,0), (0,1)\}$ is the ordered basis for $\mathbb{R}^2$. $A = \begin{pmatrix} 2 & -1 \\ 3 & -4 \end{pmatrix}$

      Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times2}(\mathbb{R}) \Rightarrow A = aE_{11} + bE_{12} + cE_{21} + dE_{22}$

      U, T are linear

      $\Rightarrow T(A) = aT(E_{11}) + bT(E_{12}) + cT(E_{21}) + dT(E_{22}) = aU(E_{11}) + bU(E_{12}) + cU(E_{21}) + dU(E_{22}) = U(A)$

      $\Rightarrow T = U$ on $M_{2\times2}(\mathbb{R})$.

      Method 2 ?

Final

Wednesday, December 13, 2017 11:31 AM

1. Prove W_1, W_2 are subspace, find β and $\dim()$ for $W_1, W_2, W_1 \cap W_2, W_1 + W_2$

$$W_1 = \left\{ \begin{pmatrix} a & b & b \\ c & d & d \\ e & 0 & e \end{pmatrix}, a, b, c, d, e \in \mathbb{R} \right\}, W_2 = \left\{ \begin{pmatrix} f & 0 & g \\ f & h & l \\ 0 & 0 & q \end{pmatrix}, f, g, h, l, q \in \mathbb{R} \right\}$$

1) W_1 is subspace
   a) Let $a = b = c = d = e = 0 \Rightarrow 0_V \in W_1$
   b) Let $A_1, A_2 \in W_1 \Rightarrow$
$$A_1 + A_2 = \begin{pmatrix} a_1 & b_1 & b_1 \\ c_1 & d_1 & d_1 \\ e_1 & 0 & e_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 & b_2 \\ c_2 & d_2 & d_2 \\ e_2 & 0 & e_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 & d_1 + d_2 \\ e_1 + e_2 & 0 & e_1 + e_2 \end{pmatrix} \in W_1$$
   c) Let $c \in F \Rightarrow$
$$cA_1 = c \begin{pmatrix} a_1 & b_1 & b_1 \\ c_1 & d_1 & d_1 \\ e_1 & 0 & e_1 \end{pmatrix} = \begin{pmatrix} ca_1 & cb_1 & cb_1 \\ cc_1 & cd_1 & cd_1 \\ ce_1 & 0 & ce_1 \end{pmatrix} \in W_1$$
   $\Rightarrow W_1$ is subspace
2) W_2 is subspace
   a) Let $g = g = h = l = q = 0 \Rightarrow 0_V \in W_2$
   b) Let $B_1, B_2 \in W_2 \Rightarrow$
$$B_1 + B_2 = \begin{pmatrix} f_1 & 0 & g_1 \\ f_1 & h_1 & l_1 \\ 0 & 0 & q_1 \end{pmatrix} + \begin{pmatrix} f_2 & 0 & g_2 \\ f_2 & h_2 & l_2 \\ 0 & 0 & q_2 \end{pmatrix} = \begin{pmatrix} f_1 + f_2 & 0 & g_1 + g_2 \\ f_1 + f_2 & h_1 + h_2 & l_1 + l_2 \\ 0 & 0 & q_1 + q_2 \end{pmatrix} \in W_2$$
   c) Let $c \in F \Rightarrow$
$$cB_1 = c \begin{pmatrix} f_1 & 0 & g_1 \\ f_1 & h_1 & l_1 \\ 0 & 0 & q_1 \end{pmatrix} = \begin{pmatrix} cf_1 & 0 & cg_1 \\ cf_1 & ch_1 & cl_1 \\ 0 & 0 & cq_1 \end{pmatrix} \in W_2$$
   $\Rightarrow W_2$ is subspace
3) W_1
   a)
$$\begin{pmatrix} a & b & b \\ c & d & d \\ e & 0 & e \end{pmatrix} = a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} + e \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$
$$= aA_1 + bA_2 + cA_3 + dA_4 + eA_5$$
   $\Rightarrow \{A_1, A_2, A_3, A_4, A_5\}$ is a generating set.
   b) Check $\{A_1, A_2, A_3, A_4, A_5\}$ linearly independence.
      Let $aA_1 + bA_2 + cA_3 + dA_4 + eA_5 = 0_V \Rightarrow a = b = c = d = e = 0$
   $\Rightarrow \{A_1, A_2, A_3, A_4, A_5\}$ is linearly independent.
   $\Rightarrow \{A_1, A_2, A_3, A_4, A_5\}$ is a basis for W_1, $\dim(W_1) = 5$
4) W_2
   a)
$$\begin{pmatrix} f & 0 & g \\ f & h & l \\ 0 & 0 & q \end{pmatrix} = f \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + g \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + h \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + l \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + q \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= fB_1 + gB_2 + hB_3 + lB_4 + qB_5$$
   $\Rightarrow \{B_1, B_2, B_3, B_4, B_5\}$ is a generating set.
   b) Check $\{B_1, B_2, B_3, B_4, B_5\}$ linearly independence.
      Let $aB_1 + bB_2 + cB_3 + dB_4 + eB_5 = 0_V \Rightarrow a = b = c = d = e = 0$
   $\Rightarrow \{B_1, B_2, B_3, B_4, B_5\}$ is linearly independent.
   $\Rightarrow \{B_1, B_2, B_3, B_4, B_5\}$ is a basis for W_2, $\dim(W_2) = 5$
5) $W_1 \cap W_2$

   Let $M = \begin{pmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix} \in (W_1 \cap W_2) \Rightarrow M \in W_1, M \in W_2 \Rightarrow$
$$M \in W_1 \Rightarrow \begin{cases} y = z \\ y_1 = z_1 \\ x_2 = z_2 \\ y_2 = 0 \end{cases}$$
$$M \in W_2 \Rightarrow \begin{cases} x = x_1 \\ y = 0 \\ x_2 = 0 \\ y_2 = 0 \end{cases}$$
$$\Rightarrow M = \begin{pmatrix} x & 0 & 0 \\ x & y_1 & y_1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow W_1 \cap W_2 = \left\{ \begin{pmatrix} x & 0 & 0 \\ x & y & y \\ 0 & 0 & 0 \end{pmatrix}, x, y \in \mathbb{R} \right\}$$

a) $\begin{pmatrix} x & 0 & 0 \\ x & y & y \\ 0 & 0 & 0 \end{pmatrix} = x \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + y \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = xM_1 + yM_2$

$\Rightarrow \{M_1, M_2\}$ is a generating set.

b) Check $\{M_1, M_2\}$ linearly independence.

Let $aM_1 + bM_2 = 0_V \Rightarrow a = b = 0$

$\Rightarrow \{M_1, M_2\}$ is linearly independent.

$\Rightarrow \{M_1, M_2\}$ is a basis for $W_1 \cap W_2$, $\dim(W_1 \cap W_2) = 2$

6) $W_1 + W_2$

$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) = 8$

If $\dim(W_1 + W_2) = 9 = \dim(M_{3\times3})$, W_1 and W_2 are subspaces of $M_{3\times3}$

$\Rightarrow (W_1 + W_2) = M_{3\times3}$, so we can choose any basis from $M_{3\times3}$ as a basis for $W_1 + W_2$

Find a basis for $(W_1 + W_2)$:

Method 1:

Extend $\beta_{W_1 \cap W_2}$ to $\beta_{W_1 + W_2}$

(1) $\beta_{W_1 \cap W_2}$ to β_{W_1}

$$\beta_{W_1 \cap W_2} = \{M_1, M_2\} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

For W_1:

$$\{A_1, A_2, A_3, A_4, A_5\} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \right\}$$

$A_4 = M_2, A_1 + A_3 = M_1$

$\Rightarrow$ remove A_4, remove A_1 or A_3

$$\Rightarrow \beta_{W_1} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \right\}$$

(2) $\beta_{W_1 \cap W_2}$ to β_{W_2}

$$\beta_{W_1 \cap W_2} = \{M_1, M_2\} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

For W_2:

$$\{B_1, B_2, B_3, B_4, B_5\} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

$B_1 = M_1, B_3 + B_4 = M_2$

$\Rightarrow$ remove B_1, remove B_3 or B_4

$$\Rightarrow \beta_{W_2} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

$\beta_{W_1 + W_2} = \beta_{W_1} + \beta_{W_1}$

$$= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

Method 2:

Reduce a generating set to a basis

$W_1 + W_2 = span\left(\beta_{W_1}\right) + span\left(\beta_{W_2}\right) = span\left(\beta_{W_1} + \beta_{W_2}\right)$

$\Rightarrow \beta_{W_1} + \beta_{W_2}$ generates $W_1 + W_2$

$\beta_{W_1} + \beta_{W_2} = \{A_1, A_2, A_3, A_4, A_5, B_1, B_2, B_3, B_4, B_5\}$

$A_1 + A_3 = B_1, B_3 + B_4 = A_4$

$\Rightarrow$ remove A_1 or A_3, remove B_3 or B_4

$$\Rightarrow \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} 0, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

2. $T: P_3(R) \to P_2(R), T\big(f(x)\big) = 2 \cdot f'(x)$;

$U: P_2 \to R^2, U\big(a + bx + cx^2\big) = (a + b, a + c)$;

U, T is linear.

$\alpha = \{1, x, x^2, x^3\}, \beta = \{1 + x, 1 + x^2, x + x^2\}, \gamma = \{e_1, e_2\}$

(a) $[T]_\alpha^\beta, [U]_\beta^\gamma, [UT]_\alpha^\gamma$

Method1:

$[T]_\alpha^\beta$

$f(x) = a + bx + cx^2 = a_1(1+x) + a_2(1+x^2) + a_3(x+x^2)$

$$\Rightarrow \begin{cases} a_1 + a_2 = a \\ a_1 + a_3 = b \\ a_2 + a_3 = c \end{cases} \Rightarrow \begin{cases} a_1 = \dfrac{a+b-c}{2} \\ a_2 = \dfrac{a-b+c}{2} \\ a_3 = \dfrac{-a+b+c}{2} \end{cases}$$

$$\Rightarrow a + bx + cx^2 = \left(\frac{a+b-c}{2}\right)(1+x) + \left(\frac{a-b+c}{2}\right)(1+x^2) + \left(\frac{-a+b+c}{2}\right)(x+x^2)$$

$$T(1) = 2 \cdot 1' = 0 \Rightarrow a = b = c = 0 \Rightarrow [T(1)]_\beta = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$T(x) = 2 \cdot x' = 2 \Rightarrow a = 2, b = c = 0 \Rightarrow [T(x)]_\beta = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$T(x^2) = 2 \cdot (x^2)' = 4x \Rightarrow a = 0, b = 4, c = 0 \Rightarrow [T(x^2)]_\beta = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$$

$$T(x^3) = 2 \cdot (x^3)' = 6x^2 \Rightarrow a = b = 0, c = 6 \Rightarrow [T(x^3)]_\beta = \begin{pmatrix} -3 \\ 3 \\ 3 \end{pmatrix}$$

$$[T]_\alpha^\beta = \begin{pmatrix} 0 & 1 & 2 & -3 \\ 0 & 1 & -2 & 3 \\ 0 & -1 & 2 & 3 \end{pmatrix}$$

$[U]_\beta^\gamma$

$1 + x \Rightarrow a = 1, b = 1, c = 0$

$U(1+x) = (2,1)$

$U(1+x^2) = (1,2)$

$U(x+x^2) = (1,1)$

$$[U]_\beta^\gamma = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

$[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$

Method2:

$$(UT)(f(x)) = U\left(T(f(x))\right) = U\left(T(a + bx + cx^2 + dx^3)\right) = U\left(2 \cdot (b + 2cx + 3dx^2)\right) = U(2b + 4cx + 6dx^2)$$

$= (2b + 4c, 2b + 6d)$

$\Rightarrow (UT)(a + bx + cx^2 + dx^3) = (2b + 4c, 2b + 6d)$

$(UT)(1) = (0,0)$

$(UT)(x) = (2,3)$

$(UT)(x^2) = (4,0)$

$(UT)(x^3) = (0,6)$

$$[UT]_\alpha^\gamma = \begin{pmatrix} 0 & 2 & 4 & 0 \\ 0 & 2 & 0 & 6 \end{pmatrix}$$

(b) $P(x) = 1 - 2x + 3x^2 - x^3 \rightarrow check\ [T(P(x))]_\beta = [T]_\alpha^\beta [P(x)]_\alpha$

$T(1 - 2x + 3x^2 - x^3) = -4 + 12x - 6x^2$

$$a + bx + cx^2 = \left(\frac{a+b-c}{2}\right)(1+x) + \left(\frac{a-b+c}{2}\right)(1+x^2) + \left(\frac{-a+b+c}{2}\right)(x+x^2)$$

$\Rightarrow 7(1+x) + 5(1+x^2) + 5(x+x^2)$

3. P116(3e) $V = P_2(R), \beta = \{x^2 - x, x^2 + 1, x - 1\}, \beta' = \{5x^2 - 2x - 3, -2x^2 + 5x + 5, 2x^2 - x - 3\}$

$$Q = \left[I_{P_2(R)}\right]_{\beta'}^\beta$$

$I_{P_2(R)}(5x^2 - 2x - 3) = 5x^2 - 2x - 3 = 5(x^2 - x) + 0(x^2 - 1) + 3(x - 1)$

$I_{P_2(R)}(-2x^2 + 5x + 5) = -2x^2 + 5x + 5 = (-6)(x^2 - x) + 4(x^2 - 1) + (-1)(x - 1)$

$I_{P_2(R)}(2x^2 - x - 3) = 2x^2 - x - 3 = 3(x^2 - x) + (-1)(x^2 - 1) + 2(x - 1)$

$$ax^2 + bx + c = r \cdot (x^2 - x) + s \cdot (x^2 - 1) + t \cdot (x - 1) \Rightarrow r = \frac{1}{2}(a - b - c), s = \frac{1}{2}(a + b + c), t = \frac{1}{2}(a + b - c)$$

$$\Rightarrow ax^2 + bx + c = \frac{1}{2}(a - b - c) \cdot (x^2 - x) + \frac{1}{2}(a + b + c) \cdot (x^2 - 1) + \frac{1}{2}(a + b - c) \cdot (x - 1)$$

$$\Rightarrow Q = [I_{R^2}]_{\beta'}^\beta = \begin{pmatrix} 5 & -6 & 3 \\ 0 & 4 & -1 \\ 3 & -1 & 2 \end{pmatrix}$$

4. Extend a linear independent subset S to a basis for V.

Step 1. Find an explicit basis β for V.

Step 2. Consider the ordered set S' consisting of the vectors in S followed by the vectors in β. Then S' generates V.

Proof: $\beta \subseteq S' \Rightarrow span(\beta) \subseteq span(S')$
$V \subseteq span(S), but\ span(S') \subseteq V \Rightarrow span(S') = V$

Step 3. Reduce S' to a basis.

e.g.:

$V = \{(x_1, x_2, x_3, x_4, x_5) \in R^5 : 2x_1 - x_3 + x_4 - 3x_5 = 0\}$ is a vector space over R.

a) Show that $S = \{(1,0,0,-2,0), (0,1,0,3,1)\}$ is a linearly indepdendent set in V.
v_1 and v_2 are not multiples of each other $\Rightarrow S$ is linearly independent.
$v_1 \in V, v_2 \in V$
$\Rightarrow S$ is a linearly independent set in V.

b) Extend S to a full basis for V.

Step 1:

$$V = \left\{\left(x_1, x_2, x_3, x_4, \frac{2}{3}x_1 - \frac{1}{3}x_3 + \frac{1}{3}x_4\right), x_1, x_2, x_3, x_4 \in R\right\}$$

$$= \left\{x_1\left(1,0,0,0,\frac{2}{3}\right) + x_2(0,1,0,0,0) + x_3\left(0,0,1,0,-\frac{1}{3}\right) + x_4\left(0,0,0,1,\frac{1}{3}\right)\right\}$$

$$= span\left\{\left(1,0,0,0,\frac{2}{3}\right), (0,1,0,0,0), \left(0,0,1,0,-\frac{1}{3}\right), \left(0,0,0,1,\frac{1}{3}\right)\right\} = span\{w_1, w_2, w_3, w_4\}$$

Check linearly independent:

$$x_1\left(1,0,0,0,\frac{2}{3}\right) + x_2(0,1,0,0,0) + x_3\left(0,0,1,0,-\frac{1}{3}\right) + x_4\left(0,0,0,1,\frac{1}{3}\right) = 0 \Rightarrow x_1 = x_2 = x_3 = x_4 = 0$$

Step 2:

$S' = \{v_1, v_2, w_1, w_2, w_3, w_4\}$ generates V.

Step 3:

S' can be reduced to a basis for V.

$\dim(V) = 4$

$[v_1]_\gamma, [v_2]_\gamma, [w_1]_\gamma, [w_2]_\gamma, [w_3]_\gamma, [w_4]_\gamma, \gamma = \{e_1, e_2, e_3, e_4, e_5\}$

$$\Rightarrow A = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \\ 0 & 1 & \frac{2}{3} & 0 & -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 3 & 2 & 0 & 0 & 1 \\ 0 & 1 & \frac{2}{3} & 0 & -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -3 & 0 & 1 \\ 0 & 0 & \frac{2}{3} & -1 & -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{2}{3} & -1 & -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & -\frac{1}{3} & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & \frac{3}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow basis = \{v_1, v_2, w_1, w_3\}$

5. Invertible:

Let $T : P_2 \to P_2$ be defined by

$$T(a + bx + cx^2) = (3a - b + c) + (a - c)x + (4b + c)x^2.$$

Using the basis $\{1, x, x^2\}$ for P_2, the matrix representation for T is

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 4 & 1 \end{bmatrix}.$$

This matrix is invertible and

$$A^{-1} = \frac{1}{17}\begin{bmatrix} 4 & 5 & 1 \\ -1 & 3 & 4 \\ 4 & -12 & 1 \end{bmatrix}.$$

Thus, T^{-1} is given by

$$T^{-1}(a + bx + cx^2) = \frac{4a+5b+c}{17} + \frac{-a+3b+4c}{17}x + \frac{4a-12b+c}{17}x^2.$$

6. Let $A = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix}$

1) Eigenvalues of A

$$\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & -2 & -3 \\ -1 & 1-\lambda & -1 \\ 2 & 2 & 5-\lambda \end{pmatrix}$$

$= (-\lambda)(1-\lambda)(5-\lambda) + (-2)(-1)(2) + (-3)(-1)(2) - (-3)(1-\lambda)(2) - (-\lambda)(-1)(2) - (-2)(-1)(5-\lambda)$

$= (-\lambda)(1-\lambda)(5-\lambda) + 10 + 6(1-\lambda) - \lambda + 2(5-\lambda) = -(\lambda-1)(\lambda-2)(\lambda-3)$

$\Rightarrow \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$

2) Eigenvectors of A

a) $\lambda_1 = 1$

$(A - \lambda_1 I)v = (A - I)v = 0$

$$\Rightarrow \begin{pmatrix} -1 & -2 & -3 \\ -1 & 0 & -1 \\ 2 & 2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -x_1 + x_4 = 0 \\ x_1 - x_4 = 0 \end{cases} \Rightarrow x_1 = x_4$$

$$\Rightarrow v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_1 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + x_2\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Eigenvectors (set of all nonzero linear combination of the vectors)

$$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

b) $\lambda_2 = 2$

$(A - \lambda_2 I)v = (A - 2I)v = 0$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 + x_4 = 0 \\ 2x_2 = 0 \\ 2x_3 = 0 \\ x_1 + x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -x_4 \\ x_2 = 0 \\ x_3 = 0 \end{cases}$$

$$\Rightarrow v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ 0 \\ -x_1 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

Eigenvectors (set of all nonzero linear combination of the vectors)

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

c) $\lambda_3 = 3$

$(A - \lambda_3 I)v = (A - 3I)v = 0$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 + x_4 = 0 \\ 2x_2 = 0 \\ 2x_3 = 0 \\ x_1 + x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -x_4 \\ x_2 = 0 \\ x_3 = 0 \end{cases}$$

$$\Rightarrow v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ 0 \\ -x_1 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

Eigenvectors (set of all nonzero linear combination of the vectors)

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

3) Q

β' is a basis of eigenvectors of A

$$\beta' = \left\{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right\}$$

Check linear independence.

$\Rightarrow \beta = \beta'$

$\Rightarrow A$ is diagonalizable

$$Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$$

4) D

$$Q^{-1}AQ = D = [L_A]_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

7. (P257, 4) Let $V = M_{2\times 2}(R)$. Let $T: V \to V$ be defined by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$.Find the eigenvalues of T and a basis B of V so that $[T]_B$ is a diagonal matrix.

1) Let $\alpha = \{E_{11}, E_{12}, E_{21}, E_{22}\}$

$$T(E_{11}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$
$$T(E_{12}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$T(E_{21}) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
$$T(E_{22}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$
$$\Rightarrow A = [T]_\alpha = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$
$$f(\lambda) = \det(A - \lambda I_4) = \det\begin{pmatrix} -\lambda & 0 & 0 & 1 \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 1 & 0 & 0 & -\lambda \end{pmatrix}$$
$$= (1-\lambda)(-1)^{2+2}\begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = (1-\lambda)(1-\lambda)(-1)^{2+2}\big((-\lambda)(-\lambda) - 1\big) = (1-\lambda)^3(1+\lambda)$$
$$\Rightarrow \begin{cases} \lambda_1 = \lambda_2 = \lambda_3 = 1, m_1 = 3 \\ \lambda_4 = -1, m_4 = 1 \end{cases} \Rightarrow 1 \le dim(E_\lambda) \le m \Rightarrow \dim\left(E_{\lambda_4}\right) = 1; \dim\left(E_{\lambda_1}\right) = ?$$

2) Eigenvectors of $A = [T]\alpha$

a) $\lambda_1 = \lambda_2 = \lambda_3 = 1$

$$(A - \lambda_1 I)v = (A - I)v = 0$$
$$\Rightarrow \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -x_1 + x_4 = 0 \\ x_1 - x_4 = 0 \end{cases} \Rightarrow x_1 = x_4$$
$$\Rightarrow v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_1 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + x_2\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Eigenvectors (set of all nonzero linear combination of the vectors)

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$
$$\Rightarrow E_{\lambda_1} = span\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right\}$$

Check linear independence

$$\Rightarrow \dim\left(E_{\lambda_1}\right) = 3 = m_1$$

b) $\lambda_4 = -1$

$$(A - \lambda_4 I)v = (A + I)v = 0$$
$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 + x_4 = 0 \\ 2x_2 = 0 \\ 2x_3 = 0 \\ x_1 + x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -x_4 \\ x_2 = 0 \\ x_3 = 0 \end{cases}$$
$$\Rightarrow v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ 0 \\ -x_1 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

Eigenvectors (set of all nonzero linear combination of the vectors)

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$
$$\Rightarrow E_{\lambda_4} = span\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}\right\}$$

Check linear independence

$$\Rightarrow \dim\left(E_{\lambda_4}\right) = 1 = m_4$$

3) B

β' is a basis of eigenvectors of A

$$\beta' = \left\{ \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\-1\end{pmatrix} \right\}$$

Check linear independence.

$\Rightarrow A$ is diagonalizable

$\Rightarrow T$ is diagonlizable and the basis of eigenvectors of T is $B = \{v_1, v_2, v_3, v_4\}$

$$v_1 = [w_1]_\alpha = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$
$$v_2 = [w_2]_\alpha = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}$$
$$v_3 = [w_3]_\alpha = \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}$$
$$v_4 = [w_4]_\alpha = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

4) $[T]_B$

$$[T]_B = \begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & -1\end{pmatrix}$$

8. (P257, 4f) Let $T: P_3(R) \to P_3(R)$ be defined by $T(f(x)) = f(x) + f(2)x$.Find the eigenvalues of T and a basis B of V so that $[T]_B$ is a diagonal matrix.

1) Eigenvalues of $T \Leftrightarrow$ eigenvalues of $[T]_\alpha = A, \alpha$ is the ordered basis for V.

Let $\alpha = \{1, x, x^2, x^3\}$

$T(1) = 1 + 1x = 1 + x$

$T(x) = x + 2x = 3x$

$T(x^2) = x^2 + 4x$

$T(x^3) = x^3 + 8x$

$$\Rightarrow A = [T]_\alpha = \begin{pmatrix}1 & 0 & 0 & 0\\1 & 3 & 4 & 8\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{pmatrix}$$

$$f(\lambda) = \det(A - \lambda I_4) = \det\begin{pmatrix}1-\lambda & 0 & 0 & 0\\1 & 3-\lambda & 4 & 8\\0 & 0 & 1-\lambda & 0\\0 & 0 & 0 & 1-\lambda\end{pmatrix} = (1-\lambda)(-1)^{1+1}(3-\lambda)(1-\lambda)(1-\lambda) = 0$$

$\Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 1, \lambda_4 = 3$

2) Eigenvectors of $A = [T]\alpha$

a) $\lambda_1 = \lambda_2 = \lambda_3 = 1$

$(A - \lambda_1 I)v = (A - I)v = 0$

$$\Rightarrow \begin{pmatrix}0 & 0 & 0 & 0\\1 & 2 & 4 & 8\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \Rightarrow x_1 + 2x_2 + 4x_3 + 8x_4 = 0$$

$$\Rightarrow v = \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}-(2x_2 + 4x_3 + 8x_4)\\x_2\\x_3\\x_4\end{pmatrix} = x_2\begin{pmatrix}-2\\1\\0\\0\end{pmatrix} + x_3\begin{pmatrix}-4\\0\\1\\0\end{pmatrix} + x_4\begin{pmatrix}-8\\0\\0\\1\end{pmatrix}$$

Eigenvectors (set of all nonzero linear combination of the vectors)

$$\begin{pmatrix}-2\\1\\0\\0\end{pmatrix}, \begin{pmatrix}-4\\0\\1\\0\end{pmatrix}, \begin{pmatrix}-8\\0\\0\\1\end{pmatrix}$$

b) $\lambda_4 = 3$

$(A - \lambda_4 I)v = (A - 3I)v = 0$

$$\Rightarrow \begin{pmatrix}-2 & 0 & 0 & 0\\1 & 0 & 4 & 8\\0 & 0 & -2 & 0\\0 & 0 & 0 & -2\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \Rightarrow \begin{cases}-2x_1 = 0\\x_1 + 4x_3 + 8x_4 = 0\\-2x_3 = 0\\-2x_4 = 0\end{cases} \Rightarrow \begin{cases}x_1 = 0\\x_3 = 0\\x_4 = 0\end{cases}$$

$$\Rightarrow v = \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}0\\x_2\\0\\0\end{pmatrix} = x_2\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

Eigenvectors (set of all nonzero linear combination of the vectors)

$$\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

3) B

$A \in M_{4\times4}(R), \beta'$ is a basis of eigenvectors of A for R^4

$$\beta' = \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -8 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

Check linear independence.

$\Rightarrow A$ is diagonalizable

$\Rightarrow T$ is diagonlizable and the basis of eigenvectors of T is $B = \{v_1, v_2, v_3, v_4\}$

$v_1 = [w_1]_\alpha = -2 + x$

$v_2 = [w_2]_\alpha = -4 + x^2$

$v_3 = [w_3]_\alpha = -8 + x^3$

$v_4 = [w_4]_\alpha = x$

4) $[T]_B$

$$[T]_B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

9. Let $A = \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}$. Find A^{500}.

$\det(A - \lambda I) = \det \begin{pmatrix} -\lambda & -2 \\ 1 & 3-\lambda \end{pmatrix} = \lambda(\lambda - 3) + 2 = (\lambda - 1)(\lambda - 2) = 0$

$\Rightarrow \lambda_1 = 1, \lambda_2 = 2$

1) $\lambda_1 = 1$

$(A - \lambda_1 I)v = 0$

$\Rightarrow \begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

2) $\lambda_2 = 2$

$(A - \lambda_2 I)v = 0$

$\Rightarrow \begin{pmatrix} -2 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

3) $Q = \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix}$

4) $D = Q^{-1}AQ = [L_A]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ (Theorem 2.23)

5) $A = QDQ^{-1}$

6) $A^n = (QDQ^{-1})^n = (QDQ^{-1})(QDQ^{-1}) \ldots (QDQ^{-1}) = QD^nQ^{-1}$

$= \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1^n & 0 \\ 0 & 2^n \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} -2 & -2^n \\ 1 & 2^n \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 - 2^n & 2 - 2^{n+1} \\ -1 + 2^n & -1 + 2^{n+1} \end{pmatrix}$

10. Prove that $T: V \to V$, a linear transformation is invertible if and only if 0 is not an eigenvalue of T.

1) $\Rightarrow$

A is not invertible $\Rightarrow \det(A) = 0 \Rightarrow \lambda_1^{m_1} \cdot \lambda_2^{m_2} \cdot \ldots \cdot \lambda_k^{m_k} = 0 \Rightarrow$ at least one is 0.

2) $\Leftarrow$

$\lambda = 0 \Rightarrow Av = \lambda v = 0$

If $A^{-1} \Rightarrow A^{-1}Av = A^{-1}0 \Rightarrow Iv = 0 \Rightarrow v = 0$

$\lambda = 0 \Rightarrow \det(A - \lambda I) = 0 \Rightarrow \det(A) = 0 \Rightarrow A$ is not invertible.