11.10 Taylor and Maclaurin Series (Omit Taylor Inequality and Binomial Series)

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Definitions & Theorems:

1. Theorem

   If f has a power series representation (expansion, centered) at a, that is, if
   $$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad |x-a| < R$$
   then its coefficients are given by the formula
   $$c_n = \frac{f^{(n)}(a)}{n!}$$
2. Theorem: Taylor series of the function f at a (or about a or centered at a)
   $$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} c_n (x-a)^n \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \end{aligned}$$
3. Theorem: Maclaurin series (Taylor series with $a = 0$)
   $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$
4. Theorem:

   If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a and
   $$\lim_{n\to\infty} R_n(x) = 0$$
   For $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.
5. Important Maclaurin Series and Their Radii of Convergence (Memorize 1-6)

   1) $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x^1 + x^2 + x^3 + x^4 + \cdots, \qquad R = 1$

   2) $e^x = \sum_{n=0}^{\infty} \dfrac{1}{n!} x^n = 1 + \dfrac{x^1}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \cdots, \qquad R = \infty$

   3) $\sin x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots, \qquad R = \infty$

   4) $\cos x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!} = \dfrac{x^0}{0!} - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots, \qquad R = \infty$

   5) $\arctan x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1} = \dfrac{x^1}{1} - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots, \qquad R = 1$

   6) $\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{n+1}}{n+1} = \dfrac{x^1}{1} - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots, \qquad R = 1$

7) $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{x} = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots, \qquad R = 1$

Proofs or Explanations:

1.

Examples:

1. Derive the Maclaurin series for $f(x) = e^x$

$$c_n = \frac{f^{(n)}(0)}{n!}$$
$$n = 0 \Rightarrow f^{(0)}(x) = f(x) = e^x \Rightarrow f^{(0)}(0) = e^0 = 1$$
$$n = 1 \Rightarrow f^{(1)}(x) = f'(x) = e^x \Rightarrow f^{(1)}(0) = 1$$
$$n = 2 \Rightarrow f^{(2)}(x) = f''(x) = e^x \Rightarrow f^{(2)}(0) = 1$$
$$c_n = \frac{1}{n!} \Rightarrow e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n, R = \infty$$

2. Derive the Maclaurin series for $f(x) = \sin x$

$$c_n = \frac{f^{(n)}(0)}{n!}$$
$$n = 0 \Rightarrow f^{(0)}(x) = f(x) = \sin x \Rightarrow f^{(0)}(0) = \sin 0 = 0$$
$$n = 1 \Rightarrow f^{(1)}(x) = f'(x) = \cos x \Rightarrow f^{(1)}(0) = \cos 0 = 1$$
$$n = 2 \Rightarrow f^{(2)}(x) = f''(x) = -\sin x \Rightarrow f^{(2)}(0) = -\sin 0 = 0$$
$$n = 3 \Rightarrow f^{(3)}(x) = f'''(x) = -\cos x \Rightarrow f^{(3)}(0) = -\cos 0 = -1$$
$$f(x) = \sin x = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$$
$$= \frac{f^{(0)}(0)}{0!} + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \cdots$$
$$= \frac{f^{(1)}(0)}{1!}x + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(5)}(0)}{5!}x^5 + \frac{f^{(7)}(0)}{7!}x^7 + \cdots$$
$$\sin x = \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

Ratio Test:

$$\lim_{n\to\infty} \left| \frac{(-1)^{(n+1)+1} x^{2(n+1)+1}}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{(-1)^{n+1} x^{2n+1}} \right| = \lim_{n\to\infty} \left| \frac{x^2}{(2n+2)(2n+3)} \right| == |x^2| \cdot 0 = 0 < 1$$

$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} x^{2n+1}$ converges for all $x \Rightarrow R = \infty$

3. Derive the Maclaurin series for $f(x) = -x^4 e^{2x}$

We know that $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ for all x

Thus, $e^{2x} = \sum_{n=0}^{\infty} \frac{1}{n!} (2x)^n$ for all x

Thus, $f(x) = -x^4 e^{2x} = -x^4 \sum_{n=0}^{\infty} \frac{1}{n!} (2x)^n = \sum_{n=0}^{\infty} \frac{-2^n}{n!} x^{n+4}$

4. $\int e^x \, dx, \int x e^{x^2} \, dx, \int x e^x \, dx, \int e^{x^2} \, dx$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \Rightarrow e^{x^2} = e^x = \sum_{n=0}^{\infty} \frac{1}{n!} (x^2)^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^{2n}$$

$$\Rightarrow \int e^{x^2}\, dx = \int \left(\sum_{n=0}^{\infty} \frac{1}{n!} x^{2n}\right) dx = \sum_{n=0}^{\infty} \int \left(\frac{1}{n!} x^{2n}\right) dx = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{x^{2n+1}}{2n+1} + C$$