11.2 Series

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Definitions & Theorems:

1. Definition: Series

   An infinite series (or just a series) is denoted by the symbol

   $$\sum_{n=1}^{\infty} a_n \quad or \quad \sum a_n$$

   which is the addition of the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$

2. Definition:

   Given a series $\sum_{a=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its nth partial sum:

   $$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

   If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called convergent and we write

   $$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = s \quad or \quad \sum_{n=1}^{\infty} a_n = s$$

   The number s is called the sum of the series. Otherwise, the series is called divergent.

   $$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i = \lim_{n\to\infty} s_n = s$$

3. Formula:

   The geometric series

   $$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots$$

   is convergent if $|r| < 1$ and its sum is

   $$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1$$

   If $|r| \geq 1$, the geometric series is divergent.

4. Theorem:

   If the series $\sum_{n=1}^{\infty} a^n$ is convergent, then $\lim_{n\to\infty} a_n = 0$

5. The Test for Divergence

   If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a^n$ is divergent.

6. Theorem:

   If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum ca_n$ (where c is a constant), $\sum(a_n + b_n)$, and $\sum(a_n - b_n)$, and

   (i) $$\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$$

   (ii) $$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

   (iii) $$\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$$

Examples:

1. In Formula3, with $a = 1$ and $r = x$, when $|x| < 1$

   $$\sum_{n=1}^{\infty} x^n = \frac{1}{1-x} \qquad |x| < 1$$

2. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$

$$\sum_{n=1}^{\infty} b_n = \frac{1}{2^n} = \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)\left(\frac{1}{2^{n-1}}\right) = \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{n-1} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

$$s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} (\frac{1}{i} - \frac{1}{i+1}) = \left(\frac{1}{1}-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \cdots + \left(\frac{1}{n}-\frac{1}{n+1}\right) = \frac{1}{1} - \frac{1}{n+1}$$

$$\Rightarrow \lim_{n\to\infty} s_n = 1 \Rightarrow \{s_n\}\ converges\ to\ 1$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n(n+1)}\ converges\ and\ \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

4. Show that $\sum_{n=1}^{\infty}\left(e^{-n} - \mathrm{e}^{-n-1}\right)$ converges.

 (i) Method 1:

$$\sum_{n=1}^{\infty}\left(e^{-n} - \mathrm{e}^{-n-1}\right) = \sum_{n=1}^{\infty}\left(1 - \mathrm{e}^{-1}\right)e^{-n} = \sum_{n=1}^{\infty}\left(1-\frac{1}{\mathrm{e}}\right)\left(\frac{1}{e}\right)^{n} = \sum_{n=1}^{\infty}\left(1-\frac{1}{\mathrm{e}}\right)\left(\frac{1}{e}\right)\left(\frac{1}{e}\right)^{n}$$

$$\left|\frac{1}{\mathrm{e}}\right| < 1 \Rightarrow \sum_{n=1}^{\infty}\left(e^{-n} - \mathrm{e}^{-n-1}\right)\ converges\ and\ converges\ to\ \frac{\left(1-\frac{1}{\mathrm{e}}\right)\left(\frac{1}{e}\right)}{1-\frac{1}{e}} = \frac{1}{e}$$

 (ii) Method 2:

$$s_n = \sum_{i=1}^{n}\left(e^{-i} - \mathrm{e}^{-i-1}\right) = \left(e^{-1} - e^{-2}\right) + \left(e^{-2} - e^{-3}\right) + \left(e^{-4} - e^{-5}\right) + \cdots + \left(e^{-n} - e^{-n-1}\right) = e^{-1} - e^{-n-1}$$

$$\Rightarrow \lim_{n\to\infty} s_n = e^{-1} \Rightarrow \{s_n\}\ converges\ to\ e^{-1}$$

$$\Rightarrow \sum_{n=1}^{\infty}\left(e^{-n} - \mathrm{e}^{-n-1}\right)\ converges\ and\ \sum_{n=1}^{\infty}\left(e^{-n} - \mathrm{e}^{-n-1}\right) = e^{-1}$$

5. $\displaystyle\sum_{n=2}^{\infty} \frac{n^2+1}{n^2-1}$

Since $\lim_{n\to\infty} \frac{n^2+1}{n^2-1} = 1 \neq 0 \Rightarrow \sum_{n=2}^{\infty} \frac{n^2+1}{n^2-1}\ diverges$

6. $\displaystyle\sum_{n=1}^{\infty} \cos n$

$\lim_{n\to\infty} \cos n$ does not exist $\Rightarrow \sum_{n=1}^{\infty} \cos n\ \ diverges$

7. $\displaystyle\sum_{n=1}^{\infty} n \arctan n$

$$\lim_{n\to\infty} n \arctan n = \infty \Rightarrow \sum_{n=1}^{\infty} n \arctan n\ \ diverges$$

★8. $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^2+1}$

$\lim_{n\to\infty} \frac{n^2+1}{n^2-1} = 0 \Rightarrow$ Test for Divergence does not apply $\Rightarrow$ no conclusion

9. If $\sum c_n$ converges and $c_n = a_n \pm b_n$, does a_n and b_n converge?

 False: $c_n = 0, a_n = 2, b_n = (\pm)2$