Sec 2.3-2.5 Functions and Cardinality

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Sec 2.3 , 2.5 Functions and Cardinality

Comp 232
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1. Recall: (i) The Cartesian Product of two sets A and B is:
   The set of ordered pairs (x, y) where x∈A and y∈B

   (ii) The Cartesian Product is denoted by A X B

2 a) Definition: A Relation R from set A to set B is a sub set of the Cartesian Product AXB

b) Relation R is denoted: R ⊆ A X B

c) There are many relations that can be formed from A × B.

Ex: If A = {1, 2,} B = {5, 6, 7} then A × B = {(1,5), (1,6), (1,7), (2,5), (2,6), (2,7)}

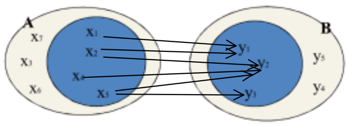
Using this A × B above one possible relation is: R = {(1,6), (2,5)}

d) If the Relation is set R and (x,y) ∈ R then: x is related to y and we write x R y

Ex: In the above Relation R: 1 R 5 and 2 R 7

3. Geometric representation of a Relation R on Cartesian product A × B

Step 1 Sketch sets A, B
Step 2 Sketch the subset of A containing the first members of the ordered pairs in R
Step 3 Sketch the subset of B containing the second members of the ordered pairs in R
Step 4 Join the related x, y values with an arrow.

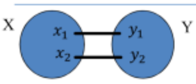


Ex: R = $\{(x_1,y_1),(x_2,y_1),(x_2,y_2),(x_4,y_2),(x_5,y_2),(x_5,y_3)\}$

How many ordered pairs are in the Cartesian Product A × B = { (x, y) | (x ∈ A) ∧ (y ∈ B) } ?
n(AXB) = n(A) * n(B) = 35

How many ordered pairs are in the Relation R ?
n(R) = 5

4. Three basic kind of Relations

		
Each x is related to a unique y Each y is related to a unique x	Each x is related to a unique y	Each y is related to a unique x
Called a one to one relation (1 to 1)	Called a many to one relation (m to 1)	Called a one to many relation (1 to m)

5. Definition of a Function.

a) A Relation from a non empty set X to a non empty set Y which is 1 to 1 or many to 1 but not 1 to many is called a Function from set X to set Y.

A Function is a special Relation because each x is related to a unique y.

b) For a Function write `f: X → Y`

c) If the Relation is a Function:

(i) x R y is written as y = f(x)

(ii) the set of x is called the Domain of the Function

(iii) the set of y is called the Range of the Function

(iv) a Function is also called a Mapping or a Transformation

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5. Additional terms and definitions

a) What does the word Onto mean ?

- Start with a set that is the Domain for a function
- Consider a second set called the Codomain
- Identify the Range of the function. If:

Function Range = Codomain	Function Range ≠ Codomain
→ Function can generate all values in the Codomain → Function is Onto	→ Function does not generate all values in the Codomain
Functions which are Onto cover all Codomain	→ Function which are Not Onto do not cover all Codomain

b) Three related terms:

Injective Function	Surjective Function	Bijective Function
A function that is 1 to 1	A function that is Onto	A function is both 1 to 1 and Onto (Also called a 1 to 1 correspondence)

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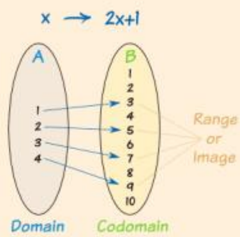
Domain, Codomain and Range

There are special name for **what can go into**, and **what can come out** of a function:

- What can go **into** a function is called the **Domain**
- What **may possibly come out** of a function is called the **Codomain**
- What **actually comes out** of a function is called the **Range**

Let us look at a simple example:



In this illustration:

- the set "A" is the Domain,
- the set "B" is the Codomain,
- and the set of elements that get pointed to in B (the actual values produced by the function) are the Range, also called the Image.

In that example:

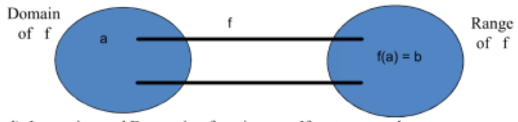
- Domain: {1, 2, 3, 4}
- Codomain: {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
- Range: {3, 5, 7, 9}

c) Definition: Inverse Function: If (i) f is a 1 to 1 correspondence (1 to 1 and onto, Bijective Function)

(ii) a second function maps the Range of f back to the original domain.

Then the second function is called the Inverse of the function f

Notation: =a iff f(a)=b



d) Increasing and Decreasing functions: If $x_2 > x_1$ and

$\forall \mathbf{x} \in (x_1, x_2)\ f(x_2) \geq f(x_1)$ then f(x) is increasing on the interval ( )

$\forall \mathbf{x} \in (x_1, x_2)\ f(x_2) \leq f(x_1)$ then f(x) is decreasing on the interval ( )

6. Operations on Functions

Operation	Symbols	Example if f(x) = x+3, g(x) = 2x-7
Addition / Subtract Multiplication Division	(f ± g) (x) = f(x) ± g(x) (f * g) (x) = f(x)g(x) $(\frac{f}{g})(x)$, $g(x) \neq 0$ = f(x)/g(x)	(f ± g) (x) = (f*g) (x) = $(\frac{f}{g})(x)$ =
Composition of Functions	(f ∘ g) (x) = f (g(x)) The domain of f must = the range of g	(f ∘ g) (x) = (2x-7) + 3

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7. Working Definition for a One to One function:

A function is 1 to 1 iff ∀a, b ∈ Domain f(a) = f(b) → a=b

Ex 1: Prove $f(x) = x^2 + 5$ is not 1 to 1

Proof (Counter example):

Show:
∃a, b ∈ Domain f(a)=f(b) ∧ a≠b

Consider:
x = 5 → f(5) = 30
x= -5 → f(-5) = 30

Ex 2: Prove f(x) = 3x + 7 is 1 to 1

Proof (Direct):

Assume $f(x_1) = f(x_2)$, x_1, x_2 any elements ∈ Domain
Show $x_1 = x_2$

Consider:

$f(x_1) = f(x_2)$	
$\rightarrow 3x_1 + 7 = 3x_2 + 7$	Definition of given function
$\rightarrow 3x_1 = 3x_2$	Algebra
$\rightarrow x_1 = x_2$	Definition of 1 to 1 function
QED	

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8. Working Definition for an Onto function:

Definition: A function is Onto iff ∀y∃x, y∈Codomain, x∈Domain, y=f(x)

(All elements y of the Codomain are images of some element x in the Domain)

Ex 1: Prove f(x) = x^2 does not map Domain **Z**+ Onto the Codomain **Z**+.

Proof (Counter example):

Show:

∃y∀x, y∈Codemain, x∈Domain ∧ y≠f(x)

Consider:

y=2 ∈ Codomain Z+

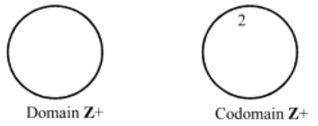
We need an x∈Domain such that f(x)=2

$\rightarrow x^2 = 2 \rightarrow x = \sqrt{2}$

But x= $\sqrt{2}$ is not in Domain Z+

QED

Geometrically:



Note: y=9 does not provide a counter example because y=9 is i n the range of the f. If we use x=3 then f(3)=9

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Ex 2: Prove f(x) = 3x+7 maps Domain **R** Onto the Codomain **R.**

Proof: (Constructive Existence)

Consider any y = b ∈ in Codomain **R**

Show ∃x ∈ Domain such that f(x) = b ?

Consider

$$x = \frac{b}{3} - \frac{7}{3}$$

$$\rightarrow f(x) = f\left(\frac{b}{3} - \frac{7}{3}\right) = 3\left(\frac{b}{3} - \frac{7}{3}\right) + 7 = b$$

QED

Note: Using backward reasoning:

If we want b = 3x+7 for some x in Domain

solve for x to get x=b/3-7/3

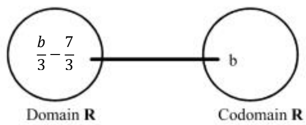
Since Codomain = Domain = **R**

b ∈ Codomain = **R** → b ∈ Domain = **R**

and Subtraction, Division are closed in **R**

$\rightarrow \frac{b}{3} - \frac{7}{3}$ ∈ Domain **R**

Geometrically:



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9. Using a One to One Correspondence to determine if two sets have the same Cardinality

a) Recall that the Cardinality of a set describes How many elements are in the set

Ex: Show that A = {1,2,3} , B = {a,b,c} have the same Cardinality:

Method 1: n(A) = 3 , n(B) = 3 → A, B have the same Cardinality

Method 2: Determine if there is a 1 to 1 correspondence (1 to 1 and onto) between A, B

1↔a
2↔b
3↔c

This establishes a 1 to 1 correspondence between A, B
→ A, B have the same Cardinality.

b) Method 2 provides a way to determine whether infinite sets C, D have the same Cardinality
We cannot use Method 1: there is no Integers to describe n(C), n(D) when C,D are infinite

Ex: Show that Z+ and the set of odd positive integers have the Same Cardinality
Consider n ϵ Z+ and form all d ϵ odd positive integers using d = f(n) = 2n -1

(i) $\forall n_1 \forall n_2$ ϵ Z+ ∧ $f(n_1) = f(n_2) \rightarrow 2n_1 - 1 = 2n_2 - 1 \rightarrow n_1 = n_2$ → f is 1 to 1

(ii) All positive odd integers d can be determined by using d = 2n – 1
→ all positive odd integers are images of an element of Z+ → f is onto

(iii) 1↔1
2↔3
3↔5
4↔7
...

This shows the 1 to 1 conrrespondence between Z+ and odd+ integers
→ Z+ and the positive odd integers have the same Cardinality.

c) Definition: Set S is Countable iff: (i) S is Finite OR
(ii) S is Infinite and it has same Cardinality as Z+

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Ex : Show that Z is Countable
We require that Z+ and Z have the Same Cardinality

Consider n ϵ Z+ and form all d ϵ Z using d = f(n) = $\begin{cases} \frac{n}{2} & \text{if n is even} \\ -\frac{n-1}{2} & \text{if n is odd} \end{cases}$

(i) It can be shown that f is 1 to 1 (each branch of f is linear)

(ii) All integers d can be determined by using f(n)

→ f maps Z onto the positive odd Integers

(iii)
1↔0
2↔1
3↔-1
. . . ↔ . . .

This shows the 1 to 1 correspondence between Z+ and Z
→ Z has same Cardinality as Z+
→ Z is Countable

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10 Given a function y = f(x) find its inverse function $f^{-1}(x)$

a) Method:

Step 1 Replace x by y and y by x

Step 2 Solve for y

Ex 1: If f (x) = 3x + 5 find $f^{-1}(x)$

y = 3x + 5	
$\rightarrow x = 3y+5$	Replace x by y and y by x
$\rightarrow y = \frac{x}{3} - \frac{5}{3}$	Algebra: solve for y
$\rightarrow f^{-1}(x) = \frac{x}{3} - \frac{5}{3}$	Notation for Inverse function

b) Note:

Ex 2: Is g(x) = x^2-5 the inverse of f (x) = $\sqrt{x+5}$

Step 1 Is f(x) one to one ?

Consider f(x_1) = f(x_2) $\rightarrow$ $\sqrt{x_1+5} = \sqrt{x_2+5} \rightarrow x_1 = x_2$

Hence: f is one to one

Step 2 Is g(x) = $f^{-1}(x)$? We must check to see if: g(f(x)) = x

Consider g(x) = x^2-5 $\rightarrow$ g(f(x)) = $f^2(x) - 5 = \left(\sqrt{x+5}\right)^2 - 5 = x$

Hence: g(x) = $f^{-1}(x)$

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11 a) Definition of Two Special Functions

(i) The <u>Floor</u> function of x is the largest integer less than or equal to x

(Floor function of x rounds x down to nearest Integer)

Also called Least Integer Function

Notation: f(x) = $\lfloor x \rfloor$

(ii) The <u>Ceiling</u> function of x is the smallest integer greater than or equal to x

(Ceiling function of x up to the nearest Integer)

Aslo called Greatest Integer Function

Notation: f(x) = $\lceil x \rceil$

Ex:

$\lfloor 3.5 \rfloor$ = 3

$\lfloor -1.5 \rfloor$ = -2

$\lceil 3.5 \rceil$ = 4

$\lceil -1.5 \rceil$ = -1

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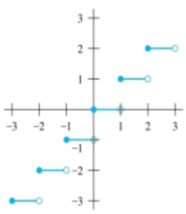
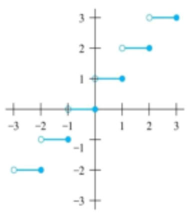
b) Graphs of Floor and Ceiling Functions

Floor Function



$y = \lfloor x \rfloor$

Ceiling Function



$y = \lceil x \rceil$

Ex: $\forall x$: $x \in \mathbb{R}$, $2 \le x < 3$, $\lfloor x \rfloor = 2$

Ex: $\forall x$: $x \in \mathbb{R}$, $2 < x \le 3$, $\lceil x \rceil = 3$

c) Recall: $\forall x \in \mathbf{R}$, $x = n + d$, $n \in \mathbf{Z}$, $d \in \mathbf{R}$

Ex: $3.14159 = 3 + .14159$

d) Proofs with Floor / Ceiling Functions

Ex 1: Prove that if x is a Real number, then $\lceil 2x + 1 \rceil = \lceil x \rceil + \lceil x + ½ \rceil$

Proof: Consider $x = n + d$ where n is the integer part and d is the decimal part $\rightarrow 0 \le d < 1$
(By Cases)

Case 1: $0 < d \le ½$

LHS $= \lceil 2x + 1 \rceil = \lceil 2(n + d) + 1 \rceil = \lceil 2n + (2d + 1) \rceil = 2n + 2$

Since $0 < d \le ½$
$\rightarrow 0 < 2d \le 1$
$\rightarrow 1 < 2d+1 \le 2$

RHS $= \lceil x \rceil + \lceil x + ½ \rceil = \lceil n + d \rceil + \lceil n + (d + ½) \rceil = (n+1) + (n+1) = 2n+2$

Since $0 < d \le ½$
$\rightarrow 1/2 < (d+1/2) \le 1$

$\rightarrow$ LHS = RHS

Case 2: $½ < d < 1$

LHS $= \lceil 2x + 1 \rceil = \lceil 2(n + d) + 1 \rceil = \lceil 2n + 1 + 2d \rceil = 2n + 3$

Since $½ < d < 1$
$\rightarrow 1 < 2d < 2$

RHS $= \lceil x \rceil + \lceil x + ½ \rceil = \lceil n + d \rceil + \lceil n + (d + ½) \rceil = (n+1)+(n+2) = 2n+3$

Since $½ < d < 1$
$\rightarrow 1 < d + \frac{1}{2} < \frac{3}{2}$

$\rightarrow$ LHS = RHS

Ex 2: Prove that if x is a Real number, then $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$

Proof: Let x = n + d where n is the integer part and d is the decimal part → $0 \le d < 1$.
(Cases)

Case 1: $0 \le d < ½$

Case 2: $½ \le d < 1$

e) Summary Floor and Ceiling Function values

TABLE 1 Useful Properties of the Floor and Ceiling Functions.
(n is an integer, x is a real number)

(1a)	$\lfloor x \rfloor = n$ if and only if $n \le x < n+1$
(1b)	$\lceil x \rceil = n$ if and only if $n-1 < x \le n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x-1 < n \le x$
(1d)	$\lceil x \rceil = n$ if and only if $x \le n < x+1$
(2)	$x-1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x+1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x+n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x+n \rceil = \lceil x \rceil + n$

Exercise 1: Let g be the function from the set {a,b,c} to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let f be the function from the set {a,b,c} to the set {1,2,3} such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$.

a) What is the composition of f and g ?

b) What is the composition of g and f ?

Exercise 2: Prove (direct method) that for all $x \in R$, $a \in Z$

$$\lfloor x+a \rfloor = \lfloor x \rfloor + a$$

Exercise 3: Prove (by cases) that for all Real numbers x

$$\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x+\tfrac{1}{2} \rfloor$$

Hint : Let $x = n + d$. Note n is the Integer part and d is decimal part.

Case 1 $0 \le d < \frac{1}{2}$

Case 2 $\frac{1}{2} \le d < 1$