

Sec 2.3 , 2.5 Functions and Cardinality

Comp 232

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1. Recall: (i) The Cartesian Product of two sets A and B is:

The set of ordered pairs (x, y) where $x \in A$ and $y \in B$

- (ii) The Cartesian Product is denoted by $A \times B$

- 2 a) Definition: A Relation R from set A to set B is

- b) Relation R is denoted: $R \subseteq A \times B$

- c) There are many relations that can be formed from $A \times B$.

Ex: If $A = \{1, 2\}$ $B = \{5, 6, 7\}$ then $A \times B = \{(1, 5), (1, 6), (1, 7), (2, 5), (2, 6), (2, 7)\}$

Using this $A \times B$ above one possible relation is: $R = \{(1, 6), (2, 5)\}$

- d) If the Relation is set R and $(x, y) \in R$ then: x is related to y and we write $x R y$

Ex: In the above Relation R: $1 R 5$ and $2 R 7$



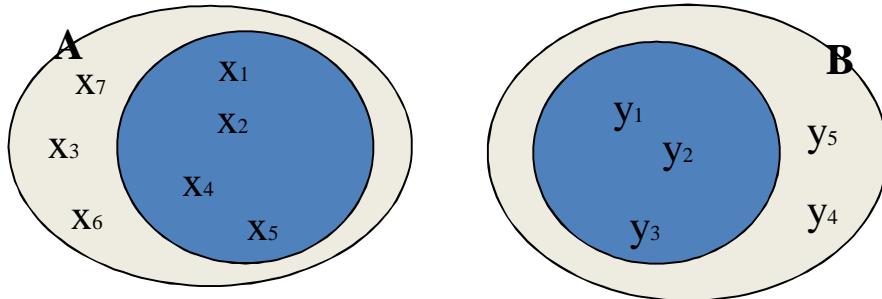
3. Geometric representation of a Relation R on Cartesian product $A \times B$

Step 1 Sketch sets A, B

Step 2 Sketch the subset of A containing the first members of the ordered pairs in R

Step 3 Sketch the subset of B containing the second members of the ordered pairs in R

Step 4 Join the related x, y values with an arrow.



Ex: $R =$

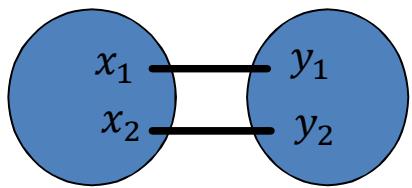
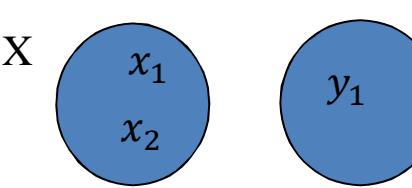
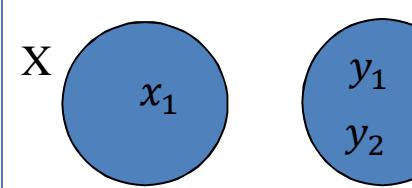
How many ordered pairs are in the Cartesian Product $A \times B = \{ (x, y) \mid (x \in A) \wedge (y \in B) \} ?$

$$n(AXB) = n(A) * n(B) = 35$$

How many ordered pairs are in the Relation R ?

$$n(R) = 5$$

4. Three basic kind of Relations

 <p>X Y</p>	 <p>X Y</p>	 <p>X Y</p>
<p>Each x is related to a unique y Each y is related to a unique x</p> <p>Called a one to one relation (1 to 1)</p>	<p>Each x is related to a unique y</p> <p>Called a many to one relation (m to 1)</p>	<p>Each y is related to a unique x</p> <p>Called a one to many relation (1 to m)</p>

5. Definition of a Function.

- a) A Relation from a non empty set X to a non empty set Y which is called a Function from set X to set Y.
- A Function is a special Relation because each x is related to a unique y.

1 to 1
or many to 1
but not 1 to many

b) For a Function write $f: X \rightarrow Y$

c) If the Relation is a Function:

- (i) $x R y$ is written as $y = f(x)$
- (ii) the set of x is called the
- (iii) the set of y is called the
- (iv) a Function is also called a

Domain of the Function
Range of the Function
Mapping or a Transformation

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5. Additional terms and definitions

a) What does the word Onto mean ?

- Start with a set that is the Domain for a function
- Consider a second set called the Codomain
- Identify the Range of the function. If:

Function Range = Codomain

- Function can generate all values in the Codomain
- Function is Onto

Functions which are Onto cover all Codomain

Function Range \neq Codomain

- Function does not generate all values in the Codomain

→ Function which are Not Onto do not cover all Codomain

b) Three related terms:

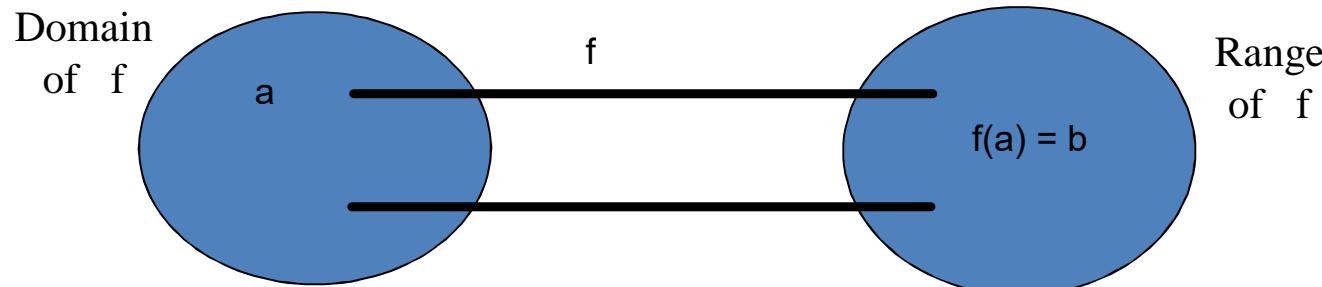


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Injective Function	Surjective Function	Bijective Function
A function that is 1 to 1	A function that is Onto	A function is both 1 to 1 and Onto (Also called a 1 to 1 correspondence)

c) Definition: Inverse Function: If (i) f is a 1 to 1 correspondence (1 to 1 and onto, Bijective Function)
(ii) a second function maps the Range of f back to the original domain.
Then the second function is called the Inverse of the function f

Notation: $a = b$ iff $f(a) = b$



d) Increasing and Decreasing functions: If $x_2 > x_1$ and

$$\forall x \in (x_1, x_2) f(x_2) \geq f(x_1)$$

then $f(x)$ is increasing
on the interval (\quad)

$$\forall x \in (x_1, x_2) f(x_2) \leq f(x_1)$$

then $f(x)$ is decreasing
on the interval (\quad)

6. Operations on Functions

Operation	Symbols	Example if $f(x) = x+3$, $g(x) = 2x-7$
Addition / Subtract	$(f \pm g)(x) = f(x) \pm g(x)$	$(f \pm g)(x) =$
Multiplication	$(f * g)(x) = f(x)g(x)$	$(f * g)(x) =$
Division	$\left(\frac{f}{g}\right)(x), g(x) \neq 0 = f(x)/g(x)$	$\left(\frac{f}{g}\right)(x) =$
Composition of Functions	$(f \circ g)(x) = f(g(x))$ The domain of f must = the range of g	$(f \circ g)(x) = (2x-7) + 3$

7. Working Definition for a One to One function:

A function is 1 to 1 iff $\forall a, b \in \text{Domain } f(a) = f(b) \rightarrow a=b$

Ex 1: Prove $f(x) = x^2 + 5$ is not 1 to 1

Proof (Counter example):

Show:

$$\exists a, b \in \text{Domain } f(a) = f(b) \wedge a \neq b$$

Consider:

$$x = 5 \rightarrow f(5) = 30$$

$$x = -5 \rightarrow f(-5) = 30$$

Ex 2: Prove $f(x) = 3x + 7$ is 1 to 1

Proof (Direct):

Assume $f(x_1) = f(x_2)$, x_1, x_2 any elements $\in \text{Domain}$

Show $x_1 = x_2$

Consider:

Definition of given function

Algebra

Definition of 1 to 1 function

8. Working Definition for an Onto function:

Definition: A function is Onto iff $\forall y \exists x, y \in \text{Codomain}, x \in \text{Domain}, y = f(x)$

(All elements y of the Codomain are images of some element x in the Domain)

Ex 1: Prove $f(x) = x^2$ does not map Domain \mathbf{Z}^+ Onto the Codomain \mathbf{Z}^+ .

Proof (Counter example):

Show:

$$\exists y \forall x, y \in \text{Codomain}, x \in \text{Domain} \wedge y \neq f(x)$$

Consider:

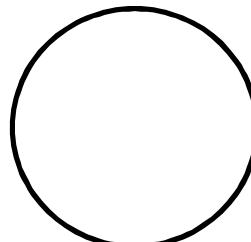
$y=2 \in \text{Codomain } \mathbf{Z}^+$

We need an $x \in \text{Domain}$ such that $f(x)=2$

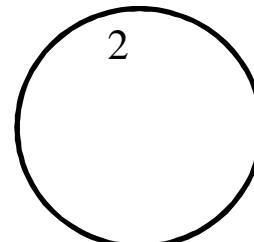
But $x=$ is not in Domain \mathbf{Z}^+

QED

Geometrically:



Domain \mathbf{Z}^+



Codomain \mathbf{Z}^+

Note: $y=9$ does not provide a counter example because $y=9$ is in the range of the f . If we use $x=3$ then $f(3)=9$

Ex 2: Prove $f(x) = 3x+7$ maps Domain \mathbf{R} Onto the Codomain \mathbf{R} .

Proof: (Constructive Existence)

Consider any $y = b \in$ in Codomain \mathbf{R}

Show $\exists x \in$ Domain such that $f(x) = b$?

Consider

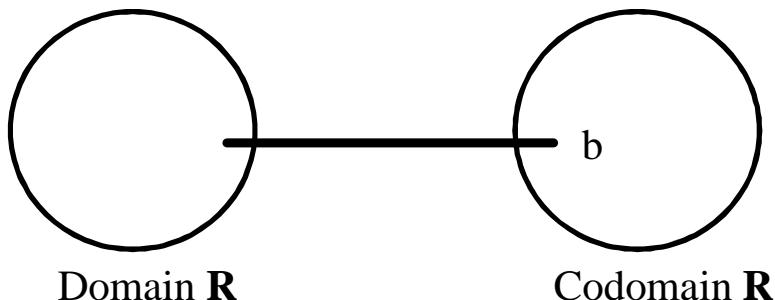
Note: Using backward reasoning:

If we want $b = 3x+7$ for some x in Domain
solve for x to get $x=b/3-7/3$

Since Codomain = Domain = \mathbf{R}

$b \in$ Codomain = $\mathbf{R} \rightarrow b \in$ Domain = \mathbf{R}
and Subtraction, Division are closed in \mathbf{R}
 $\rightarrow \frac{b}{3} - \frac{7}{3} \in$ Domain \mathbf{R}

Geometrically:



9. Using a One to One Correspondence to determine if two sets have the same Cardinality

- a) Recall that the Cardinality of a set describes How many elements are in the set

Ex: Show that $A = \{1,2,3\}$, $B = \{a,b,c\}$ have the same Cardinality:

Method 1: $n(A) = 3$, $n(B) = 3 \rightarrow A, B$ have the same Cardinality

Method 2: Determine if there is a 1 to 1 correspondence (1 to 1 and onto) between A, B

$1 \leftrightarrow a$

$2 \leftrightarrow b$

$3 \leftrightarrow c$

This establishes a 1 to 1 correspondence between A, B
 $\rightarrow A, B$ have the same Cardinality.

- b) Method 2 provides a way to determine whether infinite sets C, D have the same Cardinality

We cannot use Method 1: there is no Integers to describe $n(C), n(D)$ when C, D are infinite

Ex: Show that Z^+ and the set of odd positive integers have the Same Cardinality

Consider $n \in Z^+$ and form all $d \in$ odd positive integers using $d = f(n) = 2n - 1$

(i) $\forall n_1 \forall n_2 \in Z^+ \wedge f(n_1) = f(n_2) \rightarrow 2n_1 - 1 = 2n_2 - 1 \rightarrow f$ is 1 to 1

(ii) All positive odd integers d can be determined by using $d = 2n - 1$

\rightarrow all positive odd integers are images of an element of $Z^+ \rightarrow f$ is onto

(iii) $1 \leftrightarrow 1$
 $2 \leftrightarrow 3$
 $3 \leftrightarrow 5$
 $4 \leftrightarrow 7$

This shows the 1 to 1 correspondence between Z^+ and odd+ integers
 $\rightarrow Z^+$ and the positive odd integers have the same Cardinality.

- c) Definition: Set S is Countable iff: (i) S is Finite OR

(ii) S is Infinite and it has same Cardinality as Z^+

Ex : Show that \mathbb{Z} is Countable

We require that \mathbb{Z}^+ and \mathbb{Z} have the Same Cardinality

Consider $n \in \mathbb{Z}^+$ and form all $d \in \mathbb{Z}$ using $d = f(n) =$

if n is even

if n is odd

(i) It can be shown that f is 1 to 1 (each branch of f is linear)

(ii) All integers d can be determined by using $f(n)$

→ f maps \mathbb{Z} onto the positive odd Integers

$$1 \leftrightarrow 0$$

$$2 \leftrightarrow 1$$

$$3 \leftrightarrow -1$$

$$\dots \leftrightarrow \dots$$

This shows the 1 to 1 correspondence between \mathbb{Z}^+ and \mathbb{Z}

→ \mathbb{Z} has same Cardinality as \mathbb{Z}^+

→ \mathbb{Z} is Countable

(iii)

10 Given a function $y = f(x)$ find its inverse function $f^{-1}(x)$

a) Method:

Step 1 Replace x by y and y by x

Step 2 Solve for y

Ex 1: If $f(x) = 3x + 5$ find $f^{-1}(x)$

$y = 3x + 5$	
$\rightarrow x = 3y+5$	Replace x by y and y by x
	Algebra: solve for y
	Notation for Inverse function

b) Note:

Ex 2: Is $g(x) = x^2 - 5$ the inverse of $f(x) = \sqrt{x + 5}$

Step 1 Is $f(x)$ one to one ?

Consider $f(x_1) = f(x_2) \rightarrow$

Hence: f is one to one

Step 2 Is $g(x) = f^{-1}(x)$? We must check to see if: $g(f(x)) = x$

Consider $g(x) = x^2 - 5 \rightarrow g(f(x)) =$

Hence: $g(x) =$

11 a) Definition of Two Special Functions

(i) The Floor function of x is the largest integer less than or equal to x

(Floor function of x rounds x down to nearest Integer)

Also called Least Integer Function

Notation: $f(x) =$

(ii) The Ceiling function of x is the smallest integer greater than or equal to x

(Ceiling function of x up to the nearest Integer)

Aslo called Greatest Integer Function

Notation: $f(x) =$

Ex: $[3.5] = 3$

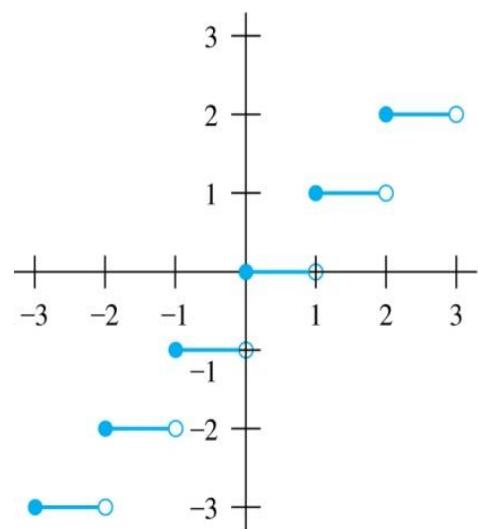
$$[-1.5] = -2$$

$$[3.5] = 4$$

$$[-1.5] = -1$$

b) Graphs of Floor and Ceiling Functions

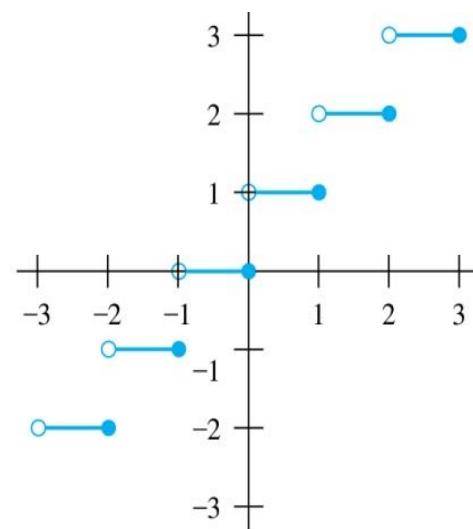
Floor Function



$$y = \lfloor x \rfloor$$

Ex: $\forall x: x \in \mathbb{R}, 2 \leq x < 3,$

Ceiling Function



$$y = \lceil x \rceil$$

Ex: $\forall x: x \in \mathbb{R}, 2 < x \leq 3,$

c) Recall: $\forall x \in \mathbb{R}, x = n + d, n \in \mathbb{Z}, d \in \mathbb{R}$

Ex: $3.14159 = 3 + .14159$

d) Proofs with Floor / Ceiling Functions

Ex 1: Prove that if x is a Real number, then $\lfloor 2x + 1 \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$

Proof: Consider $x = n + d$ where n is the integer part and d is the decimal part $\rightarrow 0 \leq d < 1$
 (By Cases)

Case 1: $0 < d \leq \frac{1}{2}$

$$\text{LHS} = \lfloor 2x + 1 \rfloor = \lfloor 2(n + d) + 1 \rfloor =$$

Since $0 < d \leq \frac{1}{2}$

$$\rightarrow 0 < 2d \leq 1$$

$$\rightarrow 1 < 2d + 1 \leq 2$$

$$\text{RHS} = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor = \lfloor n + d \rfloor + \lfloor n + (d + \frac{1}{2}) \rfloor = (n+1) + (n+1)$$

Since $0 < d \leq \frac{1}{2}$

$$\rightarrow 1/2 < (d+1/2) \leq 1$$

$$\rightarrow \text{LHS} = \text{RHS}$$

Case 2: $\frac{1}{2} < d < 1$

$$\text{LHS} = \lfloor 2x + 1 \rfloor = \lfloor 2(n + d) + 1 \rfloor =$$

Since $\frac{1}{2} < d < 1$

$$\text{RHS} = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor = \lfloor n + d \rfloor + \lfloor n + (d + \frac{1}{2}) \rfloor =$$

Since $\frac{1}{2} < d < 1$

$$\rightarrow \text{LHS} = \text{RHS}$$

Ex 2: Prove that if x is a Real number, then $[2x] = [x] + \lfloor x + \frac{1}{2} \rfloor$

Proof: Let $x = n + d$ where n is the integer part and d is the decimal part $\rightarrow 0 \leq d < 1$.
(Cases)

Case 1: $0 \leq d < \frac{1}{2}$

Case 2: $\frac{1}{2} \leq d < 1$

e) Summary Floor and Ceiling Function values

TABLE 1 Useful Properties of the Floor and Ceiling Functions.

(n is an integer, x is a real number)

(1a) $\lfloor x \rfloor = n$ if and only if $n \leq x < n + 1$

(1b) $\lceil x \rceil = n$ if and only if $n - 1 < x \leq n$

(1c) $\lfloor x \rfloor = n$ if and only if $x - 1 < n \leq x$

(1d) $\lceil x \rceil = n$ if and only if $x \leq n < x + 1$

(2) $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$

(3a) $\lfloor -x \rfloor = -\lceil x \rceil$

(3b) $\lceil -x \rceil = -\lfloor x \rfloor$

(4a) $\lfloor x + n \rfloor = \lfloor x \rfloor + n$

(4b) $\lceil x + n \rceil = \lceil x \rceil + n$

Exercise 1: Let g be the function from the set $\{a,b,c\}$ to itself such that

$g(a) = b$, $g(b) = c$, and $g(c) = a$. Let f be the function from the set $\{a,b,c\}$ to the set $\{1,2,3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$.

- a) What is the composition of f and g ?
- b) What is the composition of g and f ?

Exercise 2: Prove (direct method) that for all $x \in \mathbb{R}$, $a \in \mathbb{Z}$

$$\lfloor x+a \rfloor = \lfloor x \rfloor + a$$

Exercise 3: Prove (by cases) that for all Real numbers x

$$\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$$

Hint : Let $x = n + d$. Note n is the Integer part and d is decimal part.

Case 1 $0 \leq d < \frac{1}{2}$

Case 2 $\frac{1}{2} \leq d < 1$