## Sec 2.1 Sets

Comp 232
Robert Mearns

1. What is a Set ?
   a) Definition: A set is an un-ordered collection of objects.
   Ex: students in this class or chairs in this room
   b) Vocabulary and notation:
   (i) The objects in a set are called the elements, or members of the set.
   (ii) A set is said to contain its elements.
   (iii) A set is denoted by an uppercase letter, the elements denoted by a lowercase letter.
   (iv) The notation $a \in A$ denotes that a is an element of the set A.
   (v) The notation $\neg(a \in A) \equiv a \notin A$ ($\notin$ means "not contained in")
   (vi) Set elements are usually enclosed with brace brackets {}

2. Three ways to define the elements of a set:

List elements | Set-builder notation | Words

a) Listing the elements:
Ex:



| | |
|---|---|
| S = {a,b,c,d} = {a,b,c,d} | Order of element listing does not change set. Order does not matter. |
| S = {a,b,c,d} = {a,b,c,b,c,d} | Listing an element more than once does not change the set. |
| S = {a,b,c,d, ……, z } | Run on dots called Elipses (…), may be used to describe a set without listing all of the members when the pattern is clear. |

b) Set Builder notation:

- Specifies the property or properties that all set members must have.
- Use notation A = {x | ______________ } (fill in the blank with the properties)
- Read as: " A = set of all x such that _____________" ( | means such that )

| Set Builder | List elements |
|---|---|
| S = {x \| x is a positive integer less than 100} | S = {1,2,3,...99} |
| S = {x \| x is an odd positive integer less than 10} | S = {1,3,5,...9} |
| S = {x \| x is perfect square < 100} | S = {1,4,9,16,...81} |
| A predicate may be used: S = {x \| P(x), P(x): x<6, Domain is Z } | S = {...-1,0,1,2,3,4,5} |

c) Describe with words:

| Words | List or Set Builder |
|---|---|
| Set of Natural numbers with zero | **N** = {0,1,2,3….} |
| Set of Integers | **Z** = {…,-3,-2,-1,0,1,2,3,…} |
| Set of positive Integers | **Z+** = {1,2,3,…} |
| Set of Rational numbers | Q = {x \| x=p/q, p, q∈Z, q≠0} |
| Set of Irrational numbers | Irr = {x \| x∈R∧x∉Q} (is non repeating, non terminating when written as a decimal) |
| Set of Real numbers | R = {x \| x∈Q or x∈Irr} |
| Set of Complex numbers | **C** = {x \| x = a +b**i**, a,b ∈ **R, i** = $\sqrt{-1}$} |



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3. Universal Set
   a) Definition: The Universal set U is the Domain
      Universal set contains every element currently under consideration.
   b) The Universal set always exists. Sometimes we imply it exists without its listing or description.
   c) Sometimes we explicitly state the values in the Universal set.

4. Empty Set
   a) The Empty set is the set that contains no elements.
   b) The Empty set denoted by the Greek letter ø(phi) or by { }

5. Venn Diagram
   a) A Venn diagram is a geometric representation of a set
   b) The Venn diagram was invented by John Venn (1834-1923)

Ex: V = { a, e, i, o, u }

U = { English alphabet characters }

5. Some things to remember

a) Sets can be elements of sets.
Ex: {{1,2,3}, a, {b,c}}

b) The empty set is different from a set containing the empty set.
Ex: $\emptyset \neq \{\emptyset\}$
Why? $\emptyset = \{\ \}$ and $\{\emptyset\} = \{\{\ \}\}$

6. Subsets

a) Definition: The set A is a subset of B, if and only if every element of A is also an element of B

b) The notation $A \subseteq B$ is used to indicate that set A is a subset of set B.

c) Definition written in symbols: $A \subseteq B$ if and only if $\forall x (x \in A \rightarrow x \in B)$

Theorem: $\emptyset \subseteq S$, for every set S By definition of subset $\subseteq$, we need to show: $a \in \emptyset \rightarrow a \in S$ is T
Proof (Vacuous)

Consider the implication $a \in \emptyset \rightarrow a \in S$
$a \in \emptyset$ is False | $\emptyset$ has no elements
Hence the implication $a \in \emptyset \rightarrow a \in S$ is Ture | Truth Table for $\rightarrow$
$\emptyset \subseteq S$ | Definition of subset
QED

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7. Showing a Set is or is not a Subset of another set

a) To prove $A \subseteq B$, show that $\forall x\,(x \in A \to x \in B)$ is True
b) To prove $A \nsubseteq B$, show that $\neg\forall x\,(x \in A \to x \in B)$ is True

| | |
|---|---|
| $\exists x\,\neg\,(x \in A \to x \in B)$ is True | De Morgan for quant. |
| $\exists x\,\neg\,(\neg\, x \in A \vee x \in B)$ is True | $\to$ in terms of Or |
| $\exists x\,(\neg\neg\, x \in A \wedge \neg\, x \in B)$ is True | De Morgan |
| $\exists x\,(x \in A \wedge x \notin B)$ is True | Double neg. Equiv. for $\notin$ |
| Hence: find at least one $x \in A \wedge x \notin B$. | Counter Example |

Ex: The set of all C.S. students at Concordia is a subset of all students at Concordia
The set of integers whose squares are < 10 is not a subset of the set of **Z**+

8. Equal Sets

a) Definition: Two sets are equal if and only if they have the same elements.
b) The notation $A = B$ is used to indicate that set A equals set B
c) Definition in symbols: $A = B$ if and only if $\forall x\,(x \in A \leftrightarrow x \in B)$ .

Ex: $\{1,3,5\} = \{3, 5, 1\}$ Order does not change set
$\{1,5,5,5,3,3,1\} = \{1,3,5\}$ Repeating elements does not change set

d) Another way of expressing equality of sets. Use the logical equivalences below:

| | |
|---|---|
| $A = B$ iff $\forall x\,(x \in A \leftrightarrow x \in B)$ | Definition of = sets |
| $A = B$ iff $\forall x\,[\,(x \in A \to x \in B) \wedge (x \in B \to x \in A)\,]$ | Def. of bi-conditional |
| $A = B$ iff $\forall x\,(x \in A \to x \in B) \wedge \forall x\,(x \in B \to x \in A)$ | $\forall x$ equivalence with $\wedge$ |
| $\mathbf{A = B}$ **iff** $\mathbf{A \subseteq B \wedge B \subseteq A}$ | Definition of subset |

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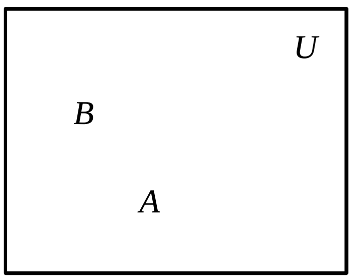
9. Proper Subsets

a) A is a proper subset of B iff $A \subseteq B$, and $A \neq B$
b) A a proper subset of B is denoted $A \subset B$
c) In symbols $A \subset B$ iff $\forall x\ [\ (x \in A \rightarrow x \in B) \wedge \forall x(x \in B \wedge x \notin A)\ ]$
d) Venn diagram for $A \subset B$:




10. Set Cardinality

a) Definition: If there are exactly n distinct (different) elements in S we say that S is finite. Otherwise S is infinite.
b) Definition: The cardinality of a finite set A is the number of distinct elements of A.
c) Cardinality of A is denoted |A| or n(A)

Ex: | ø | = 0 or n(ø) = 0
Let S be the letters of the English alphabet then | S | = 26
| {1,2,3} | = 3
| {ø} | = 1
The set of integers is infinite.

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11. Cartesian Product Set: Invented by René Descartes (1596-1650)

a) Definition: The Cartesian Product of two sets A and B, is the set of :
ordered pairs (x, y) where $x \in A$ and $y \in B$

b) The Cartesian Product is denoted by A x B

c) The set builder notation: $A \times B = \{(x, y) \mid (x \in A) \wedge (y \in B)\}$

Ex 1: If A = {1, 2, 3} and B = {5, 6} then A × B = {(1,5),(1,6),(2,5),(2,6),(3,5),(3,6)}

Ex 2**:** The Cartesian products can be expanded beyond two sets
If *A* = {0,1}, *B* = {3} and *C* = {0,1,2}
A × *B* × C = {(0,3,0),(0,3,1),(0,3,2),(1,3,0),(1,3,1),(1,3,2)}

12. Power Set

a) Definition: The power set of *A* is the set of all subsets of a set A

b) The Power set is denoted P(A)

Ex If A = {a,b} then P(A) = {ø, {a}, {b}, {a,b}}

c) If a set has *n* elements, then the cardinality of the power set is n^2

13. Truth Set

Definition: The Truth Set of P(x) is the set of elements x in Domain D for which P(x) is true.

b) The truth set of P(x) is denoted by $\{x \in D \mid P(x)\}$

c) Ex: The truth set of P(x) where the domain is the integers and P(x) is “ | *x* | = 1” is {-1, 1}

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