11.5 Alternating Series

July 25, 2016 11:48

Definitions & Theorems:

1. Definition: Alternating series

   An alternating series is a series whose terms are alternately positive and negative. The nth term of an alternating series is of the form

   $$a_n = (-1)^{n-1} b_n \quad or \quad a_n = (-1)^n b_n$$

   where b_n is a positive number. (In fact, $b_n = |a_n|$)

2. The Alternating Series Test

   If the alternating series

   $$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots b_n > 0$$

   satisfies

   (i) $\{b_n\}$ is eventually decreasing

   (ii) $\lim_{n\to\infty} b_n = 0$

   then the series is convergent.

3. Theorem: Alternating series estimation theorem

   If $s = \sum (-1)^{n-1} b_n$ is the sum of alternating series that satisifies

   (i) $0 \le b_{n+1} \le b_n$

   (ii) $\lim_{n\to\infty} b_n = 0$

   then $|R_n| = |s - s_n| \le b_{n+1}$

Examples:

1. $$\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$$

   Let $b_n = \frac{n}{n^2+1}$, $\lim_{n\to\infty} b_n = 0$

   Let $f(x) = \frac{x}{x^2+1} \Rightarrow f'(x) = \frac{1-x^2}{(x^2+1)^2} \Rightarrow f(x)$ is decreasing for $x > 1 \Rightarrow \{b_n\}$ is decreasing.

   $\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$ converges by Alternating Series Test.

2. $$\sum_{n=0}^{\infty} \frac{(-1)^{n-3}\sqrt{n}}{n+4}$$

   Let $b_n = \frac{\sqrt{n}}{n+4}$, $\lim_{n\to\infty} b_n = 0$

   Let $f(x) = \frac{\sqrt{x}}{x+4} \Rightarrow f'(x) = \frac{4-x}{2\sqrt{x}(x+4)^2} \Rightarrow f(x)$ is decreasing for $x > 4 \Rightarrow \{b_n\}$ is decreasing for $n \ge 4$.

   $\Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^{n-3}\sqrt{n}}{n+4}$ converges by Alternating Series Test.

3. $$\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}}$$

   $$\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

   Let $b_n = \frac{1}{\sqrt{n}}$, $\lim_{n\to\infty} b_n = 0$

   $n+1 > n \Rightarrow \sqrt{n+1} > \sqrt{n} \Rightarrow \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} \Rightarrow b_{n+1} < b_n \Rightarrow \{b_n\}$ is decreasing.

   $\Rightarrow \sum_{n=2}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}}$ converges by Alternating Series Test.

4. For what value(s) of p does $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converge?

(i) $p = 0 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^p} = \sum_{n=1}^{\infty} (-1)^n$

$\lim_{n\to\infty} (-1)^n$ does not exist $\Rightarrow \sum_{n=0}^{\infty}(-1)^n$ diverges by Test for Divergence.

(ii) $p < 0 \Rightarrow \lim_{n\to\infty} \frac{(-1)^n}{n^p}$ does not exist $\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ diverges by Test for Divergence.

(iii) $p > 0$

Let $b_n = \frac{1}{n^p}, \lim_{n\to\infty} b_n = 0$

Let $f(x) = \frac{1}{x^p} \Rightarrow f'(x) = -px^{-p-1}$

for $x \in [1, \infty) \Rightarrow x^{-p-1} > 0 \Rightarrow f'(x) < 0 \Rightarrow f(x)$ is decreasing for $x \geq 1 \Rightarrow \{b_n\}$ is decreasing.

$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges by Alternating Series Test.

(i)(ii)(iii)$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges for $p > 0$, and diverges otherwise.

5. $$\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$$

Compare with $\sum_{n=1}^{\infty} \frac{1}{n}$

$$\lim_{n\to\infty} \left(\frac{\frac{1}{n}}{\frac{1}{n^{1+\frac{1}{n}}}} \right) = \lim_{n\to\infty} \left(n^{\frac{1}{n}}\right) = e^{\ln \lim_{n\to\infty} \left(n^{\frac{1}{n}}\right)} = e^{\lim_{n\to\infty} \ln\left(n^{\frac{1}{n}}\right)} = e^{\lim_{n\to\infty} \frac{\ln n}{n}}$$

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1}{x} = 0 \Rightarrow e^{\lim_{n\to\infty} \frac{\ln n}{n}} = e^0 = 1$$

by Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$ has same behaviour as $\sum_{n=1}^{\infty} \frac{1}{n} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$ diverges.