Section 9.4 Closures of a Relation

Comp 232
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1. Closure of a Relation:

a) If a Relation R does not have one of the properties: (Reflexive, Symmetric, Transitive) and you add ordered pairs so R has the desired property then the new relation is Closed with respect to that property.

b) There are three Closures we will consider: Reflexive Closure, Symmetric Closure and Transitive Closure.

2. Reflexive Closure. Recall: R is Reflexive iff $\forall x\ (x,x) \in R$

Ordered Pairs Example	If R={(1,2), (2,3), (3,3)} on A×A, where A = {1,2,3}, R is not Reflexive If we add pairs (1,1), (2,2): R_R={(1,1), (1,2), (2,2), (2,3), (3,3)} is called the. Reflexive Closure of R Note: $R_R = \{R\} \cup \{ (1,1), (2,2), (3,3) \}$
Matrix Example	Recall: Identity Matrix (I) has 1 on main diagonal, 0 everywhere else: Consider: $M_R \vee I$ $= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \vee \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} = \begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{matrix}$ Now $M_R \vee I$ represents {(1,1), (1,2), (2,2), (2,3), (3,3)} = R_R Notation: $M_{R_R} = M_R \vee I$ represents the Reflexive Closure Rr of R
Set Builder Example	If $R = \{(x,y) \in Z \times Z,\ xRy \rightarrow x < y\}$ then $\forall x\ xRx \notin R$. R is not Reflexive Now change the relation from < to ⩽ . We get the Reflexive closure of R: $R_R = \{ (x,y) \in Z \times Z \mid xRy \rightarrow x \leq y \}$, now $\forall x\ xRx \in R_R$ hence Rr is Reflexive

3. Symmetric Closure   Recall: R is Symmetric iff $\forall (x, y)\ (x, y) \in R \rightarrow (y, x) \in R$

Ordered Pairs Example	If $R = \{(1,2), (2,3), (3,3)\}$ on $A \times A$, where $A=\{1,2,3\}$, R is not Symmetric If we add pairs (2,1), (3,2): $R_S = \{(1,2), (2,1), (2,3), (3,2), (3,3)\}$ is called the Symmetric Closure of R. Note: $R_S = \{R\} \cup \{(2,1), (3,2)\}$
Matrix Example	Recall: to get M_R transpose: Rows of M_R become the cols of M_R transpose Consider: $M_R \vee M_R^T$ Mrt is notation for transpose $= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \vee \begin{matrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{matrix} = \begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{matrix}$ Now $M_R \vee M_R^T$ represents $\{(1,2), (2,1), (2,3), (3,2), (3,3)\} = R_S$ Notation: $M_{R_S} = M_R \vee M_R^T$ represents the Symmetric Closure Rs of R
Set Builder Example	$R = \{(x,y) \in Z \times Z \mid xRy \rightarrow x > y\}$ then $\forall (x,y)\ (x,y) \in R$ implies $(y,x) \in R$ is False → R is not Symmetric Now change the relation from > to ≠ we get the Symmetric closure of R: $R_S = \{(x,y) \in Z \times Z \mid xRy \rightarrow x \neq y\}$. Now $[(xRy \in R_S) \rightarrow (yRx \in R_S)]$ hence Rs is Symmetric.

4. Transitive Closure Recall: R is Transitive iff $\forall (x, y, z)\ [(x, y) \in R \wedge (y, z) \in R]$ implies $(x, z) \in R$

a) The forming of the Transitive closure presents a problem:

Ex: If $R=\{(1,3), (1,4), (2,1), (3,2)\}$ on $A \times A$, where $A=\{1,2,3,4\}$, R is not Transitive

Recall definition of transitive:

$[(1,3) \wedge (3,2)]$ in R $\rightarrow$ we need (1,2) in Transitive Closure of R

$(2,1) \wedge (1,3)$ $\rightarrow$ we need (2,3)

$(2,1) \wedge (1,4)$ $\rightarrow$ we need (2,4)

$(3,2) \wedge (2,1)$ $\rightarrow$ we need (3,1)

If we do a union of sets to get missing ordered pairs: $R \cup \{(1,2), (2,3), (2,4), (3,1)\}$

We end up having: $(3,1) \wedge (1,3) \rightarrow$ now we need (3,3)

$(3,1) \wedge (1,4) \rightarrow$ now we need (3,4)

So we do not have Transitive closure. We have created some new required ordered pairs.

b) Vocabulary for Directed Graph: Consider A = { a, b }

(i) Elements of A are called the vertices

(ii) The joining of 2 different vertices is called an edge

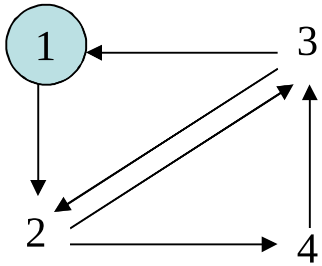
(iii) The number of edges to get from vertex **a** to vertex **b** is called the

length of the path from vertex a to vertex b (There can be more than one path from **a** to **b**)

(iv) If aRa, → the smallest path from a to a has length = 0

There may be other paths to get from **a** back to **a** . The length of these other paths > 0

Ex: Consider vertices 1,2,3,4 :



There ∃ a path from:

1 to 1: it has no edges, length of this path n=0.

1 to 1: (1 → 2 → 3 → 1) → 3 edges → length of path n=3

1 to 1: (1 → 2 → 4 → 3 → 1) → 4 edges → length of path n=

2 to 2: (2 → 3 → 2) → 2 edges → length of path n=2

2 to 2: (2 → 4 → 3 → 2) → 3 edges → length of path

2 to 3: (2 → 3) → 1 edge → length of path n=1

2 to 3: (2 → 4 → 3) → 2 edges → length of pat

c) The Directed Graph method to get Transitive Closure for R (also called the Connectivity Method)

Definition: Relation R* = {(a,b)∈ R | ∃ a path, length n ≥ 1 from a to b}

R* is called the Connectivity Relation of R

Ex: Return to our original problem to find the Transitive Closure of

R={(1,3), (1,4), (2,1), (3,2)} on A×A, where A={1,2,3,4}.

Step 1 Draw the Di-graph of R (Directed Graph)

1 4

2 3

Step 2 Form R*: Set of all pairs that have at least one path between them where length n≥1

R* = {(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3)}

Step 3 Note: R* produces the Transitive Closure of R = Rt

Notation:

d) Matrix method to get the Transitive Closure for R

If R is a Relation on A×A and A has n elements then $R_T = M_R \vee M_R^{\,2} \vee M_R^{\,3} \ldots \vee M_R^{\,n}$

Ex 1: Consider R = {(1,1),(1,3), (2,2), (3,1), (3,2) }, from A×A, A = {1,2,3}

Step 1 $M_R = \begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{matrix}$

Form matrix that represents R

Step 2 $R_T = R^* = M_R \vee M_R^{\,2} \vee M_R^{\,3}$

$$R_T = R^* = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}^2 \vee \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}^3$$

$$M_R^{\,2} = M_R \odot M_R$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{matrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{matrix} \vee \begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{matrix}$$

$$= \begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{matrix}$$

Bit-wise Or

$R_T = R^* =$ {(1,1), (1,2), (1,3), (2,2), (3,1), (3,2), (3,3)}

Form R from its matrix representation

Ex 2: There are 4 cities a,b,c,d. R = {(a,b), (a,c), (b,d), (c.a), (d,a)}
Where xRy means x "has a direct flight to" y
What flights need to be added if all cities presently connected by a flight or flights end up with a direct flight between them ?
Hint: Which Relation do we need ?

We need Rt = R*

Use Di-graph or Matrix method
(if you use a matrix multiplier each time a sum>1 appears it is replaced by 1, why?

Ex 3: There are five cities a,b,c,d,e. R = { (x,y) xRy where R means "has a direct flight to"}
The following direct flights exist. R = {(a,e), (b,c), (b,e), (c,a), (c,e), (d,a), (e,b), (e,c), (c,d)}

(i) Draw D-graph. (Directed graph)
(ii) Is it possible to get to all cities with one or more flights ?
(iii) What Relation do we need so all cities have a direct flight ?
(iv) Which cities have the greatest number of connecting flights ?
(v) What one flight needs to be added so only one connection has the maximum number of flights ?