7.4 Integration of Rational Functions by Partial Functions

July 13, 2016 14:26

Definitions & Theorems:

1. $f(x)=\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials.
   $f(x)$ is called ***proper*** if the degree of P that $\deg(P)<\deg(Q)$.

Step 1:	If $f(x)$ is ***improper***, that is $\deg(P)\geq\deg(Q)$ we should divide Q into P until a remainder $R(x)$ is obtained such that $\deg(R)<\deg(Q)$ $f(x)=\frac{P(x)}{Q(x)}=S(x)+\frac{R(x)}{Q(x)}$
Step 2:	Factor the denominator $Q(x)$
Step 3:	Express the proper rational function $\frac{R(x)}{Q(x)}$ as a sum of partial fractions of the form $\frac{A}{(ax+b)^{\mathrm{i}}}$ *or* $\frac{Ax+B}{(ax^2+bx+c)^{\mathrm{i}}}$

Case 1:	$Q(x)$ is a product of distinct linear factors. $Q(x)=(a_1x+b_1)(a_2x+b_2)\cdots(a_kx+b_k)$ $\frac{R(x)}{Q(x)}=\frac{A_1}{a_1x+b_1}+\frac{A_2}{a_2x+b_2}+\cdots+\frac{A_k}{a_kx+b_k}$
Case 2:	$Q(x)$ is a product of linear factors, some of which are repeated. Suppose the first linear factor a_1x+b_1 is repeated r times; that is, $(a_1x+b_1)^r$ occurs in the factorization of $Q(x)$. Then instead of the single term $\frac{A_1}{a_1x+b_1}$ would be $\frac{A_1}{a_1x+b_1}+\frac{A_1}{(a_1x+b_1)^2}+\cdots+\frac{A_1}{(a_1x+b_1)^r}$
Case 3:	$Q(x)$ contains irreducible quadratic factors, none of which is repeated. If $Q(x)$ has the factor ax^2+bx+c, where $b^2-4ac<0$, then, in addition to the partial fractions, the expression for $R(x)/Q(x)$ will have a term of the form $\frac{Ax+B}{ax^2+bx+c}$
Case 4:	$Q(x)$ contains a repeated irreducible quadratic factor. If $Q(x)$ has the factor $(ax^2+bx+c)^r$, where $b^2-4ac<0$, then the expression for $R(x)/Q(x)$ will have a term of the form $\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_rx+B_r}{(ax^2+bx+c)^r}$

Examples:

Case 1:

1. $\displaystyle\int\frac{x^2+2x-1}{2x^3+3x^2-2x}\mathrm{d}x$

$$2x^3+3x^2-2x=x(2x^2+3x-2)=x(2x-1)(x+2)$$
$$\frac{x^2+2x-1}{2x^3+3x^2-2x}=\frac{A}{x}+\frac{B}{2x-1}+\frac{C}{x+2}$$
$$\Rightarrow A(2x-1)(x+2)+Bx(x+2)+Cx(2x-1)=x^2+2x-1$$
$$\Rightarrow(2A+B+2C)x^2+(3A+2B-C)x-2A=x^2+2x-1$$
$$\Rightarrow\begin{cases}2A+B+2C=1\\3A+2B-C=2\\-2A=1\end{cases}\Rightarrow\begin{cases}A=\frac{1}{2}\\B=\frac{1}{5}\\C=-\frac{1}{10}\end{cases}$$
$$\int\frac{x^2+2x-1}{2x^3+3x^2-2x}\mathrm{d}x=\int(\frac{\frac{1}{2}}{x}+\frac{\frac{1}{5}}{2x-1}+\frac{-\frac{1}{10}}{x+2})\,\mathrm{d}x=\frac{1}{2}\ln|x|+\frac{1}{10}\ln|2x-1|-\frac{1}{10}\ln|x+2|+C$$

2. $\displaystyle\int\frac{1}{1-x^2}\mathrm{d}x=\int\frac{1}{(1-x)(1+x)}\mathrm{d}x=\int\frac{1}{2}\left(\frac{1}{1-x}+\frac{1}{1+x}\right)\mathrm{d}x=\frac{1}{2}\int\frac{1}{1-x}\mathrm{d}x+\frac{1}{2}\int\frac{1}{1+x}\mathrm{d}x=-\frac{1}{2}\ln|1-x|+\frac{1}{2}\ln|1+x|+C$

3. $\displaystyle\int \frac{x^4}{x-1}\,\mathrm{d}x = \int\left(x^3+x^2+x+1+\frac{1}{x-1}\right)\mathrm{d}x$

4. $\displaystyle\int \frac{x}{(x+4)(2x-1)}\,\mathrm{d}x = \int\left(\frac{A}{x+4}+\frac{B}{2x-1}\right)\mathrm{d}x$

$$\frac{x}{(x+4)(2x-1)} = \frac{A}{x+4}+\frac{B}{2x-1} \Rightarrow x = A(2x-1)+B(x+4) \Rightarrow 1x+0 = (2A+B)x+(-A+4B)$$

$$\Rightarrow \begin{cases} 2A+B=1 \\ -A+4B=0 \end{cases} \Rightarrow \begin{cases} A=\frac{4}{9} \\ B=\frac{1}{9} \end{cases} \Rightarrow \int\frac{x}{(x+4)(2x-1)}\,\mathrm{d}x = \int\left(\frac{\frac{4}{9}}{x+4}+\frac{\frac{1}{9}}{2x-1}\right)\mathrm{d}x = \frac{4}{9}\ln|x+4|+\frac{1}{18}\ln|2x-1|+C$$

Case 2:

1. $\displaystyle\int_2^3 \frac{3x-5x^2}{(3x-1)(x-1)^2}\,\mathrm{d}x$

$$\frac{3x-5x^2}{(3x-1)(x-1)^2} = \frac{A}{3x-1}+\frac{B}{x-1}+\frac{C}{(x-1)^2} \Rightarrow 3x-5x^2 = A(x-1)^2+B(3x-1)(x-1)+C(3x-1)$$

when $x=1 \Rightarrow 3(-1)-5(-1)^2 = A(1-1)^2+B(3*1-1)(1-1)+C(3*1-1) \Rightarrow -2=2C \Rightarrow C=-1$

when $x=\frac{1}{3} \Rightarrow \frac{4}{9}=\frac{4A}{9} \Rightarrow A=1$

when $x=0 \Rightarrow 0=1+B+1 \Rightarrow B=-2$

$$\Rightarrow \int_2^3 \frac{3x-5x^2}{(3x-1)(x-1)^2}\,\mathrm{d}x = \int_2^3 \left(\frac{1}{3x-1}+\frac{-2}{x-1}+\frac{-1}{(x-1)^2}\right)\mathrm{d}x = \int_2^3\frac{1}{3x-1}\,\mathrm{d}x - \int_2^3\frac{2}{x-1}\,\mathrm{d}x - \int_2^3\frac{1}{(x-1)^2}\,\mathrm{d}x$$

$$= \frac{1}{3}(\ln 8 - \ln 5) - 2\ln 2 - \frac{1}{2}$$

Case 3:

1. $\displaystyle\int \frac{10}{(x-1)(x^2+9)}\,\mathrm{d}x$

$$\frac{10}{(x-1)(x^2+9)} = \frac{A}{x-1}+\frac{Bx+C}{x^2+9} \Rightarrow 10 = A(x^2+9)+(Bx+C)(x-1)$$

when $x=1 \Rightarrow A=1$

when $x=0 \Rightarrow C=-1$

when $x=-1 \Rightarrow B=-1$

$$\Rightarrow \int\frac{10}{(x-1)(x^2+9)}\,\mathrm{d}x = \int\left(\frac{1}{x-1}+\frac{-x-1}{x^2+9}\right)\mathrm{d}x = \int\frac{1}{x-1}\,\mathrm{d}x - \int\frac{x}{x^2+9}\,\mathrm{d}x - \int\frac{1}{x^2+9}\,\mathrm{d}x$$

Let $u = x^2+9, v=\frac{x}{3} \Rightarrow \mathrm{d}u = 2x\,\mathrm{d}x,\ \mathrm{d}v=\frac{1}{3}\mathrm{d}x$

$$\Rightarrow \int\frac{1}{x-1}\,\mathrm{d}x - \int\frac{x}{x^2+9}\,\mathrm{d}x - \int\frac{1}{x^2+9}\,\mathrm{d}x = \int\frac{1}{x-1}\,\mathrm{d}x - \frac{1}{2}\int\frac{\mathrm{d}u}{u} - \left(\frac{1}{9}\right)(3)\int\frac{\mathrm{d}v}{v^2+1}$$

$$= \ln|x-1| + \frac{1}{2}\ln|u| - \frac{1}{3}\arctan(v) + C = \ln|x-1| + \frac{1}{2}\ln|x^2+9| - \frac{1}{3}\arctan\left(\frac{x}{3}\right)+C$$

Case 4:

1. $\displaystyle\int \frac{x^3+10x^2+3x+36}{(x-1)(x^2+4)^2}\,\mathrm{d}x$

$$\frac{x^3+10x^2+3x+36}{(x-1)(x^2+4)^2} = \frac{A}{x-1}+\frac{Bx+C}{x^2+4}+\frac{Dx+E}{(x^2+4)^2}$$

$x^3+10x^2+3x+36 = A(x^2+4)^2+(Bx+C)(x-1)(x^2+4)+(Dx+E)(x-1)$

$\Rightarrow A=2, B=-2, C=-1, D=1, E=0$

$$\int\frac{x^3+10x^2+3x+36}{(x-1)(x^2+4)^2}\,\mathrm{d}x = \int\left(\frac{2}{x-1}+\frac{-2x-1}{x^2+4}+\frac{x}{(x^2+4)^2}\right)\mathrm{d}x = 2\ln|x-1| - \ln(x^2+4) - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) - \frac{1}{2(x^2+4)}+C$$

Pitfalls:

1. $\displaystyle\int\frac{x^2}{x^2-1}\,\mathrm{d}x = \int\frac{x^2-1+1}{x^2-1}\,\mathrm{d}x = \int\mathrm{d}x + \int\frac{1}{x^2-1}\,\mathrm{d}x$

2. $\displaystyle\int\frac{2x-1}{x^2-x-6}\,\mathrm{d}x$

Let $u = x^2-x-6 \Rightarrow \mathrm{d}u = (2x-1)\,\mathrm{d}x$

$$\int\frac{2x-1}{x^2-x-6}\,\mathrm{d}x = \int\frac{\mathrm{d}u}{u} = \ln|u|+C = \ln|x^2-x-6|+C$$