# Probability Theory for EOR

## Expectations of functions of random variables (with discrete random variables examples) II (indicator function)

*Expectation* maps a subset of the collection of random variables to numbers (expectations/expected values of associated random variables).

We know some functions of random variables are still random variables, and we look into

**expectations of functions of random variables and how they are linked with the original random variables.**

# A special function: indicator function

$$I_A : S \mapsto \{0,1\}; A \subset S.$$

- $$I_A(s) = \begin{cases} 1, & s \in A \\ 0, & s \in A^c (= S \setminus A) \end{cases}$$
- $(I_A)^k = I_A$ for any $k \in \mathbb{N}_+$.
- $I_{A^c} = 1 - I_A$.
- $I_{A \cap B} = I_A I_B$.
- $I_{A \cup B} = I_A + I_B - I_A I_B \leq I_A + I_B$.

  $I_{A \cup B} = 1 - I_{(A \cup B)^c} = 1 - I_{A^c \cap B^c} = 1 - I_{A^c} I_{B^c} = 1 - (1 - I_A)(1 - I_B) = I_A + I_B - I_A I_B$.

  Similar to inclusion-exclusion principle with two events

  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

**$I_A$ is a function, also a random variable, if we consider $A$ a random event. It can also be a function of r.v.'s, if $A$ is a random event generated by r.v.'s.**

$I_A$ follows $\text{Bern}(P(A))$.

**Fundamental bridge:** $P(A) = EI_A$**.**

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- Let's look into events associated with arbitrary real-valued r.v. $X$ of type $\{X \leq x\}$.
- $I_{\{X \leq x\}}$ is a function of $X$ and also a random variable: takes 1 if $X \leq x$ otherwise 0.
- Expectations recover the distribution of any **real-valued** r.v. $X$: once all the expectations of $I_{\{X \leq x\}}, x \in \mathbb{R}$ are known.

Some important results derived via **indicators** and the **fundamentl bridge**.

## I. Inclusion-exclusion principle

- I.1 Two random events $A, B$.

  From $I_{A\cup B} = I_A + I_B - I_A I_B \le I_A + I_B$, we know
  $EI_{A\cup B} = EI_A + EI_B - EI_A I_B \le EI_A + EI_B$.
  Therefore, from the funcamental bridge
  $P(A\cup B) = P(A) + P(B) - P(A\cap B) \le P(A) + P(B)$.

- I.2 n random events $A_1, \cdots, A_n$.

  Note that

$$
\begin{aligned}
1 - I_{\cup_{i=1}^n A_i} &= I_{\cap_{i=1}^n A_i^c} = (1 - I_{A_1})\cdots(1 - I_{A_n}) \\
=&1 - \left(\sum_{i=1}^n I_{A_i} - \sum_{i<j} I_{A_i} I_{A_j} + \sum_{i<j<k} I_{A_i} I_{A_j} I_{A_k} - \ldots + (-1)^{n+1} \prod_{i=1}^n I_{A_i}\right) \\
=&1 - \left(\sum_{i=1}^n I_{A_i} - \sum_{i<j} I_{A_i\cap A_j} + \sum_{i<j<k} I_{A_i\cap A_j\cap A_k} - \ldots + (-1)^{n+1} I_{\cap_{i=1}^n A_i}\right)
\end{aligned}
$$

  Then from the funcamental bridge, $P(\cup_{i=1}^n A_i) =$
  $\sum_{i=1}^n P(A_i) - \sum_{i<j} P(A_i\cap A_j) + \sum_{i<j<k} P(A_i\cap A_j\cap A_k) - \ldots + (-1)^{n+1} P(\cap_{i=1}^n A_i)$.
  Similarly, we know $P(\cup_{i=1}^n A_i) \le \sum_{i=1}^n P(A_i)$ since $I_{\cup_{i=1}^n A_i} \le \sum_{i=1}^n I_{A_i}$.

II. Calculating $E(X)$ of any **non-negative integer-valued** r.v. $X$ with its survival function.

Let X be a **non-negative integer-valued** r.v. then if its expectation exists

$$\mathbb{E}X = \sum_{i=0}^{\infty} G_X(i)$$

where $G_X$ is the survival function of $X$ such that $G_X(x) = 1 - F_X(x)$.

- Proof:

  Decompose $X$ as a function of multiple indicators.

  Note that $X = I_{\{X \geq 1\}} + \cdots + I_{\{X \geq n\}} + \cdots = \sum_{i \in \mathbb{N}_+} I_{\{X \geq i\}}$.

  The above holds true since $X$ and $\sum_{i \in \mathbb{N}_+} I_{\{X \geq i\}}$ are the same function from $S$ to the set of non-negative integers.

  Linearity of expectation, fundamental bridge, and the fact that $\{X \geq i\} = \{X > i - 1\}, i \in \mathbb{N}_+$ give

$$EX = \sum_{i \in \mathbb{N}_+} \mathbb{E} I_{\{X \geq i\}} = \sum_{i=1}^{\infty} P(X \geq i) = \sum_{i=0}^{\infty} P(X > i) = \sum_{i=0}^{\infty} G_X(i).$$

III. Indicators are a useful tool to simplify questions.

There are $n$ visitors and $n$ balls with their names on respectively. If each visitor picks a ball randomly then keeps the ball, denote $X$ the number of visitors picking the one with their own names. What is $E(X)$?

- Denote $A_i$ the event that $i$th visitor gets the correct ball.
- $X = \sum_{i=1}^{n} I_{A_i}$.
- By fundamental bridge, $EX = \sum_{i=1}^{n} P(A_i)$.
- By symmetry, $P(A_i) = 1/n$.
- $E(X) = 1$.