7.1 Integration by Parts

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Definitions & Theorems:

1. Formula: the formula integration by parts
$$\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int \mathrm{g}(x)f'(x)\,\mathrm{d}x$$
2. Formula:
Let $u = f(x)$ and $v = g(x) \rightarrow \mathrm{d}u = f'(x)$ and $\mathrm{d}v = g'(x)\,\mathrm{d}x$
$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$$
3. Formula:
$$\int_a^b f(x)g'(x)\,\mathrm{d}x = f(x)g(x)]_a^b - \int_a^b \mathrm{g}(x)f'(x)\,\mathrm{d}x$$

Proofs or Explanations:

1. Formula1:
The Product Rule states that if f and g are differentiable functions, then
$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = f(x)g'(x) + \mathrm{g}(x)f'(x)$$
In the notation for indefinite integrals this equation becomes
$$\int [f(x)g'(x) + \mathrm{g}(x)f'(x)]\,\mathrm{d}x = f(x)g(x)$$
or
$$\int f(x)g'(x)\,\mathrm{d}x + \int \mathrm{g}(x)f'(x)\,\mathrm{d}x = f(x)g(x)$$
We can rearrange this equation as
$$\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int \mathrm{g}(x)f'(x)\,\mathrm{d}x$$

Extra topics:

1. How to use formula2:
For example: $\int x\mathrm{e}^x\,\mathrm{d}x$
   a. Do I see a function that differentiates "nicely" and a function that integrals "nicely"?
   b. Yes. We see x which differentiates nicely, and we see e^x integrals nicely.

   Idea: Let $u = x$ and $dv = e^x dx$
   $\rightarrow du = dx$ and $v = e^x$
   $$\rightarrow \int x\mathrm{e}^x\,\mathrm{d}x = xe^x - \int \mathrm{e}^x\,\mathrm{d}x = xe^x - e^x + C$$

Examples:

1. $\displaystyle\int x \sin x\,\mathrm{d}x$

   Method1: Using Formula 1
   Let $f(x) = x$ and $g'(x) = \sin x \rightarrow f'(x) = 1$ and $g(x) = -\cos x$
   $$\int x\sin x\,\mathrm{d}x = f(x)g(x) - \int g(x)f'(x)\,\mathrm{d}x = x(-\cos x) - \int(-\cos x)\,\mathrm{d}x = -x\cos x + \sin x + C$$
   Mehtod2: Using Formula 2
   Let $u = x$ and $dv = \sin x\,dx \rightarrow du = dx$ and $v = -\cos x$

$$\int x\sin x\,dx = \int \overbrace{x}^{u}\,\overbrace{\sin x\,dx}^{dv} = \overbrace{x}^{u}\,\overbrace{(-\cos x)}^{v} - \int \overbrace{(-\cos x)}^{v}\,\overbrace{dx}^{du}$$

$$= -x\cos x + \int \cos x\,dx$$

$$= -x\cos x + \sin x + C$$

2. $\int x^2 \cos x \, \mathrm{d}x$

   Let $u = x^2, dv = \cos x\,dx \to du = 2xdx, v = sinx$

   $\int x^2 \cos x\,\mathrm{d}x = x^2 \sin x - \int \sin x\, 2xdx = x^2 \sin x - 2(-x\cos x + \sin x) + C$

3. $\int \mathrm{e}^x \sin x \,\mathrm{d}x$

   1) Let $u_1 = \mathrm{e}^x, \mathrm{d}v_1 = \sin x\,dx \to du_1 = \mathrm{e}^x dx, v_1 = -\cos x$

      $\int \mathrm{e}^x \sin x\,\mathrm{d}x = e^x(-\cos x) - \int -\cos x\,\mathrm{e}^x\,\mathrm{d}x = -e^x\cos x + \int \cos x\,\mathrm{e}^x\,\mathrm{d}x$

   2) Let $u_2 = \mathrm{e}^x, \mathrm{d}v_2 = \cos x\,dx \to du_2 = \mathrm{e}^x dx, v_2 = \sin x$

      $\int \cos x\,\mathrm{e}^x\,\mathrm{d}x = e^x \sin x - \int \sin x\,\mathrm{e}^x\,\mathrm{d}x$

   3) $\int \mathrm{e}^x \sin x\,\mathrm{d}x = -e^x\cos x + e^x\sin x - \int \sin x\,\mathrm{e}^x\,\mathrm{d}x$

      $2\int \mathrm{e}^x \sin x\,\mathrm{d}x = -e^x \cos x + e^x \sin x$

      $\int \mathrm{e}^x \sin x\,\mathrm{d}x = \frac{1}{2}e^x(\sin x - \cos x) + C$

4. $\int \mathrm{e}^x \cos x\,\mathrm{d}x$

   Let

5. $\int x\tan^2 x\,\mathrm{d}x$

   $\int x\tan^2 x\,\mathrm{d}x = \int x(\sec^2 x - 1)\,\mathrm{d}x = \int x\sec^2 x\,\mathrm{d}x - \int x\,\mathrm{d}x$

   Let $u = x, dv = \sec^2 x \to du = dx, v = \tan x$

   $\int x\tan^2 x\,\mathrm{d}x = \int x\sec^2 x\,\mathrm{d}x - \int x\,\mathrm{d}x = x\tan x - \int \tan x\,\mathrm{d}x - \frac{x^2}{2} = x\tan x - \ln|\sec x| - \frac{x^2}{2} + C$

6. Find the error:

   $$\int \frac{\mathrm{d}x}{x} = \int \frac{1}{x} 1\,\mathrm{d}x$$

   Let $u = \frac{1}{x}, dv = 1dx \to du = -\frac{dx}{x^2}, v = x$

   $$\int \frac{\mathrm{d}x}{x} = \int \frac{1}{x} 1\,\mathrm{d}x = \frac{1}{x}x - \int x\frac{-\,\mathrm{d}x}{x^2} = 1 + \int \frac{\mathrm{d}x}{x}$$

   $\to 0 = 1$

   $\int \frac{\mathrm{d}x}{x}$ is the family of antiderivatives of $f(x) = \frac{1}{x}$,

   → It is a function or a family of functions, it is NOT a number.

   $\to \int \frac{\mathrm{d}x}{x} = F(x) + C$

   $\to \int \frac{\mathrm{d}x}{x} = F(x) + C_1$ *or* $\int \frac{\mathrm{d}x}{x} = F(x) + C_2$

   → So we cannot subtract it from both sides.

7. $\int \sin^n \theta\,\mathrm{d}\theta$

$$\int \sin^n \theta \, \mathrm{d}\theta = \int \sin^{n-1} \theta \sin \theta \, \mathrm{d}\theta$$

Let $u = \sin^{n-1} \theta \, , dv = \sin \theta \, \mathrm{d}\theta \rightarrow du = (\mathrm{n}-1)\sin^{n-2} \theta \cos \theta \, \mathrm{d}\theta, v = -\cos \theta$

$$\int \sin^n \theta \, \mathrm{d}\theta = \int \sin^{n-1} \theta \sin \theta \, \mathrm{d}\theta = \sin^{n-1} \theta \, (-\cos \theta) - \int (-\cos \theta) \, (\mathrm{n}-1)\sin^{n-2} \theta \cos \theta \, \mathrm{d}\theta$$

$$= -\sin^{n-1} \theta \cos \theta + (n-1) \int \sin^{n-2} \theta \, (1 - \sin^2 \theta) \, \mathrm{d}\theta$$

$$= -\sin^{n-1} \theta \cos \theta + (n-1) \int \sin^{n-2} \theta \, \mathrm{d}\theta - (n-1) \int \sin^n \theta \, \mathrm{d}\theta$$

$$\rightarrow n \int \sin^n \theta \, \mathrm{d}\theta = -\sin^{n-1} \theta \cos \theta + (n-1) \int \sin^{n-2} \theta \, \mathrm{d}\theta$$

$$\rightarrow \int \sin^n \theta \, \mathrm{d}\theta = -\frac{1}{n} \sin^{n-1} \theta \cos \theta + \frac{(n-1)}{n} \int \sin^{n-2} \theta \, \mathrm{d}\theta$$

8. $\int \tan^n \theta \, \mathrm{d}\theta$

$$\int \tan^n \theta \, \mathrm{d}\theta = \int \tan^{n-2} \theta \tan^2 \theta \, \mathrm{d}\theta = \int \tan^{n-2} \theta \, (\sec^2 \theta - 1) \, \mathrm{d}\theta = \int \tan^{n-2} \theta \sec^2 \theta \, \mathrm{d}\theta - \int \tan^{n-2} \theta \, \mathrm{d}\theta$$

Let $u = \tan \theta \rightarrow du = \sec^2 \theta$

$$\int \tan^n \theta \, \mathrm{d}\theta = \int u^{n-2} \, \mathrm{d}u - \int \tan^{n-2} \theta \, \mathrm{d}\theta = \frac{u^{n-1}}{n-1} - \int \tan^{n-2} \theta \, \mathrm{d}\theta = \frac{\tan^{n-1} \theta}{\tan \theta - 1} - \int \tan^{n-2} \theta \, \mathrm{d}\theta$$

9. $\int \ln x \, \mathrm{d}x$

$$\int \ln x \, \mathrm{d}x = \int \ln x \, 1 \, \mathrm{d}x$$

Let $u = \ln x \, , dv = 1dx \rightarrow du = \frac{dx}{x}, v = x$

$$\int \ln x \, \mathrm{d}x = x \ln x - \int x \frac{\mathrm{d}x}{x} = x \ln x - x + C$$

10. $\int \arctan x \, \mathrm{d}x$

$$\int \arctan x \, \mathrm{d}x = \int \arctan x \, 1 \, \mathrm{d}x$$

Let $u = \arctan x \, , dv = 1dx \rightarrow du = \frac{dx}{1+x^2}, v = x$

$$\int \arctan x \, \mathrm{d}x = x \arctan x - \int x \frac{\mathrm{d}x}{1+x^2}$$

Let $t = 1 + x^2 \rightarrow dt = 2x \, \mathrm{d}x$

$$\int \arctan x \, \mathrm{d}x = x \arctan x - \int x \frac{\mathrm{d}x}{1+x^2} = x \arctan x - \frac{1}{x} \ln\left(1 + x^2\right) + C$$

11. Let $f(x)$ be a one-to-one differentiable function with inverse $f^{-1}(x)$

$$\int f(x) \, \mathrm{d}x = \int f(x) 1 \, \mathrm{d}x$$

Let $u = f(x), dv = 1 \, \mathrm{d}x \rightarrow du = f'(x) \, \mathrm{d}x, v = x$

$$\int f(x) \, \mathrm{d}x = xf(x) - \int xf'(x) \, \mathrm{d}x = xf(x) - \int f^{-1}\left(f(x)\right) f'(x) \, \mathrm{d}x$$

Let $t = f(x) \rightarrow dt = f'(x) \, \mathrm{d}x$

$$\int f(x) \, \mathrm{d}x = xf(x) - \int f^{-1}(t) \, \mathrm{d}t$$

$$\rightarrow \int f(x) \, \mathrm{d}x = xf(x) - \int f^{-1}(u) \, \mathrm{d}u \, , where \; u = f(x)$$

Using 11 to prove 9 and 10

$$\int \ln x \, \mathrm{d}x \rightarrow f(x) = \ln x \, , f^{-1}(x) = e^x$$

$$\int \arctan x \, \mathrm{d}x \to f(x) = \arctan x \,, f^{-1}(x) = \tan x$$