7.2 Trigonometric Integrals

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Definitions & Theorems:

1. Strategy for evaluating $\int \sin^m x \cos^n x \, dx$
   (a) If the power of cosine is odd $(n = 2k + 1)$, save one consine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaing factors in terms of sine:
   $$\int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x \, (\cos^2 x)^k \cos x \, dx = \int \sin^m x \, (1 - \sin^2 x)^k \cos x \, dx$$
   Then substitute $u = \sin x$.
   (b) If the power of sine is odd $(m = 2k + 1)$, save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of consine:
   $$\int \sin^{2k+1} x \cos^n x \, dx = \int (\sin^2 x)^k \cos^n x \sin x \, dx = \int (1 - \cos^2 x)^k \cos^n x \sin x \, dx$$
   Then substitute $u = \cos x$. [Note that if the powers of both sine and cosine are odd, either (a) or (b) can be used.
   (c) If the powers of both sine and cosine are even, use the half-angle identities
   $$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$
   $$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$
   It is sometimes helpful to use the identity:
   $$\sin x \cos x = \frac{1}{2}\sin 2x$$
2. Strategy for evaluating $\int \tan^m x \sec^n x \, dx$
   (a) If the power of secant is even $(n = 2k, k \geq 2)$, save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$:
   $$\int \tan^m x \sec^{2k} x \, dx = \int \tan^m x \, (\sec^2 x)^{k-1} \sec^2 x \, dx = \int \tan^m x \, (1 - \tan^2 x)^{k-1} \sec^2 x \, dx$$
   Then substitute $u = \tan x$.
   (b) If the power of tangent is odd $(m = 2k + 1)$, save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$:
   $$\int \tan^{2k+1} x \sec^n x \, dx = \int (\tan^2 x)^k \cos^{n-1} x \sec x \tan x \, dx = \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x \, dx$$
   Then substitute $u = \sec x$.
3. To evaluate the Integrals (a) $\int \sin mx \cos nx \, dx$, (b) $\int \sin mx \sin nx \, dx$, or (c) $\int \cos mx \cos nx \, dx$, use the corresponding identity:
   (a) $\sin A \cos B = \frac{1}{2}[\sin(A - B) + \sin(A + B)]$
   (b) $\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$
   (c) $\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$

Examples:

1. $\int sin^2 \theta \, d\theta$
   a. Method 1: By Parts
   $$\int sin^2 \theta \, d\theta = \int \sin\theta \sin\theta \, d\theta$$
   Let $u = \sin\theta$, $dv = \sin\theta \, d\theta \to du = \cos\theta \, d\theta, v = -\cos\theta$
   $$\int sin^2 \theta \, d\theta = \int \sin\theta \sin\theta \, d\theta = -\sin\theta\cos\theta + \int \cos^2\theta \, \mathrm{d}\theta$$
   $$= -\sin\theta\cos\theta + \int (1 - \sin^2\theta) \, \mathrm{d}\theta = -\sin\theta\cos\theta + \theta - \int \sin^2\theta \, \mathrm{d}\theta$$
   $$\to 2\int sin^2\theta \, d\theta = -\sin\theta\cos\theta + \theta + C$$
   $$\to \int sin^2\theta \, d\theta = \frac{1}{2}(-\sin\theta\cos\theta + \theta) + C$$
   b. Method 2: Using $\sin^2 x = \frac{1}{2}(1 - \sin 2x)$
   $$\int \sin^2\theta \, \mathrm{d}\theta = \int \frac{1}{2}(1 - \cos(2\theta)) \, \mathrm{d}\theta = \frac{1}{2}\int \mathrm{d}\theta - \frac{1}{2}\int \cos(2\theta) \, \mathrm{d}\theta = \frac{1}{2}\theta - \frac{1}{4}\sin(2\theta) + C$$

2. $\int sin^3\theta\, cos^3\theta\, \mathrm{d}\theta$

a. Method 1: Power of cosine is odd

$$\int \sin^3\theta\cos^3\theta\,\mathrm{d}\theta = \int \sin^3\theta\cos^2\theta\cos\theta\,\mathrm{d}\theta = \int \sin^3\theta\left(1-\sin^2\theta\right)\cos\theta\,\mathrm{d}\theta$$

Let $u = \sin\theta \to du = \cos\theta\, d\theta$

$$\int \sin^3\theta\left(1-\cos^2\theta\right)\cos\theta\,\mathrm{d}\theta = \int u^3\left(1-u^2\right)\mathrm{d}u = \frac{u^4}{4}-\frac{u^6}{6}+C = \frac{\sin^4\theta}{4}-\frac{\sin^6\theta}{6}+C$$

b. Method 2: Power of sine is odd

$$\int \sin^3\theta\cos^3\theta\,\mathrm{d}\theta = \int \sin^2\theta\cos^3\theta\sin\theta\,\mathrm{d}\theta = \int \left(1-\cos^2\theta\right)\cos^3\theta\sin\theta\,\mathrm{d}\theta$$

Let $u = \cos\theta \to du = -\sin\theta\, d\theta$

$$\int \left(1-\cos^2\theta\right)\cos^3\theta\sin\theta\,\mathrm{d}\theta = -\int \left(1-u^2\right)u^3 du = -\frac{u^4}{4}+\frac{u^6}{6}+C = -\frac{\cos^4\theta}{4}+\frac{\cos^6\theta}{6}+C$$

3. $\int sin^2\theta\, cos^2\theta\, \mathrm{d}\theta$

a. Method 1:

$$\int sin^2\theta\, cos^2\theta\,\mathrm{d}\theta = \int sin^2\theta\,(1-sin^2\theta)\,\mathrm{d}\theta = \int \sin\theta\, sin\,\theta - \int sin^3\theta\sin\theta\,\mathrm{d}\theta$$

$$= \frac{1}{2}(-\sin\theta\cos\theta+\theta) - \left(-sin^3\theta\cos\theta + 3\int sin^2\theta\, cos^2\theta\,\mathrm{d}\theta\right)$$

$$\to 4\int sin^2\theta\, cos^2\theta\,\mathrm{d}\theta = \frac{1}{2}(-\sin\theta\cos\theta+\theta) + sin^3\theta\cos\theta + C$$

$$\to \int sin^2\theta\, cos^2\theta\,\mathrm{d}\theta = -\frac{1}{8}\sin\theta\cos\theta + \frac{1}{8}\theta + \frac{1}{4}sin^3\theta\cos\theta + C$$

b. Method 2:

$$\int sin^2\theta\, cos^2\theta\,\mathrm{d}\theta = \int \frac{1}{2}[1-\cos(2\theta)] * \frac{1}{2}[1+\cos(2\theta)]\,\mathrm{d}\theta = \frac{1}{4}\int (1-cos^2(2\theta))\,\mathrm{d}\theta$$

$$= \frac{1}{4}\int \mathrm{d}\theta - \frac{1}{4}\int cos^2(2\theta)\,\mathrm{d}\theta = \frac{\theta}{4} - \frac{1}{4}\int \frac{1}{2}[1+\cos(4\theta)]\,\mathrm{d}\theta = \frac{\theta}{4} - \frac{1}{8}\int \mathrm{d}\theta - \frac{1}{8}\int \cos(4\theta)\,\mathrm{d}\theta$$

$$= \frac{\theta}{8} - \frac{1}{8}\sin(4\theta)\left(\frac{1}{4}\right) = \frac{\theta}{8} - \frac{\sin(4\theta)}{32} + C$$

c. Method 3:

$$\int sin^2\theta\, cos^2\theta\,\mathrm{d}\theta = \int (\sin\theta\cos\theta)^2\,\mathrm{d}\theta = \int \left(\frac{\sin(2\theta)}{2}\right)^2\mathrm{d}\theta = \frac{1}{4}\int \sin^2(2\theta)\,\mathrm{d}\theta$$

$$= \frac{1}{4}\int \frac{1}{2}(1-\cos(4\theta))\,\mathrm{d}\theta = \frac{\theta}{8} - \frac{1}{8}\sin(4\theta)\left(\frac{1}{4}\right) + C = \frac{\theta}{8} - \frac{1}{32}\sin(4\theta) + C$$

4. $\int \tan^5\theta\sec^6\theta\,\mathrm{d}\theta$

a. Method 1: Power of tangent is odd

$$\int \tan^5\theta\sec^6\theta\,\mathrm{d}\theta = \int \tan^4\theta\sec^5\theta\tan\theta\sec\theta\,\mathrm{d}\theta = \int (\sec^2\theta-1)^2\sec^5\theta\tan\theta\sec\theta\,\mathrm{d}\theta$$

Let $u = \sec\theta \to du = \sec\theta\tan\theta\, d\theta$

$$\int (\sec^2\theta-1)^2\sec^5\theta\tan\theta\sec\theta\,\mathrm{d}\theta = \int \left(u^2-1\right)^2 u^5\,\mathrm{d}u \quad (u=\sec\theta)$$

b. Method 2: Power of secant is even

$$\int \tan^5\theta\sec^6\theta\,\mathrm{d}\theta = \int \tan^5\theta\sec^4\theta\sec^2\theta\,\mathrm{d}\theta = \int \tan^5\theta\,(1+\tan^2\theta)^2\sec^2\theta\,\mathrm{d}\theta$$

Let $u = \tan\theta \to du = \sec^2\theta\, d\theta$

$$\int \tan^5\theta\,(1+\tan^2\theta)^2\sec^2\theta\,\mathrm{d}\theta = \int u^5\left(1+u^2\right)^2\mathrm{d}u \quad (u=\tan\theta)$$

5. $\int \tan^2\theta\sec\theta\,\mathrm{d}\theta$

$$\int \tan^2\theta\sec\theta\,\mathrm{d}\theta = \int (\sec^2\theta-1)\sec\theta\,\mathrm{d}\theta = \int (\sec^3\theta-\sec\theta)\,\mathrm{d}\theta = \int \sec^2\theta\sec\theta\,\mathrm{d}\theta - \ln|\sec\theta+\tan\theta|$$

Let $u = \sec\theta, dv = \sec^2\theta\, d\theta \to du = \sec\theta\tan\theta\, d\theta, v = \tan\theta$

$$\int \tan^2\theta\sec\theta\,\mathrm{d}\theta = \int \sec^2\theta\sec\theta\,\mathrm{d}\theta - \ln|\sec\theta+\tan\theta| = \sec\theta\tan\theta - \int \tan\theta\sec^2\theta\,\mathrm{d}\theta - \ln|\sec\theta+\tan\theta|$$

$$\to 2\int \tan^2\theta\sec\theta\,\mathrm{d}\theta = \sec\theta\tan\theta - \ln|\sec\theta+\tan\theta| + C$$

$$\to \int \tan^2\theta\sec\theta\,\mathrm{d}\theta = \frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta+\tan\theta| + C$$

$$\rightarrow 2\int \tan^2\theta \sec\theta \,\mathrm{d}\theta = \sec\theta\tan\theta - \ln|\sec\theta + \tan\theta| + C$$

$$\rightarrow \int \tan^2\theta \sec\theta \,\mathrm{d}\theta = \frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$$