4.9 Antiderivatives

June 29, 2016 11:35

Definitions & Theorems:

1. Definition:

   A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

2. Theorem:

   If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is
   $$F(x) + C$$
   where C is an arbitrary constant.

3. Definition:

   The infinite integral of $f(x)$ is defined as
   $$\int f(x)\,\mathrm{d}x = F(x) + C$$

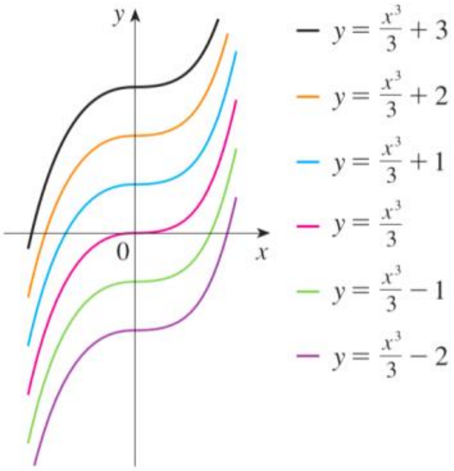
! $\int_a^b f(x)\,\mathrm{d}x$ is a number; but $\int f(x)\,\mathrm{d}x$ is a function.

$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x$	$\int \cos x\,\mathrm{d}x = \sin x + C$
$\frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x$	$\int \sin x\,\mathrm{d}x = -\cos x + C$
$\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x$	$\int \sec^2 x\,\mathrm{d}x = \tan x + C$
$\frac{\mathrm{d}}{\mathrm{d}x}\sec x = \sec x \tan x$	$\int \sec x \tan x\,\mathrm{d}x = \sec x + C$
$\frac{\mathrm{d}}{\mathrm{d}x}\csc x = -\csc x \cot x$	$\int \csc x \cot x\,\mathrm{d}x = -\csc x + C$
$\frac{\mathrm{d}}{\mathrm{d}x}\cot x = -\csc^2 x$	$\int \csc^2 x\,\mathrm{d}x = -\cot x + C$
$\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}\mathrm{d}x = \sin^{-1} x + C$
$\frac{\mathrm{d}}{\mathrm{d}x}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$	
$\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1} x = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}\mathrm{d}x = \tan^{-1} x + C$
$\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$	$\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + \mathrm{C}, n \neq -1$
$\frac{\mathrm{d}}{\mathrm{d}x}\ln\lvert x\rvert = \frac{1}{x}$	$\int \frac{1}{x}\mathrm{d}x = \ln\lvert x\rvert + C$
$\frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x$	$\int e^x\,\mathrm{d}x = e^x + C$

$\frac{\mathrm{d}}{\mathrm{d}x}a^x = a^x \ln a, a > 0$	$\int a^x \,\mathrm{d}x = \frac{a^x}{\ln a} + C$

Proofs or Explanations:

1. Theorem2:



Members of the family of antiderivatives of $f(x) = x^2$

Extra topics:

1. Even function: $f(x) = -f(x)$
$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$
2. Odd function: $f(x) = -f(-x)$
$$\int_{-a}^{a} f(x)\,dx = 0$$

Examples:

1. $\displaystyle\int\left(x^2 + 3\,sin\,x + 2^x - \frac{1}{2}csc\,x\,cot\,x\right)\mathrm{d}x = \frac{x^3}{3} - 3\cos x + \frac{2^x}{\ln 2} - \frac{1}{2}(-\csc x) + C$
2. Find $f(x)$ such that $f'(x) = 2e^x - \frac{1}{\sqrt{1-x^2}}$, and $f(0) = 4$.
$$f(x) = \int\left(2e^x - \frac{1}{\sqrt{1-x^2}}\right)dx = 2e^x - sin^{-1}\,x + C$$
$$f(0) = 2e^0 - \sin^{-1} 0 + C = 2 + C = 4 \to C = 2$$
$$f(x) = 2e^x - sin^{-1}\,x + 2$$
3. Find $f(x)$ such that $f'(x) = x - \frac{1}{1+x^2}$ and $f(1) = \frac{\pi^2}{32}$.
$$f(x) = \int\left(x - \frac{1}{1+x^2}\right)dx = \frac{x^2}{2} - tan^{-1}\,x + C$$
$$f(1) = \frac{1}{2} - \tan^{-1} 1 + C = \frac{1}{2} - \frac{\pi}{4} + C = \frac{\pi^2}{32} \to C = \frac{\pi^2}{32} + \frac{\pi}{4} - \frac{1}{2}$$
$$f(x) = \frac{x^2}{2} - tan^{-1}\,x + \frac{\pi^2}{32} + \frac{\pi}{4} - \frac{1}{2}$$