Review

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1. The joint probability density function of random variables X and Y is given by
$$f(x,y)=\frac{6}{7}\left(x^2+\frac{xy}{2}\right),\qquad 0<x<1, 0<y<2$$
   (a) Verify that this is indeed a joint density function.
   (b) Compute the density function of X.
   (c) Find $P\{X>Y\}$.

   Solution:
   (a) Verify:
   $f(x,y)\geq 0$ for $0<x<1$ and $0<y<2$.
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\,\mathrm{d}x\,\mathrm{d}y=\int_0^2\int_0^1\left[\frac{6}{7}\left(x^2+\frac{xy}{2}\right)\right]\mathrm{d}x\,\mathrm{d}y=\frac{6}{7}\int_0^2\left[\frac{x^3}{3}+\frac{x^2y}{4}\bigg|_0^1\right]\mathrm{d}y=\frac{6}{7}\int_0^2\left[\frac{1}{3}+\frac{y}{4}\right]\mathrm{d}y=\frac{6}{7}\left(\frac{y}{3}+\frac{y^2}{8}\bigg|_0^2\right)=\frac{6}{7}\left(\frac{2}{3}+\frac{1}{2}\right)=1$$
   So this is a joint density function.
   (b) $f_X(x)=\int_{-\infty}^{\infty}f(x,y)\,\mathrm{d}y=\int_0^2\frac{6}{7}\left(x^2+\frac{xy}{2}\right)\mathrm{d}y=\frac{6(2x^2+x)}{7}$
   (c) $P\{X>Y\}=\int_{-\infty}^{\infty}\int_{-\infty}^{x}f(x,y)\,\mathrm{d}y\,\mathrm{d}x=\int_0^1\int_0^x\left[\frac{6}{7}\left(x^2+\frac{xy}{2}\right)\right]\mathrm{d}y\,\mathrm{d}x=\frac{6}{7}\int_0^1\left[x^2y+\frac{xy^2}{4}\bigg|_0^x\right]\mathrm{d}x=\frac{6}{7}\int_0^1\left[x^3+\frac{x^3}{4}\right]\mathrm{d}x=\frac{6}{7}\left(\frac{5x^4}{16}\bigg|_0^1\right)=\frac{15}{56}$

2. A lot containing 7 components is sampled by a quality inspector; the lot contains 4 good components and 3 defective components. A sample of 3 is taken by the inspector. Find the expected value of the number of good components in this sample.
   Let x denote the number of good components in this sample.
$$f(x)=\frac{\binom{4}{x}\binom{3}{3-x}}{\binom{7}{3}}, x=0,1,2,3$$
$$\Rightarrow f(0)=\frac{1}{35}, f(1)=\frac{12}{35}, f(2)=\frac{18}{35}, f(3)=\frac{4}{35}$$
$$\Rightarrow E(x)=(0)\left(\frac{1}{35}\right)+(1)\left(\frac{12}{35}\right)+(2)\left(\frac{18}{35}\right)+(3)\left(\frac{4}{35}\right)=\frac{12}{7}\approx 1.7$$

3. Two ballpoint pens are selected at random from a box that contains 3 blue pens, 2 red pens, and 3 green pens. If X is the number of blue pens selected and Y is the number of red pens selected.
$$f(x,y)=\frac{\binom{3}{x}\binom{2}{y}\binom{3}{2-x-y}}{\binom{8}{2}},\qquad x=0,1,2, y=0,1,2, 0\leq x+y\leq 2$$

4. The length of time for one individual to be served at a cafeteria is a random variable having an exponential distribution with a mean of 4 minutes. What is the probability that a person is served
   in less than 3 minutes on at least 4 of the next 6 days?
$$P\{T<3\}=1-\mathrm{e}^{\left(-\frac{3}{4}\right)}=0.5276$$
$$P\{A\geq 4\}=P\{A=4\}+P\{A=5\}+P\{A=6\}$$
$$=\frac{6!}{4!\,2!}0.5276^4(1-0.5276)^2+\frac{6!}{5!\,1!}0.5276^5(1-0.5276)^1+\frac{6!}{6!\,0!}0.5276^6(1-0.5276)^0=0.3969$$

5. Jane is taking two books along on her holiday vacation. With probability 0.5 she will like the first book, with probability 0.4 she will like the second book and with probability 0.3 she will like both books. What is the probability that she dislikes both books?
   $P(A)=0.5, P(B)=0.4, P(AB)=0.3$
   $P(A^cB^c)=P\left((A\cup B)^c\right)=1-P(A\cup B)=1-[P(A)+P(B)-P(AB)]=0.4$

6. Buses arrive at a specified stop at 15-minute intervals starting at 7 A.M. That is, they arrive at 7:00, 7:15, 7:30, 7:45, and so on. If a passenger arrives at the stop at a time that is uniformly distributed between 7:00 and 7:30, what is the probability that he waits more than 10 minutes for a bus?
   $P=\frac{10}{30}$