Probability Theory for EOR

Moments

The nth moment of X: $\mathbb{E}X^n$

For a real-valued r.v. X (assume the following expectation values exist):

- ***n*th Moment**
$$\mathbb{E}X^n.$$
- ***n*th Central Moment**
$$\mathbb{E}(X-\mu)^n.$$
- ***n*th Standardized Moment**
$$\mathbb{E}((X-\mu)/\sigma)^n.$$
- **Distributions determine moments** (moments are numbers, or you may regard them as parameters of fixed numbers for given random varaibles) since they are defined based on the concept: expectation.

- Central/standarized moments are linear combinations of moments, e.g.,

	$\mathbb{E}X^0$	$\mathbb{E}X$	$\mathbb{E}X^2$	$\mathbb{E}X^3$	$\mathbb{E}X^4$	$\dots$
μ: $\mathbb{E}X$	0	1	0	0	0	0
σ^2: $\mathbb{E}X^2 - (\mathbb{E}X)(\mathbb{E}X)$	0	-μ	1	0	0	0
Skew(X): $\mathbb{E}\left(\frac{X-\mu}{\sigma}\right)^3$	-$\frac{\mu^3}{\sigma^3}$	$\frac{3\mu^2}{\sigma^2}$	$-\frac{3\mu}{\sigma}$	1	0	0
$\dots$	$\dots$					

- **Moments describe the shapes of distributions: e.g., mean.**

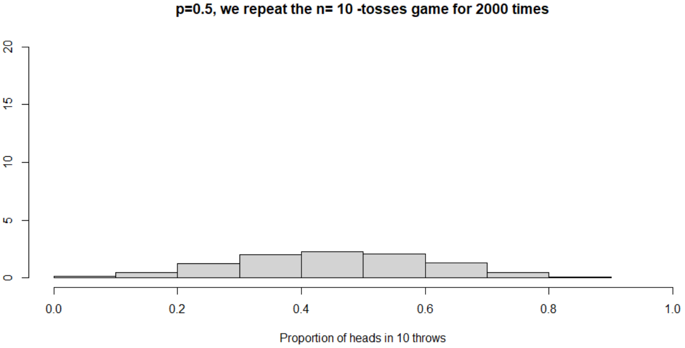
Moments are quite useful. Let's how to use random draws/obeserved data to get estimates (approximations) for our moments/parameters.

Sample moments

Toss a coin many many (n) times, denote X_i which equals one if the ith toss is a head otherwise zero for a tail,

the sample average (first sample moment) value $\frac{1}{n}\sum_{i=1}^{n} X_i$ is a random varaible, but from our daily experience, we know it is more likely to take values in a smaller and smaller neigborhood of the successful rate of getting a head when n grows, p,

which is the **1st moment** EX_i.



For each time, we throw a fair coin for n tosses, and calculate one number: the proportion of heads (one realisation of the sample average), we repeat this for 2000 times. Then we have 2000 numbers, we draw the histogram (density) plot of these 2000 numbers.

As n increases, we see the realised sample averages (random realisations) are centering towards $p = 1/2$.

We can generalise this results to all moments:
from the sample values to theoretical values!
We focus on the i.i.d. sequences.

Let $X_i, i = 1, \cdots, n$ be i.i.d. r.v.'s;

The kth sample moment:

$$M_k = \frac{1}{n} \sum_{i=1}^{n} X_i^k$$

.

- E.g., the sample mean is M_1 (the fist sample moment)

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

- **Sample moments serve as approximations (observed from realized data) to the moments (theoretical values)**.

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Let $X_i, i = 1, \cdots, n$ be i.i.d. r.v.'s with μ, σ^2.

- **Sample mean** $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$.
  By the linearity of expectation, $\mathbb{E}\bar{X}_n = \mu$;
  due to the independence, $\mathrm{Var}\bar{X}_n = \sigma^2/n \to 0$. (The only random variable type with zero variacne is the degenerated random varaible: constants.)
- $X_i^k, i = 1, \cdots, n$ **are also i.i.d. r.v.'s for any non-negative integer** k**,** so the above results also hold true if X_i^k has first and second moments.

- **Sample variance** $S_n^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2$
- From $\sum_i^n (X_i - c)^2 = \sum_i^n (X_i - \bar{X}_n)^2 + n(\bar{X}_n - c)^2$, we know $\mathbb{E}S_n^2 = \sigma^2$.