# Sec 1.6 Rules of Inference

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## Section 1.6 Rules of Inference

Comp 232
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1.What is an Argument ?

Vocabulary:

- Argument is a sequence of propositions
- Premises is another word for hypothesis (given)
- Conclusion is the final proposition in the argument
- Valid argument is one where the conclusion follows from the given premises
- Invalid argument is one where conclusion does not follow from the given premises
  (Invalid argument is called a fallacy)

Ex 1:

| | | |
|---|---|---|
| | $p \to q$ | Given |
| | $p$ | Given |
| Conclusion: q | | Valid argument |

Ex 2:

| | | |
|---|---|---|
| | $\neg p \lor q$ | Given |
| | $q$ | Given |
| Conclusion: p | | Invalid argument |

Why?
$\neg p \lor q \equiv p \to q$,
hence q is necessary but not sufficient for p

2. Rules of Inference

a) These are simple arguments.
The above Ex 1 argument is the first Rule of Inference in our list

b) There is a list of the Rules of Inferences in the following table. We have seen the ideas before that lead to this list.

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3.

| Summary of Rules of Inference | | |
|---|---|---|
| **Rule** | **Argument (Using not, and, or). Note that they are Tautologies** | **Name (related to previous results)** |
| p<br>p→q<br>Conclusion: q | $[p \wedge (p \rightarrow q)] \rightarrow q$ | Modus ponens<br>(def of implication) |
| ¬q<br>p→q<br>Conclusion: ¬p | | Modus tollens<br>(contrapositive) |
| p→q<br>q→r<br>Conclusion: p→r | | Hypothetical syllogism<br>(transitive) |
| p∨q<br>¬p<br>Conclusion: q | | Disjunctive syllogism<br>(def of or) |
| p<br>Conclusion: p∨q | | Addition<br>(def of or) |
| p∧q<br>Conclusion: p | | Simplification<br>(def of and) |
| p<br>q<br>Conclusion: p∧q | | Conjunction<br>(def of and) |
| p∨q<br>¬p∨r<br>Conclusion: q∨r | | Resolution |



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4. Resolution inference:
   This rule is based on the Tautology $[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$

Method 1: Show Tautology with logic equivalences:

| | | |
|---|---|---|
| | $[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$ | Given |
| $\equiv$ | $\neg[(p \vee q) \wedge (\neg p \vee r)] \vee (q \vee r)$ | $\rightarrow$ with Or |
| $\equiv$ | $\neg[(F \vee q) \wedge (T \vee r) \vee (T \vee q) \wedge (F \vee r)] \vee (q \vee r)$ | Only possibilities: p=F Or p=T |
| $\equiv$ | $\neg[q \wedge T \vee T \wedge r] \vee (q \vee r)$ | $F \vee x = x$, $T \vee x = T$ |
| $\equiv$ | $\neg(q \vee r) \wedge (q \vee r)$ | $x \wedge T = x$ |
| $\equiv$ | T | $\neg x \vee x = T$ |

Method 2: Show Tautology with Truth Tables:

| p | q | r | | | | | | $(p \vee q) \wedge (\neg p \vee r) \rightarrow (q \vee r)$ |
|---|---|---|---|---|---|---|---|---|
| T | T | T | | | | | | |
| T | T | F | | | | | | |
| T | F | T | | | | | | |
| T | F | F | | | | | | |
| F | T | T | | | | | | |
| F | T | F | | | | | | |
| F | F | T | | | | | | |
| F | F | F | | | | | | |

5. A valid argument (one that uses the Rules of Inference correctly) can lead to a false conclusion. This can happen if one of the given Premises is false. We say that the "structure" of the argument is correct but at least one of given Premises must not be true.

Ex: Consider the following proof:

| | |
|---|---|
| $\sqrt{2} > \frac{3}{2}$ | Premise (given) |
| $\left(\sqrt{2}\right)^2 > \left(\frac{3}{2}\right)^2$ | Square both sides |
| $2 > \frac{9}{4}$ | Simplify |
| Conclusion: 2>2.25 | $\frac{9}{4} = 2.25$ |

The conclusion is obviously false but the structure of the argument is correct.
What premise (given fact) is false ?

Answer: $\sqrt{2} > \frac{3}{2}$ is not correct.

6. Examples of arguments with one Inference Rule

Ex 1: It is freezing and raining now. Conclusion: It is freezing now.
Let p represent: it is freezing now
Let q represent: it is raining now

| | |
|---|---|
| Argument: $p \wedge q$ | Given |
| Conclusion: p | Def: And |

Replace the p in the conclusion and we get: It is freezing now

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Ex 2: If it rains today then no barbecue today
If no barbecue today then barbecue tomorrow
Conclusion: If it rains today then barbecue tomorrow

Let p represent: it rains today
Let q represent: no barbecue today
Let r represent: barbecue tomorrow

| Argument: | | |
|---|---|---|
| | $p \rightarrow q$ | Given |
| | $q \rightarrow r$ | Given |
| | Conclusion: $p \rightarrow r$ | Transitive |

Replace the variables in the conclusion and we get: If it rains today then barbecue tomorrow

7. Examples of arguments with more than one Inference rule.

Ex 1: $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, $s \rightarrow t$, Concl: t

| Argument : | | |
|---|---|---|
| | $\neg p \wedge q$ | Given |
| | $\neg p$ | Def : And |
| | $r \rightarrow q$ | Given |
| | $\neg p \rightarrow \neg r$ | Contrapositive |
| | $\neg r$ | def $\rightarrow$ |
| | $\neg r \rightarrow s$ | Given |
| | $s \rightarrow t$ | Given |
| | $\neg r \rightarrow t$ | Transitive |
| | Conclusion: t | |

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Ex 3: If you send me an email then I will finish the program. If you do not send me an email then I will go to sleep early. If I go to sleep early then I will wake up refreshed.
Conclusion: If I do not finish the program then I will wake up refreshed.

Let p represent: you send me an email
Let q represent: I will finish the program
Let r represent: I will go to sleep early
Let s represent: I will wake up refreshed: Givens are: $p \rightarrow q$, $\neg p \rightarrow r$, $r \rightarrow s$

Argument:

| | | |
|---|---|---|
| | $p \rightarrow q$ | Given |
| | $\neg q \rightarrow \neg p$ | Contrapositive |
| | $\neg p \rightarrow r$ | Given |
| | $\neg q \rightarrow r$ | Transitive |
| | $r \rightarrow s$ | Given |
| Conclusion: | $\neg q \rightarrow s$ | Transitive |

Replace the variables in the conclusion and we get: If I do not finish the program then I will wake up refreshed.

Ex 4: Using Resolution inference in an argument.

Prove the argument $(p \wedge q) \vee r$, $r \rightarrow s$, Conclusion: $p \vee s$

Argument:

| | | |
|---|---|---|
| Step 1 | $(p \wedge q) \vee r$ | Given |
| | $(p \vee r) \wedge (q \vee r)$ | Distributive |
| | $p \vee r$ | Def : AND |
| | $r \vee p$ | Commutative |
| Step 2 | $r \rightarrow s$ | Given |
| | $\neg r \vee s$ | $\rightarrow$ with or |
| Conclusion | $p \vee s$ | Resolution on step 1, step 2 ans |

8. Fallacy

a) A non valid argument is also called a fallacy. A fallacy has violated at least one of the rules of logic.

b) Two often seen fallacies:

| | |
|---|---|
| $p \rightarrow q$<br>$q$<br>Concl $p$<br>This is a fallacy because:<br>$p \rightarrow q$ means:<br>q is necessary but not sufficient for p | $p \rightarrow q$<br>$\neg p$<br>Concl $\neg q$<br>This is a fallacy because:<br>$p \rightarrow q$ is equivalent to:<br>$\neg q \rightarrow \neg p$<br>which means ¬p is necessary but not sufficient for ¬q |

9. Rules of Inference for Quantifiers

| Rules of Inference for Quantifiers | |
|---|---|
| $\forall x P(x)$<br>Concl: $P(a)$ | When a represents all elements in the Domain |
| $P(a)$<br>Concl: $\exists x P(x)$ | When a represents at least one element in the Domain |
| $P(a)$<br>Concl : $\forall x P(x)$ | When a represents all elements in the Domain |
| $\exists x P(x)$<br>Concl: $P(a)$ | When a represents at least one element in the Domain |

10. Inferences for propositions and inferences for quantifiers can appear in the same argument



| Combining Rules of inference for propositions and quantifiers | |
|---|---|
| $\forall x\ [\ P(x) \rightarrow Q(x)\ ]$<br>$P(a)$<br>Concl: $Q(a)$ | When a represents all elements in the Domain |
| $\forall x\ [\ P(x) \rightarrow Q(x)\ ]$<br>$\neg Q(a)$<br>Concl: $\neg P(a)$ | When a represents all elements in the Domain |

Ex 1: C(x): x is in Comp 232
B(x): x studies
P(x): x passes
Domain: x represents All Concordia students

Is the following argument valid ?

Given: $\exists x\,[C(x) \wedge \neg B(x)]$

$\forall x\,[C(x) \rightarrow P(x)]$

Concl: $\exists x\,[P(x) \wedge \neg B(x)]$

| | | |
|---|---|---|
| Step1 | 1. $\exists x\,[C(x) \wedge \neg B(x)]$ | Given |
| | 2. $C(a) \wedge \neg B(a)$ | x = a represents at least one student guaranteed by line 1 |
| | 3. $C(a)$ | Def And from line 2 |
| Step 2 | 4. $\forall x\,[C(x) \rightarrow P(x)]$ | Given |
| | 5. $C(a) \rightarrow P(a)$ | x = a is a student in the Domain so guarantee by line 4 |
| | 6. $P(a)$ | Def $\rightarrow$ |
| Step 3 | 7. $\neg B(a)$ | Def And from line 2 |
| | 8. $P(a) \wedge \neg B(a)$ | Def and using lines 6 and 7 |
| | Concl $\exists x\,[P(x) \wedge \neg B(x)]$ | Since there is at least one student x = a, use Def of $\exists$ |

Is the Argument valid ? Yes

What is the Conclusion in words ? There is at least one Concordia student in Comp 232 who passes and does not study.

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Ex 2: Is the following argument below valid ? If not what error(s) are made in the argument ? $[\forall x [P(x) \vee Q(x)] \rightarrow [\forall x\ P(x) \vee \forall x\ Q(x)]$

| | |
|---|---|
| 1. $\forall x [P(x) \vee Q(x)]$ | Given |
| 2. $P(a) \vee Q(a)$ | Let x = a represent all elements of Domain |
| × 3. Assume $P(a) = T$ | Def Or from line 2 |
| 4. $\forall x P(x) = T$ | Since x = a represents all elements in domain use Def $\forall$ |
| × 5. Assume $Q(a) = T$ | Def Or from line 2 |
| 6. $\forall x\ Q(x) = T$ | Since x = a represents any element in domain use Def $\forall$ |
| Concl $\forall x P(x) \vee \forall x Q(x)$ | Def Or |

The argument is not valid.

$P(a) \vee Q(a) = T$: is not sufficient to assume $P(a) = T$ in line 3.

is not sufficient to assume $Q(a) = T$ in line 5.

Ex 3: If the Or is replaced by And in Ex 2, rewrite the argument. Is it valid ? If not what error(s) are made in the argument ?

$$\forall x [P(x) \wedge Q(x)] \rightarrow \forall x P(x) \wedge \forall x Q(x)$$

This argument is valid.

In a similar way the converse can be argued: $\forall x P(x) \wedge \forall x Q(x) \rightarrow \forall x [P(x) \wedge Q(x)]$

The two implications of Ex 3 establish: $\forall x P(x) \wedge \forall x Q(x) \equiv \forall x [P(x) \wedge Q(x)]$

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Ex 4: Using the results of Ex 3, prove the following argument:

Given: $\forall x[P(x) \rightarrow Q(x)]$
$\forall x[Q(x) \rightarrow R(x)]$
Conclusion: $\forall x[P(x) \rightarrow R(x)]$

| | |
|---|---|
| $\forall x[P(x) \rightarrow Q(x)] \wedge \forall x[Q(x) \rightarrow R(x)]$ | Def And using Given |
| $\forall x\{[P(x) \rightarrow Q(x)] \wedge [Q(x) \rightarrow R(x)]\}$ | Equival. Ex 3 $[\forall x\, M(x) \wedge \forall x\, N(x)] \equiv \forall x\, [M(x) \wedge N(x)]$ |
| $\forall x\{[\neg P(x) \vee Q(x)] \wedge [\neg Q(x) \vee R(x)]\}$ | $\rightarrow$ in terms of Or |
| $\forall x\{[Q(x) \vee \neg P(x)] \wedge [\neg Q(x) \vee R(x)]\}$ | Commutative |
| $\forall x[\neg P(x) \vee R(x)]$ | Resolution inference |
| $\forall x[P(x) \rightarrow R(x)]$ | $\rightarrow$ in terms of Or |

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