Probability Theory for EOR

Story proof of bionomial coefficient equalities

The **binomial coefficient** $\binom{n}{k}$ counts the number of ways to form an unordered collection of k items chosen from a collection of n distinct items.

- We know that the binomial coefficient is defined as

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

  We set it to be zero if $n < k$.
- Many mathematical identities involve the binomial coefficient.
- Sometimes, these identities can be easily proven algebraically.

e.g.,

$$(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j}$$

Combinatorial identities: algebra

$$n\binom{n-1}{k-1} = k\binom{n}{k}$$

$$n\frac{(n-1)!}{(k-1)!\cdot(n-k)!} = \frac{n!}{\left(\frac{1}{k}\right)\cdot k!\cdot(n-k)!} = k\binom{n}{k}$$

- Algebra is ok, but
  - it is easy to make mistakes.
  - it can be boring.
- We can also make up a story (think about the underlying meaning of these coefficients) to prove identities involving the binomial coefficients.

Combinatorial identities: story proof, I

$$n\binom{n-1}{k-1} = k\binom{n}{k}$$

Ways of counting the same number is not unique.
Prove the equality by showing both sides are counting the same number!

Consider a group of n people. We need to select a k-people group consisting of a leader and $k-1$ followers.

Left hand side: There are n choices for the leader, and then $\binom{n-1}{k-1}$ ways to select the followers.

Right hand side: first select the whole group: $\binom{n}{k}$ options. Then select the leader from the k selected persons: k options.

Combinatorial identities: story proof, II

$$\binom{n}{k}\binom{n-k}{m} = \binom{n}{m}\binom{n-m}{k}$$

Count the ways of getting two separate groups of size k and m respectively with given n people.

Left hand side: First draw the group of size k and then from the remaining persons draw a group of size m.

Right hand side: First the group of size m and then from the remaining persons a group of size k.

Example: Vandermonde's identity

$$\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j} \binom{n}{k-j}$$

Left hand side: You have m apples and n pears and want to select k pieces of fruit.

Right hand side: first select j apples and then the remaining $k-j$ to be pears. You can do this for all j, so you sum over j.

Can you do this one?

$$\sum_{k=0}^{n} k^2 \binom{n}{k}^2 = n^2 \binom{2(n-1)}{n-1}.$$

Hint:

$$\sum_{k=0}^{n} k^2 \binom{n}{k}^2 = \sum_{k=1}^{n} \binom{n}{k} k \binom{n}{n-k} k$$

Form a (n+1)-people group from two groups of size n each, and each of the group need to select a leader.

Left: select k people from group one, and select the first leader from the k people, then select the remaining n-k+1 people from the group two by frist selecting n-k followers and then the second leader from the remaining k people left in group two;

Right: first select the two leaders from two groups of size n each, and then select the n-1 followers from the remaining 2(n-1) people.