Sec 1.4 Predicates and Quantifiers

Comp 232
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1 a) Definition of Propositional Function or a Predicate.

Let p represent $2 + 7 = 9$
p can be judged to be T or F

Hence p is called a proposition

Let Q(x,y) represent $x = y+3$
Q(x,y) cannot be judged to be T or F until we know values for x,y

Q(x,y) is called a Propositional Function or a Predicate
If we assign values for x, y we get a proposition

Ex: Q(1,2) represents $1 = 2 + 3$ which is F
Q(5,2) represents $5 = 2 + 3$ which is T

b) Definition: The Domain of a Predicate variable is the set of all possible values of the variable

2 a) Definition: A Quantifier designates a specific quantity of domain values

Name	Symbol	Description	Example
Universal	$\forall$	All elements	$\forall x\, P(x)$ means for all x in the Domain, P(x) = T
Existential	$\exists$	One or more	$\exists x\, Q(x)$ means there is one or more x in the Domain that makes Q(x) = T
Uniqueness	$\exists!$	Exactly one	$\exists! x\, R(x)$ means there is exactly one (unique) value of x in the Domain that makes R(x) = T

If ∀ equivalence is T, then ∃ is also T; if ∃ equivalence is F, then ∀ is also F.

2 b) Sometimes the quantifier itself is restricted:

Ex: Let P(x) represent "$x * \frac{1}{x} = 1$" and the domain of P(x) be all Real numbers

$\forall x\, P(x) = F$ Why ? P(x) = F when x=0

Note: $\forall x\ x \neq 0 : P(x) = T$

3. Order of operations: $\forall$, $\exists$ have same priority (leftmost first) then $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ as before.

Ex : Place brackets to signify the order intended in: $\forall x\, P(x) \vee Q(x) \wedge \exists x\, R(x)$

$[\forall x P(x)] \vee [Q(x) \wedge [\exists x R(x)]]$

4 a) Quantifiers are equivalent iff they have the same truth value even if predicates are changed.

Ex :

Two equivalent quantifiers:	Two non equivalent quantifiers
$\forall x\,[P(x) \wedge Q(x)] \equiv \forall x\, P(x) \wedge \forall x\, Q(x) = T$	$\exists x\,[P(x) \wedge Q(x)] \equiv \exists x\, P(x) \wedge \exists x\, Q(x) = F$
$\exists x\,[P(x) \vee Q(x)] \equiv \exists x\, P(x) \vee \exists x\, Q(x) = T$	$\forall x\,[P(x) \vee Q(x)] \equiv \forall x\, P(x) \vee \forall x\, Q(x) = F$

b) For equivalence show: LHS $\leftrightarrow$ RHS (LHS = T if and only if RHS = T)

There are two steps required: Step1: LHS $\rightarrow$ RHS
Step2: RHS $\rightarrow$ LHS

c) For non equivalence find a Counter Example:

One set of Predicates where: LHS, RHS have different Truth values.

Ex 1 : Show Equivalence: $\forall x\, [P(x) \wedge Q(x)] \equiv \forall x\, P(x) \wedge \forall x\, Q(x) = \text{True}$

Let a represent any x value in the domain of P, Q

Step 1 Show LHS $\rightarrow$ RHS

Assume $\forall x\, [P(x) \wedge Q(x)] = T$	
$[P(a) \wedge Q(a)] = T$	Definition of $\forall$
$\rightarrow P(a) = T \wedge Q(a) = T$	Truth Table for And
$\rightarrow \forall x P(x) = T \wedge \forall x Q(x) = T$	Definition of $\forall$
$\rightarrow [\forall x P(x) \wedge \forall x Q(x)] = T$	Truth Table for And
Hence LHS $\rightarrow$ RHS	

Step 2 Show RHS $\rightarrow$ LHS

Assume $[\forall x\, P(x) \wedge \forall x\, Q(x)] = T$	
$\rightarrow \forall x P(x) = T \wedge \forall x Q(x) = T$	Truth Table for And
$\rightarrow P(a) = T \wedge Q(a) = T$	Definition of $\forall$
$\rightarrow [P(a) \wedge Q(a)] = T$	Truth Table for And
$\rightarrow \forall x [P(x) \wedge Q(x)] = T$	Definition of $\forall$
Hence RHS $\rightarrow$ LHS	

QED

Ex 2 : Show Non Equivalence: $\exists x\, [P(x) \wedge Q(x)] \equiv \exists x\, P(x) \wedge \exists x\, Q(x)$ = False

Consider $P(x)$ as $x \geq 0$, $Q(x)$ as $x < 0$, Domain all Real numbers

Step 1 Show LHS = F

Either $\exists x\, [P(x) \wedge Q(x)] = T$ or $\exists x\, [P(x) \wedge Q(x)] = F$

Assume $\exists x\, [P(x) \wedge Q(x)] = T$ and let $x = a$

$\rightarrow P(a) \wedge Q(a) = T$ — Substitute

$\rightarrow a \geq 0 \wedge a < 0$ — Definition of P, Q

$\rightarrow$ Contradiction — Property of Real numbers

$\rightarrow \exists x\, [P(x) \wedge Q(x)] = F$ — Only possibility remaining for LHS

Step 2 Show RHS = T

$P(1) = T \rightarrow \exists x P(x) = T$ — Definition of $\exists$

$Q(-1) = T \rightarrow \exists x Q(x) = T$ — Definition of $\exists$

$\rightarrow \exists x P(x) \wedge \exists x Q(x) = T$ — Truth Table for And

QED

There is at least one case where LHS, RHS have different Truth values

5 a) Negation of Quantifiers

Quantified Predicate	Describe when P(x) True	Negate Quantifier	Describe when Quantified Predicate is False
$\forall x\, P(x)$	For all x in Domain P(x) = T	$\neg\forall x P(x)$	Not [for all x in Domain P(x) = T] $\rightarrow$ there is at least one x in Domain where P(x) = F This can be written: $\exists x \neg P(x)$ **Summary:** $\neg\forall x P(x) \equiv \exists x \neg P(x)$
$\exists x\, P(x)$	For at least one x in Domain P(x) = T	$\neg\exists x P(x)$	Not [For at least one x in Domain P(x) = T] $\rightarrow$ for all x in Domain P(x) = F This can be written: $\forall x \neg P(x)$ **Summary:** $\neg\exists x P(x) \equiv \forall x \neg P(x)$

5 b) The two summary statements above are called De Morgan's rules for quantifiers

Ex : Show Equivalence: $\neg \forall x\,[P(x) \rightarrow Q(x)] \equiv \exists x\,[P(x) \wedge \neg Q(x)] = T$

Step 1 Show LHS $\rightarrow$ RHS

Assume $\neg \forall x\,[\,P(x) \rightarrow Q(x)\,] = T$	
$\rightarrow \exists x \neg [P(x) \rightarrow Q(x)] = T$	De Morgan's rule for Quantifiers
$\rightarrow \exists x \neg [\neg P(x) \vee Q(x)] = T$	Implication in terms of Or
$\rightarrow \exists x [\neg\neg P(x) \wedge \neg Q(x)] = T$	De Morgan's rule (regular)
$\rightarrow \exists x [P(x) \wedge \neg Q(x)] = T$	Double negation
Hence LHS $\rightarrow$ RHS	

Step 2 Show RHS $\rightarrow$ LHS

Assume $\exists x\,[P(x) \wedge \neg Q(x)] = T$	
$\rightarrow \exists x [\neg\neg P(x) \wedge \neg Q(x)] = T$	Double negation
$\rightarrow \exists x \neg [\neg P(x) \vee Q(x)] = T$	De Morgan's rule (regular)
$\rightarrow \exists x \neg [P(x) \rightarrow Q(x)] = T$	Implication in terms of Or
$\rightarrow \neg \forall x [P(x) \rightarrow Q(x)] = T$	De Morgan's rule for Quantifiers
Hence RHS $\rightarrow$ LHS	

QED