Probability Theory for EOR

Some special continuous random variables
III (Normal)
Part B

Some **continuous random variables** are associated very special/ubiquitous **distributions (PDFs)**, they get their own names!

Definition (PDF of continuous real-valued r.v.)

The **probability density function (PDF)** of a **continuous** real-valued r.v. is a non-negative function f_X on the real line such that via the Riemann integral:

$$\int_{-\infty}^{x} f_X(s)ds = P(X \leq x).$$

For a **continuous** random variable with differentiable CDF F_X, conventionally, $f_X(x) = F_X'(x)$.

Normal/Gaussian!! Part B.

Normal distribution and MGF

Derive the MGF of $N(\mu, \sigma^2)$.

- **Derive $M_Z(t)$ for $Z \sim N(0,1)$.**

$$M_Z(t) = \mathbb{E}e^{tZ} = \int_{-\infty}^{\infty} e^{tz} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, dz = e^{t^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-t)^2}{2}} \, dz = e^{t^2/2}$$

which is finite for $t \in \mathbb{R}$, so the MGF, M_Z, is well defined.

- **Derive $M_X(t)$ for $X = \mu + \sigma Z \sim N(\mu, \sigma^2)$.**

$$M_X(t) = \mathbb{E}e^{tX} = \mathbb{E}e^{t(\mu+\sigma Z)} = e^{t\mu}\mathbb{E}e^{t\sigma Z)} = e^{t\mu}\mathbb{E}e^{(t\sigma)Z} = e^{t\mu} M_Z(t\sigma) = e^{\mu t + \frac{1}{2}\sigma^2 t^2},$$

which is finite for $t \in \mathbb{R}$, so the MGF, M_X, is well defined.

Remark: The relation $M_X(t) = e^{t\mu} M_Z(t\sigma)$ is true for general local-scale transformation relation $X = \mu + \sigma Z$.

Show that $\frac{1}{\sqrt{n}} \sum_{i=1}^{n}(X_i - 1)$ with $X_i \sim$i.i.d. Pois(1) can be nicely approximated by $N(0,1)$ when n grows to infnity.

- Derive the corresponding MGFs:

  MGF of a Pois(1)-dsitributed X:

$$M_X(t) = Ee^{tX} = e^{(e^t - 1)},$$

  which is finite for, e.g., $t \in (-1/2, 1/2)$ (so the MGF is well-defined).
  MGF of $Y = \frac{1}{\sqrt{n}} \sum_{i=1}^{n}(X_i - 1)$,

$$M_Y(t) = \prod_{i=1}^{n} M_{(X_i - 1)}(t/\sqrt{n}) = (e^{-t/\sqrt{n}} M_{X_1}(t/\sqrt{n}))^n = e^{n\left(e^{\frac{t}{\sqrt{n}}} - 1 - \frac{t}{\sqrt{n}}\right)},$$

  which is finite for $t \in \mathbb{R}$, so the MGF, M_Y, is well defined.
  MGF $M_\xi(t)$ for $\xi \sim N(\mu, \sigma^2)$.

$$M_\xi(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2},$$

  which is finite for $t \in \mathbb{R}$, so the MGF, M_ξ, is well defined.
- Note that $\lim_{n\to\infty} e^{n\left(e^{\frac{t}{\sqrt{n}}} - 1 - \frac{t}{\sqrt{n}}\right)} = \lim_{n\to\infty} e^{n\left(\sum_{i=0}^{\infty}\left(\frac{t}{\sqrt{n}}\right)^i / i! - 1 - \frac{t}{\sqrt{n}}\right)} = e^{\frac{1}{2}t^2}$.
- **The MGFs of $\frac{1}{\sqrt{n}} \sum_{i=1}^{n}(X_i - 1)$ and $N(0,1)$ are the same in the limit**, which implies in the limit these two distributions coincide.

Show that a sum of n independent $N(\mu_i, \sigma_i^2)$, $i \leq n$ is still normal.

- **MGF of $X \sim N(\mu, \sigma^2)$.**

$$M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2},$$

which is finite for $t \in \mathbb{R}$, so the MGF, M_X, is well defined.

- **MGF of $\sum_{i=1}^n X_i$** with $X_i \sim$ *independent* $N(\mu_i, \sigma_i^2)$, $i \leq n$:

$$M_{\sum_{i=1}^n X_i}(t) = \prod_{i=1}^n e^{\mu_i t + \frac{1}{2}\sigma_i^2 t^2} = e^{\left(\sum_{i=1}^n \mu_i\right)t + \frac{1}{2}\left(\sum_{i=1}^n \sigma_i^2\right)t^2}.$$

which is finite for, e.g., $t \in (-1, 1)$ (so the MGF is well-defined) and is the MGF of a $N\left(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2\right)$-dsitributed r.v.

- The MGF of $\sum_{i=1}^n X_i$ is the MGF of $N\left(\sum_{i=1}^n \mu_i, \frac{1}{2}\left(\sum_{i=1}^n \sigma_i^2\right)\right)$, then we know a sum of n independent $N(\mu_i, \sigma_i^2)$, $i \leq n$ is still Normal/Gaussian.