Section 5.3 Recursive Definitions

Comp 232
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1. Definition: A sequence is an ordered set

Ex: {1, .5, .01, .001, .0001, ……}

Order matters: The above sequence is not the same as sequence {1, .01, .5, .01, .001, .0001, …}

In a Set order does not matter. In a Sequence order does matter.

2. Three ways to define the elements of a sequence:

List the entries
Ex: Listing (see above example)

n-th term function

Recursive function

3. Functions used to define a sequence. There are two types:

n-th term function:

One equation applied to all elements in the domain

Recursive function:

Step 1 Define the initial value(s) of f(n)
Step 2 Successive function values are defined using previous function value(s)

Ex: n-th term function

$F(n) = \frac{1}{(n-1)!}$ Domain: **Z,** $n \geq 1$

$F(n) = \frac{1}{(n-1)!}$ is also written: an=1/(n-1)!, n=1,2,3,4,...

$\rightarrow$ Sequence listing: $\{ \frac{1}{0!}, \frac{1}{1!}, \frac{1}{2!}, \frac{1}{3!}, \frac{1}{4!}, \frac{1}{5!}, \ldots\ldots\ldots \}$ = {1,1,1/2,1/6,1/24,1/120,...}

Note: If asked for a_7 = 1/(7-1)! = 1/720

Ex: Recursive function

Step 1 $x_1 = f(1) = 1$

$x_2 = f(2) = 1$

Step 2 $\forall n \ \ n \geq 3, \ x_n = f(n) = x_{n-2} + x_{n-1}$

$x_3 =$ f(3) = x1 + x2 = 1 + 1 = 2

$x_4 =$ f(4) = x2 + x3 = 1 + 2 = 3

$x_5 =$ f(5) = x3 + x4 = 2 + 3 = 5

The sequence $\{x_n\} = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, \ldots\ldots\}$:

$\{x_n\} =$ {1, 1, 2, 3, 4, 8, 13, ...} Note: a8 = 21

This sequence $\{x_n\}$ is called the Fibonacci sequence

Note: If asked for a_{10}: We need a9 and a8. For a9 we need a8 and a7...

$\rightarrow$ We need all terms previous to a10

4 a) Advantage of n th term function:

If you want a specific term
substitute in the F(n) formula

Ex If $a_n = F(n) = \frac{1}{(2n-1)}$, n = 1,2,3,…

then $a_{10} =$ 1/(2*10-1) = 1/19

Disadvantage of recursive function:

If you want a specific term you need
to know all terms before it

Ex In Fibonacci sequence if we want a_{100}
we need a99 then a98 etc.

b) The disadvantage of the recursive function disappears if we use a loop structure in a program:

Ex: $x_1 = f(1) = 1$

$\forall n \; n \geq 2, \; x_n = f(n) = \dfrac{2\,(x_{n-1})^3 + P}{3\,(x_{n-1})^2}$ (P is a parameter, any positive integer)

Psuedo code:

```
p := (any positive number)
xn := 1
print xn
for counter = 1 to (number terms – 1)
    xnminus1 := xn
    xn := (function of xnminus1)
    print xn
next counter
```

The sequence produced when P = 8 is: {1, 3.3333, 2.4622, 2.0814, 2.0031, 2.0000,...}

Conjecture: The sequence gets closer and closer to 2 = 8ˆ(1/3) → Sequence approaches Pˆ(1/3)

5. Some sequences can be expressed using both function types:

Ex: Recursive: $x_1 = f(1) = a$

$\forall n \; n \geq 2, \; x_n = f(n) = x_{n-1} \times r$

Sequence listing: {xn} = {a, ar, ar^2, ar^3, ar^4, ...}

n th term formula: Since $x_1 = a$, $x_2 = ar^1$, $x_3 = ar^2$, $x_4 = ar^3$, ...

→ F(n) = ar^(n-1), a=1,2,3,...

This example is the general form of a Geometric sequence

If you add the terms of a Sequence it is called a Series

$$\sum_{n=1}^{N} ar^{n-1} = a + ar + ar^2 + ar^3 + ar^2 + \ldots\ldots + ar^{N-1}$$ = a(r^n-1)/(r-1)

6. Another often used Sequence and corresponding Series

Ex 1: Recursive: $x_1 = f(1) = a$

$\forall n \; n \geq 2, \; x_n = f(n) = x_{n-1} + d$

Sequence listing: $\{xn\} = \{a, a+d, a+2d, a+3d, ...\}$

n th term formula: Since $x_1 = a$, $x_2 = a + d$, $x_3 = a + 2d$, $x_4 = a + 3d$, ...

$\rightarrow x_n = F(n) = a + (n-1)d, n=1,2,3,...$

This example is the general form of the Arithmetic sequence

$$\sum_{n=1}^{N} a+[n-1]d = a + (a+d) + (a+2d) + (a+3d) + ... + (a+(N-1)d)$$ This is the Arithmetic Series

Ex 2: Recursive: $x_1 = f(1) = 5$

$\forall n \; n \geq 2, \; x_n = f(n) = x_{n-1} + 3$

Sequence listing: $\{xn\} = \{5, 5+3, 5+(2)3, 5+(3)3, ...\}$
$= \{5, 8, 11, 14, 17, ...\}$

n th term formula for series: $$\sum_{n=1}^{N} 5+[n-1]\,(3) \quad = \quad 5+3n-3 \quad = \quad 2+3n$$

Ex 2: $\sum_{n=1}^{N} a+[n-1]d$ = a + (a+d) + (a+2d) + (a+3d) + (a+4d) + …. (a+[N-1]d) =

$\rightarrow \sum_{n=1}^{4} 2+3n$ = 5 + 8 + 11 + 14 =

P(n): a + (a+d) + (a+2d) + ……… (a+[n-1]d) = $\frac{n}{2}$ (2a + [n-1]d) $\forall n \in Z+$

Proof: (Mathematical Induction)

Step 1 Prove: P(1)

LHS = a — 1st term of LHS

Subst. n=1

Concl: P(1) = T — LHS = RHS

Step 2 Assume P(**k**): a+(a+d)+(a+2d)+… +(a+[k-1]d) = $\frac{k}{2}$ (2a+[k-1]d), for any one $k \in Z+$

Step 3 From Step 2 assumption Prove P(**k+1**): a+(a+d)+(a+2d)+… +(a+kd) = $\frac{k+1}{2}$ (2a+kd)

Step 4 P(**k**): a+(a+d)+(a+2d)+… +(a+[k-1]d) =$\frac{k}{2}$ (2a+[k-1]d), $k \in Z+$

P(**k+1**): a+(a+d)+(a+2d)+… +(a+[k-1]d)+

$= \frac{1}{2}$ (2ak+k[k-1]d+2a+2kd)

$= \frac{1}{2}$ (2a[k+1]+d[k^2-k+2k])

$= \frac{1}{2}$ (2a[k+1]+d[k^2+k])

$= \frac{1}{2}$ (2a[k+1]+d[(k+1)k])

$= \frac{k+1}{2}$ (2a+kd)

Concl: P(k) $\rightarrow$ P(k+1) QED

7. Sequences expressed as a composition of two recursive functions: g (f (n))

Ex 1: Step 1 First recursive function: $x_1 = f(1) = a$

$\forall n \ n \geq 2$: $x_n = f(n) = x_{n-1} \times \left(-\frac{x_1^2}{(2n-2)(2n-1)}\right)$ [$a \in R$]

$x_1 = a$

x2 = f(2) = -a^3/(2*3)

x3 = f(3) = a^5/(2*3*4*5)

Sequence listing $\{x_n\} = \{ a, -\frac{a^3}{2\times3}, \frac{a^5}{2\times3\times4\times5}, -\frac{a^7}{2\times3\times4\times5\times6\times7}, ... \}$

→ n th term formula for the sequence is F(n) = [(-1)^(n+1)]a^(2n-1)/(2n-1)! and for series is

Step 2 Second recursive function: $s_1 = g(f(1)) = x_1$

$\forall n \ n \geq 2$: $s_n = g(f(n)) = s_{n-1} + x_n$

$s_1 = g(f(1)) = x_1 = a$

$s_2 = g(f(2)) = s_1 + x_2 =$

$s_3 = g(f(3)) = s_2 + x_3 =$

Sequence listing $\{s_n\} = \{ a, (a - \frac{a^3}{2\times3}), (a - \frac{a^3}{2\times3} + \frac{a^5}{2\times3\times4\times5}), (a - \frac{a^3}{2\times3} + \frac{a^5}{2\times3\times4\times5} - \frac{a^7}{2\times3\times4\times5\times6\times7}), \dots \}$

This Sequence: $\{s_n\} = s_1, s_2, s_3, s_4 \dots$ is called the

In the previous series: $a - \frac{a^3}{2\times3} + \frac{a^5}{2\times3\times4\times5} - \frac{a^7}{2\times3\times4\times5\times6\times7} + \ldots\ldots..$

if $a = \pi/6$ we get a sequence $\{s_n\}$ = {0.52359877, 0.49967417, 0.50000021, 0.49999999,………..}
$a = \pi/4$ we get a sequence $\{s_n\}$ = {0.78539816, ……………………...., 0.70710678, ………..}
$a = \pi/3$ we get a sequence $\{s_n\}$ = {1.04719755,…………………………, 0.86602540, ………..}
$a = \pi/2$ we get a sequence $\{s_n\}$ = {1.57079632,…………………………, 0.99999999, ………..}

This series is called a Taylor Series or MacLaurin Series The PSS and hence the Taylor Series converge (gets closer and closer) to the exact value called the limit of the Series.
In this Taylor Series the limit is the Sine value of the a.

Ex 2: Step 1 First recursive function: $x_1 = f(1) = 1$

$\forall n \;\; n \geq 2$: $x_n = f(n) = x_{n-1} \times \left(\frac{a}{(n-1)}\right)$ [a ϵ R]

Sequence listing: {xn} = {1, a/1, a^2/(1*2), a^3/(1*2*3), ...}

Step 2 Second recursive function: $s_1 = g(f(1)) = x_1$

(same as Ex 1) $\forall n \;\; n \geq 2$: $s_n = g(f(n)) = s_{n-1} + x_n$

Sequence listing $\{s_n\}$ = $\{ 1, (1 + \frac{a}{1}), (1 + \frac{a}{1} + \frac{a^2}{1\times2}), (1 + \frac{a}{1} + \frac{a^2}{1\times2} + \frac{a^3}{1\times2\times3}), \ldots\ldots \}$

In this series: $1 + \frac{a}{1} + \frac{a^2}{1\times2} + \frac{a^3}{1\times2\times3} + \ldots\ldots\ldots$

if a = 1 we get the PSS $\{s_n\}$ = {1, 2, 2.5, 2.66666, 2,70833, 2.71666, 2.71805, ………..}

In this Taylor Series the limit is the number e = 2.71828...

Ex 3: Step 1 First recursive function: $x_1 = f(1) = a$

$\forall n \; n \geq 2$: $x_n = f(n) = x_{n-1} \times \left(-\frac{x_1^2}{(2n-1)}\right)$ [a ϵ R, -1≤ a ≤]

Sequence listing {xn} = {a, -a^2/3, a^5/5, -a^7/7, ...

Step 2 Second recursive function: $s_1 = g(f(1)) = x_1$

(same as Ex 1,2,3) $\forall n \; n \geq 2$: $s_n = g(f(n)) = s_{n-1} + x_n$

Sequence listing $\{s_n\} = \{ a, (a - \frac{a^3}{3}), (a - \frac{a^3}{3} + \frac{a^5}{5}), (a - \frac{a^3}{3} + \frac{a^5}{5} - \frac{a^7}{7},), \ldots\ldots \}$

If we multiply the series terms by 4 and use $a = 1$ → {Pn} = 4(1-1/3+1/5-1/7+...

→ $\{p_n\}$ = { 4 , 2.66666, 3.46666, 2.89523, 3.33968, ..., 3.14159, ...

With the multiplication of this Taylor Series by 4, the limit is the number pi = 3.14159...
(the convergence to this limit is slow)

8. Why are the Recursion forms significant for digital computers and calculators ?

- In the Arithmetic unit of a digital computer or a calculator there is a circuit that can add two numbers that are represented with 0's and 1's. It can also form the negative of these bit forms.
- Subtraction is really addition of the negative. Then Algorithms exist that relate Multiplication and Division to the Add / Subtract and manipulation of bits. Hence the adder and the algorithms can evaluate the four operations of +, -, X, /.

  All other calculatoins must be expressed in terms of +, -, X, /. Recursion accomplication form.

- In the previous examples we have seen one Recursive form to express each of $\sqrt[3]{n}$, Sin x, e^x, π using only +, - , × , /. There is usually more than one Recursive form for a particular function.

  One searches for the one that is most efficient (converges faster)

4. An application of Modular arithmetic to generate pseudo random integers

Example 4 Use of the Mod function to generate Pseudo Random Integers
There are many uses for random numbers in Computer Science. To name one, they are the basis of many computer games. The only True random numbers are produced by rolling dice, picking numbers out of a hat etc. A computer can produce numbers that appear random since the pattern of their appearance is not recognizable and can be changed a large number of ways. These are called Pseudo random numbers.

Previously we stated: $\forall(a,d)\ \frac{a}{d} = q + \text{rem } r$, $a,d,q,r \in \mathbf{Z}$, $d \neq 0$. Most computer languages have functions to calculate q (div in some languages) and remainder r (mod in some languages). In this example we see how pseudo random numbers can be generated using these ideas.

We will use a recursive function that contains the mod function:

$x_1 = s$, $\forall n\ n \geq 1\ x_{n+1} = f(x_n, n)$ where $f(x_n, n) = (ax_n + c) \bmod m$

Integers s,a,c,m are called parameters (they remain constant). If we choose s = 3, a = 7, c = 4, m = 9

$x_1 = 3$

$x_2 = (7x_1 + 4) \bmod 9 = (7\times 3 + 4) \bmod 9 = 7$

$x_3 = (7x_2 + 4) \bmod 9 = (7\times 7 + 4) \bmod 9 = 8$

$x_4 = (7x_3 + 4) \bmod 9 = (7\times 8 + 4) \bmod 9 = 6$

$x_5 = (7x_4 + 4) \bmod 9 = (7\times 6 + 4) \bmod 9 = 1$

$x_6 = (7x_5 + 4) \bmod 9 = (7\times 1 + 4) \bmod 9 = 2$

x7 = 0
x8 = 4
x9 = 5. The sequence now repeats: xn = (7X5 + 4) mod 9 = 3

The previous ordered sequence of pseudo random Integers can generate Reals between 0 and 1

Form a function: $g(f(x)) = (0.1) \times \lfloor 10 \times \frac{f(x)}{m} \rfloor$ (uses the floor function)

$x1 = 0.1 \times \lfloor 10 \times \frac{3}{m} \rfloor = 0.3$

…….

We get the ordered sequence of pseudo random Reals: { 0.3, 0.7, 0.8, 0.6, 0.1, 0.2, 0, 0.4, 0.5,….}

<u>Note</u> - In practice the parameters can be changed. Often c = 0 is used with $m = 2^{31} - 1$

This gives 2^31-2 random numbers before a repeat sequence occurs.

5. Coding a message, called Cryptography, with the mod function

Example 5 First we need to examine how to solve an equation containing the mod function:
Solve the equation $7p + 3 \equiv 19 \mod 3$ where $p \in \mathbf{Z}$

Step 1	$3 \mid (7p + 3 - 19)$	by definition mod function
Step 2	$\exists a,\ (7p+3) - 19 = 3a,\ a, p \in Z$	by definition of divides
	$\rightarrow \exists a,\ p = (3a+19)/7,\ a, p \in Z$	Algebra

Step 3 Try a = 1 and solve for p: then $p = (3X1+19)/7 \rightarrow p \notin Z$

What is the smallest positive value of a that gives $p \in \mathbf{Z}$

$a=4$, then $p = (3X4+19)/7=4 \rightarrow p \in Z$

Check: If we substitute p = 4 into the original equation $7p + 3 \equiv 19 \mod 3$,

Is the equation satisfied: Does $3 \mid (7\times4 + 3) - 19 \rightarrow$ does $3 \mid 12$? $\rightarrow$ Yes

List the sequence of positive values of p that satisfy the equation:

4, 7, 10, ...

Example 6 Coding (encrypting) a message using the mod function

Message encryption:

Step 1 Function f: assign alphabet letters to an integer p: f(A)=0, f(B)=1, f(C)=2, f(D)=3…..f(Z)=25

Step 2 Function g: $g(p) = ap + c \mod 26$, $p \in \mathbf{Z}$ (a, c are parameters)

Step 3 Use f inverse to produce the coded message in alphabetic characters, transmit message

Message decipher:

Step 4 The receiver uses f to produce the integers for the code

Step 5 The receiver uses g inverse on the code to produce the original integers

Step 6 The receiver uses f inverse on the original integers to produce the original message

Suppose the message was: PARK and we use a = 7, c =3

? Message encryption:

Step 1 15 0 17 10	f (P) =15, f (A) = 0, ……..
Step 2 4 3 18 11	$g(p) = 7p+3 \mod 26 \rightarrow g(15) = 7X15 + 3 \mod 26 = 4$, $g(0) = 3$, ……..
Step 3 E D S M	f inverse (4) = E, f inverse (3) = D, ……… (this is the transmitted code)
Message Received:	
Step 4 4 3 18 11	f(E) = 4, ……………
Step 5 15 0 17 10	
Step 6 P A R K	f inverse(15) = P, ………..

Note: - Mod 26 is used here since we want all numeric codes < 26

- Various encrypting schemes can be produced by varying the parameters a, c.
- We have considered one of many encrypting schemes