6.2 Volumes (*disk / washer method)*

July 13, 2016 14:25

Definitions & Theorems:

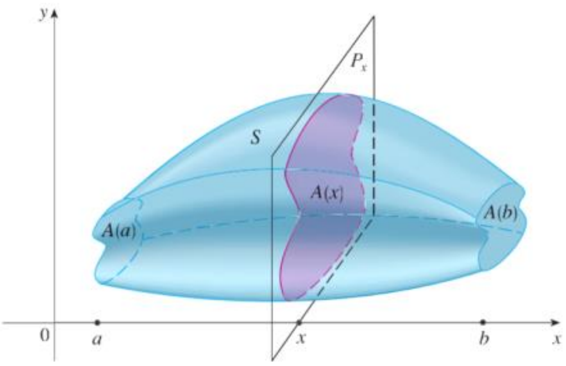
1. Definition: Volume

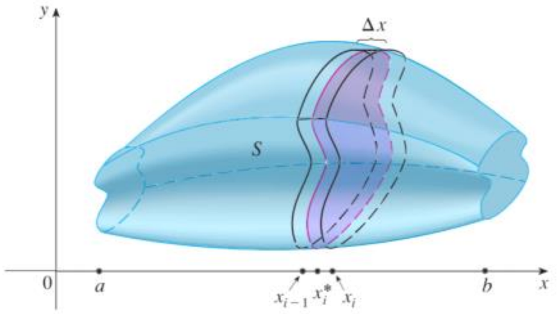
   Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S is the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the volume of S is

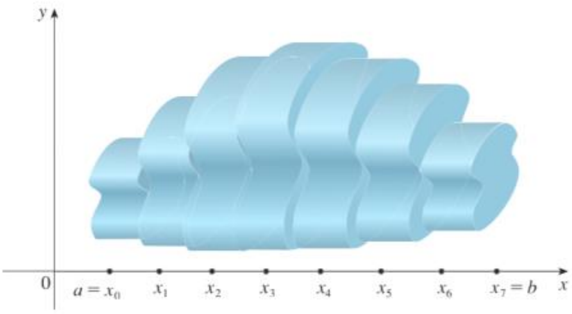
$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\,\mathrm{d}x$$

Proofs or Explanations:

1. Definition1







Examples:

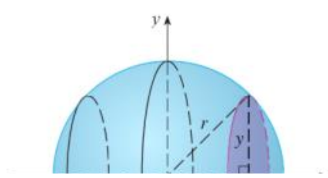
1. Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$

   The plane $P(x)$ intersects the sphere in a circle whose radius is

   $y = \sqrt{r^2 - x^2}$

   So the cross-sectional area is

   $A(x) = \pi y^2 = \pi(r^2 - x^2)$

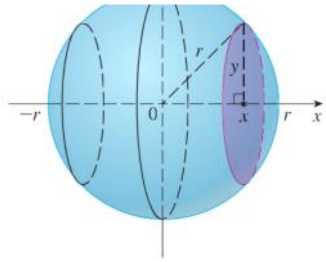


$$y=\sqrt{r^2-x^2}$$

So the cross-sectional area is

$$A(x)=\pi y^2=\pi\left(r^2-x^2\right)$$

Using the definition of volume with $a=-r$ and $b=r$, we have

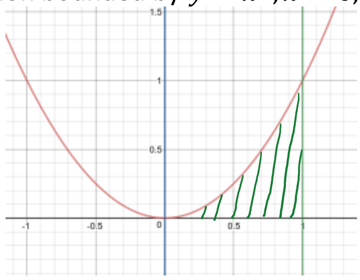


$$V=\int_{-r}^{r}A(x)\,\mathrm{d}x=\int_{-r}^{r}\pi\left(r^2-x^2\right)\mathrm{d}x$$

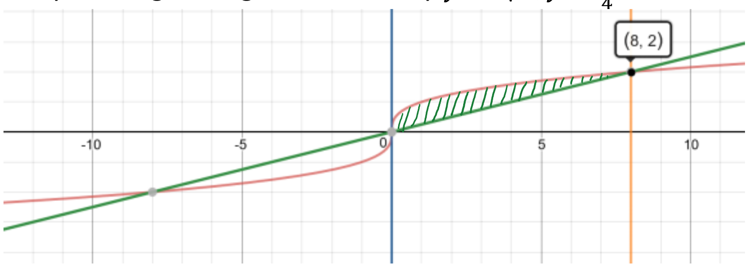
$$=2\pi\int_{0}^{r}\left(r^2-x^2\right)\mathrm{d}x=2\pi\left[r^2x-\frac{x^3}{3}\right]_0^r=\frac{4}{3}\pi r^3$$

2. Calculate the volume of the solid revolution generated by rotating the region bounded by $y=x^2, x=0, x=1$ about x-axis.

$$V=\int_0^1\pi\left[x^2\right]^2\mathrm{d}x$$



3. Calculate the volume of the solid revolution generated by rotating the region bounded by $y=\sqrt[3]{x}, y=\frac{x}{4}, x=0, x=8$ about x-axis.

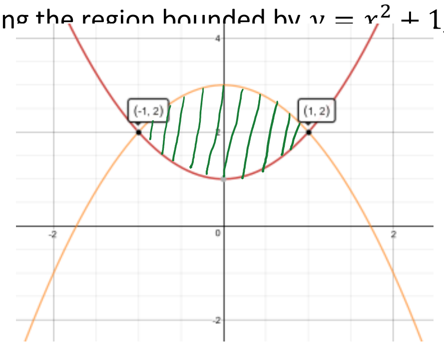


$$\sqrt[3]{x}=\frac{x}{4}\Rightarrow x=0,\pm 8$$

$$V=\int_0^8\pi\left[\sqrt[3]{\pi}\right]^2\mathrm{d}x-\int_0^8\pi\left[\frac{x}{4}\right]^2\mathrm{d}x$$

$$=\int_0^8\pi\left(\left[\sqrt[3]{\pi}\right]^2-\left[\frac{x}{4}\right]^2\right)\mathrm{d}x$$

4. Calculate the volume of the solid revolution generated by rotating the region bounded by $y=x^2+1, y=3-x^2$ about x-axis.

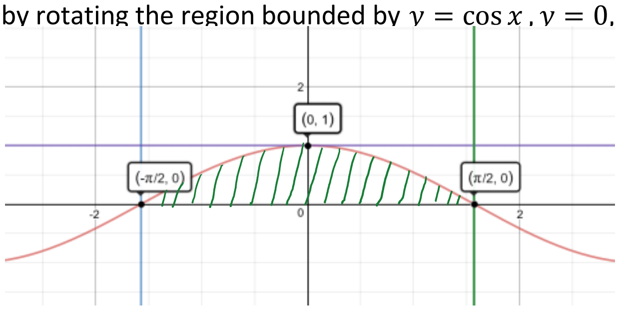


$$x^2+1=3-x^2\Rightarrow x=\pm 1$$

$$v=\int_{-1}^{1}\pi\left[3-x^2\right]^2\mathrm{d}x-\int_{-1}^{1}\pi\left[x^2+1\right]^2\mathrm{d}x$$

$$=\int_{-1}^{1}\pi\left(\left[3-x^2\right]^2-\left[x^2+1\right]^2\right)\mathrm{d}x$$

5. Calculate the volume of the solid revolution generated by rotating the region bounded by $y=\cos x, y=0, x=\frac{-\pi}{2}, x=\frac{\pi}{2}$ about $y=1$.



$$v=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\pi[1]^2\,\mathrm{d}x-\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\pi[1-\cos x]^2\,\mathrm{d}x$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\pi\left([1]^2-[1-\cos x]^2\right)\mathrm{d}x$$