Final

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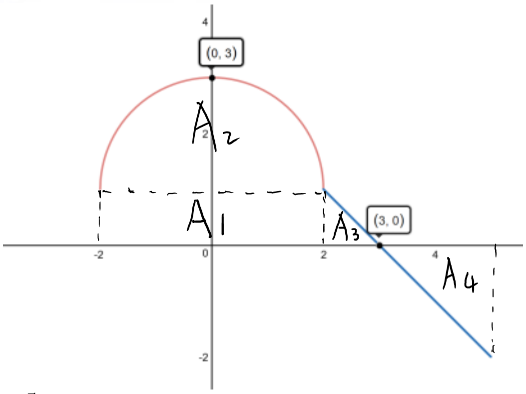
1.

a. Calculate $\int_2^5 f(x)\,dx$ in terms of singned areas, given $f(x) = \begin{cases} 1+\sqrt{4-x^2}, -2 \le 2 \le 2 \\ 3-x, x>2 \end{cases}$



$$\int_2^5 f(x)\,dx = A_1 + A_2 + A_3 - A_4 = 4 + 2\pi + \frac{1}{2} - 2 = \frac{5}{2} + 2\pi$$

b. Find a function f and a number a, such that $a + \int_4^x \frac{f(t)}{t^2}\,dt = 2\sqrt{x}$ for all $x > 0$

By FTC, $0 + \frac{f(x)}{x^2} = \frac{1}{\sqrt{x}} \Rightarrow \frac{f(x)}{x^2} = \frac{1}{\sqrt{x}} \Rightarrow f(x) = x^{\frac{3}{2}}$

$$a + \int_4^x \frac{t^{\frac{3}{2}}}{t^2}\,dt = 2\sqrt{x} \Rightarrow a + \left[2t^{\frac{1}{2}}\right]_4^x = 2\sqrt{x} \Rightarrow a = 4$$

2. Find the antiderivative $F(x)$ of $f(x) = \frac{x^2-3}{x^2+3}$, such that $F(1) = 1 - \frac{\pi}{\sqrt{3}}$

$$F(x) = \int f(x)\,dx = \int \frac{x^2-3}{x^2+3}\,dx = \int \left(1 - \frac{6}{x^2+3}\right)dx = x - 6\int \frac{1}{x^2+3}\,dx = x - 2\int \frac{1}{\left(\frac{x}{\sqrt{3}}\right)^2 + 1}\,dx$$

Let $u = \frac{x}{\sqrt{3}} \Rightarrow du = \frac{1}{\sqrt{3}}dx$

$$F(x) = x - 2\int \frac{1}{\left(\frac{x}{\sqrt{3}}\right)^2+1}\,dx = x - 2\sqrt{3}\int \frac{1}{u^2+1}\,du = x - 2\sqrt{3}\arctan u + C = x - 2\sqrt{3}\arctan\frac{x}{\sqrt{3}} + C$$

$$F(1) = 1 - 2\sqrt{3}\arctan\frac{1}{\sqrt{3}} + C = 1 - \frac{\pi}{\sqrt{3}} \Rightarrow C = -\frac{\pi}{\sqrt{3}} + 2\sqrt{3}\arctan\frac{1}{\sqrt{3}} = -\frac{\pi}{\sqrt{3}} + 2\sqrt{3}\left(\frac{\pi}{6}\right) = 0$$

$$\Rightarrow F(x) = x - 2\sqrt{3}\arctan\frac{x}{\sqrt{3}}$$

3. Calculate the following indefinite integrals.

a. $\displaystyle\int \frac{\sqrt{4x^2-1}}{4x^2}\,dx$

Let $2x = \sec\theta \Rightarrow 2dx = \sec\theta\tan\theta\,d\theta$

$$\int \frac{\sqrt{4x^2-1}}{4x^2}\,dx = \int \frac{\sqrt{sec^2\,\theta - 1}}{sec^2\,\theta}\left(\frac{1}{2}\right)\sec\theta\tan\theta\,d\theta = \frac{1}{2}\int \frac{tan^2\,\theta}{sec\,\theta}\,\mathrm{d}\theta = \frac{1}{2}\int \frac{sin^2\,\theta}{cos\,\theta}\,\mathrm{d}\theta$$

$$= \frac{1}{2}\int \frac{(1 - cos^2\,\theta)}{cos\,\theta}\,\mathrm{d}\theta = \frac{1}{2}\int (\sec\theta)\,\mathrm{d}\theta - \frac{1}{2}\int (\cos\theta)\,\mathrm{d}\theta$$

$$= \frac{1}{2}\ln|sec\,\theta + tan\,\theta| - \frac{1}{2}\sin\theta + C$$

b. $\int sin^2\,\theta\, cos^3\,\theta\, \mathrm{d}\theta$

$$\int sin^2\,\theta\, cos^3\,\theta\, \mathrm{d}\theta = \int sin^2\,\theta\, cos^2\,\theta \cos\theta\, \mathrm{d}\theta = \int sin^2\,\theta\,(1 - sin^2\,\theta)\cos\theta\, \mathrm{d}\theta$$

Let $u = \sin\theta \Rightarrow du = \cos\theta\, d\theta$

$$\int sin^2\,\theta\,(1 - sin^2\,\theta)\cos\theta\, \mathrm{d}\theta = \int u^2(1-u^2)\,\mathrm{du} = \frac{u^3}{3} - \frac{u^5}{5} + C = \frac{sin^3\,\theta}{3} - \frac{sin^5\,\theta}{5} + C$$

c. $\int cos\left(\sqrt{x}\right) dx$

Let $u = \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}} dx$

$$\int cos\left(\sqrt{x}\right) dx = \int cos\,u\, 2u\, du = 2\int u\, cos\, u\, du$$

Let $w = u, dv = \cos u\, du \Rightarrow dw = du, v = \sin u$

$$2\int u\, cos\, u\, du = 2u\sin u - 2\int sin\, u\, du = 2u\sin u + 2\cos u + C = 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$$

4. Evaluate the following definite integrals.

a. $\int_4^6 \frac{x}{\sqrt[3]{1+2x}} dx$

Let $u = 1 + 2x \Rightarrow du = 2dx$

$$\int_4^6 \frac{x}{\sqrt[3]{1+2x}} dx = \int_9^{13} \frac{\frac{1}{2}(u-1)}{\sqrt[3]{u}}\left(\frac{1}{2}\right) du = \frac{1}{4}\int_9^{13}\left(u^{\frac{2}{3}} - u^{-\frac{1}{3}}\right) du = \frac{1}{4}\left[\frac{u^{\frac{5}{3}}}{\frac{5}{3}} - \frac{u^{\frac{2}{3}}}{\frac{2}{3}}\right]_9^{13}$$

b. $\int_0^1 \frac{2}{x^2 - 100} dx$

$$\int_0^1 \frac{2}{x^2-100} dx = \int_0^1 \frac{2}{(x+10)(x-10)} dx$$

$$\frac{2}{(x+10)(x-10)} = \frac{A}{x+10} + \frac{B}{x-10} \Rightarrow A = -\frac{1}{10}, B = \frac{1}{10} \Rightarrow \frac{2}{(x+10)(x-10)} = \frac{-\frac{1}{10}}{x+10} + \frac{\frac{1}{10}}{x-10}$$

$$\int_0^1 \frac{2}{(x+10)(x-10)} dx = -\frac{1}{10}\int_0^1 \frac{1}{x+10} dx + \frac{1}{10}\int_0^1 \frac{1}{x-10} dx$$

$$= -\frac{1}{10}[\ln(x+10)]_0^1 + \frac{1}{10}[\ln(x-10)]_0^1 = \frac{1}{10}\ln 9 - \frac{1}{10}\ln 11$$

5. Evaluate the following improper integrals, or show they diverge.

a. $\int_0^\infty xe^{-x^2} dx$

$$\int_0^\infty xe^{-x^2} dx = \lim_{t\to\infty}\int_0^t \mathrm{xe}^{-x^2}\,\mathrm{dx}$$

Let $u = -x^2 \Rightarrow du = -2xdx$

$$\lim_{t\to\infty}\int_0^t \mathrm{xe}^{-x^2}\,\mathrm{dx} = \lim_{t\to\infty}\int_0^{-t^2} e^u\left(-\frac{1}{2}\right)\mathrm{du} = -\frac{1}{2}\lim_{t\to\infty}[e^u]_0^{-t^2} = -\frac{1}{2}\lim_{t\to\infty}\left(e^{-t^2} - 1\right) = \frac{1}{2}$$

$\Rightarrow \int_0^\infty xe^{-x^2} dx$ converges and $\int_0^\infty xe^{-x^2} dx = \frac{1}{2}$

b. $\int_0^{\frac{\pi}{2}} csc\,\theta\, d\theta$

$$\int_0^{\frac{\pi}{2}} csc\,\theta\, d\theta = \lim_{t\to 0^+}\int_t^{\frac{\pi}{2}} csc\,\theta\, d\theta = \lim_{t\to 0^+}\left([\ln|\csc\theta - \cot\theta|]_{\mathrm{t}}^{\frac{\pi}{2}}\right) = \lim_{t\to 0^+}(\ln 1 - \ln|\csc t - \cot t|)$$

$$= -\lim_{t\to 0^+} \ln|\csc t - \cot t| = -\lim_{t\to 0^+} \ln\left|\frac{1}{\sin t} - \frac{\cos t}{\sin t}\right| = -\lim_{t\to 0^+} \ln\left|\frac{1-\cos t}{\sin t}\right| = -\ln \lim_{t\to 0^+}\left|\frac{1-\cos t}{\sin t}\right|$$

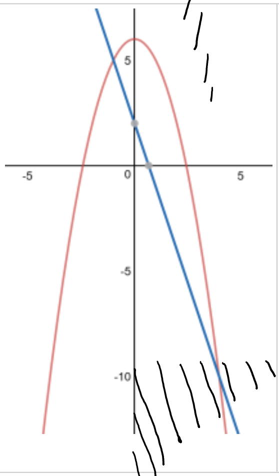
L'Hospital's Rule

$$= -\ln \lim_{t\to 0^+}\left|\frac{\sin t}{\cos t}\right| = \infty$$

$\Rightarrow \int_0^{\frac{\pi}{2}} csc\,\theta\, d\theta$ diverges

6.

a. Find the area enclosed between the curves $y = 6 - x^2, y = 2 - 3x$

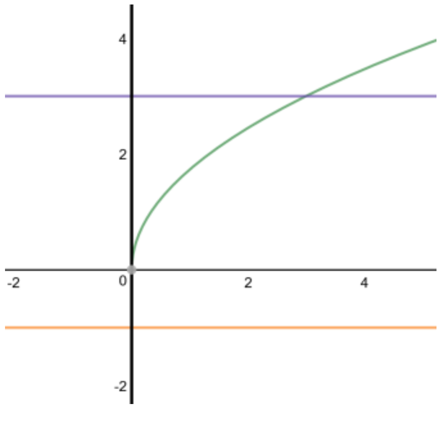


$$6 - x^2 = 2 - 3x \Rightarrow x = -1, x = 4$$

$$A = \int_{-1}^{4} [(6 - x^2) - (2 - 3x)]\, dx$$

$$= \left[-\frac{x^3}{3} + \frac{3x^2}{2} + 4x\right]_{-1}^{4} = \frac{125}{6}$$

b. Find the volume generated by rotating the are enclosed by the curve $y = \sqrt{3x}, y = 3, x = 0$ about the axis $y = -1$



$$\sqrt{3x} = 3 \Rightarrow x = 3$$

$$V = \pi \int_0^3 (R^2 - r^2)\, dx$$

$$= \pi \int_0^3 \left(4^2 - (\sqrt{3x} + 1)^2\right) dx = \frac{39\pi}{2}$$

c. Find the average value of $f(x) = x^2 e^{5x}$ on $[-1,1]$

$$f_{avg} = \frac{1}{2}\int_{-1}^{1} x^2 e^{5x}\, dx$$

Let $u = x^2, dv = e^{5x}dx \Rightarrow du = 2xdx, v = \frac{1}{5}e^{5x}$

$$\frac{1}{2}\int_{-1}^{1} x^2 e^{5x}\, dx = \frac{1}{2}\left[\frac{1}{5}e^{5x}x^2\right]_{-1}^{1} - \frac{1}{2}\int \frac{1}{5}e^{5x} 2x\, dx = \frac{1}{10}\left[e^{5x}x^2\right]_{-1}^{1} - \frac{1}{5}\int e^{5x} x\, dx$$

Let $u = x, dv = e^{5x}dx \Rightarrow du = dx, v = \frac{1}{5}e^{5x}$

$$\int e^{5x} x\, dx = \frac{1}{5}xe^{5x} - \frac{1}{5}\int e^{5x}\, dx$$

$$\Rightarrow f_{avg} = \left[\frac{1}{10}e^{5x}x^2 - \frac{1}{25}xe^{5x} + \frac{1}{125}e^{5x}\right]_{-1}^{1} = \frac{e^5}{10} - \frac{e^5}{25} + \frac{e^5}{125} - \left(\frac{e^{-5}}{10} + \frac{e^{-5}}{25} + \frac{e^{-5}}{125}\right)$$

7. Determine the limit of the following sequences.

a. $a_n = \dfrac{\pi n^3 - en + 5}{\sqrt[3]{n^9 + n^2}}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\pi n^3 - en + 5}{\sqrt[3]{n^9 + n^2}} = \lim_{n\to\infty} \frac{\pi - \frac{e}{n^2} + \frac{5}{n^3}}{\sqrt[3]{1 + \frac{1}{n^7}}} = \pi$$

b. $a_n = ln\left(\sqrt{n^2+1} - \sqrt{n}\right)$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} ln\left(\sqrt{n^2+1}-\sqrt{n}\right) = \ln \lim_{n\to\infty}\left(\sqrt{n^2+1}-\sqrt{n}\right) = \ln \lim_{n\to\infty}\frac{\left(\sqrt{n^2+1}-\sqrt{n}\right)\left(\sqrt{n^2+1}+\sqrt{n}\right)}{\left(\sqrt{n^2+1}+\sqrt{n}\right)}$$

$$= \ln \lim_{n\to\infty}\frac{\left(n+\frac{1}{n}-1\right)}{\left(\sqrt{1+\frac{1}{n^2}}+\sqrt{\frac{1}{n}}\right)} = \infty$$

$\Rightarrow$ the consequence diverges

8.

a. Find the radius of convergence and the interval of convergence if $S(x) = \sum_{n=1}^{\infty}\frac{(x+2)^n}{n4^n}$

$$\alpha = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x+2)^{n+1}}{(n+1)4^{n+1}}\frac{n4^n}{(x+2)^n}\right| = \lim_{n\to\infty}\left|\frac{(x+2)^{n+1}}{(n+1)4^{n+1}}\frac{n4^n}{(x+2)^n}\right| = \frac{|x+2|}{4}\lim_{n\to\infty}\left(\frac{n}{n+1}\right) = \frac{|x+2|}{4}$$

$\frac{|x+2|}{4} < 1 \Rightarrow R = 4, -6 < x < 2$

When $x = -6, S(x) = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$

By AST, $b_n = \frac{1}{n} > 0$, $\lim_{n\to\infty} b_n = 0, n+1 > n \Rightarrow \frac{1}{n+1} < \frac{1}{n} \Rightarrow b_{n+1} < b_n$ $\Rightarrow$decreasing

Thus, by AST, $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges.

When $x = 2, S(x) = \sum_{n=1}^{\infty}\frac{1}{n}$, it diverges ($p$-series with $p = 1$)

$\Rightarrow I = [-6,2)$

b. Derive the Maclaurine series for $f(x) = x^3\cos(-2x)$

$$\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$$

$$\Rightarrow \cos(-2x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(-2x)^{2n}$$

$$\Rightarrow f(x) = x^3\cos(-2x) = x^3\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(-2x)^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}4^n x^{2n+3}$$

9. For what values of p does $\sum_{n=2}^{\infty}\frac{1}{n(ln\, n)^p}$ converges?

10. Determine whether the following series converge or diverge; if converge, absolutely or conditionally?

a. $\sum_{n=0}^{\infty}\frac{cos(\pi n)}{n+1}$ conditionally coverge

b. $\sum_{n=1}^{\infty}\frac{1-sin(n)}{n^2}$ absolutely coverge

c. $\sum_{n=1}^{\infty}\frac{n^4+n^2-1}{n^3+6}$ diverges, Test of Divergence.