Sec 1.1-1.2-1.3 Discrete Mathematics and Applications

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Comp 232 Discrete Mathematics and Applications

Instructor: Robert Mearns

Where are we headed in this course ?

We need the rules of Logic and Discrete Mathematics to assist in the design of :

Software Systems | Hardware Systems

That are valid (give correct results ALWAYS, not just sometimes).
Comp 232 will present the rules of Logic and other Discrete Mathematics topics

Ex 1: The following three statements are equivalent: (if any one statement is true then the other two are true also)

If Susan goes to the concert then Paul will go to the concert	If Paul does not go to the concert then Susan will not go to the concert	Susan does not go to the concert or Paul goes to the concert

Ex 2: Add the integers 1 + 2 + 3 + ……. + 50

<u>Method 1</u>: Initialize the sum S = 0 Initialize the integer n = 1 (Form a loop and go through a loop 50 times) Increase the sum by n each time through the loop Increase the n by 1 each time through the loop	<u>Method 2</u>: Use a math result: There is a formula for the sum of the first n integers: Sum of: $1 + 2 + 3 + \ldots\ldots + n = \frac{n(n+1)}{2}$ So in our example since n = 50 the answer is: $1 + 2 + 3 + \ldots\ldots + 50 = \frac{50(50+1)}{2} = 1275$

1

Chap 1 Basic Logic and Proofs

Section 1.1 - 1.2 - 1.3 **Propositional Logic, Applications and Equivalences**

1 a) Definition: A Simple Declarative statement or proposition is a sentence that declares a single fact that can be judged to be: true (T) or false (F) but not both

Ex: Label the following sentences as Simple Declarative or Not Simple Declarative and state the truth value for the Declarative statements:

Sentence:	Type of sentence:	Truth Value:
1. Ottawa is the Capital city of Canada.	✓	T
2. Shut the door.	×	
3. What time is it ?	×	
4. The time now is 5 o'clock PM.	✓	F
5. $2 + 3 = 6$	✓	F
6. $x + 1 = 2$	×	

1 b) We assign a logic variable to denote a Simple Declarative proposition.

Ex: Let q represent: "Ottawa is the Capital city of Canada"
then $q = T$ (q has a truth value of true)
Let p represent: "$2 + 3 = 6$"
then $p = F$ (p has a truth value of false)

2

1 c) To Negate a Simple Declarative proposition means to change it so that it has the opposite truth value. The negation of p is written as: not p, ~p, ¬p

Ex: If q represents "Ottawa is the capital city of Canada",
then ¬ q represents " It is not true that Ottawa is the capital city of Canada".
(the language can be simplified to read "Ottawa is not the capital city of Canada)
Note in this example: ¬q = F since q = T

1 b) A truth table records the truth value of a logic variable.

Ex:

p	¬p
T	F
F	T

2 a) Definition: A Compound declarative proposition is a declarative proposition that contains two or more simple declarative propositions joined by the words AND or OR.

Ex: Ottawa is the Capital city of Canada and 2+3=6
Ottawa is the Capital city of Canada or 2+3=6

2 b) Definition: A Conditional proposition has form:
IF ***proposition* 1** THEN ***proposition* 2**. (propositions 1 and 2 may be simple or compound)

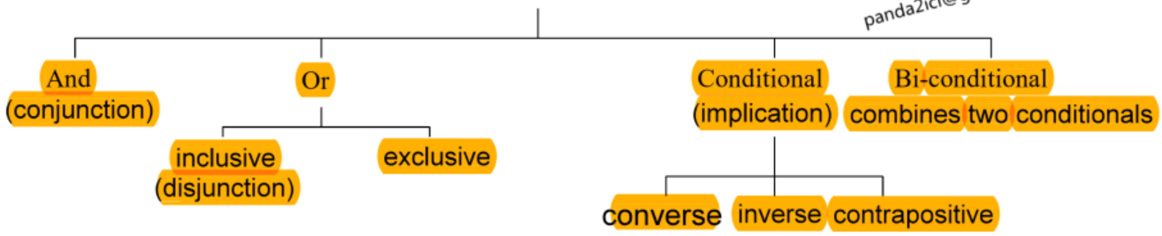
Ex: If ***the temperature is 20C*** then the snow smelts.

2 c) Definition: A Bi-conditional proposition has form:
***proposition* 1** IF AND ONLY IF ***proposition* 2** (it combines two conditional propositions)

Ex: ***A figure has four equal sides*** if and only if the figure is a square.

4

2 d) Summary of combining statements or propositions:



3. Assigning truth values to a compound statement or proposition containing <u>And</u>

Name	Symbol	Description in words	Truth table	Truth value summary
And Also called Conjunction	$p \wedge q$	p and q "p is true and at the same time q is true"	<table><tr><th>p</th><th>q</th><th>p ∧ q</th></tr><tr><td>T</td><td>T</td><td>T</td></tr><tr><td>T</td><td>F</td><td>F</td></tr><tr><td>F</td><td>T</td><td>F</td></tr><tr><td>F</td><td>F</td><td>T</td></tr></table>	$p \wedge q$ has truth value F except when both p, q have truth value T .

6

4 a) Truth value when combining propositions with Or (Inclusive Or is the default or in logic)

Name	Symbol	Description in words	Truth table	Truth value summary
Inclusive Or Disjunction	$p \vee q$	p or q "p is true or q is true or both are true	p q p ∨ q T T T T F T F T T F F F	$p \vee q$ has truth value T except when both p, q have truth value F

Ex 1: Complete a truth table with the following headings:

p	q	¬ p	¬ q	p∧q	**¬(p ∧ q)**	**¬ p∨¬ q**

Ex 2: Using the answer to Ex 1 state 2 logical forms that are Equivalent (have same truth value).
Symbol ≡ denotes equivalence.

The first De Morgan's Rule $\neg(p \wedge q) \equiv \neg p \vee \neg q$
The second De Morgan's Rule $\neg(p \vee q) \equiv \neg p \wedge \neg q$

4 b) If statement A is equivalent to statement B then statement A can be replaced by statement B when calculating truth values.

8

5 a) Truth value when combining propositions with XOR (Exclusive Or)

Name	Symbol	Description in words	Truth table	Truth value summary
Exclusive Or	p ⊕ q	p xor q "p is true or q is true But not both are true"	p q p ⊕ q T T F T F T F T T F F F	p ⊕ q has truth value F if both p, q have same truth value otherwise it is T

Ex: Complete a truth table with the following headings.

p	q	p∧q	p⊕q	**(p∧q)∧(p⊕q)**
T	T	T	F	F
T	F	F	T	F
F	T	F	T	F
F	F	F	F	F

5 b) A logic statement that is

always false is called a **Contradiction**
It is equivalent to **F**
p ∧ ¬p

always true is called a **Tautology**
It is equivalent to **T**
p ∨ ¬p

neither a Contradiction nor a Tautology is called a **Contingency**
p ∨ q

5 c) In English there is no default Or. The context of the proposition dictates Inclusive or Exclusive.
Ex: (i) Applicants must have a Computer Science course or a Calculus course. inclusive or
(ii) At 9 PM Jane will stay home or go out to a movie. exclusive or

10

6 a) Truth value when combining propositions in the form IF….THEN…. (Conditional)

Name	Symbol	Description in words	Truth table	Truth value summary
Conditional Also called Implicatoin	$p \to q$	"if p then q" "if p is true then q is true" "p true implies q true"	<table><tr><th>p</th><th>q</th><th>p → q</th></tr><tr><td>T</td><td>T</td><td>T</td></tr><tr><td>T</td><td>F</td><td>F</td></tr><tr><td>F</td><td>T</td><td>T</td></tr><tr><td>F</td><td>F</td><td>T</td></tr></table>	$p \to q$ has truth value T except when p = T and q = F

Ex 1: Complete a truth table with the following headings:

p	q	$\neg$ **p**	$\neg p \lor q$	$p \to q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Ex 2: Using the last 2 columns of Ex 1 state 2 logical forms that are equivalent . This equivalence is often used to replace an implication

$$p \to q \equiv \neg p \lor q$$

12

6 b) In the implication $p \to q$:

p is called the antecedent or the Given premise	q is called the consequent or the Conclusion premise

6 c) There are four related Conditional statements:

Name	Form	How is it formed from the original statement
Original	$p \to q$	------
Inverse	$\neg p \to \neg q$	Negate the Given and the Conclusion
Converse	$q \to p$	Switch
Contrapositive	$\neg q \to \neg p$	Switch and Negate

Ex 3: Complete one truth table containing the four above conditional statements then state which of the related conditional(s) above is (are) equivalent to the original conditional ?

				Original	Inverse	Converse	Contrapositive
p	q	$\neg p$	$\neg q$	$p \to q$	$\neg p \to \neg q$	$q \to p$	$\neg q \to \neg p$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

$p \to q \equiv \neg q \to \neg p$ $q \to p \equiv \neg p \to \neg q$

14

6 d) Alternate English wordings that describe a true conditional $p \to q$
Note: the expression "is true" is assumed by default after each logic variable

	p	q	$p \to q$
1	T	T	T
2	T	F	F
3	F	T	T
4	F	F	T

Alternate form	Meaning
If p then q	If p is true, then q is true.
p implies q	
p is sufficient for q	Is p true enough to give q true ? Yes Is p true needed to give q true ? No Conclusion: p is sufficient but not necessary for q is another form of $p \to q$
q is necessary for p	Is q true needed to give p true ? Yes Is q true enough to give p true ? No Conclusion: q is necessary but not sufficient for p is another form of $p \to q$
p only if q	This means "only q true implies p true", hence q false implies p false: $\neg q \to \neg p \equiv p \to q$ Conclusion: p only if q is another form of $p \to q$
q unless (not p)	This means "q true if not (not p) true", hence If not (not p) true then q true: $\neg(\neg p) \to q \equiv p \to q$ Conclusion: q unless (not p) is another form of $p \to q$

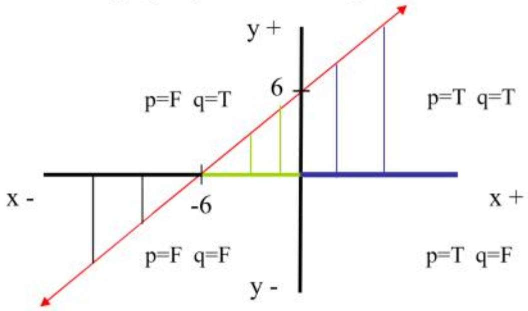
In English:

sufficient means: enough
necessary means: needed
unless means: if not

16

6 e) A Geometric example of an Implication

Consider graph $y = x + 6$. Let p represent: x is positive +. Let q represent: y is positive +



a) Write the English using x, y represented by $p \rightarrow q$ If x is positive, then y is positive.

b) On this graph is the implication $p \rightarrow q$ True or False ? T

c) Is x positive sufficient (enough) to give y positive ? Yes
Is x positive necessary (needed) to give y positive ? No

d) Using the results of c) repeat a) using the words sufficient, necessary

X positive is sufficient but not necessary for y positive.

e) Is y positive necessary (needed) to give x positive ?

Yes

Is y positive sufficient (enough) to give x positive ?

No

f) Using the results of e) repeat a) using the words sufficient, necessary

Y positive is necessary but not sufficient for x positive.

7. Truth value when combining propositions in the form ….IF AND ONLY IF…. (Bi-conditional)

Name	Symbol	Description in words	Truth table	Truth value summary
Bi-conditional Also called a Bi-implication	$p \leftrightarrow q$	"p if and only if q" or write p iff q This combines two conditionals with and: (p only if q) and (p if q) $(p \to q) \wedge (q \to p)$	p q $p \leftrightarrow q$	If p, q have same truth value then $p \leftrightarrow q$ has the truth value T otherwise $p \leftrightarrow q$ has the truth value F

Ex Complete a truth table with all columns needed to get the truth values for $p \leftrightarrow q$ then enter the result above and describe the summary of the bi-conditional truth value.

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

8. Order of logical operations:

The Logic operations are executed in the following order:

$\neg$ negation
$\wedge$ and
$\vee$ or
$\rightarrow$ conditional
$\leftrightarrow$ bi-conditional

() brackets override the above order of operations

Ex: Place brackets to describe the same order implied in proposition: $\neg p \vee p \wedge q \rightarrow p \leftrightarrow q$.

$(\neg p) \vee p \wedge q \rightarrow p \leftrightarrow q$	Negation first
$(\neg p) \vee (p \wedge q) \rightarrow p \leftrightarrow q$	And second
$[(\neg p) \vee (p \wedge q)] \rightarrow p \leftrightarrow q$	Or third
$\{[(\neg p) \vee (p \wedge q)] \rightarrow p\} \leftrightarrow q$	Conditional fourth, Bi-Conditional last

22

9. Summary of often used Equivalent Logical Statements: Enter the equivalences in column 1

$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	DeMorgan's Rules (see note 4 Ex 2)
$p \wedge q \equiv q \wedge p$ $p \vee q \equiv q \vee p$	Commutive [order does not matter]
$r \wedge (p \wedge q) \equiv r \wedge p \wedge q$ $r \vee (p \vee q) \equiv r \vee p \vee q$	Associative [brackets may be moved if operations are the same]
$p \wedge (q \vee r) \equiv p \wedge q \vee p \wedge r$ $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	Distributive [analogous to algebra distributive form $x * (a+b) = x * a + x * b$]
$p \wedge p \equiv p$ $p \vee p \equiv p$	Idempotent
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity [**T** is a truth table column with all true values] [**F** is a truth table column with all false values]
$p \wedge \mathbf{F} \equiv F$ $p \vee \mathbf{T} \equiv T$	Domination
$p \wedge \neg p \equiv F$ $p \vee \neg p \equiv T$	Negation Note also: we can have double negation: $\neg(\neg p) \equiv p$
$p \rightarrow q \equiv \neg p \vee q$ $p \rightarrow q \equiv \neg q \rightarrow \neg p$	Conditional in terms of Or (see note 6 Ex 2) Conditional in terms of Contrapositive (see note 6 Ex 3)
$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$	Bi-conditional in terms of And

Ex 1: Use the equivalences in 9 to show that $\neg p \to (q \to r) \equiv q \to (p \lor r)$ **without** truth tables.
Use the hints in the Reference column.

	References:
$\neg p \to (q \to r) \equiv \neg(\neg p) \lor (q \to r)$	Conditional with Or
$\equiv p \lor (q \to r)$	Double Negation
$\equiv p \lor (\neg q \lor r)$	Conditional with Or
$\equiv p \lor \neg q \lor r \equiv \neg q \lor p \lor r$	Associative & Commutative
$\equiv q \to (p \lor r)$	Conditional with Or
QED	Quad Erat Demonstratum (optional)

Ex 2: Use the equivalences in 9 to show whether: $[\neg q \land (p \to q)] \to \neg p$ is a Tautology, Contradiction or a Contingency **without** truth tables. Use the hints in the Reference column.

	References:
$[\neg q \land (p \to q)] \to \neg p \equiv [\neg q \land (\neg p \lor q)] \to \neg p$	Conditional with Or
$\equiv [(\neg q \land \neg p) \lor (\neg q \land q)] \to \neg p$	Distributive
$\equiv (\neg q \land \neg p) \lor F \to \neg p$	$\neg A \land A \equiv F$
$\equiv (\neg q \land \neg p) \to \neg p$	$A \lor F \equiv A$
$\equiv \neg(q \lor p) \to \neg p$	De Morgan
$\equiv (q \lor p) \lor \neg p$	Conditional with Or
$\equiv q \lor (p \lor \neg p)$	Associative
$\equiv q \lor T$	$A \lor \neg A \equiv T$
$\equiv T$	$A \lor T \equiv T$
QED	

26

Ex 3: Use the equivalences in 9) to show that: $p \wedge (p \vee q) \equiv p$
without truth tables. Use the hints in the Reference column.

	References:
$p \wedge (p \vee q) \equiv (p \vee F) \wedge (p \vee q)$	$x \equiv x \vee F$
$\equiv p \vee (F \wedge q)$	Distributive
$\equiv p \vee F$	$(F \wedge x) \equiv F$
$\equiv p$	$x \vee F \equiv x$

QED

If $\vee$ replaces $\wedge$, $\wedge$ replaces $\vee$, T replaces F and F replaces T

In a similar way it can be shown: $p \vee (p \wedge q) \equiv p$ the two statements are called Duals

Ex 4: Use the equivalences in 9) to show that: $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
without truth tables. Use the in the Reference column.

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$	Bi-conditional with And
$\equiv (\neg p \vee q) \wedge (\neg q \vee p)$	Conditional with Or
$\equiv [\neg p \wedge (\neg q \vee p)] \vee [q \wedge (\neg q \vee p)]$	Distributive
$\equiv [(\neg p \wedge \neg q) \vee (\neg p \wedge p)] \vee [(q \wedge \neg q) \vee (q \wedge p)]$	Distributive
$\equiv [(\neg p \wedge \neg q) \vee F] \vee [F \vee (q \wedge p)]$	Negation
$\equiv (\neg p \wedge \neg q) \vee (q \wedge p)$	Domination

28

$\equiv (q \wedge p) \vee (\neg p \wedge \neg q)$

Ex 5: Write the two conditional statements contained in the bi-conditional statement $\mathbf{p \leftrightarrow q}$ in each of the six alternate word forms for implication mentioned in note 6 d).

$\mathbf{p \rightarrow q}$ And $\mathbf{q \rightarrow p}$

Note: In the bi-conditional using the words sufficient, necessary we say: