# Probability Theory for EOR

## How to count the number of possible outcomes in one event

Gottfried Wilhelm Leibniz (1646-1716)

> *Music is the pleasure that human mind experiences from counting without being aware that it is counting.*

## Definition (Naive definition of probability)

Denote $A$ be an event for an experiment with a **finite*** sample spaces $S$ and each outcome is **equally likely*** to happen:

$$P_{\text{Naive}}(A) = \frac{\text{number of outcomes favorable to/in } A}{\text{number of all possible outcomes in } S} = \frac{|A|}{|S|}.$$

It is designed for common but very limited cases **(finite equally likely outcomes)**, yet, it is already complicated in certain cases.

Not always easy to count the number of outcomes in one event.

We need **binomial coefficient**: $\binom{n}{k} := \frac{n!}{k!(n-k)!}$.

.

# Problem I: hands in poker

## Example

A 5-card hand is dealt from a standard well-shuffled 52-card deck.

The hand is called a *full house* in poker if it consists of three cards from same rank and two cards of another rank, e.g. three 7's and two 10's (in any order).

What is the probability of a full house?

We need to know $|S|$ and $|A|$, where $S$ denotes the set of all possible outcomes, and $A$ denotes the set of outcomes that are a full house.

# Number of all outcomes $|S|$

**Multiplication rule (sampling without replacement)**

- We have $n$ cards.
- We select $k$ cards *one at a time*.
- We keep the selected card (without replacement).

How many outcomes ($N$) when order matters?

- Step 1: $n$ possible outcomes. Keep the card!
- Step 2: $n-1$ possible outcomes. Keep the card!

$\vdots$

- Step $k$: $n-k+1$ possible outcomes.

MR: $N = n \cdot (n-1) \cdot \ldots \cdot (n-k+1)$ *ordered* outcomes.

# Number of all outcomes $|S|$

## Definition

**Factorial**

We define $n!$ (say: $n$ factorial) as

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1$$
$$0! = 1$$

$$N = n \cdot (n-1) \cdot \ldots \cdot (n-k+1) = \frac{n!}{(n-k)!}$$

# Number of all outcomes $|S|$

## Definition

**Factorial**

We define $n!$ (say: $n$ factorial) as

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1$$
$$0! = 1$$

$$N = n \cdot (n-1) \cdot \ldots \cdot (n-k+1) = \frac{n!}{(n-k)!}$$

Alternative interpretation of how we arrive at this number of ordered outcomes:

- **We can order the cards in $n!$ ways.** Permutation of n items.
- Don't care about the ordering of the $(n-k)$ not selected cards. The not-selected cards can be ordered in $(n-k)!$ ways.

But to calculate $|S|$, we still need to adjust for over-counting!

**We don't care for the orders of selected cards either!** Divide by $k!$.

# Number of all outcomes $|S|$

We *adjust for overcounting*: $|S| = N/k! = \frac{n!}{k!(n-k)!} = \binom{n}{k}$

The **binomial coefficient** gives the number of possible outcomes of picking $k$ items out of $n$ items (where orders do not matter).

In our case: $n = 52, k = 5$.

Also, we know these outcome are equally likely, which is why we can use the naive probability.

# Number of outcomes in A, $|A|$

- Choose the rank that we have the three cards of. **13**. Keep the rank
- How many different ways of choosing three cards of a given rank (orders does not matter): we have to **choose 3 out of 4 suits** $\binom{4}{3}$.
- Choose the rank that we have the two cards of from the remaining ranks. **12**.
- How many different ways of choosing two cards of a given rank (orders does not matter): we have to **choose 2 out of 4 suits** $\binom{4}{2}$.
- $|A| = 13\binom{4}{3}12\binom{4}{2}$

# What is the probability of a full house?

$$\mathbb{P}(\text{Full house}) = |A|/|S| = 13\binom{4}{3}12\binom{4}{2}/\binom{52}{5}$$

Check examples of *Poker probability* on Wikipedia and practice the counting strategy.

It is fun.

# Problem II: full house with dice

**Example**

Suppose we roll 5 identical dice. What is the probability of a full house (three dice of a same number and two remaining dice of a same but different number, e.g., three 3's and two 6's?)?

Note that in this case, **order** matters.

# Number of all outcomes $|S|$

**Multiplication rule (sampling with replacement)**

- We have $k$ dice.
- each would have $n$ outcomes.

How many outcomes ($N$) when order matters?

- Dice 1: $n$ possible outcomes.
- Dice 2: $n$ possible outcomes.

  $\vdots$
- Dice $k$: $n$ possible outcomes.

MR: $|S| = n^k$. (23333 different from 33332)

# Number of outcomes in A, $|A|$ (ordered)

- First we select the number, $a$, that we have three dice of (a triplet). **6**.
- How many different ways of choosing three dice out 5 to have the number $a$: **choose 3 out of 5** $\binom{5}{3}$**.**
- Select the number, b($\neq a$), that we have two dice of (a pair). **5**.
- The remaining two dice must have the number b.
- Note that here we would have the number of *Ordered* outcomes: $|A| = 6\binom{5}{3}5$.

# What is the probability of a full house?

$$\mathbb{P}(\text{Full house}) = |A|/|S| = 6\binom{5}{3}5/6^5$$

# Differences in the two problems

- Without replacement or with replacement. (Or, whether the outcome number changes or not).
- **Orders**. In Problem I, there is the term "in any order"; while in Problem II there is not such term.
  We do so in order to make sure each outcome would be equally likely. **Order matters when replacement is present.**
- The way we count may not be unique.
  For example, you can also consider orders in Problem I (then $|S|$ and $|A|$ are both 5! times larger than the ones without considering the orders, think about why), you will find the probability remains the same.
- Practice more.

If you wonder why **order matters when replacement is present.** Think about the simple experiment: roll two dice and think about the following two events (a) two dice are both ones and (b) one die is 2 and another one is 3. Are they equally likely?