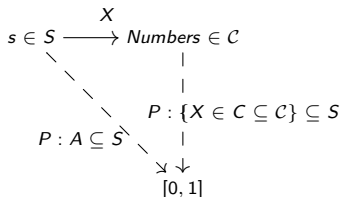


Layman's talk : what is continuous random variable

Random variables are functions



X maps elements from S to numbers! Now we can work with functions/numbers!



X essentially is a function to numbers!

E.g., X may assign random outcomes with real numbers, and thus it may take values from the real line.

To describe how the probability values are assigned to different events generated by a **real-valued** random variable X , we can look at the cumulative distribution function F_X such that

$$F_X(x) = P(X \leq x), x \in \mathbb{R}.$$

Continuous r.v. example

Let's keep throwing a ten-sided fair die ((with numbers 0,1,...,9)) without stopping!

- We have outcomes such as

$$a = 010310819.....$$

- Let's map these outcomes to numbers with $[0,1]$ via the decimal expansion:

$$X(a) = 0.a.$$

Here we follow the convention such that $0.99999 \dots = 1$.

- What is the probability of the event $\{X = 1\}$?

ZERO!

- $P(X = 1) = P(\{99999999 \dots\}) = \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} \times \dots = 0$.
- $P(X = a) = 0$.

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- We have outcomes such as

$$a = 010310819.....$$

- Let's map these outcomes to numbers with $[0,1]: X(a) = 0.a$.
- **How do we characterize the distribution of X :** e.g., how likely is the event $\{X \leq 0.5\}$.
 $\{X \leq 0.5\}$ contains all the outcomes such that the first throw is strictly smaller than five (0,1,2,3,4) and the specific outcome (only the first throw is a five the rests are all 0's)
 5000000000...:

$$P(X \leq 0.5) = 0.5 + P(\{5000000000 \dots\}) = 0.5$$

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- Let's cut $[0, 1]$ into 10^1 grids such that each grid can be expressed as

$$I_i = [x_{i-1,1}, x_{i,1}], i \in \{1, \dots, 10^1\}$$

with $x_j = j10^{-1}, j \in \{0, 1, \dots, 10^1\}$. E.g., $i = 2, [0.1, 0.2]$.

It is easy to check that

$$P(X \in \cup_{j=1}^i I_j) = P(X \leq x_i) = x_i = i * 10^{-1}, i \in \{1, \dots, 10\}$$

and thus for each grids $P(X \in I_j) = 10^{-1}$.

Let's approximate the probability value of $\{X \leq a\}$ with

$$P(X \in \cup_{j=1}^{j_a} I_j) = \sum_{i=1}^{j_a} \Delta x_i, \Delta x_i = x_i - x_{i-1}, j_a = \min\{i \in \{1, \dots, 10^1\} : a \in I_i\}.$$



$$\left| P(X \leq x) - \sum_{i=1}^{j_a} \Delta x_i \right| \leq 10^{-1}.$$

$$P(X \leq a) = a, a \in [0, 1].$$

- Let's cut $[0, 1]$ into 10^n grids such that each grid can be expressed as

$$I_i = [x_{i-1,n}, x_{i,n}], i \in \{1, \dots, 10^n\}$$

with $x_j = j10^{-n}, j \in \{0, 1, \dots, 10^n\}$. E.g., $i = 2, [10^{-n}, 2 * 10^{-n}]$.

It is easy to check that

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$$\sum_{i=1}^{j_a} \Delta x_i \rightarrow \int_0^a f_X(x) dx = a; f_X(x) = F'_X(x).$$

$$F_X(a) = P(X \leq a) = a, a \in [0, 1].$$

$$\int_{-\infty}^a f_X(x) dx = P(X \leq a) = a; f_X(x) = F'_X(x) = 1, x \in [0, 1] \text{ otherwise zero.}$$

- Two different ways describing the distribution of X , and particularly, there exists a **probability density function** f_X (integrate the (probability) *density*, we have the (probability) *mass*).

We have a probability density function f_X (integrate the (probability) density via **Riemann integral**, we have the (probability) value). If an interval has positive density function values, then there are uncountably many possible values in the interval taken by X .

$$\int_{-\infty}^a f_X(x) dx = P(X \leq a); a \in \mathbb{R}.$$

Any real-valued random variable that has a non-negative function f_X satisfying the above Riemann integral condition is a **continuous** real-valued random variable.

Let's compare.

- ▶ In the example, we see that we can choose $f_X(x) = F'_X(x)$ and $F'_X(x) = 1, x \in [0, 1]$.
There are **uncountably infinite** possible values in $[0, 1]$ taken by X .
- ▶ For any discrete random variable Y , almost everywhere F'_Y is simply **zero**, there is no f_X satisfying the above identity (via Riemann integral).
There are **at most countably many** possible values taken by Y .