Probability Theory for EOR

Expectation of continuous random variables

Expected values of r.v.'s (if exsist) are numbers.

Expectation of continuous r.v.'s

Definition (Expectation of discrete r.v.'s)

The expectation/expected value/first moment/mean/average of a **discrete** random variable X for $x_1, x_2, x_3, \ldots$ (if exists) is defined by

$$\mathrm{E}[X] = \sum_{i=1}^{\infty} x_i \mathbb{P}(X = x_i) \;\; (= \sum_{i=1}^{\infty} x_i \Delta F_X(x_i))$$

(we order all values from the support of X by the increasing order: $x_1 \leq x_2 \leq x_3, \cdots$.
We may choose $\Delta F_X(x_i) = F_X(x_i) - F_X(x_{i-1})$, let $x_0 < x_1, x_0 \notin supp(X)$ and thus $F(x_0) = 0$.
We may choose $\Delta F_X(x) = F(x) - \lim_{y \uparrow x} F(y)$, as another possible expression.
Essentially both expressions give: $\Delta F_X(x_i) = P(X = x_i)$.)

If $|\mathrm{supp}(X)|$ is finite with elements $x_1, x_2, x_3, \ldots, x_n$, then

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Example: average waiting time

On average: how long do you need to wait until the first customer?

- Denote N_t the number of arrivals within $[0,t]$, N_t follows Pois(λt). Let T_1 be the time until the first arrival: $\{T_1 > t\} = \{N_t = 0\}$ and thus $P(T_1 > t) = P(N_t = 0) = e^{-\lambda t}$.
- The CDF of T_1 then

$$F_{T_1}(t) = P(T_1 \le t) = 1 - P(T_1 > t) = 1 - e^{-\lambda t}.$$

- To calculate the average waiting time, we may consider this approximation by cutting time into pieces of unit length $t_0 = 0, t_1 = 1, \cdots$

$$\begin{aligned} &\frac{0+1}{2}P(0 < T_1 \le 1) + \frac{1+2}{2}P(1 < T_1 \le 2) + \cdots \\ =&\sum_{i=0}^{\infty} \frac{t_i + t_{i+1}}{2} \times P(t_i < T_1 \le t_{i+1}) = \sum_{i=0}^{\infty} \frac{t_i + t_{i+1}}{2} \times \left(F_{T_1}(t_{i+1}) - F_{T_1}(t_i)\right). \end{aligned}$$

- We can choose finer and finer grids such that $t_{i+1} - t_i \to 0$:

$$\sum_{i=0}^{\infty} \frac{t_i + t_{i+1}}{2} \times \left(F_{T_1}(t_{i+1}) - F_{T_1}(t_i)\right) \approx \sum_{i=0}^{\infty} t_i F'_{T_1}(t_i)\,(t_{i+1} - t_i)$$

$\to \int_0^{\infty} tF'_{T_1}(t)dt$.

- From the fact that $F_{T_1}(t) = \int_{-\infty}^{s} f_{T_1}(s)ds$, we know $\int_0^{\infty} tf_{T_1}(t)dt (= \int_{-\infty}^{\infty} tf_{T_1}(t)dt)$.

Definition (Expectation of continuous real-valued r.v.'s)

The expectation/expected value/first moment/mean/average of a **continuous** real-valued random variable X (if exists) with PDF f is defined via Riemann integral:

$$\mathrm{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

$$\left(= \int_{-\infty}^{\infty} x dF_X(x)\right)$$

the second line (color red) makes use of the Riemann–Stieltjes integral, a generalization of the Riemann integral

Law of the unconscious statistician (LOTUS)
For random variable $g(X)$, the expectation (if exists):

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

Properties of Riemann integral imply the linearity and monotinicity.