Probability Theory for EOR

Non-naive definition of probability

General definition of probability

Definition (Naive definition of probability)

Denote A be an event for an experiment with a **finite*** sample spaces S and each outcome is **equally likely*** to happen:

$$P_{\text{Naive}}(A) = \frac{\text{number of outcomes favorable to/in } A}{\text{number of outcomes in } S} = \frac{|A|}{|S|}.$$

Very limited cases with **(finite equally likely outcomes)**.

- A probability measure assigns probabilities to events.
- But how in general cases?
- After centuries of thinking, we have agreed on two

Axioms of Probability

Definition (General definition of probability)

A **probability function (measure)** P maps an event , a (well-constructed) subset A of the sample space S ($A \subseteq S$), to the probability of the event $P(A)$, a real number within $[0, 1]$. It should satisfy the *Axioms of Probability*

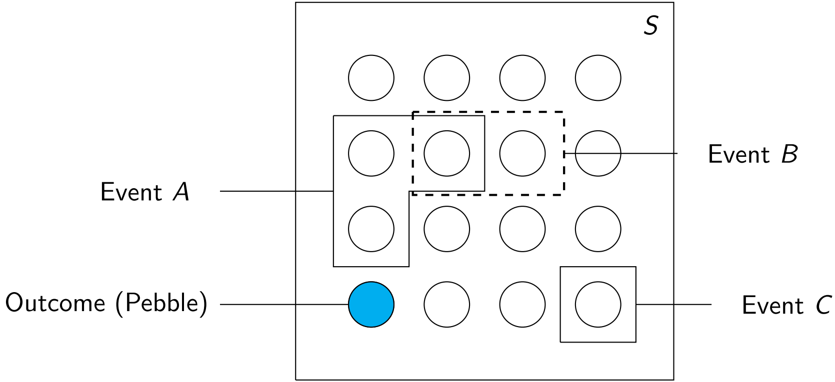
1. $P(\emptyset) = 0$, and $P(S) = 1$.
2. If $A_1, A_2, \ldots$ are disjoint (i.e. mutually exclusive, $A_i \cap A_j = \emptyset, i \neq j$) events, then

$$P\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

- **Note:** reporting probabilities outside $[0, 1]$ without the disclaimer that this cannot be the right answer will be considered a capital blunder and renders your complete answer invalid (even if that answer is 99% correct).
- The second axiom (countable additivity) involves a summation of a infinite series. It is more intuitive if you only choose finitely many non-empty A_i's: $P\left(\cup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$.

Example

The second axiom (countable additivity) involves a summation of a infinite series. It is more intuitive if you only choose finitely many non-empty A_i's: $P\left(\cup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$.



$P(A) = 3/16$, $P(B) = 1/8$, $P(C) = 1/16$, $P(A \cup B) = 1/4$, $P(A \cup C) = 1/4$.

Quiz: which one is poteintially a probability function?

Suppose we have a domain of probability: $\{A, A^c, \emptyset, S\}$ and functions P_1, P_2, P_3, P_4.

- $P_1(\emptyset) = 1$.
- $P_2(S) = 0$.
- $P_3(A) = 0, P_3(\emptyset) = 0, P_3(A^c) = 1, P_3(S) = 1$.
- $P_4(A) = 0, P_4(\emptyset) = 0, P_4(A^c) = 1/2, P_4(S) = 1$.

Only P_3!

General definition embeddes the naive one

- $P_{\text{Naive}}(\emptyset) = |\emptyset|/|S| = 0.$
- $P_{\text{Naive}}(S) = |S|/|S| = 1.$
- $P_{\text{Naive}}\left(\cup_{i=1}^{m} A_i\right) = \left|\cup_{i=1}^{m} A_i\right|/|S| = \sum_{i=1}^{m} |A_i|/|S| = \sum_{i=1}^{m} P_{\text{Naive}}(A_i)$

Now we can directly work with the general definition, and the naive one is simply a special case of the general definition.

The properties derived based on the general definition also holds for the naive one.

Properties of probability: I

A **probability function (measure)** P maps an event to a real number within $[0, 1]$. It should satisfy the *Axioms of Probability*

1. $P(\emptyset) = 0$, and $P(S) = 1$.
2. If $A_1, A_2, \ldots$ are disjoint (i.e. mutually exclusive) events, then

$$P\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Theorem

For any events A and B, we would have

- $P(A^c) = 1 - P(A)$.
- *If* $A \subseteq B$, $P(A) \leq P(B)$.
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Properties of probability: II

A **probability function (measure)** P maps an event , a (well-constructed) subset A of the sample space S ($A \subseteq S$), to a real number within $[0, 1]$. It should satisfy the *Axioms of Probability*

1. $P(\emptyset) = 0$, and $P(S) = 1$.
2. If $A_1, A_2, \ldots$ are disjoint (i.e. mutually exclusive) events, then

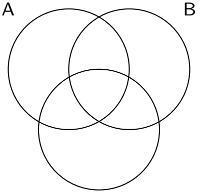
$$P\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Theorem

(Inclusion-exclusion principle: two events)
For any events A and B, we would have $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Theorem

(Inclusion-exclusion principle: three events)
For any events A, B and C, we would have
$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$.



An exercise

Example: void in at least one suit

Exercise What is the probability that a 13-card hand is void in at least one suit?