# Probability Theory for EOR

## Some special discrete random variables III (Poisson)

Some **discrete random variables** are associated very special/ubiquitous **distributions (PMFs)**, they get their own names!

## Definition

The **probability mass function (PMF)** of a **discrete** r.v., $X : S \mapsto \{x_1, x_2, \cdots\}$, is the function:

$$p_X(x) = P(X = x).$$

# Poisson!

# Poisson Process

Consider the following scenario:

**A web server would receive requests from computer A randomly.**

**How to properly model the random variable $X$,**
**the number of requests from computer A within a time interval of lenghth $t$.**

**How to properly model the random variable $X$, the number of requests from computer A within a time interval of lenghth $t$.**

- The probability of more than one hit in a very short interval is negligible.

  The probability of a single arrival in a very short interval is proportional to the length of the interval.
- The numbers of hits in non-overlapping time intervals are independent.
  E.g, what happens in interval $[a, b]$ would be independent from what happens in $[c, d]$ for all $a, b, c, d$ from the real line as long as $[a, b] \cap [c, d] = \emptyset$.

**The probability of more than one hit in a very short interval is negligible.**
**The probability of a single arrival in a very short interval is proportional to the length of the interval.**

- The arrival of one request within a very small interval $(s, s+\Delta t]$ follows $\text{Bern}(\lambda\Delta t)$, denoted as $I_{(s,s+\Delta t]}$;
- The number of total requests within interval $(s, s+t]$

$$X_{\Delta t} = \sum_{i=1}^{t/(\Delta t)} I_{(s+(i-1)\Delta t, s+i\Delta t]},$$

where for simplicity we assume that $t/(\Delta t)$ is a positive integer.

**The number of hits in non-overlapping time intervals are independent.**

- $X_{\Delta t}$ follows $\text{Bin}(t/\Delta t, \lambda\Delta t)$.
- $P(X_{\Delta t} = k) = \binom{t/(\Delta t)}{k} (\lambda\Delta t)^k (1-\lambda\Delta t)^{t/(\Delta t)-k}$.

  *Let's consider a special case such that $\Delta t = t/n$, what happens to the PMF of the random variable $X_{\Delta t}$ if we let $n \to \infty$ ($\Delta t \to 0$, we are splitting into smaller and smaller intervals).* Denote the limit random variable as $X$.
- $P(X = k) = \lim_{n\to\infty} \binom{n}{k} (\lambda t/n)^k (1-\lambda t/n)^{n-k} =$
  $\lim_{n\to\infty} (\lambda t)^k/k! \left(\prod_{i=1}^{k}(n-i+1)/n\right) (1-(\lambda t)/n)^n (1-(\lambda t)/n)^{-k} = e^{-\lambda t}(\lambda t)^k/k!$.

  **We use the fact that $(1+x/s)^s \to_{s\to\infty} e^x$.**

# Poisson distribution: Pois($\lambda$)

Denote $X$ the number of random requests within a time interval of **unit length** from the aforementioned scenario (t=1).

- The PMF of $X$ is

$$P(X = k) = e^{-\lambda}\lambda^k/k!; k = 0, 1, \cdots .$$

  where $\lambda > 0$.
- $X \sim$Pois($\lambda$).

# Expectation

$X \sim$Pois$(\lambda)$.

- The expectation $EX$.

$$\mathrm{E}[X] = \sum_{x=0}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!} = \sum_{x=1}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!} = \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$
$$= \lambda e^{-\lambda} \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} = \lambda,$$

where we use the fact that $e^x = \sum_{k=0}^{\infty} x^k / k!$.

**$\lambda$ describes the arrival rate, larger the arrival rate, larger the expected number of arrivals.**

# Variance

$X \sim$Pois$(\lambda)$.

- The variance $VX$.

$$\begin{aligned}
\mathrm{E}[X^2] &= \sum_{x=0}^{\infty} x^2 \frac{\lambda^x e^{-\lambda}}{x!} = \sum_{x=1}^{\infty} x^2 \frac{\lambda^x e^{-\lambda}}{x!} = \sum_{x=1}^{\infty} x \frac{\lambda^x e^{-\lambda}}{(x-1)!} \\
&= \sum_{x=1}^{\infty} (x-1) \frac{\lambda^x e^{-\lambda}}{(x-1)!} + \sum_{x=1}^{\infty} \frac{\lambda^x e^{-\lambda}}{(x-1)!} \\
&= \lambda \sum_{y=0}^{\infty} y \frac{e^{-\lambda}\lambda^y}{y!} + \lambda e^{-\lambda} \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} \\
&= \lambda E(X) + \lambda = \lambda^2 + \lambda,
\end{aligned}$$

$$VX = \mathrm{E}[X^2] - (\mathrm{E}[X])^2 = \lambda.$$

## From Binomial to Poisson revisits: Poisson paradigm.

**Previously,** The arrival of one request within a very small interval $(s, s+\Delta t]$ follows Bern$(\lambda\Delta t)$, denoted as $I_{(s,s+\Delta t]}$; and the number of total requests within interval $(s, s+t]$

$$X_{\Delta t} = \sum_{i=1}^{t/(\Delta t)} I_{(s+(i-1)\Delta t, s+i\Delta t]}.$$

The limit once we let $\Delta t$ go to zero follows

$$\text{Pois}(\lambda), \lambda = \sum_{i=1}^{t/(\Delta t)} \lambda\Delta t.$$

**Poisson paradigm** Let $A_i, i = 1, \cdots, n$ be **independent (or weakly dependent)** events with probability $p_i$, **n is large and $p_i$ are very small.** Let

$$X = \sum_{j=1}^{n} I_{A_j}$$

count how many of the $A_i$ occur (how many j visit one specific island within one day for example). Then **X is approximated distributed** as

$$\text{Pois}(\lambda); \lambda = \sum_j p_j.$$

- $\lambda$ as the *rate* (expected number within a certain time period) of occurrence of *rare* events.

**Sum of independent poisson is still Poisson.**

A web server would receive requests from computer A randomly, but also requests from computer B. A and B are independent.

**How to properly model the random variable $Y$,**
**the number of requests from either computer A or B within a time interval of unit lenghth**.

- Simply sum their arrival rate, and all the rest follow the same logic.
- $A \sim \text{Pois}(\lambda_A)$, $B \sim \text{Pois}(\lambda_B)$ and A,B are independent. Then
$$Y = A + B \sim \text{Pois}(\lambda_A + \lambda_B)$$
- Verify the result by looking at the distribution (PMF).

$$P(A+B=k) = \sum_{i=0}^{k} P(A=k-i|B=i)P(B=i) = \sum_{i=0}^{k} P(A=k-i)P(B=i)$$
$$= \sum_{i=0}^{k} \frac{1}{(k-i)!}\lambda_A^{k-i}e^{-\lambda_A} \cdot \frac{1}{i!}\lambda_B^{i}e^{-\lambda_B} = e^{-(\lambda_A+\lambda_B)}\frac{1}{k!}\sum_{i=0}^{k}\frac{k!}{i!(k-i)!}\lambda_A^{k-i}\lambda_B^{i}$$
$$= e^{-(\lambda_A+\lambda_B)}\frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i}\lambda_A^{k-i}\lambda_B^{i} = e^{-(\lambda_A+\lambda_B)}\frac{1}{k!}(\lambda_A+\lambda_B)^k$$

- $(a+b)^k = (a+b)(a+b)\cdots(a+b) = \sum_{i=0}^{k}\binom{k}{i}a^{k-i}b^{i}$.

## From Poisson to Binomial: Poisson given a sum of Poissons.

- **Poisson conditional on the sum is Binomials** .
- If a web server receive total $n$ requests from either computer A or B.
- Given these two computers are independent,
  **for each signal it could either be from A or B**:

  $I_{iA} \sim \text{Bern}(\frac{\lambda_A}{\lambda_A+\lambda_B})$.

- If $A \sim \text{Pois}(\lambda_A)$, $B \sim \text{Pois}(\lambda_B)$, A is independent from B, then the number of requests from computer A given $A + B = n$:

$$A \text{ given } \{A + B = n\} \text{ follows } \text{Bin}(n, \frac{\lambda_A}{\lambda_A + \lambda_B})$$

- Verify the result by looking atthe distributio (PMF).

$$\begin{aligned}
&P(A = k|A + B = n) \\
&= \frac{P(A = k, B = n - k)}{P(A + B = n)} = \frac{P(A = k)\, P(B = n - k)}{P(A + B = n)} \\
&= \frac{\left(e^{-\lambda_A} \frac{\lambda_A{}^k}{k!}\right)\left(e^{-\lambda_B} \frac{\lambda_B{}^{(n-k)}}{(n-k)!}\right)}{\left(e^{-(\lambda_A+\lambda_B)} \frac{(\lambda_A+\lambda_B)^n}{n!}\right)} = \frac{n!}{k!(n-k)!}\left(\frac{\lambda_A}{\lambda_A + \lambda_B}\right)^k \left(\frac{\lambda_B}{\lambda_A + \lambda_B}\right)^{n-k} \\
&= \binom{n}{k}\left(\frac{\lambda_A}{\lambda_A + \lambda_B}\right)^k \left(1 - \frac{\lambda_A}{\lambda_A + \lambda_B}\right)^{n-k}.
\end{aligned}$$