Probability Theory for EOR

Sample Space and Naive Definition of Probability

Probability is a function

- To quantify the uncertainty and randomness, we want a number for a random event. The larger, the higher chance for such event to happen.
  - For example, when we throw a coin, what is a proper number you would have in mind for the event getting the head of the coin?
  - Fifty-fifty (1/2).
  - What about the event of getting either head or the bottom?
  - Pretty sure 100% chances (1).
- Those are the informal/causal ways we use the idea of probability in daily life:

  probability maps **random events** to **numbers** (non-negative numbers within 0 and 1).

Probability is a function

- Essentially, the probability is a function. Like a ruler measure heights, it measures how likely one event is going to happen.
- **Let A denote a random event containing some possible random outcomes.**

$$\begin{array}{llll} \mathbb{P}: & \{A_i \subseteq S, i \in I\} & \mapsto & [0,1] \\ & A & \mapsto & x \end{array}$$

where S denotes a collection of all possible outcomes, or a sample space, ans A denotes a arbitrary (but properly chosen) subset of S.

- To understand a function (maps elements from one set to elements in another set), we need to understand: (a) domain of the function (a collection of the subsets of S, $\{A_i \subseteq S, i \in I\}$); (b) codomain of the function ([0,1]); (c) how the elements from the two sets are linked to each other.

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(a) Elements in the domain of probability

Sets of possible outcomes are elements in the domain of one probability function

Definition (Event, sample space)

The **sample space** S is the set of all possible **outcomes** of the **experiment**. An **event** A is a **subset** of the sample space S, and we say that A **occurred** if the actual outcome is in A.

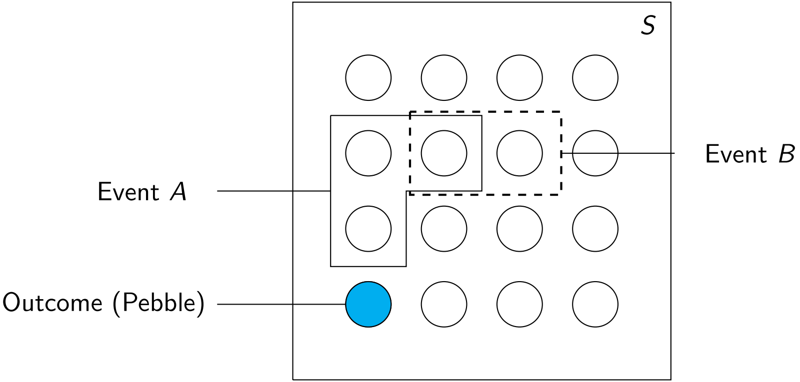
Note that:
(1) $S \subseteq S$, and we always include S as one event (the largest one as it contains all possible outcomes);
(2) if A and B are two events, then their intersections, unions are also included as events, so are their complements;
(3) $\emptyset$ is also an event (zero probability though);
(4) if A_i's are events, so is $\cup_{i=1}^{\infty} A_i$.

Example I: coin tossing for one time

- $S = \{\text{Head, Bottom}\}$;
- $A = \{\text{Head}\}$, $B = \{\text{Bottom}\}$;
- $A \cup B = S$;
- $A \cap B = \emptyset$;
- $A^c = B$.

Example II: pebble space

(all the 16 pebbles are equally likely to be chosen).



DIY: Laws

- Commutative laws
  - $A \cap B = B \cap A$
  - $A \cup B = B \cup A$
- Associative laws
  - $(A \cup B) \cup C = A \cup (B \cup C)$
  - $(A \cap B) \cap C = A \cap (B \cap C)$
- Distributive laws
  - $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
  - $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

(c) link events to real numbers within [0,1]

First attempt: naive definition of probability

- Events are sets of possible outcomes.
- Let's consider a special case: finite outcomes, all the outcomes are equally likely to happen.
- The more outcomes one event contains, the more likely it happens.

Definition (Naive definition of probability)

Denote A be an event for an experiment with a **finite*** sample spaces S and each outcome is **equally likely*** to happen:

$$P_{\text{Naive}}(A) = \frac{\text{number of outcomes favorable to } A}{\text{number of outcomes in } S} = \frac{|A|}{|S|}.$$

Example I: coin tossing for one time

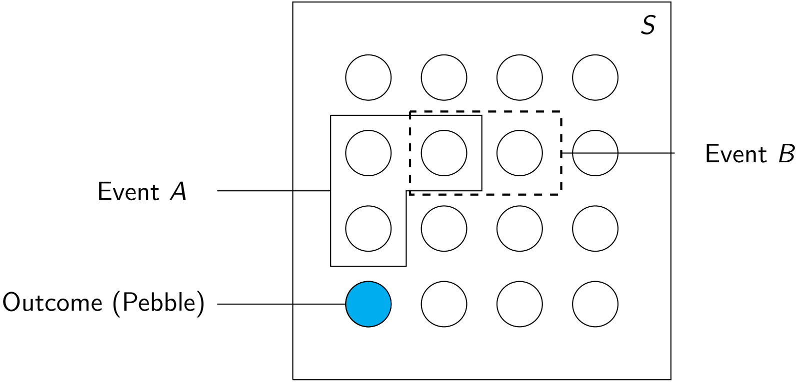
(head and bottom are equally likely to happen)

- $S = \{\text{Head, Bottom}\}$;
- $A = \{\text{Head}\}$, $B = \{\text{Bottom}\}$;

$$\mathbb{P}_{\text{Naive}}(A) = 1/2, \mathbb{P}_{\text{Naive}}(B) = 1/2, \mathbb{P}_{\text{Naive}}(A \cup B) = 1.$$

Example II: pebble space

(all the 16 pebbles are equally likely to be chosen).



$\mathbb{P}_{\text{Naive}}(A) = 3/16, \mathbb{P}_{\text{Naive}}(B) = 1/8,$
$\mathbb{P}_{\text{Naive}}(A \cup B) = 1/4, \mathbb{P}_{\text{Naive}}(A \cap B) = 1/16.$