11.4 The Comparison Test

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Definitions & Theorems:

1. The Comparison Test:

   Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

   (i) If $\sum b_n$ is convergent and $a_n \le b_n$ eventually holds, then $\sum b_n$ is also convergent.

   (ii) If $\sum b_n$ is divergent and $a_n \ge b_n$ eventually holds, then $\sum a_n$ is also divergent.

2. The Limit Comparison Test

   Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If
   $$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$
   where c is a finite number and $c > 0$, then either both series converge or both diverge.

Examples:

1. $$\sum_{n=2}^{\infty} \frac{1}{n-1}$$
   $$n-1 < n \Rightarrow \frac{1}{n-1} > \frac{1}{n}$$
   Since $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges (p-series with $p=1$), it follows that $\sum_{n=2}^{\infty} \frac{1}{n-1}$ diverges by Comparison Test.

2. $$\sum_{n=1}^{\infty} \frac{1}{3^n + n}$$
   $$3^n + n > 3^n \Rightarrow \frac{1}{3^n + n} < \frac{1}{3^n}$$
   Since $\sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ converges (geometric series with $r = \frac{1}{3}$), it follows that $\sum_{n=1}^{\infty} \frac{1}{3^n+n}$ converges by Comparison Test.

3. $$\sum_{n=1}^{\infty} \frac{1}{n^2 + 2}$$
   $$n^2 + 2 > n^2 \Rightarrow \frac{1}{n^2+2} < \frac{1}{n^2}$$
   Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series with $p=2$), it follows that $\sum_{n=1}^{\infty} \frac{1}{n^2+2}$ converges by Comparison Test.

4. $$\sum_{n=3}^{\infty} \frac{6^n}{2^n - 4}$$
   $$2^n - 4 < 2^n \Rightarrow \frac{1}{2^n-4} > \frac{1}{2^n} \Rightarrow \frac{6^n}{2^n-4} > \frac{6^n}{2^n} = (3)^n$$
   Since $\sum_{n=3}^{\infty} 3^n$ diverges (geometric series with $r=3$)$\Rightarrow \sum_{n=3}^{\infty} \frac{6^n}{2^n-4}$ diverges by Comparison Test.

5. $$\sum_{n=1}^{\infty} \frac{\arctan n}{n^{1.2}}$$
   $$\arctan n < \frac{\pi}{2} \Rightarrow \frac{\arctan n}{n^{1.2}} < \frac{\frac{\pi}{2}}{n^{1.2}}$$
   Since $\sum_{n=1}^{\infty} \frac{\frac{\pi}{2}}{n^{1.2}}$ converges (p-series with $p=1.2$) $\Rightarrow \sum_{n=1}^{\infty} \frac{\arctan n}{n^{1.2}}$ converges by Comparison Test.

6. $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n^2-\cos n}$

$$n^2-\cos n < n^2 \Rightarrow \frac{1}{n^2-\cos n} > \frac{1}{n^2} \Rightarrow \frac{n}{n^2-\cos n} > \frac{n}{n^2} = \frac{1}{n}$$

Since $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges (p-series with $p=1$) $\Rightarrow \sum_{n=1}^{\infty}\frac{n}{n^2-\cos n}$ diverges by Comparison Test.

7. $\displaystyle\sum_{n=1}^{\infty}\frac{n^2+2}{n^4+5}$

(i) Method1:

$$n^4+5 > n^4 \Rightarrow \frac{1}{n^4+5} < \frac{1}{n^4} \Rightarrow \frac{n^2+2}{n^4+5} < \frac{n^2+2}{n^4} = \frac{n^2}{n^4}+\frac{2}{n^4} = \frac{1}{n^2}+\frac{2}{n^4}$$

Since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ and $\sum_{n=1}^{\infty}\frac{2}{n^4}$ converges (p-series with $p=2,4$) $\Rightarrow \sum_{n=1}^{\infty}(\frac{n^2+2}{n^4})$ converges $\Rightarrow \sum_{n=1}^{\infty}\frac{n^2+2}{n^4+5}$ converges by Comparison Test.

(ii) Method2:

Let $a_n=\frac{n^2+2}{n^4+5}, b_n=\frac{1}{n^2} \Rightarrow \lim_{n\to\infty}\frac{a_n}{b_n}=1\in(0,\infty)$

Since $\sum_{n=1}^{\infty}b_n$ converges$\Rightarrow \sum_{n=1}^{\infty}\frac{n^2+2}{n^4+5}$ converges by Limit Comparison Test.

8. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{3^n-n}$

Let $a_n=\frac{1}{3^n-n}, b_n=\frac{1}{3^n} \Rightarrow \lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}(1-\frac{n}{3^n})$

Let $f(x)=\frac{x}{3^x} \Rightarrow \lim_{x\to\infty}\frac{x}{3^x}=\lim_{n\to\infty}\frac{1}{3^x\ln 3}=0 \Rightarrow \lim_{n\to\infty}\left(1-\frac{n}{3^n}\right)=(1-0)=1\in(0,\infty)$

Since $\sum_{n=1}^{\infty}b_n$ converges$\Rightarrow \sum_{n=1}^{\infty}\frac{n^2+2}{n^4+5}$ converges by Limit Comparison Test.

9. $\displaystyle\sum_{n=1}^{\infty}\frac{n^6-n^{\frac{1}{2}}+1}{n^7+n^6+16n}$

Let $a_n=\frac{n^6-n^{\frac{1}{2}}+1}{n^7+n^6+16n}, b_n=\frac{1}{n} \Rightarrow \lim_{n\to\infty}\frac{a_n}{b_n}=1\in(0,\infty)$

Since $\sum_{n=1}^{\infty}b_n$ diverges$\Rightarrow \displaystyle\sum_{n=1}^{\infty}\frac{n^6-n^{\frac{1}{2}}+1}{n^7+n^6+16n}$ diverges by Limit Comparison Test.