Solution to Problem 1 **First case**. We try to write them as a linear combination (in column form for brevity)

$$\begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a+b+2c \\ b+c+d \\ a+b+2c \\ a+b+2c+e \end{pmatrix} \tag{1}$$

The system clearly has no solution because the two equations marked in red are incompatible. Thus the vector v is not in the span.

Second case

$$3x^3+2x-1 = a(x^4+2)+b(2x-1)+cx^3+d(x^3+3) = ax^4+(c+d)x^3+2bx+2a-b+3d \tag{2}$$

Equating the coefficients of both sides we get the system

$$\begin{cases} a=0 \\ c+d=3 \\ 2b=2 \\ 2a-b+3d=-1 \end{cases} \Rightarrow \begin{cases} a=0 \\ c=3 \\ b=1 \\ d=0 \end{cases} \tag{3}$$

Thus

$$\mathbf{v} = w_2+3w_3 \tag{4}$$

where w_1, w_2, w_3, w_4 are the four polynomials listed in S. □

Solution to Problem 2 Consider the following linear combination of f_1, f_2, f_3 giving the zero-vector of $\mathcal{C}(\mathbb{R})$:

$$af_1+bf_2+cf_3=0. \tag{5}$$

That is,

$$af_1(x)+bf_2(x)+cf_3(x)=0 \text{ for any } x\in\mathbb{R}. \tag{6}$$

Therefore, for $x=0$, $x=1$, and $x=2$, we obtain

$$a+b=0 \tag{7}$$

$$a\mathrm{e}+b\mathrm{e}^3=0 \tag{8}$$

$$a\mathrm{e}^2+b\mathrm{e}^6+2c=0. \tag{9}$$

Using the first two equations leads to

$$b(-\mathrm{e}+\mathrm{e}^3)=0, \tag{10}$$

and hence, $b=0$ and $a=0$. Now, using these values in the third equation, we further obtain $c=0$. Thus, these vectors are linearly independent. □

Solution to Problem 3

Basis for W_1 Any matrix in W_1 is

$$M = \begin{bmatrix} a & b & a+b \\ c+d & d & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \tag{11}$$

The four matrices appearing in the linear combination above are linearly independent because the l.h.s. is zero iff $a=b=c=d=0$. Thus they form a basis and $\dim W_1 = 4$.

Basis for W_2. Any matrix in W_2 is

$$M = \begin{bmatrix} f & 2f & g \\ e & e & \ell - e \end{bmatrix} = f \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + g \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \ell \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + e \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix} \tag{12}$$

The four matrices appearing in the linear combination above are linearly independent because the l.h.s. is zero iff $f = g = \ell = e = 0$. Thus they form a basis and $\dim W_2 = 4$.

Intersection Let $M \in W_1 \cap W_2$. Then, equating the defining equations we have

$$M = \begin{bmatrix} a & b & a+b \\ c+d & d & d \end{bmatrix} = \begin{bmatrix} f & 2f & g \\ e & e & \ell - e \end{bmatrix} \tag{13}$$

which yields the system

$$\begin{cases} a = f \\ b = 2f \\ a + b = g \\ c + d = e \\ d = e \\ d = \ell - e \end{cases} \Rightarrow \begin{cases} a = f \\ b = 2f \\ g = 3f \\ c = 0 \\ d = e \\ \ell = 2e \end{cases} \Rightarrow M = \begin{bmatrix} f & 2f & 3f \\ e & e & e \end{bmatrix} = f \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} + e \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \tag{14}$$

The two matrices on the right side above are linearly independent because $M = 0$ iff $f = e = 0$. Thus $\dim(W_1 \cap W_2) = 2$ and the basis consists of the mentioned two matrices.

Sum space $W_1 + W_2$.

Method 1.

A matrix in the sum $W_1 + W_2$ has the form

$$\begin{bmatrix} a+f & b+2f & a+b+g \\ c+d+e & d+e & d+\ell-e \end{bmatrix} \tag{15}$$

We claim that any matrix in $V = Mat_{2\times 3}$ can be expressed in the above form. To see it let $M = \begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix}$. Equating the entries we have

$$\begin{bmatrix} a+f & b+2f & a+b+g \\ c+d+e & d+e & d+\ell-e \end{bmatrix} = \begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \Rightarrow \begin{cases} a + f = A \\ b + 2f = B \\ a + b + g = C \\ c + d + e = D \\ d + e = E \\ \ell + d - e = F \end{cases} \tag{16}$$

We can eye a solution by setting $f = 0 = e$ (for example) and then

$$\begin{cases} a = A \\ b = B \\ g = C - A - B \\ c = D - E \\ d = E \\ \ell = F - E \end{cases} \tag{17}$$

Thus the $\dim(W_1 + W_2) = \dim V = 6$ and a basis is for example the standard basis.

Method 2. We have that $W_1 + W_2$ is a finite-dimensional subspace of $Mat_{2\times 3}(\mathbb{R})$ with

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) = 4 + 4 - 2 = 6.$$

Therefore, since $\dim Mat_{2\times 3}(\mathbb{R}) = 6$, it results that $W_1 + W_2 = Mat_{2\times 3}(\mathbb{R})$ and a basis for $W_1 + W_2$ can be for example, the standard basis.

$\square$

Solution to Problem 4 From the nullity+rank theorem

$$nul(T) + rk(T) = \dim V \tag{18}$$

we have

$$nul(T) = \dim V - rk(T). \tag{19}$$

Since $rk(T) \leq \dim W$ and $\dim V = 2 + \dim W$, we have

$$nul(T) \geq \dim V - \dim W = 2 > 0. \tag{20}$$

The map cannot be one-to-one because the kernel (null-space) is nontrivial. $\square$

Solution to Problem 5

$$T(x+2) = \big(3+2, 1\big) = \big(5, 1\big) = 3(1,1) + 2(1,-1) \tag{21}$$

$$T(x-3) = \big(3-3, 1\big) = \big(0, 1\big) = -\frac{1}{2}(1,1) - \frac{1}{2}(1,-1) \tag{22}$$

$$T(x^2) = \big(3^2, 4\big) = \big(9, 4\big) = \frac{13}{2}(1,1) + \frac{5}{2}(1,-1) \tag{23}$$

$$[T]_\beta^\gamma = \begin{bmatrix} 3 & \frac{1}{2} & \frac{13}{2} \\ 2 & -\frac{1}{2} & \frac{5}{2} \end{bmatrix} \tag{24}$$

$$T(x^0) = \big(3^0, 0\big) = \big(1, 0\big) = \frac{1}{2}(1,1) + \frac{1}{2}(1,-1) \tag{25}$$

$$T(x) = \big(3, 1\big) = \big(3, 1\big) = 2(1,1) + 1(1,-1) \tag{26}$$

$$T(x^2) = \big(3^2, 4\big) = \big(9, 4\big) = \frac{13}{2}(1,1) + \frac{5}{2}(1,-1) \tag{27}$$

$$[T]_\alpha^\gamma = \begin{bmatrix} \frac{1}{2} & 2 & \frac{13}{2} \\ \frac{1}{2} & 1 & \frac{5}{2} \end{bmatrix} \tag{28}$$

$\square$

Solution to Problem 6 The map is linear;

$$T((p+q)(x)) = (p+q)(x^2) + 3(p+q)(x-2) = p(x^2) + 3p(x-2) + q(x^2) + 3q(x-2) = T(p(x)) + T(q(x)) \tag{29}$$

$$T(\lambda p(x)) = \lambda p(x^2) + \lambda 3p(x-2) = \lambda(p(x^2) + 3p(x-2)) = \lambda T(p(x)) \tag{30}$$

Then:

$$\begin{aligned} T(1) &= 1+3 &&= 4; && (31)\\ T(x) &= x^2+3(x-2) &&= -6+3x+x^2; && (32)\\ T(x^2) &= x^4+3(x-2)^2 &&= 12-12x+3x^2+x^4 && (33)\\ & && && (34) \end{aligned}$$

Thus

$$[T]_\beta = \begin{bmatrix} 4 & -6 & 12 \\ 0 & 3 & -12 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{35}$$

$\square$

Solution to Problem 7 If $\underline{w} \in \mathbf{R}(U+T)$ is an arbitrary vector in the indicated range, then there must exist $\underline{v} \in V$ such that $\underline{w} = (U+T)(\underline{v})$. Then

$$\underline{w} = (U+T)(\underline{v}) = U(\underline{v}) + T(\underline{v}) = \underline{w}_1 + \underline{w}_2 \tag{36}$$

where, by definition of range, $\underline{w}_1 \in \mathbf{R}(U)$, $\underline{w}_2 \in \mathbf{R}(T)$. Then, by definition of sum of vector subspaces $\underline{w} \in \mathbf{R}(U) + \mathbf{R}(T)$. Therefore we have the stated inclusion.

To give an example where the inclusion is strict consider $T : \mathbb{R}^2 \to \mathbb{R}^2$ to be the identity and $U = -T$. Then $U+T$ is the zero map, and its range is trivial. On the other hand the ranges of U, T are the whole $\mathbb{R}^2$, and hence their sum is also $\mathbb{R}^2$.

$\square$