5.4 Indefinite Integrals

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Definitions & Theorems:

1. Note: $\int_a^b f(x)\,\mathrm{d}x$ is a number, but $\int f(x)\,\mathrm{d}x$ is a function. The connection between them is
$$\int_a^b f(x)\,\mathrm{d}x = \int f(x)\,\mathrm{d}x\Big]_a^b$$
2. Properties:

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad \int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

3. Theorem: The Net Change Theorem
   The integral of a rate of change is the net change:
$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

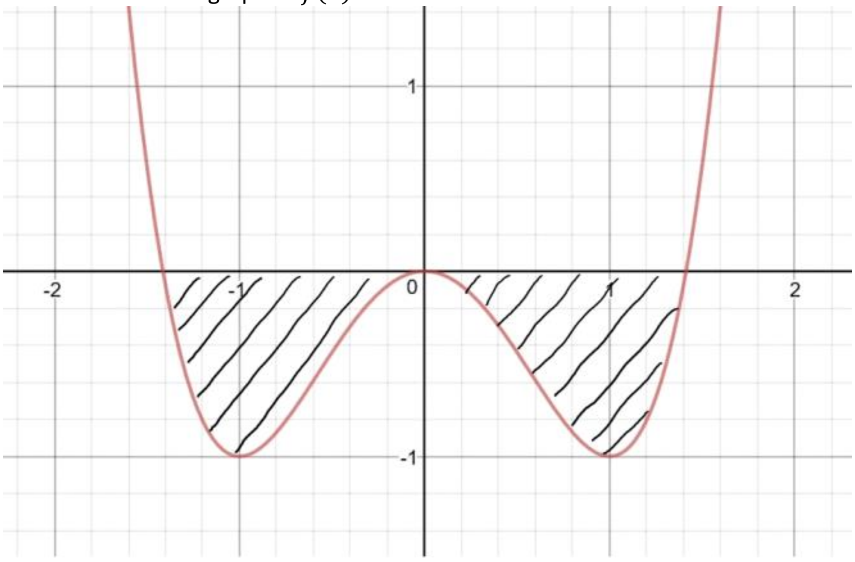
Proofs or Explanations:

1. Theorem3:
   FTC2: If f is continuous on $[a,b]$, then
$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$
   where F is any antiderivative of f. This means that $F' = f$, so the equation can be rewritten as
$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Examples:

1. $\displaystyle\int x^3\,\mathrm{d}x = \frac{x^4}{4} + C$
2. $\displaystyle\int \frac{\mathrm{d}x}{1+x^2} = \tan^{-1}x + C$
3. $\displaystyle\int 3^x\,\mathrm{d}x = \frac{3^x}{\ln 3}\mathrm{d}x + C$
4. $\displaystyle\int \frac{y+y^2+1}{y}\mathrm{d}y = \int\left(1+y+\frac{1}{y}\right)\mathrm{d}y = y + \frac{y^2}{2} + \ln|y| + C$
5. $\displaystyle\int \frac{\sin\theta}{\cos^2\theta}\mathrm{d}\theta = \int \frac{1}{\cos\theta}\frac{\sin\theta}{\cos\theta}\mathrm{d}\theta = \int \sec\theta\tan\theta\,\mathrm{d}\theta = \sec\theta + C$
6. Find the area beneath the graph of $f(b) = \frac{1}{1+x^2}$ on $[0,1]$
$$A = \int_0^1 f(x)\,\mathrm{d}x = \int_0^1 \frac{1}{1+x^2}\mathrm{d}x = \tan^{-1}x\Big]_0^1 = \frac{\pi}{4}$$

7. Find the area above the graph of $f(x) = x^4 - 2x^2$



$$A = -\int_{-\sqrt{2}}^{\sqrt{2}} f(x)\,\mathrm{d}x = -2\int_{0}^{\sqrt{2}} f(x)\,\mathrm{d}x = -2\left(\frac{x^5}{5} - \frac{2x^3}{3}\right)\Bigg]_0^{\sqrt{2}} = \frac{2^{\frac{9}{2}}}{15}$$

8. Evaluate $\int_0^4 f(x)\,\mathrm{d}x$, where $f(x) = x^2$, using Riemann sum with 4 subintervals of equal length.

$$\Delta x = \frac{4-0}{4} = 1$$

$$R_4 = \sum_{i=1}^{4} f(a + i\Delta x)\Delta x = f(1)(1) + f(2)(1) + f(3)(1) + f(4)(1) = 1^2 + 2^2 + 3^2 + 4^2 = 30$$

9. Evaluate $F(0)$ and $F'(0)$ where $F(x) = \int_{\sqrt{x}}^{3x} t^2 \sin(1 + t^2)\,\mathrm{d}t$

$$F(0) = \int_0^0 t^2 \sin(1 + t^2)\,\mathrm{d}t = 0$$

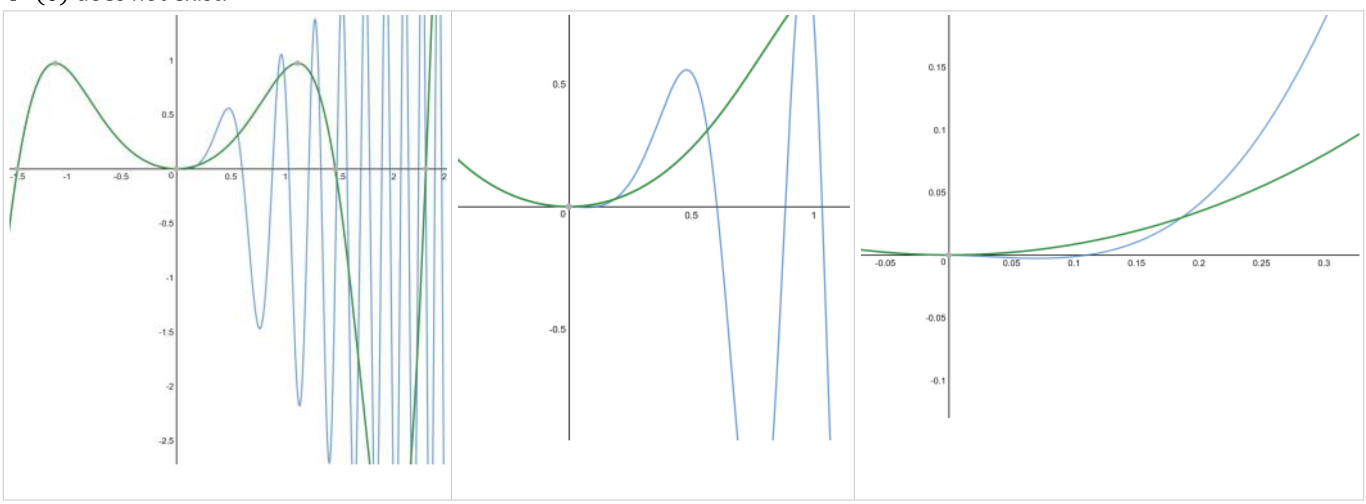
$$F'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_{\sqrt{x}}^{3x} t^2 \sin(1 + t^2)\,\mathrm{d}t$$

$$= \frac{\mathrm{d}}{\mathrm{d}x}\int_{C}^{3x} t^2 \sin(1 + t^2)\,\mathrm{d}t - \frac{\mathrm{d}}{\mathrm{d}x}\int_{C}^{\sqrt{x}} t^2 \sin(1 + t^2)\,\mathrm{d}t$$

$$= (3x)^2 \sin(1 + (3x)^2)(3) - \left(\sqrt{x}\right)^2 \sin\left(1 + \left(\sqrt{x}\right)^2\right)\left(\frac{1}{2}\right)\left(x^{-\frac{1}{2}}\right)$$

$$= 9x^2 \sin(9x^2 + 1)(3) - x\sin(x+1)\frac{1}{2\sqrt{x}}$$

$F'(0)$ does not exist.



10. Where is $G(x) = \int_{-4}^{x} \mathrm{e}^{2s}\cos^2(1 - 5s)\,\mathrm{d}s$ increasing/decreasing?

$G'(x) = \mathrm{e}^{2x}\cos^2(1 - 5x) \geq 0$

$G(x)$ is (non strictly) increasing.

11. Find the antiderivative $F(x)$ of $f(x) = \frac{4\pi}{3}x - \frac{1}{\sqrt{1-x^2}}$, $F\left(\frac{1}{2}\right) = \pi$.

$$F(x) = \int f(x)\,\mathrm{d}x = \int \left(\frac{4\pi x}{3} - \frac{1}{\sqrt{1-x^2}}\right) \mathrm{d}x = \frac{4\pi}{3}\frac{x^2}{2} - \sin^{-1} x + C$$

$$F\left(\frac{1}{2}\right) = \frac{4\pi}{3}\frac{\left(\frac{1}{2}\right)^2}{2} - \sin^{-1}\left(\frac{1}{2}\right) + C = C = \pi$$

$$F(x) = \frac{2\pi}{3}x^2 - \sin^{-1} x + \pi$$

12. A particle travels along a straight line at a velocity $v(t) = t^2 - t - 6$
    a. Find the particle's displacement during the time period $[1, 4]$
    b. Find the distance travelled during that time period.

    a. $\displaystyle\int_1^4 v(t)\,\mathrm{d}t = \int_1^4 \left(t^2 - t - 6\right) \mathrm{d}t = \frac{t^3}{3} - \frac{t^2}{2} - 6t\Big]_1^4 = -\frac{9}{2}$

    b. $t^2 - t - 6 = (\mathrm{t} - 3)(\mathrm{t} + 2) \to v(t) > 0 \text{ on } (3, \infty), v(t) < 0 \text{ on } (-2, 3)$

    $$\int_1^4 |v(t)|\,\mathrm{d}t = \int_1^3 |v(t)|\,\mathrm{d}t + \int_3^4 |v(t)|\,\mathrm{d}t = \int_1^3 -v(t)\,\mathrm{d}t + \int_3^4 v(t)\,\mathrm{d}t = \frac{61}{6}$$

13. A bee population start at 100 bees and grows at a rate of $n'(t)$ per week. What dos $100 + \int_0^{12} n'(t)\,\mathrm{d}t$ represent?

    $$100 + \int_0^{12} n'(t)\,\mathrm{d}t = 100 + \big(n(12) - n(0)\big) = 100 + (n(12) - 100) = n(12)$$

    It represents the population of bees after 12 weeks.