Sec 9.5-9.6

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Sec 9.5-9.6 Grouping the: [Reflexive, Symmetric, Anti symmetric, Transitive] Characteristics of a Relation

Comp 232
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Sections 9.5 – 9.6 include:

Review the types of Binary Relations
Review Matrix use to manipulate Binary Relations
Classification of different types of Relations and their significance
Several application where different types of Relations exist

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1 a) Review the types of Relations: Assume R is a relation on set A = {1,2,3,4}

Name	Definition	Comment	Examples
Reflexive	$\forall a \in A$ $aRa \in R$ $\exists a \in A$, $aRa \notin R$ $\to$R not reflexive (R is Irreflexive)	Pair (a, a) must be present For every $a \in A$	1. {(1,1),(1,2),(2,2),(3,3),(4,4)} 2. Ones on main diagonal of Matrix 1 1 0 0 0 1 0 0 0 0 1 0 0 0 0 1 3. $\{(x,y) \mid xRy \to x \le y \ \ x, y \in \text{Reals}\}$
Symmetric	$\forall a \forall b \in A$ $aRb \in R \to bRa \in R$ $\exists a \exists b \in A$, $aRb \wedge \neg bRa$ $\to$R not Symmetric (R is asymmetric)	We do not need $aRb \wedge bRa \in R$ For all a, $b \in A$, but IF $aRb \in R$ THEN we need $bRa \in R$	1. {(1,3),(1,4),(2,2),(3,1),(4,1)} 2. Matrix is symmetric 0 0 1 1 0 1 0 0 1 0 0 0 1 0 0 0 3. $\{(x,y) \mid xRy \to \lvert x-y \rvert < 1 \ x, y \in \text{Reals}\}$
Antisymmetric	$\forall a \forall b \in A$ If aRb and $bRa \in R$ then a = b.	The only time we have $aRb \wedge bRa \in R$ is when a=b	1. {(1,1),(1,2),(3,3),(4,4)} 2. Matrix not symmetric unless all entries off main diagonal = 0 1 1 0 0 0 0 0 0 0 0 1 0 0 0 0 1 3. $\{(x,y) \mid xRy \to x \mid y, \ \ x, y \in \text{Reals}\}$
Transitive	$\forall a \forall b \in A$ If aRb and $bRc \in R$ then $aRc \in R$	We do not need aRc For all a, $c \in A$, but IF $aRb \in R \wedge bRc \in R$ THEN we need $aRc \in R$	1. {(1,3),(1,4),(2,4),(3,4),(4,4)} 2. No general matrix form 0 0 1 1 0 0 0 1 0 0 0 1 0 0 0 1 3 3. $\{(x,y) \mid xRy \to x < y \ \ x, y \in Z\}$

1 b) Review Closure: The Closure Set for each one of Reflexive, Symmetric or Transitive means:
Alter the original set so it has the desired property.

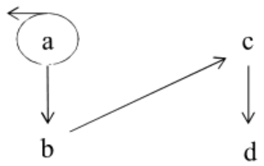
- Reflexive and Symmetric closures can be done by Inspection on the sets
- Transitive closures can be done by Inspection of the Di-graph
- All three closures can also be done With matrices

Ex: Consider the relation R = { (a,a),(a,b), (b,c),(c,d) }

Reflexive closure: R_R = {(a, a),(a, b),(b, c),(c, d),(b, b),(c, c),(d, d)}

Symmetric Closure: R_S = {(a, a),(a, b),(b, a),(b, c),(c, b),(c, d),(d, c)}

Transitive Closure:



Paths:	Length
a to b	Yes, n=1
a to c	Yes, n=2
a to d	Yes, n=3
b to a	No
b to c	Yes, n=1
b to d	Yes, n=2
c to a	No
c to b	No
c to d	Yes, n=1
d to a	No
d to b	No
d to c	No

State all paths where path length n≥1.
These pairs are added to set R to form its Transitive closure.
R_T = {(a, a),(a, b),(a, c),(a, d),(b, c),(b, d), (c, d)}

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2. Operations and Closures with matrix representation for a Relation

a) All set operations apply. Use bit-wise: And ≡ ∩, Or ≡ ∪, Compliment ≡ ∉ on corresponding entries.

Ex: Union of relations R ∪ S : $M_R \vee M_S$

$$\begin{matrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{matrix} \quad \vee \quad \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{matrix} \quad = \quad \begin{matrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{matrix}$$

$$M_R \qquad \vee \qquad M_S \qquad = \qquad M_{R \cup S}$$

b) Composition of relations S ∘ R : $M_R \odot M_S$ [Same structure as matrix Mult. Except: bit-wise And for Mult. Bit-wise Or for Add]

$$\begin{matrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{matrix} \quad \odot \quad \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{matrix} \quad = \quad \begin{matrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{matrix}$$

$$M_R \qquad \odot \qquad M_S \qquad = \qquad M_{S \circ R}$$

c) Reflexive closure: $M_{R_R} = M_V \vee$ Identity matrix I

$$\begin{matrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{matrix} \quad \vee \quad \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \quad = \quad \begin{matrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{matrix}$$

$$M_R \qquad \vee \qquad I \qquad = \qquad M_{R_R}$$

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d) Symmetric Closure: $M_{R_S} = M_R \vee M_R^T$

Note: M_R^T denotes transpose of M_R: rows of M_R become the columns of M_R^T

$$M_{R_S} = \begin{matrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{matrix} \vee \begin{matrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{matrix} = \begin{matrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{matrix}$$

$$M_R \quad \vee \quad M_R^T \quad = \quad M_{R_S}$$

e) Transitive Closure when original set has n elements: $M_{R_T} = M_R \vee M_R^2 \vee M_R^3 \vee \ldots \vee M_R^n$

Note: $M_R^2 = M_{R \circ R} = M_R \odot M_R$

$M_R^3 = M_{(R \circ R) \circ R} = M_R \odot M_R^2$

$M_R^4 = M_{(R \circ R \circ R) \circ R} = M_R \odot M_R^3$

$$M_{R_T} = \begin{matrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{matrix} \vee \begin{matrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \vee \begin{matrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \vee \begin{matrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} = \begin{matrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{matrix}$$

$$M_R \quad \vee \quad M_R^2 \quad \vee \quad M_R^3 \quad \vee \quad M_R^4 \quad = \quad M_{R_T}$$

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3. Summary of Grouping the characteristics of a Relation R on a set S

If Relation R on set S is
- Reflexive
- Symmetric
- Transitive

R is called an Equivalence Relation

If Relation R on set S is
- Reflexive
- Antisymmetric
- Transitive

R is called a Partial Order
(S, R) is called a Partially Ordered set or Poset

(It is called a Partial Order
Because there exists at least
one pair of elements in S that are
not related either way.)

$$\exists a \exists b \in S$$
$$(aRb \notin R) \wedge (bRa \notin R)$$

If every pair of elements of S
are related at least one way
it is called a Total Order

$$\forall a \forall b \in S$$
$$(aRb \in R) \vee (bRa \in R)$$

Ex 1: Equivalence Relation R on Z:	Ex 2: Partial Order R:	Ex 3: Partial Order R which is also a Total Order:
$\{(a,b) \mid aRb \rightarrow a \equiv b \bmod m,\ a,b \in Z\}$	$\{(x,y) \mid xRy \rightarrow x \mid y,\ x,y \in Z+\}$	$\{(x,y) \mid xRy \rightarrow x \leq y,\ x,y \in Z+\}$
Also denoted: (Z, mod m)	R is not a Total Order. Note: There exists 5, 7∈Z+ and $5 \nmid 7 \wedge 7 \nmid 5$ Poset is: (Z+, Divides)	Note: For all x, y∈Z+ $x \leq y \vee y \leq x$ Poset is: (Z+, ≤) and is a Totally Ordered set

4. Equivalence Relation:

Prove $\{(a,b) \mid aRb \rightarrow a \equiv b \bmod m,\ a, b \in Z$ is Reflexive, Symmetric and Transitive and hence by definition is an Equivalence Relation.

(i) Proof R is Reflexive (Direct):

Is aRa∈R for all a∈Z?

Consider a-a , a ∈ Z	
a-a = m x 0	Algebra
→ m | (a-a)	Def div
→ a ≡ a mod m	Def ≡
→ aRa	Def R

(ii) Proof R is symmetric (Direct):

Does aRb∈R → bRa∈R?

Consider aRb ∈ R	
→ a≡b mod m → m|(a-b)	Def R. Def ≡
→ (a-b) = mq, q∈Z	Def div
→ (-1)(a-b) = (-1)mq	Mult by -1, Alg
→ (b-a) = m(-q)	Algebra
→ m|(b-a)	Def div. def ≡
→ b≡a mod m	
→ bRa	Def R

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(iii) Proof R is transitive (Direct):

Proof (direct) Consider $aRb \in R \land bRc \in R$

Does $(aRb \in R \land bRc \in R) \rightarrow aRc \in R$

$aRb \rightarrow a \equiv b \bmod m \rightarrow m \mid (a-b) \rightarrow a-b = mq_1$	Def: R, ≡, div
$bRc \rightarrow b \equiv c \bmod m \rightarrow m \mid (b-c) \rightarrow b-c = mq_2$	Def: R, ≡, div
$(a-b) + (b-c) = mq_1 + mq_2$	Addition of 2 lines
$\rightarrow (a-c) = m(q_1+q_2) \rightarrow m \mid (a-c)$	Algebra, Def div
$a \equiv c \bmod m \rightarrow aRc$	Def ≡, def R

$\rightarrow \{(a,b) \mid aRb \rightarrow a \equiv b \bmod m,\ a, b \in Z\}$
is an Equivalence Relation

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5. What does an Equivalence Relation do to a set ?

a) Ex: Consider aRb $\rightarrow$ $a \equiv b \mod 4$, $a, b \in Z$

(Remainders when a, b are divided by 4 can only be: 0, 1, 2, 3

Case 1 Rem = 1: List the smallest non negative value of b:
List all possible values of a ?

b=1
a= …, -3, 1, 5, …

Case 2 Rem = 2: List the smallest non negative value of b:
List all possible values of a ?

b=2
a=…, -2, 2, 6, …

Case 3 Rem = 3: List the smallest non negative value of b:
List all possible values of a ?

b=3
a=…, -1, 3, 7, …

Case 4 Rem = 0: List the smallest non negative value of b:
List all possible values of a ?

b=0
a=…, -4, 0, 4, …

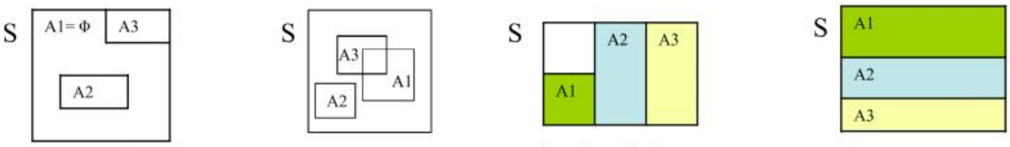
continued ……

5b) Definition: A Partition of a set S is a collection of sets $A_1, A_2, A_3, \ldots\ldots A_n$ such that:

(i) $A_i \neq \phi$ for all i — A_i are non empty

(ii) $A_i \cap A_j = \phi$ for all i, j — all A_i are disjoint

(iii) S = union of all A_i



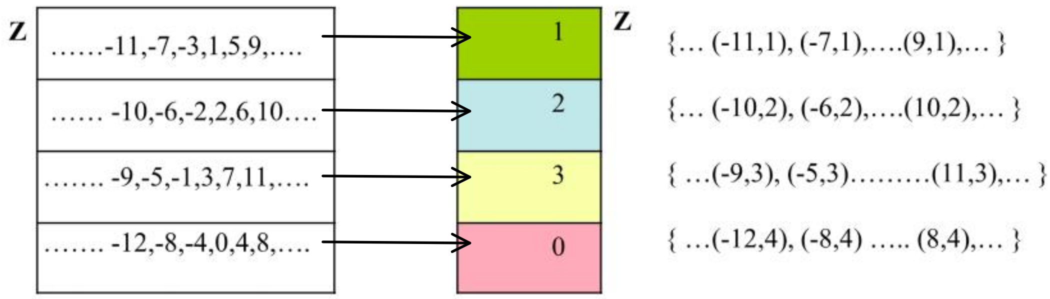
A_1, A_2, A_3 is not a partition of S
Condition (i) fails

A_1, A_2, A_3 is not a partition of S
Condition (ii) fails

A_1, A_2, A_3 is not a partition of S
Condition (iii) fails

A_1, A_2, A_3 is a partition of S

If we look at a diagram of the previous congruence mod 4 relation we have:



(i) The previous relation $a \equiv b \bmod 4$ Partitions the set of Integers Z into 4 different classes

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(ii) Since it was an equivalence relation that did the partition the classes are called Equivalence Classes

(iii) The members of each equivalence class are called equivalent elements
Ex: In the congruence mod 4 example: 5 is equivalent to 1: 5≡1, also 9≡1

(iv) Any member of the equivalence class can represent the class.
Ex: 1 or 5 or 9 or…. can represent the equivalence class in (iii)

(v) In general any equivalence relation on a set S partitions the set S

6. Partial Ordering Relation for a set:

a) Prove R ={(x,y) | xRy→ x | y, x,y ∈ Z+} is Reflexive, Anti-symmetric and Transitive

(i) Proof R is Reflexive (Direct):	Is aRa∈R for all a∈Z+?
Consider any value a ∈ Z +	
a = a x 1 → a|a → aRa	Def div, Def R
→ aRa ∈ R for all a ∈ Z+	
→ R is Reflexive	Def Reflexive
ii) Proof R is Anti-symmetric (Direct):	
Consider any values a,b ∈ Z+ where aRb, bRa	Does xRy∈R ∧ yRx ∈R → x=y?
→ a | b → b = aq1, q1∈Z+	Def R, def div
→ b | a → a = bq2, q1∈Z+	Def R, def div
→ b = (bq2)q1	Subst value of a in eq 2 for a in eq 1
→ b = bq2q1	Associative
→ 1 = 1 q2q1	Divide by b : (b ∈ Z+ → b≠0)
→ q1 = q2 = 1	q_1,q_1∈ Z+ and $q_1 q_1 = 1$
→ a=b→ R is Anti-symmetric	Sub q_1=1 in eq 1, Def Anti-sym.

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(iii) Proof R is Transitive (Direct):

Consider a, b, c $\in$ Z+ where aRb, bRc

Does (aRb∈R ∧ bRc∈R)→aRc∈R?

→ a|b → b = aq_1, q_1∈Z+	Def R, def div
→ b|c → c = bq_2, q_2∈Z+	Def R, def div
→$bq_2 = aq_1q_2$	Multiply eq 1 by q_2
→$c=a(q_1q_2)$	Substitute value c in eq 2 for b q_2 in eq 3
→a|c → aRc	Def div. Def R, Def Trans.
→R is Transitive	
QED	

b) Why is this Reflexive, Antisymmetric and Transitive Relation not called a Total Ordering Relation ?

Ex 1: Consider the above Relation R ={(x,y) | xRy→ x | y, x,y $\in$ Z+}

(i) let x = 3 and y = 8:

3 ∤ 8 and 8 ∤ 3

→ 3, 8 are not comparable using this relation R

(ii) let x = 12 and y = 3:

3|12

→ 3, 12 are comparable this relation R

In the above Reflexive, Antisymmetric and Transitive relation R on Z+ there exist elements

Where we do not have xRy and do not have yRx. (We cannot use R to relate them)

The term Total Order is used for a Relation R that is Reflexive, Antisymmetric and Transitive if

we can relate all elements in Domain with R. Hence R of Ex 1 is not a Total Ordering Relation.

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Ex 2: Is R ={(x,y) | xRy→ x < y, x,y $\in$ Z+} a Partial Order ?

No: R is not Reflexive, x not Related to y

7. Total Ordering Relation for a set:

a) Prove R ={(x,y) | xRy→ x ≤ y x,y ∈ Z+} is Reflexive, Antisymmetric and Transitive

(i) Proof R is Reflexive (Direct):	Is aRa∈R for all a∈Z+
Consider a ∈ Z +	
a≤a→aRa	Algebra, def R
(ii) Proof R is Antisymmetric (Direct):	Does (xRy∈R∧yRx∈R)→x=y
Consider a, b ∈ Z+ where aRb, bRa	
→ a≤b and b≤a	Def R
→ b=a	Algebra
(iii) Proof R is Transitive (Direct):	
Consider a, b, c ∈ Z+ where aRb, bRc	Does (xRy∈R∧yRz∈R)→xRz∈R
→a≤b and b≤c → a≤c	Def R, Alg
→aRc	Def R

b) In this Reflexive, Antisymmetric and Transitive relation R on Z+ all x, y ϵ Z+

→ x≤y or y≤x

→ for all x, y∈Z+ xRy or yRx (We can use this R to relate all x,y∈Z+)

The term Total Ordering Relation is used for Partial Order Relation if:

We can relate all elements in Domain with R. Hence the above R is a Total Ordering relation.

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8. Applications

Ex 1: Total Ordering Relation:

An Algorithm calls for the input of any Integers in any order. Repeats allowed

The Algorithm is required to produce a sequence of the Integers in order (smallest to greatest)

a) What Relation R can be used ?

$R = \{(x, y) \mid xRy \rightarrow x \leq y \; x, y \in Z\}$

b) What kind of a Relation is R ?

R is Reflexvie, Antisymmetric and Transitive and xRy or yRx for all x, y∈Z

→ R is a Total Ordering Relation

c) Describe the steps of the Algorithm using Relation R at every step

Step 1 Consider the first two input digits x, y.

xRy gives correct order for x, y

Step 2 Consider the third input digit z:

xRz gives correct order for z, x

yRz gives correct order fo z, y

Step 3 Repeat the process as in Step 2 until the remaining several digits have been placed in the correct order

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15(16)=240

140(16)=2240

2480

R={(x,y) | xRy → tweet x and tweet y have the same first 240 bits}

(xRx) → Reflexive

(xRy→yRx) → Symmetric

(xRy and yRz → xRz) → Transitive

an Equivalence Relation

R Partitions the Twitter cyber space into disjoint Equivalence classes
Each Equivalence class contains all tweets belonging to a single tag

$S = \{H, T, D, E, A, N\}$

$R = \{(T, H), (E, E), (N, D)\}$