Probability Theory for EOR

Bayes' rule and the law of total probability (LOTP)

Bayes' rule

Conditional Probability (link one to another)

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Theorem

Bayes'rule

Suppose $P(A), P(B) > 0$.

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Unless $P(A) = P(B) > 0$, $P(A|B) \neq P(B|A)$!

A bit of a digression via a simple example

Now there are two schools in statistics: Frequentists and Bayesian.

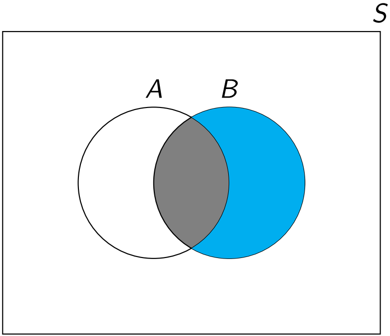
Let F be the event that a six-sided die is fair.
There are cases you encounter unfair dies, and F can be regarded as one random event as well. We want to know whether F is true or not (the hypothesis we want to test).
Let E be the event of throwing 7 times a 2.
Then you can consider E as the data we observe.

- **Frequentist**: $P(E|F)$. Data given hypothesis.
- **Bayesian**: $P(F|E)$. Hypothesis given data.
  By Bayes' rule, this requires $P(F)$: a prior belief on whether or not the die is fair, and might need some other priors as well.

The law of total probability

Law Of Total Probability (LOTP)

$$P(B) = P(B \cap A) + P(B \cap A^C)$$



$$P(B) = P(B|A)P(A) + P(B|A^C)P(A^c)$$

Law Of Total Probability (LOTP)

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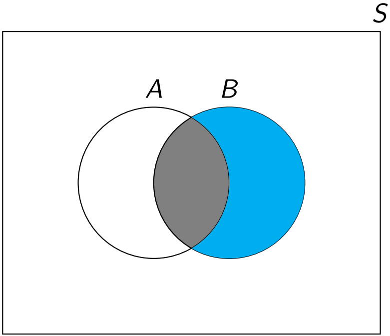

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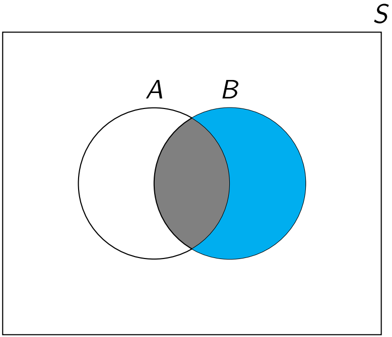
Law Of Total Probability (LOTP)



If we have a partition of S: disjoint A_i's such that $\cup_i A_i = S$ and $P(A_i) > 0$, then

$$P(B) = \sum_i P(B|A_i)P(A_i).$$

Exercise

Problem: medical tests

You decide to undergo a Corona test.

- The test is positive for 99 out of 100 people who have the virus (sensitivity)
- This test is negative for 99 out of 100 people who don't (specificity)

The current incidence of Corona is $1/1000$.

You test positive, what is the probability you have the disease?

Problem: medical tests

Bayes' rule + LOTP:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$
$$= \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^C)P(A^C)}.$$

Problem: medical tests

V: {carrying the Corona virus}.
T_+: {testing positive}.

- $P(T_+|V) = 0.99$.
- $P(T_+^C|V^C) = 0.99$.
- $P(V) = p = 0.1\%$.

$P(V|T_+) = \frac{P(T_+|V)P(V)}{P(T_+|V)P(V)+P(T_+|V^C)P(V^C)} = \frac{0.99p}{0.99p+0.01(1-p)} = \frac{0.99p}{0.98p+0.01}$.
Therefore, we know

$$P(V|T_+) = \frac{99}{98 + 1/p}.$$

$\approx$ **0.0901639344** $\approx 9\% > 0.1\%$. **If p=30%,** $\frac{99}{98+1/p} \approx 0.976973684$**!**