Probability Theory for EOR

Independence of random variables (two random variables)

Real-valued r.v.'s X, Y:

$$X, Y : S \mapsto \mathbb{R}$$

- It is natural to think that two r.v.'s are independent if knowing the events generated by one random variable does not change the distribution of another one.
- We know:

(I) the probability values of events of type $\{X \leq x\}, x \in \mathbb{R}$ would determine the whole distribution of real-valued r.v. X ;

(II) the independence of two random events $P(A \cap B) = P(A)P(B)$;

(III) look at events $\{X \leq x\}$ and $\{Y \leq y\}, x, y \in \mathbb{R}$:

$$P(\{X \leq x\} \cap \{Y \leq y\}) = P(\{X \leq x\})P(\{Y \leq y\}).$$

Sometimes, we use $P(X \leq x, Y \leq y)$ to denote $P(\{X \leq x\} \cap \{Y \leq y\})$.

Real-valued r.v.'s X, Y are independent if

$$P(X \leq x, Y \leq y) = P(X \leq x)P(Y \leq y); \forall x, y \in \mathbb{R}.$$

- **Independence means products**.
- There are many events of the type $\{X \leq x\}$ associated with real-valued r.v.'s, and independence would imply products involve all these events.

Simple example: Throw a coin twice

Denote (X_1, X_2) as the outcome of the two flips and $X_i = 1, i = 1, 2$ if ith throw get a head other wise zero.

- $S = \{(0,0), (0,1), (1,0), (1,1)\}$. Finite equally likely outcomes.
- Easy to check that:

$$P(X_1 \leq x, X_2 \leq y) = P(X_1 \leq x)P(X_2 \leq y); \forall x, y \in \mathbb{R}.$$

- And indeed, e.g.,

$$P(X_1 \leq x | X_2 \leq y) = P(X_1 \leq x), \forall x \in \mathbb{R}, y \geq 0.$$

Real-valued r.v.'s X, Y are independent if

$$P(X \leq x, Y \leq y) = P(X \leq x)P(Y \leq y); \forall x, y \in \mathbb{R}.$$

Discrete r.v.'s X, Y are independent if

$$P(X = x, Y = y) = P(X = x)P(Y = y); \forall x \in \mathrm{supp}(X), y \in \mathrm{supp}(Y).$$

Simple example revisits: Throw a coin twice

Denote (X_1, X_2) as the outcome of the two flips and $X_i = 1, i = 1, 2$ if ith throw get a head other wise zero.

- $S = \{(0,0), (0,1), (1,0), (1,1)\}$. Finite equally likely outcomes.
- Easy to check that:

$$P(X_1 = x, X_2 = y) = P(X_1 = x)P(X_2 = y); \forall x, y \in \{0, 1\}.$$

- And indeed, e.g.,

$$P(X_1 \leq x | X_2 \leq y) = P(X_1 \leq x), \forall x \in \mathbb{R}, y \geq 0.$$

Real-valued r.v.'s X, Y are independent if

$$P(X \leq x, Y \leq y) = P(X \leq x)P(Y \leq y); \forall x, y \in \mathbb{R}.$$

Discrete r.v.'s X, Y are independent if

$$P(X = x, Y = y) = P(X = x)P(Y = y); \forall x \in \text{supp}(X), y \in \text{supp}(Y).$$

Real-valued r.v.'s X, Y are **conditionally** independent given a discrete r.v. Z if

$$P(X \leq x, Y \leq y | Z = z) = P(X \leq x | Z = z)P(Y \leq y | Z = z);$$
$$\forall x, y \in \mathbb{R}, z \in \text{supp}(Z).$$

Discrete r.v.'s X, Y are are **conditionally** independent given a discrete r.v. Z if

$$P(X = x, Y = y | Z = z) = P(X = x | Z = z)P(Y = y | Z = z);$$
$$\forall x \in \text{supp}(X), y \in \text{supp}(Y), z \in \text{supp}(Z).$$

Similar to independence among events, these concepts can also be generalized to n r.v.'s.

Conditional independence $\not\leftrightarrow$ independence

Example: mystery opponent

Conditional independence $\not\to$ independence

- You play against one of two identical twins for two rounds, flip a coin to determine which game you play:

Game A. Against the first twin (A) you are evenly matched for two rounds.
Game B. Against the other you win with probability 3/4 for two rounds.

- X and Y: outcome of rounds 1 and 2. No hot hands.
- Z equal to one if you play against (A) otherwise Z equals 0.
- Conditional on $Z = 1$, X and Y are i.i.d. Bern(1/2).
- Conditional on $Z = 0$, X and Y are i.i.d. Bern(3/4).
- Without Z, $P(Y = 1|X = 1) > P(Y = 1)$.

$$P(Y = 1) = 1/2 \times 1/2 + 1/2 \times 3/4 = 5/8 = 0.625.$$

$$P(Y = 1|X = 1) = \frac{P(Y = 1, X = 1)}{P(Y = 1)} = \frac{1/2 \times (1/2)^2 + 1/2 \times (3/4)^2}{5/8} = 13/20 = 0.65.$$

Note here: by definition we should have $P(Y = 1|X = 1) = \frac{P(Y=1,X=1)}{P(X=1)}$, In this example, this number (probability value of a given event), $P(X = 1)$, happens to be equal to the probability value $P(Y = 1)$, these two numbers are equal here.

Example: coin flip

Conditional independence $\not\Leftarrow$ independence

- Cook up an example!
- Flip two coins: $X_i = I_{A_i}, i = 1, 2$, $A_i =\{i\text{th flip is a head}\}$.
- X_1, X_2 are independent.
- Denote $Z = X_1 + X_2$ then X_1, X_2 are not conditionally independent given Z.