# 11.6 Absolute Convergence and the Ratio and Root Tests

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## Definitions & Theorems:

1. Definition: Absolutely convergent

   A series $\sum a_n$ is called absolutely convergent if the series of absolute value $\sum |a_n|$ is convergent.

2. Definition: Conditionally convergent

   A series $\sum a_n$ is called conditionally convergent if it is convergent but not aboslutely convergent.

3. Theorem:

   If a series $\sum a_n$ is absolutely convergent, then it is convergent.

4. The Ratio Test:

   (i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

   (ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

   (iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

5. The Root Test:

   (i) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

   (ii) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

   (iii) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L = 1$, the Root Test is inconclusive.

## Proofs or Explanations:

1.

## Extra topics:

1. $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^p}$ converges for $p > 0$, but converges absolutely for $p > 1$.

## Examples:

1. $$\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}+16}$$

   1) $b_n = \dfrac{1}{\sqrt{n}+16}$, $\lim_{n\to\infty} b_n = 0$

   2) Let $f(x) = \frac{1}{\sqrt{x}+16}$, $f(x)$ decreases on $(0,\infty) \Rightarrow b_n$ is a decreasing sequence.

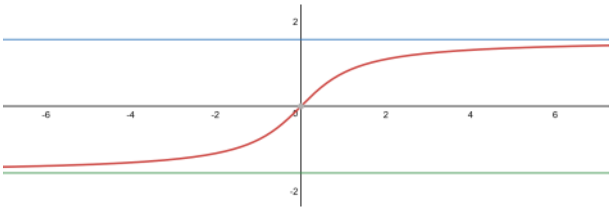
   3) By Alternating Series Test, $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^p}$ converges.

   4) Consider $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+16}$

      $\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n}+16} = 1 \Rightarrow$ by Limit Comparison Test, $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges ($p$-series with $p = \frac{1}{2}$)$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+16}$ diverges.

   $\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}+16}$ is conditional convergent.

2. $$\sum_{n=1}^{\infty}\frac{\arctan n}{n^4}$$




Since this is a series with positive terms, absolutely convergence and convergence are the same.

Compare with $\sum_{n=1}^{\infty} \frac{1}{n^4}$

$\lim_{n\to\infty} \left(\frac{\arctan n}{n^4}\right)(n^4) = \lim_{n\to\infty} \arctan n = \frac{\pi}{2}$

By Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{\arctan n}{n^4}$ converges and absolutely converges.

3. $\sum_{n=1}^{\infty} \frac{\cos n}{n^4}$

Consider $\sum_{n=1}^{\infty} \frac{|\cos n|}{n^4}$

Compare with $\sum_{n=1}^{\infty} \frac{1}{n^4}$

$\lim_{n\to\infty} \left(\frac{|\cos n|}{n^4}\right)(n^4) = \lim_{n\to\infty} |\cos n|$ ⇒Does not exist⇒Can not apply Limit Comparison Test.

$0 \le |\cos n| \le 1 \Rightarrow \frac{0}{n^4} \le \frac{|\cos n|}{n^4} \le \frac{1}{n^4}$ ⇒By Comparions Test⇒ $\sum_{n=1}^{\infty} \frac{|\cos n|}{n^4}$ converges.

⇒ $\sum_{n=1}^{\infty} \frac{\cos n}{n^4}$ absolutely converges, hence converges.

4. $\sum_{n=1}^{\infty} \frac{1}{n} \quad \sum_{n=1}^{\infty} \frac{1}{n^2}$

1) $L_1 = \lim_{n\to\infty} \left|\frac{\frac{1}{n+1}}{\frac{1}{n}}\right| = \lim_{n\to\infty} \left|\frac{n}{n+1}\right| = \lim_{n\to\infty} \left|\frac{1}{1+\frac{1}{n}}\right| = 1$

2) $L_2 = \lim_{n\to\infty} \left|\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}}\right| = \lim_{n\to\infty} \left|\frac{n^2}{(n+1)^2}\right| = 1$

3) ⇒Ratio Test did not tell anything.

5. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

$$\mathrm{L} = \lim_{n\to\infty} \left|\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}}\right| = \lim_{n\to\infty} \left|\frac{n^n}{(n+1)^n}\right| = e^{\ln \lim_{n\to\infty}\left|\frac{n^n}{(n+1)^n}\right|} = e^{\lim_{n\to\infty}\left(\frac{\ln\frac{n}{n+1}}{\frac{1}{n}}\right)}$$

$\lim_{x\to\infty} \frac{\ln\frac{x}{x+1}}{\frac{1}{x}} = -1 \Rightarrow e^{\lim_{n\to\infty}\left(\frac{\ln\frac{n}{n+1}}{\frac{1}{n}}\right)} = e^{-1} < 1$ ⇒by Ratio Test, $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges absolutely, hence converges.

6. $\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{2^n n^3}$

$$\mathrm{L} = \lim_{n\to\infty} \left|\frac{\frac{(-1)^{n+1}3^{n+1}}{2^{n+1}(n+1)^3}}{\frac{(-1)^n 3^n}{2^n n^3}}\right| = \lim_{n\to\infty} \left|\frac{3n^3}{2(n+1)^3}\right| = \frac{3}{2} > 1$$

By Ratio Test, $\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{2^n n^3}$ diverges.

7. $\sum_{n=1}^{\infty} \frac{n}{6^n}$

$$\mathrm{L} = \lim_{n\to\infty}\left|\frac{\frac{n+1}{6^{n+1}}}{\frac{n}{6^n}}\right| = \lim_{n\to\infty}\left|\frac{n+1}{6n}\right| = \frac{1}{6} < 1$$

by Ratio Test, $\sum_{n=1}^{\infty}\frac{n}{6^n}$ converges absolutely, hence converges.

8. $$\sum_{n=1}^{\infty}\frac{n^n}{3^{1+2n}}$$

$$\mathrm{L} = \lim_{n\to\infty}\left|\sqrt[n]{\frac{n^n}{3^{1+2n}}}\right| = \lim_{n\to\infty}\left|\frac{n}{3^{\frac{1}{n}+2}}\right| = \infty > 1$$

By Ratio Test, $\sum_{n=1}^{\infty}\frac{n^n}{3^{1+2n}}$ diverges.

9. $$\sum_{n=2}^{\infty}\frac{(-1)^n}{(\ln n)^n}$$

$$\mathrm{L} = \lim_{n\to\infty}\left|\sqrt[n]{\frac{(-1)^n}{(\ln n)^n}}\right| = \lim_{n\to\infty}\left|\frac{1}{\ln n}\right| = 0 < 1$$

By Ratio Test, $\sum_{n=2}^{\infty}\frac{(-1)^n}{(\ln n)^n}$ converges absolutely, hence converges.