11.1 Sequences

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Definitions & Theorems:

1. Definition: Sequences

   A sequence is a function whose domain is a subset of $\mathbb{N}$.

   We write $f(1) = a_1, f(2) = a_2, f(3) = a_3, \ldots$

   The sequence $\{a_1, a_2, a_{3,\ldots}\}$ is also denoted by $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$

   Some sequences can be defined by giving a formula for their n_{th} form.

   Ex:

   1) Even positive integers $\{2, 4, 6, 8, \ldots\} = \{2n\}_{n=1}^{\infty}$
   2) Odd positive integers $\{1, 3, 5, 7, \ldots\} = \{2n-1\}_{n=1}^{\infty}$
   3) $\left\{-\frac{1}{2}, \frac{2}{4}, -\frac{6}{8}, \frac{24}{16}, -\frac{120}{32}, \ldots\right\} = \left\{(-1)^n \frac{n!}{2^n}\right\}_{n=1}^{\infty}$

2. Definition

   A sequence $\{a_n\}$ has the limit L and we write

   $\lim\limits_{n\to\infty} a_n = L$ or $a_n \to L \ as \ n \to \infty$

   if we make the terms a_n as close to L as we like by taking n sufficiently large.

   If $\lim\limits_{n\to\infty} a_n$ exsits, we say the sequence converges (or is convergent). Otherwise, we say the sequence diverges (or is divergent).

3. Definition

   A sequence $\{a_n\}$ has the limit L and we write

   $\lim\limits_{n\to\infty} a_n = L$ or $a_n \to L \ as \ n \to \infty$

   if for every $\varepsilon > 0$ there is a corresponding integer N such that

   if $n > N$ then $|a_n - L| < \varepsilon$

4. Theorem:

   If $\lim\limits_{n\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim\limits_{n\to\infty} a_n = L$

5. Theorem:

   $$\lim_{n\to\infty} \frac{1}{n^r} = 0 \ \text{ if } r > 0$$

6. Definition

   $\lim\limits_{n\to\infty} a_n = \infty$ means that for every positive number M there is an integer N such that

   if $n > N$ then $a_n > M$

7. Properties

   If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

   $$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$
   $$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$
   $$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n$$
   $$\lim_{n\to\infty} c = c$$
   $$\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n * \lim_{n\to\infty} b_n$$
   $$\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n} \ \text{ if } \lim_{n\to\infty} b_n \neq 0$$
   $$\lim_{n\to\infty} [a_n]^p = \left[\lim_{n\to\infty} a_n\right]^p \ \text{ if } p > 0 \text{ and } a_n > 0$$

8. Theorem: Squeeze Theorem

   If $a_n \le b_n \le c_n$ for $n \ge n_0$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, then $\lim\limits_{n\to\infty} b_n = L$

9. Theorem

   If $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$

10. Theorem

    If $\lim\limits_{n\to\infty} a_n = L$ and the function f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L)$$

11. The sequence $\{r^n\}$ is convergent if $-1 < r \le 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

12. Definition

    A sequence $\{a_n\}$ is called incresing if $a_n < a_{n+1}$ for all $n \ge 1$, that is, $a_1 < a_2 < a_3 < \cdots$. It is called decreasing if $a_n < a_{n+1}$ for all $n \ge 1$. It is called **monotonic** if it is either increasing or decreasing.

13. Definition

    A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

    $$a_n \le M \qquad \text{for all } n \ge 1$$

    It is **bounded below** if there is a number m such that

    $$m \le a_n \qquad \text{for all } n \ge 1$$

    If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.

14. Theorem: Monotonic Sequence Theorem

    Every bounded, monotonic sequence is convergent.

Examples:

1. $\left\{\dfrac{2n}{n-16}\right\}_{n=17}^{\infty}$

$$\lim_{n\to\infty}\frac{2n}{n-16} = \lim_{n\to\infty}\frac{(n)2}{(n)\left(1-\frac{16}{n}\right)} = \lim_{n\to\infty}\frac{2}{1-\frac{16}{n}} = 2$$

$$\Rightarrow \left\{\frac{2n}{n-16}\right\}_{n=17}^{\infty} \quad converges$$

2. $\{(-1)^n\}_{n=1}^{\infty}$

$$\lim_{n\to\infty}(-1)^n = \begin{cases} 1, n \text{ is even} \\ -1, n \text{ is odd} \end{cases}$$

$$\Rightarrow \lim_{n\to\infty}(-1)^n \text{ does not exist}$$

$$\Rightarrow \{(-1)^n\}_{n=1}^{\infty} \; diverges$$

3. $$\lim_{n\to\infty}\left(n-\sqrt{n^2-n}\right) = \lim_{n\to\infty}\frac{\left(n-\sqrt{n^2-n}\right)\left(n+\sqrt{n^2-n}\right)}{n+\sqrt{n^2-n}} = \lim_{n\to\infty}\frac{n}{n+\sqrt{n^2-n}} = \lim_{n\to\infty}\frac{1}{1+\sqrt{1-\frac{n}{1}}} = \frac{1}{2}$$

4. $\displaystyle\lim_{n\to\infty}\frac{\ln n}{n}$

$$\lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\frac{(\ln x)'}{x'} = \lim_{x\to\infty}\frac{1}{x} = 0$$ why we need this step?

$$\Rightarrow \lim_{n\to\infty}\frac{\ln n}{n} = 0$$

($\lim_{n\to\infty}\frac{\ln n}{n}$ is the limit of a sequence, $\lim_{x\to\infty}\frac{\ln x}{x}$ is the limit of a function. We do not have **L'Hospital's Rule** for sequences but we do have for functions.)

5. $\displaystyle\lim_{n\to\infty}\frac{e^{2n}}{n}$

$$\lim_{x\to\infty}\frac{e^{2x}}{x} = \lim_{x\to\infty}\frac{2e^{2x}}{1} = \infty$$

$$\Rightarrow \lim_{n\to\infty}\frac{e^{2n}}{n} = \infty$$

$$\Rightarrow \left\{\frac{e^{2n}}{n}\right\}_{n=1}^{\infty} \quad diverges$$

6. $\displaystyle\lim_{n\to\infty}\frac{\arctan n}{n}$ $(Theorem\ 8)$

$$n \ge 1$$

$$\Rightarrow 0 \le \arctan n \le \frac{\pi}{2}$$

$$\Rightarrow \frac{0}{n} \le \frac{\arctan n}{n} \le \frac{\frac{\pi}{2}}{n}$$
$$\Rightarrow 0 \le \frac{\arctan n}{n} \le \frac{\pi}{2n}$$
$$\lim_{n\to\infty} 0 = 0, \lim_{n\to\infty} \frac{\pi}{2n} = 0 \Rightarrow \lim_{n\to\infty} \frac{\arctan n}{n} = 0$$

7. $\lim_{n\to\infty} \frac{(-1)^n}{n}$ $(Theorem\ 9)$
$$\lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty} \frac{1}{n} = 0$$
$$\Rightarrow \lim_{n\to\infty} \frac{(-1)^n}{n} = 0$$

8. $\lim_{n\to\infty} \tan\frac{\pi}{n}$ $(Theorem\ 10)$
$$\lim_{n\to\infty} \tan\frac{\pi}{n} = \tan(\lim_{n\to\infty}\frac{\pi}{n}) = \tan 0 = 0$$

9. $\lim_{n\to\infty} [\ln(n+1) - \ln n] (Theorem\ 10)$
$$\lim_{n\to\infty} [\ln(n+1) - \ln n] = \lim_{n\to\infty}\left[\ln\frac{n+1}{n}\right] = \ln\left[\lim_{n\to\infty}\frac{n+1}{n}\right] = \ln 1 = 0$$

10. Does $\{\sqrt[n]{n}\}_{n=1}^{\infty}$ converge?
$$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} n^{\frac{1}{n}} = e^{\ln\left(\lim_{n\to\infty} n^{\frac{1}{n}}\right)} = e^{\lim_{n\to\infty}\left(\ln n^{\frac{1}{n}}\right)} = e^{\lim_{n\to\infty}\left(\frac{1}{n}\right)(\ln n)} = e^{\lim_{n\to\infty}\frac{\ln n}{n}} = e^0 = 1$$

11. For what values of p, does $\int_1^{\infty} \frac{\ln x}{x^p} \mathrm{d}x$ converge?
$$\int_1^{\infty} \frac{\ln x}{x^p} \mathrm{d}x = \lim_{t\to\infty} \int_1^{t} \frac{\ln x}{x^p} \mathrm{d}x$$
Let $u = \ln x \Rightarrow du = \frac{1}{x} dx, x = e^u$
$$\int_1^{\infty} \frac{\ln x}{x^p} \mathrm{d}x = \lim_{t\to\infty} \int_1^{t} \frac{\ln x}{x^P} \mathrm{d}x = \lim_{t\to\infty} \int_1^{t} \frac{\ln x}{x^{p-1}} (\frac{\mathrm{d}x}{x}) = \lim_{t\to\infty} \int_1^{t} \frac{u}{e^{u^{P-1}}} \mathrm{d}u$$

12. $\int_2^{\infty} \frac{1}{x^p \ln x} \mathrm{d}x$

13. Bounded, Monotonic and Convergent
    a. Bounded⇒Convergent?
       False: $\{(-1)^n\}_{n=0}^{\infty}$
    b. Monotonic⇒Convergent?
       False: $\{n\}_{n=0}^{\infty}$
    c. Convergent⇒Bounded?
       True
    d. Convergent⇒Monotonic?
       False: $\left\{\frac{\sin(n)}{n}\right\}_{n=0}^{\infty}$

14. Is $\{\frac{n}{n^2+1}\}$ bounded/monotonic?
    It converges⇒it's bounded.
    Let $f(x) = \frac{x}{x^2+1} \Rightarrow f'(x) = \frac{1-x^2}{(1+x^2)^2} \Rightarrow f'(x) < 0\ for\ x > 1 \Rightarrow f(n) = a_n\ is\ decreasing\ for\ n > 1$
    Since $\lim_{n\to\infty} \frac{n}{n^2+1} = 0$, and $\left\{\frac{n}{n^2+1}\right\}_{n=1}^{\infty}$ is decreasing⇒ $0 < a_n \le \frac{1}{2}$