11.3 The Integral Test

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Definitions & Theorems:

1. The Integral Test

Suppose f is a continuous, positive, decreasing function on $[1,\infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,\mathrm{d}x$ is convergent. In other words:

(i) If $\int_1^{\infty} f(x)\,\mathrm{d}x$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\int_1^{\infty} f(x)\,\mathrm{d}x$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Note:

a. The interval $[1,\infty)$ may be $[a,\infty)$, a > 1

b. Decreasing need not be everywhere.

c. We should *not* infer from the Integer Test that the sum of the series is equal to the value of the integral.

2. The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \le 1$

Proofs or Explanations:

1. The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$

(i) $p = 0 \Rightarrow \sum_{n=1}^{\infty} 1$

diverges by Test of Divergence

(ii) $p < 0 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^p} = \infty$

diverges by Test of Divergence

(iii) $p > 0 \Rightarrow f(x) = \frac{1}{x^p}$

i. $f(x)$ is continuous on $[1,\infty)$

ii. $0 < x \Rightarrow 0 < x^p \Rightarrow \frac{1}{x^p} > 0 \Rightarrow f(x)$ is positive on $[1,\infty)$

iii. $f'(x) = \frac{-p}{x^{p+1}} < 0 \Rightarrow f(x)$ is decreasing for all $x \in [1,\infty)$

$\Rightarrow$ we can use Integral Test

$\int_1^{\infty} \frac{1}{x^p}$ converges for $p > 0$, diverges for $0 < p \le 1$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges for $0 < p \le 1$ (Integral Test)

(i)(ii)(iii)$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges otherwise.

Extra topics:

1.

Examples:

1. For what value(s) of p, does $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converge?

(i) $p = 0 \Rightarrow \sum_{n=2}^{\infty} \frac{1}{n}$ is a p-series with $p = 1$, so $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ diverges

(ii) $p \ne 0$, let $f(x) = \frac{1}{x(\ln x)^p}$

i. $f(x)$ is continuous on $[2,\infty)$

ii. $x > 0$ for $x \in [2,\infty)$, $\ln x > 0$ for $x \in [2,\infty) \Rightarrow f(x) > 0$ for all $x \in [2,\infty) \Rightarrow f(x)$ is positive

iii. $f'(x) = -[x(\ln x)^p]^2 \left[(\ln x)^p + xp(\ln x)^{p-1}\left(\frac{1}{x}\right)\right] = -[x(\ln x)^p]^2\left[(\ln x)^p + p(\ln x)^{p-1}\right]$

Let $f'(x) < 0 \Rightarrow (\ln x)^p + p(\ln x)^{p-1} > 0 \Rightarrow (\ln x)^{p-1}(\ln x + p) > 0 \Rightarrow \ln x + p > 0 \Rightarrow x > e^{-p}$

$\Rightarrow f(x)$ is decreasing on (e^{-p}, ∞)

$\int_2^\infty \frac{1}{x(\ln x)^p}\,\mathrm{d}x$ converges for $p > 1$, diverges for $p \in (-\infty, 0) \cup (0,1]$ (See section 7.8 Example 4)

$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges for $p > 1$, diverges for $p \in (-\infty, 0) \cup (0,1]$

(i)(ii)$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges for $p > 1$, diverges otherwise.

2. For what values(s) of p, does $\sum_{n=2}^{\infty} \frac{\ln p}{n^p}$ converge?
3. Dos Integral Test apply for $\sum_{n=1}^{\infty} n\mathrm{e}^{-n^2}$