5.3 The Fundamental Theorem of Calculus

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Definitions & Theorems:

1. Theorem: THE FUNDMETAL THEOREM OF CALCULUS

   Suppose f is continuous on $[a, b]$,

   1) If
   $$g(x) = \int_a^x f(t)\,\mathrm{d}t$$
   then
   $$g'(x) = f(x)$$
   2) $\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$

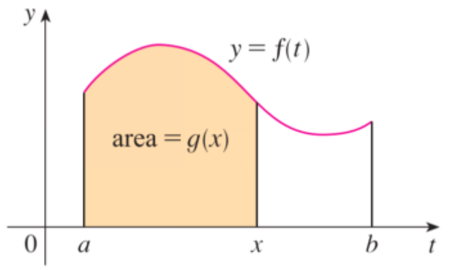
      where F is any antiderivative of f, that is, a function such that $F' = f$.

   The FTC
   1) provides a relationship between definite and indefinite integrals.
   2) gives the precise inverse relation between the derivative and the integral.
   3) provides us with a third method (the first being interpretation as an signed area; the second being interpretation as a limit of Riemann sums) of calculating definite integrals.

Proofs or Explanations:

1. Theorem 1)

   This theorem deals with functions defined by an equation of the form
   $$g(x) = \int_0^x f(t)\,dt$$
   where f is a continuous function on $[a, b]$



Examples:

1. Find $g'(x)$ where $g(x) = \int_2^x \frac{e^{t^2}}{2t^2+1} dt$
$$g'(x) = \frac{d}{dx}\int_2^x \frac{e^{t^2}}{2t^2+1}dt = \frac{e^{x^2}}{2x^2+1}$$

2. Find $g'(x)$ where $g(x) = \int_x^0 \cos t\, dt$
$$g'(x) = \frac{d}{dx}\int_x^0 \cos t\, dt = -\frac{d}{dx}\int_0^x \cos t\, dt = -\cos x$$

3. $\frac{d}{dx}\int_{2}^{x^2+1}\frac{t+1}{e^{2t}}dt=\frac{d}{du}\left(\int_{2}^{u}\frac{t+1}{e^{2t}}dt\right)\frac{du}{dx}=\frac{u+1}{e^{2u}}\frac{du}{dx}=\frac{x^2+1+1}{e^{2(x^2+1)}}\frac{d(x^2+1)}{dx}=\frac{x^2+2}{e^{2x^2+2}}(2x)$

4. Find the derivative of $\int_{-x}^{x}e^t\,dt$

$$f(x)=\int_{-x}^{x}e^t\,dt$$

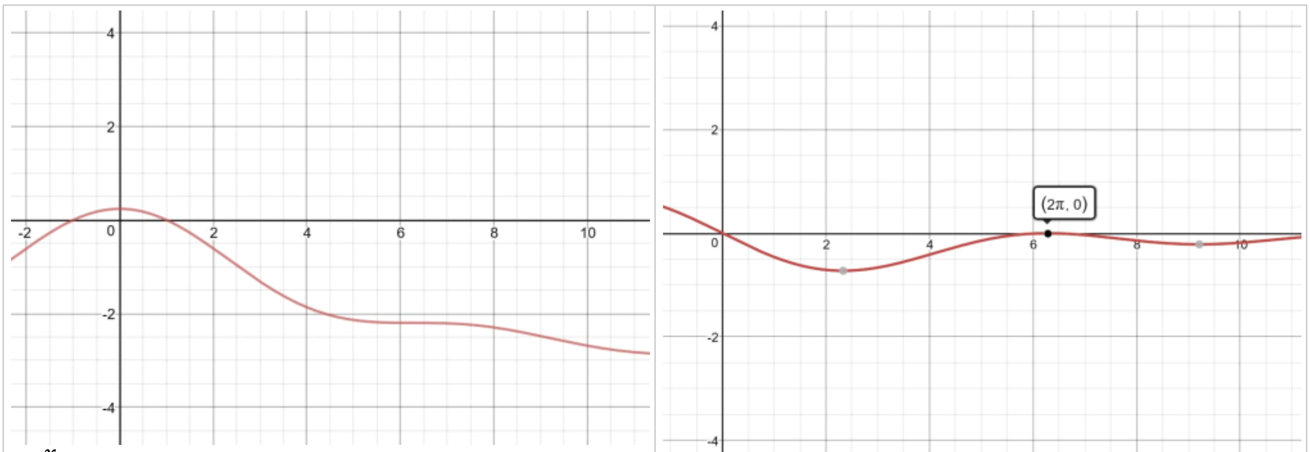
Method1: $f'(x)=F'(b)-F'(a)\rightarrow f'(x)=F'(x)-F'(-x)(-x)'\rightarrow f'(x)=F'(x)+F'(-x)\rightarrow$
$f'(x)=e^x+e^{-x}$

Method2: $f'(x)=\frac{d}{dx}\int_{-x}^{x}e^t\,dt=\frac{d}{dx}\int_{-x}^{0}e^t\,dt+\frac{d}{dx}\int_{0}^{x}e^t\,dt=-\frac{d}{dx}\int_{0}^{-x}e^t\,dt+\frac{d}{dx}\int_{0}^{x}e^t\,dt=$
$-\frac{d}{du}\left(\int_0^u e^t\,dt\right)\frac{du}{dx}+\frac{d}{dx}\int_0^x e^t\,dt=-e^u\frac{du}{dx}+e^x=-e^{(-x)}\frac{d(-x)}{dx}+e^x=e^{(-x)}+e^x$

5. Determine the local extrema of $g(x)=\int_{1}^{x}\frac{\cos t-1}{t}dt$ on $[1,3\pi]$.

$$g'(x)=\frac{d}{dx}\int_{1}^{x}\frac{\cos t-1}{t}=\frac{\cos x-1}{x}=0\rightarrow x=2\pi$$

the critical number on $[1,3\pi]$ is at 2π.



$\int_{1}^{x}\frac{\cos t-1}{t}$ $\qquad$ $\frac{\cos x-1}{x}$

Neither a local max nor a local min at 2π, so no local extrema on $[1,3\pi]$

6. $\int_a^b x\,dx=\frac{x^2}{2}\Big]_a^b=\frac{b^2-a^2}{2}$

7. $\int_0^{17}e^x\,dx=e^x]_0^{17}=e^{17}-e^0=e^{17}-1$

8. $\int_{-1}^{0}(-x^2-x)\,dx=(-\frac{x^3}{3}-\frac{x^2}{2})\Big]_{-1}^{0}=-\left(\frac{1}{3}-\frac{1}{2}\right)=\frac{1}{6}$

9. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos x\,dx=\sin x]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=2$

10. Find the area beneath $\frac{1}{x}$ from 2 to 4

$$\int_2^4\frac{1}{x}dx=\ln|x|]_2^4=\ln 4-\ln 2=\ln\frac{4}{2}=\ln 2$$