Section 5.1- 5.2 Mathematical Induction

Comp 232
Instructor: Robert Mearns

1. Mathematical Induction is a method of proof:

a) Proof methods:

Direct method	Contraposition
Contradiction	Cases (Exhaustion)
Existence	Uniqueness
Mathematical Induction	

b) Mathematical Induction works on Domain $\boldsymbol{Z}^{+}$ because it is Well Ordered.

Well Ordered Axiom states: Every non empty subset has a smallest member.

Ex: {1, 1+1=2} has a smallest member call it 1; {2, 2+1=3} has a smallest member call it 2; ...

Using the Well Ordered property we can list $\boldsymbol{Z}^{+}$ in order(smallest to greatest): $\forall k, k \in Z^{+}, k < k+1$

c) Two types of reasoning:

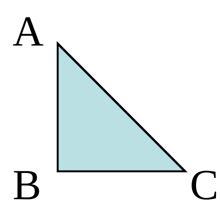
Deductive

1. State a general proposition
2. Deduce a specific case

Ex:

(i) In a right angle triangle ABC
$AC^2 = AB^2 + BC^2$

(ii) If AB = 3, BC = 4
then $AC^2 = 3^2+4^2$



Inductive

1. Start with specific cases
2. Infer (not prove) a general proposition

Ex:

(i) since $1+3+5 = 9 = 3^2$
$1+3+5+7 = 16 = 4^2$
$1+3+5+7+9 = 25 = 5^2$

(ii) Conjecture: Sum of n odd integers = n^2
But we have not proved this proposition

e) Mathematical Induction is a proof method so it is a Deductive process. Its name is miss leading

2. a) A new rule of inference to be used: $P(1) \wedge [\ P(k) \rightarrow P(k+1)\ \forall k \in \boldsymbol{Z}^+\ ] \rightarrow P(n)\ \forall n \in \boldsymbol{Z}^+$

Result of the inference: $P(1) = T$, $P(1) \rightarrow P(2)$ gives $P(2) = T$, $P(2) \rightarrow P(3)$ gives $P(3) = T$, ……

2nd part of hypothesis does <u>not</u> state $\forall k \in \boldsymbol{Z}^+\ P(k)$, it states implication: $P(k) \rightarrow P(k+1)$ must be T

b) We will break up the Mathematical Induction proofs into 4 steps as we use the above inference:

Step 1 Prove $P(1)$ — Basic step

Step 2 Assume $P(k)$ for any one $k \in Z$ — Inductive hypothesis

Step 3 Identify what has to be proved: $P(k) \rightarrow P(k+1)$ for all $k \in Z$

Step 4 Prove $P(k) \rightarrow P(k+1)$ for all $k \in Z$ — Inductive step

By Math Induction: $P(1) \wedge [\ P(k) \rightarrow P(k+1)\ \forall k \in \boldsymbol{Z}^+\ ] \rightarrow P(n)\ \forall n \in \boldsymbol{Z}^+$

3. A non mathematical illustration of what is happening in the Mathematical Induction inference;
Consider dominos standing on edge on a table. We look at three cases:

Case 1 | | | | | | | …. |

The first domino is glued to table. When we try to push first one down to right we cannot:
Equivalent to:

$\neg P(1)$

[Step 1 is False]

Case 2 | | | | | | | ……

The first domino is not glued to table but a gap exists between two of them. We push number 1 down, then 2 falls, then 3 falls but 4 does not fall:

__| | | | | | …….

Equivalent to:
$\neg[P(k) \rightarrow P(k+1)]$ for all $k \in Z$
[Step 4 is False]

Case 3 | | | | | | | | ………….

The first domino is not glued to table and no gap exists between two of them. We push number 1 down, then 2 falls, then 3 falls, then 4 falls 5 falls…………..

Equivalent to:
$P(k) \rightarrow P(k+1)$ for all $k \in Z$
[Step 1 and Step 4 are True]
All dominos are knocked down

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Case 2 | | | | | | | ……

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__ __ | | | | | …….

Equivalent to:
$\neg[P(k) \rightarrow P(k+1)]$ for all $k \in Z$
[Step 4 is False]

Case 3 | | | | | | | | ………….

The first domino is not glued to table and no gap exists between two of them. We push number 1 down, then 2 falls, then 3 falls, then 4 falls 5 falls…………..

Equivalent to:
$P(k) \rightarrow P(k+1)$ for all $k \in Z$
[Step 1 and Step 4 are True]
All dominos are knocked down

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The first domino is not glued to table but a gap exists between two of them. We push number 1 down, then 2 falls, then 3 falls but 4 does not fall:

— — — | | | | …….

Equivalent to:
$\neg [P(k) \rightarrow P(k+1)]$ for all $k \in \mathbf{Z}$
[**Step 4 is False**]

Case 3 | | | | | | | | ………….

The first domino is not glued to table and no gap exists between two of them. We push number 1 down, then 2 falls, then 3 falls, then 4 falls 5 falls…………..

— | | | | | | | | ……………..

Equivalent to:
$P(k) \rightarrow P(k+1)$ for all $k \in Z$
[Step 1 and Step 4 are True]
All dominos are knocked down

4. Mathematical induction:
   a) Does not find new propositions. It can only prove previous conjectures
   b) If Step 1 or 4 fails then Mathematical Induction proof fails. It does not say the proposition is False.

Ex 1: Recall conjecture: The sum of the first n consecutive integers = n(n+1)/2

This Conjecture came from looking at different values of n and detecting a pattern.

Proof: (by Mathematical Induction) Let P(n) represent: $1+2+3+...+n = n(n+1)/2, \ \forall n \in Z+$

1. Step 1	Prove: P(1)	
2.	LHS = 1	1st term of LHS
3.	RHS = $\frac{1(1+1)}{2} = 1$	Substitute n=1 in RHS
4.	Concl: LHS = RHS → P(1) = T	

5. Step 2 Assume P(k): $1+2+3+...+k = k(k+1)/2$, for any one $k \in Z+$

6. Step 3 Assuming Step 2 prove P(k+1): 1+2+3+...+k+(k+1)=(k+1)(k+1+1)/2

7. Step 4 Prove: P(k)→p(k+1) for any one $k \in Z+$

8.	P(k): 1+2+3+...+k = k(k+1)/2	line 5, inductive hypothesis
9.	P(k+1): 1+2+3+...+k+(k+1)=k(k+1)/2+(k+1)	add (k+1) to both sides of P(k)
10.	P(k+1): 1+2+3+...+k+(k+1)=[k(k+1)+2(k+1)]/2	Alg.(lcm) = (low. com. den.)
11.	P(k+1): 1+2+3+...+k+(k+1)=(k+1)(k+2)/2	Factor out (k+1)
12.	→P(k+1): 1+2+3+...+k+(k+1)=(k+1)(k+1+1)/2	Algebra. This is P(k+1)
13.	Conclusion: P(k)→P(k+1)	Transitive line 8 to line 12
	QED: P(n): 1+2+3+...+n=n(n+1)/2, $\forall n \in Z+$	by Mathematical Induction

Ex 2: Form a conjecture concerning the following: $1+3+5 = 9 = 3^2$

$1+3+5+7 = 16 = 4^2$

$1+3+5+7+9 = 5^2$

$1+3+5+7+9+11 =$

Conjecture: The sum of the first n odd integers = n^2

Now prove the conjecture:

Proof (by Mathematical Induction) ∀n∈Z+ P(n): 1+3+5+...+(2n−1)=n^2

1. Step 1 Prove: P(1)
2. LHS = 1 | Take 1^{st} term of LHS
3. RHS = 1^2 = 1 | Substitute n=1 in RHS
4. Concl: P(1) = T | LHS = RHS
5. Step 2 Assume P(**k**): $1+3+5\ldots\ldots+(2k-1) = \mathbf{k^2}$ for a k∈ Z+
6. Step 3 We must prove P(**k+1**): $1+3+5\ldots.+(2k-1)+(2k+1) = \mathbf{(k+1)^2}$
7. Stpe 4 Prove: P(k) → P(k+1) for a k∈Z+
8. P(k): 1+3+5+...+(2k-1)=k^2 | Line 5 assumption
9. →P(k): 1+3+5+...+(2k−1)+(2k+1)=k^2+(2k+1) | Add (2k+1) both sides
10. →P(k): 1+3+5+...+(2k−1)+(2k+1)=(k+1)^2 | Factor RHS

Conclusion: P(k) → P(k+1)

QED: ∀n∈Z+ P(n): 1+3+5+...+(2n−1)=n^2

Ex 3: Geometric Series. $\forall n \in Z+$, $r \neq 1$ P(n): $a + ar^1 + ar^2 + \ldots\ldots + ar^{n-1} = \dfrac{a\,r^n - a}{r-1}$

This is a conjecture that has many uses. It is a conjecture for a special sum. Prove this conjecture:

Proof: (Mathematical Induction)

1. Step 1 Prove: P(1)
2. LHS = a — 1st term of LHS
3. RHS = $\dfrac{a\,r^1 - a}{r-1} = \dfrac{a\,(r-1)}{r-1} = a$ — Subst. n=1 in RHS
4. Concl: P(1) = T — LHS = RHS
5. Step 2 Assume P(**k**): $a + ar^1 + a\,r^2 + \ldots\, ar^{k-1} = \dfrac{a\,r^k - a}{r-1}$, $r \neq 1$, any one $k \in Z+$
6. Step 3 Assuming Step 2 prove P(**k+1**): $a + ar^1 + a\,r^2 + \ldots\, ar^k = \dfrac{a\,r^{k+1} - a}{r-1}$, $r \neq 1$, any one $k \in Z+$
7. Step 4 Prove: $P(k) \rightarrow P(k+1)$ for any one $k \in Z+$
8. P(**k**): $a + ar^1 + a\,r^2 + \ldots\, ar^{k-1} = \dfrac{a\,r^k - a}{r-1}$, $r \neq 1$, any one $k \in Z+$ — Line 5 assumption

Add ark both sides
Common Den.
Multiply
Cancel.

Ex 4: Where could we use the previous P(n) for the Geometric Series. ?

Suppose you buy an annuity $\rightarrow$ You agree to deposit 200$ each month and receive interest compounded (get interest on previous interest earned) at the rate of 6 % per year.

How much is in your fund after deposit six ?

Step 1

After 1 month: original 200$ has grown to: 200+200X(.06/12) = 200X1.005

$\rightarrow$ after every month each previous amount

Step 2

deposit number:	#6	#5	#4	#3	#2	#1
Total =	200 +	200(1.005) +	$200(1.005)^2$ +	$200(1.005)^3$ +	$200(1.005)^4$ +	$200(1.005)^5$

(After dep 6) This is the Geometric Series with a=200, r=1.005, n=6

Step3 Using the result previously proved:

Total in your fund after deposit 6 =

This is how the value of an annuity with compound interest is calculated.

Note we could factor out a on the RHS in the Geometric series P(n) then it could be written:

$\forall n \in Z+, r \neq 1,$ P(n): $a + ar^1 + ar^2 + \ldots\ldots + ar^{n-1} = \dfrac{a\,r^n - a}{r-1} =$

Note: (i) After 20 years (240 deposits): fund = 200(1.005^240-1)/(1.005-1)=$92408.185

You deposited 48 000$, the remaining was interest: $92 408.18 – $48 000 = $**44 408.18$**

(ii) After 30 years (360 deposits): fund = $\dfrac{200(1.005^{360}-1)}{1.005-1}$ = $**200 903.01**

You deposited 72 000$: $**200 903.01** – $72 000 = $**128 903.01 (interest earned)**

Ex 5: An example with inequality:

Form a conjecture concerning the comparison of the following pairs: $1, 2^1$
$2, 2^2$
$3, 2^3$

Conjecture: ∀n∈Z+ n<2^n

Now prove the conjecture:

Proof (Mathematical Induction) ∀n∈Z+ P(n): n<2^n

1.	Step 1 Prove P(1)	
2.	LHS = 1	Substitute n = 1 in LHS
3.	RHS = 2^1 = 2	Substitute n = 1 in RHS
4.	Concl: LHS < RHS → 1 < 2^1 → P(1) = T	LHS < RHS

5. Step 2 Assume P(**<u>k</u>**): **<u>k</u>** < 2^k for any one k∈ Z+

6. Step 3 From Step 2 assumption we must prove P(**<u>k+1</u>**): **<u>k+1</u>** < 2^{k+1}

7. Step 4 Prove: P(k) → P(k+1) for any one k∈Z+ Line 5

8. P(k): k<2^k

9. →P(k+1): (2)k<(2)2^k
k+k<2^(k+1)
k+1<2^(k+1) 1≤k

Conclusion: P(k): k<2^k for any one k ∈Z+ Line 9 to 11

QED:∀n∈Z+ P(n): n<2^n

Ex 6: An example with inequality:

Form a conjecture concerning the comparison of the following pairs: $2+1, 2^2$
$3+1, 3^2$
$4+1, 4^2$

Conjecture:

Now prove the conjecture:

Proof (Mathematical Induction)

1. Step 1 Prove P(2)
2. LHS = 2+1 = 3 — Substitute n = 2 in LHS
3. RHS = 2^1 = 4 — Substitute n = 2 in RHS
4. Concl: LHS < RHS → P(1) = T

5. Step 2 Assume P(**k**): **k**+1< k^2 for any one k∈ Z+
6. Step 3 From Step 2 assumption we must prove P(**k+1**): **k+2** < $(\mathbf{k+1})^2$

Ex 7: An example with a factorial and initial n does not equal 1

Recall the Definition of the Factorial function: f(n) = n! = (1)(2)(3)...(n)
Hence: 4! = (1)(2)(3)(4) = 24
1! = 1 and 0! = 1 also by defintion

Form a conjecture concerning the comparison of the following pairs:

1^2, 1 2^2, 2 3^2, 6 4^2, 24 5^2,

Conjecture: ∀n∈Z, n>3: P(n) n^2 < n!

Proof (Mathematical Induction) ∀n∈Z, n>3, P(n):n^2<n! [Note: P(n) = F if 1≤n≤3]

1.	Step 1 Prove P(4) = T	
2.	LHS = 4^2 = 16	Substitute n= 4 in LHS
3.	RHS = 4 ! = 4×3×2×1 = 24	Substitute n= 4 in RHS
4.	16 < 24 → P(4) = T	LHS < RHS

5. Step 2 Assume P(**k**): $\mathbf{k}^2 < \mathbf{k}\,!$ for any one k ∈ Z, k > 3

6. Step 3 From Step 2 assumption prove P(**k+1**): $(\mathbf{k+1})^2 < (\mathbf{k+1})\,!$

7. Step 4 Prove: P(k) → P(k+1) for any one k∈Z, k>3

8.	P(k): k^2 < k!	Line 5 asumption
9.	→P(k+1): (k+1)k^2 < (k+1)k!	Multiply by (k+1)
10.	(k+1)k^2 < (k+1)!	Def of !
11.	(k+1)(k+1) < (k+1)k^2 < (k+1)!	(k+1) < k^2 (see Ex 6)
12.	(k+1)^2 < (k+1)!	Transitive line 10-12

Conclusion: P(k) → P(k+1) for any one k∈Z, k>3

QED: ∀n∈Z, n>3, P(n): n^2<n! By Math Induction

Ex 8: An example with a Divisibility.

Proof (Mathematical Induction) ∀n∈ Z+, P(n): $2 \mid (n^2 + n)$

1. Step 1	Prove P(1):	
2.	$2 \mid 2$	
3.	$2 \mid (1^2 + 1)$	$2 = 1^2 + 1$
4.	Concl: P(1) = T	

5. Step 2 Assume P(**k**): $2 \mid (k^2 + k)$ for any one k∈ Z+
6. Step 3 From Step 2 assumption we must prove P(**k+1**): $2 \mid [(k + 1)^2 + (k+1)]$

7. Step 4 Prove: P(k)→P(k+1) any one k∈Z+	
8. P(k): 2|(k^2+k)→(K^2+k) = 2q, q∈Z	Line 5 assumption, Def divides
9. Consider: (k^2+k)+2(k+1) = 2q+2(k+1)	Add 2(k+1) (backward reasoning below)
10. (k^2+k)+2(k+1) = 2[q+(k+1)]	Factor RHS
11. (k+1)^2+(k+1) = 2[q+(k+1)]	See backward reasoning below
12. →2|(k+1)^2+(k+1)	def divides
13. Conclusion: P(k)→P(k+1)	
QED ∀n∈Z+, P(n): 2|n^2+n	

Backward reasoning to determine what to add in line 10

In the end to get 2|(k+1)^2+(k+1) we need dividend:
(k+1)^2+(k+1) = k^2+2k+1+k+1 = (k^2+k)+2(k+1)

Ex 9: Another example with a Divisibility..

Proof (Mathematical Induction) P(n): $3 \mid (n^3 - n) \quad \forall n \in Z+$

1. Step 1 Prove P(1):
2. $3 \mid 0$ — $0 = 3 \times 0$
3. $\rightarrow 3 \mid (1^3 - 1)$ — $0 = 1^3 - 1$
4. Concl: P(1) = T

5. Step 2 Assume P(**k**): $3 \mid (k^3 - k)$ for any one k∈ Z+
6. Step 3 From Assumption in line 2 we must prove P(**k+1**): $3 \mid [(k+1)^3 - (k+1)]$

7.	Step 4 Prove: P(k)→P(k+1) any one k∈Z+	
8.	P(k): 3|(k^3-k)→(k^3-k)=3q, q∈Z	
9.	Consider (k^3-k)+3(k^2+k) = 3q+3(k^2+k)	Def. divides
10.	k^3-k+3k^2+3k = 3[q+(k^2+k)]	Add 3(k^2+k) (Backward reasoning)
11.	(k+1)^3-(k+1) = 3[q+(k^2+k)]	Expand LHS, Factor RHS
12.	→3|(k+1)^3-(k+1)	See Backward reasoning result
13.	Conclusion: P(k)→P(k+1)	Def Div
QED:	∀n∈Z+, P(n): 3|(n^3-n)	

Backward reasoning to determine what to add in line 9:

In end for dividened we want:
(k+1)^3-(k+1)=k^3+3k^2+3k+1-(k+1)=(k^3-k)+3(k^2+k)

Ex 10: Divisibility with $n \geq 0$.

Proof (Mathematical Induction) P(n): $57 \mid (7^{n+2} + 8^{2n+1})$ $\forall n \in Z$ $n \geq 0$

1. Step 1 Prove: P(0): Note if n=0 we have to prove $57 \mid (7^{0+2} + 8^{2(0)+1})$, prove $57 \mid (7^{2} + 8^{1})$
2. $49 + 8 = 57$
3. $\rightarrow 7^2 + 8^1 = 57 \times 1$
4. $\rightarrow 57 \mid (7^2 + 8^1)$ Def of Division

Concl: P(0) = T

5. Step 2 Assume P(**k**): $57 \mid (7^{k+2} + 8^{2k+1})$ for any one $k \geq 0$, $k \in Z$
6. Step 3 We must prove P(**k+1**): $57 \mid (7^{k+3} + 8^{2k+3})$

Assumption line 5
Def of Division
Multiply by 8^2
Algebra: distributive, $8^2 \times q = q_1$
$8^2 = 64 = 57 + 7$
Algebra: Distributive
Algebra, subtract & factor
Algebra: factor
Def of Division

By Math Induction

5. Strong Mathematical Induction. This is a variation of the Mathematical Induction method

a) The only difference is in Step 2 assumption and hence Step 1:

To prove by Strong Mathematical Induction that P(n) = T $\forall$n $\in$ **Z+**, n > n_0

Step 1: Prove P(k_0)…..P(k), Where k_0 is the initial n value
Step 2 Assume P(j) for j = k_0 …k for any k$\in$Z
Step 3 We must prove P(k+1) for any k$\in$Z
Step 4 Prove P(k) $\rightarrow$ P(k+1) for any k$\in$Z

b) Comparison:

Mathematical Induction
Assume P(k) for any value of k

Strong Mathematical Induction
Assume P(j) for all j $\leqslant$ any value of k

c) The two forms are logically equivalent which means that if one method works the other method works also. The reason for both methods: it is sometime easier to construct a proof with one method rather than the other.

Ex 10: All postage amounts 12 cents or more can be made using 4 and 5 cent stamps

P(n): $\forall n \in \mathbf{Z}$ $n \geq 12$ $\exists(q_1, q_2) \in \mathbf{Z}$ $q_1 \geq 0$, $q_2 \geq 0$ such that $n = 4q_1 + 5q_2$

Proof (by Stong Mathematical Induction)

Step 1 Prove P(12) :

LHS = 12 — Substitute in LHS

RHS = $4\times3 + 5\times0$ — Substitute in RHS

Concl: P(12) = T

SimiIarily P(13) = $4\times2 + 5\times1 \rightarrow$ P(13) = T

P(14) = $4\times1 + 5\times2 \rightarrow$ P(14) = T

P(15) = $4\times0 + 5\times3 \rightarrow$ P(15) = T

Step 2 Assume P(k): P(j) for j=12,…..k for any $k \in \mathbf{Z}$, $k \geq 15$

Step 3 Prove P(k+1) for any $k \in \mathbf{Z}$, $k \geq 15$

Step 4 Since we have P(j) for j=12,…..k for any $k \in \mathbf{Z}$, $k \geq 15$ — Step 2

Step 2 assumption → P(k−1)=P(k−2)=P(k−3)=T — 3 Prop. before P(k) also T

P(k−3): k−3=4q1+5q2, q1,q2∈Z — P(k-3) = T by assumption

Consider: k−3+4=4q1+5q2+4 — Add 4 both sides

k+1=4(q1+1)+5q2 — Factor. This is P(k+1)

Conclusion: P(k+1) = T

P(j) for j=12,...k ∧ [P(j) for j=12,...,k→P(k+1)]

→P(n): ∀n∈Z, n≥12 ∃(q1,q2)∈Z, q1≥0,w2≥0 such that a=4q1+5q2

Note: We could have proved the previous example with the standard Mathematical Induction but it requires 2 cases. Case 1 at least one 4-cent stamp is used, Case 2 no 4 cent stamps are used. (p 287)

6. Errors to be avoided:

Error 1: Consider the proof of P(n): $\forall n \in \mathbf{Z}^+ \;\; n > n+1$

Proof (by Mathematical Induction)

Assume P(k) for any one $k \in Z+$, Prove P(k+1)

Prove P(k+1)

P(k): $k>k+1$

$\rightarrow k+1 > k+1+1$

$k+1 > k+2$

P(k+1)

$P(k) \rightarrow P(k+1)$

QED P(n): $\forall n \in \mathbf{Z} \;\; n > n+1$

What is the error ?

Step 1 in Math Induction has not been proved. P(1)=T was not proved. In fact P(1)=F. Then you cannot make the Assumption because you have no values for n for which the P(k)=T and hence cannot get the Math Induction process started. (the frist Domino cannot be knocked down in our non Mathematical example.)

Ex 6: An example with inequality:

Form a conjecture concerning the comparison of the following pairs:

$2+1, 2^2$
$3+1, 3^2$
$4+1, 4^2$
$5+1, 5^2$

Conjecture: $\forall n \in Z+\ \ n > 1, n+1 < n^2$

Now prove the conjecture:

Proof (Mathematical Induction) $\forall n \in Z+\ n > 1, n+1 < n^2$

1.	Step 1 Prove P(2)	
2.	LHS = 2+1 = 3	Substitute n = 2 in LHS
3.	RHS = 2^1 = 4	Substitute n = 2 in RHS
4.	Concl: LHS < RHS $\rightarrow$ P(1) = T	

5. Step 2 Assume P(**k**): **k**+1< k^2 for any one k$\in$ Z+

6. Step 3 From Step 2 assumption we must prove P(**k+1**): **k+2** < $(\mathbf{k+1})^2$

7. Step 4	Prove: P(k) $\rightarrow$ P(k+1) for any one k$\in$Z+	
8.	P(**k**): k+1 < k^2	Line 5
9.	$\rightarrow$ P(**k+1**): k+1+1 < k^2 +1	Add 1
10.	k + 2 < k^2 + 2k + 1	
11.	k+2 < $(k+1)^2$	Factor
	Concl: P(k) $\rightarrow$ P(k+1)	Transitive 9 to 11
	QED $\forall n \in Z+, n>1$, P(n): $n+1 < n^2$	

Ex 10: Divisibility with n >=0.

Proof (Mathematical Induction) P(n): $57 \mid (7^{n+2} + 8^{2n+1}) \quad \forall n \in Z \ \ n \geq 0$

1. Step 1 Prove: P(0): Note if n=0 we have to prove $57 \mid (7^{0+2} + 8^{2(0)+1})$, prove $57 \mid (7^2 + 8^1)$
2. $49 + 8 = 57$
3. $\rightarrow 7^2 + 8^1 = 57 \times 1$
4. $\rightarrow 57 \mid (7^2 + 8^1)$ — Def of Division

Concl: P(0) = T

5. Step 2 Assume P(**k**): $57 \mid (7^{k+2} + 8^{2k+1})$ for any one $k \geq 0$, $k \in Z$
6. Step 3 We must prove P(**k+1**): $57 \mid (7^{k+3} + 8^{2k+3})$

7. Step 4 Prove: P(k) → P(k+1) for any one $k \geq 0$, $k \in Z$

8.	$57 \mid (7^{k+2} + 8^{2k+1})$	Assumption line 5
9.	$\rightarrow 7^{k+2} + 8^{2k+1} = 57 \times q,\ q \in Z$	Def of Division
10.	$\rightarrow 8^2\,(7^{k+2} + 8^{2k+1}) = 57 \times 8^2 \times q$	Multiply by 8^2
11.	$\rightarrow 8^2 \times 7^{k+2} + 8^2 \times 8^{2k+1} = 57 \times q_1$	Algebra: distributive, $8^2 \times q = q_1$
12.	$\rightarrow (57+7) \times 7^{k+2} + 8^{2k+3} = 57 \times q_1$	$8^2 = 64 = 57 + 7$
13.	$\rightarrow 57 \times 7^{k+2} + 7 \times 7^{k+2} + 8^{2k+3} = 57 \times q_1$	Algebra: Distributive
14.	$\rightarrow 7^{k+3} + 8^{2k+3} = 57 \times q_1 - 57 \times 7^{k+2})$	Algebra, subtract & factor
15.	$\rightarrow 7^{k+3} + 8^{2k+3} = 57(q_1 - 7^{k+2})$	Algebra: factor
16.	$\rightarrow 57 \mid (7^{k+3} + 8^{2k+3})$	Def of Division

Concl: P(k) → P(k+1)

QED P(n): $57 \mid (7^{n+2} + 8^{2n+1}) \quad \forall n \in Z \ \ n \geq 0$ — By Math Induction

Error 1: Recall the conjecture: The sum of the first n consecutive integers = $\frac{n(n+1)}{2}$.

Proof: (by Mathematical Induction) P(n): 1+2+3+4+……….+ n = $\frac{n(n+1)}{2}$, $\forall$n $\in$ Z+

1. Step 1	Prove P(1):	
2.	LHS = 1	1[st] LHS term
3.	RHS = $\frac{1(1+1)}{2}$ = 1	Substitute in RHS
4.	Concl: P(1) = T	LHS = RHS

5. Step 2 Assume P(**k**): 1+2+3+4+… + k = $\frac{k(k+1)}{2}$, for any one k$\in$ Z+

6. Step 3 We must prove P(**k+1**): 1+2+3+4+…..+ k + (k+1) = $\frac{(k+1)(k+2)}{2}$

7. Step 4	Prove: P(k) $\rightarrow$P(k+1) for any one value k $\in$ Z+	
8.	Since P(**k**): 1+2+3+4+… + k = $\frac{k(k+1)}{2}$	Line 5
9.	Substitute k+1 for k in line 5:	
10.	P(**k+1**): 1+2+3+4+… + (k+1) = $\frac{(k+1)(k+2)}{2}$	Algebra
11.	Concl: P(k) $\rightarrow$P(k+1)	
QED:	P(n): 1+2+3……….+ n = $\frac{n(n+1)}{2}$, $\forall$n$\in$ Z+	by Mathematical Induction

What is the error ?

This is the classic error (lines 9,10): If you simply substitute (k+1) for k in the assumed P(k) you are also assuming that P = T for (k+1) which is what you are trying to prove ! Instead you must alter the proposition with valid logic to get the proposition in the form of P(k+1) without assuming P(k+1) = T.