Section 9.1 and 9.3 Binary Relations
Representing Binary Relations

Comp 232
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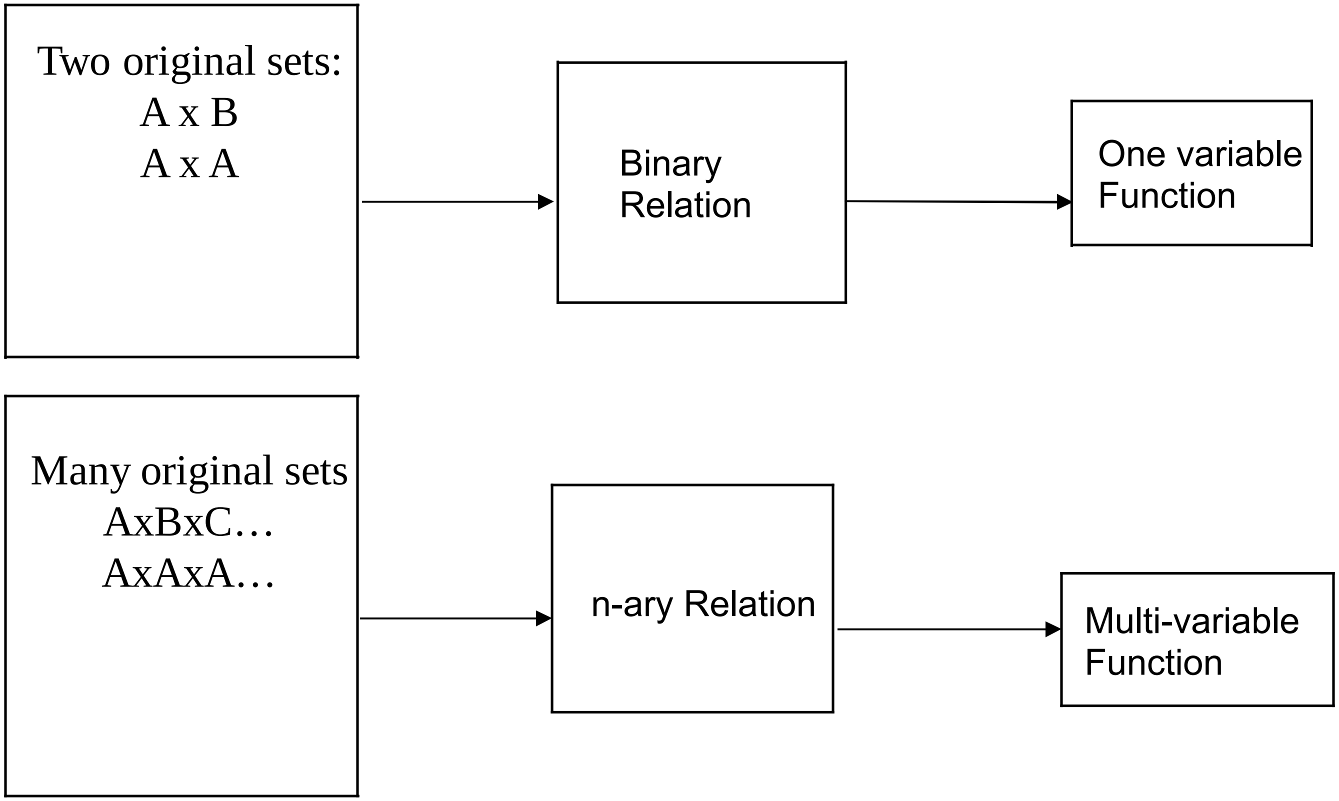
1. Where are we headed in Sections 9.1 and 9.3 ?

We introduced the notion of a relation when we covered Functions in Section 2.3-2.5



Cartesian Product

Relation
A Subset of a
Cartesion Product

Functions
Relations that are
One to One
or Many to One

Two original sets:
A x B
A x A

Binary
Relation

One variable
Function

Many original sets
AxBxC…
AxAxA…

n-ary Relation

Multi-variable
Function

In the last part of the course we look at Binary Relations of a set with itself. All Relations are subsets of A x A. The Relations may or may not be functions.

2. Methods to represent a Relation of a set with itself:

- List the ordered pairs: Consider A ={1,2}, $\overset{\text{AXA}}{R}$ ={ (1,2), (2,2) }
- Set Builder notation: R = {(x,y) ∈ ℝ × ℝ | x < y }, Note: 3R4 but $4\not{R}3$
  (R means relation, ℝ means Real numbers)
- Graphic - List elements of set twice Ex 1: 1 1
  - Join the pairs 2 2
- Directed Graph (Di-graph) - List elements of A once: (see following notes)
- Boolean Matrix (0-1 Matrix): (see following notes)

3. Directed graph or Di-graph for A→A:

Step 1 Put a point (called a vertex) for each element in set A. Each entry in A is written once

Step 2 Join the vertices that are related with arrows. Reversing arrows should be kept separate when the relation goes both ways

Step 3 If an entry is related to itself put a circular arrow around its vertex.

Ex: Consider R: A→A where A = {a, b, c} and R = { (a,a), (a,c), (b,c), (c,b), (b,b) }

We will use this form in Section 9.4

4. Matrix method to store a Relation on A × B: [A does not have to = B]

   a) Definition: A Matrix is a rectangular set of elements. (has rows and columns)
   A matrix is denoted by an uppercase letter
   Entries in Matrix are denoted by lowercase letters with subscripts for row, column
   Dimension of Matrix refers to the number of rows and number of columns

   Ex:

   Matrix M =

1	0	1
0	1	1

   b) We will use Boolean values for the matrix entries in the following work:

<table>
<tr><th>Review</th><th>Represent R with a matrix</th></tr>
<tr>
<td>Set A = {a,b,c}
Fix order of A: (order cannot change)
Form bit strings (ordered sets of 1,0)
S_1 = 1 0 1

1st bit =1 (True) → a is present
2nd bit = 0 (False) → b is not present
3rd bit =1 (True) → c is present</td>
<td>Set A = {a,b,c}
Fix order of A: (order cannot change)

Each element of R has two components so each bit entry will designate 2 positions in A: row for 1st, col for 2nd in each pair:
2nd position
1st
<table>
<tr><th></th><th>a</th><th>b</th><th>c</th></tr>
<tr><td>a</td><td>1</td><td>1</td><td>0</td></tr>
<tr><td>b</td><td>1</td><td>1</td><td>1</td></tr>
<tr><td>c</td><td>0</td><td>1</td><td>1</td></tr>
</table>
Each matrix entry designates one pair:
a, c bit = 0(False)→(a, c)∉R
......</td>
</tr>
</table>

We have R = {(a,a), (a,b), (b,a), (b,b), (b,c), (c,b), (c,c)}

In general :

We do not write the headings for the matrix entries

entry $m_{i\,j} = 1 \leftrightarrow (a_i, b_j) \in R$

entry $m_{i\,j} = 0 \leftrightarrow (a_i, b_j \not\subset R$

Ex : If A = {2,5,9}, order of A is fixed, consider $M_{R_1} = \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline 1 & 1 & 1 \\ \hline 0 & 1 & 1 \\ \hline \end{array}$

Since $m_{1\,1} = 1 \rightarrow$ (First element in A) R (First element in A) → $(2, 2) \in R$

Since $m_{1\,3} = 0 \rightarrow$ (First element in A) noR (Third element in A) → $(2, 9) \notin R$

Write the ordered pairs in Relation R_1: R1= {(2,2),(2,5),(5,2),(5,5),(5,9),(9,5),(9,9)}

5. Since a relation is a set of ordered pairs we can do set operations:

Review	**New:** Assume A = { a,b }
Negate a bit string S S = 1,0,1 becomes 0, 1, 0 called compliment of S	Negate bit Matrix: $M_R = \begin{array}{\|c\|c\|}\hline 1 & 0 \\ \hline 1 & 0 \\ \hline \end{array}$ becomes $M_{R_1} = \begin{array}{\|c\|c\|}\hline 0 & 1 \\ \hline 0 & 1 \\ \hline \end{array}$ R_1 = {(a, b), (b, b)} is the compliment of R = {(a,a), (b,a)}
Bit string And ≡ ($\wedge$, $\cap$) $S_1 = 1\ 0\ 0$ $\underline{S_1 = 1\ 1\ 0 \wedge}$ = 1 0 0	Bit string And ≡ ($\wedge$, $\cap$) Ex: Consider $M_R \wedge M_{R_1}$ $\begin{array}{\|c\|c\|}\hline 1 & 0 \\ \hline 1 & 0 \\ \hline \end{array} \wedge \begin{array}{\|c\|c\|}\hline 0 & 1 \\ \hline 0 & 1 \\ \hline \end{array} = \begin{array}{\|c\|c\|}\hline 0 & 0 \\ \hline 0 & 0 \\ \hline \end{array}$ $R \cap R_1 =$ ø (null set)

Review	**New:** Assume A = { a,b }
Bit string Or ≡ (∨, ∪) : S_1 = 1 0 0 S_2 = 1 1 0 ∨ = 1 1 0	Bit string Or ≡ (∨, ∪) Ex: Consider $M_R \vee M_{R_1}$: $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ R ∪ R_1 = A X A (All possible pairs)
Bit string Xor: S_1 = 1 0 0 S_2 = 1 1 0 ⊕ = 0 1 0 True when: (S1∨S2) ∧ ¬(S1∧S2)	Bit string Xor: Ex: Consider $M_R \oplus M_{R_2}$: $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ R ⊕ R_2 = {(b,a), (b,b)}

6. Composition of Relations: This operation has a structure similar to composition of functions.

Ex: Consider A = {3,4,5} with relations
S = { (3,4), (4,3), (4,5), (5,3) }
T = { (3,5), (4,4), (4,5), (5,4) }

We will do T ∘ S. Do relation S first . (Idea also works if it is a set related to itself as in S ∘ S)
T ∘ S ordered pairs: 3R4 (from S) then 4R4 and 4R5 (from T) so composition T ∘ S has 3R4 and 3R5

Complete the Relation T ∘ S: T ∘ S = {(3,4), (3,5), (4,5), (4,4), (5,5)}

7. Composition of two relations can be done with matrices.

a) First we need to know the structure of matrix multiplication. Multiplication of matrices does not multiply corresponding entries We will use a base ten example below to get the idea how the matrix multiplication process is defined for: $M_L \times M_R$

Step 1 “Pour” row 1 of left matrix down col 1 of right matrix multiply corresponding entries then add all the products. This gives the entry in Row 1 Col 1 of the answer matrix

Step 2 Repeat Step 1 use Row 1 with Col 2 gives the entry in Row 1 Col 2 of the answer matrix.

Step 3 Repeat …Until all rows of left matrix have operated on all columns of the right matrix

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 0 \\ \hline 3 & 0 & 4 \\ \hline 5 & 1 & 2 \\ \hline\end{array} \; x \; \begin{array}{|c|c|c|}\hline 2 & 0 & 4 \\ \hline 3 & 1 & 1 \\ \hline 5 & 1 & 3 \\ \hline\end{array} = \begin{array}{|c|c|c|}\hline 1x2+2x3+0x5 & 1x0+2x1+0x1 & 1x4+2x1+0x3 \\ \hline 3x2+0x3+4x5 & 3x0+0x1+4x1 & 3x4+0x1+4x3 \\ \hline 5x2+1x3+2x5 & 5x0+1x1+2x1 & 5x4+1x1+2x3 \\ \hline\end{array} = \begin{array}{|c|c|c|}\hline 8 & 2 & 6 \\ \hline 26 & 4 & 24 \\ \hline 23 & 3 & 27 \\ \hline\end{array}$$

b) Now to calculate T ∘ S in previous example:

Step 1 Form the matrices for the relations: M_S , M_T

Step 2 Use the matrix multiplication structure above except treat 0, 1 as Boolean values and use Boolean Multiplication(and), Addition(or) to multiply: denote as $M_S \odot M_T$

Note the order of the matrices Ms Mt. In Matrix Algebra AXB ≠ BXA

Ex: Using the previous A = {3,4,5} with relations: S = { (3,4), (4,3), (4,5), (5,3) }
T = { (3,5), (4,4), (4,5), (5,4) }

$$M_{T\circ S} = M_S \odot M_T = \begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{matrix} \odot \begin{matrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{matrix} = \begin{matrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{matrix}$$

(Put S first!)

Using the answer to matrix $M_{T\circ S}$ we get Relation T ∘ S = {(3,4),(3,5),(4,4),(4,5),(5,5)}

8. Four types of Relations R where R is a subset of the Cartesian Product A × A

R is Relexive iff $\forall x$ $(x, x) \in R$ xRx ∈ R

R is Symmetric iff $\forall (x,y)$ $(x,y) \in R \rightarrow (y,x) \in R$ xRy → yRx

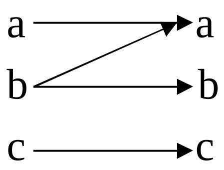
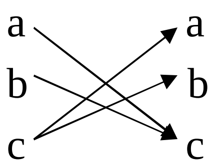
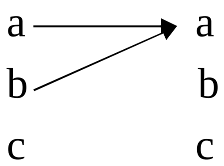
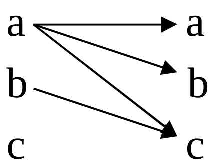
R is Anti-symmetric iff $\forall (x,y)$ $[(x,y) \in R \wedge (y,x) \in R] \rightarrow x = y$ xRy ∧ yRx → x=y

(is not opposite of Symmetric) Anti-symmetric ⟶ only time you have symmetry is when elements are equal

R is Transitive iff $\forall (x,y,z)$ $[(x,y) \in R \wedge (y,z) \in R] \rightarrow (x,z) \in R$ xRy ∧ yRz → xRz

R={(a,a)} is both Symmetric and Anti-symmetric

Ex: Let A = {a, b, c}

Type	List Relation R	Characteristic	Sketch
Reflexive	R_1={ (a,a), (b,a), (b,b), (c,c), }	aRa, bRb, cRc make it Reflexive. If cRc is missing → not Reflexive	
Symmetric	R_2={ (a,c), (b,c), (c,b), (c,a) }	aRc, cRa, bRc, cRb make it Symmetric If cRa is missing → not Symmetric	
Anti-symmetric	R_3={ (a,a), (b,a) }	aRa makes it Anti-symmetric If aRb is added → not Anti-symmetric	
Transitive	R_4={ (a,b), (b,c), (a,c), (a,a) }	aRa, bRc, aRc make it Transitive If aRc missing → not Transtive	

9. Matrix information compared to Relation information when R relates a set with itself ?

a) If xRy = { (x,y) | x ∈ A, y ∈ A, xRy}, A = {a,b,c} }
What do you know about the dimensions of M_R ?

M_{R_1} =

Number Rows = Number of Column → Mr is a Square matrix (3X3)

b) If R_1 = { (a, a), (a,b), (b,a), (b,b), (b,c), (c,b), (c,c) }
Calculate Matrix M_{R_1}

$$M_{R_1} = \begin{matrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{matrix}$$

c) Is R_1 is Reflexive ? If so what matrix characteristic do we see ?

Yes. Ones on the diagonal from top Left to bottom Right. Main Diagonal

d) Is R_1 is Symmetric ? If so what matrix characteristic do we see ?

$$M_{R_1} = \begin{matrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{matrix}$$

Yes. Matrix is Symmetric about the main diagonal. Mr1 is a Symmetric matrix.

Note: if we rewrote the matrix M_{R_1} with the rows of M_{R_1} becoming the columns of a new matrix the new matrix is called the Transpose of M_{R_1}.

Ex 1: If the matrix of a relation R is given as M_R =
(i) What relation type(s) apply ?
(ii) How many ordered pairs are there ?
(iii) If R = { (x,y) ∈ A × A | xRy, A={d,e,f,g} }, list R.

1	0	1	0
0	1	0	0
1	0	1	1
0	0	1	1

Ex 2: Consider A = {1,2,3,4} and R_1 = { (1,1), (1,4), (2,3), (3,1), (3,4) }

R_2 = { (1,1), (2,1), (3,1), (3,2), (4,1) }

(i) Find the matrices for R_1, R_2

(ii) Find the matrix for the relation $R_2 \circ R_1$

(iii) Using the matrix in (ii) write the ordered pairs in $R_2 \circ R_1$

Ex 3: If the matrix of a Relation R is $M_R = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 0 & 1 \\ \hline \end{array}$ and R relates A×A where A = {x,y}

(i) List the ordered pairs for R

(ii) Find the matrix for the relation R^3 = R ∘ R ∘ R

(iii) Using the matrix in (ii) write the ordered pairs in R^3

Ex 4: Consider R_1 = { (x,y) ∈ ℝxℝ | x < y }

R_2 = { (x,y) ∈ ℝxℝ | x > y }

Use Set-Builder notation to describe: Compliment of R1 , R1 ∩ R2, R1 ∪ R2, R1 – R2 , R2 – R1

Ex 5: $R = \{ (1,1), (2,1), (3,2), (4,3) \}$

(i) Find R^3 and R^4 without a matrix:

(ii) Make a conjecture concerning R^k, $k \geq 4$

Ex 6: $R = \{ (1,3), (4,5), (4,3) \}$, find R inverse. Inverse of a relation R has same structure as inverse of a function: If $R: A \to B$ the R inverse: $B \to A$

Ex 7: Prove: $[\, (R: A \to A) \land (S: A \to A) \land R, S$ Symmetric $] \to R \cup S$ is symmetric

Hint: A Relation is a set so the proof is similar in structure to a set builder proof.