7.8 Improper Integrals

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Definitions & Theorems:

1. Definition: Improper Integrals

   A definite integral $f = \int_a^b f(x)\,\mathrm{d}x$ is called an improper integral if the interval is infinite or f has an infinite discontinuity in $[a, b]$.

2. Definition: Definition of an improper integral of TYPE 1 (infinite interval)
   a. If $\int_a^t f(x)\,\mathrm{d}x$ exsits for every number $t \geq a$, then
   $$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{t\to\infty}\int_a^t f(x)\,\mathrm{d}x$$
   b. If $\int_t^b f(x)\,\mathrm{d}x$ exsits for every number $t \leq b$, then
   $$\int_{-\infty}^b f(x)\,\mathrm{d}x = \lim_{t\to-\infty}\int_t^b f(x)\,\mathrm{d}x$$

   The improper integrals $\int_a^\infty f(x)\,\mathrm{d}x$ and $\int_{-\infty}^b f(x)\,\mathrm{d}x$ are called convergent if the corresponding limit exists and divergent if the limit does not exist.

   c. If both $\int_a^\infty f(x)\,\mathrm{d}x$ and $\int_{-\infty}^a f(x)\,\mathrm{d}x$ are convergent, then we define
   $$\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \int_{-\infty}^a f(x)\,\mathrm{d}x + \int_a^\infty f(x)\,\mathrm{d}x\,, a \in \mathbb{R}$$

3. Definition: Definition of an improper integral of TYPE 2 (discontinuous)
   a. If f is continuous on $[a, b)$ and is discontinous at b, then
   $$\int_a^b f(x)\,\mathrm{d}x = \lim_{t\to b^-}\int_a^t f(x)\,\mathrm{d}x$$
   if this limit exists (as a finite number).

   b. If f is continuous on $(a, b]$ and is discontinous at a, then
   $$\int_a^b f(x)\,\mathrm{d}x = \lim_{t\to a^+}\int_t^b f(x)\,\mathrm{d}x$$
   if this limit exists (as a finite number).

   The improper integral $\int_a^b f(x)\,\mathrm{d}x$ is called convergent if the corresponding limit exists and divergent if the limit does not exist.

   c. If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,\mathrm{d}x$ and $\int_c^b f(x)\,\mathrm{d}x$ are convergent, then we define
   $$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$$

4. Theorem:
   $\int_1^\infty \frac{1}{x^p}\mathrm{d}x$ is convergent if $p > 1$ and divergent if $p \leq 1$.

5. Theorem: Comparison theorem

   Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.
   a. If $\int_a^\infty f(x)\,\mathrm{d}x$ is convergent, then $\int_a^\infty g(x)\,\mathrm{d}x$ is convergent.
   b. If $\int_a^\infty g(x)\,\mathrm{d}x$ is divergent, then $\int_a^\infty f(x)\,\mathrm{d}x$ is divergent.

Proofs or Explanations:

1. Theorem4:
$$\int_1^\infty \frac{1}{x^p}\mathrm{d}x = \lim_{t\to\infty}\int_1^t \frac{1}{x^p}\mathrm{d}x = \lim_{t\to\infty}\int_1^t x^{-p}\,\mathrm{d}x$$
$$= \begin{cases} p = 1 \Rightarrow \lim_{t\to\infty}\int_1^t x^{-p}\,\mathrm{d}x = \lim_{t\to\infty}[\ln|x|]_1^t = \lim_{t\to\infty}\ln t \\ p \neq 1 \Rightarrow \lim_{t\to\infty}\int_1^t x^{-p}\,\mathrm{d}x = \lim_{t\to\infty}\left[\frac{x^{1-p}}{1-p}\right]_1^t = \lim_{t\to\infty}[\frac{1}{1-p}(t^{1-p}-1)] \end{cases}$$

$$\Rightarrow \begin{cases} p=1 \Rightarrow \int_1^\infty \frac{1}{x^p}\,\mathrm{d}x = \infty \\ 1-p>0 \Rightarrow \int_1^\infty \frac{1}{x^p}\,\mathrm{d}x = \infty \\ 1-p<0 \Rightarrow \int_1^\infty \frac{1}{x^p}\,\mathrm{d}x = -\frac{1}{1-p} \end{cases} \Rightarrow \begin{cases} p=1 \Rightarrow \int_1^\infty \frac{1}{x^p}\,\mathrm{d}x = \infty \\ p<1 \Rightarrow \int_1^\infty \frac{1}{x^p}\,\mathrm{d}x = \infty \\ p>1 \Rightarrow \int_1^\infty \frac{1}{x^p}\,\mathrm{d}x = \frac{1}{p-1} \end{cases}$$

$\Rightarrow \int_1^\infty \frac{1}{x^p}\,\mathrm{d}x$ is convergent if $p>1$ and divergent if $p \le 1$

Examples:

1. $$\int_0^1 \frac{1}{x}\mathrm{d}x = \lim_{t\to 0^+}\int_t^1 \frac{1}{x}\mathrm{d}x = \lim_{t\to 0^+}[\ln x]_t^1 = \lim_{t\to 0^+}(-\ln t) = \infty$$

   So, $\int_0^1 \frac{1}{x}\mathrm{d}x$ diverges.

2. $$\int_2^3 \frac{\mathrm{d}x}{\sqrt{3-x}} = \lim_{t\to 3^-}\int_2^t \frac{\mathrm{d}x}{\sqrt{3-x}}$$

   Let $u = 3-x \Rightarrow du = -dx$

   $$\lim_{t\to 3^-}\int_2^t \frac{\mathrm{d}x}{\sqrt{3-x}} = \lim_{t\to 3^-}\int_1^{3-t} \frac{-\,\mathrm{du}}{\sqrt{\mathrm{u}}} = \lim_{t\to 3^-}\left[-2\sqrt{u}\right]_1^{3-t} = \lim_{t\to 3^-}\left(-2\sqrt{3-t}+2\right) = 2$$

3. $$\int_{-\infty}^\infty \frac{\mathrm{d}x}{1+x^2} = \int_{-\infty}^0 \frac{\mathrm{d}x}{1+x^2} + \int_0^\infty \frac{\mathrm{d}x}{1+x^2}$$

   $$\int_{-\infty}^0 \frac{\mathrm{d}x}{1+x^2} = \lim_{t\to -\infty}\int_t^0 \frac{\mathrm{d}x}{1+x^2} = \lim_{t\to -\infty}[\arctan x]_t^0 = \lim_{t\to -\infty}[-\arctan(t)] = \frac{\pi}{2}$$

   $$\int_0^\infty \frac{\mathrm{d}x}{1+x^2} = \lim_{t\to \infty}\int_0^t \frac{\mathrm{d}x}{1+x^2} = \lim_{t\to \infty}[\arctan x]_0^t = \lim_{t\to \infty}[\arctan(t)] = \frac{\pi}{2}$$

   $$\Rightarrow \int_{-\infty}^\infty \frac{\mathrm{d}x}{1+x^2} = \pi$$

4. For what values of p, $\int_2^\infty \frac{1}{x(\ln x)^p}\mathrm{d}x$ will converge.

   $$\int_2^\infty \frac{1}{x(\ln x)^p}\mathrm{d}x = \lim_{t\to\infty}\int_2^t \frac{1}{x(\ln x)^p}\mathrm{d}x$$

   Let $u = \ln x \Rightarrow du = \frac{1}{x}dx$

   $$\lim_{t\to\infty}\int_2^t \frac{1}{x(\ln x)^p}\mathrm{d}x = \lim_{t\to\infty}\int_{\ln 2}^{\ln t} \frac{1}{u^p}\mathrm{d}u = \lim_{t\to\infty}\left(\int_1^{\ln t} \frac{1}{u^p}\mathrm{d}u + \int_{\ln 2}^1 \frac{1}{u^p}\mathrm{d}u\right) = \int_1^\infty \frac{1}{u^p}\mathrm{d}u + \int_{\ln 2}^1 \frac{1}{u^p}\mathrm{d}u$$

   $\Rightarrow$ when $p>1$, $\int_2^\infty \frac{1}{x(\ln x)^p}\mathrm{d}x$ will converge