5.1 Areas and Distance

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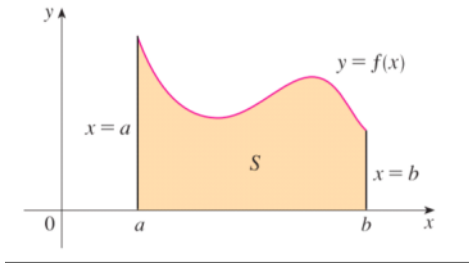
Prerequisite:

$$1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$$

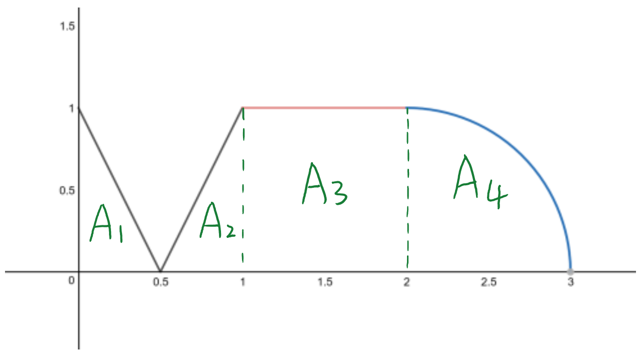
We begin by considering the area problem: Finding the area of S that lies under a non-negative continuous curve $y=f(x)$ on a closed interval $[a,b]$



Method 1: High School geometry

If the region, the area of which we wish to calculate, is a shape for which we know the area, then the area problem is solved.

Example: $f(x)=\begin{cases}|2x-1|, 0\le x<1\\ 1, 1\le x<2\\ \sqrt{1-(x-2)^2}, 2<x\le 3\end{cases}$

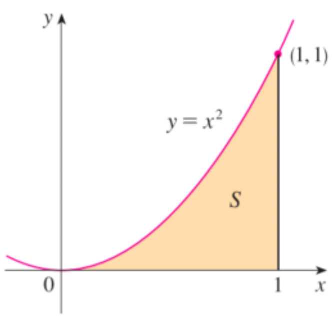


$$A_1=\frac{1}{2}\left(\frac{1}{2}\right)(1)$$
$$A_2=\frac{1}{2}\left(\frac{1}{2}\right)(1)$$
$$A_3=(1)(1)$$
$$A_4=\frac{1}{4}\pi(1)^2$$
$$A: A_1+A_2+A_3+A_4=\frac{3}{4}+\frac{1}{4}\pi$$

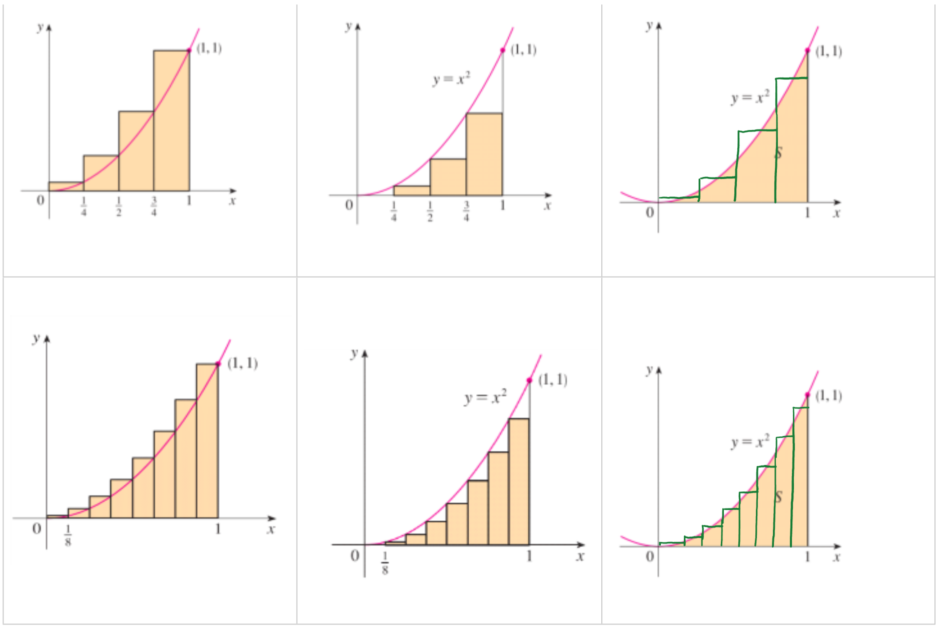
Method2: Riemann Sums

If the region, the area of which is to be calculated, in not in a shape that of which we know the area formula, than we can begin approximating the area by that of a simpler region.

Use rectangles to estimate the area under the parabola $y=x^2$ on the closed interval $[0,1]$

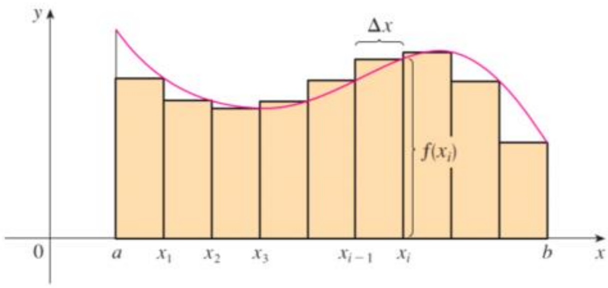


Using right endpoints R_n	Using left endpoints L_n	Using midpoints M_n



Midpoint Rule: In general, the midpoint provides a better approximation.

Denote R_{n} the Riemann sum estimation obtained by the use of n rectangles with right points;
Denote L_{n} the Riemann sum estimation obtained by the use of n rectangles with left points;
Denote M_{n} the Riemann sum estimation obtained by the use of n rectangles with middle points;



R_{n}=$\Delta x f(a+\Delta x)+\Delta x f(a+2\Delta x)+\Delta x f(a+3\Delta x)+\cdots+\Delta x f(a+n\Delta x)$
L_{n}=$\Delta x f(a+0\Delta x)+\Delta x f(a+1\Delta x)+\Delta x f(a+2\Delta x)+\cdots+\Delta x f(a+(n-1)\Delta x)$
M_{n}=$\Delta x f\left(a+\frac{1}{2}\Delta x\right)+\Delta x f\left(a+\frac{3}{2}\Delta x\right)+\Delta x f\left(a+\frac{5}{2}\Delta x\right)+\cdots+\Delta x f\left(a+\frac{2n-1}{2}\Delta x\right)$

Theorem: Let $f(x)$ a non-negative continuous function on a closed interval [a, b], then
$A=\lim_{n\to\infty} R_{\mathrm{n}}=\lim_{n\to\infty} L_{\mathrm{n}}=\lim_{n\to\infty} M_{\mathrm{n}}$

Formula for $R_{\mathrm{n}}, L_{\mathrm{n}}, M_{\mathrm{n}}$

$$R_{\mathrm{n}}=\sum_{i=1}^{n} f(a+i\Delta x)\Delta x$$

$$L_{\mathrm{n}}=\sum_{i=1}^{n} f(a+(i-1)\Delta x)\Delta x$$

$$M_{\mathrm{n}}=\sum_{i=1}^{n} f(a+\frac{2\mathrm{i}-1}{2}\Delta x)\Delta x$$

Sigma Notation:

$$a_1+a_2+a_3+\cdots+a_n=\sum_{i=1}^{n} a_i$$

Properties:

1. $\sum_{i=1}^{n} 1=1+1+1+\cdots+1=n$

2. $\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

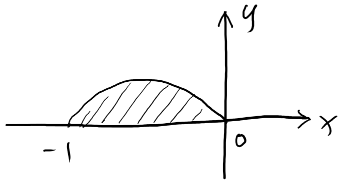
3. $\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

4. $\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

5. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i, c \in R$

6. $\sum_{i=1}^{n} [a_i \pm b_i] = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

Example: Calculate the area beneath $y = -x^2 - x$ on [-1, 0]



$$\left.\begin{matrix} a = -1 \\ b = 0 \end{matrix}\right\} \ \Delta x = \frac{b-a}{n} = \frac{1}{n}$$

$$R_n = \sum_{i=1}^{n} f\left(-1 + \frac{i}{n}\right)\frac{1}{\mathrm{n}} = \sum_{i=1}^{n} \left[\left(-1 + \frac{i}{n}\right)^2 - \left(-1 + \frac{i}{n}\right)\right]\frac{1}{\mathrm{n}}$$

$$= \sum_{i=1}^{n} \left[\frac{1}{n} - \frac{i^2}{n^2}\right]\frac{1}{n} = \sum_{i=1}^{n} \left[\frac{1}{n^2} - \frac{i^2}{n^3}\right]$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} i - \frac{1}{n^3}\sum_{i=1}^{n} i^2 = \frac{1}{n^2}\cdot\frac{n(n+1)}{2} - \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

$$= \frac{1}{2}\cdot\frac{n+I}{n} - \frac{1}{6}\cdot\frac{(n+1)(2n+1)}{n^2}$$

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[\frac{1}{2}\cdot\frac{n+1}{n} - \frac{1}{6}\cdot\frac{(n+1)(2n+1)}{n^2}\right] = \frac{1}{6}$$