# 11.9 Representation of Functions as Power Series

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## Definitions & Theorems:

1. Theorem:

   If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of converges $R > 0$, then the function $f$ defined by
   $$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$
   is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

   (i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

   (term-by-term differentiation)

   (ii) $\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$

   (term-by-term integration)

   The radii of convergence of the power series in Equation (i) and (ii) are both $R$.

## Proofs or Explanations:

1. By geometric series $\sum_{n=0}^{\infty} x^n$ converges when $|x| < 1$
   $$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} = \frac{first\ term}{1-ratio}$$

## Extra topics:

1. Conventions for power series
   a. $0^0 = 1$
   b. $(0)(\infty) = 0$

## Examples:

1. $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$

   $|-x^2| < 1 \Rightarrow -1 < x < 1 \Rightarrow R = 1$

2. $\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$

   $|-x| < 1 \Rightarrow -1 < x < 1 \Rightarrow R = 1$

3. $\frac{4x}{9+x} = (4x)\left(\frac{1}{9+x}\right) = \left(\frac{4x}{9}\right)\left(\frac{1}{1+\frac{x}{9}}\right) = \left(\frac{4x}{9}\right)\left(\frac{1}{1-(-\frac{x}{9})}\right) = \left(\frac{4x}{9}\right)\sum_{n=0}^{\infty}(-\frac{x}{9})^n = \sum_{n=0}^{\infty}\frac{(-1)^n 4x^{n+1}}{9^{n+1}}$

   $\left|-\frac{x}{9}\right| < 1 \Rightarrow -9 < x < 9 \Rightarrow R = 9$

Represent the following as a power series, and find the radii of convergence.

4. $\frac{1}{(1-x)^2}$

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \frac{d}{dx}(x^n) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

$|x| < 1 \Rightarrow -1 < x < 1 \Rightarrow R = 1$

5. $\ln(1+x)$

$$\ln(1+x) = \int \frac{1}{1+x}\,dx = \int \frac{1}{1-(-x)}\,dx = \int \sum_{n=0}^{\infty} (-x)^n\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} + C$$

$x = 0 \Rightarrow \ln(1+x) = \ln 1 = 0 + C = 0 \Rightarrow C = 0$

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$$

$R = 1$

6. $\arctan x$

$$\arctan x = \int \frac{1}{1+x^2}\,dx = \int \sum_{n=0}^{\infty} \left(-x^2\right)^n\,dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} + C$$

$x = 0 \Rightarrow \arctan 0 = 0 + C = 0 \Rightarrow C = 0$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

$R = 1$