Probability Theory for EOR

Random variables and their distributions

Distributions: probability values of events associated with random variables

Real-valued r.v. X maps elements from S to real numbers!
Now we can work with functions/numbers:

$$X : S \mapsto \mathbb{R}$$

- We can now use $\{X \in C\}$ with some subsets $C \subseteq \mathbb{R}$ to represent different random events (not all subsets C generate well-defined events, and usually we focus on different intervals: e.g., $(-\infty, a], [a, b] \cap (a, e) \cdots$).

  The distribution of a given r.v. needs to specify all probability values of all those events generated by the r.v.:

  $$P : \{\{X \in C\} : \text{some subsets } C \subseteq \mathbb{R}\} \mapsto [0, 1].$$

  For example, a distribution needs to be able to answer $P(X \in [a, b] \cup [c, d]), P(X = e), \cdots$

- Though there are so many associated events, **we only need to know probability values of some selected events** and the rest probability values can be inferred from these values.

Definition (CDF of real-valued r.v.)

The **cumulative distribution function (CDF)** of an **real-valued** r.v., $X : S \to \mathbb{R}$, is the function, usually denoted by F_X, which maps numbers to numbers within [0,1].

$$F_X(x) = P(X \leq x).$$

- We only need to know probability values of the following types of events: $\{X \leq x\}, x \in \mathbb{R}$.

Properties:

- **Increasing function from zero to one:** $0 \leq F_X(x_1) \leq F_X(x_2) \leq 1, \forall -\infty < x_1 \leq x_2 < \infty$.
  (a): $\lim_{x \to -\infty} F_X(x) = 0$; think about $\emptyset$!
  (b): $\lim_{x \to \infty} F_X(x) = 1$; think about S!
- **Right-continuous:** $F_X(a) = \lim_{x \downarrow a} F(x)$.

Simple example revisits: throw a coin 10 times

- Let X be the r.v. that returns the number of heads.
- If we know F_X, it is easy to calculate, e.g., $P(s \in S : X(s) = 6) = P(X = 6) =$
- $= F_X(6) - F_X(5)$!
- **However, for this special case, there is easier way as X at most takes 11 values (values in the support of X), we only need to know**
$$P(X = i), i = 0, 1, \cdots, 10.$$

From CDF to PMF

Definition (CDF of real-valued r.v.)

The **cumulative distribution function (CDF)** of an **real-valued** r.v., $X : S \to \mathbb{R}$, is the function, usually denoted by F_X:

$$F_X(x) = P(X \leq x).$$

Definition (PMF of discrete r.v.)

The **probability mass function (PMF)** of a **discrete** r.v., $X : S \mapsto \{x_1, x_2, \cdots\}$, is the function:

$$p_X(x) = P(X = x); x \in \mathrm{Supp}(X).$$

Definition (PMF of discrete r.v.)

The **probability mass function (PMF)** of a **discrete** r.v., $X : S \mapsto \{x_1, x_2, \cdots\}$, is the function:

$$p_X(x) = P(X = x).$$

- We only need to know probability values of the following types of events: $\{X = x\}, x \in supp(X)$.

Properties:

- **Non-negativity**: $p_X(x) > 0$ if $x \in \text{Supp}(X)$; $p_X(x) = 0$ otherwise.
- **Sum to 1.** Suppose X has support $x_1, x_2, \ldots$: $\sum_{j=1}^{\infty} p_X(x_j) = 1$.

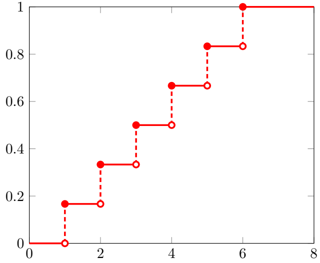
The distribution tell us almost all we need to know about a random variable.

Another simple example: Throw a fair six-sided die

What is the PMF for the random variable X denotes the number of eyes of the outcome if we throw a six-sided fair die for one time?

$$P(X = i) = 1/6, i = 1, 2, 3, 4, 5, 6.$$

From the PMF, we can also derive CDF, let's draw a graph:



Throw a fair six-sided die twice. What is the PMF of, Y, the sum of the