11.8 Power Series

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Definitions & Theorems:

1. Definition: Power series

   A power series is a series of the form
   $$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$
   where x is a variable and the c_n's are constants called the **coefficients** of the series.

2. Definition:

   A power series in $(x-a)$ or a power series centered at a or a power series about a is of the form
   $$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

3. Theorem:

   For a given power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, there are only three possibilities:

   (i) The series converges only when $x = a$.

   (ii) The series converges for all x.

   (iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

   Every power series is convergent for $x = a$.

4. Definition: Radius of convergence

   The number R in Theorem3 case (iii) is called the **radius of convergence** of the power series.

   Case (i): $R = 0$

   Case (ii): $R = \infty$

   Case (iii): $R = R$

5. Definition: interval of convergence

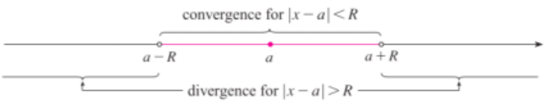
   The **interval of convergence** of a power series is the interval that consists of all values of x for which the series converges.

   Case (i): $\{a\}$

   Case (ii): $(-\infty, \infty)$

   Case (iii): four possibilities

   $(a-R, a+R) \quad (a-R, a+R] \quad [a-R, a+R) \quad [a-R, a+R]$



Examples:

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$

$$L = \lim_{n\to\infty} \left| \frac{\frac{(n+1)(x+2)^{n+1}}{3^{(n+1)+1}}}{\frac{n(x+2)^n}{3^{n+1}}} \right| = \lim_{n\to\infty} \left| \frac{n+1}{3n}(x+2) \right| = \left| \frac{x+2}{3} \right| \lim_{n\to\infty} \left| \frac{n+1}{n} \right| = \left| \frac{x+2}{3} \right|$$

(i) $L < 1 \Rightarrow \left| \frac{x+2}{3} \right| < 1 \Rightarrow -5 < x < 1$

By Ratio Test, $\sum_{n=1}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$ absolutely converges for $x \in (-5, 1)$

(ii) $L > 1 \Rightarrow \left|\frac{x+2}{3}\right| > 1 \Rightarrow x > 1, x < -5$

By Ratio Test, $\sum_{n=1}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$ diverges for $x \in (-\infty, -5) \cup (1, \infty)$

(iii) $L = 1 \Rightarrow \left|\frac{x+2}{3}\right| = 1 \Rightarrow x = 1, x = -5$

a. $x = 1, \sum_{n=1}^{\infty} \frac{n(x+2)^n}{3^{n+1}} = \sum_{n=1}^{\infty} \frac{n3^n}{3^{n+1}} = \sum_{n=1}^{\infty} \frac{n}{3}$

By Test of Divergence, $\sum_{n=1}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$ diverges on $x = 1$

b. $x = -5, \sum_{n=1}^{\infty} \frac{n(x+2)^n}{3^{n+1}} = \sum_{n=1}^{\infty} \frac{n(-3)^n}{3^{n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^n n}{3}$

By Test of Divergence, $\sum_{n=1}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$ diverges on $x = -5$

Therefore, the set of all values of x for which the series converges is $(-5, 1)$, $R = 3$

2. $\sum_{n=1}^{\infty} n!\,(2x-1)^n$

$$L = \lim_{n\to\infty} \left|\frac{(n+1)!\,(2x-1)^{n+1}}{n!\,(2x-1)^n}\right| = \lim_{n\to\infty} \left|(n+1)(2x-1)\right| = |2x-1| \lim_{n\to\infty} |n+1|$$

If $x = \frac{1}{2} \Rightarrow L = 0 < 1$

If $x \neq \frac{1}{2} \Rightarrow \mathrm{L} = \infty > 1$

Therefore, the set of all values of x for which the series converges is $\{\frac{1}{2}\}$, $R = 0$

3. $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n}$

$$L = \lim_{n\to\infty} \left|\frac{\frac{(x-5)^{n+1}}{n+1}}{\frac{(x-5)^n}{n}}\right| = \lim_{n\to\infty} \left|\frac{n}{n+1}(x-5)\right| = |x-5| \lim_{n\to\infty} \left|\frac{n}{n+1}\right| = |x-5|$$

(i) $L < 1 \Rightarrow |x-5| < 1 \Rightarrow 4 < x < 6$

By Ratio Test, $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n}$ absolutely converges for $x \in (4,6)$

(ii) $L > 1 \Rightarrow |x-5| > 1 \Rightarrow x > 6, x < 4$

By Ratio Test, $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n}$ diverges for $x \in (-\infty, 4) \cup (6, \infty)$

(iii) $L = 1 \Rightarrow |x-5| = 1 \Rightarrow x = 6, x = 4$

a. $x = 6, \sum_{n=1}^{\infty} \frac{(x-5)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$

By p-series, $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n}$ diverges on $x = 6$

b. $x = 4, \sum_{n=1}^{\infty} \frac{(x-5)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

By Alternating Series Test, $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n}$ converges on $x = 4$

Therefore, the set of all values of x for which the series converges is $[4,6)$, $R = 1$

4. $\sum_{n=1}^{\infty} \frac{x^n}{n!}$

$$L = \lim_{n\to\infty} \left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| = \lim_{n\to\infty} \left|\frac{1}{n+1}x\right| = |x| \lim_{n\to\infty} \left|\frac{1}{n+1}\right| = 0 < 1$$

Therefore, the set of all values of x for which the series converges is $(-\infty, \infty)$, $R = \infty$

5. Show that the interval of convergence of $\sum_{n=1}^{\infty} \frac{n}{b^n}(x-a)^n$ for $b > 0$ is $(a-b, a+b)$

6. Show that the interval of convergence of $\sum_{n=1}^{\infty} \frac{n}{\ln n}(x-a)^n$ for $b>0$ is $\left[a-\frac{1}{b}, a+\frac{1}{b}\right)$

★7. $\sum_{n=1}^{\infty} \frac{x^n}{3^n}$

$$\sum_{n=1}^{\infty} \frac{x^n}{3^n} = \sum_{n=1}^{\infty} \left(\frac{x}{3}\right)^n$$

$\Rightarrow \sum_{n=1}^{\infty} \left(\frac{x}{3}\right)^n$ converges when $\left|\frac{x}{3}\right| < 1$(geometric series)$\Rightarrow -3 < x < 3$