MATH 251 Midterm Fall 2014 Solutions

① (a) Let $v \in \operatorname{span}(A \cap B)$. Then

$$v = a_1 v_1 + a_2 v_2 + \dots + a_n v_n$$

where $v_1, \dots, v_n \in A \cap B$ and $a_1, \dots, a_n \in \mathbb{R}$. Since $v_1, \dots, v_n \in A$, $v \in \operatorname{span} A$. Since $v_1, \dots, v_n \in B$, $v \in \operatorname{span} B$.

Hence $v \in (\operatorname{span} A) \cap (\operatorname{span} B)$. Since v was arbitrary, we conclude

$$\operatorname{span}(A \cap B) \subseteq (\operatorname{span} A) \cap (\operatorname{span} B).$$

(b) Let $A = \{ (x, 0) \mid x \in \mathbb{R} \}$, $B = \{ (0, y) \mid y \in \mathbb{R} \}$.

Then $\operatorname{span}(A \cup B) = \mathbb{R}^2$ since $A \cup B$ contains ~~the~~ a basis $\{(1,0), (0,1)\}$ of $\mathbb{R}^2$. But $\operatorname{span} A = A$, $\operatorname{span} B = B$, so

$$\operatorname{span} A \cup \operatorname{span} B = A \cup B$$

Since $(1,1) \notin A$, $(1,1) \notin B$, this vector is <u>not</u> in $A \cup B$ & hence ~~span A ∪ span B ≠~~ $\mathbb{R}^2 = \operatorname{span}(A \cup B) \not\subseteq \operatorname{span} A \cup \operatorname{span} B$.

② (a) (i) The zero polynomial $0(x)$ satisfies $0(5) = 0$, so $0 \in W$.

(ii) If $f(x), g(x) \in W$ then $f(5) = 0$, $g(5) = 0$ so

$$(f+g)(5) = f(5) + g(5) = 0 + 0 = 0$$

hence $f + g \in W$.

(iii) If $f \in W$ and $c \in \mathbb{R}$ then $f(5) = 0$ so

$$(cf)(5) = c\, f(5) = c \cdot 0 = 0$$

So $cf \in W$.

By (i), (ii), (iii), W is a subspace.

(b) Let $f_1(x) = x-5$, $f_2(x) = x(x-5)$. Clearly $f_1, f_2 \in W$, and $\{f_1, f_2\}$ is linearly independent since f_1, f_2 have different degrees. Since $W \neq P_3(\mathbb{R})$ ($x \notin W$, for example), $\dim W \leq 2$. Since $\{f_1, f_2\}$ is linearly independent subset of W, $\dim W \geq 2$. So $\dim W = 2$, and therefore $\{x-5,\ x(x-5)\}$ is a basis.

③ (a) Since $R(T) \subseteq \mathbb{R}$, it has dimension 0 or 1. It is not 0, since $T(1) = \int_0^1 dt = t\Big|_0^1 = 1 \neq 0$. Hence $R(T) = \mathbb{R}$, $\operatorname{rank} T = 1$.

By Dimension Theorem,

$$\dim V = \operatorname{rank}(T) + \operatorname{nullity}(T)$$
$$3 = 1 + \operatorname{nullity}(T)$$

hence $\operatorname{nullity}(T) = 2$.

(b) It suffices to find two linearly independent polynomials satisfying $0 = T(f(x)) = \int_0^1 f(t)\,dt$.

We can use $x - 1/2$ and $x^2 - 1/3$ since

$$\int_0^1 (x - 1/2)\,dx = \frac{x^2}{2} - \frac{1}{2}x\Big|_0^1 = \frac{1}{2} - \frac{1}{2} = 0$$

$$\int_0^1 (x^2 - 1/3)\,dx = \frac{x^3}{3} - \frac{1}{3}x\Big|_0^1 = \frac{1}{3} - \frac{1}{3} = 0,$$

and these are lin. indep. since they have different degrees.

So $\{x - 1/2,\ x^2 - 1/3\}$ is a basis of $N(T)$.

Note: We could instead find a basis by solving $0 = \int_0^1 (a + bx + cx^2)\,dx = a + \frac{b}{2} + \frac{c}{3}$.

4 (a) $T(1,0) = T\left(\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}\right)$

$$= \frac{1}{2}T\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{2}T\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2\\3\end{pmatrix} + \frac{1}{2}\begin{pmatrix}-1\\2\end{pmatrix}$$

$$= \begin{pmatrix}2/2 - 1/2\\3/2 + 2/2\end{pmatrix} = \begin{pmatrix}1/2\\5/2\end{pmatrix}$$

$$T(0,1) = T\left(\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}\right)$$

$$= \frac{1}{2}T\begin{pmatrix}1\\1\end{pmatrix} - \frac{1}{2}T\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2\\3\end{pmatrix} - \frac{1}{2}\begin{pmatrix}-1\\2\end{pmatrix}$$

$$= \begin{pmatrix}2/2 + 1/2\\3/2 - 2/2\end{pmatrix} = \begin{pmatrix}3/2\\1/2\end{pmatrix}$$

Hence $[T]_\alpha = \frac{1}{2}\begin{pmatrix}1 & 3\\5 & 1\end{pmatrix}$

(b) $T(-3,2) = \frac{1}{2}\begin{pmatrix}1 & 3\\5 & 1\end{pmatrix}\begin{pmatrix}-3\\2\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-3+6\\-15+2\end{pmatrix} = \frac{1}{2}\begin{pmatrix}3\\-13\end{pmatrix} = \begin{pmatrix}3/2\\-13/2\end{pmatrix}$

5 Suppose $x_1(u+v) + x_2(v+w) + x_3(w+u) = \vec{0}$.

Then

$$(x_1 + x_3)u + (x_1 + x_2)v + (x_2 + x_3)w = \vec{0}.$$

Since $\{u, v, w\}$ is linearly independent, the 3 coefficients above are 0, i.e.

$$x_1 + x_3 = 0 \quad (1)$$
$$x_1 + x_2 = 0 \quad (2)$$
$$x_2 + x_3 = 0 \quad (3)$$

Then $x_3 = -x_1$ so (3) gives $x_2 - x_1 = 0$, so $x_2 = x_1$. Then (2) gives $2x_1 = 0$ hence $x_1 = 0$ so $x_2 = 0$ and $x_3 = 0$. Therefore, $\{u+v, v+w, w+u\}$ is linearly independent.