Probability Theory for EOR 2020/2021

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Solutions to Exercises

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Week 1

December 28, 2021

Counting

1. 11 distinct letters can be arranged in 11! ways. But of course, the letters of MISSISSIPPI are not all distinct: we have 1M, 4I, 4S and 2P. We therefore need to adjust for overcounting. To determine the overcounting factor, note that if we interchange the S's, nothing changes. We can interchange the S's in 4! ways. For each of these ways, we can then also interchange the I's, and still nothing changes. By the multiplication rule, we can interchange the S's and I's in $4! \cdot 4!$ ways. Adding the P's, we have $4! \cdot 4! \cdot 2!$ permutations that yield exactly the same result. We conclude that the letters of the word MISSISSIPPI can be arranged in

$$N = \frac{11!}{4! \cdot 4! \cdot 2!} \text{ ways.}$$

Alternative solution. There are 11 spots. Pick 4 spots for the S's. This can be done in $\binom{11}{4}$ ways. Then from the remaining 7 spots, pick 4 spots for the I's. This can be done in $\binom{7}{4}$ ways. Then from the remaining 3 spots, pick 2 spots for the P's. This can be done in $\binom{3}{2}$ ways. By the multiplication rule, we end up with

$$N = \binom{11}{4} \cdot \binom{7}{4} \cdot \binom{3}{2} \text{ ways.}$$

3a. If no restaurant can be the same, then on Monday Fred has 10 options, on Tuesday Fred has 9 options, on Wednesday Fred has 8 options, etc. So in total, there are $N = 10 \cdot 9 \ldots \cdot 6$ options.

3b. On Monday, Fred can choose 10 restaurants. Then on Tuesday, Fred only has 9 options, since he can't eat at the same restaurant as on Monday. On Wednesday, there are again 9 options, as the Monday restaurant is now fine, but the Tuesday restaurant is not. So we end up with $N = 10 \cdot 9^4$ options.

4a. To find the number of games that are played, you could make a roster of players

$$\begin{array}{c|cccc} & P_1 & P_2 & \dots & P_n \\ P_1 & \times & & & \\ P_2 & & \times & & \\ \vdots & & & \ddots & \\ P_n & & & & \times \end{array}$$

Now find the number of elements in the lower triangular: $\frac{n^2-n}{2}$. Put these matches in a long line. Now each of the matches can results in a Win or a Loss (doesn't matter for which player). So we can make $2^{\frac{n(n-1)}{2}}$ different strings of outcomes.

4b. See a.

6. Different solutions possible. The first player has 19 possible opponents and he can play white or black pieces, so $19 \cdot 2$ options. The next player has 17 possible opponents and can again play with white or black pieces, so $17 \cdot 2$ options. Continuing all the way down, we see that there are $N = (19 \cdot 17 \cdot 15 \cdot \ldots \cdot 3 \cdot 1) \cdot 2^{10}$ options.

   Alternatively, picture 10 boards with pieces already up. There are 20 seats for the players. There are therefore 20! arrangement possible. However, once seated, we can change the order of the boards + players. So we overcount by 10! and the total number of options is $N = \frac{20!}{10!}$.

9a. We need to set 221 steps, 110 are to the right and 111 are up. There are $N_1 = \binom{221}{111}$ ways to do so.

9b. Use the multiplication rule. From (a), there are $N_1 = \binom{221}{111}$ to get to (110,111). Then, there are $N_2 = \binom{200}{100}$ ways to get to $(210, 211)$. In total, there are $N = N_1 \cdot N_2 = \binom{221}{111} \cdot \binom{200}{100}$ ways to complete the stated path.

10a. Without restrictions there would be $N_u = \binom{20}{7}$ options. From those, we need to subtract the number of choices that do not contain statistics course, which would amount to choosing 7 courses from the 15 non-statistics courses: $N_r = \binom{15}{7}$. Then, $N = N_u - N_r = \binom{20}{7} - \binom{15}{7}$.

10b. The argument seems to be that we first choose 1 out of 5 statistics courses, so $\binom{5}{1}$ options for those, and then argue that the remaining 6 courses would be free to choose, so $\binom{19}{6}$ options for those. But now we are counting things double. Consider choosing Statistics 1 and Statistics 2, and 5 non-statistics courses. Then we are counting this options twice: once with Statistics 1 as *the* statistics course in the $\binom{5}{1}$ term and once with Statistics 2 as *the* statistics course in the $\binom{5}{1}$ term. So we are overcounting!

12a. There are 52 cards, 13 of which are dealt to Player A. The number of possible hands (when ordering within the hand is not important) is $N_A = \binom{52}{13}$.

12b. First deal to Player A. Following the answer to a., there are $N_A = \binom{52}{13}$ options. Then deal to Player B. There are 39 cards left, of which this player will receive 13. This can be done in $N_B = \binom{39}{13}$ ways. Then deal to Player C. There are 26 cards left, of which this player will receive 13. This can be done in $N_C = \binom{26}{13}$ ways. We give the remaining 13 cards to Player D. In total, we have

$$N = N_A N_B N_C = \binom{52}{13}\binom{39}{13}\binom{26}{13}$$

possibilities.

12c. If we would put the 13 cards that Player A received back into the deck before dealing to Player B, then this would be the answer.

13. Order the deck of cards by unique cards (there are 52 unique cards, with 10 replications). Then picture 52 boxes separated by 51 bars. Between each bar (as well as on the left side of the leftmost bar and the right side of the rightmost bar) we can put maximally 10 stars. So the problem is equivalent to the number of orderings of 51 bars and 10 stars. These can be ordered in $N = \binom{61}{10}$ ways.

Remark: I usually got questions about this ex 13 and the solution here could use a bit more explanations.
A first natural (**but incorrect**) thought for this question would be that we first choose one from the 520 cards and then 510 until 430 (520- (10-1)*10), because the order does not matter, we need to divide it by 10!. However, this is not correct because **(1)** the question asks for the number of different 10-card hand without considering which or the original 10 decks the cards come from (so 7 (heart) from the first deck is regarded as the same card as the 7 (heart) from the 10th deck); **(2)** the question asks for the number of different 10-card hands, but it does not require that the 10 cards in a 10-card hand have to be different cards. Is it correct then we choose the first from 52 and then until 43 and divide by 10! ? Unfortunately, it is incorrect again because point (2) mentioned above (Note that we are now having so many replications, we should have more choices for a 10-card hand, now we could even have 1111111111 all of the same suit.).

Then how do we proceed from there? Think about 51 bars (then with 51 bars we have 52 places to allocate 52 different cards each with 10 replications). Next, we can put stars to our selected cards, and the question essentially is to assign 10 stars to these 52 places, which is equivalent to select 10 positions out of (51 bars+10 stars) positions for the stars (the rest positions would be bars).

Once you draw an arbitrary figure for the bars and stars, you would determine one

10-card hand. Draw it and try to figure it out. You can do this.

Story Proofs

15. Consider n people standing in a line (1.5m apart). Each person can be selected for some test or not. So there are 2 options for the first person, 2 options for the second person etcetera. There are then 2^n possible sequences (the right hand side of the question). This gives me all possible subgroups that can be selected for testing. This means that I can also select k persons out of n for the test, and then sum over k from 0 to n (the left hand side).

16a.

$$\begin{aligned}\binom{n}{k}+\binom{n}{k-1} &= \frac{n!}{(n-k)!\cdot k!}+\frac{n!}{(n-k+1)!\cdot(k-1)!} \\ &= \frac{(n-k+1)\cdot n!}{(n-k+1)!\cdot k!}+\frac{n!\cdot k}{(n-k+1)!\cdot k!} \\ &= \frac{(n+1)!}{(n-k+1)!\cdot k!} \\ &= \binom{n+1}{k}.\end{aligned}$$

16b. I have a group of $n+1$ people and select k. We call one of these $n+1$ people president. Then there are two options: I do select the present and then select $k-1$ people from the remaining n, i.e. $\binom{n}{k-1}$, or I do not select the president and select k people from the remaining n, i.e. $\binom{n}{k}$. Adding these two options gives the left hand side.

17. On the right hand side there is a group of $2n$ people from which I select n. Divide the $2n$ people in equally sized subgroups. To select n people in total, I need to select k from the first subgroup and $n-k$ from the second subgroup. By the multiplication rule, this can be done in $\binom{n}{k}\cdot\binom{n}{n-k}=\binom{n}{k}^2$ ways. Of course, we should consider all possible values of k, and end up with $\sum_{k=0}^{n}\binom{n}{k}^2$.

Naive definition of probability

23. Define by A the event that buttons for 3 consecutive floors are pressed. Notice first that the first floor is the ground floor since this is an American book. There are in total $N_T=9^3$ different sets of preferences of the three persons. We get consecutive buttons if they want to go to floor $2,3,4,\ 3,4,5,\ 4,5,6,\ \ldots,\ 8,9,10$. These are 7 options, but the order of these number is not relevant, so there are $N_A=7\cdot 3!$ options favorable to A. Using the naive definition of probability, we find that

$$P(A)=\frac{N_A}{N_T}=\frac{7\cdot 3!}{9^3}.$$

26a. In the birthday problem, birthday's enter the room with replacement. Here survey respondents enter with replacement.

26b. From the analysis of the birthday problem, we know that it is much easier to look at the probability of no match (A).

$$P(A) = 1 \cdot \left(1 - \frac{1}{10^6}\right)\left(1 - \frac{2}{10^6}\right) \cdots \left(1 - \frac{999}{10^6}\right).$$

The probability that at least one person will be chosen more than once is equal to $1 - P(A) \approx 0.393$.

31. After the first round, the are n elk that have been captured. Suppose these are tagged. In the second round, we capture m elk. Denote by A_k the probability that exactly k of these m elk are tagged. The total number of ways to select m elk is $\binom{N}{m}$. Selecting k tagged elks and $m-k$ non-tagged elks can be done in $\binom{n}{k} \cdot \binom{N-n}{m-k}$ ways. Then,

$$P(A) = \frac{\binom{n}{k} \cdot \binom{N-n}{m-k}}{\binom{N}{m}}.$$

34a. There are $\binom{13}{5}$ ways to get 5 cards with the suit Spades (for example), so $\binom{13}{5} - 1$ ways to get 5 cards with the suit Spades excluding the royal flush. A flush can appear for all suits, so there are $N_F = 4 \cdot (\binom{13}{5} - 1)$ ways to get a flush. There are $N_T = \binom{52}{5}$ ways to draw 5 cards from a standard deck. Then, denoting by F the event of a flush,

$$P(F) = \frac{N_F}{N_T} = \frac{4 \cdot (\binom{13}{5} - 1)}{\binom{52}{5}}.$$

34b. Discussed in Tutorial Week 1.

36. Denote the event of five of each of the values as $5E$. The total number of outcomes is $N_T = 6^{30}$. Notice that in this total number of outcomes, the outcome $11111, 22222, \ldots, 66666$ is counted once, but it does include all permutations where for example a 1 and a 2 trade places. So it includes $N_{5E} = \frac{30!}{5!^6}$ sequences favorable to the outcome. Therefore $P(5E) = \frac{N_{5E}}{N_T} = \frac{30!}{5!^6 \cdot 6^{30}}$.

Remark: to derive the number N_{5E}: we consider this specific sequence 6666655555...11111 *(30 numbers containing 5 of each of the values ordered in descending order), the first 6 can be taken by one of 30 different dice, and the second 6 can be taken by one of 29 different dice, keep going and this is how the numerator comes from* $30!$. *But we are over-counting! Dice A and dice B taking the first 6 and fifth 6 respectively should be the same outcome as Dice B and dice A taking the first 6 and fifth 6 when other remain the same* ($5!$)*, this happens for all 6 numbers from 6 to 1 each with* $5!$ ($5!^6$)*, this is how is the denominator comes. By this way, we have a one to one mapping from our way* *of counting to one outcome, note here we also allows permutations, such as dice A and dice B taking a 6 and a 5 is counted as one and dice B and dice A taking a 6 and a 5 is also counted as one.*

37a. You can think of the deck as only containing kings, queens, jacks and aces, and the deck is symmetric in these four ranks. So the probability of seeing an ace first is 1/4.

37b. To start, fix an order. So consider the event $KQJA$. The probability of this event is $\frac{4}{16}\frac{4}{15}\frac{4}{14}\frac{4}{13}$. However, we can rotate KQJ in 3! ways and still have the desired event of exactly one king, one queens and one jack before the first ace (D). Therefore,

$$P(D) = 3!\frac{4}{16}\frac{4}{15}\frac{4}{14}\frac{4}{13} = \frac{16}{455}.$$

38a. Tyrion has 12 options to sit, for each of these 12 options, Cersei has 2 options, and for each of these, the remaining people at the table can be ordered in 10! ways. The total number of ways to order 12 people is 12!, so defining the event that Tyrion and Cersei have adjacent seats as A, we have by the multiplication rule and the naive definition of probability,

$$P(A) = \frac{12 \cdot 2 \cdot 10!}{12!} = \frac{2}{11}.$$

38b. The only thing that matters is Cersei's choice. Wherever Tyrion sits, there are 2 out of 11 chairs Cersei can choose to sit next to Tyrion. So $P(A) = \frac{2}{11}$.

Axioms of probability

43. $P(A \cap B) \leq P(A \cup B)$, since $A \cap B \subseteq A \cup B$. We also know from the inclusion-exclusion principle that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Hence, $P(A \cap B) = P(A)+P(B)-P(A\cup B) \geq P(A)+P(B)-1$ (where the inequality uses that probabilities are ≤ 1). Since $P(A\cap B) \geq 0$, we also have from $P(A\cap B) = P(A)+P(B)-P(A\cup B)$ that $P(A \cup B) \leq P(A) + P(B)$.

    The first inequality ($P(A\cap B) \leq P(A\cup B)$) becomes an equality when $A\cap B = A\cup B$. This means (draw a Venn diagram) that $A \cap B^C$ and $A^C \cap B$ are empty.

    For the second inequality ($P(A \cap B) \geq P(A) + P(B) - 1$), we see from the derivation above that it holds with equality if $P(A \cup B) = 1$. Also $P(A \cup B) = P(A) + P(B)$ if $P(A \cap B) = 0$.

44. The difference $\Delta = B - A = A^C \cap B$. We know that $B = (A\cap B)\cup(A^C\cap B) = (A\cap B)\cup(\Delta)$, where $A\cap B$ is disjoint from Δ. Since these are disjoint $P(B) = P(A\cap B)+P(\Delta)$. Using now that $A \subseteq B$, we have $A\cap B = A$. Hence, $P(B) = P(A)+P(\Delta)$. We conclude that $P(\Delta) = P(B) - P(A)$.

45. Use a Venn diagram.

Inclusion-Exclusion Principle

49. Denote by A_j the event that the value j never appears in n throws. The probability of A_i is $(5/6)^n$ (since we have 5 out of six options that are allowed to appear). The probability that i and j never appear, i.e. $A_i \cap A_j$ is $(4/6)^n$ (since there are now only 4 out of six options that are allowed to appear). This reasoning can be continued for the higher order intersections. Now by the inclusion exclusion principle

$$P(A_1 \cup \ldots \cup A_6) = \binom{6}{1}(5/6)^n - \binom{6}{2}(4/6)^n + \binom{6}{3}(3/6)^n \ldots + \binom{6}{5}(1/6)^n.$$

50. See lectures for a solution.

52. Define A_i to be the event that student i sits in the same seat on both days, then we are looking for $P(\cup_{i=1}^{20} A_i)$.

$$\begin{aligned}
P(\cup_{i=1}^{n} A_i) &= \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \ldots \\
&= n \cdot P(A_i) - \binom{n}{2} P(A_i \cap A_j) + \binom{n}{3} P(A_i \cap A_j \cap A_k) - \ldots \\
&= 1 - \frac{1}{2!} + \frac{1}{3!} - \ldots = -\sum_{i=1}^{n} \frac{(-1)^i}{i!} \\
&\approx -\sum_{i=1}^{\infty} \frac{(-1)^i}{i!} \\
&= -\sum_{i=0}^{\infty} \frac{(-1)^i}{i!} + 1 = 1 - e^{-1}
\end{aligned}$$

The last steps for approximating the summation when n is large (in color red) would require knowledge from later weeks: the Taylor expansion of e^x. You do not have to write that line.

Then the final result (the above is calculating the probability of the complement of the targeted event) would be

$$1 - P(\cup_{i=1}^{n} A_i) = 1 + \sum_{i=1}^{n} \frac{(-1)^i}{i!} \approx \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} = 1/e.$$