List of useful identities and PDF/PMFs

1. Useful summation tricks ($m \leq n$)[1] :

   (a) $\sum_{i=m}^{n}(f(i)+g(i)) = \sum_{i=m}^{n} f(i) + \sum_{i=m}^{n} g(i)$

   (b) $\sum_{i=m}^{n} f(i) = \sum_{i=0}^{n} f(i) - \sum_{i=0}^{m-1} f(i)$

   (c) $\sum_{i=m}^{n} cf(i) = c\sum_{i=m}^{n} f(i)$.

2. Some other useful identities[2]:

   (a) $\sum_{i=0}^{n-1} i = \frac{n(n-1)}{2}$

   (b) $\sum_{i=0}^{n-1} i^2 = \frac{n(n-1)(2n-1)}{6}$

   (c) $\sum_{i=0}^{n-1} i^3 = \left(\frac{n(n-1)}{2}\right)^2$

   (d) $\sum_{i=m}^{n} 1/(k(k+1)) = \sum_{i=m}^{n}(1/k - 1/(k+1)) = 1/m - 1/(n+1)$

   (e) $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

   (f) $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ (Basel problem, main point is to realize that this is finite, while the sum on the previous line is not)

   (g) $\sum_{x=0}^{n} z^x = \frac{1-z^{n+1}}{1-z}$ (finite geometric series)

   (h) $\sum_{x=0}^{\infty} z^x = \frac{1}{1-z}$ when $|z| < 1$ (infinite geometric series)

   (i) $\sum_{x=1}^{\infty} xz^{x-1} = \frac{1}{(1-z)^2}$ when $|z| < 1$ (easy to memorize as it is simply taking derivatives on both sides of $\sum_{x=0}^{\infty} z^x = \frac{1}{1-z}$), which also implies $\sum_{x=1}^{\infty} xz^x = \frac{z}{(1-z)^2}$ by multiplying x on both sides.

[1] 1a and 1c simply show that the summation operator also satisfies the linearity. Think about the expectation operator we have discussed.

[2] 1 and 2 are heavily used in Pois, Expo, MGF...

(j) $\sum_{x=1}^{\infty} x^2 z^{x-1} = \frac{1+z}{(1-z)^3}$ when $|z| < 1$ (easy to memorize as it is simply taking derivatives on both sides of $\sum_{x=1}^{\infty} xz^x = \frac{z}{(1-z)^2}$),which also implies $\sum_{x=1}^{\infty} x^2 z^x = \frac{z(1+z)}{(1-z)^3}$ by multiplying x on both sides.

(k) The Taylor expansion for e^x:

$$e^x = \lim_{l \to \infty} \sum_{n=0}^{l} x^n/n! = \sum_{n=0}^{\infty} x^n/n! = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots \tag{1}$$

(l) The definition of e^x:

$$e^x = \lim_{n \to \infty} (1 + \frac{x}{n})^n \tag{2}$$

We use quite often the Euler's number or exponential function (Pois, Expo, Normal). Eq(2) is covered in Math I, while for Eq (1), it is also easy to memorize once you learn the PMF of Pois(x) is simply

$$p(n) = e^{-x}\frac{x^n}{n!}, n = 0, 1, \cdots$$

and the sum of all PMFs should be one which implies

$$\sum_{n=0}^{\infty} e^{-x}\frac{x^n}{n!} = 1$$

and thus

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$$

3. Productions:

(a) $\prod_{0 \le i < j \le m} a_i a_j = \prod_{i=0}^{m} \prod_{j=i+1}^{m} a_i a_j.$

(b) $\prod_{0 \le i,j \le m} a_i a_j = \prod_{i=0}^{m} \prod_{j=0}^{m} a_i a_j.$

4. There are some conventions of when we can ignore brackets, we list some to avoid confusions:

(a) $\mathbb{E}X^p := \mathbb{E}(X^p)$

(b) $\mathbb{E}XY := \mathbb{E}(XY)$

5. Some identities concerning the choice functions (we list more than sufficient here):

(a) The Bernoulli identity $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$

(b) $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

(c) $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$

(d) $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ for all integers $n, k : 1 \le k \le n-1$

(e) $\sum_{m=0}^{n} \binom{m}{j}\binom{n-m}{k-j} = \binom{n+1}{k+1}$

(f) $\sum_{m=k}^{n} \binom{m}{k} = \binom{n+1}{k+1}$

(g) $\sum_{r=0}^{m} \binom{n+r}{r} = \binom{n+m+1}{m}$

(h) $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$

(i) $\binom{n-1}{k} - \binom{n-1}{k-1} = \frac{n-2k}{n}\binom{n}{k}$

(j) $\binom{n}{h}\binom{n-h}{k} = \binom{n}{k}\binom{n-k}{h}$

6. Some tricks concerning integration

   (a) **Integration by parts:**
   Define $F(x)$ such that $\frac{dF(x)}{dx} = f(x)$. Then

$$\int_a^b f(x)g(x)dx = F(x)g(x)|_a^b - \int_a^b F(x)\frac{dg(x)}{dx}dx$$

   Example

$$\begin{aligned} I = \int_0^\infty x^2 e^{-x} dx &= -x^2 e^{-x}|_0^\infty - \int_0^\infty 2x(-e^{-x})dx \\ &= 2\int_0^\infty xe^{-x}dx = 2\left(-xe^{-x}|_0^\infty - \int_0^\infty (-e^{-x})dx\right) \\ &= 2\int_0^\infty e^{-x}dx = 2\left[-e^{-x}\right]_0^\infty = 2 \end{aligned}$$

   (b) **Integration by substitution:**
   Suppose you need to integrate a function of the form $h(x) = f(g(x))\frac{dg(x)}{dx}$. Then you can use that

$$\int_a^b h(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

   Why?

$$\int_a^b h(x)dx = \int_a^b \frac{F(g(x))}{dx}dx = F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(u)du$$

   Example. Take $h(x) = 2x\exp(-x^2)$. Then,

$$I = \int_0^2 2x\exp(-x^2)dx = \int_0^4 exp(-u)du = 1 - e^{-4}$$

7. PDFs:

Name	Param.	PMF or PDF	Mean	Variance
Bernoulli	p	$P(X=1)=p, P(X=0)=q$	p	pq
Binomial	n, p	$\binom{n}{k}p^k q^{n-k}$, for $k \in \{0,1,\dots,n\}$	np	npq
FS	p	pq^{k-1}, for $k \in \{1,2,\dots\}$	$1/p$	q/p^2
Geom	p	pq^k, for $k \in \{0,1,2,\dots\}$	q/p	q/p^2
NBinom	r, p	$\binom{r+n-1}{r-1}p^r q^n, n \in \{0,1,2,\dots\}$	rq/p	rq/p^2
HGeom	w, b, n	$\frac{\binom{w}{k}\binom{b}{n-k}}{\binom{w+b}{n}}$, for $k \in \{0,1,\dots,n\}$	$\mu = \frac{nw}{w+b}$	$(\frac{w+b-n}{w+b-1})n\frac{\mu}{n}(1-\frac{\mu}{n})$
Poisson	λ	$\frac{e^{-\lambda}\lambda^k}{k!}$, for $k \in \{0,1,2,\dots\}$	λ	λ
Uniform	$a < b$	$\frac{1}{b-a}$, for $x \in (a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Normal	μ, σ^2	$\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)}$	μ	σ^2
Log-Normal	μ, σ^2	$\frac{1}{x\sigma\sqrt{2\pi}}e^{-(\log x-\mu)^2/(2\sigma^2)}, x > 0$	$\theta = e^{\mu+\sigma^2/2}$	$\theta^2(e^{\sigma^2}-1)$
Expo	λ	$\lambda e^{-\lambda x}$, for $x > 0$	$1/\lambda$	$1/\lambda^2$
Gamma	a, λ	$\Gamma(a)^{-1}(\lambda x)^a e^{-\lambda x}x^{-1}$, for $x > 0$	a/λ	a/λ^2
Beta	a, b	$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}x^{a-1}(1-x)^{b-1}$, for $0 < x < 1$	$\mu = \frac{a}{a+b}$	$\frac{\mu(1-\mu)}{a+b+1}$
Chi-Square	n	$\frac{1}{2^{n/2}\Gamma(n/2)}x^{n/2-1}e^{-x/2}$, for $x > 0$	n	$2n$
Student-t	n	$\frac{\Gamma((n+1)/2)}{\sqrt{n\pi}\Gamma(n/2)}(1+x^2/n)^{-(n+1)/2}$	0 if $n > 1$	$\frac{n}{n-2}$ if $n > 2$