7.3 Trigonometric Substitution

July 11, 2016 14:24

Definitions & Theorems:

1. Table of Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{(a^2-x^2)}$	$x = a\sin\theta\,, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1-\sin^2\theta = \cos^2\theta$
$\sqrt{(x^2+a^2)}$	$x = a\tan\theta\,, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1+\tan^2\theta = \sec^2\theta$
$\sqrt{(x^2-a^2)}$	$x = a\sec\theta\,, 0 \le \theta < \frac{\pi}{2}\ or\ \pi \le \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Examples:

1. $\displaystyle\int \frac{\sqrt{25x^2-4}}{x}\,\mathrm{d}x = \int \frac{\sqrt{(5x)^2-2^2}}{x}\,\mathrm{d}x$

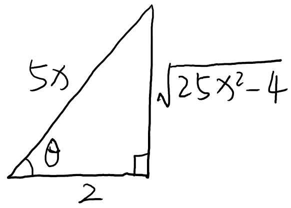
   Let $5x = 2\sec\theta \to x = \frac{2}{5}\sec\theta\,, dx = \frac{2}{5}\sec\theta\tan\theta\, d\theta$

   $$\int \frac{\sqrt{(5x)^2-2^2}}{x}\,\mathrm{d}x = \int \frac{\sqrt{(2\sec\theta)^2-2^2}}{\frac{2}{5}\sec\theta}\left(\frac{2}{5}\sec\theta\tan\theta\,\mathrm{d}\theta\right) = 2\int |\tan\theta|\tan\theta\,\mathrm{d}\theta$$

   $0 \le \theta < \frac{\pi}{2} \to \tan\theta \ge 0$

   $$2\int |\tan\theta|\tan\theta\,\mathrm{d}\theta = 2\int \tan\theta\tan\theta\,\mathrm{d}\theta = 2\int (\sec^2\theta - 1)\,\mathrm{d}\theta = 2\tan\theta - 2\theta + C$$

   $$= \sqrt{25x^2-4} - 2\,\mathrm{arcsec}\frac{5x}{2} + C$$



2. $\displaystyle\int_{\frac{2}{5}}^{\frac{4}{5}} \frac{\sqrt{25x^2-4}}{x}\,\mathrm{d}x$

   $x = \frac{2}{5}\sec\theta \Rightarrow$

   $x = \frac{2}{5} = \frac{2}{5}\sec\theta \to \sec\theta = 1 \to \theta = 0$

   $x = \frac{4}{5} = \frac{2}{5}\sec\theta \to \sec\theta = 2 \to \theta = \frac{\pi}{3}$

   $$\int_{\frac{2}{5}}^{\frac{4}{5}} \frac{\sqrt{25x^2-4}}{x}\,\mathrm{d}x = 2\int_0^{\frac{\pi}{3}} (\sec^2\theta - 1)\,\mathrm{d}\theta = 2\tan\theta - 2\theta\Big]_0^{\frac{\pi}{3}} = 2\sqrt{3} - \frac{2}{3}\pi$$

3. $\displaystyle\int e^{4x}\sqrt{1+e^{2x}}\,\mathrm{d}x$

   a. Method 1:

   Let $t = e^{2x} + 1 \to dt = 2e^{2x}dx$

   $$\int e^{4x}\sqrt{1+e^{2x}}\,\mathrm{d}x = \frac{1}{2}\int e^{2x}\sqrt{1+e^{2x}}\,2e^{2x}\,\mathrm{d}x = \frac{1}{2}\int (t-1)\sqrt{t}\,\mathrm{d}t = \frac{1}{2}\int t^{\frac{3}{2}}\,\mathrm{d}t - \frac{1}{2}\int t^{\frac{1}{2}}\,\mathrm{d}t$$

   $$= \frac{1}{2}\left(\frac{t^{\frac{5}{2}}}{\frac{5}{2}} - \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + C\right) = \frac{(e^{2x}+1)^{\frac{5}{2}}}{5} - \frac{(e^{2x}+1)^{\frac{3}{2}}}{3} + C$$

   b. Method 2:

   $$\int e^{4x}\sqrt{1+e^{2x}}\,\mathrm{d}x = \int e^{4x}\sqrt{1^2+(e^x)^2}\,\mathrm{d}x$$

   Let $e^x = 1\tan\theta \to e^x dx = \sec^2\theta\, d\theta$

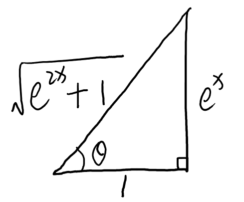
   $$\int e^{4x}\sqrt{1^2+(e^x)^2}\,\mathrm{d}x = \int e^{3x}\sqrt{1^2+(e^x)^2}\,e^x\,\mathrm{d}x = \int \tan^3\theta\sqrt{1^2+(\tan\theta)^2}\sec^2\theta\, d\theta = \int \tan^3\theta\,|\sec\theta|\sec^2\theta\, d\theta$$

   $-\frac{\pi}{2} < \theta < \frac{\pi}{2} \to \sec\theta > 0$

$$\int \tan^3\theta\,|\sec\theta|\sec^2\theta\,d\theta=\int\tan^3\theta\sec^3\theta\,d\theta=\int\tan^2\theta\sec^2\theta\tan\theta\sec\theta\,d\theta=\int(\sec^2\theta-1)\sec^2\theta\tan\theta\sec\theta\,d\theta$$

Let $u=\sec\theta\to du=\sec\theta\tan\theta\,d\theta$

$$\int\left(\sec^2\theta-1\right)\sec^2\theta\tan\theta\sec\theta\,d\theta=\int\left(u^2-1\right)u^2\,du=\frac{u^5}{5}-\frac{u^3}{3}+C=\frac{\sec^5\theta}{5}-\frac{\sec^3\theta}{3}+C$$



$$=\frac{(e^{2x}+1)^{\frac{5}{2}}}{5}-\frac{(e^{2x}+1)^{\frac{3}{2}}}{3}+C$$

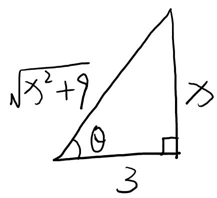
4. $\displaystyle\int\frac{\mathrm{d}x}{x^2\sqrt{x^2+9}}$

Let $x=3\tan\theta\to dx=3\sec^2\theta\,d\theta$

$$\int\frac{\mathrm{d}x}{x^2\sqrt{x^2+9}}=\int\frac{3\sec^2\theta\,d\theta}{9\tan^2\theta\sqrt{9\tan^2\theta+9}}=\int\frac{3\sec^2\theta\,d\theta}{9\tan^2\theta\sqrt{9\tan^2\theta+9}}=\frac{1}{9}\int\frac{\cos\theta}{\sin^2\theta}\mathrm{d}\theta$$

Let $u=\sin\theta\to du=\cos\theta\,d\theta$

$$\frac{1}{9}\int\frac{\cos\theta}{\sin^2\theta}\mathrm{d}\theta=\frac{1}{9}\int\frac{1}{u^2}\mathrm{d}u=-\frac{1}{9u}+C=-\frac{1}{9\sin\theta}+C$$



$$=-\frac{\sqrt{x^2+9}}{9x}+C$$

5. $\displaystyle\int_0^{\frac{1}{6}}\frac{x^5}{(36x^2+1)^{\frac{3}{2}}}\mathrm{d}x=\int_0^{\frac{1}{6}}\frac{x^5}{\left(\sqrt{(6x)^2+1^2}\right)^3}\mathrm{d}x$

Let $6x=\tan\theta\to x=\frac{1}{6}\tan\theta,\ dx=\frac{1}{6}\sec^2\theta\,d\theta$

$x=0=\frac{1}{6}\tan\theta\to\tan\theta=0\to\theta=0$

$x=\frac{1}{6}=\frac{1}{6}\tan\theta\to\tan\theta=1\to\theta=\frac{\pi}{4}$

$$\int_0^{\frac{1}{6}}\frac{x^5}{\left(\sqrt{(6x)^2+1^2}\right)^3}\mathrm{d}x=\int_0^{\frac{\pi}{4}}\frac{\left(\frac{1}{6}\tan\theta\right)^5}{(\sec\theta)^3}\left(\frac{1}{6}\sec^2\theta\,d\theta\right)=\frac{1}{6^6}\int_0^{\frac{\pi}{4}}\frac{\left(1-\cos^2\theta\right)^2}{\cos^4\theta}\sin\theta\,d\theta$$

Let $u=\cos\theta\to du=-\sin\theta\,d\theta$

$\theta=0\to\mathrm{u}=\cos\theta=1$

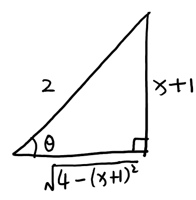
$\theta=\frac{\pi}{4}\to\mathrm{u}=\cos\theta=\frac{\sqrt{2}}{2}$

$$\frac{1}{6^6}\int_0^{\frac{\pi}{4}}\frac{\left(1-\cos^2\theta\right)^2}{\cos^4\theta}\sin\theta\,d\theta=\frac{1}{6^6}\int_1^{\frac{\sqrt{2}}{2}}\frac{\left(1-u^2\right)^2}{u^4}(-du)=\frac{1}{6^6}\int_{\frac{\sqrt{2}}{2}}^1\left(\frac{1}{u^4}-\frac{2}{u^2}+1\right)du=-\frac{1}{3u^3}+\frac{2}{u}+u\Big]_{\frac{\sqrt{2}}{2}}^1$$

6. $\displaystyle\int\frac{x}{\sqrt{3-2x-x^2}}\mathrm{d}x=\int\frac{x}{\sqrt{2^2-(x+1)^2}}\mathrm{d}x$

Let $x+1=2\sin\theta\to dx=2\cos\theta\,d\theta$

$$\int\frac{x}{\sqrt{2^2-(x+1)^2}}\mathrm{d}x=\int\frac{2\sin\theta-1}{2\sqrt{1-(\sin\theta)^2}}(2\cos\theta\,\mathrm{d}\theta)=\int(2\sin\theta-1)\mathrm{d}\theta=-2\cos\theta-\theta+C$$



$$=-2\left(\frac{\sqrt{4-(x+1)^2}}{2}\right)-\arcsin\left(\frac{x+1}{2}\right)+C$$