Probability Theory for EOR

Moment generating functions (MGFs)

Moments EX^k are pretty useful in characterizing random variables.
But there are so many of them.

Can we find something to store the information of a sequence of moments?

	$\mathbb{E}X^0$	$\mathbb{E}X$	$\mathbb{E}X^2$	$\mathbb{E}X^3$	$\mathbb{E}X^4$	$\cdots$
	$t^0/0!$	$t^1/1!$	$t^2/2!$	$t^3/3!$	$t^4/4!$	$\cdots$

All moments are uniquely encoded by one associated polynomial.
We use polynomials to store these information, label each moment with a unique polynomial! And we sum them up

$$\sum_{i=0}^{\infty} E(X^i)t^i/i!$$

Surprisingly, if the above summation is finite then it is equal to Ee^{tX}.

$$M_X(t) = Ee^{tX}.$$

Moment generating function (MGF)

$$M_X(t) = Ee^{tX}.$$

We say it is well-defined if we can find $a > 0$ such that

$$M_X(t) : (-a, a) \mapsto \mathbb{R}.$$

Namely, $M_X(t)$ is only well defined if the summation below is finite for any t from some given open interval containing 0:

$$\sum_{i=0}^{\infty} E(X^i)t^i/i! = E\sum_{i=0}^{\infty}(X^i)t^i/i! = Ee^{tX}.$$

- **Example** (MGF for $X \sim$Expo(1)): $M_X(t) = Ee^{tX} = \int_0^{\infty} e^{tx}e^{-x}dx = \frac{1}{1-t}, \forall t \neq 1$, $t = 0, M_X(t) = Ee^{tX} = 1$, **note that $\frac{1}{1-t}$ is finite for, e.g., $t \in (-0.5, 0.5)$ so we have a well-defined MGF for X.**

- **Question:** If $M_Y(t) = M_X(t), \forall t$ in some open interval containing zero, what do we know about $(\mathbb{E}(X^n))$ and $(\mathbb{E}(Y^n))$?

linearly independence of polynomials:
$\sum_{i=0}^{\infty} a_i/i!t^i = 0, \forall t$ in some open interval containing zero if and only if $a_i = 0$.

So if

$$M_X(t) = \sum_{i=0}^{\infty} E(X^i)t^i/i! = \sum_{i=0}^{\infty} E(Y^i)t^i/i! = M_Y(t) = c < \infty$$

we must have

$$E(X^i) = E(Y^i), \forall i.$$

I. We can derive moments from the MGF (if exists).
Not a surprise, as all moments are encoded in the MGF function by design.

We can derive moments from the MGF.

- **I. taking nth derivative w.r.t. t and evaluate the nth derivative function at $t = 0$:**

$$\mathbb{E}X^n = M_X^{(n)}(0)$$

Proof sketch. Notice $M_X(t) = \sum_{i=0}^{\infty} \mathbb{E}(X^n)t^i/i!$, then (under certain conditions)

$$M_X^{(n)}(t) = \left(\sum_{i=0}^{\infty} \mathbb{E}(X^i)t^i/i!\right)^{(n)} = \sum_{i=0}^{\infty} \left(\mathbb{E}(X^i)t^i/i!\right)^{(n)} = \mathbb{E}(X^n) + \sum_{i=n+1}^{\infty} (\mathbb{E}(X^i)t^{i-n}/(i-n)!)$$

- **II. use Taylor expansion/or other expansions to write the MGF as:**

$$M_X(t) = \sum_{n=0}^{\infty} a_n t^n/n!$$

then a_n is the nth moment.

Example (moments for $X \sim$Expo(1)):

- $M_X(t) = \sum_{n=0}^{\infty} \mathbb{E}(X^n) t^n/n!$ (either expansion, e.g., Taylor expansion, geometric series summation..)

$$M_X(t) = \frac{1}{1-t} = \sum_{n=0}^{\infty} t^n = \sum_{n=0}^{\infty} n! \frac{t^n}{n!}$$

- $M_X(t) = \sum_{n=0}^{\infty} M_X^{(n)}(0) t^n/n!$ (or **taking derivative and evaluate at 0**)

$$M_X^{(n)}(t) = \left(\frac{1}{1-t}\right)^{(n)} = \frac{n!}{(1-t)^{n+1}};\ M_X^{(n)}(0) = n!$$

- $EX^n = n!$
- Easy to caluclate $VX = EX^2 - (EX)^2 = 2! - (1!)^2 = 1.$

II. MGFs (if exist) determine the distributions.
If two r.v.s have the same MGF, they have the same distribution!
CDF, PMF/PDF, MGF (if exists).

Very useful when dealing with **sum of independent r.v.'s** once you notice that for independent X and Y: $E(XY) = E(X)E(Y)$ (*a way of calculating the expectation of the product*) and $e^{X+Y} = e^X e^Y$ (*a way of transforming a sum into a product*).

Independence means products:

$$M_{X+Y}(t) = Ee^{X+Y} = Ee^X Ee^Y = M_X(t)M_Y(t)$$

In order to know the distribution of $X + Y$, besides CDF, PMF/PDF, we can also look at its MGF (if exists) which is a much easier way.

Example I:

Show that a sum of two i.i.d. Expo(1)-distributed r.v.'s is not exponentially distributed.

- **MGF of a Expo(λ)-dsitributed** X:

$$M_X(t) = Ee^{tX} = \int_0^\infty e^{tx} e^{-\lambda x} dx = \frac{1}{\lambda - t},$$

which is finite for, e.g., $t \in (-\lambda/2, \lambda/2)$ **(so the MGF is well-defined)**.

- **MGF of** $Y_1 + Y_2$ with $Y_i \sim$ *i.i.d. Expo(1)*:

$$M_{Y_1+Y_2}(t) = M_{Y_1}(t) M_{Y_2}(t) = \frac{1}{(1-t)^2}.$$

which is finite for, e.g., $t \in (-1/2, 1/2)$ **(so the MGF is well-defined).**

- $\frac{1}{(1-t)^2}$ can not be written in the form of $\frac{1}{\lambda - t}$ for any λ's. We prove by contradiction, suppose they are euqal for some λ then $\frac{1}{(1-t)^2}\Big|_{t=0} = \frac{1}{\lambda-t}\Big|_{t=0}$ and thus $\lambda = 1$, however, $\frac{1}{(1-t)^2}\Big|_{t=a} \neq \frac{1}{1-t}\Big|_{t=a}$ for any $a \in \{a \neq 0, |a| <\}$.
- **The MGF of** $Y_1 + Y_2$ **is not the MGF of a exponential distribution**, then we know a sum of two i.i.d. Expo(1)-distributed r.v.'s is not exponentially distributed as MGF determines distributions (different MGF forms, different distributions).

Example II:

Bin(n,λ/n) converges to Pois(λ) with $n \to \infty$.

- Derive the corresponding MGFs:

  $X_i \sim$ i.i.d. Bern(p),
  $$M_{X_i}(t) = p(e^t - 1) + 1, t \in \mathbb{R};$$
  $Y = \sum_{i=1}^n X_i \sim$ Bin(n, p),
  $$M_Y(t) = \prod_{i=1}^n M_{X_i}(t) = (M_{X_1}(t))^n = \left(1 + p(e^t - 1)\right)^n, t \in \mathbb{R}.$$
  $Z \sim$ Pois(λ),
  $$M_Z(t) = \sum_{n=0}^{\infty} e^{tn} e^{-\lambda} \lambda^n / n! = e^{-\lambda} \sum_{n=0}^{\infty} (\lambda e^t)^n / n! = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)}, t \in \mathbb{R}.$$

- Note that $\lim_{n \to \infty, p = \lambda/n} \left(1 + p(e^t - 1)\right)^n = \lim_{n \to \infty} \left(1 + \frac{\lambda(e^t - 1)}{n}\right)^n = e^{\lambda(e^t - 1)}$.
- **The MGFs are the same in the limit**, which implies in the limit these two distributions coincide.

Example III:

Show that a sum of n independent Pois(λ_i), $i \leq n$ is still Possion.

- **MGF of a Pois(λ)-dsitributed X:**

$$M_X(t) = Ee^{tX} = e^{\lambda(e^t - 1)},$$

which is finite for, e.g., $t \in (-\lambda/2, \lambda/2)$ **(so the MGF is well-defined)**.

- **MGF of** $\sum_{i=1}^n Y_i$ with $Y_i \sim$ *independent Pois*(λ_i):

$$M_{\sum_{i=1}^n Y_i}(t) = \prod_{i=1}^n e^{\lambda_i(e^t - 1)} = e^{\sum_{i=1}^n \lambda_i(e^t - 1)}.$$

which is finite for, e.g., $t \in (-1/2, 1/2)$ **(so the MGF is well-defined) and is the MGF of a Pois($\sum_{i=1}^n \lambda_i$)-dsitributed X.**

- The MGF of $\sum_{i=1}^n Y_i$ is the MGF of Pois($\sum_{i=1}^n \lambda_i$), then we know a sum of n independent Pois(λ_i), $i \leq n$ is still Possion.