Sec 2.2 Operations on Sets

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Section 2.2 Operations on Sets

Comp 232
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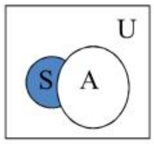
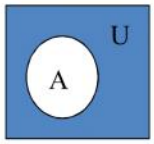
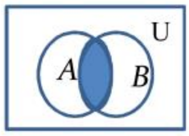
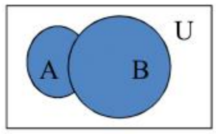
1. Define the operations on a set: Final result is shaded area below

Term	Symbol	Set Builder expression	Venn Diagram
Compliment of A compare to S	S - A	$\{x \mid (x \in S) \land (x \notin A)\}$	
Compliment of A compare to U	U - A denoted as $\bar{A}$	$\{x \mid (x \in U) \land (x \notin A)\}$ $\equiv \{x \mid x \in \bar{A}\}$ $\equiv \{x \mid \neg(x \in A)\}$	
Intersection	$A \cap B$	$\{x \mid (x \in A) \land (x \in B)\}$	
Union	$A \cup B$	$\{x \mid (x \in A) \lor (x \in B)\}$	

Ex 1: let $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{4, 5, 6, 7\}$, $B = \{0, 5, 6\}$, $C = \{8, 9\}$

$A - B = \{4, 7\}$

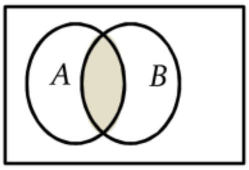
$\bar{A} = \{0, 1, 2, 3, 8, 9\}$

$A \cap B = \{5, 6\}$

$B \cap C = \{\} = ø$

$B \cup C = \{0, 5, 6, 8, 9\}$

Ex 2: The Cardinality of Two Sets

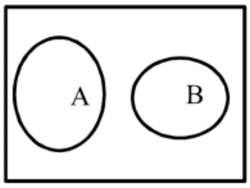


$|A \cup B| \neq |A| + |B|$ Why ?

We are counting elements in the Intersection twice.

$|A \cup B| = |A| + |B| - |A \cap B|$

or $n(A \cup B) = n(A) + n(B) - n(A \cap B)$



$|A \cap B| = 0$

A, B are called Disjoint sets

Ex 3: Let A be the set of students in the class who have taken Calculus, B be the set of students in class who have taken Matrices. $n(A) = 15$, $n(B) = 13$, $n(A \cap B) = 5$

$n(A \cup B) = 15 + 13 - 5 = 23$

$n(A \cup B)$ is the count of the number of student who have taken Calculus or Matrices or both.

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2. Equivalence between the three systems: Logic symbols, Boolean Algebra symbols, Set symbols

a) Recall from the Table of Set operations that:

<u>Compliment</u> involves <u>Not</u>: $\bar{A} = \{x \mid \neg(x \in A)\} = \{x \mid (x \notin A)\}$

<u>Intersection</u> involves <u>And</u>: $A \cap B = \{x \mid (x \in A)\} \land (x \in B)\}$

<u>Union</u> involves <u>OR</u>: $A \cup B = \{x \mid (x \in A)\} \lor (x \in B)\}$

b) This leads to the following equivalences:

Word	Set notation	Logic notation	Boolean Algebra Notation
Not	$\bar{A}$	$\neg A$	$\overline{A}$
And	$A \cap B$	$A \land B$	A*B = AB
Or	$A \cup B$	$A \lor B$	A+B

c) This leads to the equivalence of the logic properties previously developed. The properties table is summarized on the next slide.
It needs to be completed (see p 27 & 130 of Rosen Text 7th ed.)

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d) Summary of Equivalent Notations:

Logic symbols	Set symbols	Boolean Algebra	
$\neg (A \wedge B) \equiv \neg A \vee \neg B$ $\neg (A \vee B) \equiv \neg A \wedge \neg B$	$\overline{(A \cap B)} =$ $\overline{(A \cup B)} =$	$\overline{p\,q} =$ $\overline{p + q} =$	DeMorgan's Rules
$A \wedge B \equiv$ $A \vee B \equiv$	$A \cap B =$ $A \cup B =$	$p\,q = q\,p$ $p + q = q + p$	Commutative
$C \wedge (A \wedge B) \equiv$ $C \vee (A \vee B) \equiv$	$C \cap (A \cap B) =$ $C \cup (A \cup B) =$	$c\,(p\,q) = (c\,p)\,q$ $c+(p+q) = (c+p)+q$	Associative
$A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$ $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$	$A \cap (B \cup C) =$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$p\,(q + r) =$ $p + (q\,r) = (p+q)(p+r)$	Distributive
$A \wedge A \equiv$ $A \vee A \equiv$	$A \cap A =$ $A \cup A =$	$p\,p = p$ $p + p = p$	Idempotent
$A \wedge \mathbf{T} \equiv$ $A \vee \mathbf{F} \equiv$	$A \cap \mathbf{U} =$ $A \cup \Phi =$	$p * 1 = p$ $p + 0 = p$	Identites
$A \wedge \mathbf{F} \equiv$ $A \vee \mathbf{T} \equiv T$	$A \cap \Phi =$ $A \cup \mathbf{U} = U$	$p * 0 =$ $p + 1 = 1$	Domination
$A \wedge \neg A \equiv F$ $A \vee \neg A \equiv T$	$A \cap \overline{A} = \emptyset$ $A \cup \overline{A} = U$	$p\,\overline{p} = 0$ $p + \overline{p} = 1$	Negation

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3. Verification of set expressions

a) Notation: Uppercase letters and { } brackets are read as "set":

Ex: $\{ x \mid x \in (A \cap B) \}$ is "set of x such that x is contained in the intersection of sets A, B"

b) Summary of equivalent statements:

$\{x \mid x \in \bar{A} \} \equiv \{x \mid x \notin A\}$	Def. of Compliment of a set (Not)
$\{x \mid x \notin A\} \equiv \{x \mid \neg(x \in A)\}$	Def. of $\notin$
$\{x \mid x \in (A \cap B)\} \equiv \{x \mid x \in A \wedge x \in B\}$	Def. of Intersection of sets (And)
$\{x \mid x \in (A \cup B)\} \equiv \{x \mid x \in A \vee x \in B\}$	Def. of Union of sets (Or)

c) Method 1 Use: LHS set = RHS set iff [LHS set $\subseteq$ RHS set and RHS set $\subseteq$ LHS]

Ex: Show that: $A - B = A \cap \bar{B}$	
Step 1 Consider $x \in A - B$	
$\rightarrow x \in A \wedge x \notin B$	Def A-B
$\rightarrow x \in A \wedge x \in \bar{B}$	Def compliment
$\rightarrow x \in A \cap \bar{B}$	Def $\cap$
$\rightarrow A - B \subseteq A \cap \bar{B}$	Def subset $\subseteq$
Step 2 Consider $x \in A \cap \bar{B}$ In a similar way it can be shown $x \in A - B$ $A \cap \bar{B} \subseteq A - B$	
$[A \cap \bar{B} \subseteq A - B \text{ and } A - B \subseteq A \cap \bar{B}] \rightarrow A - B = A \cap \bar{B}$	Def of equal sets

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d) Method 2 Use Set-Builder notation without assuming set identities

Ex: Show that: $\overline{\bar{A} \cup B} = A \cap \bar{B}$

$\overline{\bar{A} \cup B}$	$= \{x \mid x \notin (\bar{A} \cup B)\}$	
	$= \{x \mid \neg[x \in (\bar{A} \cup B)]\}$	Def. of $\notin$
	$= \{x \mid \neg[x \in \bar{A}] \vee (x \in B)\}$	Def. of Union
	$= \{x \mid \neg [(x \notin A) \vee (x \in B)] \}$	Def of Compliment
	$= \{x \mid \neg [\neg (x \in A) \vee (x \in B)] \}$	Def of $\notin$
	$= \{x \mid [(x \in A) \wedge \neg (x \in B)] \}$	De Morgan
	$= \{x \mid (x \in A) \wedge (x \notin B)\}$	Def. of $\notin$
	$= x \in A \wedge x \in \bar{B}$	Def of Compliment
	$= A \cap \bar{B}$	Def. of Intersection

QED

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4. Applications: Representing sets with Boolean strings

Ex 1: Consider data kept by a provincial government concerning 5 cities and their total property evaluation in millions of dollars, number of properties and the tax rate per 100$ evaluation.
Universal set is U = {A,B,C,D,E}.
Sets are represented by bit strings: (i) Bit string 11111 rep. U (ii) Bit string 11001 rep.

	String	City A	City B	City C	City D	City E
1m$ < eval ≤ 5m$	B_1	1	0	0	0	0
5m$ < eval ≤ 15m$	B_2	0	0	1	0	0
eval > 15m$	B_3	0	1	1	0	1
0 < # properties ≤ 300	B_4	0	0	1	1	0
300 < # prop ≤ 1000	B_5	1	1	0	0	1
1$ < tax rate ≤ 1.30$	B_6	0	1	1	0	0
1.30$ < tax rate	B_7	1	0	0	1	1

a) Form set S_a: designates cities with (1m$< eval ≤5m$) and (300 <# prop ≤ 1000) and (tax rate >1.30$)

We need bit string B_a = bit-wise And with B_1, B_5, B_7 = 10000 → $s_a = \{A\}$

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	String	City A	City B	City C	City D	City E
1m$ < eval ≤ 5m$	B_1	1	0	0	0	0
5m$ < eval ≤ 15m$	B_2	0	0	1	0	0
eval > 15m$	B_3	0	1	1	0	1
0 < # properties ≤ 300	B_4	0	0	1	1	0
300 < # prop ≤ 1000	B_5	1	1	0	0	1
1$ < tax rate ≤ 1.30$	B_6	0	1	1	0	0
1.30$ < tax rate	B_7	1	0	0	1	1

b) Form the set S_b that designates cities with 5m$ < eval ≤ 15m$ or tax rate >1.30$

We need bit string $B_b =$ bit-wise Or with B_2 or B_7 = 1011 $\rightarrow s_b = \{A, C, D, E\}$

c) Form the set S_c that designates cities with missing total evaluation data

We need bit string $B_c =$ Not B_1 and Not B_2 and Not $B_3 =$ Not(B_1 or B_2 or B_3)

Not (bit-wise Or with B_1, B_2, B_3) = $\overline{11101} = \ 00010 \ \rightarrow s_c = \{D\}$

Ex 2: Adapt the above procedure to find the set S_d that designates cities that have more than one total evaluation

$$B_d = (B_1 \wedge B_2) \vee (B_2 \wedge B_3) \vee (B_3 \wedge B_1) = 00100 \ \rightarrow s_d = \{C\}$$

Ex 3: Consider data collected concerning the population in various postal codes designated by the Boolean strings below:

First Letter of Postal code		A	B	G	H	I	J	M	T
U designates the Universal set	U =	1	1	1	1	1	1	1	1
B_1 designates postal codes where population ≤ 1million	$B_1 =$	0	1	0	1	1	0	0	0
B_2 designates postal codes where 1 million < pop. ≤ 2 million	$B_2 =$	1	0	0	0	0	1	0	0
B_3 designates postal codes where 2 million < pop. ≤ 3 million	$B_3 =$	0	0	1	0	1	0	1	1
B_4 designates postal codes where the majority of population speak French	$B_4 =$	0	0	1	0	0	1	0	0

Form a Boolean string B_i for each condition below and using the results write the final set S_i :

a) Universal set
b) Set where (1 million < pop. ≤ 3 million)
c) (Majority speak French) and (2 million < pop. ≤ 3 million)
d) A set that designates if there is some postal code where pop. has been omitted (data error)
e) A set that designates if a postal code exists in more than one pop. size (data error)

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Answer:

a) $B_a = U, S_a = \{\mathrm{A, B, G, H, I, J, M, T}\}$
b) $B_b = B_2 \vee B_3$ = 1010 1111, $S_b = \{\mathrm{A, G, I, J, M, T}\}$
c) $B_c = B_3 \vee B_4$ = 0010 0000, $S_c = \{\mathrm{G}\}$
d) $B_d = \neg(B_1 \vee B_2 \vee B_3)$ = $\overline{1111\ 11\ 11}$ = 0000 0000, $S_d = \phi$
e) $B_e = (B_1 \wedge B_2) \vee (B_2 \wedge B_3) \vee (B_1 \wedge R_3)$ = 0000 1000, $S_e = \{\mathrm{I}\}$