Probability Theory for EOR

Random variables (discrete)

Random variables are functions

Definition (Random variables (r.v.'s) (preliminary))

A random variable (r.v.) is a function maps elements in sample space S to numbers in $\mathcal{C}$, e.g., $\mathcal{C} \subseteq \mathbb{R}$.

X **maps elements from S to numbers! Now we can work with functions/numbers!**. Usually, we consider **real-valued r.v.'s**, i.e.,

$$X : S \mapsto \mathbb{R}$$

- X essentially is a function, and when we manipulate random variables, it would be similar to manipulating functions and there are so many transformations we can work with: summation, division, max, min...
- **There are many functions from S to $\mathbb{R}$ (so there are also many random variables defined with the same S), not all functions can be regarded as random variables, but all random variables are functions**. Detailed discussion goes beyond the course scope.
- We can now use $\{X \in C\}$ with $C \subseteq \mathcal{C}$ to represent different random events, and properties derived for random events would be inherited. **This would connect random variables with probability functions** as probability functions have random events as elements in its domain.
- Given the introduction of these additional structures, we have a much larger place for our mind adventures: conditional probability/independence based on random variables, random variables with specific probability functions... **Random variables are functions, there are selected functions with nice properties, and so are some selected random variables.**

There are many functions from a sample space to a set of numbers, and we start with those map to sets of a sequence of numbers $a_1, a_2, a_3, \cdots$.

Simple example: throw a coin 10 times

- The outcomes are strings like $HHTHTHHTHT$.
- Notations:
  - define the event H_i: we see i heads.
  - define the event T_j: we see j tails.
  - $H_i = T_{n-i}$.
- **Cumbersome!**

Simple example: throw a coin 10 times

- Let X be the number of heads and Y be the number of tails.
- Improves notation: no subindices as with H_i, T_j or $H_i = T_{n-i}$.
- Improves calculation: we see $Y = n - X$.

But, what are these X and Y?

Simple example: throw a coin 10 times

Throw a coin n times. Define X to be the number of heads.

- X takes an outcome s in the sample space S, and maps that to a number on the real line.

$$
\begin{aligned}
X(HHTHTHHTHT) &= 6, \\
X(HHTHTHHTTH) &= 6, \\
X(TTTTTTTTTT) &= 0, \\
&\vdots
\end{aligned}
$$

- X is a *function* that maps each $s \in S \to \mathbb{R}$.
- Because the outcome is random, X is called a *random variable* (r.v.). But it is simply a function.

Simple example: throw a coin 10 times

- X needs a set of outcomes. An r.v. is defined *on* a sample space S.
- We can define *multiple* r.v.'s on S:
  - X: number of heads;
  - Y: number of tails;
  - Z: only equal to one if the last throw is a head.
- X assigns a number to *each* $s \in S$.

Simple example: throw a coin 10 times

- Furthermore, from X we could define events, e.g., let $\{X \in \{6\}\}$ (or $\{X = 6\}$) denote the set of outcomes $s \in S$ such that $X(s) = 6$, and it would be the event that we get 6 heads in 10 throws.
- $P(X = 6) = P(\{HHTHTHHTHT\} \cup \{HHTHTHHTTH\} \cup \ldots) = \frac{\binom{10}{6}}{2^{10}} \approx 0.205078.$
- With the defined events, it is easy to see that

$$P\left(X \in \{0, 1, 2, \cdots, 10\}\right) = 1$$

- We have a **discrete random variable**!

Discrete random variable

Definition (Discrete random variable)

A random variable X is a **discrete** random variable if there exists a set C with at most countably many numbers (so you can index all the elements within C using a set of natural numbers):

$$P(\{X \in C\}) = 1, \text{ or equivalently, } P(X \in C) = 1.$$

The **support** of a **discrete** random variable is the set of all numbers, x's, such that $P(X = x) > 0$:

$$supp(X) = \{x \in C : P(X = x) > 0\}.$$

Previous simple example revisits: throw a coin 10 times

- Previously: X takes an outcome s in the sample space S, and maps that to a number on the real line.

$$\begin{aligned} X(HHTHTHHTHT) &= 6, \\ X(HHTHTHHTTH) &= 6, \\ X(TTTTTTTTTT) &= 0, \\ &\vdots \end{aligned}$$

  X is a **real-valued** r.v.!

- $C = \{x_1, x_2, \cdots\} = \{x_i, i \in \mathbb{N}_+\}$ such that $x_i = i - 1$:

$$P(X \in C) = 1.$$

  So, X is a **discrete** r.v.!
  (The choice of such C is not unique.)

- $$supp(X) = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.$$