Probability Theory for EOR

Expectations of functions of random variables (with discrete random variables examples)

II (indicator function)

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Expectation maps a subset of the collection of random variables to numbers (expectations/expected values of associated random variables).

We know some functions of random variables are still random variables, and we look into

expectations of functions of random variables and how they are linked with the original random variables.

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A special function: indicator function

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$$I_A: S \mapsto \{0,1\}; A \subset S.$$

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$$I_A(s) = \begin{cases} 1, & s \in A \\ 0, & s \in A^c (= S \setminus A) \end{cases}$$

- $ightharpoonup (I_A)^k = I_A \text{ for any } k \in \mathbb{N}_+.$
- $\blacktriangleright I_{A^c} = 1 I_A.$
- $ightharpoonup I_{A\cap B}=I_AI_B.$
- ► $I_{A \cup B} = I_A + I_B I_A I_B \le I_A + I_B$.

$$\textit{I}_{\textit{A} \cup \textit{B}} = 1 - \textit{I}_{(\textit{A} \cup \textit{B})^{\textit{C}}} = 1 - \textit{I}_{\textit{A}^{\textit{C}} \cap \textit{B}^{\textit{C}}} = 1 - \textit{I}_{\textit{A}^{\textit{C}}} \textit{I}_{\textit{B}^{\textit{C}}} = 1 - (1 - \textit{I}_{\textit{A}}) \left(1 - \textit{I}_{\textit{B}}\right) = \textit{I}_{\textit{A}} + \textit{I}_{\textit{B}} - \textit{I}_{\textit{A}} \textit{I}_{\textit{B}}.$$

Similar to inclusion-exclusion principle with two events

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

A special function: indicator function

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Fundamental bridge

 I_A is a function, also a random variable, if we consider A a random event. It can also be a function of r.v.'s, if A is a random event generated by r.v.'s.

$$I_A$$
 follows Bern $(P(A))$.

Fundamental bridge: $P(A) = EI_A$.

Distributions determine expectations, and now expectations recover probabilities of associated random events!

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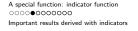
A special function: indicator function
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Fundamental bridge

Fundamental bridge: $P(A) = EI_A$.

Distributions determine expectations, and now expectations recover probabilities of associated random events!

- ▶ Let's look into events associated with arbitrary real-valued r.v. X of type $\{X \le x\}$.
- ▶ $I_{\{X \le x\}}$ is a function of X and also a random variable: takes 1 if $X \le x$ otherwise 0.
- Expectations recover the distribution of any **real-valued** r.v. X: once all the expectations of $I_{\{X \le x\}}, x \in \mathbb{R}$ are known.

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Some important results derived via **indicators** and the **fundamentl bridge**.

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A special function: indicator function 00000000000000000 Important results derived with indicators

I. Inclusion-exclusion principle

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► I.1 Two random events A, B.

From
$$I_{A\cup B}=I_A+I_B-I_AI_B\leq I_A+I_B$$
, we know $EI_{A\cup B}=EI_A+EI_B-EI_AI_B\leq EI_A+EI_B$. Therefore, from the funcamental bridge $P(A\cup B)=P(A)+P(B)-P(A\cap B)\leq P(A)+P(B)$.

► I.2 n random events A_1, \dots, A_n . Note that

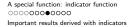
$$1 - I_{\bigcup_{i=1}^{n} A_{i}} = I_{\bigcap_{i=1}^{n} A_{i}^{c}} = (1 - I_{A_{1}}) \cdots (1 - I_{A_{n}})$$

$$= 1 - \left(\sum_{i=1}^{n} I_{A_{i}} - \sum_{i < j} I_{A_{i}} I_{A_{j}} + \sum_{i < j < k} I_{A_{j}} I_{A_{k}} - \dots + (-1)^{n+1} \prod_{i=1}^{n} I_{A_{i}}\right)$$

$$= 1 - \left(\sum_{i=1}^{n} I_{A_{i}} - \sum_{i < j} I_{A_{i} \cap A_{j}} + \sum_{i < j < k} I_{A_{i} \cap A_{j} \cap A_{k}} - \dots + (-1)^{n+1} I_{\bigcap_{i=1}^{n} A_{i}}\right)$$

Then from the funcamental bridge,
$$P(\bigcup_{i=1}^n A_i) = \sum_{i=1}^n P(A_i) - \sum_{i < j} P(A_i \cap A_j) + \sum_{i < j < k} P(A_i \cap A_j \cap A_k) - \ldots + (-1)^{n+1} P(\bigcap_{i=1}^n A_i).$$
 Similarly, we know $P(\bigcup_{i=1}^n A_i) \le \sum_{i=1}^n P(A_i)$ since $I_{\bigcup_{i=1}^n A_i} \le \sum_{i=1}^n I_{A_i}$.

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II. Calculating E(X) of any **non-negative integer-valued** r.v. X with its survival function.

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A special function: indicator function

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Important results derived with indicators

Let X be a **non-negative integer-valued** r.v. then if its expectation exists

$$\mathbb{E}X = \sum_{i=0}^{\infty} G_X(i)$$

where G_X is the survival function of X such that $G_X(x) = 1 - F_X(x)$.

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► Proof:

Decompose X as a function of multiple indicators.

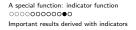
Note that
$$X = I_{\{X \ge 1\}} + \dots + I_{\{X \ge n\}} + \dots = \sum_{i \in \mathbb{N}_+} I_{\{X \ge i\}}$$
.

The above holds true since X and $\sum_{i \in \mathbb{N}_+} I_{\{X \ge i\}}$ are the same function from S to the set of non-negative integers.

Linearity of expectation, fundamental bridge, and the fact that $\{X \geq i\} = \{X > i-1\}, i \in \mathbb{N}_+$ give

$$EX = \sum_{i \in \mathbb{N}_+} \mathbb{E} I_{\{X \ge i\}} = \sum_{i=1}^{\infty} P(X \ge i) = \sum_{i=0}^{\infty} P(X > i) = \sum_{i=0}^{\infty} G_X(i).$$

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 $\ensuremath{\mathsf{III}}.$ Indicators are a useful tool to simplify questions.

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There are n visitors and n balls with their names on respectively. If each visitor picks a ball randomly then keeps the ball, denote X the number of visitors picking the one with their own names. What is E(X)?

- ightharpoonup Denote A_i the event that *i*th visitor gets the correct ball.
- $\triangleright X = \sum_{i=1}^n I_{A_i}$
- ▶ By fundamental bridge, $EX = \sum_{i=1}^{n} P(A_i)$.
- ▶ By symmetry, $P(A_i) = 1/n$.
- ► E(X) = 1.

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