Outline

September 19, 2016 12:01

CONCORDIA UNIVERSITY
FACULTY OF ENGINEERING AND COMPUTER SCIENCE
APPLIED ORDINARY DIFFERENTIAL EQUATIONS - ENGR 213

Instructor: Reinaldo Rodriguez
Office: LB 901.20
Email: reinaldo.rodriguezramos@concordia.ca
Lectures: ENGR 213/2/Lec P - Mondays and Wednesdays, 11:45-13:00 - FG C080
ENGR 213/2/Lec XX - Fridays, 17:45-20:15 - H 531
Office hours: LB 915-5 on Mondays, 3:00-4:30 Engr. 213 Lec XX
LB 912 on Wednesdays, 9:30-11:00 Engr. 213 Lec P
Course coordinator: G. Vatistas
Email: vatistas@encs.concordia.ca

Textbook: Advanced Engineering Mathematics, by Dennis G. Zill and Warren S. Wright, 5th Edition, Published by Jones and Bartlett, 2014.

Grading Scheme:

Midterm exams (2)	20%, (10% each, during tutorials, 90 min, 5 problems)
Assignments (WeBWorK)	10%
Pop-up Quizzes (5)	10% (2% each, during lectures or tutorials, 20 min, 1-2 problems)
Final exam	60% (3 hours, 10 problems including 2 problems on mathematical models)
Team projects (2)	5% (2.5% each, 1 problem on mathematical models, 1 hour; during tutorials in teams of 2)

WeBWorK: Every student will be given access to an online system called WeBWorK. Students are expected to submit assignments online using WeBWorK. Late assignments will not be accepted. Assignments contribute 10% to your final grade. Working regularly on the assignments is essential for success in this course. Students are also strongly encouraged to do as many problems as their time permits from the chapters of the textbooks listed below in this outline.

The grading scheme implies 5% bonus.

PLEASE NOTE: Electronic communication devices (including cellphones) will not be allowed in examination rooms. Only "Faculty Approved Calculators" will be allowed in examination rooms [SHARP EL-531 or CASIO FX-300MS]

Topics and recommended problems:

Week 1: 1.1 Definition and Terminology; problems: 1,3,5,6,8,10,11,13,14,21,23
1.2 Initial Value Problems; problems: 7,9,11,12,17,18
2.1 Solution curves without a solution; problems: 3, 4, 26, 27
Week 2: 2.2 Separable Equations; problems: 7,9,13,19,25,27
2.3 Linear Equations; problems: 7,9,23,27,31
2.4 Exact Equations, integrating factors; problems: 3,5,9,15,27,29,31
Week 3: 2.5 Solutions by Substitution (Bernoulli, homogeneous, linear substitution); problems: 5,7,9,13,17,19,21,25,27
Week 4: 1.3 Differential Equations as Mathematical Models; problems: 1,2,3,5,7,9,1013,15,16,19
2.7 Linear models (growth/decay, heating/cooling, circuits, mixtures); problems: 3,5,9,15,17,23,25,29,31
Week 5: 2.8 Non-linear models (Population dynamics, logistic equation, chemical reaction, leaking tank); problems: 2,3,11,13,17
17.1 Complex numbers; problems: 1,3,7,11,15,25,27,29,31,35,39
Week 6: Midterm 1 (tutorials) **on material of Weeks 1-4**
17.2 Powers and Roots; problems: 3,7,9,15,21,31,33,35
3.1 Theory of Linear Equations; problems: 1,9,23,27
Week 7: 3.3 Homogeneous Linear Equations with Constant Coefficients; problems: 3,5,9,13,15,17,21
3.4 Undetermined Coefficients; problems: 1,3,7,11,15,19,23,31
Week 8: 3.5 Variation of Parameters; problems: 1,13,15,23
3.6 Cauchy Euler Equations; problems: 5,7,11,23,45
Week 9: 3.7 Nonlinear Equations, Reduction of Order; problems: 3,7,9
3.8 Linear Models. Initial Value Problems (mass-spring systems, free motion); problems: 1,7,12,21
Week 10: 3.8 Linear Models. Initial Value Problems (driven motion and LRC-circuits); problems: 31,33,45,47,49
3.11 Non-linear models (telephone wires, rocket motion, pulled rope); problems: [Projects to be assigned]
Week 11: Midterm 2 (tutorials) **on material of Weeks 5-9**
5.1.2 Power Series Solutions; problems: 17,21,27
10.1 Theory of Linear Systems; problems: 1,3,7,18
Week 12: 10.2 Homogeneous Linear Systems; problems: 1,3,7,9,21,31,35,37,48
10.4 Non-Homogeneous Linear Systems; problems: 1,3,7,17,30
Week 13: Review

1.1 Definitions and Terminology

September 7, 2016 13:30

Definitions & Theorems:

1. **Definition 1.1.1:** Differential equation (DE)
   An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a differential equation (DE).
2. DE Classification
   a. Classification by Type
      i. If a differential equation contains only ordinary derivatives of one or more functions with respect to a single independent variable it is said to be an ordinary differential equation (ODE).

an ODE can contain more than one dependent variable

$$\frac{dy}{dx} + 6y = e^{-x}, \quad \frac{d^2y}{dx^2} + \frac{dy}{dx} - 12y = 0, \quad \text{and} \quad \underset{\downarrow}{\frac{dx}{dt}} + \underset{\downarrow}{\frac{dy}{dt}} = 3x + 2y$$

      ii. An equation involving only partial derivatives of one or more functions of two of more independent variables is called a partial differential equation (PDE).

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - \frac{\partial u}{\partial t}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Notice in thee third equation that there are two dependent variables and two independent variables in the PDE. This indicates that u and v must be functions of $two\ or\ more$ independent variables.

   b. Classification by Order: The order of a differential equation (ODE or PDE) is the order of the highest derivative in the equa tion.

The differential equations

$$\overset{\text{highest order}}{\frac{d^2y}{dx^2}} + 5\left(\frac{dy}{dx}\right)^3 - 4y = e^x, \quad 2\overset{\text{highest order}}{\frac{\partial^4 u}{\partial x^4}} + \frac{\partial^2 u}{\partial t^2} = 0$$

are examples of a second-order ordinary differential equation and a fourth-order partial differential equation, respectively. ≡

In symbols, we can express an nth-order ordinary differential equation in one dependent variable by the general form

$$F(x, y, y', \ldots, y^{(n)}) = 0, \qquad (4)$$

The normal form of (4) is

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}), \qquad (5)$$

where f is a real-valued continuous function.

   c. Classification by Linearity: An nth-order ordinary differential equation (4) is said to be linear in the variable y if F is linear in $y, y', \ldots, y^{(n)}$. This means that an nth-order ODE is linear when (4) is $a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y - g(x) = 0$ or

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x). \qquad (6)$$

Two important special cases of (6) are linear first-order (n=1) and linear second-order (n=2) ODEs:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \quad \text{and} \quad a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x). \qquad (7)$$

If the coefficients of $y, y', \ldots, y^{(n)}$ contain the dependent variable y or its derivatives or if power of $y, y', \ldots, y^{(n)}$, such as $(y')^2$, appear in the equation, then the DE is nonlinear.

(a) The equations

$$(y - x)dx + 4x\,dy = 0, \quad y'' - 2y' + y = 0, \quad x^3\frac{d^3y}{dx^3} + 3x\frac{dy}{dx} - 5y = e^x$$

are, in turn, examples of *linear* first-, second, and third-order ordinary differential equations. We have just demonstrated that the first equation is linear in y by writing it in the alternative form $4xy' + y = x$.

(b) The equations

$$\overset{\substack{\text{nonlinear term:}\\ \text{coefficient depends on } y}}{(1 - y)}y' + 2y = e^x, \quad \frac{d^2y}{dx^2} + \overset{\substack{\text{nonlinear term:}\\ \text{nonlinear function of } y}}{\sin y} = 0, \quad \frac{d^4y}{dx^4} + \overset{\substack{\text{nonlinear term:}\\ \text{power not 1}}}{y^2} = 0,$$

are examples of *nonlinear* first-, second-, and fourth-order ordinary differential equations, respectively. ≡

3. **Definition 1.1.2:** Solution of an ODE
   Any function $\emptyset$, defined on an interval I and possessing at least n derivatives that are continuous on I, which when substituted into an nth-order ordinary differential equations reduces the equation to an indentity, is said to be a solution of the equation on the interva l.
   The interval I in Definition 1.1.2 is variously called the interval of definition, the interval of validity, or the domain of the solution a nd can be an open interval (a, b), a closed interval $[a, b]$, an infinite interval (a, ∞), and so on.
4. **Definition:** Solution Curve

The graph of a solution ∅ of an ODE is called a solution curve.

5. **Definition:** Explicit Solutions
   A solution in which the dependent variable is expressed solely in terms of the independent variable and constants is said to be an explicit solution.
6. **Definition 1.1.3:** Implicit Solutions of ODE
   A relation $G(x, y) = 0$ is said to be an implicit solution of an ordinary differential equation (4) on an interval I provided there exists at least one function ∅ that satisfies the relation as well as the differential equation on I.
7. **Definition** Families of Solutions
   When solving a first-order differential equation $F(x, y, y') = 0$, we usually obtain a solution containing a single arbitrary constant or parameter c. A solution containing an arbitrary constant represents a set $G(x, y, c) = 0$ of solutions called a one-parameter family of solutions. When solving an nth-order differential equation $F(x, y, y', \dots, y^{(n)}) = 0$, we seek an n-parameter family of solutions $G(x, y, c_1, c_2, \dots, c_n) = 0$. This means that a single differential equation can possess an infinite number of solutions conrresponding to the umlimited number of choices for the parameter(s). A solutiong of a differential equation that is free of arbitrary parameters is called a particular solution.
   For example, the one-parameter family $y = cx - x\cos x$ is an explicit solution of the linear first-order equation $xy' - y = x^2 \sin x$ on the interval $(-\infty, \infty)$.

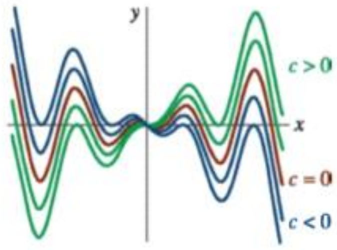


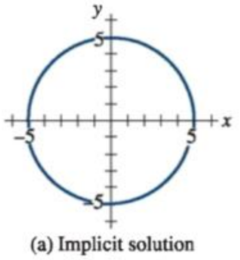
FIGURE 1.1.3 Some solutions of $xy' - y = x^2 \sin x$

   The solution $y = -x\cos x$, the red curve in the figure, is a particular solution corresponding to $c = 0$.
8. A Piecewise-Defined Solution
   The one-parameter family $y = cx^4$ is a one-parameter family of solutions of the differential equation $xy' - 4y = 0$ on the interval $(-\infty, \infty)$. The piecewise-defined differentiable function
   $$y = \begin{cases} -x^4, & x < 0 \\ x^4, & x \geq 0 \end{cases}$$
   is a particular solution of the equation but cannot be obtained from the family $y = cx^4$ by single choice of c; the solution is constructed from the family by choosing $c = -1$ for $x < 0$ and $c = 1$ for $x \geq 0$.
9. **Definition:** Singular Solution
   A solution that cannot be obtained by specializing any of the parameters in the family of solutions is called a singular solu tion.
   For example, $y = \frac{1}{16}x^4$ and $y = 0$ are solutions of the differential equation $\frac{dy}{dx} = xy^{\frac{1}{2}}$ on $(-\infty, \infty)$. But the differential equation $\frac{dy}{dx} = xy^{\frac{1}{2}}$ possesses the one-parameter family of solutions $y = \left(\frac{1}{4}x^2 + c\right)^2$. When $c = 0$, the resulting particular solution is $y = \frac{1}{16}x^4$. However, the trivial solution $y = 0$ is a singular solution since it is not a member of the family $y = \left(\frac{1}{4}x^2 + c\right)^2$; there is no way of assigning a value to the constant c to obtain $y = 0$.

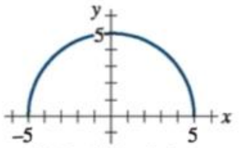
Examples:

1. Verification of an Implicit Solutions: The relation $x^2 + y^2 = 25$ is an implicit solution of the differential equation $\frac{dy}{dx} = -\frac{x}{y}$ on the interval defined by $-5 < x < 5$.

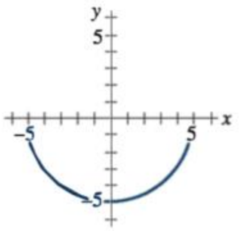
   By implicit differentiation we obtain $\frac{d}{dx}x^2 + \frac{d}{dy}y^2 = \frac{d}{dx}25$ or $2x + 2y\frac{dy}{dx} = 0$.
   Solving $x^2 + y^2 = 25$ for y in terms of x yields $y = \pm\sqrt{25 - x^2}$. The two functions $y = \phi_1(x) = \sqrt{25 - x^2}$ and $y = \phi_2(x) = -\sqrt{25 - x^2}$ satisfy the relation (that is, $x^2 + {\phi_1}^2 = 25$ and $x^2 + {\phi_2}^2 = 25$) and are explicit solutiong defined on the interval $(-5,5)$.



(a) Implicit solution
$x^2 + y^2 = 25$



(b) Explicit solution
$y_1 = \sqrt{25 - x^2}, -5 < x < 5$



(c) Explicit solution
$y_2 = -\sqrt{25 - x^2}, -5 < x < 5$

1.2 Initial-Value Problems (P28)

September 7, 2016 11:31

Definitions & Theorems:

1. **Definition:** Initial-Value Problem

   On some interval I containing x_0, the problem

$$\text{Solve: } \frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)})$$
$$\text{Subject to: } y(x_0) = y_0, y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}, \quad (1)$$

   where $y_0, y_1, \ldots, y_{n-1}$ are arbitrarily specified real constants, is called an initial-value problem (IVP).
   The values of $y(x)$ and its first $n-1$ derivatives at a single point x_0: $y(x_0) = y_0, y'(x_0) = y_1, \ldots, y^{n-1}(x_0) = y_{n-1}$, are called initial conditions.

2. First- and Second-Order IVPs

   The problem given in (1) is also called an nth-order initial-value problem. For example,

$$\text{Solve: } \frac{dy}{dx} = f(x, y)$$
$$\text{Subject to: } y(x_0) = y_0 \quad (2)$$

   and

$$\text{Solve: } \frac{d^2 y}{dx^2} = f(x, y, y')$$
$$\text{Subject to: } y(x_0) = y_0, y'(x_0) = y_1 \quad (3)$$

   are first- and second-order initial-value problems.

3. Existence and Uniqueness

   For a first-order initial-value problem, we ask:

   Existence:
   a) Does the differential equation $\frac{dy}{dx} = f(x, y)$ possess solutions?
   b) Do any of the solution curves pass through the point (x_0, y_0)?

   Uniqueness:
   a) When can we be certain that there is precisely one solution curve passing through the point (x_0, y_0)?

4. Theorem 1.2.1: Existence of a Unique Solution

   Let R be a rectangular region in the xy-plane defined by $a \le x \le b, c \le y \le d$, that contains the point (x_0, y_0) in its interior. If $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous on R, then there exists some interval $I_0: (x_0 - h, x_0 + h), h > 0$, contained in $[a, b]$, and a unique function $y(x)$ defined on I_0 that is a solution of the initial-value problem (2).

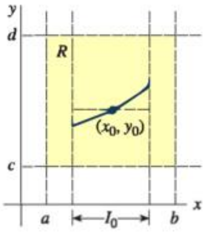
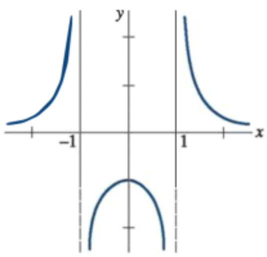


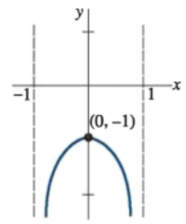
FIGURE 1.2.6 Rectangular region R

Examples:

1. (First-Order IVPs): $y = ce^x$ is a one-parameter family of solutions of the simple first-order equation $y' = y$ on the interval $(-\infty, \infty)$.
   a. If $y(0) = 3$, then $3 = ce^0 = c$
      $\Rightarrow$ Thus the function $y = 3e^x$ is a solution of the initial-value problem $y' = y, y(0) = 3$
   b. If a solution of the differential equation pass through the point $(1, -2)$, then $y(1) = -2 \Rightarrow -2 = ce \Rightarrow c = -2e^{-1}$
      $\Rightarrow$ Thus the function $y = -2e^{-1}$ is a solution of the initial-value problem $y' = y, y(1) = 2$
2. (Interval I of Definition of a Solution): $y = \frac{1}{x^2+c}$ is a one-parameter family of solutions of the first-order differential equation $y' + 2xy^2 = 0$. If we impose the initial condition $y(0) = -1$, then $c = -1$. Thus, $y = \frac{1}{x^2-1}$.
   a. Considered as a function, the domain of $y = \frac{1}{x^2-1}$ is the set of numbers x for which $y(x)$ is defined; this is the set of all numbers except $x = -1$ and $x = 1$.
   b. Considered as a solution of the differential equation $y' + 2xy^2 = 0$, the interval I of defnition of $y = \frac{1}{x^2-1}$ could be taken to be any interval which $y(x)$ is defined and differentiable. The largest intervals on which $y = \frac{1}{x^2-1}$ is a solution are $(-\infty, 1), (-1,1)$, and $(1, \infty)$.
   c. Considered as a solution of the initial-value problem $y' + 2xy^2 = 0, y(0) = -1$, the interval I of definition of $y = \frac{1}{x^2-1}$ could be taken to be any interval over which $y(x)$ is defined, differentiable, and contains the initial point $x = 0$; the largest interval for which this is true is $(-1,1)$.



(a) Function defined for all x except $x = \pm 1$



(b) Solution defined on interval containing $x = 0$

3. (Second-Order IVP): $x = c_1 \cos 4t + c_2 \sin 4t$ is a two-parameter family of solutions of $x'' + 16x = 0$. Find a solution of the initial-value problem

$$x'' + 16x = 0, x\left(\frac{\pi}{2}\right) = -2, x'\left(\frac{\pi}{2}\right) = 1$$

Solve:

$$x\left(\frac{\pi}{2}\right) = -2 \Rightarrow c_1 \cos 2\pi + c_2 \sin 4\pi = -2 \Rightarrow c_1 = -2$$

$$x'\left(\frac{\pi}{2}\right) = 1 \Rightarrow 8 \sin 2\pi + 4c_2 \cos 2\pi = 1 \Rightarrow c_2 = \frac{1}{4}$$

Hence $x = -2\cos 4t + \frac{1}{4}\sin 4t$ is a solution of the initival-value problem.

4. (An IVP Can Have Several Solutions):

Each of the function $y = 0$ and $y = \frac{1}{16}x^4$ satisfies the differential equation $\frac{dy}{dx} = xy^{\frac{1}{2}}$ and the initial condition $y(0) = 0$, and so the initial-value problem $\frac{dy}{dx} = xy^{\frac{1}{2}}, y(0) = 0$, has at least two solutions.

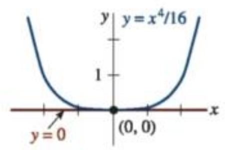


FIGURE 1.2.5 Two solutions of the same IVP in Example 4

2.1 Solution Curves Without a Solution (P49)

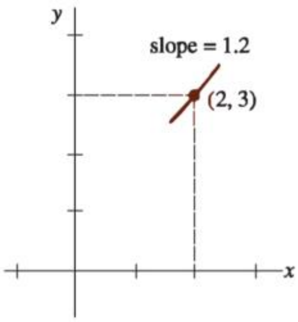
September 12, 2016 11:32

Section 2.1.1 Direction Fields

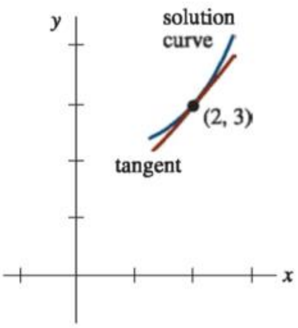
Definitions & Theorems:

1. Slope

   A derivative $\frac{dy}{dx}$ of a differentiable function $y = f(x)$ gives slopes of tangent lines at points on its graph. Because a solution $y = f(x)$ of a first-order differential equation $\frac{dy}{dx} = f(x, y)$ is necessarily a differentiable function on its interval I of definition, it must also be continuous on I. The value $f(x, y)$ that the function f assigns to the point represents the slopes of a line, or as we shall envision it, a line segment called a lineal element. For exampel, consider the equation $\frac{dy}{dx} = 0.2xy$, where $f(x, y) = 0.2xy$. At, say, the point $(2,3)$, the slope of a lineal element is $f(2,3) = 0.2(2)(3) = 1.2$. If a solution curve also passes through the point $(2,3)$, it does so tangent to this line segment.

   (a) $f(2, 3) = 1.2$ is slope of lineal element at (2, 3)

   (b) A solution curve passing through (2, 3)

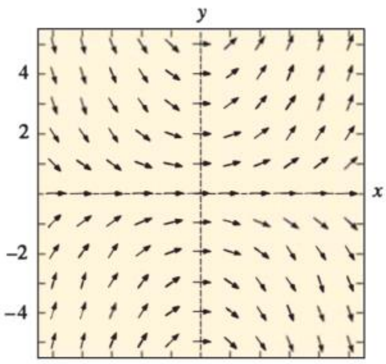
   FIGURE 2.1.1 Solution curve is tangent to lineal element at (2, 3)

2. **Definition**: Direction Field

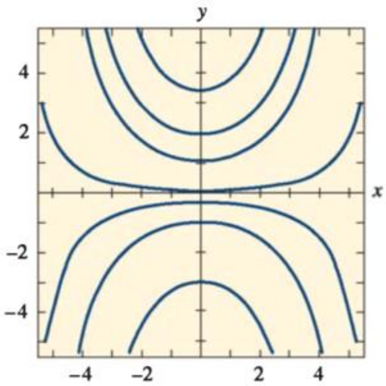
   If we systematically evaluate f over a rectangular grid of points in the xy-plane and draw a lineal element at each point (x, y) of the grid with slope $f(x, y)$, then the collection of all these lineal elements is called a direction field or a slope field of the differential equation $\frac{dy}{dx} = f(x, y)$.

Examples:

1. (Direction Field) The direction field for the differential equation $\frac{dy}{dx} = 0.2xy$. We can see that $y = e^{0.1x^2}$ is an explicit solution of the differential equation $\frac{dy}{dx} = 0.2xy$.



(a) Direction field for $dy/dx = 0.2xy$



(b) Some solution curves in the family $y = ce^{0.1x^2}$

2. (Direction Field) Use a direction field to sketch an approximate solution curve for the initial-value problem $\frac{dy}{dx} = \sin y\,, y(0) = -\frac{3}{2}$.

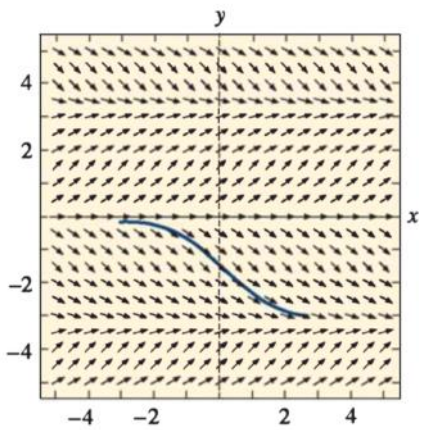


FIGURE 2.1.3 Direction field for $dy/dx = \sin y$ in Example 2

Section 2.1.2 Autonomous First-Order DEs

Definitions & Theorems:

1. DEs Free of the Independent Variable

    An ordinary differential equation in which the independent variable does not appear explicitly is said to be autonomous. If the symbol x denotes the independent variable, then an autonomous first-order differential equation can be written in general form as $F(y, y') = 0$ or in normal form as

    $$\frac{dy}{dx} = f(y)$$

    For example, the first-order equations

$$\frac{dy}{dx} = \overset{\overset{f(y)}{\downarrow}}{1 + y^2} \quad \text{and} \quad \frac{dy}{dx} = \overset{\overset{f(x,\,y)}{\downarrow}}{0.2xy}$$

are autonomous and non-autonomous, respectively.

2. **Definition**: Critical Points

   A real number c is a critical point of an autonomous differential equation of it is a zero of f, that is, $f(c) = 0$. A critical point is also called an equilibrium point or stationary point.

   If c is a critical point of an autonomous equation, then $y(x) = c$ is a constant solution of the autonomous differential equation.

   A constant solution $y(x) = c$ is called an euqilibrium solution; equilibria are the only constant solution.

3. **Definition**: Attractors and Repellers

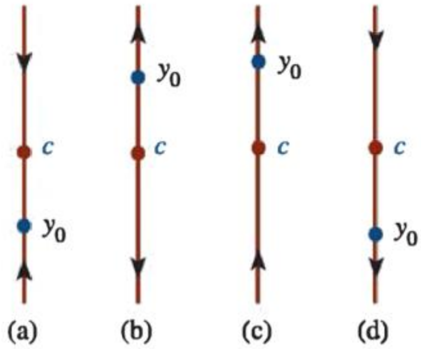


FIGURE 2.1.8 Critical point c is an attractor in (a), a repeller in (b), and semi-stable in (c) and (d)

4. Translation Property

   If $f(x)$ is a solution of an autonomous differential equation $\frac{dy}{dx} = f(y)$, then $y_1(x) = y(x-k)$, k a constant, is also a solution.

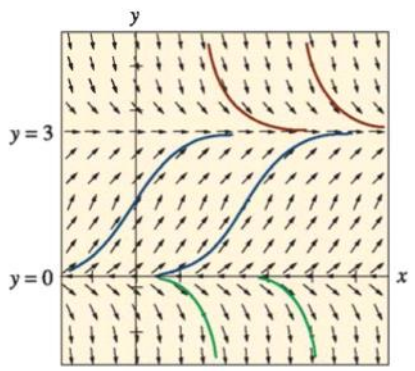


FIGURE 2.1.10 Translated solution curves of an autonomous DE

Hence, if $y(x)$ is a solution of the initial-value problem $\frac{dy}{dx} = f(x), y(0) = y_0$ then $y_1(x) = y(x - x_0)$ is a solution of the IVP $\frac{dy}{dx} = f(y), y(x_0) = y_0$.

For example, $y(x) = e^x, -\infty < x < \infty$, is a solution of the IVP $\frac{dy}{dx} = y, y(0) = 1$ and so a solution $y_1(x)$ of $\frac{dy}{dx} = y, y(4) = 1$ is $y(x) = e^x$ translated 4 unites to the right:

$$y_1(x) = y(x-4) = e^{x-4}, -\infty < x < \infty$$

Examples:

3. (An Autonomous DE) The differential equation $\frac{dP}{dt} = (a - bP)P$ where a and b are positive constants, has the normal form $\frac{dP}{dt} = f(P)$, and hence is autonomous. From $f(P) = (a - bP)P = 0$, we see that 0 and $\frac{a}{b}$ are critical points of the equation and so the equilibrium solutions are $P(t) = 0$ and $P(t) = \frac{a}{b}$.

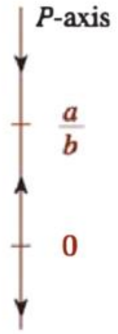


FIGURE 2.1.4 Phase portrait for Example 3

Interval	Sign of $f(P)$	$P(t)$	Arrow
$(-\infty, 0)$	minus	decreasing	points down
$(0, a/b)$	plus	increasing	points up
$(a/b, \infty)$	minus	decreasing	points down

2.2 Separable Equations (P58)

September 14, 2016 11:32

Definitions & Theorems:

1. **Definition 2.2.1**: Separable Equation

   A first-order differential equation of the form

   $$\frac{dy}{dx} = g(x)h(y)$$

   is said to be separable or to have separable variable.

   For example:

   1) separable

   $$\frac{dy}{dx} = x^2 y^4 e^{5x-3y}$$

   2) non-separable

   $$\frac{dy}{dx} = y + \cos x$$

Examples:

1. (Solving a Separable DE): Solve $(1+x)\,\mathrm{d}y - y\,\mathrm{d}x = 0$

   Dividing by $(1+x)y$, we can write

   $$\frac{\mathrm{d}y}{y} = \frac{\mathrm{d}x}{1+x}$$

   $$\Rightarrow \int \frac{\mathrm{d}y}{\mathrm{y}} = \int \frac{\mathrm{d}x}{1+x}$$

$\Rightarrow \ln\lvert y\rvert = \ln\lvert 1+x\rvert + c_1$ $\Rightarrow y = e^{\ln\lvert 1+x\rvert + c_1} = e^{\ln\lvert 1+x\rvert} e^{c_1}$ $\Rightarrow y = \lvert 1+x\rvert e^{c_1}$ $\Rightarrow y = \pm e^{c_1}(1+x)$ Relabeling $\pm e^{c_1}$ by c then gives $y = c(1+x)$	$\Rightarrow \ln\lvert y\rvert = \ln\lvert 1+x\rvert + \ln\lvert c\rvert$ $\Rightarrow \ln\lvert y\rvert = \ln\lvert c(1+x)\rvert$ $\Rightarrow y = c(1+x)$

2. (Losing a Solution) Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 - 4$

   We put the equation in the form

   $$\frac{\mathrm{d}y}{y^2-4} = \mathrm{d}x \Rightarrow \frac{1}{4}\left(\frac{1}{y-2} - \frac{1}{y+2}\right)\mathrm{d}y = \mathrm{d}x$$

   $$\Rightarrow \frac{1}{4}\ln\lvert y-2\rvert - \frac{1}{4}\ln\lvert y+2\rvert = x + c_1$$

   $$\Rightarrow \frac{y-2}{y+2} = e^{4x+c_2} \Rightarrow y = 2\frac{1+ce^{4x}}{1-ce^{4x}}$$

   Now if we factor the right side of the differential equation as $\frac{\mathrm{d}y}{\mathrm{d}x} = (y-2)(y+2)$, we know that $y=2$ and $y=-2$ are two constant (equilibrium) solutions. The solution is a member of the family of solutions defined by $y = 2\frac{1+ce^{4x}}{1-ce^{4x}}$ correspoding to the value $c=0$. However, $y=-2$ is a singular solution; it cannot be obtained from $y = 2\frac{1+ce^{4x}}{1-ce^{4x}}$ for any choice of the parameter c.

2.3 Linear Equations (P66)

September 14, 2016 11:32

Definitions & Theorems:

1. **Definition 2.3.1**: Linear Equation

   A first-order differential equation of the form
   $$a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x}+a_0(x)y=\mathrm{g}(x)$$
   is said to be a linear equation in the dependent variable y.
   When $g(x)=0$, the linear equation is said to be homogeneous; otherwise, it is non-homogeneous.
2. Standard Form

   By dividing both side of a linear equation by the lead coefficient $a_1(x)$ we obtain the standard form of a linear equation:
   $$\frac{\mathrm{d}y}{\mathrm{d}x}+P(x)y=f(x)$$
3. Guidelines for Solving a Linear First-Order Equation
   1) Put a linear equation into standard form.
   2) Determine $P(x), f(x)$ and the integrating factor $e^{\int P(x)\,\mathrm{d}x}$.
   3) Multiply the standard form by the integrating factor. The left side of the resulting equation is automatically the derivative of the integrating factor and y.
   $$\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{\int P(x)\,\mathrm{d}x}y\right]=e^{\int P(x)\,\mathrm{d}x}f(x)\Rightarrow e^{\int P(x)\,\mathrm{d}x}y=\int e^{\int P(x)\,\mathrm{d}x}f(x)\,\mathrm{d}x$$

Proofs or Explanations:

1. The Property:

   The differential equation of the standard form has the property that its solution is the sum of the two solutions, $y=y_c+y_p$, where y_c is a solution of the associated homogeneous equation
   $$\frac{\mathrm{d}y}{\mathrm{d}x}+P(x)y=0$$
   and y_p is a particular solution of the non-homogeneous equation. To see this, observe
   $$\frac{d}{dx}[y_c+y_p]+P(x)[y_c+y_p]=\underbrace{\left[\frac{dy_c}{dx}+P(x)y_c\right]}_{0}+\underbrace{\left[\frac{dy_p}{dx}+P(x)y_p\right]}_{f(x)}=f(x)$$
2. The Homogeneous DE:

   The homogeneous is separable:
   $$\frac{\mathrm{d}y}{y}+P(x)\,\mathrm{d}x=0\Rightarrow y_c=ce^{-\int P(x)\,\mathrm{d}x}$$
3. The Non-homogeneous DE:

   Our assumption for y_p is the same as $y_c=cy_1(x)$ except that c is replaced by the "variable parameter" u.
   Substitution $y_p=uy_1$ gives
   $$u\frac{dy_1}{dx}+\overset{\text{Product Rule}}{\overset{\downarrow}{y_1\frac{du}{dx}}}+P(x)uy_1=f(x)\quad\text{or}\quad u\overset{\text{zero}}{\overset{\downarrow}{\left[\frac{dy_1}{dx}+P(x)y_1\right]}}+y_1\frac{du}{dx}=f(x)$$
   so that
   $$y_1\frac{\mathrm{d}u}{\mathrm{d}x}=f(x)$$
   Separating variable and integrating then gives
   $$du=\frac{f(x)}{y_1(x)}\mathrm{d}x\qquad and\qquad u=\int\frac{f(x)}{y_1(x)}\mathrm{d}x$$

Examples:

1. (Solving a Linear DE): Solve $\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = 6$
   1) $P(x) = -3, f(x) = 6$
   2) The integrating factor is
   $$e^{\int P(x)\,\mathrm{d}x} = e^{\int(-3)\,\mathrm{d}x} = e^{-3x}$$
   3) Multiply the equation by this factor
   $$e^{-3x}\frac{\mathrm{d}y}{\mathrm{d}x} - 3e^{-3x}y = 6e^{-3x}$$
   $$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\left[e^{-3x}y\right] = 6e^{-3x}$$
   $$\Rightarrow e^{-3x}y = \int 6e^{-3x}\,\mathrm{d}x = -2e^{-3x} + c$$
   $$\Rightarrow y = -2 + ce^{3x}, -\infty < x < \infty$$

2.4 Exact Equation, integrating factors (P74)

September 19, 2016 11:33

Definitions & Theorems:

1. **Definition**: Differential of a Function of Two Variables

   If $z = f(x, y)$ is a function of two variables with continuous first partial derivatives in a region R of the xy-plane, then its differential (also called the total differential) is

   $$\mathrm{d}z = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y$$

   If $f(x, y) = c$, it follows that

   $$\frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y = 0$$

2. **Definition 2.4.1**: Exact Equation

   A differential expression $M(x, y)dx + N(x, y)dy$ is an exact differential in a region R of the xy-plane if it corresponds to the differential of some function $f(x, y)$. A first-order differential equation of the form

   $$M(x, y)dx + N(x, y)dy = 0$$

   is said to be an exact equation if the expression on the left side is an exact differential.

3. **Theorem 2.4.1**: Criterion for an Exact Differential

   Let $M(x, y)$ and $N(x, y)$ be continuous and have continuous first partial derivatives in a rectangular region R defined by $a < x < b, c < y < d$. Then a necessary and sufficient condition that $M(x, y)dx + N(x, y)dy$ be an exact differential is

   $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

4. Method of Solution

   $$f(x, y) = \int M(x, y)\, dx + g(y), g'(y) = N(x, y) - \frac{\partial}{\partial y}\int M(x, y)dx$$

   $$f(x, y) = \int N(x, y)\, dy + h(y), h'(y) = M(x, y) - \frac{\partial}{\partial x}\int N(x, y)dy$$

Examples:

1. (Solving an Exact DE) Solve $2xy\,\mathrm{d}x + (x^2 - 1)\,\mathrm{d}y = 0$

   With $M(x, y) = 2xy$ and $N(x, y) = x^2 - 1$ we have

   $$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$$

   Thus the equation is exact, and so, by Theorem 2.4.1, there exists a function $f(x, y)$ such that

   $$\frac{\partial f}{\partial x} = 2xy \text{ and } \frac{\partial f}{\partial y} = x^2 - 1$$

   Method1:

   From the first of these equations we obtain, after integrating,

   $$f(x, y) = x^2 y + g(y)$$

   Taking the partial derivative of the last expression with respect to y and setting the result equal to $N(x, y)$ gives

   $$\frac{\partial f}{\partial y} = x^2 + g'(y) \text{ and } \frac{\partial f}{\partial y} = x^2 - 1 \Rightarrow x^2 + g'(y) = x^2 - 1 = N(x, y)$$

   It follows that $g'(y) = -1$ *and* $g(y) = -y$

   Hence, $f(x, y) = x^2 y - y$

   $\Rightarrow x^2 y - y = c$

   Method 2:

   $$f_X(x, y) = \int \frac{\partial f}{\partial x}\mathrm{d}x = \int (2xy)\,\mathrm{d}x = x^2 y + c_1$$

   $$f_Y(x, y) = \int \frac{\partial f}{\partial y}\mathrm{d}y = \int (x^2 - 1)\,\mathrm{d}y = x^2 y - y + c_2$$

Then, add $f_X(x,y)$ and $f_Y(x,y)$ together, but do not write the same term twice.
$\Rightarrow x^2y - y = c$

So the solution of the equation in implicit form is $x^2y - y = c$.
The explicit form of the solution is $y = \frac{c}{1-x^2}$ and is defined on any interval not containing either $x = 1$ or $x = -1$

2. (A Non-exact DE Made Exact) The nonlinear first-order differential equation $xy\,dx + (2x^2 + 3y^2 - 20)\,dy = 0$ is not exact.
   With the identification $M = xy, N = 2x^2 + 3y^2 - 20$ we find the partial derivatives $M_y = x$ and $N_x = 4x$. The first quotient from (13) gets us nowhere since
   $$\frac{M_y - N_x}{N} = \frac{x - 4x}{2x^2 + 3y^2 - 20} = \frac{-3x}{2x^2 + 3y^2 - 20}$$
   depends on x and y. However (14) yields a quotient that depends only on y:
   $$\frac{N_x - M_y}{M} = \frac{4x - x}{xy} = \frac{3x}{xy} = \frac{3}{y}$$
   The integrating factor is then $e^{\int \frac{3}{y} dy} = e^{3\ln y} = e^{\ln y^3} = y^3$. After multiplying the given DE by $\mu(y) = y^3$ the resulting equation is
   $$xy^4\,dx + (2x^2y^3 + 3y^5 - 20y^3)dy = 0$$

2.5 Solutions by Substitution (P80)

September 21, 2016 11:33

Definitions & Theorems:

1. **Definition** Homogeneous Equations

   If a function f possesses the property $f(tx,ty) = t^\alpha f(x,y)$ for some real number α, then f is said to be a homogeneous function of degree α.

   For example, $f(x,y) = x^3 + y^3$ is a homogeneous function of degree 3 since

   $$f(tx,tx) = (tx)^3 + (ty)^3 = t^3(x^3+y^3) = t^3 f(x,y).$$

   whereas $f(x,y) = x^3 + y^3 + 1$ is seen not to be homogeneous.

   A first-order DE in differential form

   $$M(x,y)\,dx + N(x,y)\,dy = 0$$

   is said to be homogeneous if both coefficients M and N are homegeneous functions of the same degree. In other words, $M(x,y)\,dx + N(x,y)\,dy = 0$ is homogeneous if

   $$M(tx,ty) = t^\alpha M(x,y) \quad \text{and} \quad N(x,y) = t^\alpha N(x,y)$$

   If M and N are homogeneous functions of degree α, we can also write

   $$M(x,y) = x^\alpha M(1,u) \quad \text{and} \quad N(x,y) = x^\alpha N(1,u) \quad \text{where } u = \frac{y}{x},$$

   and

   $$M(x,y) = y^\alpha M(v,1) \quad \text{and} \quad N(x,y) = y^\alpha N(v,1) \quad \text{where } v = \frac{x}{y}.$$

2. Substitution:

   Either of the substitutions $y = ux$ or $x = vy$, where u and v are new dependent variables, will reduce a homogeneous equation to a separable first-order differential equation.

3. **Definition** Bernoulli's Equation

   The differential equation

   $$\frac{dy}{dx} + P(x)y = f(x)y^n$$

   where n is any real number, is called Bernoulli's equation. Note that for $n = 0$ and $n = 1$, the euqation is linear. For $n \neq 0$ and $n \neq 1$, the substitution $u = y^{1-n}$ reduces any equation to a linear equation.

4. Reduction to Separation of Variables

   A differential equation of the form

   $$\frac{dy}{dx} = f(Ax + By + C), \qquad B \neq 0$$

   can always be reduced to an equation with separable variables by means of the substitution $u = Ax + By + C, B \neq 0$.

Examples:

1. (Homogeneous DE) Solve $(x-y)dx + x\,dy = 0$

   $M(x,y) = x - y$

   $N(x,y) = x$

   $M(tx,ty) = tx - ty = t(x-y) = tM(x,y)$

   $N(tx,ty) = tx = tN(x,y)$

   Let $y = ux \Rightarrow dy = u\,dx + x\,du$

   $(x - ux)\,dx + x\,(u\,dx + x\,du) = 0$

   $x(1-u)\,dx + x\,(u\,dx + x\,du) = 0$

   $(1-u)\,dx + (u\,dx + x\,du) = 0$

   $dx + x\,du = 0$

   $\frac{dx}{x} = -du$

   By integrating $\Rightarrow \ln|x| = -u + \ln c_1 \Rightarrow \ln(c_1|x|) = -u = -\frac{y}{x}$

   $\Rightarrow y = -x\ln(cx), x > 0$

   Prove:

   $$y = -x\ln(cx), x > 0 \Rightarrow dy = \left(-\ln(cx) - x\frac{c}{cx}\right)dx$$

   $$\Rightarrow [x - (-x\ln(cx))]dx + x\left(-\ln(cx) - x\frac{c}{cx}\right)dx = 0$$

2. (Reduction to Separation of Variables) Sovle $\frac{dy}{dx} = (-2x+y)^2 + 7, y(0) = 0$

   Let $u = -2x + y$, then $\frac{du}{dx} = -2 + \frac{dy}{dx} \Rightarrow \frac{du}{dx} = -2 + \left[(-2x+y)^2 + 7\right] = 5 + u^2$

   $$\Rightarrow \frac{du}{u^2+5} = dx$$

   By integrating $\Rightarrow$

3. (Bernoulli DE) Solve $x\frac{dy}{dx} + y = x^2y^2$

1.3 & 2.7 Differential Equations as Mathematical Models (P34) & Linear Models (P88)

September 26, 2016 11:40

Definitions & Theorems:

1. The steps of the modeling process

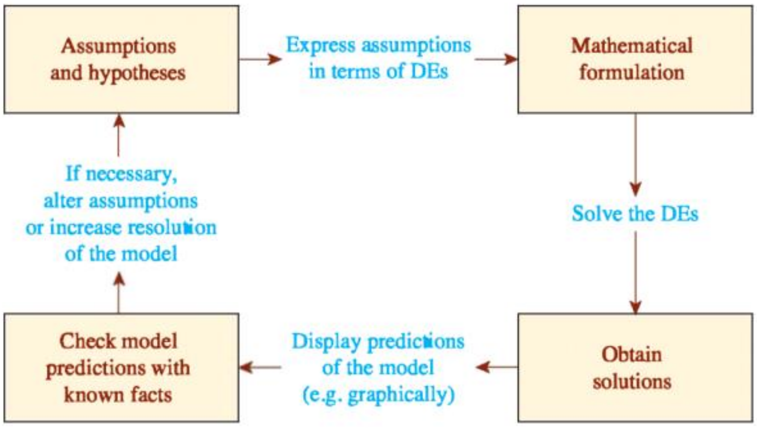


FIGURE 1.3.2 Steps in the modeling process

Proofs or Explanations:

1. Population Dynamics

   The idea of the Malthusian model is the assumption that the rate at which a population of a country grows at a certain time is proportional to the total population of the country at that time. In other words, the more people there are at time t, the more there are going to be in the future. In mathematical terms, if $P(t)$ denotes the total population at time t, then this assumption can be expressed as
   $$\frac{dP}{dt} \propto P \qquad or \qquad \frac{dP}{dt} = kP$$
   where k is a constant of proporitonality.
2. Radioactive Decay

   Assumed that the rate $\frac{dA}{dt}$ at which the nuclei of a substance decay is proporitonal to the amount $A(t)$ of the substance remaining at time t:
   $$\frac{dA}{dt} \propto A \qquad or \qquad \frac{dA}{dt} = kA$$
3. Newton's Law of Cooling/Warming

   If $T(t)$ represents the temperature of a body at time t, T_m the temperature of the surrounding medium, and $\frac{dT}{dt}$ the rate at which the temperature of the body changes, then Newton's Law of cooling/warming translates into the mathematical statement
   $$\frac{dT}{dt} \propto T - T_m \qquad or \qquad \frac{dT}{dt} = k(T - T_m)$$
   where k is a constant of proportionality.
4. Chemical Reactions

   If X denotes the amount of substance C formed and α and β are the given amounts of the first two chemicals A and B, then the instantaneous amounts not covnverted the chemical C are $\alpha - X$ and $\beta - X$, respectively. Hence the rate of formation of C is given by
   $$\frac{dX}{dt} = k(\alpha - X)(\beta - X)$$
   where k is a constant of proportionality.
5. Mixtures (P37)

   If $A(t)$ denotes the amount of salt (measured in pounds) in the tank at time t, then the rate at which $A(t)$ changes in a net rate:
   $$\frac{\mathrm{dA}}{\mathrm{dt}} = (input\ rate\ of\ salt) - (output\ rate\ of\ salt) = R_{in} - R_{out}$$
   The input rate R_{in} at which the salt enters the tank is the product of the inflow concertration of salt and the inflow rate of fluid. Note that R_{in} is measured in pounds per minute:

$$\underset{\substack{\text{concentration}\\ \text{of salt}\\ \text{in inflow}}}{\downarrow} \qquad \underset{\substack{\text{input rate}\\ \text{of brine}}}{\downarrow} \qquad \underset{\substack{\text{input rate}\\ \text{of salt}}}{\downarrow}$$

$$R_{in} = (2 \text{ lb/gal}) \cdot (3 \text{ gal/min}) = (6 \text{ lb/min}).$$

The output rate R_{out} of salt il

$$\underset{\substack{\text{concentration}\\ \text{of salt}\\ \text{in outflow}}}{\downarrow} \qquad \underset{\substack{\text{output rate}\\ \text{of brine}}}{\downarrow} \qquad \underset{\substack{\text{output rate}\\ \text{of salt}}}{\downarrow}$$

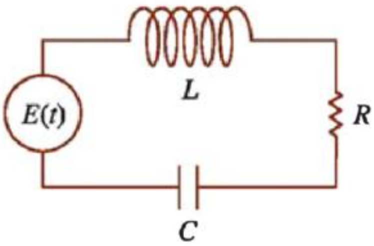
$$R_{out} = \left(\frac{A(t)}{300} \text{ lb/gal}\right) \cdot (3 \text{ gal/min}) = \frac{A(t)}{100} \text{ lb/min}.$$

The net rate then becomes

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 6 - \frac{A}{100} \qquad or \qquad \frac{\mathrm{d}A}{\mathrm{d}t} + \frac{A}{100} = 6$$

6. Series Circuits

Consider the single-loop LRC-series circuit containing an inductor, resistor, and capacitor shown in (a)



(a) *LRC*-series circuit

The letters L, R, and C are known as inductance, resistance, and capacitance, respectively, and are generally constants. According to Kirchhoff's second Law, the impressed voltage $E(t)$ on a closed loop must equal the sum of the voltage drops in the loop. Since current $i(t)$ is related to charge $q(t)$ on the capacitor by $i = \frac{dq}{dt}$, by adding the three voltage drops

Inductor	Resistor	Capacitor
$L\dfrac{di}{dt} = L\dfrac{d^2q}{dt^2}$,	$iR = R\dfrac{dq}{dt}$,	$\dfrac{1}{C}q$

and equating the sum to the impressed voltage, we obtain a second-order differential equation

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

Series Circuits (P93)

For a series circuit containing only a resistor and an inductor, Kirchhoff's second Law states that the sum of the voltage drop across the inductor $(L\frac{\mathrm{di}}{\mathrm{d}t})$ and the voltage drop across the resistor (iR) is the same as the impressed voltage $(E(t))$ on the circuit.

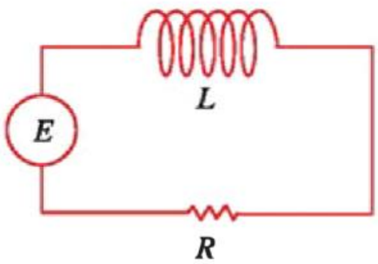


FIGURE 2.7.5 *LR*-series circuit

Thus we obtain the linear differential equation for the current $i(t)$

$$L\frac{\mathrm{di}}{\mathrm{d}t} + Ri = E(t)$$

where L and R are contants known as the inductance and the resistance, respectively.
The voltage drop across a capacitor with capacitance C is given by $\frac{q(t)}{C}$, where q is the charge on the capacitor.

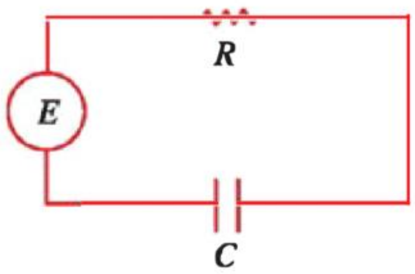


FIGURE 2.7.6 RC-series circuit

Hence, for the series circuit shown in $FIGURE$ 2.7.6, Kirchhoff's second Law gives

$$Ri + \frac{1}{C}q = E(t)$$

But current i and charge q are related by $i = \frac{\mathrm{dq}}{\mathrm{d}t}$, so it becomes the linear differential equation

$$R\frac{\mathrm{dq}}{\mathrm{d}t} + \frac{1}{C}q = E(t)$$

Examples:

1. (Bacterial Growth) A culture initially has P_0 number of bacteria. At $t = 1h$ the number of bacteria is measured to be $\frac{3}{2}P_0$. If the rate of growth is proportional to the number of bacteria $P(t)$ present at time t, determine the time necessary for the number of bacteria to triple (P89).

$P(0) = P_0 \; (t_0 = 0, P = P_0)$

$P(1) = \frac{3}{2}P_0 \; (by\ data)$

$\frac{dP}{dt} = kP \Rightarrow \frac{dP}{dt} - kP = 0$

$\frac{dP}{P} = k\,dt \Rightarrow \ln|P| = kt + c_1 \Rightarrow P = e^{kt+c_1} \Rightarrow P(t) = ce^{kt}$

At $t = 0$, then $P_0 = ce^0 = c \Rightarrow P(t) = P_0e^{kt}$

At $t = 1$, then $\frac{3}{2}P_0 = P_0e^k \Rightarrow e^k = \frac{3}{2} \Rightarrow k = \ln\frac{3}{2} = 0.4055$

$\Rightarrow P(t) = P_0e^{0.4055t}$

$\Rightarrow 3P_0 = P_0e^{0.4055t} \Rightarrow 0.4055t = \ln 3 \Rightarrow t \approx 2.71h$

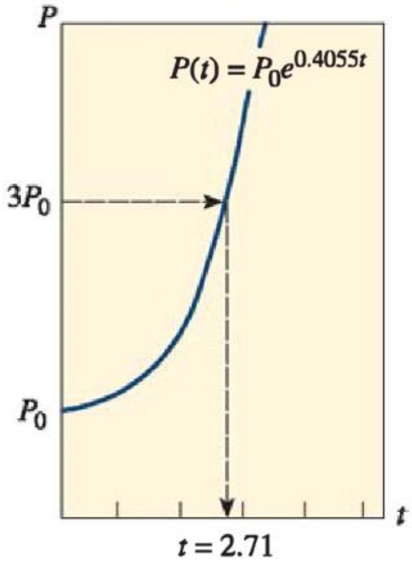


FIGURE 2.7.1 Time in which initial population triples in Example 1

$$\begin{cases} k > 0, & e^{kt} \ is \ increasing \ (growth) \\ k < 0, & e^{kt} \ is \ decreasing \ (decay) \end{cases}$$

2. (Cooling of a Cake) When a cake is removed an oven, its temperature is measured at 300^0F. Three minutes later its temperature is 200^0F. How long will it take for the cake to cool off to a room temerature of 70^0F?(P91)

$$T_m = 70, \frac{dT}{dt} = k(T - 70), T(0) = 300, T(3) = 200$$

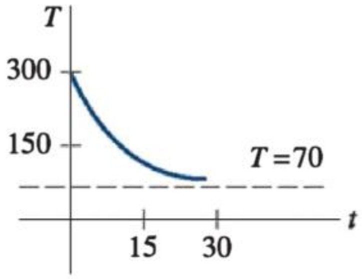
$$\frac{dT}{T - 70} = k\,dt \Rightarrow \ln|T - 70| = kt + c_1 \Rightarrow T = 70 + c_2e^{kt}$$

$$T(0) = 300 \Rightarrow 70 + c_2 = 300 \Rightarrow c_2 = 230 \Rightarrow T = 70 + 230e^{kt}$$

$$T(3) = 200 \Rightarrow 70 + 230e^{3k} = 200 \Rightarrow e^{3k} = \frac{13}{23} \Rightarrow k = \frac{1}{3}\ln\frac{13}{23} = -0.19018$$

$$\Rightarrow T(t) = 70 + 230e^{-0.19018t}$$

$$\lim_{t\to\infty} T(t) = 70$$



(a)

$T(t)$	t (in min.)
75°	20.1
74°	21.3
73°	22.8
72°	24.9
71°	28.6
70.5°	32.3

(b)

FIGURE 2.7.3 Temperature of cooling cake in Example 4

3. (Mixture of Two Salt Solutions) A large tank held 300 gallons of a brine solution. Salt was entering and leaving the tank; a brine solution was being pumped into the tank at the rate of 3 gal/min, mixed with the solution there, and then the mixture was pumped out at the rate of 3 gal/min. The concentration of the salt in the inflow, or solution entering, was 2 lb/gal, and so salt was entering the tank at the rate $R_{in} = \left(2\frac{lb}{gal}\right) \cdot \left(3\frac{gal}{min}\right) = 6\frac{lb}{min}$ and leaving the tank at the rate $R_{out} = \left(\frac{x}{300}\frac{lb}{gal}\right) \cdot \left(3\frac{gal}{min}\right) = \frac{x}{100}\frac{lb}{min}$. If there were 50 lb of salt dissolved initially in the 300 gallons, how much salt is in the tank after a long time? (P92)

   To find the amount of salt $x(t)$ in the tank at time t, we solve the initial-value problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = R_{in} - R_{out} \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = 6 - \frac{x}{100} \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{1}{100}x = 6, x(0) = 50$$

$$\Rightarrow p(t) = \frac{1}{100}, f(t) = 6 \Rightarrow e^{\int p(t)\,dt} = e^{\frac{t}{100}}$$

$$e^{\int p(t)\,dt}x = \int e^{\int p(t)\,dt} f(t)\,dt \Rightarrow e^{\frac{t}{100}}x = \int 6e^{\frac{t}{100}}\,dt = 6 \cdot e^{\frac{t}{100}} \cdot 100 + c = 600e^{\frac{t}{100}} + c$$

$$x(t) = 600 + ce^{-\frac{t}{100}}$$

   When $t = 0, x = 50 \Rightarrow c = -550$. Thus the amount of salt in the tank at any time t is given by

$$x(t) = 600 - 550e^{-\frac{t}{100}}$$

4. (Series Circuits) A 12-volt battery is connected to an LR-series circuit in which the inductance is $\frac{1}{2}$ henry and the resistance is 10 ohms. Determine the current i if the initial current is zero. (P93)

$$\frac{1}{2}\frac{\mathrm{d}i}{\mathrm{d}t} + 10i = 12, i(0) = 0$$

   First, we multiply the differential equation by 2

$$\frac{\mathrm{d}i}{\mathrm{d}t} + 20i = 24$$

   Second, read off the integrating factor e^{20t}

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{20t}i\right] = 24e^{20t}$$

   Integrating each side of the equation and solving for i gives

$$\int \left(\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{20t}i\right]\right)\mathrm{d}t = \int 24e^{20t}\,\mathrm{d}t$$

Let $u = 20t \Rightarrow du = 20dt$

$$\Rightarrow e^u i = \int \frac{24}{20} e^u \, \mathrm{d}u = \frac{6}{5} e^u + c \Rightarrow i = \frac{6}{5} + ce^{-u}$$

$$i(t) = \frac{6}{5} + ce^{-20t}$$

Now $i(0) = 0 \Rightarrow c = -\frac{6}{5} \Rightarrow i(t) = \frac{6}{5} - \frac{6}{5} e^{-20t}$

2.8 Nonlinear Models (P99)

October 3, 2016 11:41

Definitions & Theorems:

1. Definition:

Proofs or Explanations:

1. Population Dynamics

   The assumption that the rate at which a population grows (or declines) is dependent only on the number present and not on any time-dependent mechanisms such as seasonal phenomena can be stated as

   $$\frac{\frac{\mathrm{d}P}{\mathrm{d}t}}{P} = f(P) \qquad or \qquad \frac{\mathrm{d}P}{\mathrm{d}t} = Pf(P)$$

   This differential equation, which is widely assumed in models of animal populations, is called the density-dependent hypothesis.

2. Logistic Equation

   Suppose an environment is capable of sustaining no more than a fixed number of K individuals. The quantity K is called the carrying capacity of the environment. In previous differential equation, we make $f(P)$ linear, that is $f(P) = c_1 P + c_2$.

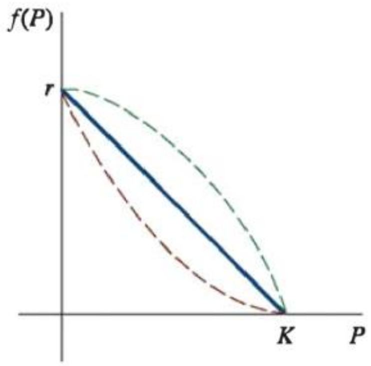
   

   FIGURE 2.8.1 Simplest assumption for $f(P)$ is a straight line

   If we use the condition $f(0) = r$ and $f(K) = 0$, we find $c_2 = r$, $c_1 = -r/K$, and so f takes on the form $f(P) = r - \left(\frac{r}{K}\right)P$.

   $$\frac{\mathrm{d}P}{\mathrm{d}t} = P\left(r - \frac{r}{K}P\right)$$

   Relabeling constants $a = r$ and $b = r/K$, the nonlinear equation is the same as

   $$\frac{\mathrm{d}P}{\mathrm{d}t} = P(a - bP)$$

   It is called the logistic equation, and its solution is called the logistic function. The graph of a logistic function is called a logistic curve.

3. Solution of the Logistic Equation (by separation of variables)

   $$\frac{\mathrm{d}P}{P(a-bP)} = \mathrm{d}t$$

   $$\Rightarrow \left(\frac{\frac{1}{a}}{P} + \frac{\frac{b}{a}}{a-bP}\right)\mathrm{d}P = \mathrm{d}t$$

   $$\Rightarrow \frac{1}{a}\ln|P| - \frac{1}{a}\ln|a-bP| = t + c_1$$

   $$\Rightarrow \ln\left|\frac{P}{a-bP}\right| = at + ac_1$$

   $$\Rightarrow \frac{P}{a-bP} = ce^{at}$$

   $$\Rightarrow P(t) = \frac{ace^{at}}{1+bce^{at}} = \frac{ac}{bc+e^{-at}}$$

   If $P(0) = P_0, P_0 \neq \frac{a}{b}$, we find $c = \frac{P_0}{a-bP_0}$

$$\Rightarrow P(t)=\frac{aP_0}{bP_0+(a-bP_0)e^{-at}}$$

4. Modifications of the Logistic Equation

$$\frac{\mathrm{d}P}{\mathrm{d}t}=P(a-bP)-h \quad \text{and} \quad \frac{\mathrm{d}P}{\mathrm{d}t}=P(a-bP)+h$$

$$\frac{\mathrm{d}P}{\mathrm{d}t}=P(a-b\ln P)$$

5. Chemical Reactions

$$\frac{dX}{dt}=k(\alpha-X)(\beta-X)$$

Examples:

1. (Logistic Growth) Suppose a student carrying a flu virus returns to an isolated college campus of 1000 students. If it is assumed that the rate at which the virus spreads is proportional not only to the number x of infected students but also to the number of students not infected, determine the number of infected students after 6 days if it is further observed that after 4 days $x(4)=50$.

   Assuming that no one leave the campus throughout the duration of the disease, we must solve the initial-value problem

   $$\frac{\mathrm{d}x}{\mathrm{d}t}=kx(1000-x), x(0)=1$$

   By making the identifications $a=1000k$ and $b=k$, we have that

   $$P(t)=\frac{aP_0}{bP_0+(a-bP_0)e^{-at}}$$

   $$\Rightarrow x(t)=\frac{1000k}{k+999ke^{-1000kt}}=\frac{1000}{1+999e^{-1000kt}}$$

   $$x(4)=50\Rightarrow 50=\frac{1000}{1+999e^{-4000k}}\Rightarrow -1000k=\frac{1}{4}\ln\frac{19}{999}=-0.9906$$

   $$\Rightarrow x(t)=\frac{1000}{1+999e^{-0.9906t}}$$

   Finally,

   $$x(6)=\frac{1000}{1+999^{-5.9436}}=276 \text{ students.}$$

2. (Second-Order Chemical Reaction) A compound C if formed when two chemicals A and B are combined. The resulting reaction between the two chemicals is such that for each gram of A, 4 grams of B is used. It is observed that 30 grams of the compound C is formed in 10 minutes. Determine the amount of C at time t if the rate of the reaction is proportional to the amounts of A and B remaining and if initially there are 50 grams of A and 32 grams of B. How much of the compound C is present at 15 minutes? Interpret the solution as $t\to\infty$

   Let $X(t)$ denote the number of grams of the compound C present at time t. Clearly $X(0)=0\text{g}$ and $X(10)=30\text{g}$. If, for example, 2 grams of compound C is present, we must have used, say, a grams of A and b grams of B so that $a+b=2$ and $b=4a$. Thus we must use $a=\frac{2}{5}=2\left(\frac{1}{5}\right)$g of chemical A and $b=\frac{8}{3}=2\left(\frac{4}{5}\right)$g of B. In general, for X grams of C we must use

   $$\frac{1}{5}X \text{ grams of } A \quad \text{and} \quad \frac{4}{5}X \text{ grams of B.}$$

   The amount of A and B remaining at any time are then

   $$50-\frac{1}{5}X \quad \text{and} \quad 23-\frac{4}{5}X$$

   respectively.

   Now we know that the rate at which compound C is formed satisfies

   $$\frac{\mathrm{d}X}{\mathrm{d}t}\propto\left(50-\frac{1}{5}X\right)\left(32-\frac{4}{5}X\right)\Rightarrow\frac{\mathrm{d}X}{\mathrm{d}t}=k(250-X)(40-X)$$

   By separation of variables and partial fractions we can write

   $$-\frac{\frac{1}{210}}{250-X}\mathrm{d}X+\frac{\frac{1}{210}}{40-X}\mathrm{d}X=k\,\mathrm{d}t$$

   Integrating gives

   $$\ln\left|\frac{250-\mathrm{X}}{40-X}\right|=210kt+c_1\Rightarrow\frac{250-\mathrm{X}}{40-X}=c_2e^{210kt}$$

   $$X(0)=0\Rightarrow c_2=\frac{25}{4}$$

$$X(10) = 30 \Rightarrow 210k = \frac{1}{10}\ln\frac{88}{25} = 0.1258$$

$$\Rightarrow X(t) = 1000\frac{1 - e^{-0.1258t}}{25 - 4e^{-0.1258t}}$$

17.1 Complex Numbers (P809)

October 5, 2016 11:24

Definitions & Theorems:

1. **Definition 17.1.1**: Complex Numbers

   A complex number is any number of the form $z = a + ib$ where a and b are real numbers and i is the imaginary unit.

2. Terminology

   The number i in Definition 17.1.1 is called the imaginary unit. The real number x in $z = x + iy$ is called the real part of z; the real number y is called the imaginary part of z. The real and imaginary parts of a complex number z are abbreviated $Re(z)$ and $Im(z)$, respectively.

3. **Definition 17.1.2**: Equality

   Complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are equal, $z_1 = z_2$, if

   $$Re(z_1) = Re(z_2) \quad \text{and} \quad Im(z_1) = Im(z_2)$$

4. Arithmetic Operations

   Complex numbers can be added, subtracted, multiplied, and divided. If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, these operations are defined as follows.

Addition:	$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$
Subtraction:	$z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2)$
Multiplication:	$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 - y_1y_2 + i(y_1x_2 + y_2x_1)$
Division:	$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \left(\frac{x_1 + iy_1}{x_2 + iy_2}\right)\left(\frac{x_2 - iy_2}{x_2 - iy_2}\right) = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\left(\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}\right)$

   The familiar commutative, associative, and distributive laws hold for complex numbers.

   Commutative laws: $z_1 + z_2 = z_2 + z_1; z_1z_2 = z_2z_1$

   Associative laws: $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3; z_1(z_2z_3) = (z_1z_2)z_3$

   Distributive law: $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$

5. Conjugate

   If z is a complex number, then the number obtained by changing the sign of its imaginary part is called the complex conjugate or, simply, the conjugate of z. If $z = x + iy$, then its conjugate is $\bar{z} = x - iy$

   a) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
   b) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$
   c) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$
   d) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$
   e) $z + \bar{z} = 2x \Rightarrow Re(z) = \frac{z + \bar{z}}{2}$
   f) $z - \bar{z} = 2iy \Rightarrow Im(z) = \frac{z - \bar{z}}{2i}$
   g) $z\bar{z} = x^2 + y^2$

6. Geometric Interpretation

   A complex number $z = x + iy$ is uniquely determined by an ordered pair of real number (x, y). In this manner we are able to associate a complex number $z = x + iy$ with a point (x, y) in a coordiante plane.

7. **Definition 17.1.3**: Modulus or Absolute Value

   The modulus or absolute value of $z = x + iy$, denoted by $|z|$, is the real number

   $$|z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$$

Proofs or Explanations:

1. Definition1

Examples:

1. (Addiction and Multiplication) If $z_1 = 2 + 4i$ and $z_2 = -3 + 8i$, find (a) $z_1 + z_2$ and (b) z_1z_2

   (a) $z_1 + z_2 = (2 + 4i) + (-3 + 8i) = -1 + 12i$

   (b) $z_1z_2 = (2 + 4i)(-3 + 8i) = -38 + 4i$

2. (Division) If $z_1 = 2 - 3i$ and $z_2 = 4 + 6i$, find (a) $\frac{\overline{z_1}}{\overline{z_2}}$ and (b) $\frac{1}{\overline{z_1}}$

   (a) $\dfrac{\overline{z_1}}{\overline{z_2}} = -\dfrac{5}{26} - \dfrac{6}{13}i$

   (b) $\dfrac{1}{\overline{z_1}} = \dfrac{2}{13} + \dfrac{3}{13}i$

17.2 Powers and Roots (P812)

October 12, 2016 11:32

Definitions & Theorems:

1. Polar Form

Let $x = r\cos\theta$, $y = r\sin\theta$, then a nonzero complex number $z = x + iy$ can be written as

$$z = r(\cos\theta + i\sin\theta)$$

which is the polar form of the complex number z.

$r = \sqrt{x^2 + y^2} = |z|$

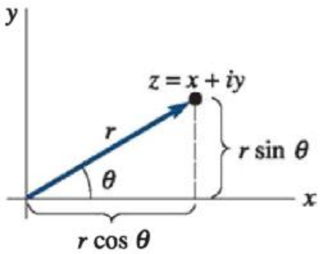


FIGURE 17.2.1 Polar coordinates

The angle θ is called an argument of z and is written $\theta = \arg z$.$\Rightarrow \tan\theta = \frac{y}{x}$

The argument of a complex number in the interval $-\pi < \theta \le \pi$ is called the principal argument of z and is denoted by $Arg\ z$. For exampel, $Arg(i) = \pi/2$.

2. Multiplication and Division, Powers of z

Suppose $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, where θ_1 and θ_2 are any arguments of z_1 and z_2, respectively. Then

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$
$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)], z_2 \ne 0$$

3. Powers of z

Suppose $z = r(\cos\theta + i\sin\theta)$. Then $z^n = r^n(\cos n\theta + i\sin n\theta)$, n is any integer.

4. DeMoivre's Formula

When $z = \cos\theta + i\sin\theta$, we have $r = \sqrt{x^2 + y^2} = |z| = 1$, and so

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

5. Roots

The nth roots of a nonzero complex number $z = r(\cos\theta + i\sin\theta)$ are given by

$$w_k = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

where $k = 0,1,2,\ldots,n-1$.

Proofs or Explanations:

1. Roots of a complex number

A number w is said to be an nth root of a nonzero complex number z if $w^n = z$. If we let $w = \rho(\cos\phi + i\sin\phi)$ and $z = r(\cos\theta + i\sin\theta)$ be the polar forms of w and z, then $w^n = z$ becomes

$$\rho^n(\cos n\phi + i\sin n\phi) = r(\cos\theta + i\sin\theta)$$
$$\Rightarrow \rho^n = r \Rightarrow \rho = r^{\frac{1}{n}}$$
$$\Rightarrow \cos n\phi = \cos\theta, \sin n\phi = \sin\theta$$
$$\Rightarrow n\phi = \theta + 2k\pi, \text{where } k \text{ is an interger}$$
$$\Rightarrow \phi = \frac{\theta + 2k\pi}{n}$$

As k takes on the successive integer values $k = 0,1,2,\ldots,n-1$, we obtain n $distinct$ roots with the same modulus but different arguments. But for $k \ge n$ we obtain the same roots because the sine and cosine are 2π-periodic.

$$\Rightarrow w_k = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

Examples:

1. (A Complex Number in Polar) Express $1 - \sqrt{3}i$ in polar form (P813).

With $x = 1$ and $y = -\sqrt{3}$, we obtain $r = \sqrt{x^2 + y^2} = |z| = 2$, $\tan\theta = \frac{y}{x} = -\sqrt{3} \Rightarrow \theta = \arg z = \frac{5\pi}{3}$, $Arg\ z = -\frac{\pi}{3}$

$$z = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right) \quad \text{or} \quad z = 2\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right]$$

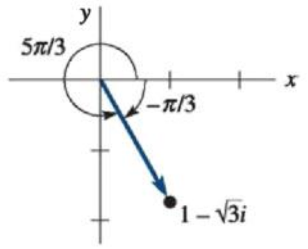


FIGURE 17.2.2 Two arguments of
$z = 1 - \sqrt{3}i$ in Example 1

2. (Roots of a Complex Number) Find the three cube roots of $z = i$.

   With $r = 1, \theta = \arg z = \pi/2$, the polar form of the given number is $z = \cos(\pi/2) + i\sin(\pi/2)$.

$$w_k = (1)^{\frac{1}{3}}\left[\cos\left(\frac{\pi/2 + 2k\pi}{3}\right) + i\sin\left(\frac{\pi/2 + 2k\pi}{3}\right)\right], k = 0,1,2.$$

   Hence, the three roots are

$$k = 0, w_0 = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$$
$$k = 1, w_0 = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$$
$$k = 0, w_0 = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -i$$

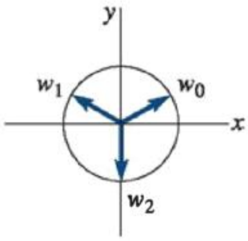


FIGURE 17.2.3 Three cube roots of i

3. (Tutorial) $w^4 + 1 = 0$

$$w^n = z \Rightarrow z = -1, n = 4$$
$$z = -1 \Rightarrow r = \sqrt{x^2 + y^2} = 1, \theta = \pi$$
$$\Rightarrow r = 1, n = 4, \theta = \pi, k = 0,1,2,3$$
$$w_k = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$
$$\Rightarrow w_k = (1)^{\frac{1}{3}}\left[\cos\left(\frac{\pi + 2k\pi}{4}\right) + i\sin\left(\frac{\pi + 2k\pi}{4}\right)\right], k = 0,1,2,3$$
$$\Rightarrow$$
$$w_0 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$$
$$w_1 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$$
$$w_2 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}$$
$$w_1 = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}$$

3.3 Homogeneous Equations with Constant Coefficients (P134)

October 19, 2016 12:23

Definitions & Theorems:

1. Auxiliary Equation:

Considering the special case of a second-order equation

$$ay'' + by' + cy = 0 \ \ (a, b, c \ are \ real \ constants)$$

Try a solution of the form $y = e^{mx}$, then after substitution

$$am^2 e^{mx} + bme^{mx} + ce^{mx} = 0$$

or

$$am^2 + bm + c = 0$$

This equation is called the auxiliary equation of the differential equation $ay'' + by' + cy = 0$

Since the two roots of the auxiliary equation are

$$m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad and \quad m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

there will be three forms of the general solution:

- m_1 and m_2 are real and distinct $(b^2 - 4ac) > 0$
- m_1 and m_2 are real and equal $(b^2 - 4ac) = 0$
- m_1 and m_2 are conjugate complex numbers $(b^2 - 4ac) < 0$

$Case \ I: Distinct \ Real \ Roots$

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$Case \ II: Repeated \ Real \ Roots$

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$$

For n-order: $y = c_1 e^{m_1 x} + c_2 x e^{m_1 x} + c_3 x^2 e^{m_1 x} + \cdots + c_n x^{(n-1)} e^{m_1 x}$

$Cse \ III: Conjugate \ Complex \ Roots$

If m_1 and m_2 are complex, then we can write $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, where $\alpha > 0, \beta > 0, i^2 = -1$.

$$y = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x \quad \text{or} \quad y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

Proofs or Explanations:

1. $Cse \ III: Conjugate \ Complex \ Roots$

If m_1 and m_2 are complex, then we can write $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, where $\alpha > 0, \beta > 0, i^2 = -1$.

$$y = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$$

Use Euler's formula:

$$e^{i\theta} = \cos\theta + i \sin\theta$$

where θ is any real number. It follows from this formula that

$$e^{i\beta x} = \cos \beta x + i \sin \beta x \quad \text{and} \quad e^{-i\beta x} = \cos \beta x - i \sin \beta x$$

$$\Rightarrow e^{i\beta x} + e^{-i\beta x} = 2\cos \beta x \quad \text{and} \quad e^{-i\beta x} - e^{-i\beta x} = 2i \sin \beta x$$

Since $y = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$ is a solution for any choice of the constant C_1 and C_2, the choice $C_1 = C_2 = 1$ and $C_1 = 1, C_2 = -1$ give, in turn, two solutions:

$$y_1 = e^{(\alpha + i\beta)x} + e^{(\alpha - i\beta)x} = e^{\alpha x}\left(e^{i\beta x} + e^{-i\beta x}\right) = 2e^{\alpha x} \cos \beta x$$

$$y_2 = e^{(\alpha + i\beta)x} - e^{(\alpha - i\beta)x} = e^{\alpha x}\left(e^{i\beta x} - e^{-i\beta x}\right) = 2ie^{\alpha x} \sin \beta x$$

Hence from Theorem 3.1.2 the last two results show that $e^{\alpha x} \cos \beta x$ and $e^{\alpha x} \sin \beta x$ are $real$ solutions. Moreover, these solutions from a fundamental set on $(-\infty, \infty)$. Consequently, the general solution is

$$y = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

Examples:

1. (Second-Order DEs) Solve the following differential equations.
   (a) $2y'' - 5y' + 3y = 0$
   (b) $y'' - 10y' + 25y = 0$
   (c) $y'' + 4y' + 7y = 0$

Solution:

(a) $a = 2, b = -5, c = -3$

$$\Rightarrow 2m_1^2 - 5m_2 - 3 = 0 \Rightarrow m_1 = -\frac{1}{2}, m_2 = 3$$

$$\Rightarrow y = c_1 e^{-\frac{x}{2}} + c_2 e^{3x}$$

(b) $a = 1, b = -10, c = 25$

$\Rightarrow m_1^2 - 10m_2 + 25 = 0 \Rightarrow m_1 = m_2 = 5$

$\Rightarrow y = c_1 e^{5x} + c_2 x e^{5x}$

(c) $a = 1, b = 4, c = 7$

$\Rightarrow m_1^2 + 4m_2 + 7 = 0 \Rightarrow m_1 = -2 + \sqrt{3}i, m_2 = -2 - \sqrt{3}i$ with $\alpha = -2, \beta = \sqrt{3}$

$\Rightarrow y = e^{-2x}\left(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x\right)$

2. (Third-Order DE) Solve $y''' + 3y'' - 4y = 0$

$m^3 + 3m^2 - 4 = 0$

$\Rightarrow (m-1)\left(m^2 + 4m^2 + 4\right) = (m-1)(m+2)^2 = 0$

$\Rightarrow m_1 = 1, m_2 = -2, m_3 = -2$

Thus the general solution is

$$y = c_1 e^{x} + c_2 e^{-2x} + c_3 x e^{-2x}$$

3.1 Theory of Linear Equations (P120)

October 12, 2016 12:43

Definitions & Theorems:

1. Initial-Value Problem:

   For a linear differential equation, an nth-order initial-value problem is

   $$\mathit{Solve}: \quad a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}} + \cdots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = g(x)$$

   $$\mathit{Subject\ to}: \ y(x_0) = y_0, y'(x_0) = y_1, \dots, y^{(n-1)}(x_0) = y_{n-1}$$

2. **Theorem 3.1.1**: Existence of a Unique Solution

   Let $a_n(x), a_{n-1}(x), \dots, a_1(x), a_0(x)$, and $g(x)$ be continuous on an interval I, and let $a_n(x) \neq 0$ for every x in this interval. If $x = x_0$ is any point in this interval, then a solution $y(x)$ of the initial-value problem exists on the interval and is unique.

3. Boundary-Value Problem:

   A problem such as

   $$\mathit{Solve}: \quad a_2(x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = g(x)$$

   $$\mathit{Subject\ to}: \ y(a) = y_0, y(b) = y_1$$

   is called a two-point boundary-value problem, or simply a boundary-value problem (BVP). The prescribed values $y(a) = y_0$ and $y(b) = y_1$ are called boundary conditions (BC).

   For a second-order differential equation, pairs of boundary conditions could be

   $$y'(a) = y_0, y'(b) = y_1$$
   $$y'(a) = y_0, y(b) = y_1$$
   $$y(a) = y_0, y'(b) = y_1$$
   $$y(a) = y_0, y(b) = y_1$$

   where y_0 and y_1 denote arbitrary constants.

   A BVP can have many, one or no solutions.

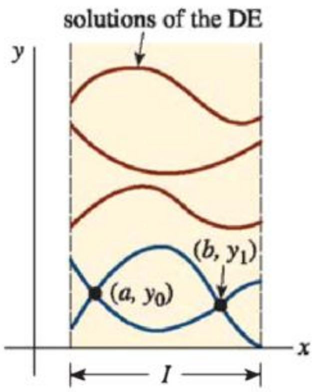


FIGURE 3.1.1 Colored curves are solutions of a BVP

4. Homogeneous Equations

   A linear nth-order differential equation of the form

   $$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}} + \cdots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = 0$$

   is said to be homogeneous, whereas an equation

   $$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}} + \cdots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = g(x)$$

   with $g(x)$ not identically zero, is said to be nonhomegeneous.

5. Differential Operators

   In calculus, differentiation is often denoted by the capital letter D; that is, $\frac{\mathrm{d}y}{\mathrm{d}x} = D$. The symbol D is called a differential operator.

   $$D(\cos 4x) = -4\sin 4x$$
   $$D(5x^3 - 6x^2) = 15x^2 - 12x$$
   $$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = D(Dy) = D^2 y$$
   $$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = D^n y$$

   In general, we define an nth-order differential operator to be

   $$L = a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x)$$

6. Differential Equations

   Any linear differential equation can be expressed in terms of the D notation. For example, the differential equations $y'' + 5y' +$

$6y = 5x - 3$ can be written as $D^2y + 5Dy + 6y = 5x - 3$ or $(D^2 + D + 6)y = 5x - 3$. Then, the linear nth-order differential equations in 4 can be written compactly as

$$L(y) = 0 \quad \text{and} \quad L(y) = g(x)$$

7. **Theorem 3.1.2**: Superposition Principle - Homogeneous Equations
   Let $y_1, y_2, \dots, y_k$ be solutions of the homogeneous nth-order differential equation on an interval I. Then the linear combination
   $$y = c_1y_1(x) + c_2y_2(x) + \cdots + c_ky_k(x)$$
   where the $c_i, i = 1,2, \dots, k$ are arbitrary constants, is also a solution on the interval.
8. **Definition 3.1.1**: Linear Dependence / Independence
   A set of functions $f_1(x), f_2(x), \dots f_n(x)$ is said to be linearly dependent on an interval I if there exist constant $c_1, c_2, \dots, c_n$, not all zero, such that
   $$c_1f_1(x) + c_2f_2(x) + \cdots + c_nf_n(x) = 0$$
   for every x in the interval. If the set of functions is not linearly dependent on the interval, it is said to be linearly independent.

   In other words, a set of functions is linearly independent on an interval if the only constants for which
   $$c_1f_1(x) + c_2f_2(x) + \cdots + c_nf_n(x) = 0$$
   for every x in the interval are $c_1 = c_2 = \cdots = c_n = 0$
9. **Definition 3.1.2**: Wronskian
   Suppose each of the function $f_1(x), f_2(x), \dots, f_n(x)$ possesses at least $n-1$ derivatives. The determinant
   $$W(f_1, f_2, \dots, f_n) = \begin{bmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{n-1} & f_2^{n-1} & \cdots & f_n^{n-1} \end{bmatrix}$$
   where the primes denote derivatives, is called the Wronskian of the functions.
10. **Theorem 3.1.3**: Criterion for Linearly Independent Solutions
    Let $y_1, y_2, \dots, y_n$ be n solutions of the homogeneous linear nth-order differential equation on an interval I. Then the set of solutions is linearly independent on I if and only if $W(y_1, y_2, \dots, y_n) \neq 0$ for every x in the interval.
11. **Definition 3.1.3**: Fundamental Set of Solutions
    Any set $y_1, y_2, \dots, y_n$ of n linearly solutions of the homogeneous linear nth-order diferential equation on an interval I is said to be a fundamental set of solutions on the interval.
12. **Theorem 3.1.5**: General Solution - Homogeneous Equations
    Let $y_1, y_2, \dots, y_n$ be a fundamental set of solutions of the homogeneous linear nth-order differential equations on an interval I. Then the general solution of the equation on the interval is
    $$y = c_1y_1(x) + c_2y_2(x) + \cdots c_ny_n(x)$$
    where $c_i, i = 1,2, \dots, n$ are arbitrary constants.

Proofs or Explanations:

1. Definition 3.1.1
   Two functions: $f_1(x)$ and $f_2(x)$
   If the functions are linearly dependent on an interval, then there exists constants c_1 and c_2 that are not both zero such that for every x in the interval $c_1f_1(x) + c_2f_2(x) = 0$. Therefore, if we assume that $c_1 \neq 0$, it follows that $f_1(x) = \left(-\frac{c_2}{c_1}\right) f_2(x)$, that is
   If two functions are linearly dependent, then one is simply a constant multiple of the other.
   Conversely, if $f_1(x) = c_2f_2(x)$ for some constant c_2, then $(-1) * f_1(x) + c_2f_2(x) = 0$ for every x on some interval. Hence the functions are linearly dependent, since at lease one of the constants is not zero. We conclude that:
   Two functions are linearly independent when neither is a constant multiple of the other on an interval.

Examples:

1. (Unique Solution of an IVP) The initial-value problem $3y''' + 5y'' - y' + 7y = 0, y(1) = 0, y'(1) = 0, y''(1) = 0$ possesses the trivial solution $y = 0$.
   1) $a_n(x) = a_3 = 3 \neq 0$
   2) $3, 5, -1, 7$ are continuous

   Then all the conditions of Theorem 3.1.1 are fulfilled. Hence $y = 0$ is the only solution on any interval containing $x = 1$.
2. (A BVP can have many, one, or no solutions) The two-parameter family of solutions of the differential equation $x'' + 16x = 0$ is $x = c_1 \cos 4t + c_2 \sin 4t$
   1) $x(0) = 0, x\left(\frac{\pi}{2}\right) = 0$
      $$x(0) = 0 \Rightarrow c_1 = 0$$
      $$x\left(\frac{\pi}{2}\right) = 0 \Rightarrow 0 = (0)(c_2)$$
      Infinitely many solutions.
   2) $x(0) = 0, x\left(\frac{\pi}{8}\right) = 0$
      $$x(0) = 0 \Rightarrow c_1 + 0 = 0 \Rightarrow c_1 = 0$$
      $$x\left(\frac{\pi}{8}\right) = 0 \Rightarrow c_1(0) + c_2(1) = 0 \Rightarrow 0 + c_2(1) = 0 \Rightarrow c_2 = 0$$

$x = 0$ is the only solution.

3) $x(0) = 0, x\left(\frac{\pi}{2}\right) = 1$

$x(0) = 0 \Rightarrow c_1 = 0$

$x\left(\frac{\pi}{2}\right) = 1 \Rightarrow 1 = 0$

No solution.

3. (Superposition - Homogeneous DE) The functions $y_1 = x^2$ and $y_2 = x^2 \ln x$ are both solutions of the homogeneous linear euqation $x^3y''' - 2xy' + 4y = 0$ on the interval $(0, \infty)$.
   By the superposition principle, the linear combination
   $$y = c_1x^2 + c_2x^2 \ln x$$
   is also a solution of the equation on the interval.
4. (Linearly Dependent Functions) The functions $f_1(x) = (\cos x)^2, f_2(x) = (\sin x)^2, f_3(x) = (\sec x)^2, f_4(x) = (\tan x)^2$ are linearly dependent on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ since
   $$c_1(\cos x)^2 + c_2(\sin x)^2 + c_3(\sec x)^2 + c_4(\tan x)^2 = 0,$$
   when $c_1 = c_2 = 1, c_3 = -1, c_4 = 1$.
5. (General Solutions of a Homogeneous DE) The functions $y_1 = e^{3x}$ and $y_2 = e^{-3x}$ are both solutions of the homogeneous linear equation $y'' - 9y = 0$ on the interval $(-\infty, \infty)$.
   Since $e^{3x} > 0, e^{-3x} > 0, \forall x \in \mathbb{R}$
   $c_1e^{3x} + c_2e^{-3x} = 0 \Rightarrow c_1 = 0, c_2 = 0$
   Therefore, the solutions are linear independent.
   $$W(e^{3x}, e^{-3x}) = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} = -6 \neq 0, \forall x \in \mathbb{R}$$
   $$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$
   We conclude that y_1 and y_2 form a fundamental set of solutions, and consequently $y = c_1e^{3x} + c_2e^{-3x}$ is the general solution of the equation on the interval.

Extra topics:

1. Abel's Theorem:

Abel's Theorem: If y_1 and y_2 are any two solutions of the equation

$$y'' + p(t)y' + q(t)y = 0,$$

where p and q are continuous on an open interval I. Then the Wronskian $W(y_1, y_2)(t)$ is given by

$$W(y_1, y_2)(t) = C\, e^{-\int p(t)dt},$$

where C is a constant that depends on y_1 and y_2, but not on t. Further, $W(y_1, y_2)(t)$ is either zero for all t in I (if $C = 0$) or else is never zero in I (if $C \neq 0$).

3.1.3 Nonhomogeneous Equations

If $y_1, y_2, \ldots, y_k$ are solutions of $L(y) = 0$ and y_p is any particular solutions of $L(y) = y$ on I, then the linear combination
$$y = c_1y_1(x) + c_2y_2(x) + \cdots + c_ky_k(x) + y_p$$
is also a solution of the nonhomogeneous equation.

Examples:

1. (General Solutions of a Non-homogeneous DE) By substitution, the function $y_p = -\frac{11}{12} - \frac{1}{2}x$ is readily shown to be a particular solution of the nonhomogeneous equation
   $$y''' - 6y'' + 11y' - 6y = 3x$$
   The general solution of the associated homogeneous equation
   $$y''' - 6y'' + 11y' - 6y = 0$$
   is
   $$y_c = c_1e^x + c_2e^{2x} + c_3e^{3x}$$
   Hence the general solution of non-homogeneous on the interval is
   $$y = y_c + y_p = c_1e^x + c_2e^{2x} + c_3e^{3x} - \frac{11}{12} - \frac{1}{2}x$$
2. (Superposition - Nonhomogeneous DE)

$$
\begin{aligned}
&y_{p_1} = -4x^2 && \text{is a particular solution of} && y'' - 3y' + 4y = -16x^2 + 24x - 8,\\
&y_{p_2} = e^{2x} && \text{is a particular solution of} && y'' - 3y' + 4y = 2e^{2x},\\
&y_{p_3} = xe^{x} && \text{is a particular solution of} && y'' - 3y' + 4y = 2xe^{x} - e^{x}.
\end{aligned}
$$

It follows from Theorem 3.1.7 that the superposition of y_{p_1}, y_{p_2}, and y_{p_3},

$$y = y_{p_1} + y_{p_2} + y_{p_3} = -4x^2 + e^{2x} + xe^{x},$$

is a solution of

$$y'' - 3y' + 4y = \underbrace{-16x^2 + 24x - 8}_{g_1(x)} + \underbrace{2e^{2x}}_{g_2(x)} + \underbrace{2xe^{x} - e^{x}}_{g_3(x)}.$$

3.4 Undetermined Coefficients (P141)

October 24, 2016 11:59

Definitions & Theorems:

1. Method of Undetermined Coefficients

   $g(x)$ is a constant, a polynomial function, exponential function $e^{\alpha x}$, sine or cosine function $\sin\beta x$ or $\cos\beta x$, or finite sums and products of these functions.

   The method of undetermined coefficients is not applicable to equations when $g(x)=\ln x, g(x)=\frac{1}{x}, g(x)=\tan x, g(x)=\sin^{-1}x$

TABLE 3.4.1 Trial Particular Solutions

$g(x)$	Form of y_p
1. 1 (any constant)	A
2. $5x+7$	$Ax+B$
3. $3x^2-2$	Ax^2+Bx+C
4. x^3-x+1	Ax^3+Bx^2+Cx+E
5. $\sin 4x$	$A\cos 4x+B\sin 4x$
6. $\cos 4x$	$A\cos 4x+B\sin 4x$
7. e^{5x}	Ae^{5x}
8. $(9x-2)e^{5x}$	$(Ax+B)e^{5x}$
9. x^2e^{5x}	$(Ax^2+Bx+C)e^{5x}$
10. $e^{3x}\sin 4x$	$Ae^{3x}\cos 4x+Be^{3x}\sin 4x$
11. $5x^2\sin 4x$	$(Ax^2+Bx+C)\cos 4x+(Ex^2+Fx+G)\sin 4x$
12. $xe^{3x}\cos 4x$	$(Ax+B)e^{3x}\cos 4x+(Cx+E)e^{3x}\sin 4x$

Proofs or Explanations:

1. Definition1

Examples:

1. (General Solution Using Undetermined Coefficients) Solve $y''+4y'-2y=2x^2-3x+6$ (P142)

   Solution:

   1) $y''+4y'-2y=0\Rightarrow m^2+4m-2=0\Rightarrow m_1=-2+\sqrt{6}, m_2=-2-\sqrt{6}$

      Hence the complementary function is

      $$y_c=c_1e^{(-2+\sqrt{6})x}+c_2e^{(-2-\sqrt{6})x}$$

   2) $g(x)$ is a quadratic polynomial, let us assume a particular solution that is also in the form of a quadratic polynomial: $y_p=Ax^2+Bx+C$

      $$\Rightarrow y_p'=2Ax+B, y_p''=2A$$
      $$\Rightarrow y_p''+4y_p'-2y_p=2A+4(2Ax+B)-2\left(Ax^2+Bx+C\right)=2x^2-3x+6$$
      $$\Rightarrow -2Ax^2+(8A-2B)x+(2A+4B-2C)=2x^2-3x+6$$
      $$\Rightarrow -2A=2, 8A-2B=-3, 2A+4B-2C=6$$
      $$\Rightarrow A=-1, B=-\frac{5}{2}, C=-9$$
      $$\Rightarrow y_p=-x^2-\frac{5}{2}x-9$$

   3) The general solution of the given equation is

      $$y=y_c+y_p=c_1e^{(-2+\sqrt{6})x}+c_2e^{(-2-\sqrt{6})x}+\left(-x^2-\frac{5}{2}x-9\right)$$

2. (Particular Solution Using Undetermined Coefficients) Find a particular solution of $y''-y'+y=2\sin 3x$

   We assume a particular solution: $y_p=A\cos 3x+B\sin 3x$

   $$y_p''-y_p'+y_p=(-9A\cos 3x-9B\sin 3x)-(-3A\sin 3x+3B\cos 3x)+(A\cos 3x+B\sin 3x)=2\sin 3x$$
   $$\Rightarrow(-8A-3B)\cos 3x+(3A-8B)\sin 3x=2\sin 3x$$
   $$\Rightarrow -8A-3B=0, 3A-8B=2$$
   $$\Rightarrow A=\frac{6}{73}, B=-\frac{16}{73}$$

A particular solution of the equation is

$$y_p = \frac{6}{73}\cos 3x - \frac{16}{73}\sin 3x$$

3. (Forming y_p by Superposition) Solve $y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}$
   1) $y'' - 2y' - 3y = 0 \Rightarrow m^2 - 2m - 3 = 0$

      The complementary solution is

      $$y_c = c_1 e^{-x} + c_2 e^{3x}$$
   2) $g(x) = g_1(x) + g_2(x) = polynomial + exponentials$

      $\Rightarrow y_p = y_{p_1} + y_{p_2}, y_{p_1} = Ax + B, y_{p_2} = (Cx + E)e^{2x}$

      $\Rightarrow y_p = y_{p_1} + y_{p_2} = Ax + B + Cxe^{2x} + Ee^{2x}$

      $\Rightarrow y_p'' - 2y_p' - 3y_p = -3Ax - 2A - 3B - 3Cxe^{2x} + (2C - 3E)e^{2x} = 4x - 5 + 6xe^{2x}$

      $\Rightarrow -3A = 4, -2A - 3B = -5, -3C = 6, 2C - 3E = 0$

      $\Rightarrow A = -\frac{4}{3}, B = \frac{23}{9}, C = -2, E = -\frac{4}{3}$

      Consequently,

      $$y_p = -\frac{4}{3}x + \frac{23}{9} - 2xe^{2x} - \frac{4}{3}e^{2x}$$
   3) The general solution of the equation is

      $$y = c_1 e^{-x} + c_2 e^{3x} - \frac{4}{3}x + \frac{23}{9} - \left(2x + \frac{4}{3}\right)e^{2x}$$

4. (General Solution Using Undetermined Coefficients) Solve $y'' - y' + \frac{1}{4}y = 3 + e^{\frac{x}{2}}$
   1) $y_c: m^2 - m + \frac{1}{4} = 0 \Rightarrow m_1 = m_2 = \frac{1}{2} \Rightarrow y_1 = e^{\frac{x}{2}}, y_2 = xe^{\frac{x}{2}}$

      $$y_c = c_1 e^{\frac{x}{2}} + c_2 x e^{\frac{x}{2}}$$
   2) y_p:

      $$Let\ y_p = A + Be^{\frac{x}{2}}, but\ y_1 = e^{\frac{x}{2}x}, so$$

      $$Let\ y_p = A + Bxe^{\frac{x}{2}}, but\ y_2 = xe^{\frac{x}{2}}, so$$

      $$Let\ y_p = A + Bx^2 e^{\frac{x}{2}}$$

      $$\Rightarrow y_p' = 2Bxe^{\frac{x}{2}} + \frac{1}{2}Bx^2 e^{\frac{x}{2}}$$

      $$\Rightarrow y_p'' = 2Be^{\frac{x}{2}} + Bxe^{\frac{x}{2}} + Bxe^{\frac{x}{2}} + \frac{1}{4}Bx^2 e^{\frac{x}{2}}$$

      $$\Rightarrow A = 12, B = \frac{1}{2}$$

      $$\Rightarrow y_p = 12 + \frac{1}{2}x^2 e^{\frac{x}{2}}$$
   3) The general solution of the equation is

      $$y = y_c + y_p$$

3.5 Variation of Parameters (P150)

October 26, 2016 11:50

Definitions & Theorems:

1. Method of Variation of Parameters:

   To solve $a_2y'' + a_1y' + a_0 = g(x)$

   1) Find the complementary function

      Write the DE in the standard form
      $$y'' + P(x)y' + Q(x)y = f(x)$$
      $$\Rightarrow y_c = c_1y_1 + c_2y_2$$

   2) Compute the Wronskian $W(y_1, y_2)$

   3) Find u_1, u_2 by integrating
      $$u_1' = \frac{W_1}{W}, u_2' = \frac{W_2}{W}$$
      where
      $$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

   4) A particular solution is
      $$y_p = u_1y_1 + u_2y_2$$

   5) The general solution is then $y = y_c + y_p$

Examples:

1. (Variation of Parameters) Solve $y'' - 4y' + 4y = (x+1)e^{2x}$

   1) $y'' - 4y' + 4y = 0 \Rightarrow m^2 - 4m + 4 = 0 \Rightarrow m_1 = m_2 = 2$

      $\Rightarrow y_c = c_1e^{2x} + C_2xe^{2x}, y_1 = e^{2x}, y_2 = xe^{2x}$

   2) $W(e^{2x}, xe^{2x}) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = e^{4x}$

   3) Since the given differential equation is already in the standard form (that is, the coefficient of y'' is 1), we identify $f(x) = (x+1)e^{2x}$
      $$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} = \begin{vmatrix} 0 & xe^{2x} \\ (x+1)e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = -(x+1)xe^{4x}$$
      $$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} == \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (x+1)e^{2x} \end{vmatrix} = (x+1)e^{4x}$$
      $$\Rightarrow$$
      $$u_1' = \frac{W_1}{W} = \frac{-(x+1)xe^{4x}}{e^{4x}} = -x^2 - x$$
      $$u_2' = \frac{W_2}{W} = \frac{(x+1)e^{4x}}{e^{4x}} = x + 1$$
      $$\Rightarrow$$
      $$u_1 = -\frac{1}{3}x^3 - \frac{1}{2}x^2, u_2 = \frac{1}{2}x^2 + x$$

   4) $y_p = \left(-\frac{1}{3}x^3 - \frac{1}{2}x^2\right)e^{2x} + \left(\frac{1}{2}x^2 + x\right)xe^{2x} = \frac{1}{6}x^3e^{2x} + \frac{1}{2}x^2e^{2x}$

   5) $y = y_c + y_p = c_1e^{2x} + c_2xe^{2x} + \frac{1}{6}x^3e^{2x} + \frac{1}{2}x^2e^{2x}$

2. (Variation of Parameters) Solve $4y'' + 36y = \csc 3x$

   Put the equation in the standard form by dividing by 4:
   $$y'' + 9y = \frac{1}{4}\csc 3x$$

   1) $y'' + 9y = 0 \Rightarrow m^2 + 9 = 0 \Rightarrow m_1 = 3i, m_2 = -3i$

      $\Rightarrow y_c = c_1 \cos 3x + c_2 \sin 3x, y_1 = \cos 3x, y_2 = \sin 3x, f(x) = \frac{1}{4}\csc 3x$

   2) $W(\cos 3x, \sin 3x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3$

   3) $W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} = \begin{vmatrix} 0 & \sin 3x \\ \frac{1}{4}\csc 3x & 3\cos 3x \end{vmatrix} = -\frac{1}{4}$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} = \begin{vmatrix} \cos 3x & 0 \\ -3\sin 3x & \frac{1}{4}\csc 3x \end{vmatrix} = \frac{1}{4}\frac{\cos 3x}{\sin 3x}$$

$\Rightarrow$

$$u_1' = \frac{W_1}{W} = \frac{-\frac{1}{4}}{3} = -\frac{1}{12}$$

$$u_2' = \frac{W_2}{W} = \frac{\frac{1}{4}\frac{\cos 3x}{\sin 3x}}{3} = \frac{1}{12}\frac{\cos 3x}{\sin 3x}$$

$\Rightarrow$

$$u_1 = -\frac{1}{12}x, u_2 = \frac{1}{36}\ln|\sin 3x|$$

4) $y_p = -\frac{1}{12}x\cos 3x + \frac{1}{36}\ln|\sin 3x|\sin 3x$

5) $y = y_c + y_p = c_1\cos 3x + c_2\sin 3x - \frac{1}{12}x\cos 3x + \frac{1}{36}\ln|\sin 3x|\sin 3x$

3.6 Cauchy-Euler Equations (P155)

October 26, 2016 12:42

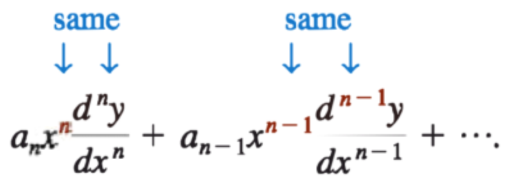
Definitions & Theorems:

1. Cauchy-Euler Equation:

Any linear differential equation of the form

$$a_n x^n \frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1} x^{n-1} \frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + a_1 x \frac{\mathrm{d}y}{\mathrm{d}x} + a_0 y = g(x)$$

where the coefficients $a_n, a_{n-1}, \ldots, a_0$ are constants, is known diversely as a Cauchy-Euler equation.
The observable characteristic of this type of equation is that the degree $k = n, n-1, \ldots, 1, 0$ of the monomial coefficients x^k matches the order of differentiation $\frac{\mathrm{d}^k y}{\mathrm{d}x^k}$:



2. Method of Solution:

We try a solution of the form $y = x^m$, where m is to be determined.

$$a_k x^k \frac{\mathrm{d}^k y}{\mathrm{d}x^k} = a_k x^k m(m-1)(m-2)\ldots(m-k+1)x^{m-k}$$
$$= a_k m(m-1)(m-2)\ldots(m-k+1)x^m$$

We start the discussion with a detailed examination of the forms of the general solutions of the homogeneous second-order equation

$$ax^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + bx \frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0$$

For example, by substituting $y = m^x$ the second-order equations becomes

$$ax^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + bx \frac{\mathrm{d}y}{\mathrm{d}x} + cy = am(m-1)x^m + bmx^m + cx^m = (am(m-1) + bm + c)x^m$$

Thus $y = x^m$ is a solution of the differential equation whenever m is a solution of the auxiliary equation

$$am^2 + (b-a)m + c = 0$$

Case I: *Distinct Real Roots*

$$y = c_1 x^{m_1} + c_2 x^{m_2}$$

Case II: *Repeated Real Roots*

$$y = c_1 x^{m_1} + c_2 x^{m_1} \ln x$$

For n-order: $y = c_1 x^{m_1} + c_2 x^{m_1} \ln x + c_3 x^{2m_1} \ln x + \cdots + c_n x^{(n-1)m_1} \ln x$

Cse III: *Conjugate Complex Roots*

If m_1 and m_2 are complex, then we can write $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, where $\alpha > 0, \beta > 0, i^2 = -1$.

$$y = x^\alpha [c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)]$$

Examples:

1. (Distinct Roots) Solve $x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2x \frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 0$

Assume $y = x^m$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = mx^{m-1}, \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = m(m-1)x^{m-2}$$

Substitute back into the differential equation:

$$x^2 m(m-1)x^{m-2} - 2xmx^{m-1} + 4x^m = 0$$
$$\Rightarrow (m^2 - m)x^m - 2mx^m + 4x^m = 0$$
$$\Rightarrow m^2 - m - 2m + 4 = 0$$
$$\Rightarrow m^2 - 3m + 4 = 0 \quad [am^2 + (b-a)m + c = 0]$$
$$\Rightarrow m_1 = -1, m_2 = 4$$

The general solution is:

$$y = c_1 x^{-1} + c_2 x^4$$

2. (Repeated Roots) Solve $4x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 8x \frac{\mathrm{d}y}{\mathrm{d}x} + y = 0$

Assume $y = x^m$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = mx^{m-1}, \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = m(m-1)x^{m-2}$$

Substitute back into the differential equation:

$$4x^2m(m-1)x^{m-2} + 8xmx^{m-1} + x^m = 0$$
$$\Rightarrow 4\left(m^2 - m\right)x^m + 8mx^m + x^m = 0$$
$$\Rightarrow 4m^2 - 4m + 8m + 1 = 0$$
$$\Rightarrow 4m^2 + 4m + 1 = 0 \quad [am^2 + (b-a)m + c = 0]$$
$$\Rightarrow (2m+1)^2 = 0$$
$$\Rightarrow m_1 = m_2 = -\frac{1}{2}$$

The general solution is:

$$y = c_1 x^{-\frac{1}{2}} + c_2 x^{-\frac{1}{2}} \ln x$$

3. (Conjugate Complex Roots, An Initial-Value Problem) Solve the initial-value problem $4x^2y'' + 17y = 0, y(1) = -1, y'(1) = -\frac{1}{2}$

Assume $y = x^m$,

$$a = 4, b = 0, c = 17$$
$$\Rightarrow am^2 + (b-a)m + c = 4m^2 - 4m + 17 = 0$$
$$\Rightarrow m_1 = \frac{1}{2} + 2i, m_2 = \frac{1}{2} = 2i$$
$$\Rightarrow \alpha = \frac{1}{2}, \beta = 2$$
$$\Rightarrow y = x^{\frac{1}{2}}[c_1 \cos(2\ln x) + c_2 \sin(2\ln x)]$$
$$y(1) = c_1 \cos 0 + c_2 \sin 0 = c_1 = -1$$
$$y' = \frac{1}{2}x^{-\frac{1}{2}}[c_1 \cos(2\ln x) + c_2 \sin(2\ln x)] + x^{\frac{1}{2}}\left[-c_1 \sin(2\ln x)\left(\frac{2}{x}\right) + c_2 \cos(2\ln x)\left(\frac{2}{x}\right)\right]$$
$$y'(1) = \frac{1}{2}\left(c_1 \cos 0 + c_2 \sin 0\right) + \left(-2c_1 \sin 0 + 2c_2 \cos 0\right) = \frac{1}{2}(-1) + 2c_2 = -\frac{1}{2} \Rightarrow c_2 = 0$$
$$\Rightarrow y = x^{\frac{1}{2}}[-\cos(2\ln x)]$$

4. (Third-Order Equation) Solve $x^3\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 5x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 7x\frac{\mathrm{d}y}{\mathrm{d}x} + 8y = 0$

Assume $y = x^m$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = mx^{m-1}, \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = m(m-1)x^{m-2}, \frac{\mathrm{d}^3y}{\mathrm{d}x^3} = m(m-1)(m-2)x^{m-3}$$

Substitute back into the differential equation:

$$x^3m(m-1)(m-2)x^{m-3} + 5x^2m(m-1)x^{m-2} + 7xmx^{m-1} + 8x^m = 0$$
$$\Rightarrow \left(m^3 - 3m^2 + 2m\right)x^m + 5\left(m^2 - m\right)x^m + 7mx^m + 8x^m = 0$$
$$\Rightarrow m^3 + 2m^2 + 4m + 8 = 0$$
$$\Rightarrow (m+2)\left(m^2 + 4\right) = 0$$
$$\Rightarrow m_1 = -2, m_2 = 2i, m_3 = -2i$$
$$\Rightarrow \alpha = 0, \beta = 2$$

The general solution is:

$$y = c_1 x^{-2} + c_2 \cos(2\ln x) + c_3 \sin(2\ln x)$$

5. (Variation of Parameters) Solve $x^2y'' - 3xy' + 3y = 2x^4e^x$

a) Solve the associated homogeneous equation

$$x^2y'' - 3xy' + 3y = 0$$
$$\Rightarrow a = 1, b = -3, c = 3$$
$$\Rightarrow m^2 - 4m + 3 = 0$$
$$\Rightarrow m_1 = 1, m_2 = 3 \Rightarrow y_1 = x, y_2 = x^3$$
$$\Rightarrow y_c = c_1 x + c_2 x^3$$

b) Put the differential equation into the standard form $y'' + P(x)y' + Q(x)y = f(x)$

$$y'' - \frac{3}{x}y' + \frac{3}{x^2}y = 2x^2e^x \Rightarrow f(x) = 2x^2e^x$$

1) $W\left(x, x^3\right) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 2x^3$

2) $W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} = \begin{vmatrix} 0 & x^3 \\ 2x^2e^x & 3x^2 \end{vmatrix} = -2x^5e^x$

$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} = \begin{vmatrix} x & 0 \\ 1 & 2x^2e^x \end{vmatrix} = 2x^3e^x$

$\Rightarrow$

$$u_1' = \frac{W_1}{W} = \frac{-2x^5e^x}{2x^3} = -x^2e^x$$
$$u_2' = \frac{W_2}{W} = \frac{2x^3e^x}{2x^2} = e^x$$

$\Rightarrow$

$$u_1 = -x^2e^x + 2xe^x - 2e^x, u_2 = e^x$$

3) $y_p = u_1y_1 + u_2y_2 = \left(-x^2e^x + 2xe^x - 2e^x\right)(x) + \left(e^x\right)\left(x^3\right)$

4) $y = y_c + y_p$

3.7 Nonlinear Equations, Reduction of Order (P162)

October 31, 2016 12:43

Definitions & Theorems:

1. Reduction of Order:
   a. $F(x, y', y'') = 0$, "y" is missing
   $$u = y'$$
   $$\frac{\mathrm{d}u}{\mathrm{d}x} = y''$$
   b. $F(y, y', y'') = 0$, "x" is missing
   $$u = y'$$
   $$\frac{\mathrm{d}u}{\mathrm{d}x} = y'' = \frac{\mathrm{d}u}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}u}{\mathrm{d}y}$$

Examples:

1. (Dependent Variable y is Missing) Solve $y'' = 2x(y')^2$
   Let $u = y'$, then $\frac{\mathrm{d}u}{\mathrm{d}x} = y''$.
   $$\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = y'' = 2x(y')^2 = 2u^2$$
   $$\Rightarrow \frac{\mathrm{d}u}{u^2} = 2x\,\mathrm{d}x$$
   $$\Rightarrow -u^{-1} = x^2 + c_1^2$$
   $$\Rightarrow -\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x^2 + c_1^2}$$
   $$\Rightarrow \mathrm{d}y = -\frac{1}{x^2 + c_1^2}\mathrm{d}x$$
   $$\Rightarrow y = -\frac{1}{c_1}\tan^{-1}\frac{x}{c_1} + c_2$$
2. (Dependent Variable x is Missing) Solve $yy'' = (y')^2$ (P163)
   Let $u = y'$
   $$y'' = \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}u}{\mathrm{d}y}$$
   $$\Rightarrow y\left(u\frac{\mathrm{d}u}{\mathrm{d}y}\right) = u^2$$
   $$\Rightarrow \frac{\mathrm{d}u}{u} = \frac{\mathrm{d}y}{y}$$
   $$\Rightarrow \ln|u| = \ln|y| + c_1 \Rightarrow u = c_2 y$$
   $$\Rightarrow c_2 y = \frac{\mathrm{d}y}{\mathrm{d}x}$$
   $$\Rightarrow \frac{\mathrm{d}y}{y} = c_2\,\mathrm{d}x$$
   $$\Rightarrow \ln|y| = c_2 x + c_3$$
   $$\Rightarrow y = c_4 e^{c_2 x}$$
3. (Taylor Series Solution of an IVP) Let us assume that a solution of the initial-value problem
   $$y'' = x + y - y^2, y(0) = -1, y'(0) = 1$$
   exists. If we further assume that the solution $y(x)$ of the problem is analytic at 0, then $y(x)$ possesses a Taylor series expansion centered at 0:
   $$y(x) = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \frac{y^{(5)}(0)}{5!}x^5 + \cdots$$
   Moreover, $y(0) = -1, y'(0) = 1, y''(0) = x + y(0) - y(0)^2 = 0 + (-1) - (-1)^2 = -2$
   We can then find expressions for the higher derivatives $y''', y^{(4)}, \ldots$, by calculating the successive derivatives of the differential equation:
   $$y'''(x) = \frac{\mathrm{d}}{\mathrm{d}x}(x + y - y^2) = 1 + y' - 2yy' \Rightarrow y'''(0) = 1 + (1) - 2(-1)(1) = 4$$
   $$y^{(4)}(x) = \frac{\mathrm{d}}{\mathrm{d}x}(1 + y' - 2yy') = y'' - 2(y')^2 - 2yy'' \Rightarrow y^{(4)}(0) = (-2) - 2(1)^2 - 2(-1)(-2) = -8$$

$$y^{(5)}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(y'' - 2\left(y'\right)^2 - 2yy''\right) = y''' - 2yy''' - 6y'y'' \Rightarrow y^{(5)}(0) = (4) - 2(-1)(4) - 6(1)(-2) = 24$$

Hence, the first six terms of a series solution of the initial-value problem are

$$y(x) = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \frac{y^{(5)}(0)}{5!}x^5 + \cdots$$

$$= -1 + x - x^2 + \frac{2}{3}x^3 - \frac{1}{3}x^4 + \frac{1}{5}x^5 + \cdots$$

3.8 Linear Models: Initial-Value Problems (P166)

November 2, 2016 11:52

Linear systems to be considered in this section:

$$a_2\frac{\mathrm{d}^2t}{\mathrm{d}t^2}+a_1\frac{\mathrm{d}y}{\mathrm{d}t}+a_0y=\ g(t),y(t_0)=y_0,y'(t_0)=\ y_1$$

3.8.1 Spring/Mass Systems: Free Undamped Motion

Definitions & Theorems:

1. Elongation:
   The amount of stretch
2. Equilibrium:
   A position at which its weight W is balanced by the restoring force ks.
3. Amplitude of free vibrations:
   $A=\sqrt{c_1^2+c_2^2}$
4. g:
   $g=32ft/s^2=9.8m/s^2$

Proofs or Explanations:

1. Hooke's Law
   By Hooke's law, the spring itself exerts a restoring force F opposite to the direction of elongation and proportional to the amount of elognation s. Simply stated,
   $$\vec{F}=ks$$
   where k is a constant of proporitonality called the spring constant.
2. Newton's Second Law
   $$\vec{F}=m\vec{a}$$
   $$ma=kx\Leftrightarrow m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=-kx$$
3. DE of Free Undamped Motion
   $$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=-kx\Rightarrow\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=-\frac{k}{m}x$$
   $$\Rightarrow\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+\omega^2x=0,\text{where }\omega=\frac{k}{m}$$
   This equation is said to describe simple harmonic motion or free undamped motion.

   Two obvious initial conditions associated with this equation:
   1) $x(0)=x_0$, the amount of initial displacment.
   2) $x'(0)=x_1$, the initial velocity of the mass.
4. Solution and Equation of Motion
   The solutions of the auxiliary equation $m^2+\omega^2=0$ are the complex number $m_1=\omega i,m_2=-\omega i,\alpha=0,\beta=1$.
   Thus the general solution is
   $$x(t)=c_1\cos\omega t+c_2\sin\omega t$$
   The period of free vibrations described by the above is $T=2\pi/\omega$, and the frequency is $f=1/T=\omega/2\pi$.
   For example, for $x(t)=2\cos3t-4\sin3t$ the period is $2\pi/3$ and the frequency is $\frac{3}{2\pi}$
5. Alternative form of $x(t)$
   $$y=A\sin(\omega t+\phi),where\sin\phi=\frac{c_1}{A},\cos\phi=\frac{c_2}{A},\tan\phi=\frac{c_1}{c_2}$$
   $$y=A\cos(\omega t+\phi),where\sin\phi=\frac{c_2}{A},\cos\phi=\frac{c_1}{A},\tan\phi=\frac{c_2}{c_1}$$

Examples:

1. (Free Undamped Motion) A mass weighing 2 pounds stretches a spring 6 inches. At $t=0$ the mass is released from a point 8 inches below the equilibrium position with an upward velocity of $\frac{4}{3}$ ft/s. Determine the equation of free motion. (P167)
   1) Convert measurement:
      $$6\ inches=\frac{1}{2}ft,8\ inches=\frac{2}{3}ft$$
   2) $x(0)=\frac{2}{3}ft,x'(0)=-\frac{4}{3}ft/s$
   3) $m=\frac{W}{g}=\frac{2\ pounds}{\frac{32ft}{s^2}}=\frac{1}{16}slug$
   4) $F=ks=k\left(\frac{1}{2}\right)=W=2\Rightarrow k=4\frac{lb}{ft}$
   5) $\frac{1}{16}\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=-4x\Rightarrow\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+64x=0\Rightarrow x=c_1\cos8t+c_2\sin8t,\omega=8$
   6) $c_1=\frac{2}{3},c_2=-\frac{1}{6},x(t)=\frac{2}{3}\cos8t-\frac{1}{6}\sin8t$
2. (Alternative Form of Solution of Example 1) (P168)
   $$A=\sqrt{c_1^2+c_2^2}=\sqrt{\left(\frac{2}{3}\right)^2+\left(-\frac{1}{6}\right)^2}=\frac{\sqrt{17}}{6}\approx0.69ft$$
   1) With $c_1=\frac{2}{3},c_2=-\frac{1}{6}$ we find $\tan\phi=-4$.
   2) With $c_1=\frac{2}{3},c_2=-\frac{1}{6}$ we find $\tan\phi=-\frac{1}{4}$.

3.8.2 Spring/Mass Systems: Free Damped Motion (P168)

Proofs or Explanations:

1. DE of Free Damped Motion

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx - \beta\frac{\mathrm{d}x}{\mathrm{d}t}$$

where β is a positive damping constant and the negative sign is a consequence of the fact that the damping force acts in a direction opposite to the motion.

$$\Rightarrow \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{\beta}{m}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{k}{m}x = 0$$

$$\Rightarrow \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\lambda\frac{\mathrm{d}x}{\mathrm{d}t} + \omega^2x = 0, \text{where } 2\lambda = \frac{\beta}{m}, \omega^2 = \frac{k}{m}$$

$$\Rightarrow m_1 = -\lambda + \sqrt{\lambda^2 - \omega^2}, m_2 = -\lambda - \sqrt{\lambda^2 - \omega^2}$$

1) $Case\ I: \lambda^2 - \omega^2 > 0$ (Overdamped)

$$x(t) = e^{-\lambda t}\left(c_1e^{\sqrt{\lambda^2-\omega^2}t} + c_2e^{-\sqrt{\lambda^2-\omega^2}t}\right)$$

2) $Case\ II: \lambda^2 - \omega^2 = 0$ (Critically damped)

$$x(t) = e^{-\lambda t}(c_1 + c_2t)$$

3) $Case\ III: \lambda^2 - \omega^2 < 0$ (Underdamped)

$$m_1 = -\lambda + \sqrt{\omega^2 - \lambda^2}i, m_2 = -\lambda - \sqrt{\omega^2 - \lambda^2}i$$

$$x(t) = e^{-\lambda t}\left(c_1\cos\sqrt{\omega^2 - \lambda^2}t + c_2\sin\sqrt{\omega^2 - \lambda^2}t\right)$$

Examples:

1. (Overdamped Motion) Solve the initial-value problem

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 5\frac{\mathrm{d}x}{\mathrm{d}t} + 4x = 0, x(0) = 1, x'(0) = 1$$

$$m^2 + 5m + 4 = 0 \Rightarrow (m+1)(m+4) = 0 \Rightarrow m_1 = -1, m_2 = -4$$

$$x(t) = c_1e^{-t} + c_2e^{-4t}$$

$$\Rightarrow x(t) = \frac{5}{3}e^{-t} - \frac{2}{3}e^{-4t}$$

3.8.3 Spring/Mass Systems: Driven Motion (P172)

Proofs or Explanations:

1. DE of Driven Motion with Damping

An external $f(t)$ acting on a vibrating mass on a spring. For example, $f(t)$ could represent a driving force causing on oscillatory vertical motion of the support of the spring.

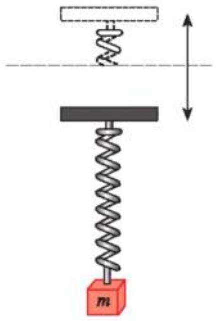


FIGURE 3.8.11 Oscillatory vertical motion of the support

The inclusion of $f(t)$ in the formulation of Newton's second law gives the differential equation of driven or forced motion:

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx - \beta\frac{\mathrm{d}x}{\mathrm{d}t} + f(t)$$

$$\Rightarrow \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{\beta}{m}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{k}{m}x = \frac{f(t)}{m}$$

$$\Rightarrow \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\lambda\frac{\mathrm{d}x}{\mathrm{d}t} + \omega^2x = F(x), \text{where } 2\lambda = \frac{\beta}{m}, \omega^2 = \frac{k}{m}, F(t) = \frac{f(t)}{m}$$

2. Transient and Steady-State Terms

When F is a periodic function, such as $F(t) = F_0\sin\gamma t$ or $F(t) = F_0\cos\gamma t$, the general solution for $\lambda > 0$ is the sum of a nonperiodic function $x_c(t)$ and a periodic function $x_p(t)$. The complementary function $x_c(t)$ is said to be a transient term of transient solution, and the function $x_p(t)$ is called a steady-state term or steady-state solution.

Examples:

1. (Interpretation of an Initial-Value Problem) Interpret and solve the initial-value problem (P173)

$$\frac{1}{5}\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 1.2\frac{\mathrm{d}x}{\mathrm{d}t} + 2x = 5\cos 4t, x(0) = \frac{1}{2}, x'(0) = 0$$

Solve:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 6\frac{\mathrm{d}x}{\mathrm{d}t} + 10x = 25\cos 4t$$

1) $m^2 + 6m + 10 = 0 \Rightarrow m_1 = -3 + i, m_2 = -3 - i$

$$\Rightarrow x_c(t) = e^{-3t}(c_1\cos t + c_2\sin t)$$

2) $x_p(t) = A\cos 4t + B\sin 4t$

$$\Rightarrow x_p'' + 6x_p' + 10x_p = (-6A + 24B)\cos 4t + (-24A - 6B)\sin 4t = 25\cos 4t$$

$$\Rightarrow \begin{cases} -6A + 24B = 25 \\ -24A - 6B = 0 \end{cases} \Rightarrow \begin{cases} A = -25/102 \\ B = 50/51 \end{cases}$$

3) $x(t) = e^{-3t}(c_1\cos t + c_2\sin t) - \frac{25}{102}\cos 4t + \frac{50}{51}\sin 4t$

4) $c_1 = \frac{38}{51}, c_2 = -\frac{86}{51}$

5) $x(t) = e^{-3t}\left(\frac{38}{51}\cos t - \frac{86}{51}\sin t\right) - \frac{25}{102}\cos 4t + \frac{50}{51}\sin 4t$

3.8.4 Series Circuit Analogue (P175)

Definitions & Theorems:

1.

Proofs or Explanations:

1. LRC-Series Circuits

If $i(t)$ denotes current in the LRC-series electrical circuit shown in FIGURE 3.8.15, then the voltage drops across the inductor, resistor, and capacitor are as shown in FIGURE 1.3.5. By Kirchhoff's second law, the sum of these voltages equals the voltage $E(t)$ impressed on the circuit; this is,

$$L\frac{\mathrm{d}i}{\mathrm{d}t}+Ri+\frac{1}{C}q=E(t)$$

But the charge $q(t)$ on the capacitor is related to the current $i(t)$ by $i=\frac{\mathrm{d}q}{\mathrm{d}t}$, and so the above DE becomes the linear second-order differential equation

$$L\frac{\mathrm{d}^2q}{\mathrm{d}t^2}+R\frac{\mathrm{d}q}{\mathrm{d}t}+\frac{1}{C}q=E(t)$$

If $E(t)=0$, the electrical vibrations of the circuit are said to be free. Since the auxiliary equation is $Lm^2+Rm+\frac{1}{C}=0$, there will be three forms of the solution with $R\neq 0$, depending on the value of the discriminant R^2-4L/C.

1) $Case\ I$: $R^2-4L/C>0$ (Overdamped)
   $i(t)=$
2) $Case\ II$: $R^2-4L/C=0$ (Critically damped)
   $i(t)=$
3) $Case\ III$: $R^2-4L/C<0$ (Underdamped)
   $i(t)=$

When $E(t)=0$ and $R=0$, the circuit is said to be undamped; the response of the circuit is simple harmonic.

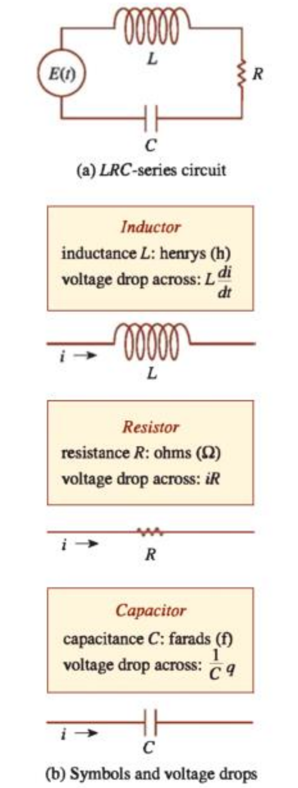


FIGURE 1.3.5 Current $i(t)$ and charge $q(t)$ are measured in amperes (A) and coulombs (C), respectively

Examples:

1. (Underdamped Series Circuit) Find the charge $q(t)$ on the capacitor in an LRC-series circuit when $L=0.25$ henry (h), $R=10$ ohms (Ω), $C=0.001$ farad (f), $E(t)=0$ volts (V), $q(0)=q_0$ coulombs (C), and $i(0)=0$ amperes (A). (P176)

Since $\frac{1}{C}=1000$

$$\frac{1}{4}q''+10q'+1000=0\quad\text{or}\quad q''+40q'+4000q=0$$

$$\Rightarrow m^2+40m+4000=0\Rightarrow m=\frac{-40\pm\sqrt{40^2-4(4000)}}{2}=-20\pm 60i$$

$$\Rightarrow m_1=-20+60i, m_2=-20-60i\Rightarrow \alpha=-20,\beta=60$$

$$\Rightarrow q(t)=e^{-20t}(c_1\cos 60t+c_2\sin 60t)$$

$$q(0)=c_1=q_0$$

$$i(t)=-20e^{-20t}(c_1\cos 60t+c_2\sin 60t)+e^{-20t}(-60c_1\sin 60t+60c_2\cos 60t)$$

$$i(0)=-20c_1+60c_2=0\Rightarrow c_2=\frac{q_0}{3}$$

$$q(t)=e^{-20t}\left(q_0\cos 60t+\frac{q_0}{3}\sin 60t\right)\Rightarrow A=\sqrt{c_1^2+c_2^2}=\frac{q_0\sqrt{10}}{3}$$

$$y=A\sin(\omega t+\phi)\text{, } where\ \sin\phi=\frac{c_1}{A},\cos\phi=\frac{c_2}{A},\tan\phi=\frac{c_1}{c_2}$$

$$q(t)=\frac{q_0\sqrt{10}}{3}e^{-20t}\sin(60t+1.249)$$

3.11 Nonlinear Models (P201)

November 9, 2016 12:42

Definitions & Theorems:

1. Definition:

Proofs or Explanations:

1. Suspended Cables

   Suppose a flexible cable, wire, or heavy rope is suspended between two vertical supports. Physical examples of this could be a long telephone wire strung between two posts as shown in red in FIGURE 1.3.8(a), or one of the two cables supporting the roadbed of a suspension bridge shown in red in Figure 1.3.8(b). Our goal is to construct a mathematical modal that describes the shape that such a cable assumes.

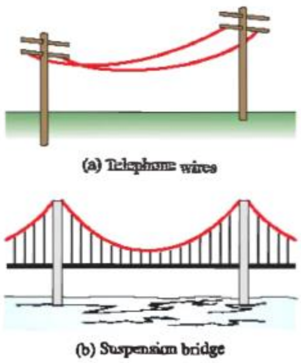


FIGURE 1.3.8 Cables suspended between vertical supports

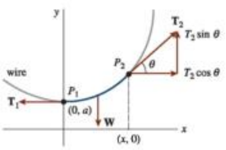


FIGURE 1.3.9 Element of cable

To begin, let's agree to examine only a portion or element of the cable between its lowest point P_1 and any arbitrary point P_2. As drawn in blue in **FIGURE 1.3.9**, this element of the cable is the curve in a rectangular coordinate system with the y-axis chosen to pass through the lowest point P_1 on the curve and the x-axis chosen a units below P_1. Three forces are acting on the cable: the tensions $\mathbf{T}_1$ and $\mathbf{T}_2$ in the cable that are tangent to the cable at P_1 and P_2, respectively, and the portion $\mathbf{W}$ of the total vertical load between the points P_1 and P_2. Let $T_1 = |\mathbf{T}_1|$, $T_2 = |\mathbf{T}_2|$, and $W = |\mathbf{W}|$ denote the magnitudes of these vectors. Now the tension $\mathbf{T}_2$ resolves into horizontal and vertical components (scalar quantities) $T_2\cos\theta$ and $T_2\sin\theta$. Because of static equilibrium, we can write

$$T_1 = T_2\cos\theta \quad \text{and} \quad W = T_2\sin\theta.$$

By dividing the last equation by the first, we eliminate T_2 and get $\tan\theta = W/T_1$. But since $dy/dx = \tan\theta$, we arrive at

$$\frac{dy}{dx} = \frac{W}{T_1}. \tag{16}$$

This simple first-order differential equation serves as a model for both the shape of a flexible wire such as a telephone wire hanging under its own weight as well as the shape of the cables that support the roadbed. We will come back to equation (16) in Exercises 2.2 and in Section 3.11.

2. Telephone Wires (P204)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{W}{T_1} = \frac{\rho s}{T_1}$$

Since the arc length between points P_1 and P_2 is given by

$$s = \int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx, \tag{9}$$

it follows from the Fundamental Theorem of Calculus that the derivative of (9) is

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{10}$$

Differentiating (8) with respect to x and using (10) leads to the nonlinear second-order equation

$$\frac{d^2y}{dx^2} = \frac{\rho}{T_1}\frac{ds}{dx} \quad \text{or} \quad \frac{d^2y}{dx^2} = \frac{\rho}{T_1}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{11}$$

$W = \rho w, \rho$ the weight per unit length of the roadbed (density), w the length.

3. Rocket Motion (P205)

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -k\frac{Mm}{y^2} \Rightarrow \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -k\frac{M}{y^2}$$

   where k is a constant of proportionality, y is the distance from the center of the Earth to the rocket, M is the mass of the Earth, and m is the mass of the rocket.

   To determine the constant k, we use the fact that when $y = R, \frac{kMm}{R^2} = mg$ or $k = gR^2/M$.

$$\Rightarrow \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -g\frac{R^2}{y^2}$$

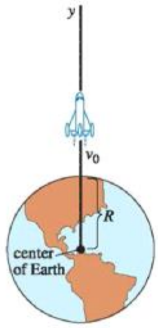


FIGURE 3.11.5 Distance to rocket is large compared to R

4. Variable Mass (P205)

$$F = \frac{\mathrm{d}}{\mathrm{d}t}(mv)$$

If m is constant, then $F = \frac{\mathrm{d}v}{\mathrm{d}t}m = ma$

Examples:

1. (An Initial-Value Problem) From the position of the y-axis in Figure 1.3.9 it is apparent that initial conditions associated with the second differential equation are $y(0) = a$ and $y'(0) = 0$. If we substitute $u = y'$, the last equation in (11) becomes $\frac{\mathrm{d}u}{\mathrm{d}x} = \sqrt{1+u^2}$

$$\int \frac{\mathrm{d}u}{\sqrt{1+u^2}} = \frac{\rho}{T_1}\int \mathrm{d}x$$
$$\Rightarrow \sinh^{-1} u = \frac{\rho}{T_1}x + c_1$$
$$y'(0) = 0 \Rightarrow u(0) = 0$$
$$\sinh^{-1} 0 = 0 \Rightarrow c_1 = 0$$
$$\Rightarrow u = \sinh\frac{\rho x}{T_1}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \sinh\frac{\rho x}{T_1} \Rightarrow y = \frac{T_1}{\rho}\cosh\frac{\rho}{T_1}x + c_2$$
$$y(0) = a, \cosh 0 = 1 \Rightarrow c_2 = a - \frac{T_1}{\rho}$$
$$\Rightarrow y = \frac{T_1}{\rho}\cosh\frac{\rho}{T_1}x + a - \frac{T_1}{\rho}$$

2. (Rope Pulled Upward by a Constant Force) A uniform 10-foot-long heavy rope is coiled loosely on the ground. One end of the rope is pulled vertically upward by means of a constant force of 5 lb. The rope weighs 1 lb per foot. Determined the height $x(t)$ of the end above ground level at time t.

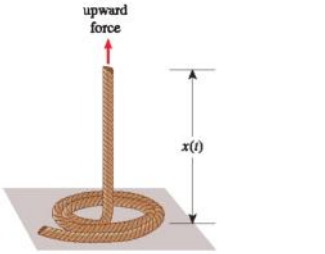


FIGURE 1.R.2 Rope pulled upward in Problem 35

SOLUTION Let us suppose that $x = x(t)$ denotes the height of the end of the rope in the air at time t, $v = dx/dt$, and that the positive direction is upward. For that portion of the rope in the air at time t we have the following variable quantities:

weight: $W = (x \text{ ft}) \cdot (1 \text{ lb/ft}) = x,$
mass: $m = W/g = x/32,$
net force: $F = 5 - W = 5 - x.$

Thus from (14) we have

$$\frac{d}{dt}\left(\frac{x}{32}v\right) = 5 - x \quad \text{or} \quad \overset{\text{Product Rule}}{\overbrace{x\frac{dv}{dt} + v\frac{dx}{dt}}} = 160 - 32x. \tag{15}$$

Since $v = dx/dt$ the last equation becomes

$$x\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + 32x = 160. \tag{16}$$

The nonlinear second-order differential equation (16) has the form $F(x, x', x'') = 0$, which is the second of the two forms considered in Section 3.7 that can possibly be solved by reduction of order. In order to solve (16), we revert back to (15) and use $v = x'$ along with the Chain Rule. From $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$ the second equation in (15) can be rewritten as

$$xv\frac{dv}{dx} + v^2 = 160 - 32x. \tag{17}$$

On inspection (17) might appear intractable, since it cannot be characterized as any of the first-order equations that were solved in Chapter 2. However, by rewriting (17) in differential form $M(x, v)\,dx + N(x, v)\,dv = 0$, we observe that the nonexact equation

$$(v^2 + 32x - 160)\,dx + xv\,dv = 0 \tag{18}$$

can be transformed into an exact equation by multiplying it by an *integrating factor*.* When (18) is multiplied by $\mu(x) = x$, the resulting equation is exact (verify). If we identify $\partial f/\partial x = xv^2 + 32x^2 - 160x$, $\partial f/\partial v = x^2v$, and then proceed as in Section 2.4, we arrive at

$$\frac{1}{2}x^2v^2 + \frac{32}{3}x^3 - 80x^2 = c_1. \tag{19}$$

From the initial condition $x(0) = 0$ it follows that $c_1 = 0$. Now by solving $\frac{1}{2}x^2v^2 + \frac{32}{3}x^3 - 80x^2 = 0$ for $v = dx/dt > 0$ we get another differential equation,

$$\frac{dx}{dt} = \sqrt{160 - \frac{64}{3}x},$$

which can be solved by separation of variables. You should verify that

$$-\frac{3}{32}\left(160 - \frac{64}{3}x\right)^{1/2} = t + c_2. \tag{20}$$

- If $(M_y - N_x)/N$ is a function of x alone, then an integrating factor for equation (11) is
$$\mu(x) = e^{\int \frac{M_y - N_x}{N}dx}. \tag{13}$$
- If $(N_x - M_y)/M$ is a function of y alone, then an integrating factor for equation (11) is
$$\mu(y) = e^{\int \frac{N_x - M_y}{M}dy}. \tag{14}$$

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Extra topics:

1. $\sinh x = \dfrac{e^x - e^{-x}}{2}$
2. $\cosh x = \dfrac{e^x + e^{-x}}{2}$

5.1.2 Power Series Solutions (P273)

November 16, 2016 11:47

Definitions & Theorems:

1. Power Series

Recall from calculus that a power series in $x - a$ is an infinite series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots.$$

Such a series is also said to be a **power series centered at *a***. For example, the power series $\sum_{n=0}^{\infty}(x+1)^n$ is centered at $a = -1$. In this section we are concerned mainly with power series in x; in other words, power series such as $\sum_{n=1}^{\infty} 2^{n-1}x^n = x + 2x^2 + 4x^3 + \cdots$ that are centered at $a = 0$. The following list summarizes some important facts about power series.

- **Convergence** A power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ is convergent at a specified value of x if its sequence of partial sums $\{S_N(x)\}$ converges; that is, if $\lim_{N\to\infty} S_N(x) = \lim_{N\to\infty}\sum_{n=0}^{N} c_n(x-a)^n$ exists. If the limit does not exist at x, the series is said to be divergent.

- **Analytic at a Point** A function f is analytic at a point a if it can be represented by a power series in $x - a$ with a positive radius of convergence. In calculus it is seen that functions such as e^x, $\cos x$, $\sin x$, $\ln(x-1)$, and so on can be represented by Taylor series. Recall, for example, that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots,\quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots,\quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots, \quad (2)$$

for $|x| < \infty$. These Taylor series centered at 0, called Maclaurin series, show that e^x, $\sin x$, and $\cos x$ are analytic at $x = 0$.

2. Ordinary and Singular Points

A Definition Suppose the linear second-order differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \quad (5)$$

is put into standard form

$$y'' + P(x)y' + Q(x)y = 0 \quad (6)$$

by dividing by the leading coefficient $a_2(x)$. We make the following definition.

Definition 5.1.1 Ordinary and Singular Points

A point x_0 is said to be an **ordinary point** of the differential equation (5) if both $P(x)$ and $Q(x)$ in the standard form (6) are analytic at x_0. A point that is not an ordinary point is said to be a **singular point** of the equation.

3. Theorem 5.1.1: Existence of Power Series Solutions

If $x = x_0$ is an oridnary point of the differential equation (5), we can always find two linearly independent solutions in the form of a power series centered at x_0; that is $y = \sum_{n=0}^{\infty} C_n(x - x_0)^n$. A series solution converges at least on some interval defined by $|x - x_0| < R$, where R is the distance from x_0 to the closest singular point.

Examples:

1. $(x^2-1)y'' + 2xy' + 6y = 0$

The only singular points of the equation are solutions of $x^2 - 1 = 0$ or $x = \pm 1$. All other finite values of x are ordinary points.

2. $\left(x^2+1\right)y''+xy'-y=0$

It has singular points at the solutions of $x^2+1=0$; namely, $x=\pm i$. All other values of x, real or complex, are ordinary points.

3. (Power Series Solutions) Solve $y''+xy=0$ (Airy's Equation) (P274)

$$y=\sum_{n=0}^{\infty} c_n x^n, a=0, |x|<\infty$$

$$y'=\sum_{n=1}^{\infty} nc_n x^{n-1}$$

$$y''=\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$$

$$\Rightarrow y''+xy=\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2}+x\sum_{n=0}^{\infty} c_n x^n=\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2}+\sum_{n=0}^{\infty} c_n x^{n+1}$$

$$=2c_2+\sum_{n=3}^{\infty} c_n n(n-1)x^{n-2}+\sum_{n=0}^{\infty} c_n x^{n+1}=0$$

Shift the summation index:

For $\sum_{n=3}^{\infty} c_n n(n-1)x^{n-2}$, let $k=n-2 \Rightarrow n=k+2$

For $\sum_{n=0}^{\infty} c_n x^{n+1}$, let $k=n+1 \Rightarrow n=k-1$

$$\Rightarrow y''+xy=2c_2+\sum_{n=3}^{\infty} c_n n(n-1)x^{n-2}+\sum_{n=0}^{\infty} c_n x^{n+1}$$

$$=2c_2+\sum_{k=1}^{\infty} c_{k+2}(k+2)(k+1)x^k+\sum_{k=1}^{\infty} c_{k-1}x^k$$

$$=2c_2+\sum_{k=1}^{\infty}\left[c_{k+2}(k+2)(k+1)+c_{k-1}\right]x^k=0$$

$$\Rightarrow 2c_2=0 \quad \text{and} \quad c_{k+2}(k+2)(k+1)+c_{k-1}=0, k=1,2,3,\ldots$$

$$\Rightarrow c_2=0, c_{k+2}=-\frac{c_{k-1}}{(k+1)(k+2)}$$

$$k = 1, \qquad c_3 = -\frac{c_0}{2\cdot 3}$$

$$k = 2, \qquad c_4 = -\frac{c_1}{3\cdot 4}$$

$$k = 3, \qquad c_5 = -\frac{c_2}{4\cdot 5} = 0 \quad \leftarrow c_2 \text{ is zero}$$

$$k = 4, \qquad c_6 = -\frac{c_3}{5\cdot 6} = \frac{1}{2\cdot 3\cdot 5\cdot 6}c_0$$

$$k = 5, \qquad c_7 = -\frac{c_4}{6\cdot 7} = \frac{1}{3\cdot 4\cdot 6\cdot 7}c_1$$

$$k = 6, \qquad c_8 = -\frac{c_5}{7\cdot 8} = 0 \quad \leftarrow c_5 \text{ is zero}$$

$$k = 7, \qquad c_9 = -\frac{c_6}{8\cdot 9} = -\frac{1}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}c_0$$

$$k = 8, \qquad c_{10} = -\frac{c_7}{9\cdot 10} = -\frac{1}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10}c_1$$

$$k = 9, \qquad c_{11} = -\frac{c_8}{10\cdot 11} = 0 \quad \leftarrow c_8 \text{ is zero}$$

and so on. Now substituting the coefficients just obtained into the original assumption

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + c_7x^7 + c_8x^8 + c_9x^9 + c_{10}x^{10} + c_{11}x^{11} + \cdots$$

we get

$$y = c_0 + c_1x + 0 - \frac{c_0}{2\cdot 3}x^3 - \frac{c_1}{3\cdot 4}x^4 + 0 + \frac{c_0}{2\cdot 3\cdot 5\cdot 6}x^6$$

$$+ \frac{c_1}{3\cdot 4\cdot 6\cdot 7}x^7 + 0 - \frac{c_0}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}x^9 - \frac{c_1}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10}x^{10} + 0 + \cdots.$$

After grouping the terms containing c_0 and the terms containing c_1, we obtain $y = c_0y_1(x) + c_1y_2(x)$, where

$$y_1(x) = 1 - \frac{1}{2\cdot 3}x^3 + \frac{1}{2\cdot 3\cdot 5\cdot 6}x^6 - \frac{1}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}x^9 + \cdots = 1 + \sum_{k=1}^{\infty}\frac{(-1)^k}{2\cdot 3\cdots(3k-1)(3k)}x^{3k}$$

$$y_2(x) = x - \frac{1}{3\cdot 4}x^4 + \frac{1}{3\cdot 4\cdot 6\cdot 7}x^7 - \frac{1}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10}x^{10} + \cdots = x + \sum_{k=1}^{\infty}\frac{(-1)^k}{3\cdot 4\cdots(3k)(3k+1)}x^{3k+1}.$$

4. (Power Series Solution) Solve $(x^2+1)y'' + xy' - y = 0$ (P276)

   The given differential equation has singular points at $x = \pm i$, and so a power series solution centered at 0 will converge at least for $|x| < 1$, where 1 is the distance in the complex plane from 0 to either i or $-i$.

$$y = \sum_{n=0}^{\infty} c_n x^n, a = 0, |x| < 1$$

$$y' = \sum_{n=1}^{\infty} nc_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$$

$$(x^2+1)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=1}^{\infty} nc_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n$$

$$= \sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} nc_n x^n - \sum_{n=0}^{\infty} c_n x^n$$

$$= 2c_2x^0 - c_0x^0 + 6c_3x + c_1x - c_1x + \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^n}_{k=n}$$

$$+ \underbrace{\sum_{n=4}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} + \underbrace{\sum_{n=2}^{\infty} nc_n x^n}_{k=n} - \underbrace{\sum_{n=2}^{\infty} c_n x^n}_{k=n}$$

$$= 2c_2 - c_0 + 6c_3x + \sum_{k=2}^{\infty}[k(k-1)c_k + (k+2)(k+1)c_{k+2} + kc_k - c_k]x^k$$

$$= 2c_2 - c_0 + 6c_3x + \sum_{k=2}^{\infty}[(k+1)(k-1)c_k + (k+2)(k+1)c_{k+2}]x^k = 0.$$

10.1 Theory of Linear Systems (P593)

November 21, 2016 11:46

Definitions & Theorems:

1. A first-order system that has normal form:

$$\begin{aligned}\frac{dx_1}{dt} &= g_1(t, x_1, x_2, \dots, x_n)\\ \frac{dx_2}{dt} &= g_2(t, x_1, x_2, \dots, x_n)\\ &\vdots \\ \frac{dx_n}{dt} &= g_n(t, x_1, x_2, \dots, x_n).\end{aligned} \tag{2}$$

2. Linear Systems

   When each of the functions $g_1, g_2, \dots, g_n$ in (2) is linear in the dependent variables $x_1, x_2, \dots, x_n$, we get the normal form of a first-order system of linear equations:

$$\begin{aligned}\frac{dx_1}{dt} &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t)\\ \frac{dx_2}{dt} &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + f_2(t)\\ &\vdots \\ \frac{dx_n}{dt} &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t).\end{aligned} \tag{3}$$

   We refer to a system of the from given in (3) simply as a linear system. We assume that the coefficients $a_{ij}(t)$ as well as the function $f_i(t)$ are continuous on a common interval I. When $f_i(t) = 0, i = 1,2, \dots, n$, the linear system is said to be homogeneous; otherwise it is nonhomogeneous.

3. Matrix Form of a Linear System

   If $X, A(t)$, and $F(t)$ denote the respective matrices

$$\mathbf{X} = \begin{pmatrix} x_1(t)\\ x_2(t)\\ \vdots \\ x_n(t)\end{pmatrix}, \quad \mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t)\\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t)\\ \vdots & & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t)\end{pmatrix}, \quad \mathbf{F}(t) = \begin{pmatrix} f_1(t)\\ f_2(t)\\ \vdots \\ f_n(t)\end{pmatrix},$$

   then the system of linear first-order differential equations (3) can be written as

$$\frac{d}{dt}\begin{pmatrix} x_1\\ x_2\\ \vdots \\ x_n\end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t)\\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t)\\ \vdots & & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t)\end{pmatrix}\begin{pmatrix} x_1\\ x_2\\ \vdots \\ x_n\end{pmatrix} + \begin{pmatrix} f_1(t)\\ f_2(t)\\ \vdots \\ f_n(t)\end{pmatrix}$$

   or simply

$$\mathbf{X}' = \mathbf{AX} + \mathbf{F}. \tag{4}$$

   If the system is homogeneous, its matrix form is then

$$\mathbf{X}' = \mathbf{AX}. \tag{5}$$

4. Definition 10.1.1 Solution Vector

   A solution vector on an interval I is any column matrix

$$\mathbf{X} = \begin{pmatrix} x_1(t)\\ x_2(t)\\ \vdots \\ x_n(t)\end{pmatrix}$$

   whose entries are differentiable functions satisfying the system (4) on the interval.

5. Theorem 10.1.2 Superposition Principle

   Let $X_1, X_2, \dots, X_k$ be a set of solution vectors of the homgeneous system (5) on an interval I. Then the linear combination

$$X = c_1X_1 + c_2X_2 + \cdots + c_kX_k$$

   where the $c_i, i = 1,2, \dots, k$ are arbitrary constants, is alos a solution on the interval.

6. Definition 10.1.2 Linear Dependence/Independence

   Let $X_1, X_2, \dots, X_k$ be a set of solution vectors of the homgeneous system (5) on an interval I. We sya that the set is linearly dependent on the interval if there exist constatns $c_1, c_2, \dots, c_k$, not all zero, such that

   $$c_1X_1 + c_2X_2 + \cdots + c_kX_k = 0$$

   for every t in the interval. If the set of vectors is ont linearly dependent on the interval, it is said to be linearly independent. That is $c_1 = c_2 = \cdots = c_k = 0$

7. Theorem 10.1.3 Criterion for Linearly Independent Solutions

   Let

   $$X_1 = \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix}, X_2 = \begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{pmatrix}, \dots, X_2 = \begin{pmatrix} x_{1\mathrm{n}} \\ x_{2n} \\ \vdots \\ x_{nn} \end{pmatrix}$$

   be n solution vectors of the homogeneous system (5) on an interval I. Then the set of solution vectors is linearly independent on I if and only if the Wronskian

   $$W(X_1, X_2, \dots, X_n) = \begin{bmatrix} x_{11}\, x_{12} \cdots x_{1n} \\ x_{21}\, x_{22} \cdots x_{2n} \\ \vdots \quad \vdots \quad \ddots \quad \vdots \\ x_{n1}\, x_{n2} \cdots x_{nn} \end{bmatrix} \neq 0$$

   for every t in the interval.

8. Definition 10.1.3 Fundamental Set of Solutions

   Any set $X_1, X_2, \dots, X_n$ of n linearly independent solution vectors of the homogeneous system (5) on an interval I is said to be a fundamental set of solutions on the interval.

9. Theorem 10.1.4 Existence of a Fundamental Set

   There exists a fundamental set of solutions for the homogeneous system (5) on an interval I.

10. Theorem 10.1.5 General Solution ---- Homogeneous Systems

    Let $X_1, X_2, \dots, X_n$ be a undamental set of solutions of the homogeneous system (5) on an interval I. Then the general solution of the system on the interval is

    $$X = c_1X_1 + c_2X_2 + \cdots + c_nX_n$$

    where the $c_i, i = 1,2, \dots n$ are arbitrary constants.

11. Nonhomogeneous Systems

    $$X' = AX + F$$

    For nonhomogeneous systems, a particular solution X_p on an interval I is any vector, free of arbitrary parameters, whose entries are functions that satisfy system $X' = AX + F$.

12. Theorem 10.1.6 General Solution ---- Nonhomogeneous Systems

    Let X_p be a given solution of the nonhomogeneous system $X' = AX + F$ on an interval I, and let

    $$X_c = c_1X_1 + c_2X_2 + \cdots + c_nX_n$$

    denote the general solution on the same interval of the associated homogeneous system $X' = AX$. Then the general solution of the nonhomogeneous system on the interval is

    $$X = X_c + X_p$$

    The general solution X_c of the associated homogeneous system is called the complementary function of the nonhomogeneous system.

Examples:

1. Systems Written in Matrix Notation

(a) If $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$, then the matrix form of the homogeneous system

$$\begin{aligned} \frac{dx}{dt} &= 3x + 4y \\ \frac{dy}{dt} &= 5x - 7y \end{aligned} \quad \text{is} \quad \mathbf{X}' = \begin{pmatrix} 3 & 4 \\ 5 & -7 \end{pmatrix}\mathbf{X}.$$

(b) If $\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, then the matrix form of the nonhomogeneous system

$$\begin{aligned} \frac{dx}{dt} &= 6x + y + z + t \\ \frac{dy}{dt} &= 8x + 7y - z + 10t \\ \frac{dz}{dt} &= 2x + 9y - z + 6t \end{aligned} \quad \text{is} \quad \mathbf{X}' = \begin{pmatrix} 6 & 1 & 1 \\ 8 & 7 & -1 \\ 2 & 9 & -1 \end{pmatrix}\mathbf{X} + \begin{pmatrix} t \\ 10t \\ 6t \end{pmatrix}$$

2. Using the Superposition Principle (P595)

$$X_1 = \begin{pmatrix} \cos t \\ -\frac{1}{2}\cos t + \frac{1}{2}\sin t \\ -\cos t - \sin t \end{pmatrix} \quad \text{and} \quad X_2 = \begin{pmatrix} 0 \\ e^t \\ 0 \end{pmatrix}$$

are solutions of the system

$$\mathbf{X}' = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{pmatrix}\mathbf{X}.$$

By the superposition principle, Theorem 10.1.2, the linear combination

$$\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 = c_1\begin{pmatrix} \cos t \\ -\frac{1}{2}\cos t + \frac{1}{2}\sin t \\ -\cos t - \sin t \end{pmatrix} + c_2\begin{pmatrix} 0 \\ e^t \\ 0 \end{pmatrix}$$

is yet another solution of the system.

3. General Solution (P597)

The vectors

$$X_1 = \begin{pmatrix} \cos t \\ -\frac{1}{2}\cos t + \frac{1}{2}\sin t \\ -\cos t - \sin t \end{pmatrix}, X_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^t, X_3 = \begin{pmatrix} \sin t \\ -\frac{1}{2}\sin t + \frac{1}{2}\cos t \\ -\sin t + \cos t \end{pmatrix}$$

are solution of the system

$$X' = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{pmatrix} X$$

Now

$$W(X_1, X_2, X_3) = \begin{bmatrix} \cos t & 0 & \sin t \\ -\frac{1}{2}\cos t + \frac{1}{2}\sin t & 1 & -\frac{1}{2}\sin t + \frac{1}{2}\cos t \\ -\cos t - \sin t & 0 & -\sin t + \cos t \end{bmatrix} = e^t \neq 0$$

for all real values of t. We conclude that X_1, X_2 and X_3 form a fundamental set of solutions on $(-\infty, \infty)$. Thus the general solution of the system on the interval is the linear combination $X = c_1X_1 + c_2X_2 + c_3X_3$; that is,

$$X = c_1\begin{pmatrix} \cos t \\ -\frac{1}{2}\cos t + \frac{1}{2}\sin t \\ -\cos t - \sin t \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^t + c_3\begin{pmatrix} \sin t \\ -\frac{1}{2}\sin t + \frac{1}{2}\cos t \\ -\sin t + \cos t \end{pmatrix}$$

10.2 Homogeneous Linear Systems (P599)

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Definitions & Theorems:

1. Eigenvalues and Eigenvectors

   If

   $$X = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} e^{\lambda t} = K e^{\lambda t}, k_1, k_2, \dots, k_n, \lambda \text{ are constants}$$

   is to be a solution vector of the system $X' = AX$, then

   $$X' = K\lambda e^{\lambda t}$$

   so that

   $$K\lambda e^{\lambda t} = AKe^{\lambda t}$$

   $$\Rightarrow AK - \lambda K = 0$$

   $$K = IK \Rightarrow (A - \lambda I)K = 0$$

   It is equivalent to the system of linear algebraic equations

   $$\begin{aligned} (a_{11} - \lambda)k_1 + \quad a_{12}k_2 + \cdots + \quad a_{1n}k_n &= 0 \\ a_{21}k_1 + (a_{22} - \lambda)k_2 + \cdots + \quad a_{2n}k_n &= 0 \\ \vdots \qquad\qquad\qquad & \vdots \\ a_{n1}k_1 + \quad a_{n2}k_2 + \cdots + (a_{nn} - \lambda)k_n &= 0. \end{aligned}$$

   A nontrivial solution of $(A - \lambda I)K = 0$ is determined by

   $$\det(A - \lambda I) = 0$$

   This polynomial equation in λ is called the characteristic equation of the matrix A; its solutions are the eigenvalues of A. A solution $K \neq 0$ of $(A - \lambda I)K = 0$ corresponding to an eigenvalue λ is called an eigenvector of A. A solution of homogeneous system $X' = AX$ is then $X = Ke^{\lambda t}$.

   Three cases:

   $Case\ I: Distinct\ Real\ Eigenvalues$

   $Case\ II: Repeated\ Eigenvalues$

   $Cse\ III: Complex\ Eigenvalues$

10.2.1 Distinct Real Eigenvalues (P600)

Definitions & Theorems:

1. Distinct Real Eigenvalues:

   When the n X n matrix A possesses n distinct real eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, then a set of n linearly independent eigenvectors $K_1, K_2, \dots, K_n$ can always be found and

   $$X_1 = K_1 e^{\lambda_1 t}, X_2 = K_2 e^{\lambda_2 t}, \dots, X_n = K_n e^{\lambda_n t}$$

   is a fundamental set of solutions of $X' = AX$ on $(-\infty, \infty)$.
2. Theorem 10.2.1 General Solution ---- Homogeneous Systems

   Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be n distinct real eigenvalues of the coefficient matrix A of the homogeneous system $X' = AX$, and let $K_1, K_2, \dots, K_n$ be the corresponding eigenvectors. Then the genral solutions on the interval $(-\infty, \infty)$ is given by

   $$X = c_1 K_1 e^{\lambda_1 t} + c_2 K_2 e^{\lambda_2 t} + \cdots + c_n K_n e^{\lambda_n t}$$
3. Repeller, Attractor

Examples:

1. (Distinct Eigenvalues, P600) Solve

   $$\frac{dx}{\mathrm{dy}} = 2x + 3y$$

   $$\frac{dx}{\mathrm{dy}} = 2x + y$$

$A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}, a_{11} = 2, a_{12} = 3, a_{21} = 2, a_{22} = 1$

$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 4 = (\lambda+1)(\lambda-4) = 0$

$\Rightarrow$ The eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 4$

For $\lambda_1 = -1$, $(A - \lambda I)K = 0 \Rightarrow$

$3k_1 + 3k_2 = 0$
$2k_1 + 2k_2 = 0$
$\Rightarrow k_1 = -k_2$

When $k_2 = -1$, then related eigenvectors is

$$K_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

For $\lambda_1 = 4$, $(A - \lambda I)K = 0 \Rightarrow$

$-2k_1 + 3k_2 = 0$
$2k_1 - 3k_2 = 0$
$\Rightarrow k_1 = \frac{3}{2}k_2$

When $k_2 = 2$, then related eigenvectors is

$$K_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Since the matrix of coefficient A is a 2 X 2 matrix, and since we have found two linearly independent solutions

$$X_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} \quad \text{and} \quad X_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t}$$

We conclude that the general solution of the system is

$$X = c_1X_1 + c_2X_2 = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t}$$

$$\Rightarrow x = c_1e^{-t} + 3c_2e^{4t}, y = -c_1e^{-t} + 2c_2e^{4t}$$

2. (Distinct Eigenvalues P602) Solve

$$\frac{dx}{dt} = -4x + y + z$$
$$\frac{dy}{dt} = x + 5y - z$$
$$\frac{dz}{dt} = y - 3z$$

10.2.1 Repeated Eigenvalues (P603)

If m is a positive integer and $(\lambda - \lambda_1)^m$ is a factor of the characteristic equation while $(\lambda - \lambda_1)^{m+1}$ is not a facotr, then λ_1 is said to be an eigenvalue of multiplicity m.

i) Linearly independent eigenvectors $K_1, K_2, \ldots, K_m$ corresponding to an eigenvalue λ_1. Then the general solution of the system contains the linear combination

$$c_1K_1e^{\lambda_1 t} + c_2K_2e^{\lambda_2 t} + \cdots c_nK_ne^{\lambda_n t}$$

ii) Only one eigenvector corresponding to the eigenvalue λ_1 of multiplicity m, then m linearly independent solutions of the form

$$X_1 = K_{11}e^{\lambda_1 t}$$
$$X_2 = K_{21}te^{\lambda_1 t} + K_{22}e^{\lambda_1 t}$$
$$\vdots$$
$$X_m = K_{m1}\frac{t^{m-1}}{(m-1)!}e^{\lambda_1 t} + K_{m2}\frac{t^{m-2}}{(m-2)!}e^{\lambda_1 t} + \cdots + K_{mm}e^{\lambda_1 t}$$

Definitions & Theorems:

1. Second Solution:

Second Solution Now suppose that λ_1 is an eigenvalue of multiplicity two and that there is only one eigenvector associated with this value. A second solution can be found of the form

$$\mathbf{X}_2 = \mathbf{K}te^{\lambda_1 t} + \mathbf{P}e^{\lambda_1 t}, \tag{12}$$

where $$\mathbf{K} = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} \quad \text{and} \quad \mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix}.$$

To see this we substitute (12) into the system $\mathbf{X}' = \mathbf{AX}$ and simplify:

$$(\mathbf{AK} - \lambda_1\mathbf{K})te^{\lambda_1 t} + (\mathbf{AP} - \lambda_1\mathbf{P} - \mathbf{K})e^{\lambda_1 t} = 0.$$

Since this last equation is to hold for all values of t, we must have

$$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{K} = \mathbf{0} \tag{13}$$

and $$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{P} = \mathbf{K}. \tag{14}$$

Equation (13) simply states that $\mathbf{K}$ must be an eigenvector of $\mathbf{A}$ associated with λ_1. By solving (13), we find one solution $\mathbf{X}_1 = \mathbf{K}e^{\lambda_1 t}$. To find the second solution $\mathbf{X}_2$ we need only solve the additional system (14) for the vector $\mathbf{P}$.

Proofs or Explanations:

1. Definition1

Examples:

1. (Repeated Eigenvalues, P604) Solve

$$X' = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} X$$

SOLUTION Expanding the determinant in the characteristic equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & -2 & 2 \\ -2 & 1-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{vmatrix} = 0$$

yields $-(\lambda+1)^2(\lambda-5) = 0$. We see that $\lambda_1 = \lambda_2 = -1$ and $\lambda_3 = 5$.

For $\lambda_1 = -1$, Gauss–Jordan elimination immediately gives

$$(\mathbf{A} + \mathbf{I}|\mathbf{0}) = \left(\begin{array}{ccc|c} 2 & -2 & 2 & 0 \\ -2 & 2 & -2 & 0 \\ 2 & -2 & 2 & 0 \end{array}\right) \overset{\text{row operations}}{\Longrightarrow} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The first row of the last matrix means $k_1 - k_2 + k_3 = 0$ or $k_1 = k_2 - k_3$. The choices $k_2 = 1$, $k_3 = 0$ and $k_2 = 1, k_3 = 1$ yield, in turn, $k_1 = 1$ and $k_1 = 0$. Thus two eigenvectors corresponding to $\lambda_1 = -1$ are

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Since neither eigenvector is a constant multiple of the other, we have found, corresponding to the same eigenvalue, two linearly independent solutions

$$\mathbf{X}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{-t} \quad \text{and} \quad \mathbf{X}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^{-t}.$$

Last, for $\lambda_3 = 5$, the reduction

$$(\mathbf{A} - 5\mathbf{I}|\mathbf{0}) = \left(\begin{array}{ccc|c} -4 & -2 & 2 & 0 \\ -2 & -4 & -2 & 0 \\ 2 & -2 & -4 & 0 \end{array}\right) \overset{\text{row operations}}{\Longrightarrow} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

implies $k_1 = k_3$ and $k_2 = -k_3$. Picking $k_3 = 1$ gives $k_1 = 1$, $k_2 = -1$, and thus a third eigenvector is

$$\mathbf{K}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

We conclude the general solution of the system is

$$X = c_1X_1 + c_2X_2 + c_3X_3 = c_1\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{-t} + c_2\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^{-t} + c_3\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{5t}$$

2. (Repeated Eigenvalues P605) Find the general solution of the system

$$X' = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix} X$$

$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -18 \\ 2 & -9-\lambda \end{vmatrix} = (\lambda+3)^2 = 0$

$\Rightarrow \lambda_1 = \lambda_2 = -3$ is a root of multiplicity two.

One solution is $X_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{-3t}$

Identifying $K = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $P = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$

$\Rightarrow (A - \lambda_1 I)P = (A + 3I)P = K \Rightarrow \begin{pmatrix} 6 & -18 \\ 2 & -6 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\Rightarrow 6p_1 - 18p_2 = 3, 2p_1 - 6p_2 = 1 \Rightarrow 2p_1 - 6p_2 = 1$

$\Rightarrow \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$

$\Rightarrow X_2 = K_1te^{\lambda_1 t} + Pe^{\lambda_1 t} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} te^{-3t} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} e^{-3t}$

$\Rightarrow X = c_1X_1 + c_2X_2 = c_1\begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{-3t} + c_2\left[\begin{pmatrix} 3 \\ 1 \end{pmatrix} te^{-3t} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} e^{-3t}\right]$

10.2.3 Complex Eigenvalues (P607)

Definitions & Theorems:

1. Theorem 10.2.2 Solutions Corresponding to a Complex Eigenvalue

   Let A be the coefficient matrix having real entries of the homogeneous system $X' = AX$, and let K_1 be an eigenvector corresponding to the complex eigenvalue $\lambda_1 = \alpha + i\beta$, α and β are real. Then

   $$K_1 e^{\lambda_1 t} \quad and \quad \overline{K_1} e^{\overline{\lambda_1} t}$$

   are solutions of $X' = AX$

2. Theorem 10.2.3 Real Solutions Corresponding to a Complex Eigenvalue

   Let $\lambda_1 = \alpha + \beta i$ be a complex eigenvalue of the coefficient matrix A in the homogeneous system $X' = AX$, and let

   $$B_1 = \frac{1}{2}\left(K_1 + \overline{K_1}\right), \qquad B_2 = \frac{i}{2}\left(-K_1 + \overline{K_1}\right)$$

   Then an independent solutions of $X' = AX$ on $(-\infty, \infty)$ are

   $$X_1 = e^{\alpha t}\left[B_1 \cos\beta t - B_2 \sin\beta t\right]$$
   $$X_2 = e^{\alpha t}\left[B_2 \cos\beta t + B_1 \sin\beta t\right]$$

   The matrices B_1 and B_2 are often denoted by

   $$B_1 = Re(K_1), \qquad B_2 = Im(K_1)$$

Proofs or Explanations:

1. Definition1

Examples:

1. (Complex Eigenvalues, P609) Solve

   $$X' = AX = \begin{pmatrix} 2 & 8 \\ -1 & -2 \end{pmatrix} X$$

   $$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 8 \\ -1 & -2-\lambda \end{vmatrix} = \lambda^2 + 4 = 0$$

   $$\Rightarrow \lambda_1 = 2i, \lambda_2 = -2i \Rightarrow \alpha = 0, \beta = 2$$

   $$\Rightarrow \begin{pmatrix} 2-2i & 8 \\ -1 & -2-2i \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow k_1 = -(2+2i)k_2$$

   $$\Rightarrow K_1 = \begin{pmatrix} 2+2i \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + i\begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

   $$\Rightarrow B_1 = Re(K_1) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \qquad B_2 = Im(K_1) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

   $\Rightarrow$

   $$X_1 = e^{\alpha t}\left[B_1 \cos\beta t - B_2 \sin\beta t\right] = \left[\begin{pmatrix} 2 \\ -1 \end{pmatrix}\cos 2t - \begin{pmatrix} 2 \\ 0 \end{pmatrix}\sin 2t\right]$$

   $$X_2 = e^{\alpha t}\left[B_2 \cos\beta t + B_1 \sin\beta t\right] = \left[\begin{pmatrix} 2 \\ 0 \end{pmatrix}\cos 2t + \begin{pmatrix} 2 \\ -1 \end{pmatrix}\sin 2t\right]$$

   $$X = c_1 X_1 + c_2 X_2$$

10.4 Nonhomogeneous Linear Systems (P615)

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$$\boldsymbol{X}' = \boldsymbol{AX} + \boldsymbol{F}(\boldsymbol{t})$$

General Solution: $\boldsymbol{X} = \boldsymbol{X_c} + \boldsymbol{X_p}$

$X_c = c_1X_1 + c_2X_2 + \cdots + c_nX_n$ is the complementary function

X_p is any particular solution of the nonhomogeneous system.

10.4.1 Undetermined Coefficients (P615)

Examples:

1. (Undetermined Coefficients, P615) Solve the system $X' = \begin{pmatrix}-1 & 2\\ -1 & 1\end{pmatrix}X + \begin{pmatrix}-8\\ 3\end{pmatrix}$ on the interval $(-\infty, \infty)$
   1) X_c

   $$X' = \begin{pmatrix}-1 & 2\\ -1 & 1\end{pmatrix}X \Rightarrow \det(A - \lambda i) = \begin{pmatrix}-1-\lambda & 2\\ -1 & 1-\lambda\end{pmatrix} = \lambda^2 + 1 = 0 \Rightarrow \lambda_1 = i, \lambda_2 = -i$$

   $$X_c = c_1\begin{pmatrix}\cos t + \sin t\\ \cos t\end{pmatrix} + c_2\begin{pmatrix}\cos t - \sin t\\ -\sin t\end{pmatrix}$$

   2) X_p

   $$F(t) = \begin{pmatrix}-8\\ 3\end{pmatrix} \Rightarrow \text{Let } X_p = \begin{pmatrix}a_1\\ b_1\end{pmatrix} \Rightarrow X_p' = \begin{pmatrix}0\\ 0\end{pmatrix}$$

   $$\Rightarrow \begin{pmatrix}0\\ 0\end{pmatrix} = \begin{pmatrix}-1 & 2\\ -1 & 1\end{pmatrix}\begin{pmatrix}a_1\\ b_1\end{pmatrix} + \begin{pmatrix}-8\\ 3\end{pmatrix} \Rightarrow \begin{pmatrix}a_1\\ b_1\end{pmatrix} = \begin{pmatrix}14\\ 11\end{pmatrix}$$

   $$\Rightarrow X_p = \begin{pmatrix}14\\ 11\end{pmatrix}$$

   3) $X = X_c + X_p = c_1\begin{pmatrix}\cos t + \sin t\\ \cos t\end{pmatrix} + c_2\begin{pmatrix}\cos t - \sin t\\ -\sin t\end{pmatrix} + \begin{pmatrix}14\\ 11\end{pmatrix}$

2. (Undetermined Coefficients, P616) Solve the system $X' = \begin{pmatrix}6 & 1\\ 4 & 3\end{pmatrix}X + \begin{pmatrix}6t\\ -10t + 4\end{pmatrix}$ on the interval $(-\infty, \infty)$
   1) X_c

   $$X' = \begin{pmatrix}6 & 1\\ 4 & 3\end{pmatrix}X \Rightarrow \lambda_1 = 2, \lambda_2 = 7, K_1 = \begin{pmatrix}1\\ -4\end{pmatrix}, K_2 = \begin{pmatrix}1\\ 1\end{pmatrix}$$

   $$X_c = c_1\begin{pmatrix}1\\ -4\end{pmatrix}e^{2t} + c_2\begin{pmatrix}1\\ 1\end{pmatrix}e^{7t}$$

   2) X_p

   $$F(t) = \begin{pmatrix}6\\ -10\end{pmatrix}t + \begin{pmatrix}0\\ 4\end{pmatrix} \Rightarrow \text{Let } X_p = \begin{pmatrix}a_2\\ b_2\end{pmatrix}t + \begin{pmatrix}a_1\\ b_1\end{pmatrix} \Rightarrow X_p' = \begin{pmatrix}a_2\\ b_2\end{pmatrix}$$

   $$\Rightarrow \begin{pmatrix}a_2\\ b_2\end{pmatrix} = \begin{pmatrix}6 & 1\\ 4 & 3\end{pmatrix}\left[\begin{pmatrix}a_2\\ b_2\end{pmatrix}t + \begin{pmatrix}a_1\\ b_1\end{pmatrix}\right] + \left[\begin{pmatrix}6t\\ -10t + 4\end{pmatrix}\right] \text{(!!!!!not } \begin{pmatrix}a_2\\ b_2\end{pmatrix}t + \begin{pmatrix}a_1\\ b_1\end{pmatrix}\text{)}$$

   $$\Rightarrow \begin{pmatrix}a_2\\ b_2\end{pmatrix} = \begin{pmatrix}6 & 1\\ 4 & 3\end{pmatrix}\begin{pmatrix}a_2\\ b_2\end{pmatrix}t + \begin{pmatrix}6 & 1\\ 4 & 3\end{pmatrix}\begin{pmatrix}a_1\\ b_1\end{pmatrix} + \begin{pmatrix}6\\ 10\end{pmatrix}t + \begin{pmatrix}0\\ 4\end{pmatrix}$$

   $$\Rightarrow \begin{pmatrix}a_2\\ b_2\end{pmatrix} = \begin{pmatrix}6a_2t + b_2t\\ 4a_2t + 3b_2t\end{pmatrix} + \begin{pmatrix}6a_1 + b_1\\ 4a_1 + 3b_1\end{pmatrix} + \begin{pmatrix}6\\ 10\end{pmatrix}t + \begin{pmatrix}0\\ 4\end{pmatrix}$$

   $$\Rightarrow \begin{pmatrix}a_2\\ b_2\end{pmatrix} = \begin{pmatrix}6a_2t + b_2t + 6a_1 + b_1 + 6\\ 4a_2t + 3b_2t + 4a_1 + 3b_1 + 10t + 4\end{pmatrix}$$

   $$\Rightarrow \begin{pmatrix}0\\ 0\end{pmatrix} = \begin{pmatrix}(7a_2 + b_2 + 6)t + (6a_1 + b_1 - a_2)\\ (4a_2t + 3b_2t + 10t) + (4a_1 + 3b_1 + 4 - b_2)\end{pmatrix}$$

   $$\Rightarrow \begin{cases}7a_2 + b_2 + 6 = 0\\ 6a_1 + b_1 - a_2 = 0\\ 4a_2 + 3b_2 + 10 = 0\\ 4a_1 + 3b_1 + 4 - b_2 = 0\end{cases}$$

   $$\Rightarrow X_p = \begin{pmatrix}-2\\ 6\end{pmatrix}t + \begin{pmatrix}-4/7\\ 10/7\end{pmatrix}$$

   3) $X = X_c + X_p = c_1\begin{pmatrix}1\\ -4\end{pmatrix}e^{2t} + c_2\begin{pmatrix}1\\ 1\end{pmatrix}e^{7t} + \begin{pmatrix}-2\\ 6\end{pmatrix}t + \begin{pmatrix}-4/7\\ 10/7\end{pmatrix}$

10.4.2 Variation of Parameters (P617)

If $X_1, X_2, \ldots, X_n$ is a fundamental set of solutions of the homogeneous system $X' = AX$ on an interval I, then its general solution on the interval is the linear combination $X = c_1X_1 + c_2X_2 + \cdots + c_nX_n$ or

$$\mathbf{X} = c_1\begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix} + c_2\begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{pmatrix} + \cdots + c_n\begin{pmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{pmatrix} = \begin{pmatrix} c_1x_{11} + c_2x_{12} + \cdots + c_nx_{1n} \\ c_1x_{21} + c_2x_{22} + \cdots + c_nx_{2n} \\ \vdots \\ c_1x_{n1} + c_2x_{n2} + \cdots + c_nx_{nn} \end{pmatrix}. \qquad (1)$$

The last matrix in (1) can be written as the product

$$X = \phi(t)C$$

where C is the $n \times 1$ column vector of arbitrary constants $c_1, c_2, \ldots, c_n$ and c_n, and $\phi(t)$ is then $n \times n$ matrix whose columns consist of the entries the solution vectors of the system $X' = AX$:

$$\mathbf{\Phi}(t) = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}$$

The matrix $\phi(t)$ is called a fundamental matrix of the system on the interval.

$$X_p = \phi(t)\int \phi^{-1}(t)F(t)\,dt$$

$$X = \phi(t)c + \phi(t)\int \phi^{-1}(t)F(t)\,dt$$

Extra topics:

1. Inverse of a 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\underbrace{ad-bc}_{\text{determinant}}}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}^{-1} = \frac{1}{4\text{x}6-7\text{x}2}\begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}$$

How do we know this is the right answer?

Remember it must be true that: $A \times A^{-1} = \mathbf{I}$

It should **also** be true that: $A^{-1} \times A = \mathbf{I}$

Examples:

1. (Variation of Parameters, P618) Solve the system $X' = \begin{pmatrix} -3 & 1 \\ 2 & -4 \end{pmatrix}X + \begin{pmatrix} 3t \\ e^{-t} \end{pmatrix}$ on the interval $(-\infty, \infty)$

   1) X_c

$$X' = \begin{pmatrix} -3 & 1 \\ 2 & -4 \end{pmatrix}X \Rightarrow \lambda_1 = -2, \lambda_2 = -5$$

$$X_c = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-2t} + c_2\begin{pmatrix} 1 \\ -2 \end{pmatrix}e^{-5t}$$

$$\Rightarrow X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-2t} = \begin{pmatrix} e^{-2t} \\ e^{-2t} \end{pmatrix}, X_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}e^{-5t} = \begin{pmatrix} e^{-5t} \\ -2e^{-5t} \end{pmatrix}$$

$$F(t) = \begin{pmatrix} 3t \\ e^{-t} \end{pmatrix}$$

2) X_p

$$\phi(t)=\begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix} \Rightarrow \phi^{-1}(t)=\begin{pmatrix} 2/3e^{2t} & 1/3e^{2t} \\ 1/3e^{5t} & -1/3e^{5t} \end{pmatrix}$$

$$X_p=\phi(t)\int \phi^{-1}(t)F(t)\,\mathrm{d}t=\begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix}\int \begin{pmatrix} 2/3e^{2t} & 1/3e^{2t} \\ 1/3e^{5t} & -1/3e^{5t} \end{pmatrix}\begin{pmatrix} 3t \\ e^{-t} \end{pmatrix}\mathrm{d}t$$

$$\Rightarrow X_p=\begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix}\int \begin{pmatrix} 2te^{2t}+1/3e^{t} \\ te^{5t}-1/3e^{4t} \end{pmatrix}\mathrm{d}t$$

$$\Rightarrow X_p=\begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix}\begin{pmatrix} te^{2t}-1/2e^{2t}+1/3e^{t} \\ 1/5te^{5t}-1/25e^{5t}-1/12e^{4t} \end{pmatrix}=\begin{pmatrix} \frac{6}{5}t-\frac{27}{50}+\frac{1}{4}e^{-t} \\ \frac{3}{5}t-\frac{21}{50}+\frac{1}{2}e^{-t} \end{pmatrix}$$

3) $X=X_c+X_p=c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-2t}+c_2\begin{pmatrix} 1 \\ -2 \end{pmatrix}e^{-5t}+\begin{pmatrix} \frac{6}{5}t-\frac{27}{50}+\frac{1}{4}e^{-t} \\ \frac{3}{5}t-\frac{21}{50}+\frac{1}{2}e^{-t} \end{pmatrix}$

$$=c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-2t}+c_2\begin{pmatrix} 1 \\ -2 \end{pmatrix}e^{-5t}+\begin{pmatrix} 6/5 \\ 3/5 \end{pmatrix}t-\begin{pmatrix} 27/50 \\ 21/50 \end{pmatrix}+\begin{pmatrix} 1/4 \\ 1/2 \end{pmatrix}e^{-t}$$

Final

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1. Separable Equations
$$\frac{dy}{dx} = g(x)h(y)$$
2. Linear Equations
$$\frac{\mathrm{dy}}{\mathrm{dx}} + P(x)y = f(x)$$
$$e^{\int P(x)\,\mathrm{dx}}\boldsymbol{y} = \int e^{\int P(x)\,\mathrm{dx}}\boldsymbol{f(x)}\,\mathrm{dx}$$
3. Exact Equations
$$M(x,y)dx + N(x,y)dy = 0, \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$
$$f(x,y) = \int M(x,y)\,dx + g(y), g'(y) = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)dx$$
$$f(x,y) = \int N(x,y)\,dy + h(y), h'(y) = M(x,y) - \frac{\partial}{\partial x}\int N(x,y)dy$$
4. Substitution
   a. Homogeneous Equations
   $$M(x,y)\,dx + N(x,y)\,dy = 0, \qquad M(tx,ty) = t^{\alpha}M(x,y), N(x,y) = t^{\alpha}N(x,y)$$
   $$y = ux$$
   $$x = vy$$
   b. Bernoulli's Equation
   $$\frac{dy}{dx} + P(x)y = f(x)y^n$$
   $$u = y^{1-n}$$
   c. Reduction to Separation of Variables
   $$\frac{dy}{dx} = f(Ax + By + C), \qquad B \neq 0$$
   $$u = Ax + By + C$$

5. Complex Number
$$z = x + iy$$
$$r = |z| = \sqrt{x^2 + y^2}$$
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z^n = r^n(\cos n\theta + i\sin n\theta)$$
$$w^n = z \Rightarrow w_k = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right], \qquad k = 0,1,2,\dots,n-1$$

6. Homogeneous Equation with Constant Coefficients
$$ay'' + by' + cy = 0$$
$$y = e^{mx} \Rightarrow am^2 + bm + c = 0$$
$$m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad and \quad m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$
$Case\ I: Distinct\ Real\ Roots$
$$y = c_1e^{m_1x} + c_2e^{m_2x}$$
$Case\ II: Repeated\ Real\ Roots$
$$y = c_1e^{m_1x} + c_2xe^{m_1x}$$
For n-order: $y = c_1e^{m_1x} + c_2xe^{m_1x} + c_3x^2e^{m_1x} + \cdots + c_nx^{(n-1)}e^{m_1x}$

$Cse\ III: Conjugate\ Complex\ Roots$

If m_1 and m_2 are complex, then we can write $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, where $\alpha > 0, \beta > 0, i^2 = -1$.
$$y = c_1e^{\alpha x}\cos\beta x + c_2e^{\alpha x}\sin\beta x \quad \text{or} \quad y = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$$
7. Undetermined Coefficients

TABLE 3.4.1 Trial Particular Solutions

$g(x)$	Form of y_p
1. 1 (any constant)	A
2. $5x+7$	$Ax+B$
3. $3x^2-2$	Ax^2+Bx+C
4. x^3-x+1	Ax^3+Bx^2+Cx+E
5. $\sin 4x$	$A\cos 4x+B\sin 4x$
6. $\cos 4x$	$A\cos 4x+B\sin 4x$
7. e^{5x}	Ae^{5x}
8. $(9x-2)e^{5x}$	$(Ax+B)e^{5x}$
9. x^2e^{5x}	$(Ax^2+Bx+C)e^{5x}$
10. $e^{3x}\sin 4x$	$Ae^{3x}\cos 4x+Be^{3x}\sin 4x$
11. $5x^2\sin 4x$	$(Ax^2+Bx+C)\cos 4x+(Ex^2+Fx+G)\sin 4x$
12. $xe^{3x}\cos 4x$	$(Ax+B)e^{3x}\cos 4x+(Cx+E)e^{3x}\sin 4x$

8. Variation of Parameters

$$y''+P(x)y'+Q(x)y=f(x)$$
$$y_c=c_1y_1+c_2y_2$$
$$u_1'=\frac{W_1}{W},u_2'=\frac{W_2}{W},\qquad W=\begin{vmatrix}y_1 & y_2\\ y_1' & y_2'\end{vmatrix},W_1=\begin{vmatrix}0 & y_2\\ f(x) & y_2'\end{vmatrix},W_2=\begin{vmatrix}y_1 & 0\\ y_1' & f(x)\end{vmatrix}$$
$$y_p=u_1y_1+u_2y_2$$
$$y=y_c+y_p$$

9. Cauchy-Euler Equations

$$a_nx^n\frac{\mathrm{d}^ny}{\mathrm{d}x^n}+a_{n-1}x^{n-1}\frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}}+\cdots+a_1x\frac{\mathrm{d}y}{\mathrm{d}x}+a_0y=g(x)$$
$$y=x^m$$
$$ax^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+bx\frac{\mathrm{d}y}{\mathrm{d}x}+cy=0\Rightarrow am^2+(b-a)m+c=0$$

Case I: Distinct Real Roots

$$y=c_1x^{m_1}+c_2x^{m_2}$$

Case II: Repeated Real Roots

$$y=c_1x^{m_1}+c_2x^{m_1}\ln x$$

For n-order: $y=c_1x^{m_1}+c_2x^{m_1}\ln x+c_3x^{2m_1}\ln x+\cdots+c_nx^{(n-1)m_1}\ln x$

Cse III: Conjugate Complex Roots

If m_1 and m_2 are complex, then we can write $m_1=\alpha+i\beta$ and $m_2=\alpha-i\beta$, where $\alpha>0,\beta>0,i^2=-1$.

$$y=x^\alpha[c_1\cos(\beta\ln x)+c_2\sin(\beta\ln x)]$$

10. Reduction of Order
    a. $F(x,y',y'')=0$, "y" is missing

    $$u=y'$$
    $$\frac{\mathrm{d}u}{\mathrm{d}x}=y''$$

    b. $F(y,y',y'')=0$, "x" is missing

    $$u=y'$$
    $$\frac{\mathrm{d}u}{\mathrm{d}x}=y''=\frac{\mathrm{d}u}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x}=u\frac{\mathrm{d}u}{\mathrm{d}y}$$

11. Power Series
12. Linear Systems
13. Linear Models
14. Nonlinear Models
15. Linear System Models