Section 1.5 Nested Quantifiers

Comp 232
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1. What are Nested Quantifiers ?

a) Think of Nested Loops:

Step 1 Choose one x in its Domain

Step 2 Using above x, loop through all y in its Domain and form P(x,y) for each combination

Step 3 Repeat Step 1, Step 2 until all x in its Domain have been paired with all y in its Domain

b)

Words	Notation
(i) If for every x and every y the P(x,y) = T	$\forall x \forall y\ P(x, y)$
(ii) If for every x we find at least one y that makes the P(x,y) = T	$\forall x \exists y\ P(x, y)$
(iii) If there is at least one y when paired with all x makes the P(x,y) = T	$\exists y \forall x\ P(x, y)$
(iv) If there is at least one x and at least one y that makes the P(x,y) = T	$\exists x \exists y\ P(x, y)$

Note: (ii) and (iii) are not the same. (see 2 b)
There may be a different y that works with each x when: $\forall x \exists y\ P(x, y)$
There is only one y that works with all the x when: $\exists y \forall x\ P(x, y)$

c) If the domains of the variables are infinite, then this process can not actually be carried out but you can think of the results as if it is completed.

2. Order in nested quantifiers:

a) Same Quantifiers

$\forall(x, y)$

(i) $\forall x \forall y\ P(x,y)$, $\forall y \forall x\ P(x,y)$ imply all possible pairs (x, y) that make predicate $P(x,y) = T$

In the nested loop interpretation the only difference is the order of selection of pairs (x, y)

Hence LHS→RHS and RHS→LHS

$\rightarrow [\forall x \forall y\ P(x,y) \equiv \forall y \forall x\ P(x,y)] = T$

$\exists(x, y)$

(ii) $\exists x \exists y\ P(x,y)$, $\exists y \exists x\ P(x,y)$ imply the existence of at least one pair (x, y) that makes $P(x,y) = T$

In the nested loops the only difference is the order of selection of this pair or pairs (x, y)

Hence LHS→RHS and RHS→LHS

$\rightarrow [\exists x \exists y\ P(x,y) \equiv \exists y \exists x\ P(x,y)] = T$

b) Different Quantifiers

Show that $[\forall x \exists y\ P(x,y) \equiv \exists y \forall x\ P(x,y)] = F$

Consider $P(x,y)$, $x \epsilon \{a,b\}$, $y \epsilon \{c,d\}$ where $P(a,c) = T$, $P(a,d) = F$
$P(b,c) = F$, $P(b,d) = T$

Step 1 Show LHS = T:

$P(a,c)=T$ and $P(b,d)= T \rightarrow \forall x \exists y\ P(x,y)$
$\rightarrow$ LHS=T

Given, Def $\forall x \exists y$

Step 2 Show RHS = F:

$P(a,d)=F$ and $P(b,c)=F \rightarrow \neg \exists y \forall x\ P(x,y)$
$\rightarrow$ RHS=F

A single y does not make $P(x,y)=T$ for all x

$\rightarrow \forall x \exists y\ P(x,y) \equiv \exists y \forall x\ P(x,y) = F$

A Predicate $P(x,y)$ exists where LHS, RHS have different truth values

Counter example

$P(x,y)$: $y-x=2$, $x \in \{1, 2\}$, $y \in \{3, 4\}$

Ex 1: Let Domain be Real numbers, Define P(x,y): $x + y = 0$

What is the truth value for each of the following ?

(i) $\forall x \forall y$ P(x,y) F, Counter example: Select x=y=1

(ii) $\forall x \exists y$ P(x,y) T, If x=a, select y=−a, $a \in R$

(iii) $\exists y \forall x$ P(x,y) F, No single y makes x+y=0 for all $x, y \in R$

(iv) $\exists x \exists y$ P(x,y) T, Select x=1, y=-1

Ex 2: Let Domain be Real numbers, Define P(x,y): $\frac{x}{y} = 1$

What is the truth value for each of the following ?

(i) $\forall x \forall y$ P(x,y) F, Counter example: Select y=0

(ii) $\forall x \exists y$ P(x,y) F, Counter exmaple: When x=0, $y \in R$ does not exist

(iii) $\exists y \forall x$ P(x,y) F, No single $y \in R$ makes x/y = 1 for all $x \in R$

(iv) $\exists x \exists y$ P(x,y) T, Select any x=a, y=a where $a \in R, a \neq 0$

3. Translating Nested Quantifiers into English

Ex 1: Translate the statement $\forall x\, [\, C(x) \vee \exists y\, \{\, C(y) \wedge F(x, y)\, \}\,]$ into English.

$C(x)$ is "x has a computer"

$F(x,y)$ is "x and y are friends"

Where x, y represent students in the school.

For all students x in the school

x has a computer

or there exists at least one student y in the school who has a computer and x, y are friends.

Simplified version:

Every student has a computer or has a friend who has a computer.

Ex 2: Translate the statement $\exists x\, \forall y\, \forall z, (y \neq z) : [\, F(x, y) \wedge F(x,z) \wedge \neg F(y,z)\,]$ into English

Same definitions as in Ex 1. x, y, z represent students in the school.

There exist a student x in the school such that for any students y, z in the school, where y, z are not the same,

x,y are friends and x,z are friends and y,z are not friends.

4. Translating English Statements into Nested Quantifiers

Ex 1: Translate "The sum of two positive integers is always positive" into a logical expression.

For every two integers,
if these integers are both positive,
then the sum of these integers are positive.

$\forall x \forall y \; x, y \in Z+: \; x+y>0$

Sometimes a rewrite of the statement makes the quantifiers and domains more explicit (clear):

Z^+ denotes positive Integers

Ex 2: Translate "There is a woman who has taken at least one flight on every airline in the world." into a logical expression

Let P(w, f) represent: woman w has taken flight f
Let Q(f, a) represent: flight f is on airline a

w represents women

Predicate definitions

$\exists w \forall a \exists f \; [P(w, f) \; \wedge \; Q(f, a)]$

Logical expression

Ex 3: Consider the following predicates and domains:
B(x,y) represents x and y are brothers.
S(x,y) represents x and y are siblings (siblings are two people with same mother)
L(x,y) represents x and y are lovers.
Domain for x, y is all people.

Translate each statement into symbols using quantifiers.	Answers
a) Brothers are siblings.	$\forall x \forall y\ [B(x, y) \rightarrow S(x, y)]$
b) Siblinghood is symmetric. (x related y ≡ y related to x)	$\forall x \forall y\ [S(x, y) \rightarrow S(y, x)]$
c) Everybody loves somebody.	$\forall x \exists y\ L(x, y)$
d) There is someone who is loved by everyone	$\exists y \forall x\ L(x, y)$ or $\exists x \forall y\ L(x, y)$
e) There is someone who loves someone	$\exists x \exists y\ L(x, y)$
f) Everyone loves himself	$\forall x\ L(x, x)$

7. Negating Nested Quantifiers

Ex 1: Consider the statement: $\neg\exists w\ \forall a\ \exists f\ [\ P(w,f) \wedge Q(f,a)\ ]$. Use De Morgan's Rules to move the negation inside the square brackets. Show all steps.

$\neg\exists w\forall a\exists f\ [\ P(w, f) \wedge Q(f, a)\ ]$	Given
$\equiv\ \forall w\neg\forall x\exists f[\ P(w, f) \wedge Q(f, a)\ ]$	De Morgan's rule for $\neg\exists$
$\equiv\ \forall w\exists a\neg\exists f[\ P(w, f) \wedge Q(f, a)\ ]$	De Morgan's rule for $\neg\forall$
$\equiv\ \forall w\exists a\forall f\ \neg[\ P(w, f) \wedge Q(f, a)\ ]$	De Morgan's rule for $\neg\exists$
$\equiv \forall w\exists a\forall f\ [\neg P(w,f)\ \vee\ \neg Q(f,a)]$	De Morgan's rule for $\wedge$

Ex 2: Show $[\ \forall x\forall y: P(x,y)\rightarrow Q(x,y) \equiv \forall x\forall y: P(x,y) \rightarrow \forall x\forall y: Q(x,y)\ ] = F$

Step 1 Express $\rightarrow$ in terms of Or, then use De Morgan for quantifiers:

$\forall x\forall y\ [\neg P(x,y) \vee Q(x,y)]\ \equiv\ \neg\forall x\forall y\ P(x,y)\ \vee\ \forall x\forall y\ Q(x,y)\ = F$

$\forall x\forall y\ [\neg P(x,y) \vee Q(x,y)]\ \equiv\ \exists x\exists y\ \neg P(x,y)\ \vee\ \forall x\forall y\ Q(x,y)\ = F$

Step 2 Counter example: Consider predicates $P(x,y)$, $Q(x,y)$: $x,y \in \{a,b\}$, $P(a,b) = T$, $Q(a,b) = F$
$P(b,a) = F$, $Q(b,a) = T$

Show LHS = F	
$[\neg\ P(a,b) = F \wedge Q(a,b) = F] \rightarrow \neg P(a,b) \vee Q(a,b)\ = F$	Given, truth value for Or
$\rightarrow \forall x\forall y\ [\neg p(x,y) \vee Q(x,y)]\ = F\ \rightarrow$ LHS=F	Def $\forall$: a,b only values for x, y
Show RHS = T	
$\neg\ P(b,a) = T \rightarrow\ \exists x\exists y\ \neg P(x,y)\ = T$	Def $\exists$
$Q(a,b) = F \wedge\ Q(b,a) = T \rightarrow\ \forall x\forall y\ Q(x,y)\ = F$	Def $\forall$
$\exists x\exists y\ \neg\ P(x,y) = T \wedge \forall x\forall y\ Q(x,y) = F$	
$\rightarrow\ \exists x\exists y\ \neg P(x,y)\ \vee\ \forall x\forall y\ Q(x,y)\ = T\ \rightarrow$ RHS = T	Truth value for Or
	LHS, RHS different T values