Probability Theory for EOR

Functions of random variables

Functions of random variables

Random variables are functions.

X **maps elements from** S **to numbers! Now we can work with functions/numbers!**: e.g.,

$$X : S \mapsto \mathbb{R}$$

There are transformations on **functions** (composite functions), e.g., from f, g to get **new functions**:
$(f)^2$, $(f)^3$, $f + 1$, fg, $f + g$, $\max\{f, g, 0\} \cdots$

We can manipulate **random variables** in a similar way:
$(X)^2$, $(X)^3$, $X + 1$, XY, $X + Y$, $\max\{X, Y, 0\} \cdots$

Functions of a real-valued r.v.'s

For an r.v. $X : S \mapsto \mathbb{R}$ with sample space S, and a proper function $g : \mathbb{R} \to \mathbb{R}$, $g(X)$ is the r.v. that maps s to $g(X(s)) \in \mathbb{R}$ for all $s \in S : S \mapsto \mathbb{R}$.
Example:

$X \sim$Bern(p), then $Y = 2(X+1)$ is also an r.v. with PMF:

$$P(Y=2) = 1-p; P(Y=4) = p.$$

For a discrete r.v. X, the PMF of $Y = g(X)$:

$$P(Y=y) = \sum_{x:g(x)=y} P(X=x); y \in \{g(x) : x \in \text{supp}(X)\}.$$

Functions of multi real-valued r.v.'s

For multiple r.v.'s $X_i : S \mapsto \mathbb{R}, i = 1, 2, 3 \cdots, n$ with sample space S, and a proper function $g : \mathbb{R}^n \mapsto \mathbb{R}$, $g(X_1, X_2, \cdots, X_n)$ is the r.v. that maps s to $g(X_1(s), X_2(s), \cdots, X_n(s)) \in \mathbb{R}$ for all $s \in S : S \mapsto \mathbb{R}$.

Example:
$X_1, X_2 \sim$Bern(0.5) and they are independent, then $Y = X_1 + X_2$ is also an r.v. with PMF:

$$P(Y=0) = 1/4; P(Y=1) = 1/2; P(Y=2) = 1/4.$$

Practice and some new results

Throw a fair six-sided die

What is the PMF for the random variable X denotes the number of eyes of the outcome if we **throw a six-sided fair die for one time**?

$$P(X = i) = 1/6, i = 1, 2, 3, 4, 5, 6.$$

Throw a fair six-sided die twice. What is the PMF of, Y, the sum of the two **independent** die rolls?

$$P(Y = i), i = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.$$

Denote X_1 the number of eyes of the outcome in the first throw, and X_2 the number of eyes of the outcome in the second throw ($Y = X_1 + X_2$):

$$\begin{aligned} P(Y = 3) = & \sum_{x_1, x_2 : x_1 + x_2 = 3} P(X_1 = x_1, X_2 = x_2) \\ = & \sum_{x_1, x_2 : x_1 + x_2 = 3} P(X_1 = x_1) P(X_2 = x_2) \\ = & P(X_1 = 1) P(X_2 = 2) + P(X_1 = 2) P(X_2 = 1) \\ = & 1/6 \times 1/6 + 1/6 \times 1/6 = 1/18. \end{aligned}$$

Throw a fair six-sided die three times. Denote by X, Y, Z the eyes by the first, second and third die. Suppose X, Y and Z are **independent identically distributed (i.i.d.).** Calculate $P(\max(X, Y, Z) > 5)$:

Results: Binomial and Hypergeometric

- $X \sim$Bin(n, p), $Y \sim$Bin(m, p). X and Y are independent:

  (a) $Z = X + Y$, $Z \sim$Bin$(n + m, p)$.

  (b) The conditional distribution of X given $Z = r$ is HGeom(n, m, r):

$$P(X = k|Z = r) = \frac{\binom{n}{k}\binom{m}{r-k}}{\binom{n+m}{r}}$$

- $X \sim$HGeom(w, b, n), if $N = w + b \to \infty$ and $p = \frac{w}{w+b}$:

  The PMF of X converges to the Bin(n, p) PMF:

$$P\left(X = k\right) = \binom{n}{k} \frac{\prod_{i=0}^{k-1}(p - \frac{i}{N}) \prod_{j=0}^{n-k-1}(q - \frac{j}{N})}{\prod_{l=1}^{n-1}(1 - \frac{l}{N})} \to \binom{n}{k} p^k q^{n-k}$$