Probability Theory for EOR

Random variables (discrete)

Random variables are functions

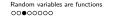
Definition (Random variables (r.v.'s) (preliminary))

A random variable (r.v.) is a function maps elements in sample space S to numbers in C, e.g., $C \subseteq \mathbb{R}$.

X maps elements from S to numbers! Now we can work with functions/numbers!. Usually, we consider real-valued r.v.'s, i.e.,

$$X:S\mapsto \mathbb{R}$$

- X essentially is a function, and when we manipulate random variables, it would be similar to manipulating functions and there are so many transformations we can work with: summation, division, max, min...
- ▶ There are many functions from S to \mathbb{R} (so there are also many random variables defined with the same S), not all functions can be regarded as random variables, but all random variables are functions. Detailed discussion goes beyond the course scope.
- ▶ We can now use $\{X \in C\}$ with $C \subseteq C$ to represent different random events, and properties derived for random events would be inherited. **This would connect random variables with probability functions** as probability functions have random events as elements in its domain.
- Given the introduction of these additional structures, we have a much larger place for our mind adventures: conditional probability/independence based on random variables, random variables with specific probability functions... Random variables are functions, there are selected functions with nice properties, and so are some selected random variables.



Discrete random variable

There are many functions from a sample space to a set of numbers, and we start with those map to sets of a sequence of numbers a_1, a_2, a_3, \cdots

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- ► The outcomes are strings like *HHTHTHTHT*.
- ► Notations:
 - ightharpoonup define the event H_i : we see i heads.
 - ightharpoonup define the event T_j : we see j tails.
 - $\blacktriangleright \ H_i = T_{n-i}.$
- **▶** Cumbersome!

- ▶ Let X be the number of heads and Y be the number of tails.
- ▶ Improves notation: no subindices as with H_i , T_j or $H_i = T_{n-i}$.
- ▶ Improves calculation: we see Y = n X.

But, what are these X and Y?

Throw a coin n times. Define X to be the number of heads.

X takes an outcome s in the sample space S, and maps that to a number on the real line.

$$X(HHTHTHHTHT) = 6,$$

 $X(HHTHTHHTTH) = 6,$
 $X(TTTTTTTTTT) = 0,$
 \vdots

- ▶ X is a function that maps each $s \in S \to \mathbb{R}$.
- ▶ Because the outcome is random, *X* is called a *random variable* (r.v.). But it is simply a function.

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- \blacktriangleright X needs a set of outcomes. An r.v. is defined on a sample space S.
- ► We can define *multiple* r.v.'s on *S*:
 - ► X: number of heads;
 - ► Y: number of tails:
 - ► Z: only equal to one if the last throw is a head.
- ▶ X assigns a number to each $s \in S$.

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- ▶ Furthermore, from X we could define events, e.g., let $\{X \in \{6\}\}$ (or $\{X = 6\}$) denote the set of outcomes $s \in S$ such that X(s) = 6, and it would be the event that we get 6 heads in 10 throws.
- ► $P(X = 6) = P(\{HHTHTHHTHT\} \cup \{HHTHTHHTTH\} \cup ...) = \frac{\binom{10}{6}}{\frac{10}{210}} \approx 0.205078.$
- ▶ With the defined events, it is easy to see that

$$P(X \in \{0, 1, 2, \cdots, 10\}) = 1$$

► We have a discrete random variable!

Discrete random variable

Definition (Discrete random variable)

A random variable X is a **discrete** random variable if there exists a set C with at most countably many numbers (so you can index all the elements within C using a set of natural numbers):

$$P({X \in C}) = 1$$
, or equivalently, $P(X \in C) = 1$.

The **support** of a **discrete** random variable is the set of all numbers, x's, such that P(X = x) > 0:

$$supp(X) = \{x \in C : P(X = x) > 0\}.$$

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Previous simple example revisits: throw a coin 10 times

▶ Previously: *X* takes an outcome *s* in the sample space *S*, and maps that to a number on the real line.

$$X(HHTHTHHTHT) = 6,$$

 $X(HHTHTHHTTH) = 6,$
 $X(TTTTTTTTTT) = 0,$

X is a **real-valued** r.v.!

►
$$C = \{x_1, x_2, \dots\} = \{x_i, i \in \mathbb{N}_+\}$$
 such that $x_i = i - 1$:
 $P(X \in C) = 1$.

So, X is a **discrete** r.v.! (The choice of such C is not unique.)

$$supp(X) = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.$$

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