Probability Theory for EOR

Properties of expectation
(proofs with discrete random variables)

Expectation maps a subset of the collection of random variables to numbers (expectations/expected values of associated random variables).

If we regard *expectation* as a special function from random variables to numbers, what are its properties?

Properties of expectation (proofs with discrete r.v.'s)

Linearity. Assume all expectations are well defined.

For any **real-valued** r.v.'s X, Y and any constant c,

$$\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y)$$
$$\mathbb{E}(cX) = c\mathbb{E}(X)$$

A linear function!

Proof I with discrete r.v.'s:

$$\mathbb{E}(X+Y) = \sum_c cP(X+Y=c) = \sum_c c\left(\sum_{a\in\text{supp}(X)} P(X=a, Y=c-a)\right)$$

$$= \sum_{a\in\text{supp}(X)} \sum_c cP(X=a, Y=c-a) = \sum_{a\in\text{supp}(X)} \sum_c ((a)+(c-a))P(X=a, Y=c-a)$$

$$= \sum_{a\in\text{supp}(X)} \sum_{b\in\text{supp}(Y)} (a+b)P(X=a, Y=b)$$

$$= \sum_{a\in\text{supp}(X)} \sum_{b\in\text{supp}(Y)} aP(X=a, Y=b) + \sum_{a\in\text{supp}(X)} \sum_{b\in\text{supp}(Y)} bP(X=a, Y=b)$$

$$= \sum_{a\in\text{supp}(X)} a \sum_{b\in\text{supp}(Y)} P(X=a, Y=b) + \sum_{b\in\text{supp}(Y)} b \sum_{a\in\text{supp}(X)} P(X=a, Y=b)$$

$$= \sum_{a\in\text{supp}(X)} aP(X=a) + \sum_{b\in\text{supp}(Y)} bP(Y=b)$$

$$= E(X) + E(Y)$$

Proof II with discrete r.v.'s and finite outcomes $|S| < \infty$:

$$
\begin{aligned}
\mathbb{E}(X+Y) &= \sum_c cP(X+Y=c) \\
&= \sum_c c \sum_{s\in S:(X+Y)(s)=c} P(\{s\}) \\
&= \sum_c \sum_{s\in S:(X+Y)(s)=c} cP(\{s\}) \\
&= \sum_c \sum_{s\in S:(X+Y)(s)=c} (X(s)+Y(s))P(\{s\}) \\
&= \sum_{s\in S} (X(s)+Y(s))P(\{s\}) \\
&= \left(\sum_{s\in S} X(s)P(\{s\})\right) + \left(\sum_{s\in S} Y(s)P(\{s\})\right) \\
&= \left(\sum_{a\in \mathrm{supp}(X)} \sum_{s\in S:X(s)=a} X(s)P(\{s\})\right) + \left(\sum_{b\in \mathrm{supp}(Y)} \sum_{s\in S:Y(s)=b} Y(s)P(\{s\})\right) \\
&= \left(\sum_{a\in \mathrm{supp}(X)} aP(X=a)\right) + \left(\sum_{b\in \mathrm{supp}(Y)} bP(Y=b)\right) \\
&= E(X) + E(Y)
\end{aligned}
$$

Example

Throw a coin for one time (X denotes the number of heads), what is $E(X)$? 1/2.

Throw a coin for two time (Y denotes the number of heads), what is $E(Y)$? 1.

Throw a coin for n time (Z_n denotes the number of heads), what is $E(Z_n)$? n/2.

Monotonicity. Assume all expectations are well defined.

For any **real-valued** r.v.'s X, Y such that $X \geq Y$ with probability one ($P(X \geq Y) = 1$), then

$$\mathbb{E}(X) \geq \mathbb{E}(Y)$$

with equality holding iif $\mathbb{P}(X = Y) = 1$.

A linear monotone function!

The results can be extended to more general cases, e.g., any **real-valued** r.v.'s $\tilde{X}, \tilde{Y}$ have identical distributions with X, Y respectively such that $X \geq Y$ with probability one, then $E(\tilde{X}) \geq E(\tilde{Y})$.

Proof with discrete r.v.'s:

Note that $Z = X - Y$ would be non-negative with probability one, such that $Z(s) \geq 0$ for all $s \in S$, and thus $E(Z)$ is a weighted sum of non-negative values (≥ 0), and by linearity

$$E(X) - E(Y) = E(Z) \geq 0.$$

If $E(X) = E(Y)$, then $E(Z) = \sum_{z_i \in \text{supp}(Z)} z_i P(Z = z_i) = 0$. Since $z_i \geq 0, P(Z = z_i) > 0$, we know $z_i = 0$ (otherwise $E(Z) > 0$).

Example

Throw a fair coin for two time.

X_1 denotes the number of head of the first throw, X_2 denotes the number of head of the second throw, $Z = X_1 + X_2$. $P(Z \geq X_1) = P(Z - X_1 \geq 0) = P(X_2 \geq 0) = 1$.

$E(Z) \geq E(X_1)$ **(via monotonicity directly).**

Y denotes the number of tails. Note that there is outcome (s) such that $Y(s) < X_1(s)$ (s : two heads, $P(\{s\}) = 1/4$). Y and Z have the identical distribution, we still have

$E(Y) = 1 \geq E(X_1) = 0.5$ **(not via monotonicity directly, but through the fact that** $E(Y) = E(Z)$ **and the fact that** $E(Z) \geq E(X_1)$ **via monotonicity).**

Distributions would determine expectations: $E(Y) = E(Z)$.