Probability Theory for EOR

Some special continuous random variables II (Uniform)

Some **continuous random variables** are associated very special/ubiquitous **distributions (PDFs)**, they get their own names!

Definition (PDF of continuous real-valued r.v.)

The **probability density function (PDF)** of a **continuous** real-valued r.v. is a non-negative function f_X on the real line such that via the Riemann integral:

$$\int_{-\infty}^{x} f_X(s)ds = P(X \leq x).$$

For a **continuous** random variable with differentiable CDF F_X, conventionally, $f_X(x) = F'_X(x)$.

Uniform distribution: Unif((a, b))

- A continuous real-valued r.v. U is said to have the uniforma distribution on (a, b) (or follows Unif((a, b))), if its PDF is

$$f_U(x) = \begin{cases} \frac{1}{b-a} & x \in (a, b) \\ 0 & \text{otherwise} \end{cases}$$

- U takes values in (a,b).
- Probability is proportional to length (e.g., $(c, d) \subseteq (a, b)$):

$$P(c \leq U \leq d) = \int_c^d \frac{1}{b-a} dx = \frac{c-d}{b-a}.$$

If U follos Unif(0,1), what is the distribution of $X = a + (b-a)U$?

- Unif(a,b).
- Verify by CDF ($x \in (a, b)$):

$$\begin{aligned} F_X(x) &= P(a + (b-a)U \le x) \\ &= P(U \le (x-a)/(b-a)) = (x-a)/(b-a). \end{aligned}$$

Expectation and Varaince of Unif(a,b)

$Y \sim$Unif(a,b), $X \sim$Unif(0,1)

- The expectation EX.

$$\mathrm{E}[X] = \int_0^1 x dx = 1/2.$$

- The varaince VX.

$$\mathrm{E}[X^2] = \int_0^1 x^2 dx = 1/3.$$
$$\mathrm{V}[X] = \mathrm{E}[X^2] - (\mathrm{E}[X])^2 = 1/12.$$

- $EY = E\left(a + (b-a)X\right) = a + \frac{b-a}{2} = \frac{a+b}{2}, VY = V\left(a + (b-a)X\right) = (b-a)^2/12.$

I. Conditional on a unif is a unif

Let $U \sim$Unif(a,b), $(c,d) \subseteq (a,b)$. Then the conditional distribution of U given $U \in (c,d)$ is Unif(c,d).

- Proof: for $x \in (c,d)$

$$\begin{aligned} & P\left(U \leq x | U \in (c,d)\right) \\ =& P\left(U \in (c,x]\right)/P\left(U \in (c,d)\right) = \frac{x-c}{d-c} \end{aligned}$$

which is exactly the CDF of Unif(c,d).

II. Universality of the Unif

Let F be a strictly increasing (so F^{-1} the inverse function is well defined) CDF of a continuous r.v., we then have

- **Let $U\sim$Unif(0,1), $F^{-1}(U)$ is an r.v. with CDF F.**

  Proof:

$$F_{F^{-1}(U)}(x) = P(F^{-1}(U) \leq x)$$
$$= P(U \leq F(x)) = F(x)$$

- **Let X be an r.v. with CDF F, then $F(X) \sim Unif(0,1)$.**

  Proof:

$$F_{F(X)}(x) = P(F(X) \leq x)$$
$$= P(X \leq F^{-1}(x)) = F\left(F^{-1}(x)\right) = x.$$

If you have a uniformly distributed random varaible you can generate many random varaibles: e.g., $\log\left(\frac{1}{1-U}\right) \sim$Expo(1).