Notes for Advanced Economics

The First Semester

Wei Ling

School of Finance, Renmin University of China

To Teachers and Classmates

Preface

Without clear and systematic documentation, knowledge and ideas do not belong to you really, at least, in the sense of application efficiency. Lecture notes from dear teachers show an elegant and powerful mastering of knowledge, but we are far from it.

Mathematical Analysis Methods, *Advanced Macroeconomics*, *Advanced Microeconomics* and *Advanced Econometrics* are the most important compulsory courses for postgraduate students majoring in economics or finance and of course, you have to work day and night to master them in a clear and systematic manner. Unluckily, these courses are established in delicate but tedious mathematics knowledge, e.g., real analysis, which is out of my range. But I refuse to give up and decide to master them step by step, and this document is the best witness. None of advanced mathematics beyond your knowledge will show in the contents, but if you are unfamiliar with the common conceptions, e.g., stationary, my another book may be helpful, which summarizes the minimal knowledge of undergraduate economics, available on https://gitee.com/lingweir/rue.

The Mathematical Analysis Methods (MAM) course is presented by Shunming Zhang in the autumn semester, 2022, that is a compulsory course for the finance students from School of Finance, Remin University of China. This note is created to document the core knowledge and my understanding of this course. Actually, MAM is my favorite course as it tells me how to think economics from a mathematical expert's perspective.

The key words which can summarize economics are equilibrium, optimum and arbitrage. Equilibrium describes the final state, optimum decides the target of agents and arbitrage limits the possible actions. Two most important mathematical theorems in economics is fixed-point theorem and separating hyperplane theorem. Them may appear in latter chapters. There are many famous economists who make lots of contribution in economics, including Smith, Courtnet, Walras, Edgeworth, Pareto, Neumann, Kakutani, Nash, Arrow, Debreu and so on.

The reference textbook used in MAM is *Theory of value, An Axiomatic Analysis of Economics Equilibrium* by Gerad Debreu in 1959.

Advanced Macroeconomics (AMa) is one of the most important course for academic students, and also in my interesting field (I desire to understand how real economy works). AMa is hold by Rong Li, and the reference textbook is *Real Macroeconomic Theory* by Per Krusell, which dose not include monetary factors in models. Macroeconomics theories are various and everyone holds a different view to it. Therefore, understanding the generation alteration of macroeconomic theories and why is crucial for students who must form a board view to real economics. Macroeconomics theories have developed four versions, Neoclassical, Keyneasian, Neo Keyneasian and New classical.

Advanced Microeconomics (AMi) is a crucial course for any student who desires to understand how people behave and how to analyze them. AMi is presented by Wang Xiang, who is vigorous, hardworking and beloved. I am full of confidence with great acquirement in this course. AMi includes the classical microeconomics theory, mainly focusing on people's react to exogenous environment. As the general equilibrium part of AMi coincides with MAM and Prof. Wang has provided an fully explained lecture note, so the general part will be omitted.

Without econometrics, you cannot find strong empirical evidence of economic theories. Econometrics is an necessary tool, not the aim for us who desire to know about the world. Advanced Econometrics (AE) is presented by Jinghua Lei, a great econometrics professor, who deliver the course in a clear, elegant and systematic manner. AE introduces preliminary mathematics and classic econometrics models and estimations, all of which are the basis to understand state-of-art econometrics methods in future. Anyway, AE is in my interest and matters a lot for further researches, so we must master it!

Beyond four advanced courses mentioned above, notes for English and Continuous Time Finance: Mathematical Tools and Application are also included in this document.

The newest version of this document is available on https://gitee.com/lingweir/collection. Any comments and suggestions are appreciated, and please contact me through Email: lingwei3418@163.com.

Wonderful work of respective teachers and generous help from dear classmates have influenced this note. To those who aid me in learning advanced economics and achieving this text, I am most grateful.

Wei Ling
The Third Red House, Beijing
January 1, 2023

This page intentionally left blank

Contents

Part I
Mathematical Analysis Methods

1 Sets

1.1 Definitions and Operations of Sets

1.1.1 Definitions of Sets

To understand the power of sets, look at Example 1.1.

Example 1.1. *We have $(0,1)$ and $[0,1]$, which contains more elements?*

Let $N(A)$ denotes the number of elements of set A.The compare of number of elements is obtained by one-to-one correspondence. Let $y = arctan(x - \frac{1}{2})\pi, x \in (0,1)$, where $arctan(\cdot)$ is one-to-one correspondence, which indicates that $y \in (-\infty, +\infty)$. Thus $N((0,1)) = N((-\infty, +\infty))$. As $(0,1) \subset [0,1] \subset (-\infty, +\infty)$, we have $N((0,1)) \leq N([0,1]) \leq N((-\infty, +\infty))$. For $N((0,1))$ equals to $N((-\infty, +\infty))$, we have $N((0,1)) = N([0,1]) = N((-\infty, +\infty))$, which reaches the conclusion.

Set Theory is created by Geory Cantor in 1880s and it is to explore various properties of sets. Cantor's original statement of the definition of set is given as follows.

1.1.2 Definitions and Operations of Sets

DEFINITION 1.1. ***Set*** *refers to all objects that have certain characteristics or meet certain properties as a whole, this whole is called a set, and these objects are called elements of the set.*

Therefore, the so-called given set or the existence of a set means that a certain constraint has been determined, and it can determine whether any object belongs to the set. Notice that, all natural numbers form a set, written as $\mathbb{N}$, $\mathbb{Q}$ denotes the set formed by all rational numbers and $\mathbb{R}$ denotes the set formed by all real numbers.

A set S is a collection of objects of any kind, which are called the *elements or points* of S; sometimes one says a class / system / family of elements. The sets which constitute the universe of discourse must always be explicitly listed at the outset.

DEFINITION 1.2. *$x \in S$ expresses that x denotes a certain elements of S; it is read, x belongs to S, or x is an elements of S, or x is in S, or S owns x.*

DEFINITION 1.3. *If x and x' denote elements of S, then $x = x'$(x equals x') expresses that they denote the same elements, and $x \neq x'$ (x different from x') expresses that they denote different elements.*

DEFINITION 1.4. *Let $\mathscr{P}$ be a property which any element x of S has or does not have. $\{x \in S | x \text{ has property} \mathscr{P}\}$ denotes the set of all the elements of S which have the property $\mathscr{P}$; it may be read: the set of x in S such that x has property $\mathscr{P}$.*

To understand the basic definitions of Set, see Example 1.2

Example 1.2. *We know $\{x,y,z\}, \{x^2,y^2,z^2\}$ and $\{xy,yz,zx\}$ denote the same set A. What is A?*

Let the summation of elements in A is k, so $x+y+z = x^2+y^2+z^2 = xy+yz+zx = k$. $(x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+zx)$ indicates that $k^2 = 3k$. h denotes the multiply of all elements in A, we have $xyz = h = (xyz)^2 = h^2$. If any element in A equals to zero, two elements of xy, yz, zx equal to zero, which is impossible. Thus, we get $h = 1$.

With $xyz = 1$ and $xy+yz+zx = x+y+z = k, k = 0,1$, according to Vieta theorem, we have x,y,z is the solution of $t^3 - 3t^3 + 3t - 1 = 0$ or $t^3 - 1 = 0$. The solution of the first equation if $t_1 = t_2 = t_3 = 1$, not desirable. The solution for $t^3 - 1 = 0$ is $t_1 = 1, t_2 = \frac{-1+\sqrt{3}i}{2}, t_3 = \frac{-1-\sqrt{3}i}{2}$, which is reasonable. Thus, the elements of set A includes $1, w, w^2$ where $w = \frac{-1+\sqrt{3}i}{2}$.

With clear definitions of sets, here are the relation and operation laws of sets, including subset, unions, intersections, complements and symmetric difference.

DEFINITION 1.5. *For two sets A and A', if every member of set A is also a number of set A', i.e., $x \in A \Rightarrow x \in A'$, we write $A \subseteq A'$ (A is contained in A') and we say that A is a subset of A' or A' contains A and write $A' \supseteq A'$. If $A \subseteq A'$ and A' is not contains in A, we say that A is a proper subset of A' and write $A \subset A'$ (A is strictly contained in A'); if $A \subseteq A'$ and $A' \subseteq A$, we write $A = A'$ (A is equal to A'). In this case $x \in A$ is equivalent to $x \in A'$.*

We write $x \notin A$ if $x \in A$ is not true; $A' \supsetneq A$ if $A' \supseteq A$ is not true; $A \neq A'$ if $A = A'$ is not true, etc. A subset of A is said to be empty if it contains no elements, which is denoted by $\varnothing$. A subset of A is said to be full if it is equal to the set A. The set consisting of the elements, $x_1, x_2, \cdots, x_N$ is denoted by $\{x_1, x_2, \cdots, x_N\}$. More generally, the set of elements of A which satisfy a property $\mathscr{P}$ is denoted by $\{x \in A | x \text{ satisfies } \mathscr{P}\}$. Notice! The element x of S and the subset $\{x\}$ of S is not the same thing.

A property $\mathscr{P}$ defines a subset of S, namely the set of elements of S having that property. When no element of S has the property $\mathscr{P}$, one says that P defines the empty subset of S denoted $\varnothing$. This convention is necessary if a property is always define a subset of S.

Let Λ be a given set. For each $\lambda \in \Lambda$, we specify a set A_λ. In this way, we get many sets, whose total is called the set family, which is written as $\{A_\lambda | \lambda \in \Lambda\}$ or $\{A_\lambda\}_{\lambda \in \Lambda}$. Here, Λ is often referred to as an indicator set. When $\Lambda = \mathbb{N}$, a set family is also called a set sequence, which is abbreviated as $\{A_n\}$, etc.

1.1.3 Unions and Intersections

Two sets can be "added" together.

DEFINITION 1.6. *The **union** of A_1 and A_2, denoted by $A_1 \cup A_2$, is the set of all things that are members of either A_1 or A_2, i.e. $A_1 \cup A_2 = \{x | x \in A_1 \text{ or } x \in A_2 \text{ or both}\}$*

Some basic properties of unions are listed as follows,

1. $A_1 \cup A_2 = A_2 \cup A_1$(Ablian or Commutativity);
2. $A_1 \cup (A_2 \cup A_3) = (A_1 \cup A_2) \cup A_3$(Associativity);
3. $A_1 \subseteq A_1 \cup A_2$;
4. $A \cup A = A$;
5. $A \cup S = S$;
6. $A \cup \varnothing = A$;
7. $A_1 \subseteq A_2$ if and only if $A_1 \cup A_2 = A_2$.

We extend the union of two sets to general case of multiple sets. If $\mathcal{F}$ is a family of sets. the union of all sets in $\mathcal{F}$ is denoted by

$$\cup \mathscr{F} = \{x | x \in A \text{ for at least one } A \in \mathscr{F}\} = \{x | \text{ there exists some } A \in \mathscr{F} \text{ such that } x \in A\} \tag{1.1}$$

For a set Λ. suppose that for each $\lambda \in \Lambda$, there is defined a set A_λ. Let $\mathscr{F}$ be the collection of sets $\{A_\lambda | \lambda \in \Lambda\}$. We write $\mathscr{F} = \{A_\lambda | \lambda \in \Lambda\}$ and refer to this as an indexing (or parametrization) of $\mathscr{F}$ by the index set (or parameter set) Λ. The union of all sets in $\mathcal{F}$ is defined by

$$\cup_{\lambda \in \Lambda} A_\lambda = \{x | x \in A_\lambda \text{ for at least one } \lambda \in \Lambda\} = \{x | \text{there exists } \lambda \in \Lambda \text{ such that } x \in A_\lambda\} \tag{1.2}$$

We often set Λ as $\{1, 2, \cdots, N\}$ or $\{1, 2, \cdots\}$.

A new set can also be constructed by determining which members two sets have in common.

DEFINITION 1.7. *The **intersection** of A_1 and A_2, denoted by $A_1 \cap A_2$, is the set of all things that are members of both A_1 and A_2, i.e., $A_1 \cap A_2 = \{x | x \in A_1 \text{ and } x \in A_2\}$*

Some basic properties of intersections are listed as follows,

1. $A_1 \cap A_2 = A_2 \cap A_1$(Ablian or Commutativity);
2. $A_1 \cap (A_2 \cap A_3) = (A_1 \cap A_2) \cap A_3$(Associativity);
3. $A_1 \cap A_2 \subseteq A_1$;
4. $A \cap A = A$;
5. $A \cap S = A$;
6. $A \cap \varnothing = \varnothing$;

7. $A_1 \subseteq A_2$ if and only if $A_1 \cap A_2 = A_1$.

We extend the intersection of two sets to general case of multiple sets. If $\mathscr{F}$ is a family of sets, the intersection of all sets in $\mathscr{F}$ is defined by

$$\cap \mathscr{F} \{x | x \in A \text{ for every } A \in \mathscr{F}\} \tag{1.3}$$

For a set Λ. suppose that for each $\lambda \in \Lambda$, there is defined a set A_λ. Let $\mathscr{F}$ be the collection of sets $\{A_\lambda | \lambda \in \Lambda\}$. We write $\mathscr{F}\{A_\lambda | \lambda \in \Lambda\}$ and refer to this as an indexing of $\mathscr{F}$ by the index set Λ. The intersection of all sets in $\mathscr{F}$ is defined by

$$\cap_{\lambda \in \Lambda} = \{x | x \in A_\lambda \text{ for every } \lambda \in \Lambda\} \tag{1.4}$$

We often set Λ as $\{1, 2, \cdots, N\}$ or $\{1, 2, \cdots\}$.

If $A_1 \cap A_2 = \varnothing$, then A_1 and A_2 are said to be **disjoint** and we have $A_1 \cap A_2 = A_1 + A_2$. The sets belonging to a family of sets $\mathscr{F}$ are pairwise disjoint if any two distinctly indexed sets in $\mathscr{F}$ are disjoint. Let $\{A_\lambda | \lambda \in \Lambda\}$ be the collection of sets. If for any $\lambda_1 \in \Lambda$ and $\lambda_2 \in \Lambda$ with $\lambda_1 \neq \lambda_2$, $A_1 \cap A_2 = \varnothing$, then this collection of sets is **pairwise disjoint**. If the condition holds, we have $\cap_{\lambda \in \Lambda} A_\lambda = \sum_{\lambda \in \Lambda} A_\lambda$. Λ can be $\{1, 2, \cdots\}$ or $\{1, 2, \cdots\}$. With precise definitions of unions, intersections and set family, some theorems can be obtained easily as follows.

THEOREM 1.1. *Let A_1, A_2 and A_3 be sets.*

1. *Ablian or Commutativity:*
   (a) $A_1 \cup A_2 = A_2 \cup A_2$;
   (b) $A_1 \cap A_2 = A_2 \cap A_1$.
2. *Associativity:*
   (a) $A_1 \cup (A_2 \cup A_3) = (A_1 \cup A_2) \cup A_3$;
   (b) $A_1 \cap (A_2 \cap A_3) = (A_1 \cap A_2) \cap A_3$.
3. *Distributivity:*
   (a) $A_1 \cup (A_2 \cap A_3) = (A_1 \cup A_2) \cap (A_1 \cup A_3)$;
   (b) $A_1 \cap (A_2 \cup A_3) = (A_1 \cap A_2) \cup (A_1 \cap A_3)$.
4. *Absorption:*
   (a) $A_1 \cup (A_1 \cap A_2) = A_1$;
   (b) $A_1 \cap (A_1 \cup A_2) = A_1$.
5. *Idempotence:*
   (a) $A_1 \cap A_1 = A_1$;
   (b) $A_1 \cup A_2 = A_2$.

THEOREM 1.2.

1. *If $A_\lambda \subseteq A'_\lambda$ for $\lambda \in \Lambda$, then $\cup_{\lambda \in \Lambda} A_\lambda \subseteq \cup_{\lambda \in \Lambda} A'_\lambda$. In particular, if $A_\lambda \subseteq A'$ for $\lambda \in \Lambda$, then $\cup_{\lambda \in \Lambda} A_\lambda \subseteq A'$.*
2. *If $A_\lambda \subseteq A'_\lambda$ for $\lambda \in \Lambda$, then $\cap_{\lambda \in \Lambda} A_\lambda \subseteq \cup_{\lambda \in \Lambda} A'_\lambda$. In particular, if $A \subseteq A'_\lambda$ for $\lambda \in \Lambda$, then $A \subseteq \cap_{\lambda \in \Lambda} A'_\lambda$.*
3. $\cup_{\lambda \in \Lambda} (A_\lambda \cup A'_\lambda) = (\cup_{\lambda \in \Lambda} A_\lambda) \cup (\cup_{\lambda \in \Lambda} A'_\lambda)$.
4. $\cap_{\lambda \in \Lambda} (A_\lambda \cap A'_\lambda) = (\cap_{\lambda \in \Lambda} A_\lambda) \cap (\cap_{\lambda \in \Lambda} A'_\lambda)$.
5. $A \cap (\cup_{\lambda \in \Lambda} A_\lambda) = \cup_{\lambda \in \Lambda} (A \cap A_\lambda)$.
6. $A \cup (\cap_{\lambda \in \Lambda} A_\lambda) = \cap_{\lambda \in \Lambda} (A \cup A_\lambda)$.

1.1.4 Complements

Two sets can also be "subtracted". The **relative complement** of A_2 in A_1 (also called the set-theoretic difference of A_1 and A_2), denoted by $A1\backslash A_2$, is the set of all elements that are members of A_1 but not members of A_2, i.e., $A_1\backslash A_2 = \{x|x \in A_1 \text{ and } x \notin A_2\}$. Notice! It is valid to "subtract" members of a set that are not in the set, like, $(A_1\backslash A_2) \cup A_2 = A_1 \cup A_2$.

In certain settings all sets under discussion are considered to be subsets of a given universal set S. If A is a subset of S, and x an element of S, then $x \notin A$ expresses that x is not an element of A. The elements of S which do not belong to A form a set called the complement of A in S (the absolute complement or simply complement of A relative to S) and denoted by $S\backslash A = \mathbf{C}_s A$. When there can be no ambiguity about S one says the complement of A and one writes $\mathbf{C}A$.

THEOREM 1.3. *1. $\mathbf{C}S = \varnothing$ and $\mathbf{C}\varnothing = S$;*

2. $A \cup (\mathbf{C}A) = S$ and $A \cap (\mathbf{C}A) = \varnothing$;

3. $\mathbf{C}(\mathbf{C}A) = A$;

4. If $A_1 \supseteq A_2$, then $\mathbf{C}A_1 \subseteq \mathbf{C}A_2$;

5. If $A_1 \cap A_2 = \varnothing$, then $A_1 \subseteq \mathbf{C}A_2$ and $A_2 \subseteq \mathbf{C}A_1$.

THEOREM 1.4. *De Morgan's Law.*

1. $\mathbf{C}(A_1 \cup A_2) = \mathbf{C}A_1 \cap \mathbf{C}A_2$ and $\mathbf{C}(A_1 \cap A_2) = \mathbf{C}A_1 \cup \mathbf{C}A_2$.

2. $\mathbf{C}(\cup_{n=1}^{N} A_n) = \cap_{n=1}^{N} \mathbf{C}A_n$ and $\mathbf{C}(\cap_{n=1}^{N} A_n) = \cup_{n=1}^{N} \mathbf{C}A_n$.

3. $\mathbf{C}(\cup_{n=1}^{\infty} A_n) = \cap_{n=1}^{\infty} \mathbf{C}A_n$ and $\mathbf{C}(\cap_{n=1}^{\infty} A_n) = \cup_{n=1}^{\infty} \mathbf{C}A_n$.

4. $\mathbf{C}(\cup_{\lambda\in\Lambda} A_\lambda) = \cap_{\lambda\in\Lambda} \mathbf{C}A_\lambda$ and $\mathbf{C}(\cap_{\lambda\in\Lambda} A_\lambda) = \cup_{\lambda\in\Lambda} \mathbf{C}A_\lambda$.

1.1.5 Symmetric Difference

DEFINITION 1.8. *In mathematics, the **symmetric difference**, also knoen as the disjunctive union, of two sets is the set of elements which are in either the sets and not in their intersection. The symmetric difference of the sets A_1 and A_2 is commonly denoted by $A_1 \Delta A_2$, which is equivalent to the union of both relative complements,i.e., $A_1 \Delta A_2 = (A_1\backslash A_2) \cup (A_2 \cap A_1)$.*

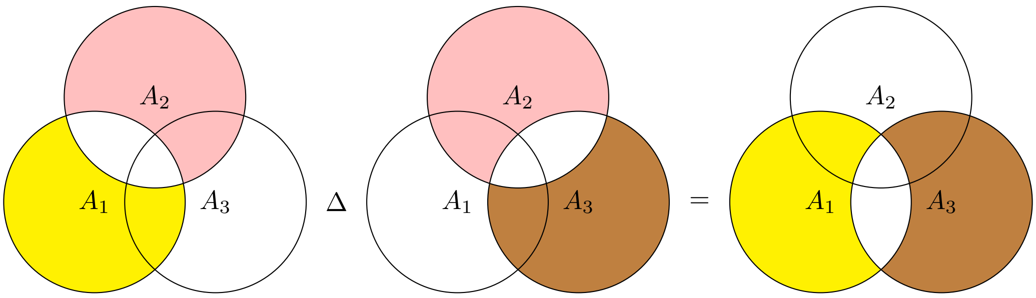
In particular, $A_1 \Delta A_2 \subseteq A_1 \cup A_2$; the equality in this non-strict inclusion occurs of and only if A_1 and A_2 are disjoint sets,i.e., $A_1 \Delta A_2 = A_1 \cup A_2$ if and only if $A_1 \cap A_2 = \varnothing$. Furthermore, $A_1 \Delta A_2$ and $A_1 \cap A_2$ are always disjoint, so $A_1 \Delta A_2$ and $A_1 \cap A_2$ partition $A_1 \cup A_2$.

The symmetric difference is commutative and associative: $A_1 \Delta A_2 = A_2 \Delta A_1$ and $(A_1 \Delta A_2) \Delta A_3 = A_1 \Delta (A_2 \Delta A_3)$. The empty set is neutral, and every set is its own inverse: $A \Delta \varnothing = A$, $A \Delta A = \varnothing$, $A \Delta \mathbf{C}A = S$, $A \Delta S = \mathbf{C}A$.

From the property of inverses in a Boolean group, it follows that the symmetric difference of two repeated symmetric differences is equivalent to the repeated symmetric difference of the join of the two multisets, where for each double set both can re removed. In particular, $(A_1 \Delta A_2) \Delta (A_2 \Delta A_3) = A_1 \Delta A_3$. This implies triangle inequality: the symmetric difference of A_1 and A_3 is contained in the union of the symmetric difference of A_1 and A_2 and that of A_2 and A_3. See the illustrated figure below.

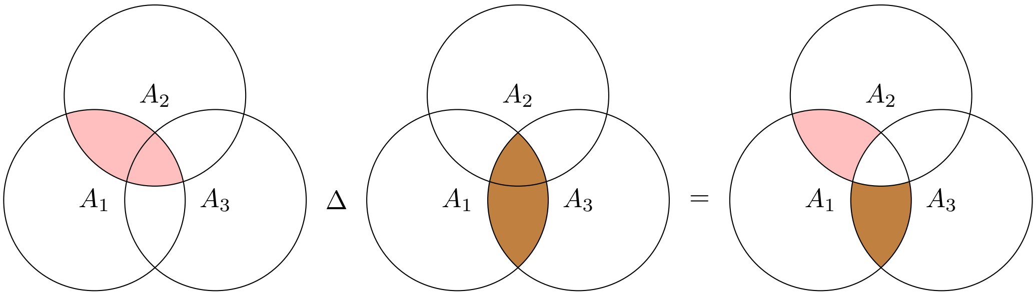
THEOREM 1.5. $(A_1 \Delta A_2) \Delta (A_2 \Delta A_3) = A_1 \Delta A_3$

With associativity, $A_1 \Delta A_2$ can be viewed as a set, then $(A_1 \Delta A_2) \Delta (A_2 \Delta A_3) = [(A_1 \Delta A_2) \Delta A_2] \Delta A_3 = [A_1 \Delta (A_2 \Delta A_2)] \Delta A_3 = (A_1 \Delta \varnothing) \Delta A_3 = A_1 \Delta A_3$.



Intersection distributions over symmetric difference (see illustration):

$$A_1 \cap (A_2 \Delta A_3) = (A_1 \cap A_2)\Delta(A_1 \cap A_3) \tag{1.5}$$



and this shows that the power set of S becomes a ring with symmetric difference as addition and intersection as multiplication. This is the prototypical example of a Boolean ring. Further properties of the symmetric difference:

1. $A_1\Delta A_2 = A_1^c \Delta A_2^c$;
2. $A_1 = A_1\Delta A_2$ if and only if $A_2 = \varnothing$;
3. For any sets A and A', there is a unique set E such that $E\Delta A = A'$, in fact, $E = A\Delta A'$;
4. $(\cup_{\lambda\in\Lambda} A_\lambda)\Delta(\cup_{\lambda\in\Lambda} A'_\lambda) \subseteq \cup_{\lambda\in\Lambda}(A_\lambda \Delta A'_\lambda)$.

One by one proof is as follows.

THEOREM 1.6. $A_1\Delta A_2 = A_1^c\Delta A_2^c$

First, with definition of complement, we say $A^c\backslash B^c = \{x|x\in A^c \text{ but } x \notin B^c\} = \{x \in B \text{ but } x\notin A\}$. With De Morgan's Law,

$$\begin{aligned} A_1^c\Delta A_2^c &= (A_1^c \cup A_2^c)\backslash(A_1^c \cap A_2^c) \\ &= (A_1\cap A_2)^c\backslash(A_1\cup A_2)^c \\ &= (A_1\cup A_2)\backslash(A_1\cap A_2) \\ &= A_1\Delta A_2 \end{aligned} \tag{1.6}$$

THEOREM 1.7. $A_1 = A_1\Delta A_2$ *if and only if* $A_2 = \varnothing$

Sufficiency: if $A_2 = \varnothing$, $A_1\cup A_2 = A_1$, $A_1\cap A_2 = \varnothing$. So, $A_1\Delta A_2 = (A_1\cup A_2)\backslash(A_1\cap A_2) = A_1\backslash\varnothing = A_1$.

Necessity (counter-evidence): we assume $A_2 \neq \varnothing$, i.e., $\exists x \in A_2$ and $x\in A_1$ or $x\notin A_1$. (1) If $x\in A_1$, $x\in A_1\cap A_2$, so $x\notin A_1\Delta A_2 = (A_1\cup A_2)\backslash(A_1\cap A_2)$. But with $A_1\Delta A_2 = A_1$, x should belong to $A_1\Delta A_2$. (2) If $x\notin A_1$, $x\in A_2\cap A_1$. Thus, $x\in(A_1\backslash A_2)\cup(A_2\backslash A_1) = A_1\Delta A_2 = A_1$, which is conflicted to $x\notin A_1$. In this way, if $A_1 = A_1\Delta A_2$, the assumption $A_2\neq\varnothing$ cannot hold, which indicates that $A_2 = \varnothing$.

THEOREM 1.8. *For any sets A and A', there is a unique set E such that $E\Delta A = A'$, in fact, $E = A\Delta A'$.*

Sufficiency: If $E = A\Delta A'$, $E\Delta A = A'$.

$$A\Delta A'\Delta A = [(A\backslash A')\cup(A'\backslash A)]\Delta A = X\backslash Y \tag{1.7}$$

$$\begin{aligned} X &= [(A\backslash A')\cup(A'\backslash A)]\cup A = (A\backslash A')\cup[(A'\backslash A)\cup A] \\ &= (A\backslash A')\cup(A\cup A') = A\cup A' \end{aligned} \tag{1.8}$$

$$\begin{aligned} Y &= [(A\backslash A')\cup(A'\backslash A)]\cap A = [(A\backslash A')\cap A]\cup[(A'\backslash A)\cap A] \\ &= (A\backslash A')\cup\varnothing = A\backslash A' \end{aligned} \tag{1.9}$$

$$\Rightarrow A\Delta A'\Delta A = (A\cup A')\backslash(A\backslash A') = A' \tag{1.10}$$

Notice that $A\cup A' = \{x|x\in A \text{ or } x\in A'\}$, $A\backslash A' = \{x|x\in A \text{ but } x\notin A'\}$, so $(A\cup A')\backslash(A\backslash A') = \{x|x\in A\} = A$. To prove that clearly, we need $A\backslash(B\backslash C) = (A\backslash(B\cup C))\cup(A\cap C)$, which is easy to obtain from the venn figure (not shown).

Necessity (counter-evidence) : we assume $E\neq A\Delta A'$. If $E=\varnothing$, $E\Delta A = A\neq A'$. If $E\neq\varnothing$, $\exists x\in E$ but $x\notin A\Delta A'$ or $\exists x\in A\Delta A'$ but $x\notin A$.

If $\exists x \in E$ *but* $x \notin A\Delta A'$ *holds,* x *must belong to* $(A\Delta A')^c = (A\cap A')\cup(A\cup A')^c$*. If* $x \in A\cap A' \subseteq A$*, we have* $x \in E\cap A$*, so* $x \notin E\Delta A = A'$*. But* $x \in A\Delta A' \subseteq A'$*, which is conflicted to* $x \notin A'$*. If* $x \in (A\cup A')^c$*, which is disjoint with* A*, we have* $x \notin A$*. With* $x \in E, x \notin A$*,* x *must belong to* $x \in E\Delta A = A'$*, which is conflicted to* $x \notin A'(([A\cup A')^c]\cap A' = \varnothing)$*. In this way, if* $E\Delta A = A'$*, the assumption* $E \neq A\Delta A'$ *cannot hold, which indicates that* $E = A\Delta A'$*.*

If $\exists x \in A\Delta A'$ *but* $x \notin E$ *holds. If* $x \in A$*, we have* $x \in E\Delta A = A'$*, so* $x \notin A\Delta A'$*, which is conflicted to* $x \in A \in A'$*. If* $x \notin A$*,* x *must belong to* A' *because* $x \in A\Delta A'$*. Then* $x \in A' = E\Delta A$*, so that* x *must belong to* E *under the condition* $x \notin A$*, which is conflicted to* $x \notin E$*.*

Three possible situation induce the same outcome conflicted to the assumption. Therefore, the necessity are proved.

THEOREM 1.9. $(\cup_{\lambda\in\Lambda}A_\lambda)\Delta(\cup_{\lambda\in\Lambda}A'_\lambda) \subseteq \cup_{\lambda\in\Lambda}(A_\lambda\Delta A'_\lambda)$

For $\forall x \in (\cup_{\lambda\in\Lambda}A_\lambda)\Delta(\cup_{\lambda\in\Lambda}A'_\lambda)$*,* x *satisfies* $x \in A_i, x \notin \cup_{\lambda\in\Lambda}A'_\lambda$ *or* $x \in A'_j, x \notin \cup_{\lambda\in\Lambda}A_\lambda$*(*$i,j \in \Lambda$*). Therefore, we get* $x \in A_i, x \notin A'_i \Rightarrow x \in (A_i\Delta A'_i)$ *or* $x \in A'_j, x \notin A_j \Rightarrow x \in (A_j\Delta A'_j)$*. So* $x \in \cup_{\lambda\in\Lambda}(A_\lambda\Delta A'_\lambda)$*.*

With operations of set, specified set can be expressed elegantly. Let A_1, A_2 and A_3 be three sets, then

1. $(A_1\cap A_2)\cup(A_2\cap A_3)\cup(A_3\cap A_1)$ represents a set formed by all elements belonging to at least two sets in A_1, A_2 and A_3;
2. $(A_1\cup A_2\cup A_3)\backslash(A_1\Delta A_2\Delta A_3)$ indicates a set formed by all elements belonging to at least two sets by no three sets in A_1, A_2 and A_3;
3. $(A_1\Delta A_2\Delta A_3)\backslash(A_1\cap A_2\cap A_3)$ indicates a set formed by all elements belonging to one set but not two sets in A_1, A_2 and A_3.

There are some examples to reinforce understanding of sets.

THEOREM 1.10. *Let* A *and* A *be two subsets of the whole set* S*. If, for any* $E \subseteq S, E\cap A = E\cup A'$*, then* $A = S$ *and* $A' = \varnothing$*.*

If $E = S$*, then* $A = S$*; Take* $E = \mathbf{C}A$*, then it is assumed that* $\varnothing = \mathbf{C}A\cap A = \mathbf{C}A\cup A' \supseteq A'$*, that is* $A' = \varnothing$*.*

THEOREM 1.11. *Let* A_1 *and* A_1 *be two sets, then* $A_1 = A_2$ *if and only if set* A_3 *such that* $A_1\cap A_3 = A_2\cap A_3$ *and* $A_1\cup A_3 = A_2\cup A_3$*.*

The necessity is obvious. Now we prove the sufficiency.

$$A_1 = A_1\cup(A_3\cap\mathbf{C}A_3) = (A_1\cap A_3)\cup(A_1\cap\mathcal{C}A_3) \tag{1.11}$$
$$A_2 = A_2\cup(A_3\cap\mathbf{C}A_3) = (A_2\cap A_3)\cup(A_2\cap\mathcal{C}A_3) \tag{1.12}$$

then $A_1\cup A_3 = (A_1\cap\mathbf{C}A_3)\cup A_3$ *and* $A_2\cup A_3 = (A_2\cap\mathbf{C}A_3)\cup A_3$*. Note that* $A_3\cap(A_1\cap\mathbf{C}A_3) = \varnothing = A_3\cap(A_2\cap A_3)$*, then* $A_1\cap\mathcal{C}A_3 = A_2\cap\mathbf{C}A_3$*. Thus* $A_1 = (A_1\cap A_3)\cup(A_1\cap\mathbf{C}A_3) = (A_2\cap A_3)\cup(A_2\cap\mathbf{C}A_3) = A_2$*.*

THEOREM 1.12. *Let* A, A' *and* E *be three subsets of the whole set* S*, then* $A' = \mathbf{C}(E\cap A)\cap(\mathbf{C}E\cup A)$ *if and only if* $\mathbf{C}A' = E$*.*

We can rewrite E *with* A *and implement De Morgan's Law to alter its expression equivalently.*

$$E = E\cup(A\cap\mathbf{C}A) = (E\cap A)\cup(E\cap\mathbf{C}A) \tag{1.13}$$
$$\begin{aligned}\mathbf{C}E &= \mathbf{C}(E\cap A)\cap\mathbf{C}(E\cap\mathbf{C}A) \\ &= \mathbf{C}(E\cap A)\cap(\mathbf{C}E\cup A) \\ &= A'\end{aligned} \tag{1.14}$$
$$\mathbf{C}A' = E \Leftrightarrow \mathbf{C}E = A$$

Note that operations above (including complement) are reversible, so the sufficiency and necessity are proved simultaneously.

THEOREM 1.13. *If* $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots, A'_1 \subseteq A'_2 \subseteq \cdots \subseteq A'_n \subseteq \cdots$*, then* $(\cup_{n=1}^\infty A_n)\cap(\cup_{n=1}^\infty A'_n) = \cup_{n=1}^\infty(A_n\cap A'_n)$*.*

Induction: Let M_k *denote* $\cup_{n=1}^k A_n$*,* N_k *denote* $\cup_{n=1}^k A'_n$ *and* P_k *denote* $\cup_{n=1}^k(A_n\cap A'_n)$*. When* $s = 1$ $M_1\cap N_1 = P_1$ *is obvious. When* $s = k$*, we assume that* $M_k\cap N_k = P_k$ *holds. When* $s = k+1$*,* $M_{k+1}\cap N_{k+1} = (M_k\cup A_{k+1})\cap(N_k\cup A'_{k+1}) = [(M_k\cup A_{k+1})\cap N_k]\cup[(M_k\cup A_{k+1})\cap A'_{k+1}] = (M_k\cap N_k)\cup(A_{k+1}\cap A'_{k+1}) = P_{k+1}$*(Note that* $M_k \subseteq A_{k+1}$*). In the way, we proved that* $\forall s \in \mathbb{N}^+, M_s\cap N_k = P_k$*. So* $(\cup_{n=1}^\infty A_n)\cap(\cup_{n=1}^\infty A'_n) = \cup_{n=1}^\infty(A_n\cap A'_n)$ *holds.*

THEOREM 1.14. *If $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots, A'_1 \supseteq A'_2 \supseteq \cdots \supseteq A'_n \supseteq \cdots$, then $(\cap_{n=1}^{\infty} A_n) \cup (\cap_{n=1}^{\infty} A'_n) = \cap_{n=1}^{\infty}(A_n \cup A'_n)$.*

Induction: Let M_k denote $\cap_{n=1}^{k} A_n$, N_k denote $\cap_{n=1}^{k} A'_n$, P_k denote $\cap_{n=1}^{k}(A_n \cup A'_n)$. When $s = 1$, $M_1 \cup N_1 = P_1$ is obvious. When $s = k$, we assume that $M_k \cup N_k = P_k$ holds. When $s = k+1$, $M_{k+1} \cup N_{k+1} = (M_k \cap A_{k+1}) \cup (N_k \cap A'_{k+1}) = [(M_k \cap A_{k+1}) \cup N_k] \cap [(M_k \cap A_{k+1}) \cup A'_{k+1}] = (M_k \cup N_p) \cap (A_{k+1} \cup A'_{k+1}) = P_{k+1}$(note that $A_{k+1} \subseteq M_k$). In the way, we proved that $\forall s \in \mathbb{N}^+, M_s \cup N_k = P_k$. So $(\cap_{n=1}^{\infty} A_n) \cup (\cap_{n=1}^{\infty} A'_n) = \cap_{n=1}^{\infty}(A_n \cup A'_n)$ holds.

THEOREM 1.15. *Let A and A' and E and E' be three subsets of the whole set S.(1) If $A \cup A' = E \cup E'$, $A \cap E' = \varnothing$ and $A' \cap E = \varnothing$, then $A = E$ and $A' = E'$;(2) If $A \cup A' = E \cup E'$, let $A_1 = A \cap E, A_2 = A \cap E'$, then $A_1 \cup A_2 = A$.*

(1) If $A = \varnothing$, we get $A \cup A' = A' = E \cup E'$. Then $A' \cap E = (E \cup E') \cap E = E = \varnothing$, so that $A = E$. If $A \neq \varnothing$, $\exists x \in A \subseteq A \cup A' = E \cup E'$. If $x \in E'$, we obtain $x \in A \cap E'$, which is conflicted to $A \cap E' = \varnothing$. So $x \in E$, $A \subseteq E$. In the same way, we get $E \subseteq A$. So $A = E$. Similarly, we can prove $A' = E'$(not shown).

(2) $A_1 \cup A_2 = (A \cap E) \cup (A \cap E') = A \cap (E \cup E') = A \cap (A \cup A') = A$

1.2 Limit of Sequence of Sets

DEFINITION 1.9. *Let $\{A_n : n \in \mathbb{N}\}$ be a set sequence. If $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$, this sequence is called a* ***decreasing sequence of sets****. In this case, we call its intersection $\bigcap\limits_{n=1}^{\infty} A_n$ as the limit sets of the set sequence $\{A_n : n \in \mathbb{N}\}$, denoted as $\lim\limits_{n\to\infty} A_n$.*

If $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$, this sequence is called a ***increasing sequence of sets****. In this case, we call its intersection $\bigcup\limits_{n=1}^{\infty} A_n$ as the limit sets of the set sequence $\{A_n : n \in \mathbb{N}\}$, denoted as $\lim\limits_{n\to\infty} A_n$.*

Example 1.3. *If $A_n = [n, \infty)$ for $n = 1, 2, \cdots$, then $\lim\limits_{n\to\infty} A_n = \varnothing$.*

THEOREM 1.16. *Let $\{f_n : n \in \mathbb{N}\}$ be an increasing sequence of real-value functions on $\mathbb{R}$, $f_1(x) \leq f_2(x) \leq \cdots \leq f_n(x) \leq \cdots$ and $\lim\limits_{n\to\infty} f_n(x) = f(x)$. For any given real number t, define a sequence of sets $E_n = \{x \in \mathbb{R} | f_n(x) > t\}, n \in \mathbb{N}$ then $E_1 \subseteq E_2 \subseteq \cdots \subseteq \cdots$, and*

$$\lim_{n\to\infty} E_n = \bigcup_{n=1}^{\infty}\{x \in \mathbb{R} | f_n(x) > t\} = \{x \in R | f(x) > t\} \tag{1.15}$$

that is,

$$\lim_{n\to\infty}\{x \in \mathbb{R} | f_n(x) > t\} = \{x \in \mathbb{R} | f_n(x) > t\} \tag{1.16}$$

DEFINITION 1.10. *Let $\{A_n : n \in \mathbb{N}\}$ be a sequence of sets. Define*

$$B_N = \bigcup_{n\geq N} A_n = \bigcup_{n=N}^{\infty} A_n, N \in \mathbb{N} \tag{1.17}$$

then $B_1 \supseteq B_2 \supseteq \cdots \supseteq B_n \supseteq \cdots$, $\{B_n : n \in \mathbb{N}\}$ is a decreasing sequence of sets,

$$\lim_{n\to\infty} B_n = \bigcap_{N=1}^{\infty} B_N = \bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty} A_n \tag{1.18}$$

is called the upper limit sets of sequence of sets $\{A_n : n \in \mathbb{N}\}$, and is recorded as $\overline{\lim\limits_{n\to\infty}} A_n = \bigcap\limits_{N=1}^{\infty}\bigcup\limits_{n=N}^{\infty} A_n$.

Similarly, define

$$B_N = \bigcap_{n\geq N} A_n = \bigcap_{n=N}^{\infty} A_n, N \in \mathbb{N} \tag{1.19}$$

then $B_1 \subseteq B_2 \subseteq \cdots \subseteqq B_n \subseteq \cdots$, $\{B_n : n \in \mathbb{N}\}$ is an increasing sequence of sets,

$$\lim_{n\to\infty} B_n = \bigcup_{N=1}^{\infty} B_N = \bigcup_{N=1}^{\infty}\bigcap_{n=N}^{\infty} A_n \tag{1.20}$$

is called the lower limit sets of sequence of sets $\{A_n : n \in \mathbb{N}\}$, and is recorded as $\underline{\lim\limits_{n\to\infty}} A_n = \bigcup\limits_{N=1}^{\infty}\bigcap\limits_{n=N}^{\infty} A_n$.

If the upper and lower limit sets of sequence of sets $\{A_n : n \in \mathbb{N}\}$ are equal, we call that its limit set exists and is equal to the upper and lower limit sets, $\lim\limits_{n\to\infty} A_n$.

THEOREM 1.17. *Let $\{A_n : n \in \mathbb{N}\}$ be a sequence of sets, then*

1. $\overline{\lim_{n\to\infty}} A_n = \{x : \forall N \in \mathbb{N}, \exists n \geq N, x \in A_n\}$;

2. $\underline{\lim_{n\to\infty}} A_n = \{x : \exists N \in \mathbb{N}, \forall n \geq N, x \in A_n\}$.

That is, the upper limit set of $\{A_n : n \in \mathbb{N}\}$ is formed by elements belonging to an infinite number of sets in $\{A_n : n \in \mathbb{N}\}$, the lower limit set of $\{A_n : n \in \mathbb{N}\}$ is formed by elements belonging to all but finite many sets in $\{A_n : n \in \mathbb{N}\}$. Therefore,

$$\overline{\lim_{n\to\infty}} A_n \supseteq \underline{\lim_{n\to\infty}} A_n \tag{1.21}$$

Let us solve some problems.

THEOREM 1.18. *Let $A_n = \{\frac{m}{n} | m \in \mathbb{Z}\}$ and $B_n = (\frac{1}{n}, 1 + \frac{1}{n})$ for $n \in \mathbb{N}$, then*

$$\overline{\lim_{n\to\infty}} A_n = \mathbb{Q}, \underline{\lim_{n\to\infty}} A_n = \mathbb{Z}, \lim_{n\to\infty} B_n = (0, 1]. \tag{1.22}$$

Proof. (1) First we show $\overline{\lim_{n\to\infty}} A_n \supseteq \mathbb{Q}$. Let $B_N = \bigcup\limits_{n=N}^{\infty} A_n$, $\overline{\lim_{n\to\infty}} A_n = \bigcap\limits_{N=1}^{\infty} B_N$. $\forall x \in \mathbb{Q}, \exists m \in \mathbb{Z}, n \in \mathbb{N}_+$ such that $x = \frac{m}{n}$. $\forall N \in \mathbb{N}_+, x = \frac{m}{n} = \frac{m \cdot N}{n \cdot N} \in A_{n \cdot N} \subseteq B_N$, i.e., $x \in B_N$. Therefore, $x \in \overline{\lim_{n\to\infty}} A_n$.

Next we show $\overline{\lim_{n\to\infty}} A_n \subseteq \mathbb{Q}$, $\forall x \in \overline{\lim_{n\to\infty}} A_n$ implies $\forall N \in \mathbb{N}, x \in B_N = \bigcup\limits_{n=N}^{\infty} A_n$, i.e., $x = \frac{m}{n}, n > N, m \in \mathbb{Z}$. Thus $x \in \mathbb{Q}$. Hence, we reach the conclusion.

(2) First we show $\underline{\lim_{n\to\infty}} A_n \supseteq \mathbb{Z}$. Let $B_N = \bigcap\limits_{n=N}^{\infty} A_n$, $\underline{\lim_{n\to\infty}} A_n = \bigcup\limits_{N=1}^{\infty} B_N$. $\forall x \in \mathbb{Z}, n \in \mathbb{N}_+$, we have either $x = 0 = \frac{0}{n} \in A_n$ or $x = \frac{x \cdot n}{n} \in A_n$, i.e, $\forall N \in \mathbb{N}_+, x \in B_N$ (stronger than $\exists N, x \in B_N$). Therefore, we have $x \in \underline{\lim_{n\to\infty}} A_n$.

Next we show $\underline{\lim_{n\to\infty}} A_n \subseteq \mathbb{Z}$. Suppose $\exists x \notin \mathbb{Z}$ but $x \in \underline{\lim_{n\to\infty}} A_n$, i.e., $\exists N \in \mathbb{N}_+, \forall n > N, x \in A_n$. Then $x = \frac{m}{n}, m \in \mathbb{Z}, n \in \mathbb{N}_+ \backslash \{1\}$ and n is not n factor of m (or otherwise, $x \in \mathbb{Z}$). If $\exists N \in \mathbb{N}_+, x \in A_N$, n must be a factor or N (or otherwise one cannot eliminate n to get a pure $z \in \mathbb{Z}$). As n is a factor of N and $n > 1$, then n cannot be a factor of $N + 1$ as $(N + 1) - N < n$. Therefore $x \notin A_{N+1}$, which contradicts to $x \in \underline{\lim_{n\to\infty}} A_n$. Hence we reach the conclusion.

(3) We will show $\overline{\lim_{n\to\infty}} B_n = \underline{\lim_{n\to\infty}} B_n = (0, 1]$. Note that $\forall n < N, B_n \supseteq B_N$. Let $C_N = \bigcup\limits_{n=N}^{\infty} B_n = (\frac{1}{N}, 1 + \frac{1}{N})$. As $\lim\limits_{N\to\infty} \frac{1}{N} = 0$, $\bigcap\limits_{N=1}^{\infty} C_N = (0, 1]$, i.e., $\overline{\lim_{n\to\infty}} B_n = (0, 1]$. Let $D_N = \bigcap\limits_{n=N}^{\infty} A_n = (0, 1]$, then $\underline{\lim_{n\to\infty}} B_n = \bigcup\limits_{N=1}^{\infty} D_N = (0, 1]$.

We used $\bigcap\limits_{n=1}^{\infty} (\frac{1}{n}, 1 + \frac{1}{n}) = (0, 1]$ (obvious). ☐

THEOREM 1.19. *Let $\{f_n\}$ and f be real-valued functions defined on $\mathbb{R}$, and*

$$\lim_{n\to\infty} f_n(x) = f(x), x \in \mathbb{R}, \tag{1.23}$$

then, for $t \in \mathbb{R}$,

$$\{x \in \mathbb{R} | f(x) \leq t\} = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{x \in \mathbb{R} | f_n(x) < t + \frac{1}{k}\}. \tag{1.24}$$

Proof. Let $M = \{x \in \mathbb{R} | f(x) \leq t\}, A_n = \{x \in \mathbb{R} | f_n(x) < t + \frac{1}{k}\}, B_N = \bigcap\limits_{n=N}^{\infty} A_n, C_k = \bigcup\limits_{N=1}^{\infty}, D = \bigcap\limits_{k=1}^{\infty} C_k$. We first show that $M \subseteq D$. $\forall x \in M, f(x) \leq t$. As $\forall k \in \mathbb{N}_+, \frac{1}{2k} > 0$, then $f(x) < t + \frac{1}{2k}$. As $\lim\limits_{n\to\infty} f_n(x) = f(x)$, we have for $\varepsilon = \frac{1}{2k}, \exists N \in \mathbb{N}_+, \forall n > N, |f_n(x) - f(x)| < \varepsilon$, i.e., $f_n(x) < f(x) + \frac{1}{2k} < t + \frac{1}{k}$, i.e., $x \in C_k, \forall k \in \mathbb{N}_+$. Therefore, $x \in D$.

Then we show $D \subseteq M$. $\forall x \in D$, i.e., $\forall k \in \mathbb{N}_+, x \in C_k$. Thus, $\exists N \in \mathbb{N}_+, \forall n > N, f_n(x) < t + \frac{1}{k}$. From $\lim\limits_{n\to\infty} f_n(x) = f(x), \forall N \in \mathbb{N}$, we can find a $\epsilon > 0$ satisfying $|f_n - f(x)| < \varepsilon$, i.e., $f_n(x) - f(x) > -\varepsilon$ (or otherwise, $f_n(x) - f(x)$ is not bounded which contradicts to this sequence has a limit 0). Therefore, $-\varepsilon < f_n(x) - f(x) < t + \frac{1}{k} - f(x), f(x) < t + \frac{1}{k} + \varepsilon$. Obviously $\lim\limits_{N\to\infty} \varepsilon = 0$. So $f(x) < t + \frac{1}{k}, \forall k \in \mathbb{N}_+$, i.e., $f(x) \leq t$. Then $x \in M$. Hence We reach the conclusion. ☐

THEOREM 1.20. *If $a_n \to a(n \to \infty)$, then*

$$\bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} (a_n - \frac{1}{k}, a_n + \frac{1}{k}) = \{a\}. \tag{1.25}$$

Proof. With Example 1.19, it is easy to prove this. First let $f_n(x) = a_n, \forall n \in \mathbb{N}_+, A_1 = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{t \in \mathbb{R} | a_n < t + \frac{1}{k}\}$. Then $A_1 = [a, -\infty)$. Next let $f_n(x) = -a_n, \forall n \in \mathbb{N}_+, A_2 = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{t \in \mathbb{R} | -a_n < -t + \frac{1}{k}\}$. Then $-A_2 = [-a, \infty), A_2 = (-\infty, a]$. With $A_1 \cap A_2 = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} (a_n - \frac{1}{k}, a_n + \frac{1}{k})$ and $A_1 \cap A_2 = \{a\}$, we reach the conclusion. □

DEFINITION 1.11. *A collection $\mathscr{X}$ of subsets of S forms a **partition** of S if they are **pairwise disjoint** (i.e., if any two different subsets belonging to $\mathscr{X}$ are disjoint) and if their union is S. That is, if each element of S belongs to one and only one of the subsets in $\mathscr{X}$. A partition of a set corresponds to the familiar ides of a classification of its elements.*

A collection $\mathscr{X} = \{X_\gamma | \gamma \in \Gamma\}$ of subsets of S forms a partition of S if

1. $\forall \gamma \in \Gamma, \varnothing \neq X_\gamma \subseteq S$;
2. $X_\gamma \cap X_{\gamma'} = \varnothing, \gamma \neq \gamma'$;
3. $S = \bigcup_{\gamma \in \Gamma} X_\gamma$.

DEFINITION 1.12. *Consider two sets S_1 and S_2; the set of pairs (x_1, x_2) where $x_1 \in S_1, x_2 \in S_2$, is called their **Cartesian product** $S_1 \times S_2$. The order is essential.*

*More generally, consider N sets $S_1, S_2, \cdots, S_N$. The set of N-tuples $(x_1, x_2, \cdots, x_N)$, where $x_n \in S_n, n = 1, 2, \cdots, N$, is the **Cartesian product** $S_1 \times S_2 \times \cdots \times S_N$, also denoted $\prod_{n=1}^{N} S_n$. The order is essential. The N-tuple $(x_1, x_2, \cdots, x_N)$ is denoted by (x_n), called the nth **coordinate** (or the $n$the component) of (x_n).*

THEOREM 1.21. *For family $\mathscr{A} = \{A_\lambda | \lambda \in \Lambda\}$ we have*

$$A_\lambda \neq \varnothing, \forall \lambda \in \Lambda \Leftrightarrow \prod_{\lambda \in \Lambda} A_\lambda \neq \varnothing. \tag{1.26}$$

Proof. Necessity: $\forall \lambda \in \Lambda, A_\lambda \neq \varnothing$ implies $\exists a_\lambda \in A_\lambda$. Then we have $(a_{\lambda_1}, a_{\lambda_2}, \cdots, a_{\lambda_k})$ where $\lambda_i \neq \lambda_j, i \neq j, k = |\Lambda|$ and $\lambda_i \in \Lambda, i = 1, 2, \cdots, k$, which is just one element of $\prod_{\lambda \in \Lambda} A_\lambda$. Thus, $\prod_{\lambda \in \Lambda} A_\lambda \neq \varnothing$.

Sufficiency: $\prod_{\lambda \in \Lambda} A_\lambda \neq \varnothing$ implies $\exists (a_{\lambda_1}, a_{\lambda_2}, \cdots, a_{\lambda_k})$ where $a_{\lambda_i} \in A_{\lambda_i}, i = 1, 2, \cdots, k$, i.e., $A_\lambda \neq \varnothing, \forall \lambda \in \Lambda$. □

THEOREM 1.22. *If two families $\mathscr{A} = \{A_\lambda | \lambda \in \Lambda\}$ and $\mathscr{A}' = \{A'_\lambda | \lambda \in \Lambda\}$ have the same index set, then*

$$\begin{cases} A_\lambda \neq \varnothing, \forall \lambda \in \Lambda \\ \prod_{\lambda \in \Lambda} A_\lambda \subseteq \prod_{\lambda \in \Lambda} A'_\lambda \end{cases} \Leftrightarrow \varnothing \subset A_\lambda \subseteq A'_\lambda, \forall \lambda \in \Lambda \tag{1.27}$$

and

$$\prod_{\lambda \in \Lambda} A_\lambda \cup \prod_{\lambda \in \Lambda} A'_\lambda \subseteq \prod_{\lambda \in \Lambda} (A_\lambda \cup A'_\lambda) \tag{1.28}$$

$$\prod_{\lambda \in \Lambda} A_\lambda \cap \prod_{\lambda \in \Lambda} A'_\lambda = \prod_{\lambda \in \Lambda} (A_\lambda \cap A'_\lambda). \tag{1.29}$$

Proof. (1) We firstly show that 1.27 holds. Necessity: a. $A_\lambda \neq \varnothing, \forall \lambda \in \Lambda$ implies $\varnothing \subset A_\lambda, \forall \lambda \in \Lambda$. b. With $\prod_{\lambda \in \Lambda} A_\lambda \subseteq \prod_{\lambda \in \Lambda} A'_\lambda, \forall x \in \prod_{\lambda \in \Lambda} A_\lambda, x = (a_1, a_2, \cdots, a_k), a_i = A_{\lambda_i}, i = 1, 2, \cdots, k, k = |\Lambda|$, we have $x \in \prod_{\lambda \in \Lambda} A'_\lambda$, i.e., $a_i \in A'_{\lambda_i}, i = 1, 2, \cdots, k$. Thus, $A_\lambda \subseteq A'_\lambda, \forall \lambda \in \Lambda$ as $\forall x$ covers all elements in $\mathscr{A}$.

Sufficiency: a. $\varnothing \subset A_\lambda, \forall \lambda \in \Lambda$ implies there is at least element in every element of $\mathscr{A}$, i.e., $A_\lambda \neq \varnothing, \forall \lambda \in \Lambda$. b. With $A_\lambda \subseteq A'_\lambda, \forall \lambda \in \Lambda$, we derive a partition of A'_λ with (A_λ, A^c_λ). Then, $\prod_{\lambda \in \Lambda} A'_\lambda = \prod_{\lambda \in \Lambda} (A_\lambda \cup A^c_\lambda) =$

$(\prod_{\lambda\in\Lambda} A'_\lambda) \cup (\prod_{\lambda\in\Lambda} A^c_\lambda) \cup ((\prod_{\lambda\in\Lambda_1} A_\lambda) \times (\prod_{\lambda\in\Lambda_1^c} A^c_\lambda))$ where (Λ_1, Λ_1^c) is a causal partition of Λ. Thus, we have $\varnothing \subset A_\lambda \subseteq A'_\lambda, \forall\lambda \in \Lambda$. Hence we reach the conclusion.

(2) Next we prove 1.28. $\forall x \in \prod_{\lambda\in\Lambda} A_\lambda \cup \prod_{\lambda\in\Lambda} A'_\lambda$, $x = (a_1, a_2, \cdots, a_k), k = |\Lambda|$, we have either $a_i \forall A_{\lambda_i}, \forall i = 1, 2, \cdots, k$ or $a_i \forall A'_{\lambda_i}, \forall i = 1, 2, \cdots, k$. Then $a_i \in A_{\lambda_i} \cup A'_{\lambda_1}, i = 1, 2, \cdots, k$, i.e., $x \in\subseteq \prod_{\lambda\in\Lambda} (A_\lambda \cup A'_\lambda)$. Hence we reach the conclusion.

(3) Lastly we prove 1.29. $\prod_{\lambda\in\Lambda} A_\lambda \cap \prod_{\lambda\in\Lambda} A'_\lambda = \prod_{\lambda\in\Lambda} (A_\lambda \cap A'_\lambda) \Leftrightarrow \forall x = (a_1, a_2, \cdots, a_k), k = |\Lambda|$ we have $a_i \in A_{\lambda_i} \cap A'_{\lambda_i} \Leftrightarrow \prod_{\lambda\in\Lambda} (A_\lambda \cap A'_\lambda)$. □

1.3 Functions and Correspondences

DEFINITION 1.13. *Let S and T be two sets; if with each element $x \in S$ in associated one and only one element $y \in T$, a **function** from S to T is defined. f is also called a **transformation** of S into T. x is the **variable**, y is the $image$ of x by f, or the **transform** of x by f, or thr value of f at x, and one writes $y = f(x)$ (read y equals f of x), or $x \to f(x)$.*

DEFINITION 1.14. *Consider in $S \times T$ the set of elements (x, y) for which $y = f(x)$. This subsets of $S \times T$ is called the **graph** of the function f.*

$$G(f) = \{(x, y) \in S \times T | y = f(x)\} \tag{1.30}$$

DEFINITION 1.15. *Let X be a subset of S; take the image $y = f(x)$ of each $x \in X$. The set of the images so obtained as a subset of T called the **image** of X and denoted $f(X)$.*

$$f(X) = \{f(x) \in T | x \in T\} \tag{1.31}$$

*If $f(S)$ consists of one element of T, in other words, if all the elements of S have the same image in T, the function is said to be **constant**. If $f(S) = T$, in other words, if each element of T is rhe image of some elements of S, f is said to be a function from S **onto** T, i.e., function $f : S \to T$ is a **surjective (onto)** mapping.*

THEOREM 1.23. *If f is a function from a set S to a set T, and if $\mathscr{X}$ is a collection of subsets of S, then*

1. $f(\bigcup_{X\in\mathscr{X}} X) = \bigcup_{X\in\mathscr{X}} f(X)$
2. $f(\bigcap_{X\in\mathscr{X}} X) \subseteq \bigcap_{X\in\mathscr{X}} f(X)$

Proof. (1) $f(\bigcup_{X\in\mathscr{X}} X) = \{f(x) | x \in X, X \in \mathscr{X}\} = \bigcup_{X\in\mathscr{X}} \{f(x) | x \in X\} = \bigcup_{X\in\mathscr{X}} f(X)$.

(2) $\forall y \in f(\bigcap_{X\in\mathscr{X}} X)$, $\exists x \in \bigcap_{X\in\mathscr{X}} X$ satisfying $y = f(x)$. Thus $y \in \bigcap_{X\in\mathscr{X}} f(X)$. Then we have $f(\bigcap_{X\in\mathscr{X}} X) \subseteq \bigcap_{X\in\mathscr{X}} f(X)$. □

Remark: If $\exists x_1 \sim X_1, x_2 \sim X_2, x_1 \notin X_2, x_2 \notin X_1, X_1, X_2 \subseteq \mathscr{X}$, and $x_1 \neq x_2, f(x_1) = f(x_2) = f_0 =\neq f(x), \forall x \in \mathscr{X}$. Thus, $x_1, x_2 \notin X_1 \cap X_2, f_0 \notin f(\bigcap_{X\in\mathscr{X}})$ but $f_0 \in \cap_{X\in\mathscr{X}} f(X)$.

DEFINITION 1.16. *Conversely, let Y be a subset of T; the set of $x \in S$ which have their images in Y is a subset of S called the **inverse image** of Y denoted $f^{-1}(Y)$.*

THEOREM 1.24. *If f is a function from a set S to a set T, and if $\mathscr{Y}$ is a collection of subsets of T, then is is easy to prove that:*

1. $f^{-1}(\bigcup_{Y\in\mathscr{Y}}) = \bigcup_{Y\in\mathscr{Y}} f^{-1}(Y)$.
2. $f^{-1}(\bigcap_{Y\in\mathscr{Y}}) = \bigcap_{Y\in\mathscr{Y}} f^{-1}(Y)$.
3. $f^{-1}(Y \backslash Y') = f^{-1}(Y) \backslash f^{-1}(Y')$ *for* $Y' \subseteq Y \subseteq T$.

Proof. (1) This is immediate from the theorem 1.23.

(2) $f^{-1}(\underset{Y\in\mathscr{Y}}{\cap}) \subseteq \underset{Y\in\mathscr{Y}}{\cap} f^{-1}(Y)$ is immediate from the theorem 1.23. $\forall x \in \underset{Y\in\mathscr{Y}}{\cap} f^{-1}(Y)$, $\exists x \in S$ satisfying $y = f(x), y \in \underset{Y\in\mathscr{Y}}{\cap} Y$. Thus, $x \in f^{-1}(\underset{Y\in\mathscr{Y}}{\cap})$, so $f^{-1}(\underset{Y\in\mathscr{Y}}{\cap}) \supseteq \underset{Y\in\mathscr{Y}}{\cap} f^{-1}(Y)$. Therefore, we have $f^{-1}(\underset{Y\in\mathscr{Y}}{\cap}) = \underset{Y\in\mathscr{Y}}{\cap} f^{-1}(Y)$.

(3) $(\forall) x \in f^{-1}(Y\backslash Y'), \exists y \in Y\backslash Y', y = f(x) \Leftrightarrow (\forall) x \in f^{-1}(Y)\backslash f^{-1}(Y')$ (and vice versa). Therefore, we have $f^{-1}(Y\backslash Y') \subseteq (\supseteq) f^{-1}(Y)\backslash f^{-1}(Y')$, so $f^{-1}(Y\backslash Y') = f^{-1}(Y)\backslash f^{-1}(Y')$ for $Y' \subseteq Y \subseteq T$. □

Remark: The difference between f and f^{-1} is the latter must be one-to-one mapping, but the former may not.

DEFINITION 1.17. *If each $y \in T$ is associated with one and only one element of S, f is said ro establish a **one-to-one correspondence** between S and T, i.e., function $f : S \to T$ is an **injective (one-to-one)** mapping.*

DEFINITION 1.18. *Let f be a function from a set F to a set T, and let G be a set containing F. A function g from G to T is said to be an **extension** of f to G if one has $f(x) = g(x)$, for every x in F.*

*Let f be a function from a set F to a set T, and let G be a set contained in F. A function g from G to T is said to be an **restriction** of f to G if one has $f(x) = g(x)$, for every x in G.*

DEFINITION 1.19. *Consider N sets $S_1, S_2, \cdots, S_N$ and their product $\prod_{n=1}^{N} S_n$. The function which associates with the generic element (x_n) of $\prod_{n=1}^{N} S_n$, its nth coordinates x_n in S_n is called the nth **projection** (or the projection on S_n). The image of an element (resp. of a set) by a projection function is called the **projection** of that element (resp. of that set).*

DEFINITION 1.20. *If with each element x in a given set S is associated a **non-empty** subset Y of a given set T, a function φ, from S to the set of subsets of T is defined. It is sometimes preferable to consider φ as a **correspondence** from S to T. One writes $Y = \varphi(x)$.*

DEFINITION 1.21. *Let S and T bw two sets. If with each element s of S we associate a subset $\varphi(s)$ of T, we say that the correspondence $x \to \varphi(x)$ is a **mapping** of S into T; the set $\varphi(x)$ is called the **image** of x under the mapping φ. The set $S^* = \{x \in S | \varphi(x) \neq \varnothing\}$ is called the **domain** (or set of definition) of φ and $T^* = \underset{x\in S}{\cup} \varphi(x)$ is called the **range** (or set of values) of φ; we also say that φ is defined on S^* and that it is a mapping of S **onto** T^*, i.e., correspondence $\varphi : S^* \to T^*$ is a **surjective (onto)** mapping.*

*If the mapping φ of S into T is such that the set $\varphi(x)$ always consists of a single element, we say that φ is a **single-valued function** or a **single-valued mapping** of S into T.*

A mapping is called **semi-single-valued** if $\varphi(x) \cap \varphi(x') \neq \varnothing \Rightarrow \varphi(x) = \varphi(x')$.

Clearly a single-valued mapping is also a semi-single-valued. A mapping is called **injective (one-to-one)** if $x \neq x' \Rightarrow \varphi(x) \cap \varphi(x') = \varnothing$.

An injective mapping is evidently semi-single-valued.

If φ is a mapping of S into T and A is a non-empty subset of S, we write $\varphi(A) = \underset{x\in A}{\cup} \varphi(x)$.

If $A = \varnothing$, we write $\varphi(\varnothing) = \varnothing$. The set $\varphi(A)$ is called the image of A under the mapping φ.

If $\mathscr{A} = \{A_\gamma | \gamma \in \Gamma\}$ is a family of sets, we write $\varphi(\mathscr{A}) = \{\varphi(A_\gamma) | \gamma \in \Gamma\}$ and call $\varphi(\mathscr{A})$ the image of $\mathscr{A}$ under the mapping φ.

About the mapping, we give some properties.

THEOREM 1.25. *$A \subseteq A'$ implies that $\varphi(A) \subseteq \varphi(A')$.*

Proof. $\forall y \in \varphi(A), \exists x \in A$, we have $y \in \varphi(x)$. As $x \in A \subseteq A', x$ must belong to A'. Thus, we always can find a $x \in A'$ which satisfies $y \in \varphi(x)$. So $y \in \varphi(A'), \varphi(A) \subseteq \varphi(A')$. □

THEOREM 1.26. $\varphi(\underset{\gamma\in\Gamma}{\cup} A_\gamma) = \underset{\gamma\in\Gamma}{\cup} \varphi(A_\gamma)$

Proof. With definition of $\varphi(\cdot)$, we have the following equivalent chain: $\varphi(\underset{\gamma\in\Gamma}{\cup} A_\gamma) \Leftrightarrow \varphi(\underset{x_\gamma^i \in A_\gamma, \gamma\in\Gamma}{\cup} x_\gamma^i) \Leftrightarrow \underset{x_\gamma^i \in A_\gamma, \gamma\in\Gamma}{\cup} \varphi(x_\gamma^i) \Leftrightarrow \underset{\gamma\in\Gamma}{\cup}[\underset{x_\gamma^i \in A_\gamma}{\cup} \varphi(x_\gamma^i)] \Leftrightarrow \underset{\gamma\in\Gamma}{\cup} \varphi(A_\gamma)$ □

THEOREM 1.27. $\varphi(\underset{\gamma\in\Gamma}{\cap} A_i) \subseteq \underset{\gamma\in\Gamma}{\cap} \varphi(A_i)$

Proof. $\forall y \in \varphi(\bigcap_{\gamma\in\Gamma} A_i)$, it is a must that $\exists x \in \bigcap_{\gamma\in\Gamma}$, which satisfies $y \in \varphi(x)$. As $x \in A_\gamma, \forall\gamma \in \Gamma$, y must belong to $\varphi(A_\gamma)$ as long as $\gamma \in \Gamma$, that indicates $y \in \bigcap_{\gamma\in\Gamma} \varphi(A_i)$. So we get $\varphi(\bigcap_{\gamma\in\Gamma} A_i) \subseteq \bigcap_{\gamma\in\Gamma} \varphi(A_i)$. □

THEOREM 1.28. *If $A \subseteq S$ and if φ is a mapping of S with range T, then $\mathbf{C}(\varphi(A)) \subseteq \varphi(\mathbf{C}A)$. If, further, φ is injective, then $\mathbf{C}(\varphi(A)) = \varphi(\mathbf{C}A)$.*

Proof. As $T = \bigcup_{x\in S} \varphi(x)$ (range), $\forall y \in T$, $\exists x \in A$ satisfies $y = \varphi(x)$. $\forall y \in \mathbf{C}(\varphi(A))$, $\exists x \in \mathbf{C}A$ satisfies $y = \varphi(x)$, which indicates $y \in \varphi(\mathbf{C}A)$.

Moreover, if φ is injective, we suppose $\exists y \in \varphi(\mathbf{C}A)$ but $y \notin \mathbf{C}(\varphi(A))$. $y \in \varphi(\mathbf{C}A)$ implies $\exists x \in \mathbf{C}A, y = \varphi(x)$. $y \notin \mathbf{C}(\varphi(A))$ indicates $y \in \varphi(A)$, that needs $\exists x \in A, y = \varphi(x)$. In this way, $y = \varphi(x), x \in A$ and $x \in \mathbf{C}A$ must hold simultaneously, that is contradictory to φ is injective. □

DEFINITION 1.22. *The **graph** of the correspondence φ is a subset of $S \times T$, namely*

$$G(\varphi) = \{(x, y) \in S \times T | y \in \varphi(x)\} \tag{1.32}$$

DEFINITION 1.23. *A **binary operation** $\top$ on a set S associates with each pair (x, y) of elements of S (the order of which is essential) a unique element z of S. One writes $z = x\top y$. Thus a binary operation on S is nothing else than a function from $S \times S$ to S.*

We can define properties of a binary operation like correspondence. Consider $\top, \perp, \forall x, y, z \in S$, we say $\top$ is

- associate: $(x\top y)\top z = z\top(y\top z)$;
- commutative: $x\top y = y\top x$;
- distributive: $x\perp(y\top z) = (x\perp y)\top(x\perp z)$;

If φ_1 and φ_2 are two mapping of S into T, their **union** is the mapping $(\varphi_1 \cup \varphi_2)$ of S into T defined By

$$(\varphi_1 \cup \varphi_2)(x) = \varphi_1(x) \cup \varphi_2(x). \tag{1.33}$$

The intersection of φ_1 and φ_2 is the mapping$(\varphi_1 \cap \varphi_2)$ of S into T defined By

$$(\varphi_1 \cap \varphi_2)(x) = \varphi_1(x) \cap \varphi_2(x). \tag{1.34}$$

The Cartesian product of φ_1 and φ_2 in the mapping $(\varphi_1 \times \varphi_2)$ of S into $T \times T$ defined by

$$(\varphi_1 \times \varphi_2)(x) = \varphi_1(x) \times \varphi_2(x). \tag{1.35}$$

More generally, given any operations $\vee$ and $\wedge$ on sets, we write

$$(\varphi_1 \vee \varphi_2)(x) = \varphi_1(x) \vee \varphi_2(x); (\varphi_1 \wedge \varphi_2)(x) = \varphi_1(x) \wedge \varphi_2(x). \tag{1.36}$$

If φ_1 is a mapping of S into T and φ_2 is a mapping of T into U, the **composition product** of φ_1 is the mapping $(\varphi_2 \cdot \varphi_1)$ of S into U defined by

$$(\varphi_2 \cdot \varphi_1)(x) = \varphi_2(\varphi_1(x)). \tag{1.37}$$

We summarize the operations of mapping:

1. $(\varphi_1 \cup \varphi_2)(A) = \varphi_1(A) \cup \varphi_2(A)$;
2. $(\varphi_1 \cap \varphi_2)(A) = \varphi_1(A) \cap \varphi_2(A)$;
3. $(\varphi_1 \times \varphi_2)(A) = \varphi_1(A) \times \varphi_2(A)$;
4. $(\varphi_2 \cdot \varphi_1)(A) = \varphi_2(\varphi_1(A))$.

A mapping Δ is said to be **constant** if there exists a subset C of T such that $\Delta(x) = C$ for all S. A constant mapping satisfies the following property:

$$(\Delta \cap \varphi)(A) = \Delta(A) \cap \varphi(A). \tag{1.38}$$

Another important mapping is the **identity** mapping, which is the single-valued mapping I of S onto S defined by

$$I(x) = \{x\} \tag{1.39}$$

If $\varphi_1, \varphi_2, \varphi_3$ are mapping of S into S, we have associativity of composition product

$$\varphi_2 \cdot (\varphi_2 \cdot \varphi_1)(x) = (\varphi_3 \cdot \varphi_2) \cdot \varphi_1(x). \tag{1.40}$$

Thus, if φ is a mapping of S into S, we can write $\varphi^2(x) = \varphi(\varphi(x))$, $\varphi^3(x) = \varphi(\varphi^2(x)) = \varphi^2(\varphi(x))$, etc.

The **transitive closure** of φ is a mapping $\hat{\varphi}$ of S into S defined by

$$\hat{\varphi}(x) = \{x\} \cup \{\varphi(x)\} \cup \{\varphi^2(x)\} \cup \cdots \tag{1.41}$$

The correspondence $A \Rightarrow \varphi(\hat{A})$ is a closure operation, for

1. $\hat{\varphi}(A) \supseteq A$,
2. $A \supseteq A' \Rightarrow \varphi(\hat{A}) \supseteq \varphi(\hat{A'})$,
3. $\hat{\varphi}(\hat{\varphi})(A) = \varphi(\hat{A})$.

THEOREM 1.29. *If Γ_1 and Γ_2 are two semi-single-valued mappings, $\Gamma_1 \cap \Gamma_2$ and $\Gamma_1 \times \Gamma_2$ are semi-single-valued mappings.*

Proof. $(\Gamma_1 \cap \Gamma_2)(x) \cap (\Gamma_1 \cap \Gamma_2)(x') = [\Gamma_1(x) \cap \Gamma_2(x)] \cap [\Gamma_1(x') \cap \Gamma_2(x')] = [\Gamma_1(x) \cap \Gamma_1(x')] \cap [\Gamma_2(x) \cap \Gamma_2(x')] \neq \varnothing$. Then $\Gamma_1(x) \cap \Gamma_1(x') \neq \varnothing$ and $\Gamma_2(x) \cap \Gamma_2(x') \neq \varnothing$. As Γ_1, Γ_2 are semi-single-valued mappings, one has $x = x'$. Therefore, $\Gamma_1 \cap \Gamma_2$ is a semi-single-valued mapping.

$(\Gamma_1 \times \Gamma_2)(x) \cap (\Gamma_1 \times \Gamma_2)(x') = \varnothing = [\Gamma_1(x) \times \Gamma_2(x)] \cap [\Gamma_1(x') \times \Gamma_2(x')]$, that indicates $\Gamma_1(x) \cap \Gamma_1(x') \neq \varnothing$ and $\Gamma_2(x) \cap \Gamma_2(x') \neq \varnothing$. As Γ_1, Γ_2 are semi-single-valued mappings, one has $x = x'$. Therefore, $\Gamma_1 \times \Gamma_2$ is a semi-single-valued mapping. □

THEOREM 1.30. *If one of the mappings Γ_1 and Γ_2 is injective, the mappings $\Gamma_1 \cap \Gamma_2$ and $\Gamma_1 \times \Gamma_2$ are injective.*

Proof. As Γ_1, Γ_2 are injective, $\forall x \neq x', \Gamma_1(x) \cap \Gamma_1(x') = \varnothing, \Gamma_2(x) \cap \Gamma_2(x') = \varnothing$. $[(\Gamma_1 \cap \Gamma)(x)] \cap [(\Gamma_1 \cap \Gamma_2)(x')] = [\Gamma_1(x) \cap \Gamma_1(x)] \cap [\Gamma_2(x') \cap \Gamma_2(x')] = \varnothing \cap \varnothing = \varnothing$, i.e., $\Gamma_1 \cap \Gamma_2$ is injective.

$[(\Gamma_1 \times \Gamma_2)(x)] \cap [(\Gamma_1 \times \Gamma_2)(x')] = [\Gamma_1(x) \cap \Gamma_1(x')] \times [\Gamma_2(x) \cap \Gamma_2(x')] = \varnothing \times \varnothing = \varnothing$, i.e., $\Gamma_1 \times \Gamma_2$ is injective. □

DEFINITION 1.24. *If φ is a mapping of S into T, its* ***lower inverse*** *is the mapping φ^- of T into S defined by*

$$\varphi^-(y) = \{x \in S | y \in \varphi(x)\}. \tag{1.42}$$

This is a mapping whose domain is T^* and range is S^*; for any non-empty subset B of T we have

$$\varphi^-(B) = \{x \in S | \varphi(x) \cap B \neq \varnothing\} \tag{1.43}$$

and we also write $\varphi^-(\varnothing) = \varnothing$. It is clear that the inverse of φ^- is $(\varphi^-)^- = \varphi$ and that $y \in \varphi(x)$ is equivalent to $x \in \varphi^-(y)$.

THEOREM 1.31. *If φ is singe-valued, φ^- is injective; if φ is injective, φ^- is single-valued; if φ is semi-single-valued, φ^- is semi-singe-valued.*

Proof. (1)(counter-evidence) We suppose φ^- is not injective. Then $\exists y \neq y', \varphi^-(y) \cap \varphi^-(y') \neq \varnothing$. Let x denotes an element of $\varphi^-(y) \cap \varphi^-(y')$, we have $y \in \varphi(x), y' \in varphi(x), y \neq y'$, that is contradictory to φ is single-valued. Thus, we have φ^- is injective.

(2)(counter-evidence) We suppose φ^- is not single-valued. There exists $x_1, x_2 \in S, x_1 \neq x_2, y \in T$ satisfying $y \in \varphi(x_1), y \in \varphi(x_2)$, which is contradictory to φ is injective. Thus, we have φ^- is single-valued.

(3)(counter-evidence) We suppose φ^- is not semi-single-valued. $\exists y, y' \in T, \varphi^-(y) \cap \varphi^-(y') \neq \varnothing$ and $\varphi^-(y) \neq \varphi^-(y')$. That implies that $\exists x \neq x'$ satisfy $y \in \varphi(x), y' \in \varphi(x)$ and $y \in \varphi(x'), y' \notin \varphi(x')$ (or vice versa). Then $y \in \varphi(x) \cap \varphi(x') \neq \varnothing$, but $\varphi(x) \neq \varphi(x')$. That is contradictory to φ is semi-single-valued. Thus, we have φ^- is semi-single-valued. □

DEFINITION 1.25. *Apart from the mapping φ^-, it is sometimes useful to consider another correspondence of a similar kind, which we call the **upper inverse** of φ; in this mapping to each subset B of T there corresponds the set*

$$\varphi^+(B) = \{x \in S | \varphi(x) \subseteq B\}. \tag{1.44}$$

In particular, if $B \neq \varnothing$, we have $\varphi^+(\varnothing) = \varnothing$. We observe that for all subsets B we have $\varphi^+(B) \subseteq \varphi^-(B)$.

THEOREM 1.32. *If B_1 and B_2 are subsets of T, then we have*

1. $\mathbf{C}\varphi^+(B) = \varphi^-(\mathbf{C}B)$ *and* $\mathbf{C}\varphi^-(B) = \varphi^+(\mathbf{C}B)$;
2. $\varphi^+(B_1) \cup \varphi^+(B_2) \subseteq \varphi^+(B_1 \cup B_2)$;
3. $\varphi^-(B_1) \cup \varphi^-(B_2) = \varphi^-(B_1 \cup B_2)$.

Proof. (1)$\mathbf{C}\varphi^+(B) = \{x \in S | \varphi(x) \nsubseteq B\} = \{x \in S | \varphi(x) \cap \mathbf{C}B \neq \varnothing\} = \varphi^-(\mathbf{C}B)$. $\mathbf{C}\varphi^-(B) = \{x \in S | \varphi(x) \cap B = \varnothing\} = \{x \in S | \varphi(x) \subseteq \mathbf{C}B\} = \varphi^+(\mathbf{C}B)$.

(2) $\forall x \in \varphi^+(B_1) \cup \varphi^+(B_2), \varphi(x) \subseteq B_1, \varphi(x) \subseteq B_2$. Then $\varphi(x) \subseteq B_1 \cup B_2$, we get $x \in \varphi^+(B_1 \cup B_2)$. Then $\varphi^+(B_1) \cup \varphi^+(B_2) \subseteq \varphi^+(B_1 \cup B_2)$.

(3) $\varphi^-(B_1) \cup \varphi^-(B_2) = \{x \in S | \varphi \cap B_1 \neq \varnothing$ or $\varphi(x) \cap B_2 \neq \varnothing\} = \{x \in S | \varphi(x) \cap (B_1 \cup B_2) \neq \varnothing\} = \varphi^-(B_1 \cup B_2)$. □

DEFINITION 1.26. *A set S is **countable** if it has at most as many elements as $\mathbb{N}$. In a precision fashion, a set S is defined as countable if it is can be put in one-to-one correspondence with a subset of $\mathbb{N}$. When the counter set S is not empty one can always choose the corresponding subsets of N so that it owns 1. and so that, whenever it owns two positive integers, it owns every positive integer between them. The image of $x \in S$ in the correspondence is then called the **rank** of x. Thus a set is countable if and only if all its elements can be ranked, no two different elements having the same rank.*

THEOREM 1.33. *The product of finite countable set is countable.*

Proof. $\forall$ countable X, let $x_i, i = 1, 2, \cdots, k$ denote elements of X. We can let $\forall (x_i, x_j) \in X \times X$ corresponds to $(j-1)k + i \in \{1, 2, \cdots, k^2\}$(one-to-one correspondence). so $X \times X$ is countable. □

THEOREM 1.34. *The set of subsets of $\mathbb{N}$ is not countable.*

Proof. We show that there is no surjection from a set A to its power set 2^A. Consider any function $f : A \to 2^A$ and let

$$B = \{a \in A | a \notin f(a)\} \tag{1.45}$$

This makes sense because of $f(a) \subseteq A$ and $B \subseteq A$. We claim that there is no $b \in A$ such that $f(b) = B$ (that you cannot label the special subset B with an element of A, in other word, you cannot label a special subset of $\mathbb{N}$ with an element of $\mathbb{N}$). Indeed, assume $f(b) = B$ for some $b \in A$. Then either $b \in B$ hence $b \notin f(b)$ which is a contradiction, or $b \notin B$ implying that $b \in B$ which is again a contradiction. Hence the map f is not surjective as claimed. □

1.4 Binary Relation, Preorderings and Preference

DEFINITION 1.27. *Let $\mathscr{R}$ be a **binary relation** in which any two elements x_1 and x_2 of S (the order of which is essential) stand or do not stand. If they do, one writes x_1.*

To define with full generality an ordering relation on a set, one preserves certain properties of $\leq$ on $\mathbb{N}$.

DEFINITION 1.28. *In precise manner, a binary relation on $\mathscr{R}$ on S which satisfies*

1. $x\mathscr{R}x$ *for every* $x \in S$ *(reflexivity),*
2. $x_1\mathscr{R}x_2$ *and* $x_2\mathscr{R}x_1$ *implies* $x_1\mathscr{R}x_3$ *(transitivity)*

*is called a **preordering** (also quasi-ordering). Often the symbols $\succsim$ will be used (in place of $\mathscr{R}$) to denote a preordering. By definition, $x_1 \succsim x_2$ means $x_1 \precsim x_2$. Moreover, $x_1 \precsim x_2$ and $x_2 \succsim x_1$ is written $x_1 \sim x_2$. $x_1 \precsim x_2$ and not $x_2 \precsim x_1$ is written $x_1 \prec x_2$.*

DEFINITION 1.29. *When one necessarily has $x_1 \precsim x_2$ or $x_2 \precsim x_1$ (or both), rhe preordering is called **complete** (to emphasize that a certain is not necessarily complete, one calls it **partial**). The $x_1 \precsim x_2$ is the negation of $x \succ x_2$. Note that partial preordering is more general than complement ordering.*

Here comes two important definitions.

DEFINITION 1.30. *Let S be a partial preordered set. When $y \in S$ and there is no $x \in S$ such that $x \succ y$ (resp. $x \prec y$), y is called a **maximal** (resp. **minimal**) element of S. When $y \in S$ and $\forall x \in S$ one has $x \precsim y$ (resp. $x \succsim y$), y is called a **greatest** (resp. **least**) element of S.*

When the preordering in complete the converse is also true and the distinction disappears. When the preordering is an ordering there is at most one greatest element and at most one least element.

Note the difference between $maximal$ and $greatest$. The greatest element is defined in the comparable set, i.e., the fixed-return security and we can decide which is best. While the maximal element is defined in a more general set, i.e., including the fixed-return security and the risk security, we can decide whether the risk security is better than the fixed-return security. That indicates that the maximal element must be proved through counter-evidence while the greatest element can be obtained with simple compares. In economics, the greatest is the best we can reach while the maximal is the best we want reach but may cannot.

DEFINITION 1.31. *Let S be a preordered set, and consider a subset X of S. An element $y \in S$ such that $\forall x \in X$ one has $x \precsim y$ (resp. $x \succsim y$) is called an **upper bound** (resp. a **lower bound** of X).*

*Consider the set Y of the upper bounds of X; the set Y has the property: $y \in Y$ and $y' \succsim y$ implies $y' \in Y$. A least element of Y is a **least upper bound** of X. If X has a greatest element y, y is clearly a **least upper bound**. Consider the set Y' of the lower bounds of X; the set Y' has the property: $y \in Y'$ and $y \succsim y'$ implies $y' \in Y'$. A greatest element of Y' is a **greatest lower bound** of X. If X has least element y, y is clearly a **greatest lower bound**.*

DEFINITION 1.32. *Let S be a preordered set. A subset I of S is an **interval** if $x \in I, y \in I$ and $x \precsim z \precsim y$ implies $z \in I$.*

Let a, b be two elements of S such that $a \precsim b$; particular cases of intervals are:

- $[a, b] = \{x \in S | a \precsim x \precsim y\}$,
- $(a, b) = \{x \in S | a \prec x \prec y\}$,
- $[a, b) = \{x \in S | a \precsim x \prec y\}$,
- $(a, b] = \{x \in S | a \prec x \precsim y\}$,
- $[a, \rightarrow) = \{x \in S | a \precsim x\}$,
- $(\leftarrow, b] = \{x \in S | x \precsim y\}$,

The define the preordering on the set family.

DEFINITION 1.33. *Denote by $S_1, S_2, \cdots, S_N$ N preordered sets, by $\precsim_n$ the preordering on S_n, by x_n a generic element of S_n. A preordering $\precsim$ is defined on the product $S = \prod_{n=1}^{N} S_n$ by $(x_n) \precsim (x'_n)$ if $x_n \precsim_n x'_n$ for every $n = 1, 2, \cdots, N$. $(x_n) \prec (x'_n)$ means that (α) for all n, $x_n \precsim_n x'_n$ and (β) not for all n, $x'_n \precsim_n x_n$, i.e., for at least one n, $x_n \prec_n x'_n$. The notation $(x_n) \prec\prec (x'_n)$ will express that, for all n, $x_n \prec_n x'_n$. With n the exprction of trivial cases, the preordering $\precsim$ on S cannot be complete. (example: $(1, 2, 3) and (2, 3, 0)$ are not comparable)*

DEFINITION 1.34. *Let S and T the two preordered sets, and denote by $\precsim_s$ (resp. $\precsim_T$) the preordering on S (resp. on T). A function f from S to T is said to be **increasing** (or to be a **representation** of S in T) if $x \sim_S x'$ implies $f(x) \sim_T f(x')$ and $x \prec_S x'$ implies $f(x) \prec_T f(x')$.*

DEFINITION 1.35. *For a given set S, let $S \times S$ denote the usual Cartesian product of all ordered pairs (x_1, x_2), where both x_1 and x_2 are from S. A **binary relation** $\mathscr{R}$ on the set S is formally defined as a subset of $S \times S$, written as $\mathscr{R} \subseteq S \times S$, and $(x_1, x_2) \in \mathscr{R}$ if the ordered pair is in the relation $\mathscr{R}$. Another, quickly way to write $(x_1, x_2) \in \mathscr{R}$ is $x_1 \mathscr{R} x_2$, which can be read as "x_1 stands in the relation $\mathscr{R}$ to x_2". If $(x_1, x_2) \notin \mathscr{R}$, we'll write "$x_1 \not\mathscr{R} x_2$".*

A binary relation $\mathscr{R}$ on a set S is:

- reflexive if $x \mathscr{R} x$ for all $x \in S$;
- irreflexive if $x \not\mathscr{R} x$ for all $x \in S$;
- symmetric if $x_1 \mathscr{R} x_2$ implies $x_2 \mathscr{R} x_1$;

- asymmetric if $x_1\mathscr{R}x_2$ implies $x_2\not\mathscr{R}x_1$;
- antisymmetric if $x_1\mathscr{R}x_2$ and $x_2\mathscr{R}x_1$ implies $x_1 = x_2$;
- transitive if $x_1\mathscr{R}x_2$ and $x_2\mathscr{R}x_3$ implies $x_1\mathscr{R}x_3$;
- negatively transitive if $x_1\not\mathscr{R}x_2$ and $x_2\not\mathscr{R}x_3$ implies $x_1\not\mathscr{R}x_3$;
- complete or connected if for all $x_1 \in S$ and $x_2 \in S$, $x_1\mathscr{R}x_2$ or $x_2\mathscr{R}x_1$ (or both);
- weakly connected if for all $x_1 \in S$ and $x_2 \in S, x_1 = x_2$ or $x_1\mathscr{R}x_2$ or $x_2\mathscr{R}x_1$;
- acyclic if $x_1\mathscr{R}x_2, \cdots, x_{N-1}\mathscr{R}x_N$ imply $x_1 \neq x_N$.

Strict preference is a binary relation on S. Consider the following properties that this binary relation might possess: asymmetry, transitivity, irreflexivity, negative transitivity. Then we study the relations between them.

LEMMA 1.1. *A binary relation $\mathscr{R}$ is negatively transitive if and only if $x_1\mathscr{R}x_2$ implies that, for all $x \in S$, $x_1\mathscr{R}x$ or $x\mathscr{R}x_2$.(converse-negative proposition)*
Proof:

$$\begin{aligned}
& \textit{Binary relation } \mathscr{R} \textit{ is negatively transitive} && (1.46)\\
\Leftrightarrow\ & \textit{If } x_1\not\mathscr{R}x \textit{ and } x\not\mathscr{R}x_3 \textit{ for some } x \in S,\ \textit{then } x_1\not\mathscr{R}x_2 && (1.47)\\
\Leftrightarrow\ & \textit{If not } x_1\not\mathscr{R}x_2,\ \textit{then not } [x_1\not\mathscr{R}x \textit{ and } x\not\mathscr{R}x_3] \textit{ for all } x \in S && (1.48)\\
\Leftrightarrow\ & \textit{If not } x_1\not\mathscr{R}x_2,\ \textit{then not } x_1\not\mathscr{R}x \textit{ or } x\not\mathscr{R}x_3 \textit{ for all } x \in S && (1.49)\\
\Leftrightarrow\ & \textit{If } x_1\mathscr{R}x_2,\ \textit{then } x_1\mathscr{R}x \textit{ or } x\mathscr{R}x_2 \textit{ for all } x \in S. && (1.50)
\end{aligned}$$

Here comes the most important definitions about preference.

DEFINITION 1.36. *A binary relation $\succ$ on a set S is called a **strict preference relation** if it is asymmetric and negatively transitive.*

- *asymmetric: if $x_1 \succ x_1$, then $x_2 \not\succ x_1$;*
- *negatively transitive: if $x_1 \not\succ x_2$ and $x_2 \not\succ x_3$, then $x_1 \not\succ x_3$.*

Many properties can be induced by these basic conditions.

PROPOSITION 1.1. *If $\succ$ is a strict preference relation, then it is irreflexive, transitive, and acyclic.*
Proof: [1] Asymmetry directly implies irreflexivity.
[2] Suppose $x_1 \succ x_3$ and $x_2 \succ x_3$. By negative transitivity and Lemma 1.1, $x_1 \succ x_2$ implies either $x_1 \succ x_3$ or $x_3 \succ x_2$. But $x_3 \succ x_2$ is impossible because $x_2 \succ x_3$ is assumed and $\succ$ is asymmetric. Thus $x_1 \succ x_3$. which is transitivity. Analyzing $x_2 \not\succ x_3$ gives the same outcome.
[3] If $x_1 \succ x_2, \cdots, x_{N-1} \succ x_N$, then by transitivity $x_1 \succ x_N$. Since $\succ$ is irreflexive, then this implies $x_1 \neq x_N$. Thus $\succ$ is acyclic.

With strict preference relation, we can define its negation.

DEFINITION 1.37. *We define **preference** $\succsim$ as $x_1 \succsim x_3$ if $x_2 \not\succ x_1$. Indifference is defined as $x_1 \sim x_2$ if $x_1 \not\succ x_2$ and $x_2 \not\succ x_1$. Indifference expresses the absence of strict preference in either direction.*

Properties of preference and indifference can be obtained with strict preference.

PROPOSITION 1.2. *If $\succ$ is a strict preference relation, then*

1. *For all x_1 and x_2, exactly one of $x_1 \succ x_2, x_2 \succ x_1$ or $x_1 \sim x_2$ holds.*
2. *$\succsim$ is complete and transitive.*
3. *$\sim$ is reflexive, symmetric and transitive.*
4. *$x_1 \succ x_2, x_2 \succ x_3, x_3 \succ x_4$ imply $x_1 \succ x_3$ and $x_2 \succ x_4$.*
5. *$x_1 \succsim x_2$ if and only if $x_1 \succ x_2$ or $x_1 \sim x_2$.*

6. *$x_1 \succsim x_2$ and $x_2 \sim x_1$ imply $x_1 \sim x_2$.*

Proof: [1] follows the definition of $\sim$ and the fact $\succ$ is asymmetric.
[2] By the asymmetry of $\succ$,either $x_1 \not\succ x_2$ and $x_2 \not\succ x_1$ (or both) for all x_1 and x_2, thus $\succsim$ is complete. For the transitivity of $\succsim$, note that this follows immediately the negatively transitivity of $\succ$.
[3] $\sim$ is reflexive because $\succ$ is irreflexive. $\sim$ is symmetric because the definition of $\sim$ is symmetric. For the transitivity, suppose $x_1 \sim x_2 \sim x_3$. Then $x_1 \not\succ x_2 \not\succ x_3$ and $x_3 \not\succ x_2 \not\succ x_1$. By negatively transitivity of $\succ$, $x_1 \not\succ x_3 \not\succ x_1$, or $x_1 \sim x_3$.
[4] According to Proposition 1.1, this is obvious.
[5] $x_1 \succsim x_3$ if and only if $x_2 \not\succ x_1$ or $x_1 \sim x_2$ if and only if $x_1 \succ x_2$ or $x_1 \sim x_2$.
[6] This is immediate from the definition of $\succsim$ and $\sim$.

With definition of $\succsim$, we can induce the definitions and properties of $\succ$ and $\sim$.

DEFINITION 1.38. *A binary relation $\succsim$ on a set S is called a **preference relation** if it is complete and transitive.*

- *complete: for all $x_1, x_2 \in S$, $x_1 \succsim x_2$ or $x_2 \succsim x_1$ (or both).*
- *transitive: if $x_1 \succsim x_2$ and $x_2 \succsim x_3$, then $x_1 \succsim x_3$.*

In this way, strict preference $\succ$ and indifference is defined as $x_1 \succ x_2$ if $x_2 \not\succsim x_1$ and $x_1 \sim x_2$ if $x_1 \succsim x_2$ and $x_2 \succsim x_1$.

PROPOSITION 1.3. *If $\succsim$ is a preference relation, then*

1. *For all x_1 and x_2, $x_1 \succsim x_2$ or $x_2 \succsim x_1$ (or both) holds.*
2. *$\succ$ is asymmetric and negatively transitive.*
3. *$\sim$ is reflexive, symmetric and transitive.*
4. *$x_1 \succ x_2$ if and only if $x_1 \succsim x_2$ but $x_2 \not\succsim x_1$.*
5. *$x_1 \succ x_2$ if and only if $x_1 \succsim x_2$ but $x_2 \not\succ x_1$.*

Proof: [1] This is immediate from the complete of $\succsim$.
[2] With complete of $\succsim$, $\forall x_1, x_2 \in S$, $x_1 \succsim x_2$ or $x_2 \succsim x_1$ (or both). If $x_1 \succ x_2$, $x_2 \succsim x_1$ does not hold. So we get $x_1 \succsim x_2$, which indicates $x_2 \succ x_1$ cannot hold. Therefore $\succ$ is asymmetric. For the negatively transitivity of $\succ$, note that this follows immediately the transitivity of $\succsim$.
[3] $\sim$ is reflexive because $\succsim$ is reflexive (from the completion, $x \succsim x$ must hold). $\sim$ is symmetric because the definition of $\sim$ is symmetric. For the transitivity, suppose $x_1 \sim x_2 \sim x_3$, then $x_1 \succsim x_2 \succsim x_3$ and $x_3 \succsim x_2 \succsim x_1$. By transitivity of $\succsim$, $x_1 \succsim x_3 \succsim x_1$, or $x_1 \sim x_3$.
[4] $x_1 \succ x_2$ if and only if $x_2 \not\succsim x_1$ if and only if $x_1 \succsim x_2$ but $x_2 \not\succsim x_1$.
[5] $x_1 \succ x_2$ if and only if $x_2 \not\succsim x_1$ if and only if $x_1 \succsim x_2$ but $x_2 \not\succsim x_1$ if and only if $x_2 \not\succ x_1$.

Lastly, a question arises: whether the correspondence of $\succ, \sim, \succsim$ is one-by-one?

PROPOSITION 1.4. *Given a preference relation $\succsim'$ on a set S, define two new binary relation $\succ'$ and $\sim'$ from $\succsim'$ by $x_1 \succ' x_2$ if $x_2 \not\succsim' x_1$ and $x_1 \sim' x_2$ if $x_1 \succsim' x_2$ and $x_2 \succsim' x_1$. Then we define $\succsim, \sim$ from $\succ'$ by $x_1 \succsim x_2$ if $x_2 \not\succ' x_1$ and $x_1 \sim x_2$ if $x_1 \not\succ' x_2$ and $x_2 \not\succ' x_1$. Then $\succsim'$ and $\succsim$ will agree, as wil $\sim'$ and $\sim$.*
$x_1 \succsim' x_2$ implies $x_2 \not\succ' x_1$ from the completeness of $\succsim'$, then $x_1 \succsim x_2$ from the definition of $\succsim$. $x_1 \succsim x_2$ implies $x_2 \not\succ' x_1$ from the definition of $\succsim$, then $x_1 \succsim' x_2$from the completeness of $\succsim'$.
The proof of that $\sim'$ and $\sim$ are equivalent is from the definitions of the indifference.

1.5 Summary

This chapter has introduced the fundamental ideas of topological spaces, including sets, limits of sets, functions (correspondence) and binary relations. We list the most important conceptions and theorems below, so you are able to review and establish a clear frame of this chapter quickly.
Definitions:

- set A;
- element $a, a \in A$ or $a \notin A$;

- subset $A \subseteq B, A \nsubseteq B, A \subsetneq B$;
- a decreasing sequence of sets;
- an increasing sequence of sets;
- the upper limit sets of sequence of sets $\overline{\lim\limits_{n\to\infty}} A_n = \bigcap\limits_{N=1}^{\infty} \bigcup\limits_{n=N}^{\infty} A_n$;
- the lower limit sets of sequence of sets $\underline{\lim\limits_{n\to\infty}} A_n = \bigcup\limits_{N=1}^{\infty} \bigcap\limits_{n=N}^{\infty} A_n$;
- cartesian product $\prod\limits_{\lambda\in\Lambda} A_\lambda$;
- a binary relation $x_1 \mathscr{R} x_2$;
- a preordering (reflexivity and transitivity);
- a maximal (minimal) element;
- a greatest (least) element;
- an upper (lower) bound;
- interval;
- representation;
- a strict preference relation (symmetric and negatively transitive $Leftrightarrow$ irreflexive, transitive and acyclic);
- preference (complete and transitive);
- function, variable, transform;
- image, inverse image;
- injective, surjective;
- extension, restriction;
- projection;
- correspondence;
- mapping, domain, range;
- single-valued function, semi-single-valued;
- graph;
- lower inverse, upper inverse;

Operations of sets:

- union $A \cup B$;
- intersection $A \cap B$; (disjoint $A \cap B = \varnothing$)
- complement $\mathbf{C}A$;
- symmetric difference $A\Delta B = (A_1\backslash A_2) \cup (A_2\backslash A_1) = (A_1 \cup A_2)\backslash(A_1 \cap A_2)$;

Properties of a binary relation may own:

- reflexivity $x\mathscr{R}X, \forall x \in S$;
- transitivity $x_1\mathscr{R}x_2, x_2\mathscr{R}x_2 \Rightarrow x_1\mathscr{R}x_3$;
- completeness $\forall x_1, x_2 \in S$, either $x_1\mathscr{R}x_2$ or $x_2\mathscr{R}x_1$ (or both); (partial, if not);

- negatively transitive $x_1\mathscr{R}x_2 \Rightarrow \forall x \in S$, either $x_1\mathscr{R}x$ or $x\mathscr{R}x_2$ (or both);

Useful equations:
Unions' properties,

1. $A_1 \cup A_2 = A_2 \cup A_1$(Ablian or Commutativity);
2. $A_1 \cup (A_2 \cup A_3) = (A_1 \cup A_2) \cup A_3$(Associativity);
3. $A_1 \subseteq A_1 \cup A_2$;
4. $A \cup A = A$;
5. $A \cup S = S$;
6. $A \cup \varnothing = A$;
7. $A_1 \subseteq A_2$ if and only if $A_1 \cup A_2 = A_2$.

properties of intersections,

1. $A_1 \cap A_2 = A_2 \cap A_1$(Ablian or Commutativity);
2. $A_1 \cap (A_2 \cap A_3) = (A_1 \cap A_2) \cap A_3$(Associativity);
3. $A_1 \cap A_2 \subseteq A_1$;
4. $A \cap A = A$;
5. $A \cap S = A$;
6. $A \cap \varnothing = \varnothing$;
7. $A_1 \subseteq A_2$ if and only if $A_1 \cap A_2 = A_1$.

Properties of complements,

1. $\mathbf{C}S = \varnothing$ and $\mathbf{C}\varnothing = S$;
2. $A \cup (\mathbf{C}A) = S$ and $A \cap (\mathbf{C}A) = \varnothing$;
3. $\mathbf{C}(\mathbf{C}A) = A$;
4. If $A_1 \supseteq A_2$, then $\mathbf{C}A_1 \subseteq \mathbf{C}A_2$;
5. If $A_1 \cap A_2 = \varnothing$, then $A_1 \subseteq \mathbf{C}A_2$ and $A_2 \subseteq \mathbf{C}A_1$.

Properties of symmetric difference,

1. $A_1 \Delta A_2 = A_2 \Delta A_1$;
2. $(A_1 \Delta A_2)\Delta A_3 = A_1 \Delta (A_2 \Delta A_3)$;
3. $A\Delta\varnothing = A, A\Delta A = \varnothing$;
4. $A\Delta\mathbf{C}A = S, A\Delta S = \mathbf{C}A$;
5. $(A_1 \Delta A_2)\Delta(A_2 \Delta A_3) = A_1 \Delta A_3$;
6. $A_1 \cap (A_2 \Delta A_3) = (A_1 \cap A_2)\Delta(A_1 \cap A_3)$;
7. $A_1 \Delta A_2 = A_1^c \Delta A_2^c$;
8. $A_1 = A_1 \Delta A_2$ if and only if $A_2 = \varnothing$;
9. For any sets A and A', there is a unique set E such that $E\Delta A = A'$, in fact, $E = A\Delta A'$;
10. $(\cup_{\lambda\in\Lambda} A_\lambda)\Delta(\cup_{\lambda\in\Lambda} A'_\lambda) \subseteq \cup_{\lambda\in\Lambda}(A_\lambda \Delta A'_\lambda)$;
11. $A_1 = A_2$ if and only if $\forall A_3, A_1 \cap A_3 = A_2 \cap A_3$ and $A_1 \cup A_3 = A_2 \cup A_3$;

12. $\forall A, A' = \mathbf{C}(E \cap A) \cap (\mathbf{C}E \cup A)$ if and only if $\mathbf{C}A' = E$.

Properties of a binary relation, a binary relation $\mathscr{R}$ on a set S is:

- reflexive if $x\mathscr{R}x$ for all $x \in S$;
- irreflexive if $x\not\mathscr{R}x$ for all $x \in S$;
- symmetric if $x_1\mathscr{R}x_2$ implies $x_2\mathscr{R}x_1$;
- asymmetric if $x_1\mathscr{R}x_2$ implies $x_2\not\mathscr{R}x_1$;
- antisymmetric if $x_1\mathscr{R}x_2$ and $x_2\mathscr{R}x_1$ implies $x_1 = x_2$;
- transitive if $x_1\mathscr{R}x_2$ and $x_2\mathscr{R}x_3$ implies $x_1\mathscr{R}x_3$;
- negatively transitive if $x_1\not\mathscr{R}x_2$ and $x_2\not\mathscr{R}x_3$ implies $x_1\not\mathscr{R}x_3$;
- complete or connected if for all $x_1 \in S$ and $x_2 \in S$, $x_1\mathscr{R}x_2$ or $x_2\mathscr{R}x_1$ (or both);
- weakly connected if for all $x_1 \in S$ and $x_2 \in S, x_1 = x_2$ or $x_1\mathscr{R}x_2$ or $x_2\mathscr{R}x_1$;
- acyclic if $x_1\mathscr{R}x_2, \cdots, x_{N-1}\mathscr{R}x_N$ imply $x_1 \neq x_N$.

Proof Methods:

- $A \subseteq B \leftarrow \forall x \in A, x \in B$;
- $A = B \Leftarrow A \subseteq B, B \subseteq A$;

Important theorems:

THEOREM 1.35. *Let A_1, A_2 and A_3 be sets.*

1. *Ablian or Commutativity:*

   (a) $A_1 \cup A_2 = A_2 \cup A_2$;
   (b) $A_1 \cap A_2 = A_2 \cap A_1$.

2. *Associativity:*

   (a) $A_1 \cup (A_2 \cup A_3) = (A_1 \cup A_2) \cup A_3$;
   (b) $A_1 \cap (A_2 \cap A_3) = (A_1 \cap A_2) \cap A_3$.

3. *Distributivity:*

   (a) $A_1 \cup (A_2 \cap A_3) = (A_1 \cup A_2) \cap (A_1 \cup A_3)$;
   (b) $A_1 \cap (A_2 \cup A_3) = (A_1 \cap A_2) \cup (A_1 \cap A_3)$.

4. *Absorption:*

   (a) $A_1 \cup (A_1 \cap A_2) = A_1$;
   (b) $A_1 \cap (A_1 \cup A_2) = A_1$.

5. *Idempotence:*

   (a) $A_1 \cap A_1 = A_1$;
   (b) $A_1 \cup A_2 = A_2$.

THEOREM 1.36. 1. *If $A_\lambda \subseteq A'_\lambda$ for $\lambda \in \Lambda$, then $\cup_{\lambda\in\Lambda} A_\lambda \subseteq \cup_{\lambda\in\Lambda} A'_\lambda$. In particular, if $A_\lambda \subseteq A'$ for $\lambda \in \Lambda$, then $\cup_{\lambda\in\Lambda} A_\lambda \subseteq A'$.*

2. *If $A_\lambda \subseteq A'_\lambda$ for $\lambda \in \Lambda$, then $\cap_{\lambda\in\Lambda} A_\lambda \subseteq \cup_{\lambda\in\Lambda} A'_\lambda$. In particular, if $A \subseteq A'_\lambda$ for $\lambda \in \Lambda$, then $A \subseteq \cap_{\lambda\in\Lambda} A'_\lambda$.*

3. $\cup_{\lambda\in\Lambda}(A_\lambda \cup A'_\lambda) = (\cup_{\lambda\in\Lambda} A_\lambda) \cup (\cup_{\lambda\in\Lambda} A'_\lambda)$.

4. $\cap_{\lambda\in\Lambda}(A_\lambda \cap A'_\lambda) = (\cap_{\lambda\in\Lambda} A_\lambda) \cap (\cap_{\lambda\in\Lambda} A'_\lambda)$.

5. $A \cap (\cup_{\lambda \in \Lambda} A_\lambda) = \cup_{\lambda \in \Lambda}(A \cap A_\lambda)$.

6. $A \cup (\cap_{\lambda \in \Lambda} A_\lambda) = \cap_{\lambda \in \Lambda}(A \cup A_\lambda)$.

THEOREM 1.37. *De Morgan's Law.*

1. $\mathbf{C}(A_1 \cup A_2) = \mathbf{C}A_1 \cap \mathbf{C}A_2$ *and* $\mathbf{C}(A_1 \cap A_2) = \mathbf{C}A_1 \cup \mathbf{C}A_2$.

2. $\mathbf{C}(\cup_{n=1}^N A_n) = \cap_{n=1}^N \mathbf{C}A_n$ *and* $\mathbf{C}(\cap_{n=1}^N A_n) = \cup_{n=1}^N \mathbf{C}A_n$.

3. $\mathbf{C}(\cup_{n=1}^\infty A_n) = \cap_{n=1}^\infty \mathbf{C}A_n$ *and* $\mathbf{C}(\cap_{n=1}^\infty A_n) = \cup_{n=1}^\infty \mathbf{C}A_n$.

4. $\mathbf{C}(\cup_{\lambda \in \Lambda} A_\lambda) = \cap_{\lambda \in \Lambda} \mathbf{C}A_\lambda$ *and* $\mathbf{C}(\cap_{\lambda \in \Lambda} A_\lambda) = \cup_{\lambda \in \Lambda} \mathbf{C}A_\lambda$.

THEOREM 1.38. *If* $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots, A_1' \subseteq A_2' \subseteq \cdots \subseteq A_n' \subseteq \cdots$, *then* $(\cup_{n=1}^\infty A_n) \cap (\cup_{n=1}^\infty A_n') = \cup_{n=1}^\infty (A_n \cap A_n')$.

If $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots, A_1' \supseteq A_2' \supseteq \cdots \supseteq A_n' \supseteq \cdots$, *then* $(\cap_{n=1}^\infty A_n) \cup (\cap_{n=1}^\infty A_n') = \cap_{n=1}^\infty (A_n \cup A_n')$.

THEOREM 1.39. *Let* $\{A_n : n \in \mathbb{N}\}$ *be a sequence of sets, then*

1. $\overline{\lim_{n\to\infty}} A_n = \{x : \forall N \in \mathbb{N}, \exists n \geq N, x \in A_n\}$;

2. $\underline{\lim_{n\to\infty}} A_n = \{x : \exists N \in \mathbb{N}, \forall n \geq N, x \in A_n\}$.

THEOREM 1.40. *Let* $\{f_n\}$ *and* f *be real-valued functions defined on* $\mathbb{R}$, *and*

$$\lim_{n\to\infty} f_n(x) = f(x), x \in \mathbb{R}, \tag{1.51}$$

then, for $t \in \mathbb{R}$,

$$\{x \in \mathbb{R} | f(x) \leq t\} = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{x \in \mathbb{R} | f_n(x) < t + \frac{1}{k}\}. \tag{1.52}$$

THEOREM 1.41. *For family* $\mathscr{A} = \{A_\lambda | \lambda \in \Lambda\}$ *we have*

$$A_\lambda \neq \varnothing, \forall \lambda \in \Lambda \Leftrightarrow \prod_{\lambda \in \Lambda} A_\lambda \neq \varnothing. \tag{1.53}$$

THEOREM 1.42. *If two families* $\mathscr{A} = \{A_\lambda | \lambda \in \Lambda\}$ *and* $\mathscr{A}' = \{A_\lambda' | \lambda \in \Lambda\}$ *have the same index set, then*

$$\begin{cases} A_\lambda \neq \varnothing, \forall \lambda \in \Lambda \\ \prod\limits_{\lambda \in \Lambda} A_\lambda \subseteq \prod\limits_{\lambda \in \Lambda} A_\lambda' \end{cases} \Leftrightarrow \varnothing \subset A_\lambda \subseteq A_\lambda', \forall \lambda \in \Lambda \tag{1.54}$$

and

$$\prod_{\lambda \in \Lambda} A_\lambda \cup \prod_{\lambda \in \Lambda} A_\lambda' \subseteq \prod_{\lambda \in \Lambda} (A_\lambda \cup A_\lambda') \tag{1.55}$$

$$\prod_{\lambda \in \Lambda} A_\lambda \cap \prod_{\lambda \in \Lambda} A_\lambda' = \prod_{\lambda \in \Lambda} (A_\lambda \cap A_\lambda'). \tag{1.56}$$

THEOREM 1.43. *A binary relation* $\mathscr{R}$ *is negatively transitive if and only if* $x_1 \mathscr{R} x_2$ *implies that, for all* $x \in S$, $x_1 \mathscr{R} x$ *or* $x \mathscr{R} x_2$. *(converse-negative proposition)*

THEOREM 1.44. *If* $\succ$ *is a strict preference relation, then*

1. *For all* x_1 *and* x_2, *exactly one of* $x_1 \succ x_2, x_2 \succ x_1$ *or* $x_1 \sim x_2$ *holds.*

2. $\succsim$ *is complete and transitive.*

3. $\sim$ *is reflexive, symmetric and transitive.*

4. $x_1 \succ x_2, x_2 \succ x_3, x_3 \succ x_4$ *imply* $x_1 \succ x_3$ *and* $x_2 \succ x_4$.

5. $x_1 \succsim x_2$ *if and only if* $x_1 \succ x_2$ *or* $x_1 \sim x_2$.

6. $x_1 \succsim x_2$ *and* $x_2 \sim x_1$ *imply* $x_1 \sim x_2$.

THEOREM 1.45. *If* f *is a function from a set* S *to a set* T*, and if* $\mathscr{X}$ *is a collection of subsets of* S*, then*

1. $f(\bigcup\limits_{X\in\mathscr{X}} X) = \bigcup\limits_{X\in\mathscr{X}} f(X)$;

2. $f(\bigcap\limits_{X\in\mathscr{X}} X) \subseteq \bigcap\limits_{X\in\mathscr{X}} f(X)$;

THEOREM 1.46. *If* f *is a function from a set* S *to a set* T*, and if* $\mathscr{Y}$ *is a collection of subsets of* T*, then is is easy to prove that:*

1. $f^{-1}(\bigcup\limits_{Y\in\mathscr{Y}}) = \bigcup\limits_{Y\in\mathscr{Y}} f^{-1}(Y)$.

2. $f^{-1}(\bigcap\limits_{Y\in\mathscr{Y}}) = \bigcap\limits_{Y\in\mathscr{Y}} f^{-1}(Y)$.

3. $f^{-1}(Y\backslash Y') = f^{-1}(Y)\backslash f^{-1}(Y')$ *for* $Y' \subseteq Y \subseteq T$.

THEOREM 1.47. $A \subseteq A'$ *implies that* $\varphi(A) \subseteq \varphi(A')$.
$\varphi(\bigcup\limits_{\gamma\in\Gamma} A_\gamma) = \bigcup\limits_{\gamma\in\Gamma} \varphi(A_\gamma)$.
$\varphi(\bigcap\limits_{\gamma\in\Gamma} A_i) \subseteq \bigcap\limits_{\gamma\in\Gamma} \varphi(A_i)$.
If $A \subseteq S$ *and if* φ *is a mapping of* S *with range* T*, then* $\mathbf{C}(\varphi(A)) \subseteq \varphi(\mathbf{C}A)$*. If, further,* φ *is injective, then* $\mathbf{C}(\varphi(A)) = \varphi(\mathbf{C}A)$.
Operations of mapping:

1. $(\varphi_1 \cup \varphi_2)(A) = \varphi_1(A) \cup \varphi_2(A)$;

2. $(\varphi_1 \cap \varphi_2)(A) = \varphi_1(A) \cap \varphi_2(A)$;

3. $(\varphi_1 \times \varphi_2)(A) = \varphi_1(A) \times \varphi_2(A)$;

4. $(\varphi_2 \cdot \varphi_1)(A) = \varphi_2(\varphi_1(A))$.

THEOREM 1.48. *If* Γ_1 *and* Γ_2 *are two semi-single-valued mappings,* $\Gamma_1 \cap \Gamma_2$ *and* $\Gamma_1 \times \Gamma_2$ *are semi-single-valued mappings.*
If one of the mappings Γ_1 *and* Γ_2 *is injective, the mappings* $\Gamma_1 \cap \Gamma_2$ *and* $\Gamma_1 \times \Gamma_2$ *are injective.*
If φ *is singe-valued,* φ^- *is injective; if* φ *is injective,* φ^- *is single-valued; if* φ *is semi-single-valued,* φ^- *is semi-singe-valued.*
If B_1 *and* B_2 *are subsets of* T*, then we have*

1. $\mathbf{C}\varphi^+(B) = \varphi^-(\mathbf{C}B)$ *and* $\mathbf{C}\varphi^-(B) = \varphi^+(\mathbf{C}B)$;

2. $\varphi^+(B_1) \cup \varphi^+(B_2) \subseteq \varphi^+(B_1 \cup B_2)$;

3. $\varphi^-(B_1) \cup \varphi^-(B_2) = \varphi^-(B_1 \cup B_2)$.

2 Real Numbers

2.1 Definitions and Fundamental Theorems of Real Numbers

2.1.1 Definitions of Real Numbers

The set of $\mathbb{R}$ of (finite, real) numbers is defined as a set of elements having the following properties:

1. There are on $\mathbb{R}$ two associative and commutative binary operations (addition $+$ and multiplication $\cdot$). There is an element, 0, which, if added to any element x, gives x. Any element x has a negative, i.e., an element which, if added to x, gives 0. Any element x different from 0 has an inverse, i.e., an element which, if

multiplied by x, gives 1. multiplication is distributive with respect to addition. 0 is different from 1. That is, $\forall a, b, x \in \mathbb{R}$

$$(a+b)+c = a+(b+c) \tag{2.1}$$
$$a+b = b+a \tag{2.2}$$
$$(ab)c = a(bc) \tag{2.3}$$
$$ab = ba \tag{2.4}$$
$$x+0 = x \tag{2.5}$$
$$x+(-x) = 0 \tag{2.6}$$
$$1 \cdot x = x \tag{2.7}$$
$$x \cdot \frac{1}{x} = 1, x \neq 0 \tag{2.8}$$
$$a(b+c) = ab+ac \tag{2.9}$$

2. There is on $\mathbb{R}$ a complete ordering denoted $\leq$. If z is any element and $x_1 \leq x_2$, then $x_1 + z \leq x_2 + z$. If $0 \leq x_1$ and $0 \leq x_2$, then $0 \leq x_1 \cdot x_2$. That is,

$$x_1 \leq x_2 \Rightarrow x_1 + z \leq x_2 + z \tag{2.10}$$
$$0 \leq x_1, 0 \leq x_2 \Rightarrow 0 \leq x_1 \cdot x_2 \tag{2.11}$$

3. Finally, every non-empty subset X of $\mathbb{R}$ which has an upper bound has a least upper bound, and every non-empty subset X of $\mathbb{R}$ which has a lower bound has a greatest lower bound.

4. $x_1 + x_2$ is the sum of x_1 and x_2, $x_1 \cdot x_2$ is their product. If $x_1, x_2, \cdots, x_N$ are N numbers of $\mathbb{R}$, their sum is denoted $\sum_{n=1}^{N} x_n$. The product of N elements equal to x is denoted x^N and called the Nth power of x. $0(1)$ is the only element having the property which defines $0(1)$. Any $x \in \mathbb{R}$ has a unique negative denoted $-x$. One writes $x_1 - x_2$ for $x_1 + (-x_2)$. The corresponding binary operation is called subtraction, and the result difference. Any $x \in \mathbb{R}$ different from 0 has a unique inverse denoted $\frac{1}{x}$. One writes $\frac{x_1}{x_2}$ for $x_1 \cdot \frac{1}{x_2}$. The corresponding binary operation is called division, and the result quotient. For any $x \in \mathbb{R}$ one has $0 \cdot x = 0$ and $(-1) \cdot x = -x$. The multiplication dot will now always be dropped.

5. $x_1 \leq x_2$ (resp. $x_1 < x_2$) is read x_1 less than or equal to x_2 (resp. x_1 less than x_2). $x \leq 0$ (resp. $x < 0$) is read x non-positive (resp. x negative). The expressions are transposed in an obvious way if the inequality sign is inverted. One has $0 < 1$.

6. A greatest (resp. least) element of a subset X of $\mathbb{R}$, if it exists, is unique; it is called the maximum (resp. minimum) of X and denoted max X (resp. min X). One defines the absolute value $|x|$ of a number of x by $x = max\{x, -x\}$, and the sign of a number $x \neq 0$ by sign $x = \frac{x}{|x|}$. One has $|x_1 + x_2| \leq |x_1| + |x+2|$.

7. A least upper bound of a subset X of $\mathbb{R}$, if it exists, is unique; it is called the supremum of X and denoted sup$X$. Every non-empty subset X of $\mathbb{R}$ which has a lower bound has a unique greatest lower bound called the infimum of X and denoted inf X.

8. The infimum (resp. supremum) of a real interval is also called its origin (resp. extremity). The length of an interval with origin a and extremity b is $b - a$.

9. By repeated addition of 1 to 0, and repeated subtraction of 1 from 0, one obtains the set $\mathbb{J}$ os integers. (non-negative and non-positive) as a subset of $\mathbb{R}$.

10. A real number of form $\frac{p}{q}$, where $p, q \neq 0 \in \mathbb{J}$, is called a rational number. The set of rationals, a subset of $\mathbb{R}$, is denoted by $\mathbb{Q}$.

11. One can prove that the set $\mathbb{R}$ is not countable.

12. Given a number $x \geq 0$ and a positive integer N, one can prove that there is a unique $y \geq 0$ such that $y^N = x$. One calls y the Nth root of x, and one writes $y = x^{\frac{1}{N}}$.

13. Consider a sequence (x^q) of numbers. Intuitively, one says that (x^q) converges (or tends) to the number x^0 if x^q is as close to x^0 as one wishes provided that q is large enough. In a precise fashion, (x^q) converges to x^0 if, for any number $\varepsilon > 0$, there is an integer $q' > 0$ (depending on ε) such that $q > q'$ implies $|x^q - x^0| < \varepsilon$. One writes $x^q \to x^0$.

14. A sequence which tends to a number is called convergent. It is clear that $x^q \to x^0$ and $x^q \to x^*$ implies $x^0 = x^*$; thus a convergent sequence tends to a unique number called its limit.

15. The elements of $\mathbb{R}$ are also called points. The set $\mathbb{R}$ may be visualized as follows. On a horizontal straight line, two different points are chosen; they will represent 0 and 1, 1 being to the right of 0. An element x of $\mathbb{R}$ is then represented by a point og the straight line at distance $|x|$ from 0, to the right (resp. left) of 0 if x is positive (resp. negative).

16. The letters $\mathbb{N}, \mathbb{J}, \mathbb{Q}, \mathbb{R}$ have throughout this volume the meaning will be introduced latter.

About the real number, we must remember two important theorems below.

THEOREM 2.1. *The set $\mathbb{Q}$ is countable.*

Proof. $Q = \{\frac{p}{q} | p, q \neq 0 \in \mathbb{J}\}$ is one-to-one correspondent to $\{(p, q) | p, q \neq 0 \in \mathbb{J}\}$. As $\mathbb{N} \times \mathbb{N}$ is countable, we have $\{(p, q) | p, q \in \mathbb{J}\}$ is countable which is the supset of $\{(p, q) | p, q \neq 0 \in \mathbb{J}\}$. Therefore, $\mathbb{Q}$ is countable. □

THEOREM 2.2. *If $x, y \in \mathbb{R}, x < y$, there is a rational r such that $x < r < y$.*

The left question is how to construct $\mathbb{R}$ and two approaches are available.

1. Start with the set of natural numbers N, construct the set $\mathbb{Z}$ of all integers $(\mathbb{N} \cup \{0\} \cup (-\mathbb{N}))$, and next, the set $\mathbb{Q}$ of rationals and then the set of $\mathbb{R}$ of all numbers either as the set of all Cauchy sequences of rationals or as Dedekin cuts. The step going from $\mathbb{Q}$ to $\mathbb{R}$ via Cauchy sequences is also available for completing any incomplete metric space.

2. Define the set of real numbers, $\mathbb{R}$, as a set that satisfies three sets of axioms. The first set is algebraic involving addition and multiplication. The second set if on ordering that, with the first, makes $\mathbb{R}$ an ordered field. The third set is a single axiom known as the completeness axiom. Thus $\mathbb{R}$ is defined as complete ordered field.

Here comes definitions of Group, Ring and Field.

DEFINITION 2.1. *A group is a set G, together with an operation $\cdot$ (called the group law of G) that combines any two elements a_1 and a_2 to form another element, denoted $a_1 \cdot a_2$. To qualify as a group, the set and operation, $(G, \cdot)$, must satisfy four requirements known as the group axioms:*

1. *Closure: $\forall a_1, a_2 \in G$, the result of operation is also in G, i.e., $a_1 \cdot a_2 \in G$.*

2. *Associativity: $\forall a_1, a_2, a_3 \in G$, we have $(a_1 \cdot a_2) \cdot a_3 = a_1 \cdot (a_2 \cdot a_3)$.*

3. *Identity element: $\exists e \in G$ such that, $\forall a \in G$, the equation $e \cdot a = a \cdot e = a$ holds. Such an element is unique, called identity element.*

4. *Inverse element: $\forall a \in G$, $\exists a^{-1}$ (an inverse element) such that $a \cdot a^{-1} = a^{-1}a = e$.*

A group with the property that $a_1 \cdot a_2 = a_2 \cdot a_1, \forall a_1, a_2 \in G$ is called Abelian or commutative and called non-Abelian or non-commutative otherwise.

DEFINITION 2.2. *A ring is a set R required with two binary operations $+$ and $\cdot$ satisfying three sets of axioms, called the ring axioms:*

1. *R is an Ablian group under addition $+$, meaning that: Commutativity, Associativity, Additive identity and Additive inverse.*

2. *R is a monoid under multiplication $\times$, meaning that: Closure, Associativity, Multiplicative identity. (inverse element unnecessary)*

3. *Multiplication is distributive with respect to addition: $a_1 \times (a_2 + a_3) = a_1 \times a_2 + a_1 \times a_3, (a_2 + a_3) \times a_1 = a_2 \times a_1 + a_3 \times a_1$.*

If we add commutativity to the second axioms of ring ,i.e, $\forall a_1, a_2 \in R, a_1 \times a_2 = a_2 \times a_1$, we get the Abeian Ring (or Commutative Ring).

DEFINITION 2.3. *With an Abelian Ring, if we add the multiplicative inverse addition to it, we will get a field F. That is,*

1. *F is an Ablian group under addition $+$: commutativity, associativity, addition identity, additive inverse.*
2. *F is a monoid under multiplication $\times$: commutativity, associativity, multiplicative identity, multiplicative inverse.*
3. *Multiplication is distributive with respect to addition.*

If there is an element 1 in R such that $1 \neq 0$ and $1 \times a = a \times 1 = a$ for each element $a \in R$, we say that R is a ring with unity or identity. A commutative ring R with identity is called an integral domain if, for every $a_1, a_2 \in R$ such that $a_1 \times a_2 = 0$, either $a_1 = 0$ or $a_2 = 0$. A division ring is a ring R, with an identity, in which every non-zero element in R is a unit; that is, for each a in R with $a \neq 0$, there exists a unique element a^{-1} such that $a^{-1} \times a = a \times a^{-1} = 1$. A commutative division ring is called a field.

$$\text{Ring: } \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C} \tag{2.12}$$

$$\text{Field: } \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C} \tag{2.13}$$

DEFINITION 2.4. *A vector space over a field F is a set V together with two operations that satisfy the eight axioms listed below.*

1. *The first operation, called vector addition or simply addition $+ : V \times V \to V$, takes any two vectors v_1 and v_2 and assigns to them a third vector which is commonly written as $v_1 + v_2$, and called the sum of these two vectors.*
2. *The second operation, called scalar multiplication $\cdot : F \times V \to V$, takes any scalar a and any vector v and gives another vector $a \cdot v$.*

*Elements of V are commonly called **vectors**. Elements of F are commonly called **scalars**.*

1. *Commutativity: $\forall v_1, v_2 \in V, v_1 + v_2 = v_2 + v_1 \in V$.*
2. *Associativity: $\forall v_1, v_2, v_3 \in V, (v_1 + v_2) + v_3 = v_1 + (v_2 + v_3) \in F$.*
3. *Additive identity: $\exists 0 \in V, \forall v \in V, 0 + v = v + 0 = v$.*
4. *Additive inverse: $\forall v \in V, \exists - v \in F$ such that $v + (-v) = (-v) + v = 0$.*
5. *Multiplicative identity in field F for scalar multiplication: $\forall v \in V, 1 \cdot v = v$.*
6. *Compatibility of scalar multiplication with field multiplication: $\forall a_1, a_2 \in F, \forall v \in V, a_1 \cdot (a_2 \cdot v) = (a_1 \cdot a_2) \cdot v$.*
7. *Distributivity of scalar multiplication with respect to vector addition: $\forall a \in F, \forall v_1, v_2 \in V, a \cdot (v_1 + v_2) = a \cdot v_1 + a \cdot v_2$.*
8. *Distributivity of scalar multiplication with respect to field addition: $\forall a_1, a_2 \in F, \forall v \in V, (a_1 + a_2) \cdot v = a_1 \cdot v + a_2 \cdot v$.*

Axiom 2.1. *(Order Axioms) There is a subset $\mathbb{P}$ of the set $\mathbb{R}$ of real numbers, called the set of strictly positive numbers such that:*

1. *For any real number $a \in \mathbb{R}$, exactly one of the following holds: $a = 0$ or $a \in \mathbb{P}$ or $-a \in \mathbb{P}$.*
2. *If a_1 and $a_2 \in \mathbb{P}$, $a_1 + a_2 \in \mathbb{P}$.*
3. *If a_1 and $a_2 \in \mathbb{P}$, $a_1 \cdot a_2 \in \mathbb{P}$.*

DEFINITION 2.5. *(The Less Than Relation) We now define*

- *$a_1 < a_2$ to mean $a_2 - a_1 \in \mathbb{P}(a_2 - a_1 \in \mathbb{R}_{++})$*
- *$a_1 \leq a_2$ to mean $a_2 - a_1 \in \mathbb{P}$ or $a_1 = a_2(a_2 - a_1 \in \mathbb{R}_{+})$*
- *$a_1 > a_2$ to mean $a_1 - a_2 \in \mathbb{P}(a_1 - a_2 \in \mathbb{R}_{++})$*
- *$a_1 \geq a_2$ to mean $a_1 - a_2 \in \mathbb{P}$ or $a_1 = a_2(a_1 - a_2 \in \mathbb{R}_{+})$*

where $\mathbb{P} = \mathbb{R}_{+} \ \{0\} = \mathbb{R}_{++}$. $a_1 < a_2$ iff (if and only if) $a_2 > a_1$; $a_1 \leq a_2$ iff $a_2 \geq a_1$.

DEFINITION 2.6. *(Ordered Field) Any field F, together with two operations $\oplus$ and $\otimes$ and two members $0_{\oplus}$ and $0_{\otimes}$ of F, and a subset P of F, which satisfies the Order Axioms, is called an ordered field. For example, $\mathbb{Q}, \mathbb{R}$ are ordered fields.*

2.1.2 Fundamental Theorems of Real Numbers

Axiom 2.2. *(Dedekind Completeness Axiom) Suppose A and B are two (non-empty) sets of real numbers with the properties:*

1. *if $a \in A$ and $b \in B$, then $a < b$.*
2. *every real number is in either A or B ($A \cup B = \mathbb{R}$).*

Then, there is a unique real number c such that

1. *if $a < c$ then $a \in A$;*
2. *if $b > c$ then $b \in B$.*

*Hence if $c \in A$ then $A = (-\infty, c], B = (c, +\infty)$; if $c \in B$ then $A = (-\infty, c), B = [c, \infty)$. The pair of sets $\{A, B\}$ is called a **Dedekind Cut**. The intuition idea of Dedekind Completeness Axiom is that there is no hole in the real numbers.*

Axiom 2.3. *(Dedekind - Cantor Axiom of Continuity of a Real Line) Suppose the pair of sets (A, A^c) is a Dedekind Cut of $\mathbb{R}$ with the property: if $a \in A$ and $b \in A^c$. Then either there exists a maximum in A ($\Rightarrow A = (-\infty, c], A^c = (c, \infty)$) or there exists a minimum in A^c ($\Rightarrow A = (-\infty, c), A^c = [c, \infty)$).*

DEFINITION 2.7. *Let S be a set of real numbers. u is an upper bound for S if $x \leq u, \forall x \in S$. A set S if bounded above if it has an upper bound. b is the least upper bound (or l.u.b. or supremum or sup) for S if b is an upper bound, and moreover $b \leq u$ whatever u is any upper bound for S.*

Let S be a set of real numbers. d is an lower bound for S if $x \geq d, \forall x \in S$. A set S if bounded below if it has an lower bound. I is the greatest lower bound (or g.l.b. or infimum or inf) for S if I is an upper bound, and moreover $I \geq d$ whatever d is any lower bound for S.

THEOREM 2.3. *(Theorem of Supremum) Suppose S is an non-empty set of real numbers which is bounded above. Then S has a supremum in $\mathbb{R}$.*

THEOREM 2.4. *(Theorem of Infimum) Suppose S is an non-empty set of real numbers which is bounded below. Then S has a infimum in $\mathbb{R}$.*

THEOREM 2.5. *Prove: Dedekind - Cantor Axiom and Supremum and Infimum Completeness Axiom are equivalent.*

Proof. Necessity: Suppose that S is an non-empty set of real numbers which is bounded above. Let

$$B = \{x \in \mathbb{R} | x \geq y, \forall y \in S\}, A = \mathbb{R} \backslash B \tag{2.14}$$

Note that $B \neq \varnothing$; and if $x \in S$ then $x - 1 \notin S$, so $A \neq \varnothing$.

Suppose $a \in A, b \in B$, then $a < b$. If $a \geq b$. then a would also be an upper bound for S, i.e., $a \in B$, which contradicts to $a \in R \backslash B$.

Let c be the real number given by Dedekind - Cantor Axiom. We claim that $c = \sup S$.

If $c \in A$ then c is not an upper bound for S and so there exists $x \in S$ with $c < x$. But then $a = \frac{c+x}{2}$ is not an upper bound for S, i.e., $a \in A$, contradicting the fact that from the conclusion of Dedekind - Cantor Axiom that $a \leq c$ for all $a \in A$. Hence $c \in B$. But if $c \in B$ then $c \leq b$ for all $b \in B$; i.e., c is less than any upper bound for S. (The proof for theorem of infimum is similar).

Sufficiency: Suppose $\{A, B\}$ is a Dedekind cut. Then A is bounded above (by every element of B). Let $c = \sup B$. We claim that $a < c \Rightarrow a \in A$ and $b > c \Rightarrow b \in B$.

- Suppose $a < c$. Now every element of B is an upper bound for A, from the first property of Dedekind cut; hence $a \notin B$, as otherwise a would be an upper bound for A which is less than the least upper bound c. Hence $a \in A$.
- Suppose $b > c$. Since c is an upper bound for A, it follows we cannot have $b \in A$, and thus $b \in B$.

(The proof for theorem of infimum is similar) ☐

PROPOSITION 2.1. *Suppose S is an non-empty set of real numbers. Then $I = \inf S$ iff*

1. *$x \geq I, \forall x \in S$;*

2. $\forall \varepsilon > 0, \exists x \in S$ *such that* $x < I + \varepsilon$.

Suppose S *is an non-empty set of real numbers. Then* $b = sup\ S$ *iff*

1. $x \leq b, \forall x \in S$;
2. $\forall \varepsilon > 0, \exists x \in S$ *such that* $x > I - \varepsilon$.

Two definitions of supremum and infimum are equivalent.

THEOREM 2.6. *Suppose* S *is a non-empty set of real numbers. The* $b = sup\ S$ *iff*

1. $x \leq b, \forall x \in S[1]$;
2. $b \leq u, \forall u \in U \equiv \{u | u \geq x, \forall x \in S\}[2]$ *or* $\forall \varepsilon > 0, \exists x \in S, x > b - \epsilon[3]$.

$[1] + [2] \Rightarrow [3]$, *suppose not,* $\exists \varepsilon > 0, \forall x \in S, x \leq b - \varepsilon$, *then* $b - \varepsilon$ *is an upper bound of* S, *contradicting to* $b - \varepsilon < b = sup\ S$.
$[1] + [3] \Rightarrow [2]$, *suppose not,* $\exists u \in U, u < b$, *then from* $[3]$ *we know* $u \notin U$, *which contradicts to* $u \in U$.
The proof for the definitions of infimum is similar.

DEFINITION 2.8. *A sequence is bounded if the set of terms from the sequence is bounded.*

DEFINITION 2.9. *A sequence* $\{x_n, n \in \mathbb{N}\} \subseteq \mathbb{R}$ *is*

1. *increasing if* $x_n \leq x_{n+1}, \forall n \in \mathbb{N}$;
2. *decreasing if* $x_n \geq x_{n+1}, \forall n \in \mathbb{N}$;
3. *strictly increasing if* $x_n < x_{n+1}, \forall n \in \mathbb{N}$;
4. *strictly decreasing if* $x_n > x_{n+1}, \forall n \in \mathbb{N}$;

THEOREM 2.7. *(Theorem on Limit of BOunded Monotone Sequence) Every bounded monotone sequence in* $\mathbb{R}$ *has a limit in* $\mathbb{R}$.

The theorem can be divided into two similar theorems.

THEOREM 2.8. *(Theorem of Limit of Increasing Sequence) Suppose* $\{x_n, n \in \mathbb{N}\}$ *is an increasing sequence of real numbers which is bounded above. Then there exists a limit of* $\{x_n, n \in \mathbb{N}\}$, *or* $\lim\limits_{n\to\infty} x_n$ *exists.*

Proof. Suppose $\{x_n : n \in \mathbb{N}\} \subseteq \mathbb{R}$ is increasing. Since the set of its terms is bounded above, it has a supremum x. We claim that $\lim\limits_{n\to\infty} x_n = x$.

Note that $x_n \leq x, \forall n \in \mathbb{N}$; but if $\varepsilon > 0$ then $\exists N \in \mathbb{N}$ such that $x_N > x - \varepsilon$, as otherwise $x - \varepsilon$ would be ann upper bound. Choose such $N = N(]varepsilon)$. Since $x_N > x - \varepsilon$, then $x_n > x - \varepsilon, \forall n > N$. Hence

$$x - \varepsilon < x_N \leq x \tag{2.15}$$

for all $n > N$. Thus $|x - x_n| < \varepsilon, \forall n > N$ and so $\lim\limits_{n\to\infty} = x$ (ε is arbitrary). □

THEOREM 2.9. *(Theorem of Limit of Decreasing Sequence) Suppose* $\{x_n, n \in \mathbb{N}\}$ *is an decreasing sequence of real numbers which is bounded below. Then there exists a limit of* $\{x_n, n \in \mathbb{N}\}$, *or* $\lim\limits_{n\to\infty} x_n$ *exists.*

Proof. The proof is similar to the last one. □

Then we introduce four Cantor Nested Intervals Theorems.

THEOREM 2.10. *If* $\{I_n = [d_n, u_n], n \in \mathbb{N}\}$ *is a sequence of closed bounded nested intervals, then there exists a real number* $D = sup\{d_n : n \in \mathbb{N}\}$ *such that* $D \in \bigcap\limits_{n=1}^{\infty}$.

Proof. A sequence of nested intervals implies that $I_n \supseteq I_{n'}, \forall n < n', n, n' \in \mathbb{N}$ and thus $d_n < d_{n'}, u_n > u_{n'}$. We know $d_i < u_j, \forall i, j \in \mathbb{N}$, or otherwise, $\exists i, j \in \mathbb{N}, d_i \geq u_j$, then $u_i > d_i \geq u_j > d_j$, i.e., $u_i > u_j, d_i > d_j$, implying $i < j$ and $j > i$, nonsense!

We consider $A \equiv \{d_n | n \in \mathbb{N}\}$. A is bounded above because $\forall n \in \mathbb{N}, d_n < u_1$, i.e., $D = supA = \lim\limits_{n\to\infty} d_n$. So $\forall n \in \mathbb{N}, d_n \leq D$. Moreover, we know $u_n > d, \forall d \in A$, i.e., u_n is the upper bound of A. $D = supA \leq u_n$. So $\forall n \in \mathbb{N}, D_n \in I_n$, i.e., $D \in \bigcap\limits_{n=1}^{\infty} I_n$. □

THEOREM 2.11. *If $\{I_n = [d_n, u_n], n \in \mathbb{N}\}$ is a sequence of closed bounded nested intervals, then there exists a real number $U = sup\{u_n : n \in \mathbb{N}\}$ such that $U \in \bigcap_{n=1}^{\infty}$.*

Proof. The proof is similar to the last one. □

THEOREM 2.12. *(Nested Intervals Theorem) If $\{I_n = [d_n, u_n], n \in \mathbb{N}\}$ is a sequence of closed bounded nested intervals, then there exists two real numbers $D = inf\{d_n : n \in \mathbb{N}\} = \lim_{n\to\infty} d_n, U = sup\{u_n : n \in \mathbb{N}\} = \lim_{n\to\infty} u_n$ such that $[D, U] = \bigcap_{n=1}^{\infty}$.*

Proof. We first prove $[D, U] \subseteq \bigcap_{n=1}^{\infty} I_n$. Let $x \in [D, U]$, then $D \leq x \leq U$. We know that $d_n \leq D, U \leq u_n, \forall n \in \mathbb{N}$ and so $d_n \leq D_n \leq U \leq u_n, \forall n\mathbb{N}$. Therefore, $x \in [d_n, u_n], \forall n \in \mathbb{N}$, i.e., $x \in \bigcap_{n=1}^{\infty} I_n$, $[D, U] \subseteq \bigcap_{n=1}^{\infty} I_n$.

Then we prove $[D, U] \supseteq \bigcap_{n=1}^{\infty} I_n$. Let $x \in \bigcap_{n=1}^{\infty} I_n$. Then $d_n \leq x \leq u_n, \forall n \in \mathbb{N}$. Also $d_n \leq D < U \leq u_n$, we need to show that $D \leq x \leq U$. If $x < D$, then $\exists d_x \in \{d_n : n \in \mathbb{N}\}, d_x > x$ because $D = sup\{d_n : n \in \mathbb{N}\}$ which contradicts to $d_n \leq x, \forall n \in \mathbb{N}$. Thus $D \leq x$, and we have $x \leq U$ in the same way. Therefore $x \in [D, U]$, i.e., $[D, U] \supseteq \bigcap_{n=1}^{\infty} I_n$. Hence we reach the conclusion. □

THEOREM 2.13. *(Nested Intervals Theorem) If $\{I_n = [d_n, u_n]\ n \in \mathbb{N}\}$ is a sequence of closed bounded nested intervals and $\lim_{n\to\infty} |I_n| = \lim_{n\to\infty} (u_n - d_n) = 0$, then $\lim_{n\to\infty} d_n = \lim_{n\to\infty} u_n \in \bigcap_{n=1}^{\infty} I_n$.*

Proof. The non-empty set $A = \{d_n : n \in \mathbb{N}\}$ is an increasing sequence of real numbers which is bounded above by u_1, then it has a limit $D = \lim_{n\to\infty} d_n$. We have $U = \lim \ltimes_{n\to\infty} u_n$ in the same way.

Thus $d_n \leq D \leq U \leq u_n, \forall n \in \mathbb{N}$, and $0 \leq U - D \leq u_n - d_n$. Note $\lim_{n\to\infty} |I_n| = \lim_{n\to\infty} (u_n - d_n) = 0$, then $D = U \in \cap_{n=1}^{\infty} I_n$. □

THEOREM 2.14. *(Bolzano Theorem) Each bounded sequence in $\mathbb{R}$ has a convergent subsequence; each bounded sequence in $\mathbb{R}$ has a limit point.*

Proof. Let $\{x_n : n \in \mathbb{N}\}$ be a bounded sequence, we assume $\{x_n : n \in \mathbb{N}\} \subseteq [D, U]$. We divide closed interval $[D, U]$ into two equal-length parts $[D, \frac{D+U}{2}], [\frac{D+U}{2}, U]$. There exists an interval including infinite items of sequence $\{x_n : n \in \mathbb{N}\}$, this interval is written as $[D_1, U_1]$ and we take $x_{n_1} \in [D_1, U_1]$. We repeat the process and obtain a sequence $\{[D_k, U_k] : k \in \mathbb{N}\}$ of closed bounded nested intervals:

$$[D_1, U_1] \supseteq [D_2, U_2] \supseteq \cdots \supseteq [D_k, U_k] \supseteq \cdots \text{ with } U_k - D_k = \frac{U - D}{2^k} \tag{2.16}$$

with a subsequence $\{x_{n_k} : k \in \mathbb{N}\}$ with $x_{n_k} \in [D_k, U_k], \forall k \in \mathbb{N}$. From Cantor Nested Intervals Theorem there exists a unique point $a \in \bigcap_{k=1}^{\infty} [D_k, U_k]$ and $|x_{n_k} - a| < U_k - D_k = \frac{U-D}{2^k}$, then $\lim_{k\to\infty} x_{n_k} = a$. □

THEOREM 2.15. *(Weierstrass Theorem) Every bounded infinite set in $\mathbb{R}$ has an accumulation point ($A \subseteq X, a \in X, \forall \varepsilon > 0, \exists a_0 \in A$ satisfying $a_0 \in N(a, \varepsilon) \equiv \{x \in X : |x - a| < \varepsilon\}$, then a is a accumulation point of A).*

Proof. We just need to construct an convergent sequence from the bounded infinite set. Let $E \in \mathbb{R}$ be a bounded infinite set and $\forall x \in E, |x| \leq M$. With the same practice from the proof of the last theorem, we construct a infinite sequence of intervals $\{I_n : n \in \mathbb{N}\}$, satisfying

$$I_0 \supseteq I_1 \supseteq \cdots \supseteq I_n \supseteq \cdots, I_n \cap E \text{ is an infinite set} \tag{2.17}$$

where the length of I_n is $\frac{M}{2^{n-1}}, \forall n \in \mathbb{N}$. Note $\lim_{n\to\infty} \frac{M}{2^{n-1}} = 0$, then from Cantor Nested Intervals Theorem, there exists a unique point $x \in \bigcap_{n=1}^{\infty} I_n$. We claim that x is a accumulation point of E.

Note that $\lim_{n\to\infty} \frac{M}{2^{n-1}} = 0$, then $\forall \delta > 0$, $\exists n_0 \in \mathbb{N}$, $\frac{M}{2^{n_0}} < \delta$. The length of I_{n_0+1} is strictly less thn δ, then $I_{n_0+1} \subseteq N(x, \delta)$. Hence $E \cap I_{n_0+1} \subseteq E \cap N(x, \delta)$, i.e., x is an accumulation point of E. □

DEFINITION 2.10. *(Cauchy Sequence) $\{x_n : n \in \mathbb{N}\} \in \mathbb{R}$ is a Cauchy sequence if $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that when $n_1 > N, x_2 > N, |x_{n_1} - x_{n_2}| < \varepsilon$.*

THEOREM 2.16. *(Cauchy Convergence Criterion) The sequence $\{x_n : n \in \mathbb{N}\} \subseteq \mathbb{R}$ is convergent iff it is a Cauchy sequence.*

Proof. Necessity: Let $\{x_n : n \in \mathbb{N}\} \subseteq \mathbb{R}$ be a convergent sequence, and suppose $x = \lim_{n\to\infty} x_n$. $\forall\varepsilon > 0, \exists N \in \mathbb{N}$ such that when $n > N, |x_n - x| < \varepsilon$. It follows that $\forall n_1, n_2 > N$, we have $|x_{n_1} - x_{n_2}| \leq |x_{n_1} - x| + |x_{n_2} - x| < 2\varepsilon$. Thus $\{x_n : n \in \mathbb{N}\}$ is a Cauchy sequence.

Sufficiency: Three methods are available for the proof.

(1) Let $\{x_n : n \in \mathbb{N}\} \subseteq \mathbb{R}$ be a Cauchy sequence. $\forall\varepsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n_1, n_2 > N, |x_{n_1} - x_{n_2}| < \varepsilon$, Then $\forall n > N, |x_n - x_{N+1}| < \varepsilon$ or $|x_n| \leq |x_n - x_{N+1}| + |x_{N+1}| < \varepsilon + |x_{N+1}| \leq M$ where $M = max\{|x_n| : n = 1, 2, \cdots, N + 1\} + \varepsilon$, thus $\{x_n : n \in \mathbb{N}\}$ is bounded.

From Bolzano-Weierstrass Theorem, the bounded sequence has a convergent subsequence $\{x_{n_k} : k \in \mathbb{N}\}$ with $\lim_{k\to\infty} = a$. $\forall\varepsilon > 0, \exists K \in \mathbb{N}$ such that $\forall k > K, |x_{n_k} - a| < \varepsilon$.

Take $k_0 = max\{K + 1, N + 1\}$, then $n_{k_0} \geq n_{N+1} \geq N + 1 > N$. Then, when $n > N, |x_n - x_{n_{k_0}}| < \varepsilon$, and hence $|x_n - a| \leq |x_n - x_{n_{k_0}}| + |x_{n_{k_0}} - a| < 2\varepsilon$, that is $\lim_{n\to\infty} = a$.

(2) From (1) we know $\{x_n : n \in \mathbb{N}\}$ is bounded. Define $y_n = inf\{x_n, x_{n+1}, \cdots\}$ for each $n \in \mathbb{N}$. It follows that $y_{n+1} \geq y_n$ since y_{n+1} is the infimum over a subset of the et corresponding to y_n, i.e., $\{y_n : n \in \mathbb{N}\}$ is an increasing sequence. Moreover, since $\{x_n : n \in \mathbb{N}\}$ is bounded, $\{y_n : n \in \mathbb{N}\}$ is bounded. From Theorem on monotone sequence. there exists $a \in \mathbb{R}$ such that $\lim_{n\to\infty} y_n = a$. $\forall\varepsilon > 0, \exists N_1 \in \mathbb{N}$ such that when $n_1 > N$, $|y_n - a| \leq \frac{1}{2}\varepsilon$ or $a - \frac{1}{2}\varepsilon < y_n < a + \frac{1}{2}\varepsilon$. We claim that $\lim_{n\to\infty} x_n = a$.

As $\{x_n : n \in \mathbb{N}\}$ is a Cauchy sequence, we have $\forall\varepsilon > 0, \exists N_2 \in \mathbb{N}$ such that $\forall n_1 > N_1, n_2 > N_2, |x_{n_1} - x_{n_2}| < \frac{1}{2}\varepsilon$, or $x_{n_1} - \frac{1}{2}\varepsilon < x_{n_2} < x_{n_1} + \frac{1}{2}\varepsilon$. From the definition of $\{y_n : n \in \mathbb{N}\}$, we have $y_n \leq x_n, \forall n \in \mathbb{N}$, then $y_{n_1} - \frac{1}{2}\varepsilon \leq x_{n_1} - \frac{1}{2}\varepsilon < x_{n_2} < x_{n_1} + \frac{1}{2}\varepsilon$. $x_{n_2} \leq inf\{x_{n_1} + \frac{1}{2}\varepsilon : n_1 \geq N_2\} = inf\{x_{n_1} : n_1 \geq N_2\} + \frac{1}{2}\varepsilon = y_{n_1} + \frac{1}{2}\varepsilon$. Thus there exists $N_2 \in \mathbb{N}$ such that when $n_1 > N_2, n_2 > N_2$, $y_{n_1} - \frac{1}{2}\varepsilon < x_{n_2} \leq y_{n_1} + \frac{1}{2}\varepsilon$.

Take $N = max\{N_1, N_2\}$, then, when $n_1 > N$ and $n_2 > N$,

$$a - \varepsilon \leq y_{n_1} - \frac{1}{2}\varepsilon < x_{n_2} \leq y_{n_1} + \frac{1}{2}\varepsilon < a + \varepsilon \tag{2.18}$$

Since $\varepsilon > 0$ is arbitrary, it follows that $\lim_{n\to\infty} = a$.

(3) Redefine $y_n = sup\{x_n, x_{n+1}, \cdots\}$ and the left is just like (2). □

DEFINITION 2.11. *(Open Covering) A collection $\mathscr{H}$ of open sets is an open covering of a set S if every point in S is contained in a set H belonging to $\mathscr{H}$; that is, $S \subseteq \{H \in \mathscr{H}\}$.*

THEOREM 2.17. *Heine - Borel Theorem If $\mathscr{H}$ is an open covering of a closed and bounded subset S of the real line (real number); then S has an open Covering $\tilde{\mathscr{H}}$ consisting of finitely many open sets belonging to $\mathscr{H}$ (Closed and bounded $S \subseteq R^k$ is compact).*

Proof. (Reduction to Absurdity) Let $\mathscr{H}$ be an open covering of $[D, U]$. Assume that $[D, U]$ cannot be covered by finite many open sets belonging to $\mathscr{H}$. We divide closed interval $[D, U]$ into two equal-length parts $[D, \frac{D+U}{2}], [\frac{D+U}{2}, U]$, There exists at least one small interval that cannot be covered by finite many open sets belonging to $\mathscr{H}$, written as $[D_1, U_1]$. We repeat this step and obtain a sequence $\{D_n, U_n : n \in \mathbb{N}\}$ of closed bounded nested intervals that cannot be covered by finite many open sets belonging to $\mathscr{H}$:

$$[D_1, U_1] \supseteq [D_2, U_2] \supseteq \cdots \supseteq [D_n, U_n] \text{ with } U_n - D_n = \frac{U - D}{2^n} \tag{2.19}$$

From Cantor Nested Intervals Theorem there exists a unique point $a \in \bigcap_{n=1}^{\infty} [D_n, U_n]$. Note that $\mathscr{H}$ is an open covering of $[D, U]$, then there exists $H_0 \in \mathscr{H}$ such that $a \in H_0$. H_0 is a open set, then there exists $\delta > 0$ such that $(x - \delta, x + \delta) \subseteq H_0$.

Note that $\lim_{n\to\infty} D_n = a = \lim_{n\to\infty}$, then there exists an integer N such that, when $n > N$, $D_n \in (x - \delta, x + \delta), U_n \in (x - \delta, x + \delta)$, i.e., $[D_n, U_n] \subseteq (x - \delta, x + \delta) \subseteq H_0, \forall n > N$, which contradicts to that $[D_n, U_n]$ cannot be covered by finite many open sets belonging to $\mathscr{H}$. □

we summarize the basic theorems of real number space in Table 1

Table 1: Basis of real number space

Theorem	Content
Dedekind - Cantor Axiom of Continuity of a Real Line	Dedekind cut $(A, A^c)(i.e., \forall a \in A, b \in A^c \Rightarrow a < b) \Rightarrow \exists$ a maximum in A or a minimum in A^c
Theorem of Supremum and Infimum	bounded above (resp. below) non-empty $E \subseteq \mathbb{R} \Rightarrow \exists$ a supremum (resp. infimum) of E in $\mathbb{R}$
Theorem of Limit of Monotone Sequence	an increasing (resp. decreasing) sequence $\{x_n : n \in \mathbb{N}\} \subseteq \mathbb{R}$ is bounded above (resp. below) $\Rightarrow \exists \lim\limits_{n\to\infty} x_n$
Nested Intervals Theorem	a sequence of closed bounded nested intervals $\{I_n = [d_n, u_n] : n \in \mathbb{N}\}$ $\Rightarrow \exists D = sup\{d_n : n \in \mathbb{N}\}, U = inf\{u_n : n \in \mathbb{N}\}$ such that $[D, U] \subseteq \bigcap\limits_{n=1}^{\infty} I_n$
Bolzano Theorem	Each bounded sequence in $\mathbb{R}$ has a convergent subsequence
Weierstrass Theorem	Every bounded infinite set in $\mathbb{R}$ has an accumulation point
Cauchy Convergence Criterion	the sequence $\{x_n : n \in \mathbb{N}\}$ in $\mathbb{R}$ is convergent $\Leftrightarrow$ it is a Cauchy sequence
Heine - Borel Theorem	$\mathscr{H}$ is an open covering of a closed and bounded subset E of the real line $\Rightarrow E$ has an finite open covering $\tilde{\mathscr{H}}$

2.2 Limits in Real Numbers

2.2.1 Properties and Points of Sets

The set of $\mathbb{R}^N$ is the product of N sets equal to $\mathbb{R}$, i.e., $\prod\limits_{i=1}^{N} \mathbb{R}$. An element of $\mathbb{R}^N$ is an N-tuple of real numbers, $x = (x_N) = (x_1, x_2, \cdots, x_N)$. The nth number in the N-tuple is the nth coordinate of x. $\mathbb{R}^2$ can be visualized in the oxy plane.

$\mathbb{R}^N$ is called the N-dimensional **Euclidean** space. Let $\mathscr{N}$ be a subset $\{1, 2, \cdots, N\}$,then set $\{x \in \mathbb{R}^N | x_n = 0, n \notin \mathscr{N}\}$ is called a **coordinate space** of $\mathbb{R}^N$, with dimension $|\mathscr{N}|$. When $\mathscr{N} = \varnothing, \{(0, 0, \cdots, 0)\}$ is the **origin** of $\mathbb{R}^N$.

Consider a sequence (x^q) of points of $\mathbb{R}^N$. One says that (x^q) **converges** (or tend) to a point x^0 (written as $x^q \to x^0$) if $\forall n = 1, 2, \cdots, N, x^q_n \to x^0_n$. And we say (x^q) is **convergent** and x^0 is its **limit** (obviously, unique).

DEFINITION 2.12. *(Adherence) Let $X \subseteq \mathbb{R}^N$, $x \in \mathbb{R}^N$ is adherent to X if $\exists (x^q) \in X, x^q \to x$, i.e., $\forall \varepsilon > 0, \exists x_0 \in X, |x_0 - x| < \varepsilon$. The set of points of $\mathbb{R}^N$ adherent to X is called the **adherence** of X, and denoted as $\overline{X}$.*

Obviously, $\forall x \in X,$, x is adherent to X as we can take $x_0 = x$, i.e., $X \subseteq \overline{X}$. Moreover, $X \subseteq X' \Rightarrow \overline{X} \subseteq \overline{X'}$($\forall x \in \overline{X}, \forall \varepsilon > 0, \exists x_0 \in X, |x_0 - x| < \varepsilon$, as $x_0 \in X'$ also, so $x \in \overline{X'}$).

Further, we obtain $\overline{\mathbb{Q}} = \mathbb{R}$ as (1) $\mathbb{Q} \subseteq \mathbb{R} \Rightarrow \overline{\mathbb{Q}} \subseteq \overline{\mathbb{R}} = \mathbb{R}$; (2) $\forall x \in \mathbb{R}, \forall \varepsilon > 0, \exists x_0 \in \mathbb{Q}, |x_0 - x| < \varepsilon$ ($\mathbb{Q}$ is dense in $\mathbb{R}$) $\Rightarrow \mathbb{R} \subseteq \overline{\mathbb{Q}}$.

DEFINITION 2.13. *(Closed) Let $X \in \mathbb{R}^N, X$ is **closed** is $\overline{X} \subseteq X$, i.e., $X = \overline{X}$. In $\mathbb{R}^N$, we have*

1. *$\forall X \in \mathbb{R}^N, \overline{X}$ is closed;*
2. *$\forall X \in \mathscr{X}$ is closed $\Rightarrow \bigcap\limits_{X \in \mathscr{X}} X$ is closed;*
3. *$\forall X \in \mathscr{X}$ is closed and $|\mathscr{X}|$ is finite $\Rightarrow \bigcup\limits_{X \in \mathscr{X}} X$ is closed;*
4. *$\forall S \in \mathbb{R}^N, \exists X \in S$ such that $S \subseteq \overline{X}$ and X is countable.*

THEOREM 2.18. *$\forall X \in \mathbb{R}^N, \overline{X}$ is closed.*

Proof. We just need to prove $\overline{\overline{X}} \subseteq \overline{X}$. $\forall \varepsilon > 0, \forall x \in \overline{\overline{X}}, \exists x_0 \in \overline{X}$ such that $d(x, x_0) < \frac{\varepsilon}{2}$ and $\exists x_1 \in X$ such that $d(x_0, x_1) < \frac{\varepsilon}{2}$. Hence $d(x, x_1) \leq d(x, x_0) + d(x_0, x_1) < \varepsilon$, implying $x \in \overline{X}$. Then $\overline{\overline{X}} \subseteq \overline{X}$. □

We can define above connections in a specific set: let $S \in \mathbb{R}^N$ and $X \subseteq S$, we define the adherence of X in S as $\overline{X} \cap S$ and X is closed in S if $\overline{X} \cap S \subseteq X$. Moreover, above properties still hold if we substitute S for $\mathbb{R}^N$.

THEOREM 2.19. *$\forall S \in \mathbb{R}^N, \exists X \in S$ such that $S \subseteq \overline{X}$ and X is countable.*

Proof. We construct such a set from $\mathbb{Q}^N$. Q^N is countable and $\overline{\mathbb{Q}^N} = \mathbb{R}^N$. Let $\mathbb{Q}^N = \{q_1, q_2, \cdots\}$ and

$$T = \{(n,k) \in \mathbb{N}^2 | S \cap B(q_n, \frac{1}{k}) \neq \varnothing\} \tag{2.20}$$

$\forall (n,k) \in T$ we can take $y_{n,k} \in S \cap B(q_n, \frac{1}{k})$ and obtain a countable set $X = \{y_{n,k} | (n,k) \in T\}$. Then we claim $S \subseteq \overline{X}$. Suppose $\forall x \in S, \forall \varepsilon > 0$ such that $\frac{2}{k} < \varepsilon$. As $\overline{\mathbb{Q}^N} = \mathbb{R}^N, \exists n \in \mathbb{N}$ such that $q_n \in B(x, \frac{1}{k})$. That implies $x \in B(q_n, \frac{1}{k})$ and $(n,k) \in T$. Then, $y_{n,k} \in X$ and $d(x, y_{n,k}) < d(x, q_n) + d(q_n, y_{n,k}) < \frac{2}{k} < \varepsilon$. Hence $S \subseteq \overline{X}$. □

Then we can say, an arbitrary subset S of $\mathbb{R}^N$ contains a countable set X which is dense in S, i.e., $\forall x \in S, \exists x_0 \in X$ such that $\forall \varepsilon > 0, |x_0 - x| < \varepsilon$. Further, $\mathbb{Q}^N$ is dense in $\mathbb{R}^N$.

DEFINITION 2.14. *(Interior) Let $X \subseteq \mathbb{R}^N$, a point $x \in S$ is **interior** to X if $x \notin \overline{\mathbf{C}X}$, i.e., x is completely surrounded by points of X. The interior of X is the set of points interior to X, i.e., $\mathbf{C}(\overline{\mathbf{C}X})$. Moreover, let $X \subseteq S$, $x \in S$ is said to be interior to X in S if $x \in \mathbf{C}(\overline{\mathbf{C}_S X})$.*

DEFINITION 2.15. *(Boundary Point) Let $X \subseteq \mathbb{R}^N$, $x \in \mathbb{R}$ is a **boundary point** of X if $x \in \overline{X} \cap \overline{\mathbb{C}X}$. Interestingly, boundary points of $\mathbb{Q}^N$ is $\mathbb{R}^N$. ($\mathbb{R}^N \backslash \mathbb{Q}^N$ is dense in $\mathbb{R}^N$)*

DEFINITION 2.16. *(Exterior) Let $X \subseteq \mathbb{R}^N$, $x \in \mathbb{R}^N$ is a **exterior** point of X if $x \in \mathbf{C}\overline{X}$. Actually, the interior, the boundary and the exterior from a partition of $\mathbb{R}^N$; the interior, the boundary form a partition of the adherence of X, which indicates X is closed iff it contains its boundary.*

DEFINITION 2.17. *(Cube) Let $x = (x_1, x_2, \cdots, x_N) \in \mathbb{R}^N$, $|x| = \max\{x_1, x_2, \cdots, x_N\}$ and $M \in \mathbb{R}_+$, the set $\mathbb{M} = \{x \in \mathbb{R}^N | |x| \leq M\}$ is a closed **cube** of $\mathbb{R}^N$ with center $(0, 0, \cdots, 0)$, edge $2M$.*

DEFINITION 2.18. *(Bounded) A set $X \subseteq \mathbb{R}^N$ is **bounded** if it is contained in some closed cube $\mathbb{M}$. Otherwise, there is a point in X which is arbitrarily far from the origin.*

THEOREM 2.20. *A subset S of $\mathbb{R}^N$ is said to be compact if it is closed and bounded. If $S_1, S_2, \cdots, S_K$ are closed (resp. compact) subsets of $\mathbb{R}^{N_1}, \mathbb{R}^{N_2}, \cdots, \mathbb{R}^{N_K}$, then $\prod_{k=1}^{K} S_k$ is a closed (resp. compact) subset of $\mathbb{R}^{\sum_{k=1}^{K} N_k}$.*

Proof. We just prove the second part. Let $x = (x_1, x_2, \cdots, X_K) \in \prod_{k=1}^{K} S_k$ where $x_k \in S_k, k = 1, 2, \cdots, K$.

(1) $\forall x \in \overline{\prod_{k=1}^{K} S_k}$, there exists $\{x^q, q \in \mathbb{N}\} \prod_{k=1}^{K} S_k$ such that $x^q \to x$. Hence $\forall k = 1, 2, \cdots, K$, we have $x_k^q \to x_k$. As S_k is closed, $x_k \in S_k$. Hence $x \in \prod_{k=1}^{K} S_k$, i.e., $\prod_{k=1}^{K} S_k$ is closed.

(2) If $S_k, k = 1, 2, \cdots, K$ are compact, then they are bounded in cubes with edge $2M_1, 2M_2, \cdots, 2M_K$ respectively. Let $M = \max\{M_1, M_2, \cdots, M_K\}$, then $\forall x \in \prod_{k=1}^{K}$, we have $|x| < M$. Thus, $\prod_{k=1}^{K} S_k$ is bounded. As it is also closed, so it is compact. □

DEFINITION 2.19. *(Connected) Let $X \subseteq \mathbb{R}^N$ is said to be **connected** if it is not a union of two non-empty, disjoint subsets closed in S. In other words, S is connected if it cannot be partitioned into two non-empty subsets closed in S (it is of one piece).*

THEOREM 2.21. *A subset of $\mathbb{R}$ is connected iff it is an interval.*

Proof. (1) First show only if part. Suppose A is connected and not an interval. Then we can take $a_1, a_2, a \in \mathbb{R}$ such that $a_1 < a < a_2$ and $a_1 \in A, a_2 \in A, a \notin A$. Let $A_1 = A \cap (-\infty, a], A_2 = A \cap [a, \infty)$, then $A = A_1 \cup A_2$ and $A_1 \cap A_2 = \varnothing$. As $(-\infty, a], [a, \infty)$ are non-empty and closed, A_1, A_2 must be closed and non-empty. Hence A is not connected.

(2) Next show the if part. Let A be an interval of $\mathbb{R}$. Suppose A is not connected, then $\exists A_1, A_2$ such that $A_1 \cup A_2 = A, A_1 \cap A_2 = \varnothing$ and they are non-empty and closed. Then we take $a_1 \in A_1$ and $a_2 \in A_2$ and let $a_1 < a_2$ without loss of generality. Define $a = sup\{t \in A_1 | t < a_2\}$ and note that $a_1 \leq a \leq a_2$ and then $a \in A$ (interval).

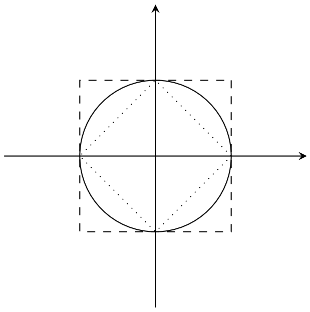


Figure 1: A_1(dashed line), A_2(real line) and A_∞(dotted line)

If $a \in A_1$, then $a \neq a_2$, so $a < a_2$. As $a \notin A_2$ and A_2 is closed, then $\exists \Delta > 0, \forall x \in A_2, |x-a| > \varepsilon$, i.e., $\forall x < a-\varepsilon, x \notin A_2$. $\forall x > a, x \notin A_1$ (or otherwise, $a = x$ as a is the supremum of A_1). Therefore $\forall x \in (a, a+\varepsilon), x \notin A_1 \cup A_2 = A$, contradicting to $x \in [a_1, a_2] \Rightarrow x \in A$.

If $a \in A_2$, then $a \notin A_1$ and $a_1 < a$. As $a \notin A_1$ and A_1 is closed, we know $\exists \varepsilon > 0, \forall x \in A_1, |x-a| > \varepsilon$. Hence $\forall x \in (a-\varepsilon, a), x \notin A$ (the same logic) but x should belong to A as $a_1 \leq x \leq a_2$, contradiction. □

DEFINITION 2.20. *(Metric) $\forall x^1, x^2 \in \mathbb{R}^N$, we define the **metric** (distance) between them is*

$$\rho_p(x^1, x^2) = \{\sum_{n=1}^{N} |x_n^1 - x_n^2|^p\}^{\frac{1}{p}}, p \geq 1 \tag{2.21}$$

Usually, we take $p = 1, 2, \infty$: $\rho_1(x^1, x^2) = \sum_{n=1}^N |x_n^1 - x_n^2|, \rho_2(x^1, x^2) = \{\sum_{n=1}^N |x_n^1 - x_n^2|^2\}^{\frac{1}{2}}, \rho_\infty(x^1, x^2) = \max\{|x_n^1 - x_n^2|, \forall n = 1, 2, \cdots, N\}$.

THEOREM 2.22. *Plot $A_1 = \{x \in \mathbb{R}^2 | \rho_1(x, 0) \leq 1\}, A_2 = \{x \in \mathbb{R}^2 | \rho_2(x, 0) \leq 1\}, A_\infty = \{x \in \mathbb{R}^N | \rho_1(x, 0) \leq 1\}$ and show $A_\infty \subseteq A_2 \subseteq A_1$.*

Moreover, let $B_1 = \{x \in \mathbb{R}^2 | \rho_1(x, 0) \leq 2\}, B_2 = \{x \in \mathbb{R}^2 | \rho_2(x, 0) \leq 2\}, B_\infty = \{x \in \mathbb{R}^N | \rho_1(x, 0) \leq 2\}$, we have $A_2 \subseteq B_\infty, A_1 \subseteq B_2 \subseteq B_1$, which tell us that these metrics are the same if we extend 2 to ∞.

Metrics have the following properties:

1. Positivity: $\rho(x^1, x^2) \geq 0$; $\rho(x^1, x^2) = 0$ iff $x^1 = x^2$.
2. Symmetry: $\rho(x^1, x^2) = \rho(x^2, x^1)$;
3. Triangle inequality: $\rho(x^1, x^2) \leq \rho(x^1, x^3) + \rho(x^2, x^3)$.

Conversely, any function satisfying the three properties is a metric. From the triangle inequality, $\rho(x^1, x^2)$ is a continuous function on (x^1, x^2) if $\lim\limits_{n\to\infty} \rho(x_n^1, x^1) = 0$ and $\lim\limits_{n\to\infty} \rho(x_n^2, x^1) = 0$ then $\lim\limits_{n\to\infty} \rho(x_n^1, x_n^2) = \rho(x^1, x^2)$. In particular, we have $\lim\limits_{n\to\infty} \rho(x^1, x_n^2) = \rho(x^1, x^2) = \lim\limits_{n\to\infty} \rho(x_n^1, x^2)$.

Especially, usually we define the metric with $p = 2$, denoted as $||x||$. Then $\forall x = (x_1, x_2, \cdots, x_N) \in R^N$, its metric is $||x|| = \rho(x, 0) = \sqrt{\sum_{n=1}^N (x_n^2)}$ which satisfying

1. Positivity: $||x|| \geq 0$; $||x|| = 0$ iff $x = 0$;
2. Homogeneity: $||\alpha x|| = |\alpha| ||x||$, α is a constant;
3. Triangle inequality: $||x^1 + x^2|| \leq ||x^1|| + ||x^2||$.

Then we come to some more fancy conceptions and theorems about neighborhood, accumulation point and etc.

DEFINITION 2.21. *(Neighborhood) The set of all points in $\mathbb{R}^N$ whose distance to some given point x_0 is strictly less than δ is called **neighborhood** with center x_0 and radius δ (without boundary), written as $N(x_0, \delta) = \{x \in \mathbb{R}^N | \rho(x, x_0) < \delta\}$.*

*When it must not mention point x_0 or radius δ, we just say a neighborhood. The set of all points in $\mathbb{R}^N$ whose distance to some given point x_0 is strictly less than δ is called **neighborhood** with center x_0 and radius δ (with boundary), written as $B(x_0, \delta) = \{x \in \mathbb{R}^N | \rho(x, x_0) \leq \delta\}$.*

Sometimes we delete the center x_0 and get neighborhood with center x_0 and radius δ without center x_0 written as $D(x_0, \delta) = \{x \in \mathbb{R}^N | 0 < \rho(x, x_0) < \delta\} = \{x \in \mathbb{R}^N | \rho(x, x_0) < \delta, x \neq x_0\} = N(x_0, \delta) \backslash \{x_0\}$.

Then we define the interior, exterior, and boundary in a different way.

DEFINITION 2.22. *(Interior and Exterior) Let $E \subseteq \mathbb{R}^N$ and $p_0 \in \mathbb{R}^N$. If $\exists N(P_0, \delta)$ such that $N(p_0, \delta) \subseteq E$ (resp. $N(p_0, \delta) \cap E = \varnothing$), then p_0 is one of the **interior** (resp. **exterior**) points of E. The set of all interiors is called the interior, written as E°.*

DEFINITION 2.23. *(Boundary) Let $E \subseteq \mathbb{R}^N$ and $p_0 \in \mathbb{R}^N$. If $\exists N(P_0, \delta)$ such that some $p \in N(p_0, \delta), p \in E$ and some $p' \in N(p_0, \delta), p' \notin E$, then p_0 is one of the **boundary** points of E*

DEFINITION 2.24. *(Accumulation) Let $E \subseteq \mathbb{R}^N, p_0 \in \mathbb{R}^N$. If $\forall N(p_0, \delta)$ with center p_0 there exists infinite points in $N(p_0, \delta)$ belonging to E, then p_0 is one of **accumulation** (cluster, limit) points of E. Any interior is accumulation point of the same set from definitions.*

DEFINITION 2.25. *(Derived Set) The set of all accumulation points of $E \subseteq \mathbb{R}^N$ is called the **derived set**, written as E'.*

DEFINITION 2.26. *(Closure) Let $E \subseteq \mathbb{R}^N$. $E \cup E'$ is called **closure** of E, written as $\overline{E} = E \cup E'$ (note that the closure is equivalent to the adherence from definitions, written in the the same way).*

Then we give some theorems about derived sets and interiors.

THEOREM 2.23. *Let $E \subseteq \mathbb{R}^N$ and $p_0 \in \mathbb{R}^N$. $p_0 \in E'$ iff p_0 is one of the limit points of E, i.e., there exists a sequence of different points $p_n \in E$ such that $\lim\limits_{n\to\infty} \rho(p_n, p_0) = 0$.*

Proof. Sufficiency: According to $\exists(p_n)$, $\lim\limits_{n\to\infty} \rho(p_n, p) = 0, \forall N(p_0, \delta)$ with center p_0, we have $\exists N \in \mathbb{N}, \forall n > N$ such that $\rho(p_n, p_0) < \delta$, then $p_n \in N(p_0, \delta)$ and such p_n is infinite.

Necessity: We need to construct such a sequence with limit p_0 from $p_0 \in E'$, and we just need to narrow the neighborhood and take different points in every neighborhood.

Define $\delta_n = \frac{1}{n} > 0$. $p_0 \in E'$, then there exists infinite points in $N(p_0, \delta_1)$ belonging to E. We take p_1 from it arbitrarily. $p_0 \in E'$, and there still exists infinite points in $N(p_0, \delta_1) \backslash \{p_1\}$ belonging to E, then we take p_2 arbitrarily and $p_1 \neq p_2$. Repeat this step (remember to delete points chosen) and we get a sequence of points $(p_n) \subseteq E$ with $|p_n - p_0| < \frac{1}{n}$. So $\lim\limits_{n\to\infty} \rho(p_n, p) = 0$. □

THEOREM 2.24. $A_1 \subseteq A_2 \Rightarrow A_1' \subseteq A_2', \overline{A_1} \subseteq \overline{A_2}$. *(Obvious)*

THEOREM 2.25. $(A_1 \cup A_2)' = A_1' \cup A_2'$.

Proof. (1) $A_1, A_2 \subseteq A_1 \cup A_2 \Rightarrow A_1', A_2' \subset (A_1 \cup A_2)'$, then $A_1' \cup A_2' \subseteq (A_1 \cup A_2)'$.

(2) If $p \in (A_1 \cup A_2)'$, then $\exists (p_n) subseteq A_1 \cup A_2$ such that $\lim\limits_{n\to\infty} \rho(p_n, p) = 0$. If $p \in A_1'$, then $p \in A_1' \cup A_2'$. If $p \notin A_1'$, then infinite points of (p_n) belong to A_2, i.e., $p \in A_2'$. Therefore, $p \in A_1' \cup A_2'$, $A_1' \cup A_2' \subseteq (A_1 \cup A_2)'$. □

An intuitive extension of this theorem is $(\bigcup\limits_{i=1}^{N} A_n)' = \bigcup\limits_{i=1}^{N} A_n'$.

THEOREM 2.26. $A_1 \subseteq A_2 \Rightarrow A_1^\circ \subseteq A_2^\circ$. *(Obvious)*

THEOREM 2.27. $(A_1 \cap A_2)^\circ = A_1^\circ \cap A_2^\circ$.

Proof. (1) $A_1 \cap A_2 \subseteq A_1, A_2 \Rightarrow (A_1 \cap A_2)^\circ \subseteq A_1^\circ, A_2^\circ \Rightarrow (A_1 \cap A_2)^\circ \subseteq A_1^\circ \cap A_2^\circ$.

(2) $\forall p \in A_1^\circ \cap A_2^\circ$, there exists $N(p, \delta)$ with center p such that $N(p, \delta) \subseteq A_1, A_2$ thus $N(p, \delta) \subseteq A_1 \cap A_2$. Therefore $p \in (A_1 \cap A_2)^\circ, A_1^\circ \cap A_2^\circ \subseteq (A_1 \cap A_2)^\circ$. □

An intuitive extension is $(\bigcap\limits_{n=1}^{N} A_n)^\circ = \bigcap\limits_{n=1}^{N} A_n^\circ$.

THEOREM 2.28. *(Bolzano-Weierstrass Theorem) Let E be a bounded and infinite set in $\mathbb{R}^N$. Then there exists at least an accumulation point p in E, i.e., $E' \neq \varnothing$.*

Proof. The proof is somewhat like the proof of "a bound and closed set must be compact". Let E is bounded in a cube with edge $2M$, then we let $R_0 = \{(x_1, x_2, \cdots, x_N) | |x_n| \leq M, \forall n = 1, 2, \cdots, N\}, E \subseteq R_0$. Then we cut R_0 into 2^N equal little cubes and find one cube containing infinite points of E, denoted as R_1 (or otherwise E is finite). Repeat this step and we obtain a sequence:

$$R_0 \supseteq R_1 \supseteq \cdots \supseteq R_n \supseteq \cdots \tag{2.22}$$

where the edge of R_n is $\frac{M}{2^{n-1}}$ and $R_n \cap E$ has infinite points for $n = 1, 2, \cdots$.

Note $\lim\limits_{n\to\infty} \frac{M}{2^{n-1}} = 0$, then from Cantor Nested Intervals Theorem, there exists a unique point $p \in \bigcap\limits_{n=1}^{\infty} R_n$. We claim p is an accumulation point of E.

$\forall N(p, \delta)$ with center p, $\exists n \in \mathbb{N}$ such that $\sqrt{N}\frac{M}{2^{n-1}} < \delta$. With $p \in R_n, \forall x_1, x_2 \in R_n, |x_1 - x_2| < \sqrt{N}\frac{M}{2^{n-1}}$, we have $\forall x \in R_n, \rho(x, p) < \sqrt{N}\frac{M}{2^{n-1}} < \delta$, thus $R_n \subseteq N(p, \delta)$. Therefore, there exists infinite points of $N(p, \delta)$ in E. □

DEFINITION 2.27. *(Isolated Point) A point p in $E \subseteq \mathbb{R}^N$ is an* ***isolated*** *pint of E if $\exists N(p, \delta), \forall x \in N(p, \delta), x \notin E$. A boundary point is either an isolated point of an accumulation point. A set is isolated if its all points are isolated.*

THEOREM 2.29. *Each isolated set is countable (finite or denumerable).*

Proof. We can establish a mapping from isolated sets to $\mathbb{Q}^N$. $\forall p \in E, \forall \delta > 0$, we take a rational point r_p in a $N(p, \delta_p/2)$. We claim that if $p_1 \neq p_2, r_{p_1} \neq r_{p_2}$.

Suppose not, $\exists r_p \in \mathbb{Q}^N, r_p \in N(p_1, \delta/2) \cap N(p_2, \delta/2)$, then $\rho(p_1, p_2) \leq \rho(p_1, r_p) + \rho(p_2, r_p) < \delta/2 + \delta/2 = \delta$, i.e., $N(p_1, \delta) \cap E = \{p_1, p_2\}$ which contradicts to $\forall p_1 \in E, \exists N(p_1, \delta)$ such that $N(p_1, 2\delta) \cap E = \{p_1\}$.

Therefore there exists one-to-one correspondence form $p \in E$ to $r_p \in \mathbb{Q}^N$. As $\mathbb{Q}^N$ is countable, E is countable. □

THEOREM 2.30. *E is an isolated set iff $E \cap E' = \varnothing$.*

Proof. (1) Sufficiency: Suppose $E \cap E' = \varnothing$ but E is not isolated. Then $\exists p \in E, \forall \delta > 0, \exists p_1 \in E, \rho(p_1, p) < \varnothing$, which implies $p \in E'$, i.e., $p \in E \cap E'$, contradicting to $E \cap E' = \varnothing$.

(2) Necessity: Suppose E is isolated but $E \cap E' \neq \varnothing$. Then $\exists p \in E \cap E'$ such that $\forall \delta > 0, \exists p_1 \in E, \rho(p, p_1) < \delta$, which indicates $p \in E$ is not isolated, contradicting to E is isolated. □

E is a discrete set if $E' = \varnothing$. Any discrete set is isolated but an isolated set may not be discrete, e.g., $\{\frac{1}{n}, n = 1, 2, \cdots\}$ with an accumulation point $\{0\}$.

THEOREM 2.31. *Let $E \subseteq \mathbb{R}^N, p_0 \in \mathbb{R}^N$. $p_0 \in E'$ iff $\forall N(p, \delta)$ of p_0 (i.e., $p_0 \in N(p, \delta)$), there exists a points $p' \neq p_0$ such that $p' \in E$. (that implies such p' is infinite)*

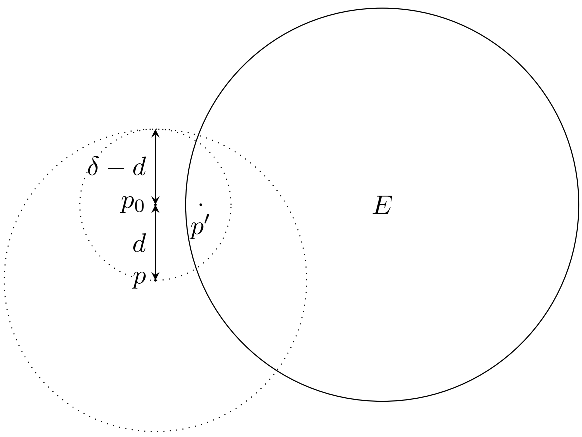


Figure 2: $p_0, N(p, \delta)$ and $p' \in E$

Proof. (1) Necessity: If $p_0 \in E'$, then $\forall N(p, \delta), \exists p' \neq p_0$ such that $p' \in E$. $\forall N(p, \delta), p_0 \in N(p, \delta)$ we have $0 \leq \rho(p, p_0) = d < \delta$. As $p_0 \in E', \forall \delta - d > 0$, there exists infinite points in $N(p_0, \delta - d) \cap E$, and we take $p' \neq p_0$. Then we have $\rho(p, p') \leq \rho(p, p_0) + \rho(p_0, p') < d + \delta - d$. Therefore, $\exists p' \in E \cap N(p, \delta)$ and $p' \neq p_0$. Note that such p' is infinite.

(2) Sufficiency: We need to construct a sequence $(p_n) \subseteq E$ such that $p_n \to p$. Take $\delta_n = \frac{1}{n}$, then $\forall N(p, \delta_n), p_0 \in N(p, \delta_n), n = 1, 2, \cdots$, we have $\exists p'_n \in E, p'_n \in N(p, \delta_n)$, then $\rho(p_0, p'_n) < \frac{2}{n}$. We need add constraints to (q'_n).

As $n = 1$, we take such $\exists p'_1 \in E, p'_1 \in N(p, \delta_1)$ and $0 < d_1 = \rho(p_0, p'_1) < 2\delta_1$. Then let $\delta_2 = \min\{d_1, \frac{1}{n}\}, n = 2$, and take such $p'_2 \in\in N(p, \delta_2) \cap E$ with $p'_2 \neq p_0$. Repeat this step and we obtain a sequence $(p'_n) \subseteq E$ with $p'_i \neq p'_j, \forall i \neq j$ and $p'_n \to p_0$ as $\lim\limits_{n\to} \rho(p_0, p'_n) = d_n < \lim\limits_{n\to\infty} \frac{1}{n} = 0$. Therefore $p_0 \in E'$. □

THEOREM 2.32. *Let $E \subseteq \mathbb{R}^N, p_0 \in \mathbb{R}^N$. Then $p_0 \in E^\circ$ iff $\exists N(p,\delta), p_0 \in N(p,\delta)$ such that $N(p,\delta) \subseteq E$.*

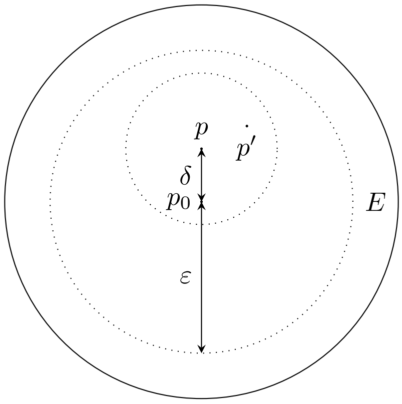


Figure 3: $p_0, N(p,\delta)$ and E

Proof. The proof is similar to the last theorem.

(1) Necessity: If $p_0 \in E^\circ$, then $\exists \varepsilon > 0, N(p_0,\varepsilon) \subseteq E$, we take $p \in N(p_0,\delta), \delta < \varepsilon/2$ and $0 < \rho(p,p_0) = d < \delta$. Then $\forall p' \in N(p,\delta)$ we have $\rho(p',p_0) \le \rho(p',p) + \rho(p,p_0) < 2\delta < \varepsilon$. Thus $N(p,\delta) \subseteq N(p_0,\varepsilon) \subseteq E$.

(2) Sufficiency: $\exists N(p,\delta) \subseteq E, p_0 \in N(p,\delta)$, then $N(p_0, \delta - \rho(p_0,p)) \subseteq N(p,\delta) \subseteq E$. Hence $p_0 \in E^\circ$. □

Let $E \subseteq \mathbb{R}^N$, if $E' \subseteq E$, then E is a closed set. As $\overline{E} = E \cup E'$, we say $\overline{E}$ is the closure of E (E is closed iff $\overline{E} = E$).

THEOREM 2.33. *The closed rectangular in $\mathbb{R}^N$ is a closed set. $\mathbb{R}^N$ is closed, and the closure of $\mathbb{Q}$ is $\mathbb{R}$. (Obvious)*

THEOREM 2.34. *If $f(x)$ is defined on $\mathbb{R}^N$, then $f \in C(\mathbb{R}^N)$ (i.e., continuous function) iff $\forall t \in \mathbb{R}$, sets $E_1 = \{x \in \mathbb{R}^N | f(x) \ge t\}$ and $E_2 = \{x \in \mathbb{R}^N | f(x) \le t\}$ are closed.*

(1) Necessity: $\forall x_0 \in E_1'$, $\exists \{x_k | k \in \mathbb{N}\} \subseteq \mathbb{E}$ and $\lim\limits_{k\to\infty} x_k = x_0$, from the continuity of $f(x)$ and $f(x_k) \ge t$, then $f(x_0) = \lim\limits_{k\to\infty} f(x_k) \ge t$, i.e., $x_0 \in E_1$, E_1 is closed. We can obtain E_2 is closed in the same way.

(2) Sufficiency: Suppose $\exists x_0 \in \mathbb{R}^N$ and $f(x)$ is not continuous at x_0. Then $\forall \{x_k, k \in \mathbb{N}\}$ such that $\lim\limits_{k\to\infty} x_k = x_0$ and $\exists \varepsilon > 0$ such that $\forall k \in \mathbb{N}$, either $f(x_k) \le f(x_0) - \varepsilon$ or $f(x_k) \ge f(x_0) + \varepsilon$. We can take a infinite subsequence of (x_k') that satisfying $f(x_k') \le f(x_0) - \varepsilon$ (or $f(x_k') \le f(x_0) - \varepsilon$, at least one of them holds) and $\lim\limits_{k\to\infty} x_k' = x_0$. Then $f(x_0) \le f(x_0) - \varepsilon$ (or $f(x_0) \ge f(x_0) + \varepsilon$), contradiction!

THEOREM 2.35. *The closure of the open ball $N(x_0,r)$ in $\mathbb{R}^N$ is the closed ball $B(x_0,r)$, i.e., $B(x_0,r) = \overline{N(x_0,r)}$.*

Proof. As $N(x_0,r) \subseteq B(x_0,r)$, we have $\overline{N(x_0,r)} \subseteq \overline{B(x_0,r))} = B(x_0,r)$ ($B(x_0,r)$ is closed). Then we need to show $B(x_0,r) \subseteq \overline{N(x_0,r)}$. $\forall x \in B(x_0,r)$, we define $x_k = \frac{1}{k}x_0 + (1 - \frac{1}{k}x), k = 1,2,\cdots$, then $(x_k) \subseteq N(x_0,r)$ and $x_k \to x$ as $\lim\limits_{k\to\infty} |x - x_k| = \frac{1}{k}(x_0 - x) < \lim\limits_{k\to\infty} \frac{1}{k} r = 0$. Therefore $x \in N'(x_0,r) \subseteq \overline{N(x_0,r)}, B(x_0,r) \subseteq \overline{N(x_0,r)}$. □

2.2.2 Closure and Open Operations

We can define closure operation and any operation satisfying these definitions will produce a closure.

DEFINITION 2.28. *(Closure Operation) If with each subset X of set S, we associate a subset $\overline{X}$ of S such that*

1. $X \subseteq \overline{X}$;
2. $X_1 \subseteq X_2 \Rightarrow \overline{X_1} \subseteq \overline{X_2}$;
3. $\overline{\overline{X}} = \overline{X}$;
4. $\overline{\varnothing} = \varnothing$,

*then we call the correspondence $X \to \overline{X}$ a **closure operation**.*

THEOREM 2.36. *E' and $\overline{E}$ are closed sets.*

Proof. As E' and $\overline{E}$ satisfy the four conditions but the third obviously as we see before. We just need to prove that $E'' = E'$ and $\overline{\overline{X}} = \overline{X}$.

(1) $\forall p_0 \in E''$, then $\forall N(p,\delta), p_0 \in N(p,\delta), \exists p_1 \in E', p_1 \neq p_0, p_1 \in N(p,\delta)$ as we proved before. Then $\forall N(p',\delta'), p_1 \in N(p',\delta'), \exists p_2 \in E, p_2 \neq p_1, p_2 \neq p_0$ such that $p_2 \in E$. Thus, $\rho(p_0,p_1) < \rho(p_0,p_1) + \rho(p_1,p_2) < 2\delta + 2\delta'$, i.e., $p_2 \in N(p_0, 2\delta+2\delta'), p_0 \in E'$, therefore E' is closed.

(2) $\overline{\overline{E}} = (E\cup E')\cup(\overline{E\cup E'}) = (E\cup E')\cup(\overline{E}\cup\overline{E'}) = (E\cup E')\cup(E\cup E'\cup E'\cup E'') = E\cup E'\cup E'' = E\cup E' = \overline{E}$. Note that here we use $\overline{A\cup B} = (A\cup B)\cup(A\cup B)' = (A\cup B)\cup(A'\cup B') = (A\cup A')\cup(B\cup B') = \overline{A}\cup\overline{B}$. An intuitive extension is $\overline{\bigcup\limits_{n=1}^{N} A_n} = \bigcup\limits_{n=1}^{N} \overline{A_n}$. ☐

THEOREM 2.37. *The union of finite closed set is a closed set.*

Proof. Let $A = \bigcup\limits_{n=1}^{N} A_n$. As we know $\overline{E}$ and E' are closed set. We just need to prove that $A' \subseteq A$ or $\overline{A} \subseteq A$ (equivalently).

(1) Note that $A_n' \subseteq A_n$ as A_n closed for all n. Then $A' = (\bigcup\limits_{n=1}^{N} A_n)' = \bigcup\limits_{n=1}^{N} A_n' \subseteq \bigcup\limits_{n=1}^{N} A_n = A$.

(2) Note that $\overline{A_n} \subseteq A_n$ as A_n closed for all n. Then $\overline{A} = \overline{\bigcup\limits_{n=1}^{N} A_n} = \bigcup\limits_{n=1}^{N} \overline{A_n} \subseteq \bigcup\limits_{n=1}^{N} A_n = A$. ☐

THEOREM 2.38. *The intersection of multiple closed sets is a closed set.*

Proof. Let $A = \bigcap\limits_{n=1}^{N} A_n$ and A_n are closed for all n. Similarly, we have two approaches.

(1) $\forall n = 1, 2, \cdots$, we have $A' \subseteq A$ and then $A' = (\bigcap\limits_{n=1}^{N} A_n)' \subseteq \bigcap\limits_{n=1}^{N} A_n' \subseteq \bigcap\limits_{n=1}^{N} A_n = A_n = A$. Then A is a closed set.

(2) $\forall n = 1, 2, \cdots$, we have $\overline{A} \subseteq A$ and then $\overline{A} = \overline{\bigcap\limits_{n=1}^{N} A_n} \subseteq \bigcap\limits_{n=1}^{N} \overline{A_n} \subseteq \bigcap\limits_{n=1}^{N} A_n = A_n = A$. Then A is a closed set.

Moreover, we can substitute $\{A_\gamma\}, \gamma \in \Gamma$ for $\{A_n\}_{n=1}^{\infty}$ and get same conclusions. ☐

Note that we used the following equations:

- $(\bigcup\limits_{n=1}^{N} A_n)' = \bigcup\limits_{n=1}^{N} A_n'$, $\overline{\bigcup\limits_{n=1}^{N} A_n} = \bigcup\limits_{n=1}^{N} \overline{A_n}$;
- $(\bigcap\limits_{n=1}^{\infty} A_n)' \subseteq \bigcap\limits_{n=1}^{\infty} A_n'$, $\overline{\bigcap\limits_{n=1}^{\infty} A_n} \subseteq \bigcap\limits_{n=1}^{\infty} \overline{A_n}$,

But $(\bigcup\limits_{n=1}^{\infty} A_n)' \subseteq \bigcup\limits_{n=1}^{\infty} A_n'$, $\overline{\bigcup\limits_{n=1}^{\infty} A_n} \subseteq \bigcup\limits_{n=1}^{\infty} \overline{A_n}$, e.g., $A_n = \{1/n\}, A' = \{0\}$ but $\{0\} \notin A$.

THEOREM 2.39. *The closure $\overline{A}$ of A is the intersection of all closed sets which contains A.*

Proof. Let $E = \bigcap\limits_{\gamma\in\Gamma} A_\gamma$,and $\forall \gamma \in \Gamma$, A_γ is closed, $E \subseteq A_\gamma$. Then E is closed. Since $\overline{A}$ contains A and is closed, then $E \subseteq \overline{A}$. As $A \subseteq E$, we have $\overline{A} \subseteq \overline{E} = A$, so $E = \overline{A}$. ☐

THEOREM 2.40. *(Cantor Nested Closed Sets Theorem) If $\{A_k, k \in \mathbb{N}\}$ is a sequence of non-empty bounded closed sets in $\mathbb{R}^N$ with $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$, then $\bigcap\limits_{n=1}^{\infty} A_k \neq \varnothing$.*

Proof. We need to construct a sequence (x_q) and prove its limit is in $A = \bigcap\limits_{n=1}^{\infty} A_k$.

We take $B_1 = A_1$, and if $A_1 \backslash A_2 \neq \varnothing$, we take $B_2 = A_1$. If not, we take $B_2 = \varnothing$. Then if $A_2 \neq A_3$, we take $B_3 = A_3$ and if not, we take $B_3 = \varnothing$. Note that $B_2 \supsetneqq B_3$ if $B_2 \neq \varnothing$. Repeat this step and we obtain $\{B_n\}_{n=1}^{\infty}$, and delete all $\varnothing$. Then we have a set sequence $C_1 \supsetneqq C_2 \cdots \supsetneqq \cdots$. Actually, A contains (more precisely, equals) to the intersection of all C_n. Now we take $x_n \in C_n \backslash c_{n+1}$ arbitrarily, if $\{C_n\}$ is finite, we have $\exists N \in \mathbb{N}, \forall n > N, A_n = A_N$. Then $A = A_n$ is non-empty. If $\{C_n\}$ is infinite, we have (x_n) is decreasing and bounded by A_1, then from Bolzano-Weierstrass Theorem, we have $\exists x \in \mathbb{R}^N$, $x_n \to x$. As C_n is closed, so $x \in C_n$ for all n. Then $x \in A$. Therefore A is non-empty. ☐

THEOREM 2.41. *Let $E \subseteq \mathbb{R}$ be a non-empty set. If any subset of E is a closed set, is E a finite set?*

Proof. Not necessarily, e.g., $E = \mathbb{N}$. $\square$

THEOREM 2.42. *Let $E_1, E_2 \subseteq \mathbb{R}$, whether must the equation $\overline{E_1 \cap E_2} = \overline{E_1} \cap \overline{E_2}$ hold?*

Proof. We can reach $\overline{E_1 \cap E_2} \subseteq \overline{E_1} \cap \overline{E_2}$ with $E_1 \cap E_2 \subseteq E_1, E_2$ but not that far, e.g., $E_1 = (0, 1], E_2 = (1, 2)$, then $\overline{E_1 \cap E_2} = \varnothing \neq \{1\} = \overline{E_1} \cap \overline{E_2}$. $\square$

THEOREM 2.43. *Let $E_k \subseteq \mathbb{R}^N, k \in \mathbb{N}$, define $E = \bigcup\limits_{k=1}^{\infty} E_k$. If $x_0 \in E'$, whether must $\exists E_{k_0}, x \in E_{k_0}$?*

Proof. No. That holds just for finite set sequences. E.g., $E_k = \{\frac{1}{k}\}, k = 1, 2, \cdots$. We have $x_0 = 0 \in E'$ but $E'_k = \varnothing, k = 1, 2, \cdots$. $\square$

THEOREM 2.44. *Let $E \subseteq \mathbb{R}^N$. Prove that $\overline{E} = \bigcap\limits_{F \supseteq E} F$ where F is closed.*

This is just the same we have proved before: The closure $\overline{A}$ of A is the intersection of all closed sets which contains A.

Proof. Let $E = \bigcap\limits_{\gamma \in \Gamma} A_\gamma$,and $\forall \gamma \in \Gamma$, A_γ is closed, $E \subseteq A_\gamma$. Then E is closed. Since $\overline{A}$ contains A and is closed, then $E \subseteq \overline{A}$. As $A \subseteq E$, we have $\overline{A} \subseteq \overline{E} = A$, so $E = \overline{A}$. $\square$

THEOREM 2.45. *Let $F \subseteq \mathbb{R}$ be a bounded and closed set and $f(x)$ be a real-valued function defined on F. If $\forall x_0 \in F'$, $f(x) \to +\infty$ as $x \to x_0$ and $x \in F$, prove that F is a countable set.*

Proof. To prove F is countable, we need to establish a one-to-one correspondence from $\mathbb{Q}^N$ to F as we usually do. First we want to show that F is isolated. If not, $\exists x \in F, \forall \delta_1 > 0, \exists x_1 \neq x, x_1 \in F$ such that $x_1 \in N(x, \delta_1)$. Then we can find $x_2 \in N(x, \delta_2), \delta < \min\{\delta_1, \rho(x, \delta_1/2)\}$ and get $x_2 < x_1$. Repeat this step, we have a bounded and decreasing sequence $(x_n - x)$ with limit 0 as $(x_n - x) < \frac{\delta}{2^{(n-1)}}$ for each n. Then $x_n \to x$, so x is an accumulation point, $f(x) \to +\infty$, contradicting to $f(x)$ is real-valued.

Thus, F is isolated, then $\forall x \in F, \exists N(q, \frac{1}{k}), q \in \mathbb{Q}$ such that $\forall y \in F, y \neq x, y \notin N(q, \frac{1}{k})$. As $\mathbb{Q}$ is dense on $\mathbb{R}$, the first part is obvious. The second part is immediately from F is isolated ($\forall x \in F, \exists N(x, \delta), \forall y \in F, y \neq x, y \notin N(x, \delta)$, we take $q \in (x, x + \delta/3)$ and $\frac{1}{k} < \delta/2$). We just need one pair (n, k) for one element of F. And we collect them in T. Therefore, we establish a one-to-one correspondence from F to T, as T is countable, then F is countable. $\square$

THEOREM 2.46. *Let $f \in C(R)$, prove that $F = \{(x, y), f(x) \geq y\}$ is a closed set in $\mathbb{R}^2$.*

Proof. Suppose not, $\exists (x, y) \notin F$, but there exists a sequence $(x_n, y_n) \subseteq F$ and $(x_n, q_n) \to (x, y)$. Then $x_n \to x, y_n \to y$. First x be in the domain of F as $x \in \mathbb{R}$. As $x_n \to x$, from the continuity of f, we have $f(x_n) \to f(x)$, i.e., $y_n \to y \leq f(x)$ as $y_n \leq f(x_n)$. That is $(x, y) \in F$, contradicting to $(x, y) \notin F$. $\square$

THEOREM 2.47. *Try to list countable collection of disjoint dense countable sets in $\mathbb{R}$.*

Proof. $\mathscr{A} = \bigcap\limits_{\gamma \in -\mathbb{N} \cup \mathbb{N}} \{A_\gamma\} \cup \mathbb{N} \cup (-\mathbb{N} \backslash \{0\}), A_\gamma = \{\frac{p}{q} + \gamma, p < q, p, q \in \mathbb{N}_+\}$. $\square$

Then we discuss the open sets. If $E^\circ = E$, every point in E is an interior, then E is an open set.

THEOREM 2.48. *If G is an non-empty set in $\mathbb{R}^N$, then G is an open set iff $\forall x \in G, \exists \delta > 0$ such that $B(x, \delta) \subseteq G$. The empty set $\varnothing$ and the whole space $\mathbb{R}$ are open.*

DEFINITION 2.29. *(Interior Operation) A correspondence $X \to X^\circ$ is called an interior operation if*

1. $X^\circ \subseteq X$;
2. $X_1 \subseteq X_2 \Rightarrow X_1^\circ \subseteq X_2^\circ$;
3. $(X^\circ)^\circ = X^\circ$;
4. $S^\circ = S$.

Then we want to repeat the theorem for open sets we has proved in a similar manner.

THEOREM 2.49. *If $E \subseteq \mathbb{R}^N$, then E° is an open set.*

Proof. To prove a set is open, we need to show that for every point of it, there exists a neighborhood with it as center being contained in the set. Let $E \subseteq \mathbb{R}^N, \forall x \in E^\circ$. Then $\exists \delta > 0, N(x, \delta) \subseteq E$. We claim $N(x, \delta) \subseteq E^\circ$.

$\forall x_1 \in N(x, \delta)$, let $d = \rho(x_1, x)$. As we have proved before, $N(x_1, \delta - d) \subseteq N(x, \delta) \subseteq E$, then x_1 is an interior point, i.e., $N(x, \delta) \subseteq E^\circ$. □

THEOREM 2.50. *The intersection of finite open sets is an open set.*

Proof. Let $G = \bigcap_{n=1}^{N} G_n$ where G_n is open for all n. (1) As G_n is open, we have $G_n^\circ = G$. Then $G^\circ = (\bigcap_{n=1}^{N} G_n)^\circ = \bigcap_{n=1}^{N} G_n^\circ = \bigcap_{n=1}^{N} G_n = G$. Therefore G is an open set.

Note that we use $G^\circ = G$ iff G is open and $(\bigcap_{n=1}^{N} G_n)^\circ = \bigcap_{n=1}^{N} G_n^\circ$, that need more explain.

(2) Suppose $x \in G$, then $x \in G_n$ for all n and there exists $\delta_n > 0$ such that $N(x, \delta_n) \subseteq G_n$. Let $\delta = \min\{\delta_1, \delta_2, \cdots\}$, then $N(x, \delta_n) \subseteq G$, i.e., G is open. □

THEOREM 2.51. *The union of multiple open sets is an open set.*

Proof. Let $G = \bigcup_{n=1}^{\infty} G_n$ where G_n is open. (1) As $G_n \subseteq G$, then $G_n^\circ \subseteq G^\circ$. Thus, $\bigcup_{n=1}^{\infty} G_n^\circ \subseteq G^\circ$. Note that G_n is an open set, $G_n^\circ = G_n$. Then $G = \bigcup_{n=1}^{\infty} G_n = \bigcup_{n=1}^{\infty} G_n^\circ \subseteq (\bigcup_{n=1}^{\infty} G_n)^\circ = G^\circ$.

(2) Suppose $x \in G$, then $\exists n \in \mathbb{N}, x \in G_n$. So $\exists \delta > 0$ such that $N(x, \delta) \subseteq G_n$. Then $N(x, \delta) \subseteq G$. So G is open.

Similarly, we can substitute Γ for $\mathbb{N}$ in this theorem. □

THEOREM 2.52. *(1)The non-empty set in $\mathbb{R}$ is the union of countable disjoint open intervals (certainly including $(-\infty, a), (a, \infty), (-\infty, \infty)$);*

(2) The non-empty open sets G in $\mathbb{R}^N$ is the union of countable disjoint semi-open and semi-closed rectangulars.

Proof. (1) To prove that, we just need to construct a sequence of disjoint interval partitioning the open set. Then disjoint interval is countable as we can take isolated point fom its two end, and as we proved before, isolated points are countable.

Let $G \in \mathbb{R}$ be open. We start the proof by choosing an interior point a of G. Then $\exists \delta > 0$ such that $(a - \delta, a + \delta) \subseteq G$. Then we find the disjoint interval it belongs by $a' = inf\{x : (x, a) \subseteq G\}, a'' = sup\{x : (a, x) \subseteq G\}$. Then $(a', a'') \subseteq G$ and $a' < a < a''$. We call such an open interval (a', a'') the constituent interval I_a of G about a.

If I_{a_1}, I_{a_2} are constituent intervals of G, then they are coincident or disjoint. If not, $I_{a_1} \cap I_{a_2} \neq \varnothing$, then $a_1'' > a_2'$ we take $x_0 < a_2$ from $I_{a_1} \cap I_{a_2}$. Assume $a_1' < a_2'$ (taking a_1'', a_2'' or $>$ makes no difference), then $(a_1', x_0) \subseteq (a_1', a_1'') \subseteq G$. Then $(a_1', a_2) = (a_1', x) \cup (x, a_2) \subseteq G$. So $a_1' \in \{x : (x, a_2) \subseteq G\}$, contradicting to $a_1' < a_2' = inf\{x : (x, a_2) \subseteq G\}$. Thus, I_{a_1}, I_{a_2} must be coincident, and we can just delete one of it. In this way, we construct a disjoint open partition of G.

(2) Firstly, $\mathbb{R}^N$ is divided into countable semi-open and semi-closed cubes with side length of 1 by lattice points (coordinates are all integers), and the while sets is denoted as Γ_0.

Each side of each cube in Γ_0 is bisected, then each cube can be divided into 2^N semi-open and semi-closed cubes with a side length $\frac{1}{2}$. In Γ_0, all the sub-cubes thus created are denoted Γ_1. Continue to divide by this method to obtain the sequence $\{\Gamma_k, k \in \mathbb{N}\}$, where the side length of each cube in Γ_k is 2^{-k}, and this cube is the union of the corresponding 2^k disjoint cubes in Γ_{k+1}. We call the cube thus divided a **dyadic cube**.

Now all cubes in Γ_0 included in G are taken out and written as $\mathscr{H}_0$. Then all cubes in Γ_1 included in $G \backslash \bigcup_{H \in \mathscr{H}} H$ are taken out and written as $\mathscr{H}_0$. Repeat this step and we obtain $\{\mathscr{H}_n\}_{n=1}^{\infty}$.

Since G is an open set, $\forall x \in G, \exists \delta > 0$ such that $B(x, \delta) \subseteq G$. The diameter of the cub in Γ_k approaches to zero as $K \to \infty$, so x will eventually fall into a certain cube in Γ_k. Thus $G = \bigcup_{k=1}^{\infty} \bigcup_{H \in \mathscr{H}} H$.

Another approach: it is an important fact that there exists an open set family Γ, which contains of countable open sets in $\mathbb{R}^N$, such that any open set in $\mathbb{R}^N$ is the union of some open sets in Γ. In fact, Γ can be taken as

$$\{B(q, \frac{1}{k}), q \in \mathbb{Q}^N, k \in \mathbb{N}\} \tag{2.23}$$

Firstly, Γ is a countable set. Secondly, for any point x in the open set G in $\mathbb{R}^N$, there must be $\delta > 0$ such that $B(x, \delta) \subseteq G$. Now take $x' \in \mathbb{Q}$ such that $\rho(x, x') < \frac{1}{k} < \delta/2$ such that $x \in B(x', \frac{1}{k}) \subseteq B(x, \delta) \subseteq G$. Obviously, the union of such $B(x', \frac{1}{k})$ is G. □

THEOREM 2.53. *If $N \geq 2, F \subseteq R^N$ is a closed set, then ∂F is an infinite set.*

Proof. As $N = 1$, and let $F = [0,1]$, we have $F' = \{0,1\}$, finite. But $N \geq 2$. We take $p_1 \in F, p_2 \in \mathbf{C}F$ and obtain a closed set $P = \{\alpha p_1 + (1-\alpha)p_2 | \alpha \in (0,1)\}$. As F is closed, so $P \cap F$ is closed. We can find $\alpha_0 = inf\{\alpha p_1 + (1-\alpha)p_2 \in F, \alpha \in (0,1)\}$. Then we have $p_0 = \alpha_0 + (1-\alpha_0)p_1$ is in ∂F. $\forall \delta > 0, N(p_0, \delta)$ contain two desired points, $x_1 = \beta p_0 + (1-\beta)p_1 \in F, x_2 = \gamma p_0 + (1-\gamma)p_2 \in \mathbf{C}F$ with $\beta, \gamma \in (0,1)$. Since p_0 is decided uniquely by p_1, p_2 and at least one of $F, \mathbf{C}F$ contains infinite elements, so such p_0 is infinite, i.e., ∂F is infinite. □

THEOREM 2.54. *Let $E \subseteq \mathbb{R}^N$, prove that $E^\circ = \mathbf{C}(\overline{\mathbf{C}X})$ and $\partial E = \overline{E} \backslash E^\circ$.*

Proof. (1) The first is immediately from the definition. $x \in E^\circ$ if $\exists B(x,\delta) \subseteq E$ iff $x \notin E^\circ$ if $\forall \delta > 0$, there exists infinite $x_0 \in B(x,\delta), x_0 \notin E$ iff $x \notin E^\circ$ if $\exists \delta > 0$, there exists infinite $x_0 \in B(x,\delta), x \in \mathbf{C}E$ iff $x \notin E^\circ$ if $x \in \overline{\mathbf{C}X}$ iff $x \in E^\circ$ if $x \in \mathbf{C}(\overline{\mathbf{C}X})$.

(2) First we show $\partial F \subseteq \overline{E}$. $\forall x \in F, \forall \delta > 0, \exists x_0 \in E$ such that $x_0 \in N(x,\delta)$. So $x \in \overline{E}$. Then we say $\forall x \in \overline{E}$, we have either $x \in E^\circ$ or $x \in \partial E$ and not both. As $x \in \overline{E} = E \cup E'$. If $x \in E'$, $\forall \delta > 0$ there exists infinite points of E belonging to E. If all, $x \in E^\circ$, if not all, $x \in \partial E$. If $x \in E$ but $x \notin E'$, then x is an isolated point and must in ∂E as $\forall N(x,\delta)$ contains only one $x \in E$ and infinite elements of $\mathbf{C}E$. So we proved that $\partial E \subseteq \overline{E}$ and all elements of $\overline{E}$ can be divided into ∂E and E°. So $\partial E = \overline{E} \backslash E^\circ$. □

THEOREM 2.55. *Prove that the set of discontinuous points of function*

$$f(x_1, x_2) = \begin{cases} x_1 sin(\frac{1}{x_2}), x_2 \neq 0, \\ 0, x_2 = 0 \end{cases}$$

is not a closed set.

Proof. This is easy. We just need to prove that the set is $A = \{(x,0) | x \neq 0, x \in \mathbb{R}\}$, which is not closed as the accumulation point $\{(0,0)\}$ is not in A. We next show that $(0,0)$ is a continuous point of f. Let (x_n, y_n) be any sequence satisfying $x_n \to 0, y_n \to 0$, then $\lim\limits_{n\to\infty} |f(x_n,y_n)| = \lim\limits_{n\to\infty} |x_n sim(\frac{1}{y_n})| < \lim\limits_{n\to\infty} |x_n| = 0$. So $\lim\limits_{n\to\infty} f(x_n,y_n) = f(0,0) = 0$. Thus $(0,0) \notin A$. □

THEOREM 2.56. *Prove that $G \subseteq \mathbb{R}^N$ is an open set iff $G \cap \partial G = \varnothing$; $F \subseteq \mathbb{R}^N$ is a closed set iff $\partial F \subseteq F$.*

Proof. (1) G is open iff $\forall x \in G, \exists N(x,\delta) \subseteq G$ iff $\forall x \in G, \exists N(x,\delta) \subseteq G$ and not $\exists x \in G, \exists N(x,\delta)$ such that $\exists x_0 \in N(x,\delta), x_0 \in \mathbf{C}F$ iff $\forall x \in G, x \notin \partial F$ iff $G \cap \partial G = \varnothing$.

(2) F is closed iff $\overline{F} \subseteq F$ iff $\overline{F} = \partial F \cup F, \partial F \subseteq F$ iff $\partial F \subseteq F$. We just need to show $\overline{F} = \partial F \cup F$. As we show before $\partial F = \overline{F} \backslash F^\circ$, with $F^\circ \subseteq F$, so $\partial F \cup F = \overline{F}$. □

THEOREM 2.57. *Let $G \subseteq \mathbb{R}^N$ be a non-empty open set, $r_0 > 0$. If $\forall x \in G$, define a closed sphere $\overline{B(x,r_0)}$, prove that $A = \bigcup\limits_{x\in G} B(x,r_0)$ is an open set.*

Proof. Actually, $\overline{B(x,r_0)} = B(x,r_0)$ as $B(x,r_0)$ is closed. $\forall x_0 \in A, \exists x \in G$, we have $x_0 \in B(x,r_0)$. Then $B(x_0, r_0 - \rho(x,x_0)) \subseteq B(x,r_0) \subseteq A$, so $x_0 \in A$. Therefore A is open. □

THEOREM 2.58. *Let $F \subseteq \mathbb{R}$ be a closed set, prove that there is a countable subset of E such that $\overline{E} = F$.*

Proof. We just need to offer the E. We claim $E = F \cap \mathbb{Q}$, and E is countable actually. As $E \subseteq F$, $\overline{E} \subseteq \overline{F} = F$. And $\forall x \in F$, we know $\forall \delta > 0, N(x,\delta)$ contains infinite points of $\mathbb{Q} \cap F$ as $\mathbb{Q}$ is dense in $\mathbb{R}$. So $F \subseteq \overline{E}$. Therefore, $\overline{E} = F$. □

THEOREM 2.59. *Let every point in $E \subseteq \mathbb{R}^N$ be an isolated point of E, prove that E is the intersection of an open set and a closed set.*

Proof. We just need to show E is closed and let $E = \mathbb{R}^N \cap E$.

Then we just need to show $E' \subseteq E$. As E is isolated, $\forall x \in E, \exists \delta > 0, N(x,\delta) \cap E = \{x\}$, which indicates we cannot find infinite different points of E staying in a small neighborhood of any point of $\mathbb{R}^N$ to obtain an accumulation point. Therefore, $E' = \varnothing, E' \subseteq E$. Then E is closed. □

THEOREM 2.60. *Let $f \in C([a,b])$ and $f'(x) > 0$ except $\{x_n\} \subseteq [a,b]$, prove that $f(x)$ is strictly increasing.*

Proof. We need to show $\forall x, y \in [a,b], x > y$ implies $f(x) > f(y)$. Obviously, that holds for $x, y \in [a, x_n]$ or $x, y \in [x_n, b]$. If $x \in [a, x_n)$, $y \in (x_n, b]$ we claim $f(x) < f(y)$. $f(y) - f(x) = \int_x^y f'(t)dt = \int_x^{x_n} f'(t)dt + \int_{x_n}^y f'(t)dt > 0$. Then we reach the conclusion. □

Now we discuss the open covering again, and you will know it better.

DEFINITION 2.30. *(Open Covering) Let $E \subseteq \mathbb{R}^N$, Γ is an family of open sets in $\mathbb{R}^N$. If $\forall x \in E$, $\exists G \in \Gamma$ such that $x \in G$, then Γ is called an* ***open covering*** *of E. If $\Gamma' \subseteq \Gamma$ is still an open covering of E, then Γ' is a subcovering of Γ about E.*

LEMMA 2.1. *Any open cover Γ of $E \subseteq \mathbb{R}^N$ contains a countable subcovering.*

Proof. This is an extension of 2.52 and we just need to prove that the subcovering of Γ can be divided into many disjoint open set. Then we can choose an rational number from each one of them, like labeling them in a countable way. □

THEOREM 2.61. *(Heine-Borel Finite Covering Theorem) An open covering of a bounded closed set in $\mathbb{R}^N$ contains a finite subcovering.*

Proof. Let F be a bounded closed set in $\mathbb{R}^N$, and Γ is an open covering of F. From the above Lemma, it can be assumed that Γ consists of countable open sets: $\Gamma = \{G_1, G_2, \cdots\}$. Define $H_k = \overset{K}{\underset{k=1}{\cup}}, L_K = F \cap \mathbf{C}K, K = 1, 2, \cdots$. Obviously, H_K is an open set, L_K is a closed set with $L_K \supseteq L_{K+1}, K = 1, 2, \cdots$. There are two cases:

1. There is K_0 such that L_{K_0} is an empty set, that is $\mathbb{C}H_{K_0}$ does not contains any point of F, then $F \subseteq H_{K_0}$, the theorem is proved;

2. If all L_K is not an empty set, then from Cantor's closed set theorem, there is a point $x_0 \in L_K, K = 1, 2, \cdots$, i.e., $x_0 \in F, x_0 \in \mathbf{C}H_K$ for $K = 1, 2, \cdots$. Thus there is a point $x_0 \in F$ does not belong to every H_K, which is inconsistent with the original assumption, so this case does not exist.

□

Then we give another "neighborhood" expression of this important theorem.

THEOREM 2.62. *(Heine-Borel Finite Covering Theorem) If $\mathscr{H}$ is an open covering of a closed and bounded subset E of $\mathbb{R}$ (for any x in E there exists a neighborhood $H \in \mathscr{H}$ such that $x \in H$); then E has open covering $\tilde{\mathscr{H}}$ consisting of finitely many open sets belonging to $\mathscr{H}$.*

Proof. The proof method belongs to Lebesgue. We complete the proof by two steps.

(1) $\exists \delta > 0$ such that $N(x, \delta)$ with center x in E is contained in some open neighborhood of $\mathscr{H}$. Suppose such δ does not exist. Then $\forall n \in \mathbb{N}_+$, we take $\delta = \frac{1}{n}$, then $\exists x_n \in E$ such that $N(x_n, \frac{1}{n})$ is not contained in any open neighborhood of $\mathscr{H}$. E is bounded and $(x_n) = \{x_n, n = 1, 2, \cdots\} \subseteq E$ is bounded. If (x_n) has infinite different points, then there exists an accumulation point from Bolzano-Weierstrass theorem. If (x_n) has finite points, then there exists at least a point x_0 presented infinite times. In a word, (x_n) has a convergent subsequence (x_{n_k}) with limit x_0. Note that E is closed, so $x_0 \in E$. Since $\mathscr{H}$ is an open covering of a closed and bounded set E, then $\exists N = N(y_0, \eta) \subseteq \mathscr{H}$ such that $x \in N$ (a neighborhood). Then we let $0 < \eta' < \eta - \rho(x_0, y_0)$ such that $N(x_0, \eta') \subseteq N(y_0, \eta)$. Note that $x_{n_k} \to x_0$, then $\exists M \in \mathbb{N}$ such that $\forall k > M$, $\rho(x_{n_k}, x_0) < \frac{\eta'}{2}$ and $\frac{1}{n_k} < \frac{\eta'}{2}$. So $N(x_{n_k}, \frac{1}{n}) \subseteq N(x_0, \eta') \subseteq N(y_0, \eta) \in \mathscr{H}$, which contradicts to $N(x_{n_k}, \frac{1}{n_k})$ does not belong to any neighborhood of $\mathscr{H}$. Therefore there exists such $\delta > 0$, which is called **Lebesgue number**.

(2) Since E is bounded, we can divide E into finite small cubes by using hyperplane parallel axises such that the length in each cube is strictly less than $\delta > 0$. Set these small cubes are $E_1, E_2, \cdots, E_K$ with $E = \overset{K}{\underset{k=1}{\cup}} E_k$, we take arbitrary $x_n \in E_k$ such that $E_k \subseteq N(x_k, \delta)$, then $\exists N_k \in \mathscr{H}$ such that $N(x_k, \delta) \subseteq N_k$. Thus we obtain finite open neighborhood $N_1, N_2, \cdots, N_k$ with $E = \overset{K}{\underset{k=1}{\cup}} E_k \subseteq \overset{K}{\underset{k=1}{\cup}} N(x_k, \delta) \subseteq \overset{K}{\underset{k=1}{\cup}} N_k$. □

THEOREM 2.63. *If F is a bounded closed set in $\mathbb{R}^N$ and G is an open set in $\mathbb{R}^N$ and $F \subseteq G$, then there is $\delta > 0$ such that when $|x| < \delta$, $F + \{x\} \equiv \{y + x | y \in F\} \subseteq G$.*

Proof. $\forall y \in G$, $\exists \delta_y > 0$ such that $B(y, \delta_y) \subseteq G$. As $\{B(y, \frac{\delta_y}{2}) | y \in F\}$ constitutes an open covering of F, according to the finite subcovering theorem, there is $y_1, y_2, \cdots, y_K \in F$ such that $F \subseteq \bigcup_{k=1}^{K} b(y_k, \frac{\delta_{y_k}}{2})$.

Thus, each $y \in F$ belongs to at least one $B(y_k, \frac{\delta_{y_k}}{2})$, and the distance between y and any point in z in $\mathbf{C}G$ is $\rho(y, z) \geq \rho(z, y_k) - \rho(y_k, y) > \delta_{y_k} - \frac{\delta_{y_k}}{2} = \frac{\delta_{y_k}}{2}$. Take $\delta = \frac{1}{2}\min\{\delta_{y_1}, \delta_{y_2}, \delta, \delta_{y_K}\}$, then when $\rho(x, 0) < \delta, y + x \in G$, that is $F + \{x\} \subseteq G$. □

THEOREM 2.64. *Let $E \subseteq \mathbb{R}^N$. If any open covering of F contains finite subcovering, then E is a bounded and closed set.*

Proof. We first show that E is closed. Suppose not, we take $x \in E'$ and $x \notin E$. Then $\exists (x_n) \subseteq E, x_n \to x$. We focus on the open covering of (x_n) and we can take them out from any open covering with its number infinite or finite unchanging. Let $\{N(x_n, \rho(x_n, x_{n+1})/2), n = 1, 2, \cdots\}$ be the open covering of (x_n). Then if we take out any neighborhood from it, the left neighborhood cannot cover all (x_n). The secrete is its limit is out of E. Thus, E is closed.

But the following proof does not need E to be closed first, which is elegant!

We take $y \in \mathbf{C}E$, then $\forall x \in E$, $\exists \delta_x > 0$ such that $N(x, \delta_x) \cap N(y, \delta_x) = \varnothing$ (that is $\delta_x < \rho(x, y)/2$). Then $\{N(x, \delta_x) | x \in E\}$ is an open covering of E. Then there are a finite subcovering, written as $N(x_k, \delta_{x_k}), k = 1, 2, \cdots, K$. Then E is bounded (as K is finite). Now define $\delta_0 = \min\{\delta_{x_1}, \delta_{x_2}, \cdots, \delta_{x_K}\}$, then $N(y, \delta_0) \cap E = \varnothing$, that is $y \notin E'$. Thus $E' \subseteq E$. E is a closed set. □

THEOREM 2.65. *In $\mathbb{R}^N$, $\{G_\gamma | \gamma \in \Gamma\}$ is an open covering of E, can $\{\overline{G_\gamma} | \gamma \in \Gamma\}$ cover $\overline{E}$?*

Proof. Certainly, from Lebesgue's number, $\exists \delta > 0$, $\forall x \in E, \exists \Gamma, N(x, \delta) \subseteq N(y, \eta) \subseteq G_\Gamma$. Then as $N' = B$ with the same center and radius, we have $B(x, \delta) \subseteq B(y, \eta) \subseteq \overline{G_\Gamma}$. As $\forall x \in E'$, x must belongs to some neighborhood (with boundary) of E (especially x is an accumulation points, x stays one the boundary of neighborhood at least). So $\{\overline{G_\gamma} | \gamma \in \Gamma\}$ can cover $\overline{E}$. □

THEOREM 2.66. *Let $\Gamma = \{[a_\alpha, b_\alpha] | \alpha \in [0, 1]\}$ and any two closed intervals in Γ must intersect. Prove $A = \bigcap_{\alpha \in [0,1]} [a_\alpha, b_\alpha] \neq \varnothing$.*

Proof. Let $a_0 = sup\{a_\alpha | \alpha \in [0, 1]\}, b_0 = inf\{b_\alpha | \alpha \in [0, 1]\}$. As $\forall m, n \in [0, 1]$ we have $a_m \leq b_n$, then $a_0 \leq b_0$. If $a_0 = b_0$, then with $a_\alpha \leq b_\alpha, \alpha \in [0, 1]$, we have $a_\alpha = b\alpha = a_0 = b_0, \forall \alpha \in [0, 1]$. Then $A = \{a_0\}$. If $a_0 < b_0$, then $\forall m, n \in [0, 1], a_m \leq a_0 < b_0 \leq b_n$, so $\forall [a_0, b_0] \subseteq [a_m, b_m] \cap [a_n, b_n]$, i.e., $A = [a_0, b_0]$. □

THEOREM 2.67. *Let $F \subseteq \mathbb{R}$ be a non-empty countable closed set, prove that F must contain isolated points.*

Proof. We can reach further: F is isolated. Suppose not, $\exists x \in F$ such that $\forall \delta, N(x, \delta)$ contains infinite points of F, and we can take $(x_n), x_n \to x, x_n \in \mathbb{Q}$ from it. As $\mathbb{Q}$ is dense in $\mathbb{R}$, then (x_n) is dense in $\mathbb{R}$, or we cannot obtain (x_n). So $\overline{(x_n)} \subseteq F$ is an uncountable set in $\mathbb{R}$, contradicting to F is countable and closed. □

THEOREM 2.68. *Let $\{f_n(x)\}$ be a sequence of non-negative decreasing continuous functions on $\mathbb{R}$. If $f_n(x) \to 0$ as $n \to \infty$ on a bounded and closed set F, prove that $f_n(x)$ uniformly converges to zero on F.*

Then we want to discuss the relation between closure operations and interior operations.

THEOREM 2.69. *A one-to-one correspondence can be established between closure operations and interior operations by means of the formula: $X^\circ = \mathbf{C}(\overline{\mathbf{C}X}), \overline{X} = \mathbf{C}(\mathbf{C}X)^\circ$.*

Proof. We need to show they satisfy each four conditions. (1) First $X^\circ = \mathbf{C}(\overline{\mathbf{C}X})$ is an interior operation.

1. $X^\circ = \mathbf{C}(\overline{\mathbf{C}X}) \subseteq \mathbf{C}(\mathbf{C}X)$ (as $\mathbf{C}X \subseteq \overline{\mathbf{C}X}$) $= X$;
2. $X_1 \subseteq X_2 \Rightarrow \mathbf{C}X_1 \supseteq \mathbf{C}X_2 \Rightarrow \overline{\mathbf{C}X_1} \supseteq \overline{\mathbf{C}X_2} \Rightarrow X_1^\circ \subseteq X_2^\circ$;
3. $(X^\circ)^\circ = \mathbf{C}(\overline{\mathbf{C}(\mathbf{C}(\overline{\mathbf{C}X}))}) = \mathbf{C}(\overline{\overline{\mathbf{C}X}}) = \mathbf{C}(\overline{\mathbf{C}X}) = X^\circ$;
4. $S^\circ = \mathbf{C}(\overline{\mathbf{C}S}) = \mathbf{C}\overline{\varnothing} = \mathbf{C}\varnothing = S$.

(2) Then we show $\overline{X} = \mathbf{C}(\mathbf{C}X)^\circ$ is an closure operation.

1. $\overline{X} = \mathbf{C}(\mathbf{C}X)^\circ \supseteq \mathbf{C}(\mathbf{C}X) = X$ as $(\mathbf{C}X)^\circ \subseteq \mathbf{C}X$;
2. $X_1 \subseteq X_2 \Rightarrow \mathbf{C}X_1 \subseteq \mathbf{C}X_2 \Rightarrow (\mathbf{C}X_1)^\circ \supseteq (\mathbf{C}X_2)^\circ \Rightarrow \mathbf{C}(\mathbf{C}X_1)^\circ \subseteq \mathbf{C}(\mathbf{C}X_2)^\circ \Rightarrow \overline{X_1} \subseteq \overline{X_2}$;

3. $\overline{\overline{X}} = \mathbf{C}(\mathbf{C}(\mathbf{C}(\mathbf{C}X)^\circ))^\circ = \mathbf{C}((\mathbf{C}X)^\circ)^\circ = \mathbf{C}(\mathbf{C}X)^\circ = \overline{X}$;
4. $\overline{\varnothing} = \mathbf{C}S^\circ = \mathbf{C}S = \varnothing$.

$\square$

The complement of a closed set is an open set (vice versa) fir $\mathbf{C}F = \mathbf{C}\overline{F} = \mathbf{C}(\overline{\mathbf{C}(\mathbf{C}F)}) = (\mathbf{C}F)^\circ$. Then we can just study properties of closure operations and those of interior operations being deducible immediately on appealing to duality for $\mathbf{C}G = \mathbf{C}G^\circ = \mathbf{C}(\mathbf{C}(\overline{\mathbf{C}G})) = \overline{\mathbf{C}G}$.

Then we can reprove the following theorem in an elegant manner.

THEOREM 2.70. *The intersection of a family $\mathscr{F} = \{F_\gamma | \gamma \in \Gamma\}$ of closed sets is a closed set.*

Proof. Let $F = \bigcap\limits_{\gamma\in\Gamma} F_\gamma$. As $\overline{F}$ is a closure operation, then $\overline{F} \subseteq \overline{F}_\gamma = F_\gamma, \forall\gamma \in \Gamma$. Therefore $\overline{F} \subseteq F$, then $\overline{F} = F$. $\square$

THEOREM 2.71. *The union of a family $\mathscr{G} = \{G_\gamma | \gamma \in \Gamma\}$ of open sets is an open set.*

Proof. Let $G = \bigcup\limits_{\gamma\in\Gamma} G_\gamma$. As G° is an interior operation, then $G_\gamma = G_\Gamma^\circ \subseteq G^\circ, \forall\gamma \in \Gamma$. Therefore $G \subseteq G^\circ$, hence $G^\circ = G$. $\square$

THEOREM 2.72. *The closure $\overline{X}$ of X is the intersection of all the closed sets which contains X.*

Proof. Let E be the intersection of all closed sets which contains X; by the proceeding theorem, E is closed. Since $\overline{X}$ is a closed set which contains X, then $E \subseteq \overline{X}$. But we also have $\overline{X} \subseteq \overline{E} = E$ as $X \subseteq E$. Hence $E = \overline{X}$. $\square$

DEFINITION 2.31. *(Connectedness) The two sets A_1, A_2 in metric space X are said to be separated if $(A_1 \cap \overline{A_2}) \cup (A_2 \cap \overline{A_1}) = \varnothing$. If there exist two separated sets A_1, A_2 such that $X = A_1 \cup A_2$ (certainly $A_1 \cap A_2 = \varnothing$), then X is a **disconnected** space. Otherwise, S is a **connected** space. In other words, a set X is said to be connected if it cannot be partitioned into two non-empty separated sets in S (it is of one piece, perhaps with holes).*

THEOREM 2.73. *For a metric space X, the following statements are equivalent:*

1. *X cannot be divided into two disjoint non-empty open sets.*
2. *X cannot be divided into two disjoint non-empty closed sets.*
3. *The only clopen (closed and open) subsets of X are X and the empty sets.*
4. *The only subsets of X with empty boundary are X and the empty set.*
5. *X cannot be written as the union of two non-empty separated sets.*
6. *All continuous functions from X to $\{0,1\}$ are constant, where $\{0,1\}$ is the two-point space endowed with the discrete metric.*

Proof. We prove this following the logic of Figure 4. $(1) \Rightarrow (2)$: Suppose not (2), then there exist two non-empty,

$$\begin{tikzcd}
(1) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & (3) \arrow[r, leftrightarrow] \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & (4) \\
(2) \arrow[r, leftrightarrow] & (5) \arrow[u] & (6)
\end{tikzcd}$$

Figure 4: Proof logic of equivalent conditions of connected sets

disjoint and closed sets A_1, A_2 such that $X = A_1 \cup A_2$, Then $A_1 \cap A_2 = \varnothing$, hence $X = \mathbf{C}\varnothing = \mathbf{C}A_1 \cup \mathbf{C}A_2$ but $\mathbf{C}A_1$ and $\mathbf{C}A_2$ are non-empty, disjoint and open sets, thus (1) is false. (Note that we use the complement of close sets are open sets, and certainly, vice versa)

$(2) \Rightarrow (1)$: the same logic of proof of $(1) \Rightarrow (2)$.

$(1) \Rightarrow (3), (2) \Rightarrow (3)$: If the clopen subset S of X is a proper subset, then $X = S \cup \mathbf{C}S$, where $\mathbf{C}S$ is an clopen proper subset of X. Thus X is not connected.

$(3) \Rightarrow (1)$: If (1) is false, then there exist two non-empty disjoint and opens sets A_1, A_2 such that $X = A_1 \cup A_2$, hence $A_2 = \mathbf{C}A_1$ is closed. Thus A_2 is a clopen proper subset of X, that is (3) is false.

$(3) \Rightarrow (2)$: the same logic of proof of $(3) \Rightarrow (2)$;

$(3) \Leftrightarrow (4)$: We need to show $A \subseteq X, \partial A = \varnothing$ iff A is a clopen subset of X. $A \subseteq X, \partial A = \varnothing$ iff $\overline{A} = A^\circ \cup \partial A, \partial A = \varnothing$ iff $\overline{A} = A^\circ$ iff $\overline{A} = A = A^\circ$ iff A is a clopen subset of X.

$(2) \Rightarrow (5)$: If (5) is false, then there exist two non-empty separated sets A_1, A_2 such that $A_1 \cup A_2 = X$. Since $\overline{A_1} \cap A_2 = \varnothing, \overline{A_2} \cap A_1 = \varnothing$, then $\overline{A_1} \subseteq A_1, \overline{A_2} \subseteq A_2$, i.e., A_1, A_2 are non-empty closed sets. Then (2) is false.

$(5) \Rightarrow (3)$: If (3) is false, then there exists a non-empty clopen proper subset A of X, hence $X = A \cup \mathbf{C}A$ where $\mathbf{C}A$ is a clopen proper subset of X. Thus $A, \mathbf{C}A$ are non-empty separated sets, that is (5) is false.

$(3) \Rightarrow (6)$: As $\{0\}, \{1\}$ are clopen proper subsets in the space $\{0, 1\}$, two-point space endowed with discrete metric. Then if f is continuous, $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ are clopen proper subsets, then $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ are either X or $\varnothing$, which implies f is constant. (Note that we use: if f is continuous, then if G is open (close), we have $f^{-1}(G)$ is open (close); next proof also uses this).

$(6) \Rightarrow (3)$: As $\{0\}, \{1\}$ are clopen proper subsets in the space $\{0, 1\}$, two-point space endowed with the discrete metric. Then if f is continuous, $f^{-1}(\{0\}), f^{-1}(\{1\})$ are clopen subsets. Then if f is constant, $f^{-1}(\{0\}), f^{-1}(\{1\})$ are either X or $\varnothing$. Therefore, the only clopen subsets of X are X and the empty set. □

2.3 Continuous Functions

There are three equivalent definitions of continuity.

DEFINITION 2.32. *(Continuous functions using sequences) Let f be a function from S to T, (x_q) be a sequence of points of S and consider a point $x_0 \in S$. The function f is continuous at the point x_0 if $\lim\limits_{x_q \to x_0} f(x_q) = f(x_0)$.*

DEFINITION 2.33. *(Continuous functions using neighborhood) A function $f : X \to Y$ is continuous at x if for every neighborhood V of $f(x)$, its inverse image $f^{-1}(V)$ is a neighborhood of x. That is, for every open set V with $f(x) \in V$, there exists an open set U with $x \in U$ such that $z \in U \Rightarrow f(z) \in V$, i.e., $f(U) \subseteq V$.*

DEFINITION 2.34. *(Continuous functions using metrics) A function $f : X \to Y$ is continuous at x if $\forall \varepsilon > 0$, there exists some $\delta > 0$ such that $d_X(x, z) < \delta \Rightarrow d_Y(f(x), f(z)) < \varepsilon$. Here d_X, d_Y can be different metrics, e.g., $d_X = I_1, d_Y = I_2$.*

DEFINITION 2.35. *Let S_1, S_2, T be subsets of $\mathbb{R}^{K_1}, \mathbb{R}^{K_2}, \mathbb{R}^L$ respectively, f be a function from S_1 to S_2 and g be a function from S_2 to T. Define a function h from S_1 to T by $h(x) = g(f(x)), \forall x \in S_1$.*

$$\begin{tikzcd} S_1 \arrow[r, "f"] \arrow[rr, bend right, "h"'] & S_2 \arrow[r, "g"] & T \end{tikzcd}$$

It is immediate that, if f is continuous at the point x of S_1, and g if continuous at the point $f(x)$ of S_2, then h is continuous at x.

$\forall n = 1, 2, \cdots, N$, let T_n be a subset of $\mathbb{R}^{L_n}$ and consider the product $T = \prod\limits_{n=1}^{N} T_n$. It is immediate that the projection on T_n is continuous on T.

Let f_n be a function from S to T_n, and define a function from S to T by $f(x) = (f_n(x)), \forall x \in S$ where $(f_n(x)) = (f_1(x), f_2(x), \cdots, f_N(x))$. It is immediate that, if every f_n is continuous at the point x of S, then f is continuous at x.

THEOREM 2.74. *A mapping f is from a metric space X to a metric space Y (note that mapping is a correspondence from an element $x_0 \in X$ to a set $Y_0 \subseteq Y$). The following statements are equivalent:*

1. *f is continuous;*
2. *$\forall$ open $G \subseteq Y, f^{-1}(G)$ is an open set in X;*
3. *$\forall$ closed $F \subseteq Y, f^{-1}(F)$ is a closed set in X;*

4. $\forall A \subseteq X, f(\overline{A}) \subseteq \overline{f(A)}$;

5. $\forall B \subseteq Y, \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$.

Proof. We use the topological definition (neighborhood) of continuous functions here with proof following the Figure 5.

$$\begin{tikzcd}
(1) \arrow[r, leftrightarrow] & (3) \arrow[r] \arrow[dl, leftrightarrow] \arrow[d, leftrightarrow] & (5) \arrow[dl, leftrightarrow] \\
(2) & (4) &
\end{tikzcd}$$

Figure 5: Proof logic of equivalent conditions of continuous functions

(1) $\Rightarrow$ (2): $\forall x_0 \in f^{-1}(G)$, we have $f(x_0) \in G$. As G is open, there exists a neighborhood $V \subseteq G$ of $f(x_0)$ with $f(x_0) \in V$. From the definition of continuous functions, we can find a neighborhood U of x_0 with $x_0 \in U$ such that $f(U) \subseteq V$. Then $x_0 \in U, U \subseteq f^{-1}(V) \subseteq f^{-1}(G)$. That is, we find a neighborhood of x_0 staying in the $f^{-1}(G)$, so $f^{-1}(G)$ is open. (Note that we use $A \subseteq B \Leftrightarrow f^{-1}(A) \subseteq f^{-1}(B), A \subseteq B \Leftrightarrow f(A) \subseteq f(B)$, which is important for the proof of this theorem).

(2) $\Rightarrow$ (1): $\forall$ open G, we know $f^{-1}(G)$ is open. Then $\forall$ open neighborhood V of $f(x_0)$ with $f(x_0) \in V$, we have $f^{-1}(V)$ is open. As V is open, we can find an open neighborhood $V' \subseteq V$ of x_0 with $x_0 \in V'$. From (2), we know $f^{-1}(V')$ is open and $f^{-1}(V') \subseteq f^{-1}(V)$. That is, we find neighborhood $f^{-1}(V')$ of x_0 such that $f(f^{-1}(V')) = V' \subseteq V, f(x_0) \in V$. Therefore, f is continuous.

(2) $\Rightarrow$ (3): We know the complement of an open set is a closed set and vice versa. Let $G \subseteq Y$ be an open set, then $F \equiv Y \backslash G$ is closed and $f^{-1}(G)$ is open. We need to show $f^{-1}(F) = f^{-1}(Y \backslash G)$ is closed. As $f^{-1}(Y \backslash G) = f^{-1}(Y) \backslash f^{-1}(G) = X \backslash f^{-1}(G)$ and $f^{-1}(G)$ is open, we have $f^{-1}(Y \backslash G)$ is closed.

(3) $\Rightarrow$ (2): The logic is similar to the proof of (2) $\Rightarrow$ (3) (dual problems).

(3) $\Rightarrow$ (4): With (3), we know $\forall F \subseteq Y$ (closed), we have $f^{-1}(F)$ is closed. As $f(A) \subseteq \overline{f(A)}$, then $A \subseteq f^{-1}(\overline{f(A)})$. As $\overline{f(A)}$ is closed, then $f^{-1}(\overline{f(A)})$ is closed. Thus $\overline{A} \subseteq \overline{f^{-1}(\overline{f(A)})} = f^{-1}(\overline{f(A)})$. So $f(\overline{A}) \subseteq \overline{f(A)}$.

(4) $\Rightarrow$ (3): We have $\forall A \subseteq X, f(\overline{A}) \subseteq \overline{f(A)}$. Let $F = f(A)$ be closed. So $f(\overline{A}) \subseteq \overline{f(A)} = f(A)$, then $f^{-1}(f(\overline{A})) \subseteq f^{-1}(f(A))$, i.e., $\overline{A} \subseteq A$. Then $A = f^{-1}(F)$ is closed. (Note that we do not have $f^{-1}(f(A)) = A$, while we just use $f(A) \subseteq f(B) \Leftrightarrow A \subseteq B$)

(3) $\Rightarrow$ (5): We know any inverse image of a closed set is closed. Then $\forall B \subseteq Y$, $\overline{B}$ is closed and $f^{-1}(B)$ is closed. So $\overline{f^{-1}(B)} = f^{-1}(B)$. As $B \subseteq \overline{B}$, we have $f^{-1}(B) \subseteq f^{-1}(\overline{B})$. So $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$.

(5) $\Rightarrow$ (3): We know $\forall B \subseteq Y$, $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$. Let B be closed, then $\overline{B} = B$, and $f^{-1}(\overline{B}) = f^{-1}(B)$. As $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B}) = f^{-1}(B)$, we have $f^{-1}(B)$ is closed, which is just the inverse image of the closed set B.

(4) $\Leftrightarrow$ (5): Let $A = f^{-1}(B)$, then $B = f(A)$. Then $f(\overline{A}) \subseteq \overline{f(A)}$ iff $\overline{A} \subseteq f^{-1}(\overline{f(A)})$ iff $\overline{f(B)} \subseteq f^{-1}(\overline{B})$. □

From this theorem, we have the following lemma.

LEMMA 2.2. *A function f from S to $\mathbb{R}$ (i.e., f is real-valued on S) is continuous iff the inverse image of every interval of $\mathbb{R}$ of the form $(-\infty, y]$ or $[y, \infty)$ is closed in S.*

Proof. Necessity: As $(-\infty, y], [y, \infty)$ is closed and f is continuous, so the inverse image must be closed from the last theorem.

Sufficiency: We need to show that any inverse image of open sets in $\mathbb{R}$ is open, and we can prove the inverse image of an open interval is open first and prove any open set can be partitioned into many disjoint open intervals next (that has been proved before).

We say, there are three types of open intervals: $(-\infty, a), (a, b), (b, \infty)$ and we have

$$f^{-1}(-\infty, a) = f^{-1}(\mathbb{R} \backslash [a, \infty)) = f^{-1}(\mathbb{R}) \backslash f^{-1}[a, \infty) \tag{2.24}$$

$$f^{-1}(b, \infty) = f^{-1}(\mathbb{R} \backslash (-\infty, b]) = f^{-1}(\mathbb{R}) \backslash f^{-1}(-\infty, b] \tag{2.25}$$

$$f^{-1}(a, b) = f^{-1}((-\infty, b) \backslash (-\infty, a]) = f^{-1}(-\infty, b) \backslash f^{-1}(-\infty, a] \tag{2.26}$$

are all open in S. We know every open set in $\mathbb{R}$ is the countable union of disjoint open intervals. So for any open set $O \subseteq \mathbb{R}$, there exists countable open interval $\{I_n, n \in \mathbb{N}\}$ such that $O = \bigcup_{n=1}^{\infty} I_n$. So $f^{-1}(O) = \bigcup_{n=1}^{\infty} f^{-1}(I_n)$ are also open in S. Then f is continuous. $\square$

LEMMA 2.3. *Let f be a functions from S to T. If f is continuous on S, and S is compact, then $f(S)$ is compact.*

Proof. Let $\bigcup_{\gamma\in\Gamma} U_\gamma$ be an open covering of $f(S)$ on Y. The the set $f^{-1}(U_\gamma)$ are open and $f^{-1}(\bigcup_{\gamma\in\Gamma} U_\gamma)$ form an open covering of S since we have show $f^{-1}(\bigcup_{\gamma\in\Gamma} U_\gamma) = \bigcup_{\gamma\in\Gamma} f^{-1}(U_\gamma)$. Since S is compact, then there exists a finite subcovering Γ' of Γ such that $\bigcup_{\gamma\in\Gamma'} f^{-1}(U_\gamma)$ forms an open covering of S. Then $f(S) \subseteq f(\bigcup_{\gamma\in\Gamma'} f^{-1}(U_\gamma)) = f(f^{-1}(\bigcup_{\gamma\in\Gamma'} U_\gamma)) = \bigcup_{\gamma\in\Gamma'} U_\gamma$. (Note that $f(f^{-1}(U)) = U$ holds) $\square$

Applying this lemma to the particular case of f is real-valued, then we have the Weierstrass's Theorem.

THEOREM 2.75. *(Weierstrass' Theorem) Let f be a function from S to $\mathbb{R}$. If f is continuous on S which is non-empty and compact, then $f(S)$ has a maximum and minimum.*

Proof. From the last lemma, we know $f(S)$ is compact. We need to show that any compact set F in $\mathbb{R}$ must obtain both maximum and minimum.

As F is compact, so F is bounded and closed. As F is bounded, there exist the least upper bound sup x and the greatest lower bounded inf y of F and they belongs to F'. As F is closed, so $F' \subseteq F$, then $x, y \in F$, that is sup $F \in F$ and inf $F \in F$, which is exactly the maximum and minimum. $\square$

DEFINITION 2.36. *(Maximizer and Minimizer) Let f be a functions from S to $\mathbb{R}$. f has a maximum and a minimum implies $\exists x^M \in S$ such that $\forall x \in S, f(x) \leq f(x^M)$ and $\exists x^m \in S$ such that $\forall x \in S, f(x) \geq f(x^m)$. We call x^M and x^m a maximizer and a minimizer of f respectively.*

LEMMA 2.4. *Let f be a function from S to T. If f is continuous on S, and S is connected, then $f(S)$ is connected.*

Proof. Suppose $f(S)$ is not connected and f is continuous, then $\exists V_1, V_2 \subseteq T$ such that $V_1 \cup V_2 = f(S)$ and V_1, V_2 are non-empty, disjoint and open. Then $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are non-empty open subsets of S and they are disjoint (or otherwise V_1, V_2 are joint). Then $f^{-1}(V_1) \cup f^{-1}(V_2) = f^{-1}(V_1 \cup V_2) = S$. Thus S is not connected, contradicting to S is connected. $\square$

THEOREM 2.76. *(Bolzano's Theorem) Let f be a functions from S to $\mathbb{R}$. If f is continuous on S, and S is connected, then $f(S)$ is an interval.*

Proof. From the last lemma, $f(S)$ is connected. With any connected set in $\mathbb{R}$ is an interval, we reach the conclusion. $\square$

From Bolzano's theorem, we immediately have if $f : S \to \mathbb{R}$ is continuous and S is connected, then $\forall f(x_1) \leq \alpha \leq f(x_2)$, we have $\exists x \in [x_1, x_2]$ such that $f(x) = \alpha$. (The Intermediate Value Theorem)

The continuity condition is strict sometimes, and we want to loose it, turing out two general conditions lower semi-continuous and upper semi-continuous and study their properties.

DEFINITION 2.37. *(Lower semi-continuous and Upper semi-continuous) A numerical function f defined on X is said to be* ***lower semi-continuous*** *(l.s.c.) at a point x_0 if $\forall \varepsilon > 0$, there exists a neighborhood $U(x_0)$ such that $x \in U(x_0) \Rightarrow f(x) > f(x_0) - \varepsilon$.*

A numerical function f defined on X is said to be ***upper semi-continuous*** *(u.s.c.) at a point x_0 if $\forall \varepsilon > 0$, there exists a neighborhood $U(x_0)$ such that $x \in U(x_0) \Rightarrow f(x) < f(x_0) + \varepsilon$.*

Obviously, f is continuous at x iff it is both l.s.c. and u.s.c. at x.

We say that a function is l.s.c. (u.s.c.) in X if it is l.s.c. (u.s.c.) at each point of X. If f is l.s.c., then $-f$ is u.s.c. at the same domain. They any properties of u.s.c. can be induce form l.s.c. and vice versa (duality).

THEOREM 2.77. *A numerical function f be l.s.c. iff $\forall \lambda$, the set $S_\lambda^- = \{x \in X | f(x) > \lambda\}$ is open iff $\forall \lambda$ iff $\forall \lambda$, the lower contour set $T_\lambda^- = \{x \in X | f(x) \leq \lambda\}$ is closed.*

A numerical function f be u.s.c. iff $\forall \lambda$, the set $S_\lambda^+ = \{x \in X | f(x) < \lambda\}$ is open iff $\forall \lambda$ iff $\forall \lambda$, the upper contour set $T_\lambda^+ = \{x \in X | f(x) \geq \lambda\}$ is closed.

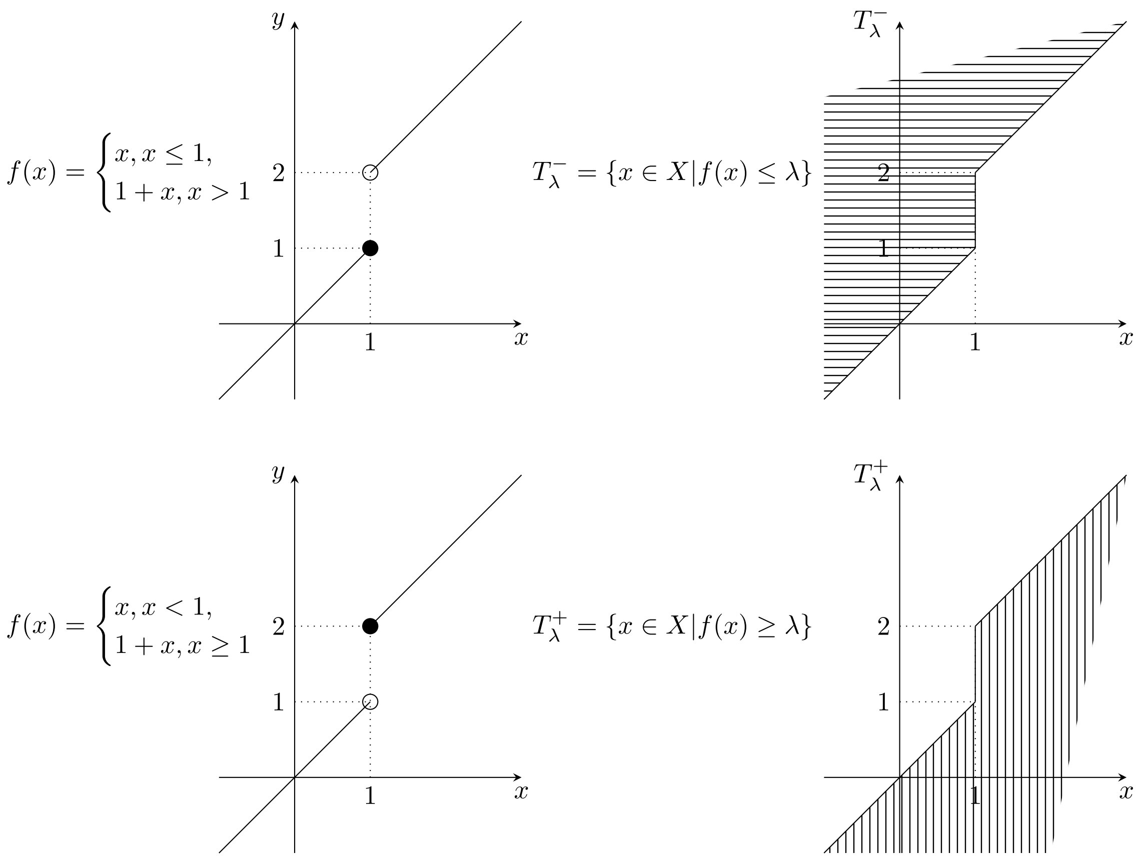


Figure 6: Counter sets of l.s.c. functions and u.s.c. functions

Proof. First we show what $S_\lambda^-, S_\lambda^+, T_\lambda^-$ and T_λ^+ denote with illustration of Figure 6.

The theorem follows intuition, and we just need to prove one conclusion of them, the left is self-evident following the same logic and duality. We show that f is l.s.c. iff S_λ^- is open. We can reach this conclusion with two approaches.

(1) Topological method. Necessity: Let $f(x_0) > \lambda, x_0 \in S_\lambda^-$. Then f is l.s.c. if $\forall \varepsilon > 0, \exists U$ of x_0 with $x_0 \in U$ such that $\forall x \in U, f(x) > f(x_0) - \varepsilon$. Let $0 < \varepsilon < f(x_0) - \lambda$, then $\forall x \in U, f(x) > f(x_0) - \varepsilon > \lambda$, i.e., $U \subseteq S_\lambda^-$.

Sufficiency: We know for any λ, S_λ^- is open. Then $\forall x_0 \in X, \varepsilon > 0$, $S_{f(x_0)-\varepsilon}^-$ is an open set and $x_0 \in S_{f(x_0)-\varepsilon}^-$. Then $\exists U$ of x_0 with $x_0 \in U, U \subseteq S_{f(x_0)-\varepsilon}$, then $\forall x \in U, f(x) > f(x_0) - \varepsilon$. That implies f is l.s.c.

(2) Abstract but elegant: The family $\mathscr{D}^-$ of intervals $\{(\lambda, +\infty), \lambda \in \mathbb{R}\}$ determines a topology in $\mathbb{R}$. Then from the definition of l.s.c. that f is l.s.c. iff it is a continuous mapping of X into the topological space $(\mathbb{R}, \mathscr{D}^-)$. Since $S_\lambda^- = f^{-1}((\lambda, \infty))$, the theorem stated is equivalent to this result. $\square$

THEOREM 2.78. *If K is a compact subset of X, a l.s.c. function f attains in K the value $m = \inf\limits_{x \in K} f(x)$ (m is a minimum).*

If K is a compact subset of X, a u.s.c. function f attains in K the value $M = \sup\limits_{x \in K} f(x)$ (M is a maximum, f is bounded below).

Proof. We just need to prove the first statement when f is l.s.c. and the second is self-evident then.

K is compact implies it is bounded and closed. We have two approaches.

(1) Let μ be a number such that $\mu > m$. Then $S_\mu^- = \{x \in K | f(x) \leq \mu\}$ is not empty and is a closed subset of the compact set K and $S_\lambda^- \subseteq S_\mu^-$. Moreover, $\forall \mu_1 > \mu_2 > \lambda$, we have $S_{\mu_2}^- \subseteq S_{\mu_1}^-$. Hence, by the finite intersection axiom, we have $\bigcap\limits_{\mu > m} S_\mu^- \neq \varnothing$. $\forall x \in \bigcap\limits_{\mu > m} S_\mu^-$, we have $f(x) = m$.

(2) We first prove that f is bounded with subcovering and then show the infimum is contained in its range. Let $U_\lambda^- = (\lambda, +\infty)$. $\{U_\lambda, \lambda \in \Lambda\}$ be an open covering of $f(X)$. Then $\{f^{-1}(U_\lambda^-), \lambda \in \Lambda\}$ form an open covering of X since we have $f^{-1}(\bigcup\limits_{\lambda \in \Lambda} U_\lambda^-) = \bigcup\limits_{\lambda \in \Lambda} f^{-1}(U_\lambda^-)$. Since X is compact, then $\exists$ a finite subcovering $\Lambda' \subseteq \Lambda$

still cover X. Then $X \subseteq \bigcup_{\lambda\in\Lambda'} f^{-1}(U_\lambda^-), f(X) \subseteq f(\bigcup_{\lambda\in\Lambda'} f^{-1}(U_\lambda^-)) = f(f^{-1}(\bigcup_{\lambda\in\Lambda'} U_\lambda^-)) = \bigcup_{\lambda\in\Lambda'} U_\lambda^-$. Hence $f(x) > \lambda, \forall x \in X, \forall \lambda \in \Lambda'$,i.e., f is bounded below.

Let $m = \inf_{x\in X} f(x)$, there exist a sequence $x_n \in X$ such that $f(x_n) \to m$. Since $x_n \in X$ and X is compact, we have some $x_0 \in X$ and some subsequence such that $x_{n_k} \to x_0$ as X is compact. As f is l.s.c., then $\forall \varepsilon > 0$, $\exists U$ of $x_0, x_0 \in U$ (neighborhood) such that $f(x) > f(x_0) - \varepsilon, \forall x \in U$. As $x_{n_k} \to x_0$, $\exists K \in \mathbb{N}$ such that $\forall k > K, x_{n_k} \in U$, hence $f(x_{n_k}) > f(x_0) - \varepsilon$. Let $k \to \infty$, we see $m \geq f(x_0) - \varepsilon$. As ε is arbitrary, we have $m \geq f(x_0)$, and by definition of m, we have $f(x_0) \geq m$, hence $f(x_0) = m$, x_0 is a minimizer. □

THEOREM 2.79. *If $\{f_i | i \in I\}$ is a family of l.s.c. functions, then the function $g = \sup_{i\in I} f_i(x)$ is l.s.c.*

If $\{f_i | i \in I\}$ is a family of u.s.c. functions, then the function $g = \inf_{i\in I} f_i(x)$ is u.s.c.

Proof. We just need to show the first one. $\forall x_0$, as $g(x_0)$ is the supremum of $\{f_i, i \in I\}$, so $\forall \varepsilon > 0, \exists i \in I$ such that $f_i(x_0) > g(x_0) - \frac{\varepsilon}{2}$. Since f_i is l.s.c., then $\exists$ a neighborhood U of $x_0, x_0 \in U$ such that $\forall x \in x_0, f(x) > f(x_0) - \varepsilon/2$. Then $g(x) \geq f_i(x) > f_i(x_0) - \varepsilon/2 > g(x_0) - \varepsilon$. Hence g is l.s.c. since x_0, ε is arbitrary. □

THEOREM 2.80. *If functions $\{f_i, i = 1, 2, \cdots, N\}$ are l.s.c., then $g(x) = \inf_{n=1,2,\cdots,N} f_n(x)$ is l.s.c.*

If functions $\{f_i, i = 1, 2, \cdots, N\}$ are u.s.c., then $g(x) = \sup_{n=1,2,\cdots,N} f_n(x)$ is u.s.c. (Note that here f_i is finite while it is arbitrary in the last theorem)

Proof. Please notice how the finite condition works. $\forall x_0, \forall \varepsilon > 0$, we have $\exists$ a neighborhood $U_i(x_0)$ such that $\forall x \in x_0, f_i(x) > f_i(x_0)$ for $i = 1, 2, \cdots, N$. Then we take $U = \bigcap_{i=1,2,\cdots,N} U_i(x_0)$, as N if finite, we say U is a neighborhood instead of a point (as N infinite). So $\forall x \in U, f_i(x) > f_i(x_0) - \varepsilon$, then we take infimum in each side,

$$g(x) = \inf_{i=1,2,\cdots,N} f_i(x) \geq \inf_{i=1,2,\cdots,N} f_i(x_0) - \varepsilon = g(x_0) - \varepsilon \tag{2.27}$$

So g is l.s.c. □

THEOREM 2.81. *If f_1, f_2 are two l.s.c. numerical functions, the function $f_1 + f_2$ are l.s.c.*
If f_1, f_2 are two u.s.c. numerical functions, the function $f_1 + f_2$ are u.s.c.
If f_1, f_2 are two positive l.s.c. numerical functions, the function $h(x) = f_1(x)f_2(x)$ are l.s.c.
If f_1, f_2 are two positive u.s.c. numerical functions, the function $h(x) = f_1(x)f_2(x)$ are u.s.c.

Proof. The statements are intuitive but useful.

(1) The set

$$\{x | f_1(x) + f_2(x) > a\} = \bigcup_\lambda \{x | f_1(x) > a - \lambda\} \cap \{x | f_2(x) > \lambda\} \tag{2.28}$$

is a union of open sets and so is open.

(2) The set

$$\{x | f_1(x) + f_2(x) < a\} = \bigcup_\lambda \{x | f_1(x) < a - \lambda\} \cap \{x | f_2(x) < \lambda\} \tag{2.29}$$

is a union of open sets and so is open.

(3) The set

$$\{x | f_1(x)f_2(x) > \alpha\} = \bigcup_{\lambda>0} \{x | f_1(x) > \lambda\} \cap \{x | f_2(x) > \alpha/\lambda\} \tag{2.30}$$

is a union of open sets and so is open.

(3) The set

$$\{x | f_1(x)f_2(x) < \alpha\} = \bigcup_{\lambda>0} \{x | f_1(x) < \lambda\} \cap \{x | f_2(x) < \alpha/\lambda\} \tag{2.31}$$

is a union of open sets and so is open. □

2.4 Continuous Correspondence

Let Γ be a correspondence from X to Y, written as $\Gamma : X \rightrightarrows Y$.

DEFINITION 2.38. *(Upper Inverse) The **upper** (strong) **inverse** of E under Γ, denoted as $\Gamma^+ E$, is defined as:*

$$\Gamma^+ E = \{x \in X | \Gamma x \subseteq E\} \tag{2.32}$$

DEFINITION 2.39. *(Lower Inverse) The **lower** (weak) **inverse** of E under Γ, denoted as $\Gamma^- E$, is defined as:*

$$\Gamma^- E = \{x \in X | \Gamma x \cap E \neq \varnothing\} \tag{2.33}$$

For a single-valued mapping $\Gamma : X \to Y$, define $\Gamma^{-1} y = \{x \in X | y \in \Gamma x\} = \{x \in X | y = \Gamma x\}$, then $\Gamma^{-1} y = \Gamma^- \{y\} = \Gamma^+ \{y\}$.

DEFINITION 2.40. *(Lower semi-continuous) Let $\Gamma : X \rightrightarrows Y$. For a given $x_0 \in X$, if $\forall G$, G is open and $\Gamma x_0 \cap G \neq \varnothing$, we have $\exists U(x_0)$ (a neighborhood of x_0) such that*

$$\forall x \in U(x_0) \Rightarrow \Gamma x \cap G \neq \varnothing \tag{2.34}$$

i.e., $\Gamma U(x_0) \cap G \neq \varnothing$, or equivalently

$$U(x_0) \subseteq \Gamma^- G \tag{2.35}$$

*then we say Γ is **lower semi-continuous** (l.s.c.) at x_0. We say Γ is lower semi-continuous in X if it is l.s.c. at every point of X.*

DEFINITION 2.41. *(Upper semi-continuous) Let $\Gamma : X \rightrightarrows Y$. For a given $x_0 \in X$, if $\forall G$, G is open and $\Gamma x_0 \subseteq G$, we have $\exists U(x_0)$ (a neighborhood of x_0) such that*

$$\forall x \in U(x_0) \Rightarrow \Gamma x \subseteq G \tag{2.36}$$

i.e., $\Gamma U(x_0) \subseteq G$, or equivalently

$$U(x_0) \subseteq \Gamma^+ G \tag{2.37}$$

*then we say Γ is **upper semi-continuous** (u.s.c.) at x_0. We say Γ is upper semi-continuous in X if it is u.s.c. at every point of X.*

If Γ is both l.s.c. and u.s.c., then we say it is **continuous** in X. Then we give equivalent condition of l.s.c and u.s.c. to shed light on the meaning of these properties.

THEOREM 2.82. *Let $\Gamma : X \rightrightarrows Y$, then the following statements are equivalent:*

(1) Γ is l.s.c.;

(2) $\Gamma^- G$ is open if $G \subseteq Y$ is open;

(3) $\Gamma^+ F$ is closed if $G \subseteq F$ is closed.

Proof. We just prove (1) $\Leftrightarrow$ (2) and (2) $\Leftrightarrow$ (3).

(1) $\Leftrightarrow$ (2): Let $x_0 \in X$ and $\Gamma x_0 \cap G \neq \varnothing$, i.e., $x_0 \in \Gamma^- G$ where G is open. Γ is l.s.c. at x_0 iff $\exists U(x_0), \Gamma U(x_0) \cap G \neq \varnothing$ iff $\exists U(x_0), U(x_0) \subseteq \Gamma^- G$ iff $x_0 \in \Gamma^- G$ and $\Gamma^- G$ is open.

(2) $\Leftrightarrow$ (3): We first show $\mathbf{C}_X(\Gamma^- E) = \Gamma^+(\mathbf{C}_Y E)$ and $\Gamma^-(\mathbf{C}_Y E) = \mathbf{C}_X(\Gamma^+ E)$. As the second equation is immediately from the first one (by complement), we just show the first one.

$$\begin{aligned} \Gamma^+(\mathbf{C}_Y E) &= \{x \in X | \Gamma x \subseteq \mathbf{C}_Y E\} && (2.38)\\ &= \{x \in X | \Gamma x \cap E = \varnothing\} && (2.39)\\ &= \mathbf{C}_X \{x \in X | \Gamma x \cap E \neq \varnothing\} && (2.40)\\ &= \mathbf{C}_X(\Gamma^- E) && (2.41) \end{aligned}$$

Then let $F = \mathbf{C}_Y E$, we have $\Gamma^+ F = \mathbf{C}_X(\Gamma^- \mathbf{C}_Y F)$, then $\mathbf{C}_X(\Gamma^+ F) = \Gamma^-(\mathbf{C}_Y F)$.

Then (2) holds iff $\forall G$ is open, then $\Gamma^-(G)$ is open iff $\mathbf{C}_Y G$ is closed, then $\Gamma^+(\mathbf{C}_Y G) = \mathbf{C}_X(\Gamma^- E)$ is closed iff (3) holds. (Note that a complement of an open set is closed and vice versa) $\square$

THEOREM 2.83. *Let $\Gamma : X \rightrightarrows Y$, then the following statements are equivalent:*

(1) Γ *is u.s.c.;*

(2) Γ^+G *is open if* $G \subseteq Y$ *is open;*

(3) Γ^-F *is closed if* $G \subseteq F$ *is closed.*

Like the last theorem, we prove $(1) \Leftrightarrow (2)$ *and* $(2) \Leftrightarrow (3)$. *Actually, we just need to prove* $(1) \Leftrightarrow (2)$ *as* $(2) \Leftrightarrow (3)$ *is from* $\mathbf{C}_X(\Gamma^-E) = \Gamma^+(\mathbf{C}_Y E)$ *and* $\Gamma^-(\mathbf{C}_Y E) = \mathbf{C}_X(\Gamma^+E)$ *similarly.*

$(1) \Leftrightarrow (2)$*: Let* $x_0 \in X$ *and* $x_0 \in \Gamma^+G$*, i.e.,* $\Gamma x_0 \subseteq G$ *where* G *is open. Then* Γ *is u.s.c. iff* $\exists U(x_0)$ *s.t.* $\Gamma U(x_0) \subseteq G$ *iff* $\exists U(x_0)$ *s.t.* $U(x_0) \subseteq \Gamma^+G$ *iff* $x_0 \in \Gamma^+G$ *and* Γ^+G *is open.*

THEOREM 2.84. *If* $\Gamma : X \rightrightarrows Y$ *is a compact-valued u.s.c. correspondence, then the image* ΓK *of a compact subset* K *of* X *is also compact.*

Proof. This follows intuition. Let $\mathscr{G}$ be an open cover of ΓK. If $x \in K$, the set Γx is compact as Γ is compact-valued, so $\exists \mathscr{G}_x \subseteq \mathscr{G}$ which can cover Γx and is finite, i.e., $\Gamma x \subseteq \mathscr{G}_x$ and then $x \in \Gamma^+\mathscr{G}_x$. Then $\{\Gamma^+\mathscr{G}_x | x \in K\}$ is an open covering of K so it contains a finite subcovering $\bigcup\limits_{n=1}^{N} \Gamma^+\mathscr{G}_{x_n}$ coving K. So $\bigcup\limits_{n=1}^{N} \mathscr{G}_{x_n}$ covers ΓK, i.e., $\Gamma K \subseteq \bigcup\limits_{n=1}^{N} \mathscr{G}_{x_n}$. (Note that we use $A \subseteq B \Rightarrow \Gamma A \subseteq \Gamma B$, surely hold) ☐

Then we introduce closed mapping.

DEFINITION 2.42. *(Closed Mapping) Let* $\Gamma : X \rightrightarrows Y$. *We say* Γ *is a closed mapping one of these statements hold (equivalently):*

(1) $\forall x_0 \in X, y_0 \in Y, y_0 \notin \Gamma x_0$*, then* $\exists U(x_0), V(y_0)$ *(neighborhoods) s.t.* $x \in U(x_0) \Rightarrow \Gamma x \cap V(y_0) = \varnothing$*;*

(2) $\forall x_0, y_0$*, if* $\forall U(x_0), V(y_0)$ *we have* $x \in U(x_0) \Rightarrow \Gamma x \cap V(y_0) \neq \varnothing$*;*

(3) $\forall x_0, y_0$*, if* $\forall U(x_0), V(y_0)$ *s.t.* $x \in U(x_0), y \in V(y_0)$ *and* $y \in \Gamma x$*, then* $y_0 \in \Gamma x_0$*;*

(4) If $x_q \to x_0, y_q \to y_0, y_q \in \Gamma x_q$*, then* $y_0 \in \Gamma x_0$.

The graphical representation $\sum_{x \in X} \Gamma x$ of Γ in $X \times Y$:

$$G(\Gamma) \equiv \sum_{x \in X} \Gamma x = \{(x, y) \in X \times Y | y \in \Gamma x\} \tag{2.42}$$

this is a closed set iff Γ is a closed mapping, equivalently, if $y_0 \notin \Gamma x_0$, then $\exists U(x_0), V(y_0)$ s.t. $U(x_0) \times V(y_0) \subseteq \mathbf{C}(G(\Gamma))$. We see, if Γ is a closed mapping, then Γx is closed in Y. The next lemma shows that.

LEMMA 2.5. *Let* $\Gamma : X \rightrightarrows Y$. Γ *is a closed mapping iff* $G(\Gamma)$ *is closed in* $X \times Y$.

Proof. $G(\Gamma)$ is closed iff $\forall x_q \to x_0, y_q \to y_0, y_q \in \Gamma(x_q)$, equivalently ,$(x_q, y_q) \in G(\Gamma), (x_q, y_q) \to (x, y)$, then $(x, y) \in G(\Gamma)$ iff $\forall (x_q, y_q) \in G(\Gamma), (x_q, y_q) \to (x, y)$, then $y \in \Gamma x$ iff Γ is a closed mapping. ☐

In particular, if λ is a continuous numerical function in a metric space (X, d), the mapping $\Gamma x = B_{\lambda(x)}(x) = \{y \in X | d(x, y) - \lambda(x) \leq 0\}$ is closed.

LEMMA 2.6. *The Cartesian product* $\Gamma = \prod\limits_{n=1}^{N} \Gamma_n$ *of a finite number of closed mapping* Γ_n *(of* X *into* Y_n*) is a closed mapping of* X *into* $Y = \prod\limits_{n=1}^{N} Y_n$.

Proof. If $x_q \to x_0, (y_{1,q}, \cdots, y_{N,q}) = y_q \to y_0 = (y_{1,0}, \cdots, y_{N,0})$ and $y_q \in \Gamma x_q = \prod\limits_{n=1}^{N} \Gamma_n x_q$, then $y_{n,q} \in \Gamma_n x_q$ for $n = 1, 2, \cdots, N$. If every Γ_n is closed at the point x_0, then $y_{n,0} \in \Gamma_n x_0, n = 1, 2, \cdots, N$ and hence $y_0 \in \prod\limits_{n=1}^{N} \Gamma_n x_0 = \Gamma x_0$. Thus $\Gamma = \prod\limits_{n=1}^{N} \Gamma_n$ is a closed mapping at x_0. ☐

LEMMA 2.7. *Let* $\Gamma : X \rightrightarrows Y$ *be a closed mapping at* x_0*, then* Γx *is a closed set.*

Proof. Let $x_q \equiv x_0, y_q \to y, y_q \in \Gamma x_q = \Gamma x_0$. Then we have $y \in \Gamma x$, i.e., Γx_0 is a closed set. (the limit of any convergent sequence is in it) ☐

THEOREM 2.85. *If $\{\Gamma_i | i \in I\}$ is a family of closed mapping of X into Y, then $\Gamma = \bigcap_{i\in I} \Gamma_i$ is also a closed mapping.*

Proof. We have two approaches.

(1) If $y_0 \notin \Gamma x_0$, then $\exists i_0$ s.t. $y_0 \notin \Gamma_{i_0} x_0$, then $\exists U(x_0), V(y_0)$ s.t. $\Gamma_{i_0} U(x_0) \cap V(y_0) = \varnothing$. Thus, as $\Gamma x_0 \subseteq \Gamma_{i_0} x_0$, then $\Gamma x_0 \cap V(y_0) = \varnothing$, i.e., Γ is a closed mapping at x_0.

(2) If $x_q \to x_0, y_q \to y_0, y_q \in \Gamma x_q = \bigcap_{i\in I} \Gamma_i x_q$, then $y_q \in \Gamma_i x_q, \forall i \in I$. If every Γ_i is a closed mapping at x_0, then $y_0 \in \Gamma_i x_0, \forall i \in I$, i.e., $y_0 \in \bigcap_{i\in I} \Gamma_i x_0 = \Gamma x_0$, i.e., Γ is a closed mapping at x_0. □

THEOREM 2.86. *The union of a finite family of closed mappings $\Gamma = \bigcup_{n=1}^{N} \Gamma_n$ is a closed mapping.*

Proof. We have two approaches.

(1) If $y_0 \notin \Gamma x_0 = \bigcup_{n=1}^{N} x_0$, then $\forall n = 1, 2, \cdots, N$, we have $y_0 \notin \Gamma_n x_n$. Then $\forall n = 1, 2, \cdots, N$ we have $\exists U_n(x_0), V_n(y_0)$ s.t. $\Gamma_n U_n(x_0) \cap V_n(y_0) = \varnothing$. Let $U(x_0) = \bigcap_{n=1}^{N} U_n(x_0), V(y_0) = \bigcap_{n=1}^{N} V_n(y_0)$, we have $\Gamma U(x_0) \cap V(y_0) = \bigcup_{n=1}^{N} [\Gamma_n U(x_0) \cap V(y_0)] = \varnothing$. Hence Γ is a closed mapping at x_0. (Note that $U(x_0), V(y_0)$ is open for they are finite intersections of open set, where the finite condition functions)

(2) If $x_q \to x_q, y_q \to y_0, y_q \in \Gamma x_0 = \bigcup_{n=1}^{N} \Gamma_n x_q$, then exists $n_0 \in \{1, 2, \cdots, N\}$ s.t. $y_q \in \Gamma_{n_0} x_q$. If $\forall n, \Gamma_n$ is a closed mapping at x_0, then Γ_{n_0} is a closed mapping at x_0, then $y_0 \in \Gamma_{n_0} x_0$, then $y_0 \in \bigcup_{n=1}^{N} \Gamma_n x_0 = \Gamma x_0$. Thus, $\Gamma = \bigcup_{n=1}^{N} \gamma_n$ is a closed mapping at x_0. (Note that if N is infinite, we cannot find such n_0, e.g., $y_0 = 1 \in [0, 1] = \bigcup_{n=1}^{\infty} (1/n, 1 - 1/n)$) □

THEOREM 2.87. *Let $\Gamma : X \rightrightarrows Y$. If Γ is compact-valued and u.s.c., then Γ is closed.*

Proof. Let $x_0 \in X, y_0 \in Y, y_0 \notin \Gamma x_0$, we want to show $\exists U(x_0), V(y_0)$ s.t. $\Gamma U(x_0) \cap V(y_0) = \varnothing$ (closed). Since Γ is compact-valued, then Γx_0 is closed, i.e., $\exists G \supseteq \Gamma x_0$ s.t. $G \cap V(y_0)$ where G is open ($y_0 \notin \Gamma x_0$). Since Γ is u.s.c., then $\exists U(x_0)$ s.t. $\Gamma U(x_0) \subseteq G$, then, $\Gamma U(x_0) \cap V(y_0) = \varnothing$. So we reach the conclusion. □

THEOREM 2.88. *Let $\Gamma_1, \Gamma_2 : X \rightrightarrows Y$. If Γ_1 is a closed mapping and Γ_2 is compact-valued and u.s.c., then $\Gamma = \Gamma_1 \cap \Gamma_2$ is u.s.c.*

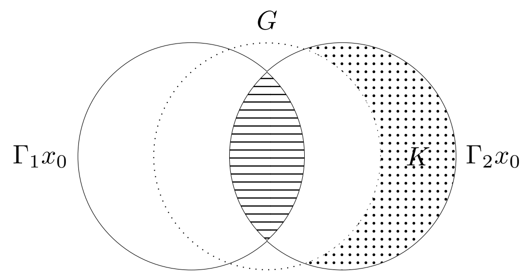


Figure 7: Intersection of a closed mapping and a compact-valued u.s.c. mapping

Proof. $\forall x_0 \in X, \forall G, \Gamma x_0 \in G$, i.e., $\Gamma_1 \cap \Gamma_2 \subseteq G$ where G is open, we want to show $\exists U(x_0)$ s.t. $\Gamma U(x_0) \subseteq G$, i.e., $\Gamma_1 U(x_0) \cap \Gamma_2 U(x_0) \subseteq G$. Generally, it is rather difficult to prove the type $A \cap B \subseteq C$, but if we can find D, s.t. $A \cap B \subseteq C \cup D$ and $(A \cap B) \cap D = \varnothing$, loosely, if $A \subseteq C \cup D$ and $B \cap D = \varnothing$, we have $A \cap B \subseteq C$.

First, $\forall x \in X$, Γx is compact because it is a closed set contained in a compact set $\Gamma_2 x$. Thus Γ is compact-valued.

Let G be an open set and $\Gamma x_0 = \Gamma_1 x_0 \cap \Gamma_2 x_0 \subseteq G$. If $\Gamma_2 x_0 \subseteq G$, then $\exists U(x_0)$ s.t. $\Gamma_2 U(x_0) \subseteq G$ as Γ_2 is u.s.c. Hence $\Gamma U(x_0) = \Gamma_1 U(x_0) \cap \Gamma_2 U(x_0) \subseteq \Gamma_2 U(x_0) \subseteq G$. So we have Γ is u.s.c.

If $\Gamma_2 x_0 \cap \mathbf{C}G = K \neq \varnothing$, then $\exists y \in K$ s.t. $y \notin \Gamma_1 x_0$. Then $\exists V(y), U_y(x_0)$ s.t. $\Gamma_1 U_y(x_0) \cap V(y) = \varnothing$ because Γ_1 is a closed mapping at x_0.

Since K is compact ($\Gamma_2 x_0$ is compact), then $\exists\{y_1, y_2, \cdots, y_N\}$ s.t. $\{V(y_1), V(y_2), \cdots, V(y_N)\}$ cover K. Let $K = \bigcup\limits_{n=1}^{N} V(y_n) \equiv V(K)$, then $\Gamma_2 x_0 \subseteq G \cup V(K)$.

As Γ_2 is u.s.c., then $\exists U'(x_0)$ s.t. $\Gamma_2 U'(x_0) \subseteq G \cup V(K)$. Let $U(x_0) = U_{y_1}(x_0) \cap U_{y_2}(x_0) \cap \cdots \cap U_{y_N}(x_0) \cap U'(x_0)$, so $\Gamma_1 U(x_0) \cap V(K) = \varnothing$ (Note that $\Gamma_1 U_y(x_0) \cap V(y) = \varnothing, \forall y \in K$, so $[\Gamma_1 U_{y_1}(x_0) \cap U_{y_2}(x_0) \cap \cdots \cap U_{y_N}(x_0)] \cap V(K) = \varnothing$).

And $U(x_0) \subseteq U'(x_0)$, so $\Gamma_2 U(x_0) \subseteq G \cup V(K)$. Therefore, we have $\begin{cases} \Gamma_1 U(x_0) \cap V(K) = \varnothing \\ \Gamma_2 U(x_0) \subseteq G \cup V(K) \end{cases}$, so $\Gamma U(x_0) = \Gamma_1 U(x_0) \cap \Gamma_2 U(x_0) \subseteq G$, i.e., Γ is u.s.c. (also compact). ☐

LEMMA 2.8. *If Y is a compact space and $\Gamma : X \rightrightarrows Y$ is a closed-valued mapping, then Γ is closed iff it is u.s.c.*

Proof. First we show Γ is compact-valued. As Y is a compact space and Γ is a closed-valued mapping, then Γx is a closed set for $\forall x \in X$, thus Γx is a compact set (We use a closed set in a compact space is compact). Thus Γ is compact-valued.

Since Γ is compact-valued, then if Γ is u.s.c., we have Γ is closed.

Conversely, let $\Delta : X \rightrightarrows Y$ is a mapping such that $\Delta x = Y, \forall$. Then Δ is a compact-valued u.s.c. mapping, if Γ is closed, then $\Gamma = \Gamma \cap \Delta$ is u.s.c. ☐

THEOREM 2.89. *Let $\Gamma : X \rightrightarrows X$ be a closed-valued u.s.c. mapping. If X is a compact space, and $\forall x \in X, \Gamma x \neq \varnothing$, then $\exists K \subseteq X$ which is non-empty and compact s.t. $\Gamma K = K$.*

Proof. First we have Γ is a closed mapping and compact-valued mapping.

Since $\Gamma X \subseteq X$, then $\Gamma^2 X \subseteq \Gamma X$, and generally $\Gamma^n X \subseteq \Gamma^{n-1} X$. So $X \supseteq \Gamma X \supseteq \cdots \supseteq \Gamma^n X \supseteq \cdots$. If $\exists n$ s.t. $\Gamma^n X = \Gamma^{n-1} X$, we have $K = \Gamma^{n-1} X$.

If $\Gamma^n X$ is distinct for all n, then $\{\Gamma^n X\}$ is decreasing, so we have $K = \bigcap\limits_{n=1}^{\infty} \Gamma^n X \neq \varnothing$ as $\Gamma^n X$ is compact for $\forall n$ (generally, if A is compact and Γ is compact-valued and u.s.c., then we have ΓA is compact)

Then $\forall n$, we have $K \subseteq \Gamma^{n-1} X$ so $\Gamma K \subseteq \Gamma^n X$, then $\Gamma K \subseteq K$.

Next we prove $K \subseteq \Gamma K$. Let $a \in K$, then $\forall n$, $\exists x_n \in \Gamma^n X$ s.t. $a \in \Gamma x_n$ as $K \subseteq \Gamma(\Gamma^n X)$. The sequence (x_n) admits a cluster point x_0 and so a sub-sequence (x_{k_n}) converges to x_0. Since at most $n-1$ points of this sub-sequence are outside $\Gamma^n X$ and $\Gamma^n X$ is compact, we have $x_0 \in \Gamma^n X$ so $x_0 \in K$. Also $(x_{k_n}, a) \to (x_0, a)$ and $(x_{k_n}, a) \in G(\Gamma) = \sum_{x \in X} \Gamma x$ so since Γ is a closed mapping, and $a \in \Gamma x_0$. So $a \in \Gamma K$, i.e., $K \subseteq \Gamma K$. ☐

2.5 Properties of Semi-continuous Mappings

Before introducing theorems of semi-continuous, we offer some general equations which will help a lot.

- $(\Gamma_2 \Gamma_1)^- A = \Gamma_1^-(\Gamma_2^- A)$;
- $(\Gamma_2 \Gamma_1)^+ A = \Gamma_1^+(\Gamma_2^+ A)$.
- $\Gamma = \bigcup\limits_{i \in I} \Gamma_i, \Gamma^- A = \bigcup\limits_{i \in I} \Gamma_i^- A$;
- $\Gamma = \bigcup\limits_{i \in I} \Gamma_i, \Gamma^+ A = \bigcap\limits_{i \in I} \Gamma_i^+ A$.

Now prove them,

- $(\Gamma_2 \Gamma_1)^- A = \Gamma_1^-(\Gamma_2^- A)$.

$$
\begin{aligned}
(\Gamma_2 \Gamma_1)^- A &= \{x \in X | (\Gamma_2 \Gamma_1) x \cap A \neq \varnothing\} && (2.43) \\
&= \{x \in X | \Gamma_2(\Gamma_1 x) \cap A \neq \varnothing\} && (2.44) \\
&= \{x \in X | \Gamma_1 x \subseteq \Gamma_2^- A\} && (2.45) \\
&= \{x \in X | \Gamma_1 x \cap \Gamma_2^- A = \Gamma_1 x \neq \varnothing\} && (2.46) \\
&= \Gamma_1^-(\Gamma_2^- A) && (2.47)
\end{aligned}
$$

If $\Gamma_1 x = \varnothing$, then $\Gamma_2(\Gamma_1 x) = \varnothing$, $\Gamma_2(\Gamma_1 x) \cap A = \varnothing$, contradiction.

- $(\Gamma_2\Gamma_1)^+A = \Gamma_1^+(\Gamma_2^+A)$.

$$(\Gamma_2\Gamma_1)^+A = \{x \in X | (\Gamma_2\Gamma_1)x \subseteq A\} \tag{2.48}$$
$$= \{x \in X | \Gamma_2(\Gamma_1 x) \subseteq A\} \tag{2.49}$$
$$= \{x \in X | \Gamma_1 x \subseteq \Gamma_2^+ A\} \tag{2.50}$$
$$= \Gamma_1^+(\Gamma_2^+A) \tag{2.51}$$

- $\Gamma = \bigcup\limits_{i\in I} \Gamma_i$, $\Gamma^- A = \bigcup\limits_{i\in I} \Gamma_i^- A$.

$$\Gamma^- A = \{x \in X | (\bigcup_{i\in I} \Gamma_i)x \cap A \neq \varnothing\} \tag{2.52}$$
$$= \{x \in X | \bigcup_{i\in I} (\Gamma_i x) \cap A \neq \varnothing\} \tag{2.53}$$
$$= \bigcup_{i\in I} \{x \in X | \Gamma_i x \cap A \neq \varnothing\} \tag{2.54}$$
$$= \bigcup_{i\in I} \Gamma_i^- A \tag{2.55}$$

- $\Gamma = \bigcup\limits_{i\in I} \Gamma_i$, $\Gamma^+ A = \bigcap\limits_{i\in I} \Gamma_i^+ A$.

$$\Gamma^+ A = \{x \in X | (\bigcup_{i\in I} \Gamma_i)x \subseteq A\} \tag{2.56}$$
$$= \{x \in X | \bigcup_{i\in I} (\Gamma_i x) \subseteq A\} \tag{2.57}$$
$$= \bigcap_{i\in I} \{x \in X | \Gamma_i x \subseteq A\} \tag{2.58}$$
$$= \bigcap_{i\in I} \Gamma_i^+ A \tag{2.59}$$

THEOREM 2.90. *If Γ_1 is a l.s.c. mapping of X into Y and Γ_2 is a l.s.c. mapping of Y into Z, then composition product $\Gamma_2\Gamma_1$ is a l.s.c. mapping of X into Z.*

Proof. Let G be open, then $\Gamma_2^- G$ is open, then $\Gamma^- G = (\Gamma_2\Gamma_1)^- G = \Gamma_1^-(\Gamma_2^- G)$ is open.
Let F be closed, then $\Gamma_2^+ F$ is closed, then $\Gamma^+ F = (\Gamma_2\Gamma_1)^+ F = \Gamma_1^+(\Gamma_2^+ F)$ is closed. □

THEOREM 2.91. *If Γ_1 is a u.s.c. mapping of X into Y and Γ_2 is a u.s.c. mapping of Y into Z, then composition product $\Gamma_2\Gamma_1$ is a u.s.c. mapping of X into Z.*

Proof. Let G be open, then $\Gamma_2^+ G$ is open, then $\Gamma^+ G = (\Gamma_2\Gamma_1)^+ G = \Gamma_1^+(\Gamma_2^+ G)$ is open.
Let F be closed, then $\Gamma_2^- F$ is closed, then $\Gamma^- F = (\Gamma_2\Gamma_1)^- F = \Gamma_1^-(\Gamma_2^- F)$ is closed. □

THEOREM 2.92. *Let $\Gamma = \bigcup\limits_{i\in I} \Gamma_i$ where $\Gamma_i : X \rightrightarrows Y$ is l.s.c for $\forall i \in I$, then Γ is l.s.c.*

Proof. We have two approaches.
(1) Let $G \subseteq Y$ be open, then $\Gamma_i^- G$ is open for $\forall i \in I$. So $\Gamma^- G = \bigcup\limits_{i\in I} \Gamma_i^- G$ is open (an arbitrary union of open sets is open), so Γ is l.s.c.
(2) Let $F \subseteq Y$ be closed, then $\Gamma_i^+ F$ is closed for $\forall i \in I$. So $\Gamma^+ F = \bigcup\limits_{i\in I} \Gamma_i^+ F$ is closed (an arbitrary union of closed sets is closed), so Γ is l.s.c. □

THEOREM 2.93. *Let $\Gamma = \bigcup\limits_{n=1}^{N} \Gamma_i, \Gamma_i : X \rightrightarrows Y$ is u.s.c. $\forall i = 1, 2, \cdots, N$. Then Γ is u.s.c.*

Proof. We have two approaches.

(1) Let G be an open set, then $\Gamma_i^+ G$ is open as Γ_i is l.s.c. for $\forall i = 1, 2, \cdots, N$. So $\Gamma^+ G = \bigcap\limits_{n=1}^{N} \Gamma_i^+ G$ is open as a finite intersection of open sets is open.

(2) Let F be a closed set, then $\Gamma_i^- F$ is closed as Γ_i is l.s.c. for $\forall i = 1, 2, \cdots, N$. So $\Gamma^- F = \bigcup\limits_{n=1}^{N} \Gamma_i^- F$ is closed as a finite union of closed sets is closed. □

THEOREM 2.94. *The intersection $\Gamma = \bigcap\limits_{i\in I} \Gamma_i$ of a family of compact-valued u.s.c. mappings Γ_i if X into Y is also a compact-valued u.s.c. mapping of X into Y.*

Proof. Since Γ_i is compact-valued and u.s.c., Γ_i is closed $\forall i \in I$. Then Γ is closed. (Note that an arbitrary intersection of closed mappings is closed)

Since Γ_i is compact-valued for $\forall i \in I$, so $\Gamma = \bigcap_{i\in I} \Gamma_i$ is compact-valued from the non-empty finite intersection theorem.

Since Γ is compact-valued, so $Y = \Gamma X$ is compact (if not, take $Y = \Gamma X$). Thus, Y is a compact space and Γ is compact-valued, and Γ is closed, so Γ is u.s.c. □

First we show two equations and then give two theorem of cartesian product of mapping. Let $\Gamma = \prod_{n=1}^{N} \Gamma_n$ and $E^k = \prod_{n=1}^{N} A_n^k$, then $\Gamma^- E^k = \bigcap_{n=1}^{N} \Gamma_n^- A_n^k, \Gamma^+ E^k = \bigcap_{n=1}^{N} \Gamma_n^+ A_n^k$.

$$\Gamma^- E^k = \{x \in X | \Gamma x \cap E^k \neq \varnothing\} \tag{2.60}$$
$$= \{x \in X | \prod_{n=1}^{N} (\Gamma_n x) \cap \prod_{n=1}^{N} A_n^k \neq \varnothing\} \tag{2.61}$$
$$= \{x \in X | \prod_{n=1}^{N} (\Gamma_n x \cap A_n^k) \neq \varnothing\} \tag{2.62}$$
$$= \bigcap_{n=1}^{N} \{x \in X | \Gamma_n x \cap A_n^k \neq \varnothing\} \tag{2.63}$$
$$= \bigcap_{n=1}^{N} \Gamma_n^- A_n^k \tag{2.64}$$

$$\Gamma^+ E^k = \{x \in X | \Gamma x \subseteq E^k\} \tag{2.65}$$
$$= \{x \in X | \prod_{n=1}^{N} \Gamma_n x \subseteq \prod A_n^K\} \tag{2.66}$$
$$= \bigcap_{n=1}^{N} \{x \in X | \Gamma_n x \subseteq A_n^k\} \tag{2.67}$$
$$= \bigcap_{n=1}^{N} \Gamma_n^+ A_n^k \tag{2.68}$$

THEOREM 2.95. *Let* $\Gamma = \prod_{n=1}^{N} \Gamma_n : X \rightrightarrows \prod_{n=1}^{N} Y_n$ *and* $\Gamma_n : X \rightrightarrows Y$ *is l.s.c. for* $\forall n$*, then* Γ *is l.s.c.*

Proof. We have two approaches.

(1) Let $E^k = \prod_{n=1}^{N} G_n^k$ where $G_n^k \subseteq Y_n$ is open, so E^k is open in Y. As Γ_i is l.s.c., we have $\Gamma_n^- G_n^k$ is open for $n = 1, 2, \cdots, N$. Then $\Gamma^- E^k = \bigcap_{n=1}^{N} \Gamma_n^- G_n^k$ is open as a finite intersection of open sets is open, so Γ is l.s.c.

(2) Let $E^k = \prod_{n=1}^{N} F_n^k$ where $F_n^k \subseteq Y_n$ is closed, so E^k is closed in Y. As Γ_i is l.s.c., we have $\Gamma_n^+ F_n^k$ is closed for $n = 1, 2, \cdots, N$. Then $\Gamma^+ E^k = \bigcap_{n=1}^{N} \Gamma_n^+ F_n^k$ is closed as an arbitrary intersection of closed sets is closed, so Γ is l.s.c. □

THEOREM 2.96. *Let* $\Gamma = \prod_{n=1}^{N} \Gamma_n : X \rightrightarrows \prod_{n=1}^{N} Y_n$ *and* $\Gamma_n : X \rightrightarrows Y$ *is u.s.c. for* $\forall n$*, then* Γ *is u.s.c.*

Proof. We have two approaches.

(1) Let $E^k = \prod_{n=1}^{N} G_n^k$ where $G_n^k \subseteq Y_n$ is open, so E^k is open in Y. As Γ_i is u.s.c., we have $\Gamma_n^+ G_n^k$ is open for $n = 1, 2, \cdots, N$. Then $\Gamma^+ E^k = \bigcap_{n=1}^{N} \Gamma_n^+ G_n^k$ is open as a finite intersection of open sets is open, so Γ is u.s.c.

(2) Let $E^k = \prod_{n=1}^{N} F_n^k$ where $F_n^k \subseteq Y_n$ is closed, so E^k is closed in Y. As Γ_i is u.s.c., we have $\Gamma_n^- F_n^k$ is closed for $n = 1, 2, \cdots, N$. Then $\Gamma^- E^k = \bigcap_{n=1}^{N} \Gamma_n^- F_n^k$ is closed as an arbitrary intersection of closed sets is closed, so Γ is u.s.c. □

THEOREM 2.97. *Let $\Gamma = \prod_{n=1}^{N} \Gamma_n : X \rightrightarrows \prod_{n=1}^{N} Y_n$ and $\Gamma_n : X \rightrightarrows Y$ is compact-valued u.s.c. for $\forall n$, then Γ is compact-valued u.s.c.*

Proof. Since Γ_n is compact-valued and u.s.c., then Γ_n is a closed mapping for $n = 1, 2, \cdots, N$. Then Γ is a closed mapping. Obviously, it is compact-valued, so it is u.s.c. □

THEOREM 2.98. *Let $\Gamma_n : X \rightrightarrows Y$ be a compact-valued u.s.c. mapping for $n = 1, 2, \cdots, N$, then $\Gamma = \sum_{n=1}^{N} \Gamma_n$ is a compact-valued u.s.c. mapping.*

Proof. (1) we first show Γ is compact-valued. Define $f : Y^N \to Y, f(y_1, y_2, \cdots, y_N) = \sum_{n=1}^{N} y_n$, then f is a continuous function. $f(\Gamma_1 x_0, \Gamma_2 x_0, \cdots, \Gamma_N x_0) = \sum_{n=1}^{N} \Gamma_n x_0$. As $\Gamma_n x_0$ is compact for $n = 1, 2, \cdots, N$, then Γx_0 is compact. (continuous function on a compact domain)

(2) $\forall G \subseteq Y$ (open) s.t. $\Gamma x_0 = \sum_{n=1}^{N} \Gamma_n x_0 \subseteq G$, then $\exists V = U(\mathbf{0})$ of origin s.t. $\sum_{n=1}^{N} \Gamma_n x_0 + V \subseteq G$ as G is open. Then $\exists V_1, V_2, \cdots, V_N$ s.t. $\sum_{n=1}^{N} V_n \subseteq V$, so $\sum_{n=1}^{N} [\Gamma_n x_0 + V_n] = \sum_{n=1}^{N} \Gamma_n x_0 + V \subseteq G$.

Since Γ_n is open, so $\exists U_n(x_0)$ s.t. $\Gamma_n U_n(x_0) \subseteq \Gamma_n x_0 + V_n$. Let $U(x_0) = \bigcap_{n=1}^{N} U_n(x_0)$, so $\forall x \in U(x_0), \Gamma x = \sum_{n=1}^{N} \Gamma_n x \subseteq \sum_{n=1}^{N} [\Gamma_n x_0 + V_n] \subseteq \sum_{n=1}^{N} \Gamma_n x_0 + V \subseteq G$. Therefore, Γ is u.s.c. □

DEFINITION 2.43. *(Inner Semi-continuous, Outer Semi-continuous) Let X, Y be a metric space, and $\Gamma : X \rightrightarrows Y$ be a correspondence. We say Γ is inner semi-continuous (i.s.c.) if $\forall x_q \to x, y \in \Gamma$, there is a sequence $y_q \in \Gamma x_q, \forall q$ s.t. $y_q \to y$.*

We say Γ is outer semi-continuous (o.s.c.) if $\forall x_q \to x, y_q \in \Gamma x_q, \forall q$, there is a convergent subsequence y_{q_k} s.t. $y_{q_k} \to y, y \in \Gamma x$.

THEOREM 2.99. *Let X, Y be two metric spaces. A correspondence $\Gamma : X \rightleftarrows Y$ is l.s.c. iff it is i.s.c.*

Proof. (1) $i.s.c. \Rightarrow l.s.c.$. Suppose Γ is i.s.c. but not l.s.c. at $x_0 \in X$. Then $\exists G$(open) s.t. $G \cap \Gamma x_0 \neq \varnothing$ and $\forall U(x_0)$, we have $\exists x \in U(x_0), G \cap \Gamma x = \varnothing$. Let $y \in G \cap \Gamma x_0$, then as Γ is i.s.c, $\forall x_q \to x, \exists y_q \to y$ s.t. $y_q \in \Gamma x_q$. But we can choose (x_q) s.t. $G \cap \Gamma x_q = \varnothing, \forall q$, i.e., $y_q \notin G, \forall q$. With G is open and $y \in G$, $y_q \to q$ contradicts to $y_q \notin G, \forall G$.

(2) $l.s.c. \Rightarrow i.s.c.$ Suppose Γ is l.s.c. then given $x_q \to x_0, y \in \Gamma x_0$, we need to construct a sequence (y_q) s.t. $y_q \to y, y_q \in \Gamma x_q$. Let $G = N_\epsilon(y)$, then $\exists U_\epsilon(x_0)$ s.t. $\forall x \in U_\epsilon(x_0)$, we have $\Gamma x \cap N_\epsilon(y) \neq \varnothing$, i.e., $\exists y_\epsilon \in N_\epsilon(y) \cap \Gamma x$. Then, let $\epsilon_q = 1/q, q = 1, 2, \cdots$, so $U_{\epsilon_1}(x_0) \supseteq U_{\epsilon_2}(x_0) \supseteq \cdots$. Now we construct the sequence.

a. if $x_q \notin U_{\epsilon_1}(x_0)$, take $y_q \in \Gamma x_q$ arbitrarily;

b. otherwise, $x_q \in U_{\epsilon_i}(x_0) \backslash U_{\epsilon_{i+1}}(x_0), i \in \mathbf{N}_{++}$, take $y_q \in \Gamma x_q \cap N_{\epsilon_i}(y)$.

As $x_q \to x_0$, such x_q s.t. b. is infinite, so (y_q) is infinite. As $x_q \to x_0$, we have $\epsilon_i \to 0$, so $y_q \to y$, i.e., Γ is i.s.c. □

THEOREM 2.100. *Let X, Y be two metric spaces. Let $\Gamma : X \rightrightarrows Y$:*

- *if Γ is compact-valued and u.s.c. at x, then it is o.s.c. at x;*
- *if Γ is o.s.c. at x, then it is u.s.c. at x.*

Proof. (1) compact-valued u.s.c. $\Rightarrow$ o.s.c. Suppose Γ is compact-valued and u.s.c. at x_0, but not o.s.c. Then $\exists (x_q), (y_q)$ s.t. $x_q \to x_0, y_q \in \Gamma x_q$ but $\forall y \in \Gamma x_0$ we can not find a subsequence of (y_q) which converges to y. Then $\exists \epsilon > 0, N \in \mathbb{N}$ s.t. $\forall n > N, y \notin N_\epsilon(y), \forall y \in \Gamma x_0$.

As Γ is compact-valued at x_0, then Γx_0 is compact. Let $G = \bigcup_{y \in \Gamma x_0} N_\epsilon(y)$ be a cover of Γx_0, then there is a finite set $I \subseteq \Gamma x_0$ s.t. $G' = \bigcup_{y \in I} N_\epsilon(y)$ covers Γx_0. As Γ is u.s.c., $\exists U(x_0)$ s.t. $\Gamma U(x_0) \subseteq G'$. Let $x'_q \in \{x_N, x_{N+1}, \cdots\} \cap U(x_0)$ ($x_q \to x_0$ ensure it is nonempty), so $\Gamma x'_q \subseteq G'$. Then $y'_q \in G' = \bigcap_{y \in I} N_\epsilon(y)$. Then $\exists y'$ s.t. $y'_q \in N_\epsilon(y')$, violating $y_q \notin N_\epsilon(y), \forall y \in \Gamma x_0$. (if Γ is not compact-valued, we can not ensure such ϵ exists)

(2) o.s.c. $\Rightarrow$ u.s.c. Suppose Γ is o.s.c. but not u.s.c., then $\exists G$(open) s.t. $\Gamma x_0 \subseteq G$ but $\forall U(x_0)$ we have $\Gamma U(x_0) \not\subseteq G$. Then $\forall U_\epsilon(x_0)$, we say $\exists x', y'$ s.t. $x' \in U(x_0), y' \in \Gamma x', y' \notin G$.

Let $\epsilon_1 = 1$, then $\exists x'_1 \in U_{\epsilon_1}(x_0), y'_1 \in \Gamma x'_1, y'_1 \notin G$; let $\epsilon_2 = \min\{1/2, ||x_0 - x'_1||\}$, then $\exists x'_2 \in U_{\epsilon_2}(x_0), y'_2 \in \Gamma x'_2, y'_2 \notin G$. Repeat the operation we obtain two sequences $(x'_q), (y'_q)$ s.t. $x'_q \to x_0, y'_q \in \Gamma x'_q, y'_q \notin G$. Then as G is open, we can not find a subsequence of (y'_q) which converges to an element of G, contradicting to Γ is o.s.c. □

THEOREM 2.101. *Let $\Gamma_1 : X \rightrightarrows Y, \Gamma_2 : Y \rightrightarrows Z$. If Γ_1, Γ_2 are i.s.c., the composition product $\Gamma = \Gamma_1\Gamma_2$ is an i.s.c mapping of X into Z.*

Proof. $\forall x_q \to x, z \in \Gamma x$, we need to show $\exists(z_q)$ s.t. $z_q \in \Gamma x_q, \forall q$ and $z_q \to z$. As Γ_1 is i.s.c., then $\forall y \in \Gamma_1, \exists(y_q)$ s.t. $y_q \in \Gamma_1 x_q, \forall q$ and $y_q \to y$. As Γ_2 is i.s.c., and $\exists y \in Y$ s.t. $z \in \Gamma_2 y$, so with the (y_q) we have found, we say $\exists(z_q)$ s.t. $z_q \in \Gamma_2 y_q$ and $z_q \to z$. Thus, with $z_q \in \Gamma_2 y_q, y_q \in \Gamma_1 x_q$, we have $z_q \in \Gamma_2 y_q \subseteq \Gamma_2\Gamma_1 x_q = \Gamma x_q, \forall q$. So we find the (z_q) s.t. $z_q \in \Gamma x_q, \forall q$ and $z_q \to z$. ☐

THEOREM 2.102. *Let $\Gamma_1 : X \rightrightarrows Y, \Gamma_2 : Y \rightrightarrows Z$. If Γ_1, Γ_2 are o.s.c., the composition product $\Gamma = \Gamma_2\Gamma_1$ is an o.s.c mapping of X into Z.*

Proof. $\forall x_q \to x, z_q \in \Gamma x_q, \forall q$ we need to show $\exists(z_{q_k})$ s.t. $z_{q_k} \to z$ and $z \in \Gamma x$. $\forall z_q \in \Gamma x_q$ we have $\exists y_q \in \Gamma_1 x_q$ s.t. $z_q \in \Gamma_2 y_q, \forall q$. As Γ_1 is o.s.c. and $x_q \to x, y_q \in \Gamma_1 x_q$, then $\exists y, (y_{q_k})$ s.t. $y_{q_k} \to y$ and $y \in \Gamma x$. As Γ_2 is o.s.c. and $y_{q_k} \to y, z_{q_k} \in \Gamma_2 y_{q_k}$, then $\exists z, (z_{q'_k})$ s.t. $z_{q'_k} \to z$ and $z \in \Gamma_2 y \subseteq \Gamma_2\Gamma 1x = \Gamma x$. So we have find a subsequence $(z_{q'_k})$ which converges to Γx. ☐

THEOREM 2.103. *Let $\Gamma_n : X_n \rightrightarrows Y_n, n = 1, 2, \cdots, N$ and $\Gamma = \prod_{n=1}^{N} \Gamma_n$. If Γ_n is i.s.c. $\forall n$, then Γ is i.s.c.*

Proof. $\forall x_q \to x, y \in \Gamma x$, we need to show $\exists(y_q)$ s.t. $y_q \in \Gamma x_q, \forall q$ and $y_q \to y$. Let $x = (x_1, x_2, \cdots, x_N), y = (y_1, y_2, \cdots, y_N)$, then $y_n \in \Gamma x_n, \forall n$. As Γ_n is i.s.c., $\exists(y_{n,q})$ s.t. $y_{n,q} \in \Gamma x_{n,q}, \forall q$ and $y_{n,q} \to y_n$ for $\forall n$. So let $y_q = (y_{1,q}, y_{2,q}, \cdots, y_{N,q})$, we have $y_q \in \Gamma x_q$ and $y_q \to y$, i.e., Γ is i.s.c. ☐

THEOREM 2.104. *Let $\Gamma_n : X_n \rightrightarrows Y_n, n = 1, 2, \cdots, N$ and $\Gamma = \prod_{n=1}^{N} \Gamma_n$. If Γ_n is o.s.c. $\forall n$, then Γ is o.s.c.*

Proof. $\forall x_q \to x, y_q \in \Gamma x_q, \forall q$, we want to show $\exists y_{q_k}, y$ s.t. $y_{q_k} \to y, y \in \Gamma x$. As Γ_1 is o.s.c., with $x_{1,q} \to x_1, y_{1,q} \in \Gamma_1 x_{1,q}$ we have $\exists y_{1,q_k^1}$ s.t. $y_{1,q_k^1} \to y_1, y_1 \in \Gamma_1 x_1$. Then, $x_{q_k^1} \to x$ and $y_{q_k^1} \in \Gamma x_{q_k^1}, \forall k$.

As Γ_2 is o.s.c., with $x_{2,q_k^2} \to x_2, y_{2,q_k^1} \in \Gamma_1 x_{2,q_k^1}$ we have $\exists y_{2,q_k^2}$ s.t. $y_{2,q_k^2} \to y_2, y_2 \in \Gamma_2 x_2$. (Also, $y_{1,q_k^2} \to y_1$) Repeat the operation in N times, we obtain a infinite (N is finite) subsequence $(y_{q_k^N})$ and $y = (y_1, y_2, \cdots, N)$ s.t. $y_{k^N} \to y, y \in \Gamma x$. Therefore Γ is o.s.c. ☐

2.6 Berge's Maximization Theorem

Table 2: Berge's Maximization Theorems

ϕ	Γ	$M(x)$		maximizer/minimizer
l.s.c.	l.s.c.	$\sup\{\phi(x,y)\|y \in \Gamma x\}$	l.s.c.	
u.s.c.	compact-valued & u.s.c.	$\max\{\phi(x,y)\|y \in \Gamma x\}$	u.s.c.	
continuous	continuous	$\max\{\phi(x,y)\|y \in \Gamma x\}$	continuous	compact-valued & u.s.c.
l.s.c.	compact-valued & u.s.c.	$\inf\{\phi(x,y)\|y \in \Gamma x\}$	l.s.c.	
u.s.c.	l.s.c.	$\inf\{\phi(x,y)\|y \in \Gamma x\}$	u.s.c.	
continuous	continuous	$\min\{\phi(x,y)\|y \in \Gamma x\}$	continuous	compact-valued & u.s.c.

Berge's Maximization Theorems are of great importance in economics as summarized in Table 2.

THEOREM 2.105. *If ϕ is a l.s.c. numerical function in $X \times Y$ and Γ is a l.s.c. mapping of X into Y s.t. $\forall x, \Gamma x \neq \varnothing$, the numerical function M defined by*

$$M(x) = \sup\{\phi(x,y)|y \in \Gamma x\} \tag{2.69}$$

is l.s.c.

Proof. Let $x_0 \in X$. $\forall \epsilon > 0$, we need to show $\exists U(x_0)$ s.t. $\forall x \in U(x_0)$ we have $M(x) \geq M(x_0) - \epsilon$. Given ϵ, we now search a $U(x_0)$.

$M(x_0) = \sup\{\phi(x_0,y)|y \in \Gamma x_0\}$, then $\exists(x_0,y_0), y_0 \in \Gamma x_0$ s.t. $\phi(x_0,y_0) \geq M(x_0) - \frac{\epsilon}{2}$. As ϕ is l.s.c, then $\exists U'(x_0) \times V'(y_0)$ s.t. $\forall (x',y') \in U'(x_0) \times V'(y_0)$, we have $\phi(x',y') \geq \phi(x_0,y_0) - \frac{\epsilon}{2} \geq M(x_0) - \epsilon$. As Γ is l.s.c., then for $V'(y_0), y_0 \in \Gamma x_0$, we have $\exists U''(x_0)$ s.t. $\forall x'' \in U''(x_0)$, we have $\Gamma x'' \cap V'(y_0) \neq \varnothing$. Then, let $U(x_0) = U'(x_0) \cap U''(x_0)$, we have $\forall x \in U(x_0)$

$$\begin{aligned} M(x) &= \sup\{\phi(x,y)|y \in \Gamma x\} && (2.70)\\ &\geq \sup\{\phi(x,y)|y \in \Gamma x \cap V'(y_0)\} && (2.71)\\ &\geq \phi(x,y), x \in U'(x_0), y \in V'(y_0) && (2.72)\\ &\geq M(x_0) - \epsilon && (2.73) \end{aligned}$$

□

THEOREM 2.106. *If ϕ is an u.s.c. numerical function in $X \times Y$ and Γ is a compact-valued u.s.c. mapping of X into Y s.t. $\forall x, \Gamma x \neq \varnothing$, the numerical function M defined by*

$$M(x) = \max\{\phi(x,y)|y \in \Gamma x\} \tag{2.74}$$

is u.s.c. (Note that we ensure the maximizer is attainable as $\phi(x,y)$ is u.s.c. and compact-valued)

Proof. Let $x_0 \in X$, we need to show $\forall \epsilon > 0$, there exists $U(x_0)$ s.t. $\forall x \in U(x_0)$ we have $M(x) \leq M(x_0) + \epsilon$.

As $M(x_0) = \max\{\phi(x_0,y)|y \in \Gamma x_0\}$, $\forall y_0 \in \Gamma x_0$, we have $\phi(x_0,y_0) \leq M(x_0)$. As ϕ is u.s.c., for (x_0,y_0), there exists $U'(x_0) \times U'(y_0)$ s.t. $\forall (x',y') \in U'(x_0) \times U'(y_0)$, $\phi(x',y') \leq \phi(x_0,y_0) + \epsilon \leq M(x_0) + \epsilon$. As Γ is compact-valued u.s.c., then for $V'(y_0)$, there exists $U''(x_0)$ s.t. $\forall x'' \in U''(x_0)$, $\Gamma x'' \subseteq V'(y_0)$. Let $U(x_0) = U'(x_0) \cap U''(x_0)$, we have $\forall x \in U(x_0)$

$$\begin{aligned} M(x) &= \max\{\phi(x,y)|y \in \Gamma x\} && (2.75)\\ &\leq \max\{\phi(x,y)|y \in V'(y_0)\} && (2.76)\\ &\leq \max\{\phi(x,y)|x \in U'(x_0), y \in V'(y_0)\} && (2.77)\\ &\leq M(x_0) + \epsilon && (2.78) \end{aligned}$$

□

THEOREM 2.107. *If ϕ is a continuous numerical function in $X \times Y$ and Γ is a compact-valued continuous mapping of X into Y s.t. $\forall x, \Gamma x \neq \varnothing$, the numerical function M defined by*

$$M(x) = \max\{\phi(x,y)|y \in \Gamma x\} \tag{2.79}$$

is continuous. The mapping $\Phi : X \rightrightarrows Y$ defined by

$$\Phi x = \{y \in \Gamma x | \phi(x,y) = M(x)\} \tag{2.80}$$

is nonempty compact-valued u.s.c. (the maximizer is attainable as ϕ is continuous and the domain is compact)

Proof. With the last two theorems, we know $M(x)$ is continuous. Now we show Φ is u.s.c. Let

$$\Delta x = \{y \in Y | M(x) - \phi(x,y) \leq 0\} \tag{2.81}$$

and then $\Phi = \Gamma \cap \Delta$. Since Γ is compact-valued u.s.c., if Δ is closed, then Φ is compact-valued u.s.c. Now we show Δ is closed.

$\forall x_q \to x_0, y_q \to y_0, y_q \in \Delta x_q$, we need to show $y_0 \in \Delta x_0$. As $x_q \to x_0$ and $M(x)$ is continuous, so $M(x_q) \to M(x_0)$. As $y_q \to y_0, x_q \to x_0$ and $\phi(x,y)$ is continuous, so $\phi(x_q,y_q) \to \phi(x_0,y_0)$. As $y_q \in \Delta x_q$, so $\phi(x_q,y_q) \geq M(x_q), \forall q$. Thus, $\phi(x_0,y_0) \geq M(x_0)$, i.e., $y_0 \in \Delta x_0$. (We use if $x_q \to x, y_q \to y, x_q \geq y_q$, then $x \geq y$) □

DEFINITION 2.44. *Selective Let ϕ be a continuous numerical function defined in a topological space Y. A family compact sets $\mathscr{K} = \{K_i, i \in I\}$ in Y is called selective (with respect to ϕ) if for each i, there exists one and only one a_i s.t.*

$$a_i \in K_i, \phi(x_i) = \max |\phi(y)| y \in K_i \tag{2.82}$$

In other words, the maximum of ϕ is attained at only one point of the set K_i. For example, in $\mathbb{R}^N$, every family of balls is selective w.r.t. $\phi(y) = \pi_i y$; in $\mathbb{R}$, every family of compact sets is selective w.r.t. $\phi(y) = y$.

THEOREM 2.108. *Let $\Gamma : X \rightrightarrows Y$ be continuous and $\Gamma x \neq \varnothing, \forall x \in X$. If the family sets $\{\Gamma x | x \in X\}$ is selective, there is a single-valued continuous mapping σ of X into Y s.t. $\forall x, \sigma x \in \Gamma x$.*

Proof. Let ϕ be a continuous numerical function in Y for which $\{\Gamma x | x \in X\}$ is selective. Let $\Phi x = \{y \in \Gamma x | \phi(y) = M(x)\}$, then Φ is single-valued and u.s.c., so it is continuous. And the mapping $\sigma x = \Phi x$ s.t.

$$\sigma x \in \Gamma x, \forall x \in X \tag{2.83}$$

□

LEMMA 2.9. *If Γ is a continuous mapping of X into $\mathbb{R}$ s.t. $\forall x \in X, \Gamma x \neq \varnothing$, there exists a continuous single-valued mapping σ s.t. $\forall x, \sigma x \in \Gamma x$.*

Proof. It is trivial, let $\phi(y) = y$, then $\sigma x = \max \Gamma x$. □

THEOREM 2.109. *If ϕ is an l.s.c. numerical function in $X \times Y$ and Γ is a compact-valued u.s.c. mapping of X into Y s.t. $\forall x, \Gamma x \neq \varnothing$, the numerical function M defined by*

$$M(x) = \inf\{\phi(x, y) | y \in \Gamma x\} \tag{2.84}$$

is l.s.c.

Proof. Let $x_0 \in X$. We need to show $\forall \epsilon > 0$, there exists a $U(x_0)$ s.t. $\forall x \in U(x_0), M(x) \geq M(x_0) - \epsilon$.

Given $\epsilon > 0$. $\forall y \in \Gamma x_0$, as ϕ is l.s.c., we say $\exists U'_y(x_0) \times V'(y)$ (note that we first choose y, then choose $U'_y(x_0)$ and $V'(y)$ according to y, suppose we do it for all y), s.t. $\forall (x', y') \in U'_y(x_0) \times V'(y)$ we have $\phi(x', y') \geq \phi(x_0, y) - \epsilon \geq M(x_0) - \epsilon$.

Now $\underset{y \in \Gamma x_0}{\cup} V'(y)$ covers Γx_0, as Γ is compact-valued, so Γx_0 is compact, then $\exists I \subseteq \Gamma x_0$ (finite) s.t. $\underset{y \in I}{\cup} V'(y)$ covers Γx_0, i.e.,

$$\Gamma x_0 \subseteq \underset{y \in I}{\cup} V'(y) \tag{2.85}$$

As Γ is u.s.c., so $\exists U_0(x_0)$ s.t. $\forall x \in U_0(x_0)$ we have $\Gamma x \subseteq \underset{y \in I}{\cup} V'(y)$.

Now let $U(x_0) = U_0(x_0) \cap (\underset{y \in I}{\cap} U'_y(x_0))$ (infinite as I is finite), so $\forall x \in U(x_0)$,

$$\begin{aligned} M(x) &= \inf\{\phi(x, y) | y \in \Gamma x\} && (2.86)\\ &\geq \inf\{\phi(x, y) | y \in \underset{y \in I}{\cup} V'(y)\} && (2.87)\\ &\geq \inf\{\phi(x, y) | (x, y) \in \underset{y \in I}{\cup} [U'_y(x_0) \times V'(y)]\} && (2.88)\\ &\geq M(x_0) - \epsilon && (2.89) \end{aligned}$$

□

THEOREM 2.110. *If ϕ is an u.s.c. numerical function in $X \times Y$ and Γ is a compact-valued l.s.c. mapping of X into Y s.t. $\forall x, \Gamma x \neq \varnothing$, the numerical function M defined by*

$$M(x) = \inf\{\phi(x, y) | y \in \Gamma x\} \tag{2.90}$$

is u.s.c.

Proof. Let $x_0 \in X$. We need to show $\forall \epsilon > 0$, there exists $U(x_0)$ s.t. $\forall x \in U(x_0), M(x) \leq M(x_0) + \epsilon$.

Given $\epsilon > 0$. As $M(x_0) = \inf\{\phi(x_0, y) | y \in \Gamma x_0\}$, $\exists y_0$ s.t. $\phi(x_0, y_0) \leq M(x_0) + \frac{\epsilon}{2}$. As ϕ is u.s.c., then $\exists U'(x_0) \times V'(y_0)$ s.t. $\forall (x', y') \in U'(x_0) \times V'(y_0)$, we have $\phi(x', y') \leq \phi(x_0, y_0) + \frac{\epsilon}{2} \leq M(x_0) + \epsilon$. As Γ is l.s.c. and $y_0 \in \Gamma x_0 \cap V'(y_0) \neq \varnothing$, so $\exists U''(x_0)$ s.t $\forall x \in U''(x_0), \Gamma x \cap V'(y_0) \neq \varnothing$, i.e.,

$$\inf\{\phi(x, y) | y \in \Gamma x\} \leq \inf\{\phi(x, y) | y \in \Gamma x \cap V'(x_0)\} \tag{2.91}$$

Let $U(x_0) = U'(x_0) \cap U''(x_0)$, we have, $\forall x \in U(x_0)$,

$$\begin{aligned} M(x) &= \inf\{\phi(x, y) | y \in \Gamma x\} && (2.92)\\ &\leq \inf\{\phi(x, y) | y \in \Gamma x \cap V'(y_0)\} && (2.93)\\ &\leq \phi(x_0, y_0) + \frac{\epsilon}{2} && (2.94)\\ &\leq M(x_0) + \epsilon && (2.95) \end{aligned}$$

□

THEOREM 2.111. *If ϕ is a continuous numerical function in $X \times Y$ and Γ is a compact-valued continuous mapping of X into Y s.t. $\forall x, \Gamma x \neq \varnothing$, the numerical function M defined by*

$$M(x) = \min\{\phi(x, y) | y \in \Gamma x\} \tag{2.96}$$

is continuous. The mapping $\Phi : X \rightrightarrows Y$ defined by

$$\Phi x = \{y \in \Gamma x | \phi(x, y) = M(x)\} \tag{2.97}$$

is nonempty compact-valued u.s.c. (the minimizer is attainable as ϕ is continuous and the domain is compact)

2.7 Vectors in $\mathbb{R}^N$ and Asymptotic Cone

We introduce vectors in $\mathbb{R}^N$ and their operations and then develop properties of asymptotic cones.

The elements of $\mathbb{R}^L$ are called vectors.

Let $x^1 = (x_i^1), x^2 = (x_i^2)$ be two elements of $\mathbb{R}^L$. We define their sum $x^1 + x^2$ as $(x_i^1 + x_i^2)$, i.e., the ith coordinate of $x^1 + x^2$ is the sum of the ith coordinates of x^1 and x^2. The element **0** has been defined by the condition that all its coordinates are equal to 0. The negative of x is $-x = -(x_i)$. One writes $x^1 - x^2$ for $x^1 + (-x^2)$. If $x^1, x^2, \cdots, x^N$ are N elements of $\mathbb{R}^L$, their sum is denoted as $\sum_{n=1}^N x^n$.

Let $x \in \mathbb{R}^L$ and $t \in \mathbb{R}$, a real number; one defines their product their product tx or xt as (tx_i). It is clear that the functions $(x^1, x^2) \to x^1 + x^2$ from $\mathbb{R}^L \times \mathbb{R}^L$ to $\mathbb{R}^L$ and $(t, x) \to tx$ from $\mathbb{R} \times \mathbb{R}^L$ are continuous.

DEFINITION 2.45. *(a-translation) Given a vector a of $\mathbb{R}^L$, the transformation of $\mathbb{R}^L$ into itself defined by $x \to x + a$ is called the a-translation in $\mathbb{R}^L$.*

Let X^1, X^2 be two subseteq of $\mathbb{R}^L$; one defines their sum $X^1 + X^2$ as the set of elements of $\mathbb{R}^L$ of the form $x^1 + x^2$ where $x^1 \in X^1, x^2 \in X^2$, i.e.,

$$X^1 + X^2 = \{x^1 + x^2 \in \mathbb{R}^L | x^1 \in X^1, x^2 \in X^2\} \tag{2.98}$$

One defines $-X$ as the set of elements of $\mathbb{R}^L$ of the form $-x$ where $x \in X$.

$$-X = \{-x \in \mathbb{R}^L | x \in X\} \tag{2.99}$$

One writes $X^1 - X^2$ for $X^1 + (-X^2)$. If $X^1, X^2, \cdots, X^N$ are N subsets of $\mathbb{R}^L$, their sum is denoted as $\sum_{n=1}^N X^n$.

THEOREM 2.112. *If $X_1, X_2, \cdots, X_N \subseteq \mathbb{R}^L$, then $\sum_{n=1}^N \overline{X_n} \subseteq \overline{\sum_{n=1}^N X_n}$. (The sum of their adherence is contained in the adherence of their sum)*

Proof. $\forall x \in \sum_{n=1}^N \overline{X_n}$ we want to show $x \in \overline{\sum_{n=1}^N X_n}$. As $x \in \sum_{n=1}^N \overline{X_n}$, then $x = \sum_{n=1}^N x_n$ where $x_n \in \overline{X_n}, \forall n$. $\forall n, x_n \in \overline{X_n}$ implies $\exists (x_n^q)$ s.t. $x_n^q \to x_n$ and $x_n^q \in X_n, \forall q$. Let $x^q = \sum_{n=1}^N x_n^q$, so $x^q \in \sum_{n=1}^N X_n, \forall q$ and $x^q \to x$, which implies $x \in \overline{\sum_{n=1}^N X_n}$. □

THEOREM 2.113. *Let $X_1, X_2, \cdots, X_N \subseteq \mathbb{R}^L$, and $X = \sum_{n=1}^N X_n$. If $X_1, X_2, \cdots, X_N$ are compact, then X is compact.*

Proof. X_n is compact, so it is closed and bounded for $\forall n$. So X is bounded as N is finite. Now we need to show X is closed, i.e., $\forall x^q \to x, x^q \in X$, we have $x \in X$.

$x^q \in X$ implies $x^q = \sum_{n=1}^N x_n^q$ where $x_n^q \in X_n, \forall n, q$. As X_1 is compact and (x_1^q) is infinite, so with the Weierstrass theorem, we have there is an accumulation point for (x_1^q), i.e., there is a subsequence $(x_1^{q_1})$ s.t. $x_1^{q_1} \to x_1$ and $x_1 \in X_1$. As X_2 is compact, now $(x_2^{q_1})$ is infinite, so we know there is a subsequence $(x_2^{q_2})$ of $(x_2^{q_1})$ s.t. $x_2^{q_2} \to x_2$. Repeat the trick N times , we obtain a sequence (q_N) s.t. $x_n^{q_N} \to x_n$ and $x_n^{q_N} \in X_n, \forall n, q$. So

$$\lim_{q_N \to \infty} \sum_{x_n^{q_N}} = \sum_{n=1}^N x_n \in \sum_{n=1}^N X_n \tag{2.100}$$

Thus, there is a subsequence (x^{q_N}) of (x^q) s.t.

$$\lim_{q_N \to \infty} x^{q_N} = \lim_{q_N \to \infty} \sum_{n=1}^N x_n^{q_N} = \sum_{n=1}^N x_n = x \in \sum_{n=1}^N X_n = X \tag{2.101}$$

(if $x_q \to x$, then x_{q_k} also converges at x) So X is compact. □

THEOREM 2.114. *Let $S \subseteq \mathbb{R}^L$, and $\forall n = 1, 2, \cdots, N$, let f_n be a function from S to $\mathbb{R}^L$. Define a function f from S to $\mathbb{R}^L$ by $f(x) = \sum_{n=1}^N f_n(x), \forall x \in S$. Then, if every f_n is continuous at $x \in S$, then f is continuous at x.*

Proof. $\forall x^q \to x, x \in S$ and $\forall \epsilon > 0$ we want to show $\exists q'$ s.t. $\forall q > q'$, we have $|f(x^q) - f(x)| < \epsilon$. Let $0 < \epsilon_n < \frac{\epsilon}{N}$, as f_n is continuous at x, so $\exists q'_n$ s.t. $\forall q > q'_n$, we say $|f_n(x^q) - f_n(x)| < \epsilon_n$ for $\forall n$. Let $q' = \max\{q'_1, q'_2, \cdots, q'_N\}$, so $\forall q > q'$, we say $|f_n(x^q) - f_n(x)| < \epsilon_n, \forall n$. Then $|f(x^q) - f(x)| = |\sum_{n=1}^N f_n(x^q) - \sum_{n=1}^N f_n(x)| = |\sum_{n=1}^N (f_n(x^q) - f_n(x))| \leq \sum_{n=1}^N |f_n(x^q) - f_n(x)| \leq \sum_{n=1}^N \epsilon_n < \epsilon$, i.e., f is continuous at x. □

THEOREM 2.115. *Let $S \subseteq \mathbb{R}^K$ and $\forall n = 1, 2, \cdots, N$. Let T_n be a subset of $\mathbb{R}^L$ and φ_n be a correspondence from S to T_n. Define $T = \sum_{n=1}^N T_n$ and a correspondence φ from S to T by $\varphi(x) = \sum_{n=1}^N \varphi_n(x), \forall x \in S$. If every T_n is compact, then T is compact. If φ_n is compact-valued u.s.c. (resp. l.s.c.) for $\forall n$ at the point $x \in S$, then φ is compact-valued u.s.c. (resp. l.s.c.) at x.*

Proof. We prove with o.s.c. and i.s.c. instead of u.s.c. and l.s.c.

(1) φ is o.s.c. $\forall x^q \to x, y^q \in \varphi(x^q)$, then $\exists y^{q_k}$ s.t. $y^{q_k} \to y, y \in \varphi(x)$. Now we construct a y^{q_k}. Since $y^q \in \varphi(x^q)$, so $y^q = \sum_{n=1}^N y_n^q, y_n^q \in \phi_n(x^q), \forall n, q$. As φ_1 is o.s.c. and $x^q \to x, y_1^q \in \varphi_1(x^q), \forall q$, so there is a sequence $y_1^{q_{k_1}}$ s.t. $y_1^{q_{k_1}} \to y_1, y_1 \in \varphi_1(x)$. In this way, we select a subsequence (q_{k_1}) from index set (q). Now $x^{q_{k_1}} \to x$ also and $y_2^{q_{k_1}} \in \varphi(x^{q_{k_1}}), \forall q$, with φ_2 is o.s.c., we say there is a subsequence $(y_2^{q_{k_2}})$ s.t. $y_2^{q_{k_2}} \to y_2, y_2 \in \varphi_2(x)$. (we also have $y_1^{q_{k_2}} \to y_1$). Repeat the step N times and we obtain a subsequence (q_{k_N}) of (q) s.t. $y_n^{q_{k_N}} \to y_n, y_n \in \varphi_n(x), \forall n$. Let $y^{q_{k_N}} = \sum_{n=1}^N y_n^{k_N}$, we have $y^{q_{k_N}} \to \sum_{n=1}^N y_n = y \in \varphi(x)$ as desired.

(2) φ is i.s.c. $\forall x_q \to x, y \in \varphi(x)$, we want to show $\exists y_q, y_q \in \varphi(x^q), \forall q$ s.t. $y_q \to y$.

$y \in \varphi(x)$ implies $y = \sum_{n=1}^N y_n, y_n \in \varphi_n(x), \forall n$. As $x^q \to x, y_n \in \varphi_n(x)$ and φ_n is i.s.c., we have $\exists y_n^q \in \varphi_n(x^q), \forall q$ s.t. $y_n^q \to y_n$ for $\forall n$. So let $y^q = \sum_{n=1}^N y_n^q \in \varphi(x^q)$, we have $y^q \to y$ as desired. □

Let x^1, x^2 be two points of $\mathbb{R}^L, t^1, t^2$ be two real numbers s.t. $t^1 + t^2 = 1$. The point $t^1x^1 + t^2x^2$ is called the weighted average of x^1, x^2 with weights t^1 and t^2 respectively. Let $x, y \in \mathbb{R}^L, x \neq y$,

- the straight line x, y is $\{z \in \mathbb{R}^L | t \in \mathbb{R}^L, z = (1-t)x + ty\}$;
- the closed half-line x, y (origin x) is $\{z \in \mathbb{R}^L | t \in \mathbb{R}_+, z = (1-t)x + ty\}$;
- the open half-line x, y is $\{z \in \mathbb{R}^L | t \in \mathbb{R}_{++}, z = (1-t)x + ty\}$.

We can rewrite $z = x + t(y - x)$ to show the origin explicitly.

Let $x, y \in \mathbb{R}^L$ (may be the same). The closed segment x, y, denoted $[x, y]$ is $\{z \in \mathbb{R}^L | t \in \mathbb{R}, t \in [0, 1], z = (1-t)x + ty\}$ (the sequence is not important). When $x = y$, the closed segment $[x, y]$ is said to be degenerate.

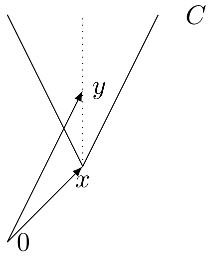


Figure 8: Cone

DEFINITION 2.46. *(Cone) Given a subset C of $\mathbb{R}^L$ and a point $x \in C$. C is called a cone with vertex x if it contains the half-line x, y whenever it owns the point $y (y \neq x)$. That implies*

$$y \in C \Rightarrow x + t(y - x) \in C, \forall t \geq 0 \tag{2.102}$$

See Figure 8.

In particular, if C is a cone with the origin $\mathbf{0}$, then $y \in C$ implies $ty \in C, \forall t \geq 0$.

DEFINITION 2.47. *(Positivity Semi-independent) Given N cones $C_1, C_2, \cdots, C_N$ with vertex $\mathbf{0}$, they are said to be positivity semi-independent (p.s.i.) if $x_n \in C_n, \forall n$ and $\sum_{n=1}^{N} x_N = \mathbf{0}$ implies $x_n = \mathbf{0}, \forall n$. In other words, it is impossible to take a vector in each cone so that their sum is $\mathbf{0}$ unless there all equal to $\mathbf{0}$.*

THEOREM 2.116. *Given two cones C_1, C_2 with vertex $\mathbf{0}$. C_1, C_2 are independent if and only if $C_1 \cap (-C_2) = \{\mathbf{0}\}$.*

Proof. (1) p.s.i. $\Rightarrow C_1 \cap (-C_2) = \{\mathbf{0}\}$. Suppose not, $\exists a \in C_1 \cap (-C_2)$ and $a \neq \mathbf{0}$, then $a \in C_1, -a \in C2$. So we find $a + (-a) = \mathbf{0}, a \in C_1, -a \in C_2$ but $a, -a \neq \mathbf{0}$, contradicting to p.s.i.

(2) $C_1 \cap (-C_2) = \{\mathbf{0}\} \Rightarrow$ p.s.i. Suppose not, C_1, C_2 are not p.s.i., then $\exists a_1 \in C_1, a_2 \in C_2$ s.t. $a_1 + a_2 = \mathbf{0}, a_1, a_2 \neq \mathbf{0}$. So $a_1 = -a_2 \in (-C_2)$, i.e., $a_1 \in C_1 \cap (-C_2)$ and $a_1 \neq \mathbf{0}$, violating $C_1 \cap (-C_2) = \{\mathbf{0}\}$. ☐

DEFINITION 2.48. *(Asymptotic Cone) Let $S \subseteq \mathbb{R}^L$ and M be a non-negative real number. Let*

$$S^M = \{x \in S | ||x|| \geq M\} \tag{2.103}$$

denoted the set of vectors in S whose norm is greater than or equal to M. Let $\Gamma(S^M)$ be the least closed cone with vertex $\mathbf{0}$ containing S^M (the intersection of all closed cones with vertex $\mathbf{0}$ containing S^M). The asymptotic cone of S, denoted as $\mathbf{A}S$, is defined as the intersection of all $\Gamma(S^M)$,

$$\mathbf{A}S = \bigcap_{M \geq 0} \Gamma(S^M) \tag{2.104}$$

which is a closed cone with vertex $\mathbf{0}$.

THEOREM 2.117. *Let $S_1, S_2 \subseteq \mathbb{R}^L$. If $S_1 \subseteq S_2$, then $\mathbf{A}S_1 \subseteq \mathbf{A}S_2$.*

Proof. $\forall M \geq 0$, with $S_1 \subseteq S_2$, we say $S_1^M \subseteq S_2^M$. So $\bigcap_{M \geq 0} \Gamma(S_1^M) \subseteq \bigcap_{M \geq 0} \Gamma(S_2^M)$, i.e., $\mathbf{A}S_1 \subseteq \mathbf{A}S_2$. ☐

If $S, T \subseteq \mathbf{R}^L$ and $T \neq \varnothing$, then $\mathbf{A}S \subseteq \mathbf{A}(S+T)$.

DEFINITION 2.49. *(Convex Sets) Let $G \subseteq \mathbb{R}^L$. If $\forall x, y \in G$, the closed segment $[x, y]$ is contained in G, then G is convex, i.e.,*

$$x, y \in G \Rightarrow \forall t \in [0,1], tx + (1-t)y \in G, \text{ i.e.,} [x, y] \subseteq G \tag{2.105}$$

DEFINITION 2.50. *Convex Polyhedral Cone A convex polyhedral cone is the sum of N closed half-lines.*

THEOREM 2.118. *The intersection of convex sets are convex.*

Proof. Let $S_i \subseteq \mathbb{R}^L, i \in I$ and S_i is convex for $\forall i \in I$. We want to show $S = \bigcap_{i \in I} S_i$ is convex.

$\forall x, y \in S$, then $x, y \in S_i, \forall i \in I$. As S_i is convex, we say $[x, y] \subseteq S_i$ for $\forall S_i$. So $[x, y] \subseteq \bigcap_{i \in I} S_i = S$, i.e., S is convex. ☐

THEOREM 2.119. *The sum and the product of N convex sets are convex.*

Proof. Let $S_n \subseteq \mathbb{R}^L, n = 1, 2, \cdots, N$ and S_n is convex for $\forall n$. We want to show $S = \sum_{n=1}^{N} S_n$ and $G = \prod_{1}^{N} S_n$ are convex.

(1) $S = \sum_{n=1}^{N} S_n$ is convex. $\forall x, y \in S$, we say $x = \sum_{n=1}^{N} x_n, y = \sum_{n=1}^{N} y_n, x_n, y_n \in S_n, \forall n$. As S_n is convex, $[x_n, y_n] \subseteq S_n$. $[x, y] = \{z \in \mathbb{R}^L | tx + (1-t)y, t \in [0,1]\} = \{z \in \mathbb{R}^L | \sum_{n=1}^{N} (tx_n + (1-t)y_n), t \in [0,1]\}$, so

$$[x, y] \subseteq \sum_{n=1}^{N} [x_n, y_n] \subseteq \sum_{n=1}^{N} S_n \tag{2.106}$$

i.e., S is convex.

(2) $G = \prod_{n=1}^{N} S_n$ is convex. $\forall x, y \in G$, then $x = \prod_{n=1}^{N} x_n, \prod_{n=1}^{N} y_n, x_n, y_n \in S_n, \forall n$. As S_n is convex, $[x_n, y_n] \subseteq S_n$. $[x, y] = \{z \in \mathbb{R}^L | tx + (1-t)y, t \in [0,1]\} = \{z \in \mathbb{R}^L | \prod_{n=1}^{N} (tx_n + (1-t)y_n), t \in [0,1]\}$, so

$$[x, y] \subseteq \prod_{n=1}^{N} [x_n, y_n] \subseteq \prod_{n=1}^{N} S_n \tag{2.107}$$

i.e., G is convex. ☐

THEOREM 2.120. *The adherence of a convex set is convex.*

Proof. Let $S \subseteq \mathbb{R}^L$ be convex, we want to show $\overline{S}$ is convex. Let $x, y \in \overline{S}$, then there are $(x^q), (y^q)$ s.t. $x^q \to x, y^q \to y$ and $x^q, y^q \in S, \forall q$. $\forall t \in [0,1]$,as S is convex, then $tx^q + (1-t)y^q \in S$. As $x^q \to x, y^q \to y$, so $tx + (1-t)y = \lim_{q\to\infty} tx^q + (1-t)y^q \in \overline{S}$. □

THEOREM 2.121. *A convex set is connected.*

Proof. Let $S \subseteq \mathbb{R}^L$ be convex. Suppose S it not connected, then $\exists S_1, S_2$ (both open) partition S, i.e., $S_1 \cup S_2 = S, S_1 \cap S_2 = \varnothing$. Let $x_1 \in S_1, x_2 \in S_2$, then $[x_1, x_2] \subseteq S$ we define

$$G_1 = [x_1, x_2] \cap S_1, G_2 = [x_1, x_2] \cap S_2 \tag{2.108}$$

So $G_1 \cap G_2 = \varnothing$ but $G_1 \cup G_2 = ([x_1, x_2] \cap S_1) \cup ([x_1, x_2] \cap S_2) = [x_1, x_2] \cap S = [x_1, x_2]$. Then G_1, G_2 partition $[x_1, x_2]$ but G_1, G_2 are both open, so $[x_1, x_2]$ is not connected, contradiction. □

Let $x^1 = (x_k^1), x^2 = (x_k^2)$ be two elements of $\mathbb{R}^L$. We define their inner product $x^1 \cdot x^2$ as the number $\sum_{k=1}^L x_k^1 x_k^2$. It is clear that the function $(x^1, x^2) \to x^1 \cdot x^2$ from $\mathbb{R}^L \times \mathbb{R}^L$ to $\mathbb{R}^L$ is continuous. When $x^1 \cdot x^2 = 0$, we say x^1 and x^2 are orthogonal.

DEFINITION 2.51. *(Hyperplane) Let $p \in \mathbb{R}^L, p \neq \mathbf{0}$ and $c \in \mathbb{R}$. The set*

$$H = \{z \in \mathbb{R}^L | p \cdot z = c\} \tag{2.109}$$

is a hyperplane with normal p. If $z' \in H$ (H goes through z'), we have $p \cdot z' = c$, so H can be write as $H = \{z \in \mathbb{R}^L | p \cdot (z - z') = c\}$. See Figure 9

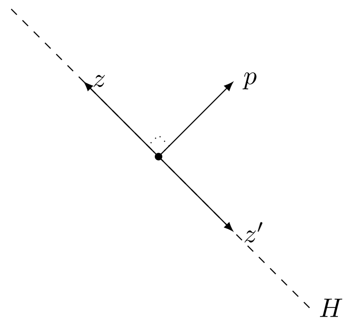


Figure 9: Hyperplane

The hyperplane H is the set of points z of $\mathbb{R}^L$ s.t. $z - z'$ is orthogonal to p; p and H are said to be orthogonal. If p and c are multiplied by the same real number different from 0, the hyperplane is unchanged. An intersection of the hyperplane is called a linear manifold.

Given a hyperplane H with normal p, the point z of $\mathbb{R}^L$ is said to be above H if $p \cdot z > c$. The closed half-space above H is $\{z \in \mathbb{R}^L | p \cdot z \geq c\}$. One obtains similar definitions replacing above $>, \geq$ be below $<, \leq$. A closed half-space is easily seen to be closed and convex. SO is a hyperplane, since it is the intersection of two closed half-spaces, and hence a linear manifold.

A hyperplane H is said to be bounding for a subset S of $\mathbb{R}^L$ if S is contained in one of the two closed half-spaces determined by H. In other words, H is bounding for S if S is entirely on one side of H with possibly, points in it.

THEOREM 2.122. *(Minkowski's Theorem) Let K be a convex subset of $\mathbb{R}^L$ and $z \in \mathbb{R}^L$. There is a hyperplane H through z and bounding K if and only if z it not interior to K.*

Proof. We first show a stronger statement. Let $T = \overline{K - \{z\}}$ (convex), then we show there is a hyperplane H through $\mathbf{0}$ such that H properly bounds T if and only if $\mathbf{0} \notin T$.

Let $U = \{x \in \mathbf{R}^L \,|\, ||x|| \leq \alpha\}$ s.t. $U \cap T \neq \varnothing$. As T, U are closed, so $U \cap T$ is closed. So $U \cap T$ is closed and bounded (i.e., compact), then the minimizer of $f(x) = ||x||$ is attainable, denoted as x_0 ($x_0 \in U \cap T$Weierstrass' Theorem). Thus, $||x_0|| \leq ||x||, \forall x \in U \cap T$. Moreover $||x|| > \alpha \geq ||x_0||, \forall x \in T \backslash U$. Thus

$$||x_0|| < ||x||, \forall x \in T \tag{2.110}$$

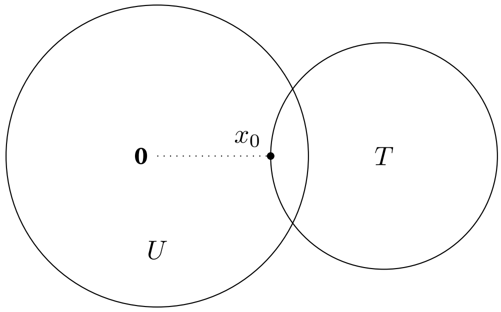


Figure 10: Hyperplane bounding a convex set

As T is convex, so $\forall x \in T, \forall \lambda \in (0, 1]$ (drop 0), we have $\lambda x + (1-\lambda)x_0 = \lambda(x - x_0) + x_0 \in T$, thus

$$||lambda(x - x_0) + x_0||^2 \geq ||x_0||^2 \tag{2.111}$$

$$\lambda^2(x - x_0) \cdot (x - x_0) + 2\lambda x_0 \cdot (x - x_0) \geq 0 \tag{2.112}$$

$$(x - x_0) \cdot (x - x_0) + 2\frac{1}{\lambda} x_0 \cdot (x - x_0) \geq 0 \tag{2.113}$$

If $\exists x$ s.t. $x_0 \cdot (x - x_0) < 0$, we can let $\lambda \to 0$, resulting a violation of Equation 2.113. So

$$x_0 \cdot (x - x_0) \geq 0 \Rightarrow x_0 \cdot x \geq x_0 \cdot x_0 \tag{2.114}$$

Since $x_0 \in T$, so $x_0 \neq \mathbf{0}$ and $x_0 \cdot x_0 > 0$. Let

$$H = \{x \in \mathbb{R}^L | x_0 \cdot x = 0\} \tag{2.115}$$

So H properly bounds T as desired, i.e., $\forall x \in T$, we have $x \cdot x_0 > 0$ (strictly greater than).

Conversely, if H properly bounds T, we show $\mathbf{0} \notin T$. Suppose not, $\mathbf{0} \in T$, however, $\mathbf{0} \in T$, violating the definition of properly bound.

The remaining task is trivial. H properly bounds T is equivalent to

$$\{x \in \mathbb{R}^L | x \cdot x_0 = z \cdot x_0\} \tag{2.116}$$

bounds K ($\forall x \in K, x \cdot x_0 = (x' + z) \cdot x_0 = x' \cdot x_0 + z \cdot x_0 > z \cdot x_0$ where $x' = x - z \in T$;may not properly bound as K may not be closed) and $z \notin int(K)$ is equivalent to $\mathbf{0} \notin T$. □

LEMMA 2.10. *Let* , $\forall x^q \to x, x^q \in \mathbb{R}^N, \forall q$, *we have* $\forall y \in \mathbb{R}^N$, $\lim_{q\to\infty} x^q y = xy$, *i.e.,* $\forall \delta > 0, \exists q'$ *s.t.* $\forall q > q'$, $|x^q y - xy| < \delta$.

Proof. If $y = \mathbf{0}$, the statement is evident. If $y \neq \mathbf{0}$, let $y = (y_1, y_2, \cdots, y_N)$, then $\sum_{i=1}^N |y_i| > 0$. $\forall \delta > 0$, let $0 < e < \frac{\delta}{\sum_{i=1}^N |y_i|}$. As $x^q \to x$, so $\exists q'$ s.t. $\forall q > q'$, $|x^q - x| < e$, thus $|x_i^q - x_i| < e, \forall i, q$. So

$$|x^q y - xy| = |\sum_{i=1}^N (x_i^q - x_i) y_i| \leq \sum_{i=1}^N |(x_i^q - x_i) y_i| \leq \sum_{i=1}^N |(x_i^q - x_i)||y_i| \leq \sum_{i=1}^N e|y_i| = e\sum_{i=1}^N |y_i| < \delta \tag{2.117}$$

□

In particular, if $x^q \to x, x^q y \leq 0$, then $xy \leq 0$.

DEFINITION 2.52. *(Polar) Let* C *be a cone with vertex* **0**. *Its polar is the set*

$$C^\circ \{x \in \mathbb{R}^L | x \cdot y \leq 0, \forall y \in C\} \tag{2.118}$$

THEOREM 2.123. *A polar* C^0 *is a closed, convex cone with vertex* **0**.

Proof. (1) Let $x^q \to x, x^q \in C^\circ$, then $\forall y \in C$, we have $x^q y \leq 0$. Then $\forall y \in C$, we have $xy \leq 0$, so $x \in C^\circ$, i.e., $\overline{C^\circ} \subseteq C^\circ$, C° is closed.

(2) Let $\forall x_1, x_2 \in C^\circ$, and $\forall y \in C$, then $x_1 y \leq 0, x_2 y \leq 0$, so $\forall t \in [0, 1], (tx_1 + (1-t)x_2)y = t(x_1 y) + (1-t)(x_2 y) \leq 0$, i.e., $tx_1 + (1-t) \in C^\circ$, so C° is convex. □

The non-negative orthant of $\mathbb{R}^L$ is the set $\Omega = \{x \in \mathbb{R}^L | x \geq 0\}$.

Let $S \in \mathbb{R}^L$, the asymptotic cone can be rewritten as

$$\mathbf{A}S = \bigcap_{M \geq 0} \Gamma(S^M) = \lim_{M \to \infty} \Gamma(S^M) \tag{2.119}$$

Thus $\mathbf{A}S$ is the cone generated by the points in S which are at infinite. As $\Gamma(S^M)$ is the least closed cone with vertex $\mathbf{0}$, we say

$$\Gamma(S^M) = \overline{\{\alpha x \in \mathbb{R}^L | \alpha \in \mathbb{R}^L_{++}, x \in S^M\}} \tag{2.120}$$

$$= \{\alpha x \in \mathbb{R}^L | \alpha \in \mathbb{R}^L_{++}, x \in \overline{S^M}\} \tag{2.121}$$

$$= \{\alpha x \in \mathbb{R}^L | \alpha \in \mathbb{R}^L_{++}, x \in \overline{S}^M\} \tag{2.122}$$

Another definition of asymptotic cone is given by Auslender and Marc(2003). Let $\{x_k, k \in \mathbb{N}\}$ be a sequence in $\mathbb{R}^L$, we are interested in how to handle situations when (x_k) is unbounded. To derive some convergent properties, we consider directions $d_k = \frac{x_k}{||x_k||}, x_k \neq \mathbf{0}, \forall k$. The Bolzano-Weierstrass Theorem implies that we can extract a convergent subsequence (d_{k_q}) s.t. $d = \lim_{q \to \infty} d_{k_q}$ and $d \neq \mathbf{0}(||d_k|| = 1, \forall k)$.

Suppose the sequence $(x_k) \subseteq \mathbb{R}^L$ s.t. $||x_k|| \to \infty$, then $\exists t_k = ||x_k||, k \in \mathbb{K} \subseteq \mathbb{N}$ s.t. $\lim_{k \to \infty} t_k = \infty$ and $\lim_{k \to \infty} \frac{x_k}{t_k} = d$.

A sequence $(x_k) \subseteq \mathbb{R}^L$ is said to converge to a direction $d \in \mathbb{R}^L$ if $\exists (t_k) \subseteq \mathbb{R}$ s.t. $t_k \to \infty$ s.t. $\lim_{k \to \infty} \frac{x_k}{t_k} = d$.

DEFINITION 2.53. *(Asymptotic Cone) Let C be a non-empty set in $\mathbb{R}^L$. Then the asymptotic cone of C, denoted by C_∞, is the set of vectors $d \in \mathbb{R}^L$ that are limits in directions of the sequences $\{x_k, k \in \mathbb{N}\} \subseteq C$, i.e.,*

$$C_\infty = \{d \in \mathbb{R}^L | \exists \{x_k, k \in \mathbb{N}\} \subseteq C, \{t_k, k \in \mathbb{N}\} \subseteq \mathbb{R}, \text{ with } t_k \to +\infty \text{ s.t. } \lim_{k \to \infty} \frac{x_k}{t_k} = d\} \tag{2.123}$$

Note that $t_k \to +\infty$, we can take $t'_k = |t_k|$ also satisfies the condition, so we won not stress $t_k > 0$ in most situations unless it is crucial.

The second definition is from R. Tyrrell Rockafellae (1970).

DEFINITION 2.54. *(Recession Cone and Asymptotic Cone) Let C be a nonempty convex set in $\mathbb{R}^L$. We say C recedes in the direction D if C includes all the half lines in the direction D which start at points of C. In other words, C recedes in the direction of y, where $y \neq \mathbf{0}$, iff $x + \lambda y \in C, \forall x \in C, \lambda \geq 0$.*

The set of all vectors $y \in \mathbb{R}^L$ satisfying the latter condition, including $y = \mathbf{0}$, will be called the recession cone of C, i.e.,

$$\mathbf{0}^+ C = \{y \in \mathbb{R}^L | \forall x \in C, \lambda \geq 0, x + \lambda y \in C\} \tag{2.124}$$

Directions in which C recedes will also referred to as directions of recession of C. The recession cone of $\overline{C}$ is exactly the asymptotic cone of C,

$$\mathbf{A}S = \mathbf{0}^+ \overline{C} \tag{2.125}$$

THEOREM 2.124. *Let $C \subseteq \mathbb{R}^L$ be nonempty, then*

(1) C_∞ is a closed cone;

(2) $(\overline{C})_\infty = C_\infty$;

(3) if C is a cone with vertex $\mathbf{0}$, then $C_\infty = \overline{C}$.

Proof. Let $C_\infty = \{d \in \mathbb{R}^L | \exists (x_k) \subseteq C, (t_k) \subseteq \mathbb{R}, \text{ with } t_k \to \infty \text{ s.t. } \lim_{k \to \infty} \frac{x_k}{t_k} = d\}$.

(1) We first show C_∞ is cone with vertex $\mathbf{0}$ then show it is closed. Let $x_k = d \in C, t_k = k, \forall k \subseteq \mathbb{R}$. Then $t_k \to \infty$ and $\lim_{k \to \infty} \frac{x_k}{t_k} = \mathbf{0}$, so $\mathbf{0} \in C_\infty$. Now $\forall d \in C_\infty$, we want to show $td \in C_\infty, \forall t \geq 0$. $d \in C_\infty$ implies $\exists (x_k) \subseteq C, (t_k) \subseteq \mathbb{R}^L$ with $t_k \to \infty$ and $\lim_{k \to \infty} \frac{x_k}{t_k} = d$. Let $t'_k = \frac{t_k}{t}, \forall k$, then $t'_k \to \infty$ and $\lim_{k \to \infty} \frac{x_k}{t'_k} = \lim_{k \to \infty} \frac{x_k}{t_k / t} = t \lim_{k \to \infty} \frac{x_k}{t_k} = td$, so $td \in C_\infty$, i.e., C_∞ is c cone with vertex $\mathbf{0}$.

Then we show C_∞ is closed, i.e., $\forall d^n \to d, d^n \in C_\infty$, we have $d \in C_\infty$. That is we want to find suitable $(x_k), (t_k)$.

$d^n \in C_\infty$ implies, $\exists(x_k^n) \subseteq C, (t_k^n) \subseteq \mathbb{R}$ s.t. $t_k^n \to \infty$ and $\lim_{k\to\infty} \frac{x_k^n}{t_k^n} = d^n$. As $\lim_{k\to\infty} \frac{x_k^n}{t_k^n} = d^n$, so $\exists k_n'$ s.t. $\forall k > k_n'$, we have $|\frac{x_k^n}{t_k^n} - d^n| < \frac{1}{n}$, for $\forall n$. Let $x_n = x_{k_n'}^n, t_n = t_{k_n'}^n$, then

$$\lim_{n\to\infty} |\frac{x_n}{t_n} - d| \le \lim_{n\to\infty} |\frac{x_n}{t_n} - d_n| + \lim_{n\to\infty} |d_n - d| \le \lim_{n\to\infty} \frac{1}{n} + \lim_{n\to\infty} |d_n - d| = 0 \tag{2.126}$$

So we construct such desired $(x_n), (t_n)$.

(2) $(\overline{C})_\infty = C_\infty = \overline{C_\infty}$. First we show if $A \subseteq B$, then $A_\infty \subseteq B_\infty$. $\forall d \in A_\infty$, then $\exists(x_k) \subseteq A, (t_k) \subseteq \mathbb{R}$ s.t. $t_k \to \infty$ and $\lim_{k\to\infty} \frac{x_k}{t_k} = d$. As $A \subseteq B$, such $\exists(x_k) \subseteq B, (t_k) \subseteq \mathbb{R}$ also give $d \in B_\infty$. So $A_\infty \subseteq B_\infty$.

As $C \subseteq \overline{C}$, so $C_\infty \subseteq (\overline{C})_\infty$.

Now we show $(\overline{C})_\infty \subseteq C_\infty$. $\forall d \in (\overline{C})_\infty$, we have $\exists(x_k) \subseteq \overline{C}, (t_k) \subseteq \mathbb{R}$ s.t. $t_k \to \infty$ and $\lim_{k\to\infty} \frac{x_k}{t_k} = d$. As $x_n \in \overline{C}$, then $\exists(x_q^n) \subseteq C$ s.t. $x_q^n \to x_n$, so $\exists q_1$ s.t. $\forall q > q_1$, we have $|\frac{x_q^n - x_n}{t_n}| < \frac{1}{n}$. Let $y_n = x_{q_n}^n$, then

$$\lim_{n\to\infty} |\frac{y_n}{t_n} - d| \le \lim_{n\to\infty} |\frac{x_{q_n}^n - x_n}{t_n}| + \lim_{n\to\infty} |\frac{x_n}{t_n} - d| \le \lim_{n\to\infty} \frac{1}{n} + \lim_{n\to\infty} |\frac{x_n}{t_n} - d| = 0 \tag{2.127}$$

So $d \in C_\infty$.

(c) Let C be a cone with vertex $\mathbf{0}$. We first show $C_\infty \subseteq \overline{C}$. Let $\forall d \in C_\infty$, then $\exists(x_k) \subseteq C, (t_k) \subseteq \mathbb{R}$ s.t. $t_k \to \infty$ and $\lim_{k\to\infty} \frac{x_k}{t_k} = d$. As $x_k C$ and C is a cone with vertex $\mathbf{0}$, so $\frac{x_k}{t_k} \in C, \forall k$, i.e., $d \in \overline{C}$.

Then we show $\overline{C} \subseteq C_\infty$. $\forall d \in \overline{C}$, so $\exists(d_k) \subseteq C$ s.t. $d_k \to d$. As C is a cone with vertex $\mathbf{0}$, then let $x_k = d_k k \in C$. Thus, $\lim_{k\to\infty} \frac{x_k}{k} = \lim_{k\to\infty} d_k = d$, so $d \in C_\infty$. ☐

Now we prove the first and second definitions of asymptotic cone are equivalent.

THEOREM 2.125. *Let $S \subseteq \mathbb{R}^L$. We have*

$$\mathbf{A}S = S_\infty \tag{2.128}$$

Proof. (1) $\mathbf{A}S \subseteq S_\infty$, i.e.,$\forall d \in \mathbf{A}S$, we need to construct $(x_k) \subseteq S, (t_k) \subseteq \mathbb{R}$ s.t. $t_k \to \infty$ and $\lim_{k\to\infty} \frac{x_k}{t_k} = d$. $d \in \mathbf{A}S$ implies $\forall M \ge 0$, we have $d \in \Gamma(S^M) = \overline{\{\alpha x \in \mathbb{R}^L | \alpha \in \mathbb{R}_+, x \in S^M\}}$. Then $\exists \alpha_M > 0, x_M \in S^M$ s.t. $||d - \alpha_M x_M|| < \frac{1}{M}$ for $\forall M$. Let $t_M = \frac{1}{\alpha_M}, \forall M$. As $||\alpha_M x_M|| < ||d|| + ||d - \alpha_M x_M||$, then

$$|\alpha_M| \le \frac{||d|| + ||d - \alpha_M x_M||}{||x_M||} \le \frac{||d|| + \frac{1}{M}}{M} \tag{2.129}$$

So $\lim_{M\to\infty} |\alpha_M| = 0$, i.e., $t_M \to \infty$. So $\lim_{M\to\infty} \frac{x_M}{t_M} = \lim_{M\to\infty} a_M x_M = d$, then $d \in S_\infty$.

(2) $S_\infty \subseteq \mathbf{A}S$. $\forall d \in S_\infty$, if $d = \mathbf{0}$, then $d \in \mathbf{A}S$ is trivial. If $d \neq \mathbf{0}$, then $\exists(x_k) \subseteq S, (t_k) \subseteq \mathbb{R}$ with $t_k \to \infty$ and $\lim_{k\to\infty} \frac{x_k}{t_k} = d$. $\forall M \ge 0$, as $t_k \to \infty$, then $\exists k_M$ s.t. $\forall k > k_M$, $||x_k|| \ge M$. So $x_k \in S^M, \forall k > k_M$. Then $\frac{x_k}{t_k} \in \Gamma(S^M)$ which is a closed cone with vertex $\mathbf{0}$. So $d = \lim_{k\to\infty} \frac{x_k}{t_k} \in \Gamma(S^M)$ for $\forall M$. Thus, $d \in \bigcap_{M\ge 0} \Gamma(S^M)$ as desired. ☐

DEFINITION 2.55. *(Normalized Set) Let $C \subseteq \mathbb{R}^L$ be a nonempty set. The normalized set is defined as*

$$c_L = \{d \in \mathbb{R}^L | \exists(x_k) \subseteq C \textit{ with } ||x_k|| \to \infty, \textit{ s.t. } d = \lim_{k\to\infty} \frac{x_k}{||x_k||}\} \tag{2.130}$$

THEOREM 2.126. *Let $C \subseteq \mathbb{R}^L$ be a nonempty set and*

$$Cone(C_L) = \{\lambda x \in \mathbb{R}^L | x \in C_L, \lambda \ge 0\} \tag{2.131}$$

where C_L is the normalized set of C. We have $C_\infty = Cone(C_L)$.

Proof. (1) $Cone(C_L) \subseteq C_\infty$. Let $\forall d \in C_L$, then $\exists(x_k) \subseteq \mathbb{R}^L$ s.t. $||x_k|| \to \infty$ and $\lim_{k\to\infty} \frac{x_k}{||x_k||} = d$. Thus, $t_k = ||x_k||, t_k \to \infty$ and $\lim_{k\to\infty} \frac{x_k}{t_k} = d$, so $d \in C_\infty$, i.e., $C_L \subseteq C_\infty$. $\forall d' \in Cone(C_L)$, $\exists d \in C_L, \lambda \ge 0$ s.t. $d' = \lambda d$. As $d \in C_\infty$, so $d' = \lambda d \in C_\infty$ (let $x_k' = \lambda x_k, t_k' = t_k, \forall k$). So $Cone(C_L) \subseteq C_\infty$.

(2) $C_\infty \subseteq Cone(C_L)$. Let $d \in C_L, \lambda = 0$, then $\mathbf{0} \in Cone(C_L)$. $\forall d \in C_\infty$, if $d = \mathbf{0}$, then $d \in Cone(C_L)$. If $d \neq \mathbf{0}$, then $\exists(x_k) \subseteq \mathbb{R}^L, (t_k) \subseteq \mathbb{R}$ s.t. $|t_k| \to \infty$ and $\lim_{k\to\infty} \frac{x_k}{t_k} = d$. So, $d = \lim_{k\to\infty} \frac{x_k}{t_k} = \lim_{k\to\infty} \frac{x_k}{||x_k||} \frac{||x_k||}{t_k}$

with $||x_k|| \to \infty$. Since $||\frac{x_k}{||x_k||}|| = 1, \forall k$, so $(\frac{||x_k||}{t_k})$ is a bounded infinite sequence. By the Bolzano-Weierstrass Theorem, there is a subsequence $(\frac{||x_{k_q}||}{t_{k_q}})$ s.t. $\lim\limits_{q\to\infty} \frac{||x_{k_q}||}{t_{k_q}} = \lambda \geq 0 (t_k \to +\infty)$. So,

$$d = \lim_{k\to\infty} \frac{x_k}{||x_k||}\frac{||x_k||}{t_k} = \lim_{q\to\infty} \frac{x_{t_q}}{||x_{k_q}||}\frac{||x_{k_q}||}{t_{k_q}} = \lim_{q\to\infty} \frac{x_{t_q}}{||x_{k_q}||} \lim_{q\to\infty} \frac{||x_{k_q}||}{t_{k_q}} = \lambda d_L \tag{2.132}$$

where $d_L = \lim\limits_{q\to\infty} \frac{x_{t_q}}{||x_{k_q}||} \in C_L$, so $d \in Cone(C_L)$. □

THEOREM 2.127. *A set $C \subseteq \mathbb{R}^L$ is bounded iff $C_\infty = \{\mathbf{0}\}$.*

Proof. (1) bounded $\Rightarrow C_\infty = \{\mathbf{0}\}$. Suppose not, then $\exists (x_k) \subseteq C$ s.t. $||x_k|| \to \infty$, implies C is not bounded, contradiction.

(2) $C_\infty = \{\mathbf{0}\} \Rightarrow$ bounded. Suppose not, C is unbounded, then $\exists (x_k) \subseteq C$ s.t. $||x_k|| \to \infty$. So the infinite bounded sequence $(\frac{x_k}{||x_k||})$ have an accumulation point (Weierstrass Theorem), so there is a subsequence $(\frac{x_{k_q}}{||x_{k_q}||})$ s.t. $d = \lim\limits_{q\to\infty} (\frac{x_{k_q}}{||x_{k_q}||})$ with $||d|| = 1$, so $d \in C_\infty$, but $d \neq \mathbf{0}$, contradiction. □

THEOREM 2.128. *Let $S \subseteq \mathbb{R}^L$, and $x \in \mathbb{R}^L$, then*

$$\mathbf{A}S = \mathbf{A}(S + \{x\}) \tag{2.133}$$

in other words, $S_\infty = (S + \{x\})_\infty$, a-translation will not change the asymptotic cone.

Proof. We just need to show $(S+\{x\})_\infty \subseteq S_\infty$ (then $S_\infty = (S+\{x\}-\{x\})_\infty \subseteq (S+\{x\})_\infty$). $\forall d = (S+\{x\})_\infty$, we have $\exists (x_k) \subseteq S + \{x\}, (t_k) \subseteq \mathbb{R}$ with $t_k \to \infty$, s.t. $\lim\limits_{k\to\infty} \frac{x_k}{t_k} = d$. As $x_k \in S + \{x\}$, then $x_k - x \in S, \forall k$. So

$$\lim_{k\to\infty} |\frac{x_k - x}{t_k} - d| \leq \lim_{k\to\infty} |\frac{x_k}{t_k} - d| + \lim_{k\to\infty} |\frac{x}{t_k}| = 0 + 0 = 0 \tag{2.134}$$

Thus, $d \in S$ as desired. (We can also show $S_\infty \subseteq (S + \{x\})_\infty$ in the same way) □

THEOREM 2.129. *Let $S_n \subseteq \mathbb{R}^L, n = 1, 2, \cdots, N$, then $\mathbf{A}(\prod\limits_{n=1}^{N} S_n) \subseteq \prod\limits_{n=1}^{N} \mathbf{A}S_n$.*

Proof. $\forall d \in \mathbf{A}(\prod\limits_{n=1}^{N} S_n)$, then $\exists (x^k) \subseteq \prod\limits_{n=1}^{N} S_n, (t^k) \subseteq \mathbb{R}$ s.t. $t^k \to \infty$ and $\lim\limits_{k\to\infty} \frac{x^k}{t^k} = d$. $x^k \in \prod\limits_{n=1}^{N} S_n$ implies $x^k = \lim\limits_{n=1}^{N} x_n^k, x_n^k \in S_n, \forall n$, for $\forall k$. Let $d = \prod\limits_{n=1}^{N} d_n$, then $\lim\limits_{k\to\infty} \frac{x_n^k}{t_k} = d_n$, i.e., $d_n \in \mathbf{A}S_n, \forall n$, so $d \in \subseteq \prod\limits_{n=1}^{N} \mathbf{A}S_n$. □

THEOREM 2.130. *Let $S_\gamma \in \mathbb{R}^L, \forall \gamma \in \Gamma$, if $\bigcap\limits_{\gamma\in\Gamma} \mathbf{A}S_\gamma = \{\mathbf{0}\}$, then $\bigcap\limits_{\gamma\in\Gamma} S_\gamma$ is bounded, i.e., $\mathbf{A}(\bigcap\limits_{\gamma\in\Gamma} S_\gamma) = \{\mathbf{0}\}$.*

Proof. As $\bigcap\limits_{\gamma\in\Gamma} S_\gamma \subseteq S_\gamma, \forall \gamma \in \Gamma$, then

$$\mathbf{A}(\bigcap_{\gamma\in\Gamma} S_\gamma) \subseteq \mathbf{A}S_\gamma, \forall \gamma \in \Gamma \tag{2.135}$$

So

$$\mathbf{A}(\bigcap_{\gamma\in\Gamma} S_\gamma) \subseteq \bigcap_{\gamma\in\Gamma} \mathbf{A}S_\gamma = \{\mathbf{0}\} \tag{2.136}$$

Thus, $\mathbf{A}(\bigcap\limits_{\gamma\in\Gamma} S_\gamma) = \{\mathbf{0}\}$. □

DEFINITION 2.56. *(Linear Mapping) Let $L : \mathbb{R}^L \to \mathbb{R}^K$ s.t.*

(1) $L(x + y) = Lx + Ly, \forall x, y \subseteq \mathbb{R}^L$;

(2) $L(tx) = tLx, \forall x \subseteq \mathbb{R}^L, \forall t \in \mathbb{R}$.

Then L is called a linear mapping (a function).

As linear mapping L is a function, we let $L^{-1}x$ denote the inverse image of $x, \forall x \in \mathbb{R}^K$. It is trivial that L is continuous (Need CHECK).

THEOREM 2.131. *For a linear mapping $L : \mathbb{R}^L \to \mathbb{R}^K$ and a closed set $C \subseteq \mathbb{R}^L$, if $L^{-1}\{\mathbf{0}\} \cap C_\infty = \{\mathbf{0}\}$, then*

(1) LC is closed;

(2) $L(C_\infty) = (LC)_\infty$.

Proof. (1) L is a continuous function. We first show if (x_q) is unbounded, then $y_q = Lx_q, \forall q$, (y_q) is unbounded. Suppose not $(x_q) \subseteq C, ||x_k|| \to \infty$ but (y_q) is bounded. As $(\frac{x_k}{||x_k||})$ is bounded and infinite, then with Bolzano-Weierstrass Theorem, we say $\exists(\frac{x_{k_q}}{||x_{k_q}||})$ s.t. $\frac{x_{k_q}}{||x_{k_q}||} \to x$ thus $x \in C_\infty$. But

$$Lx = \lim_{q\to\infty} L\frac{x_{k_q}}{||x_{k_q}||} = \lim_{q\to\infty} \frac{1}{||x_{k_q}||} Lx_{k_q} = \lim_{q\to\infty} \frac{y_{k_q}}{||x_{k_q}||} = \mathbf{0} \tag{2.137}$$

Thus, $x \in L^{-1}\{\mathbf{0}\}$ but $x \neq \mathbf{0}$ ($||x|| = 1$), contradiction to $L^{-1}\{\mathbf{0}\} \cap C_\infty = \{\mathbf{0}\}$.

$\forall y_q \to y, y_q \in LC$, we need to show $y \in LC$. Since $y_q \in LC$, then $\exists x_q \in C$ s.t. $y_q = Lx_q, \forall q$. As $y_q \to y$, then (y_q) is bounded, with L is continuous, then (x_q) is bound. From Bolzano-Weierstrass Theorem, we know there is an accumulation point x s.t. $\exists(x_{q_k}) \to x$ (subsequence). With C is closed, so $x \in C$. As $y_q \to y$, so $y_{q_k} \to y$. Then $y = Lx$ as L is continuous, so $y \in LC$, i.e., LC is closed. (why not: the image of a continuous function over a closed set is closed? directly)

(2) $L(C_\infty) \subseteq (LC)_\infty$. $\forall y \in L(C_\infty)$, then $x \in C_\infty$ s.t. $y = Lx$. $x \in C_\infty$ implies $\exists(x_q) \subseteq C, (t_q) \subseteq \mathbb{R}$ s.t. $t_q \to \infty$ and $\lim\limits_{q\to\infty} \frac{x_q}{t_q} = x$. Let $y_q = L(x_q)$, then $\lim\limits_{q\to\infty} \frac{y_q}{t_q} = \lim\limits_{q\to\infty} \frac{Lx_q}{t_q} = \lim\limits_{q\to\infty} L\frac{x_q}{t_q} = L(\lim\limits_{q\to\infty} \frac{x_q}{t_q}) = Lx = y$ (L is continuous). So we $y \in C_\infty$.

$(LC)_\infty \subseteq L(C_\infty)$. $\forall y \in (LC)_\infty$, then $\exists(y_q) \subseteq LC, (t_q) \subseteq \mathbb{R}$ s.t. $t_q \to \infty$ and $\lim\limits_{q\to\infty} \frac{y_q}{t_k} = y$. $y_q \in LC$ implies $y_q = Lx_q, x_q \in C, \forall q$. So $\lim\limits_{q\to\infty} \frac{y_q}{t_q} = \lim\limits_{q\to\infty} \frac{Lx_q}{t_q} = \lim\limits_{q\to\infty} L\frac{x_q}{t_q} = d$, so $(L\frac{x_q}{t_q})$ is bounded, then $(\frac{x_q}{t_q})$ is bounded and infinite. With Bolzano-Weierstrass Theorem, we say $\exists(x_{q_k}), (t_{q_k})$ s.t. $\lim\limits_{k\to\infty} \frac{x_{q_k}}{t_{q_k}} = x$, so $x \in C_\infty$. With L is continuous, we say $y = Lx$, then $y \in L(C_\infty)$. □

THEOREM 2.132. *(!!!) Let $C_n \subseteq \mathbb{R}^L$ be closed for $n = 1, 2, \cdots, N$. If $(C_1)_\infty, (C_2)_\infty, \cdots, (C_N)_\infty$ is p.s.i., then*

(1) $\sum_{n=1}^N C_n$ is closed;

(2) $(\sum_{n=1}^N C_n)_\infty \subseteq \sum_{n=1}^N (C_n)_\infty$.

Proof. (1) Let $C = \prod\limits_{n=1}^N C_n$ and $L : \mathbb{R}^{NL} \to \mathbb{R}^L, L(x_1, x_2, \cdots, x_N) = \sum_{n=1}^N x_n, \forall x_n \in C_n, \forall n$. Then L is a linear mapping. As $(C_1)_\infty, (C_2)_\infty, \cdots, (C_N)_\infty$ are p.s.i., i.e., $\forall x \in \prod\limits_{n=1}^N (C_n)_\infty$, if $Lx = \mathbf{0}$, then $x = \{\mathbf{0}\}$, i.e., $L^{-1}\{\mathbf{0}\} \cap \prod\limits_{n=1}^N (C_n)_\infty = \mathbf{0}$.

So $C_\infty \subseteq \prod\limits_{n=1}^N (C_n)_\infty$, so $L^{-1} \cap C_\infty = \{\mathbf{0}\}$. □

DEFINITION 2.57. *(Asymptotically Regular) Let $C \subseteq \mathbb{R}^L$ be nonempty and define*

$$C_\infty^1 = \{d \in \mathbb{R}^L | \forall(t_k) \text{ s.t. } t_k \to \infty, \exists(x_k) \subseteq C \text{ s.t. } \lim_{k\to\infty} \frac{x_k}{t_k} = d\} \tag{2.138}$$

If $C_\infty = C_\infty^1$, we say C is asymptotically regular.

Trivially, $\forall C \subseteq \mathbb{R}^L$, we have $C_\infty^1 \subseteq C_\infty$.

THEOREM 2.133. *Let $C \subseteq \mathbb{R}^L$ be nonempty. If C is convex, then C is asymptotically regular, i.e., $C_\infty = C_\infty^1$.*

Proof. We need to show $C_\infty \subseteq C_\infty^1$. $\forall d \in C_\infty$ and $\forall(t'_k), t'_k \to \infty$, we need to find $(x'_k) \subseteq C$ s.t. $\lim\limits_{k\to\infty} \frac{x'_k}{t'_k} = d$.

$d \in C_\infty$ implies $\exists (x_k) \subseteq C, (t_k) \subseteq \mathbb{R}$ s.t. $t_k \to \infty$ and $\lim_{k\to\infty} \frac{x_k}{t_k} = d$. C is nonempty, we take $x \in C$ arbitrarily. $\forall k$, we find a q_k s.t. $t_{q_k} > t'_k$, so $\frac{t'_k}{t_{q_k}} \in (0,1)$ ($t'_k \to \infty$). Let $x'_k = \frac{t'_k}{t_{q_k}} x_{q_k} + (1 - \frac{t'_k}{t_{q_k}})x$, so $x'_k \in C$ as C is convex. Now

$$\lim_{k\to\infty} |\frac{x'_k}{t'_k} - d| \le \lim_{k\to\infty} |\frac{x'_k}{t'_k} - \frac{x_{q_k}}{t_{q_k}}| + \lim_{k\to\infty} |\frac{x_{q_k}}{t_{q_k}} - d| \tag{2.139}$$

$$= \lim_{k\to\infty} |\frac{1}{t'_k} - \frac{1}{t_{q_k}}|||x|| + \lim_{k\to\infty} |\frac{x_{q_k}}{t_{q_k}} - d| \tag{2.140}$$

$$\le (\lim_{k\to\infty} |\frac{1}{t'_k}| + \lim_{k\to\infty} |\frac{1}{t_{q_k}}|)||x|| + \lim_{k\to\infty} |\frac{x_{q_k}}{t_{q_k}} - d| \tag{2.141}$$

$$= 0 \tag{2.142}$$

So $\lim_{k\to\infty} \frac{x'_k}{t'_k} = d$, i.e., $d \in C^1_\infty$. □

THEOREM 2.134. *Let C be a nonempty convex set in $\mathbb{R}^L$, then the asymptotically cone C_∞ is a closed convex cone.*

Proof. We have shown C_∞ is closed if C is closed. Now we prove the left part. As C is convex, so $C_\infty = C^1_\infty$. $\forall x, y \in C^1_\infty$ and $\forall t \in [0,1]$, we want to show $z^t = tx + (1-t)y \in C_\infty$. For $\forall (t_k) \subseteq \mathbb{R}^L$ s.t. $t_k \to \infty$, we say: $x \in C^1_\infty$ implies $\exists (x_q) \subseteq C$ s.t. $\lim_{k\to\infty} \frac{x_k}{t_k} = x$ and so does y and (y_q). Let $z^t_q = tx_q + (1-t)y_q$, then $z^t_q \in C$ as C is convex. So

$$\lim_{k\to\infty} \frac{z^t_q}{t_k} = \lim_{k\to\infty} t\frac{x_k}{t_k} + (1-t)\lim_{k\to\infty} t\frac{y_k}{t_k} = tx + (1-t)y = z^t \tag{2.143}$$

So $z^t \in C^1_\infty$. So C^1_∞ is convex, i.e., C_∞ is convex. □

THEOREM 2.135. *Let $C \subseteq \mathbb{R}^L$ be nonempty, then*

$$\mathbf{0}^+ C \subseteq C_\infty \tag{2.144}$$

where $\mathbf{0}^+C$ is a recession cone. If C is convex and closed, so $\mathbf{0}^+C = C_\infty$. As $(\overline{C})_\infty = C_\infty$, so generally, if C is convex, we say $\mathbf{0}^+\overline{C} = C_\infty$.

Proof. (1) $\mathbf{0}^+C \subseteq C_\infty$. $\forall y \in \mathbf{0}^+C$, then $\forall x \in C, \forall \lambda \ge 0$, we have $x + \lambda y \in C$. To show $y \in C_\infty$, we need to find suitable $(x_k), (t_k)$. Pick $x \in C$ and $(\lambda_k), \lambda_k \to \infty$ arbitrarily, then let $x_k = x + \lambda_k y$, then $x_k \in C$, so

$$\lim_{k\to\infty} |\frac{x_k}{\lambda_k} - y| = \lim_{k\to\infty} |\frac{x}{\lambda_k}| = 0 \tag{2.145}$$

So $y \in C_\infty$.

(2) if C is convex and closed, we show $C_\infty \subseteq \mathbf{0}^+C$. $\forall y \in C_\infty$, we want to show $\forall x \in C, \lambda \ge 0$, we have $x + \lambda y \in C$. $y \in C_\infty$ implies $\exists (x_k) \subseteq C, (t_k) \subseteq \mathbb{R}$ s.t. $t_k \to \infty$ and $\lim_{k\to\infty} \frac{x_k}{t_k} = y$. Given $x \in C, \lambda \ge 0$, then let $t_{q_k} \ge \lambda, \forall k$ and $y_k = (1 - \frac{\lambda}{t_{q_k}})x + \frac{\lambda}{t_{q_k}} x_{q_k}$, $\frac{\lambda}{t_{q_k}} \in (0,1)$. Then $y_k \in C$ as C is convex,

$$\lim_{k\to\infty} y_k = \lim_{k\to\infty} (1 - \frac{\lambda}{t_{q_k}})x + \frac{\lambda}{t_{q_k}} x_{q_k} = \lim_{k\to\infty} (1 - \frac{\lambda}{t_{q_k}})x + \lim_{k\to\infty} \frac{\lambda}{t_{q_k}} x_{q_k} = x + \lambda y \tag{2.146}$$

So $x + \lambda y \in C$ as C is closed. Then $d \in \mathbf{0}^+C$. □

THEOREM 2.136. *Let $C \subseteq \mathbb{R}^L$ be nonempty, closed and convex, then*

$$C = C + C_\infty \tag{2.147}$$

Proof. (1) $C \subseteq C + C_\infty$. As $\mathbf{0} \in C_\infty$, so $C = C + \mathbf{0} \subseteq C + C_\infty$.

(2) $C + C_\infty \subseteq C$. $\forall z \in C + C_\infty$, let $z = x + y, x \in C, y \in C_\infty$, we want to show $z \in C$. $y \in C_\infty$ implies $\exists (y_k) \subseteq C, (t_k) \subseteq \mathbb{R}$ s.t. $t_k \to \infty$ and $\lim_{k\to\infty} \frac{y_k}{t_k} = y$. Pick a subsequence (k_q) s.t. $t_{k_q} > 1$, so let $z^q = (1 - \frac{1}{t_{k_q}})x + \frac{1}{t_{k_q}} y_{k_q}$ with $\frac{1}{t_{k_q}} \in (0,1)$, so $z^q \in C$ as C is convex. Then

$$\lim_{q\to\infty} z^q = \lim_{q\to\infty} (1 - \frac{1}{t_{k_q}})x + \frac{1}{t_{k_q}} y_{k_q} = \lim_{q\to\infty} (1 - \frac{1}{t_{k_q}})x + \lim_{q\to\infty} \frac{1}{t_{k_q}} y_{k_q} = x + y \tag{2.148}$$

As C is closed, so $x + y \in C$, i.e., $C + C_\infty \subseteq C$. Thus, $C = C + C_\infty$. □

THEOREM 2.137. *A closed convex set $C \subseteq \mathbb{R}^L$ owning $\mathbf{0}$, contains its asymptotic cone.*

Proof. As $\mathbf{0} \in C$, then $C_\infty = \{\mathbf{0}\} + C_\infty \subseteq C + C_\infty = C$, so $C_\infty \subseteq C$. □

THEOREM 2.138. *Let $S \subseteq \mathbb{R}^L$ be a closed convex set, for $x_0 \in S$, we define the cone*

$$C = \max\{C' | x_0 + C' \subseteq S\} \tag{2.149}$$

Then C is an asymptotic cone, i.e.,

(1) $C = \bigcap_{\lambda \geq 0} \lambda(S - x_0)$;

(2) C is independent with x_0;

(3) $C = \mathbf{A}S$.

Proof. □

Lastly, we add properties of convex hull.

LEMMA 2.11. *Let S be a convex set, $x_1, x_2, \cdots, x_N \in S$. We have $\forall a_1, a_2, \cdots, a_N \in [0,1], \sum_{n=1}^N a_n = 1$, then $\sum_{n=1}^N a_n x_n \in S$.*

Proof. We use the mathematical induction method. Let $N = 1, 2$, surely we have $\sum_{n=1}^N a_n x_n \in S$ form the definition of convex sets.

Let $N = k$, suppose we have $y_k = \sum_{n=1}^k a_n x_n \in S, \sum_{n=1}^k a_n = 1, x_n \in S, a_n \in [0,1] \forall n = 1, 2, \cdots, k$. Then when $N = k+1, a_{k+1} \neq 1$ (otherwise degenerate to $N = 1$), we have $y_{k+1} = \sum_{n=1}^{k+1} a_n x_n = a_{k+1} y_{k+1} + (1 - a_{k+1}) \sum_{n=1}^k \frac{a_n}{1-a_{k+1}} x_n$ where $\sum_{n=1}^k \frac{a_n}{1-a_{k+1}} = 1, \frac{a_n}{1-a_{k+1}} \in [0,1], n = 1, 2, \cdots, k$. So $y_{k+1} = a_{k+1} y_{k+1} + (1 - a_{k+1}) y_k, y_k \in S$, so $y_{k+1} \in S$. □

LEMMA 2.12. *Let $S \subseteq \mathbb{R}^L, CH(S) = \bigcap_{S \subseteq C} C$ be the convex hull of S, where C is a convex set. Define $A(x_1, x_2, \cdots, x_N) = \{\sum_{n=1}^N a_n x_n | \sum_{n=1}^N a_n = 1, a_n \in [0,1], n = 1, 2, \cdots, N\}$ (A is a convex set from Lemma 2.11) and $B = \bigcup_{x_1, x_2, \cdots, x_N \in S} A(x_1, x_2, \cdots, x_N)$. Then, $B = CH(S)$.In other words, $x \in CH(S)$ if and only if $\exists x_1, x_2, \cdots, x_N \in S, a_1, a_2, \cdots, a_N \in [0,1]$ s.t. $x = \sum_{n=1}^N a_n x_n, \sum_{n=1}^N a_n = 1$.*

Proof. (1) $B \subseteq CH(S)$. Let $\forall x \in S$, then we have $x = \sum_{n=1}^N a_n x_n, \sum_{n=1}^N a_n = 1, x_1, x_2, \cdots, x_N \in S, a_1, a_2, \cdots, a_N \in [0,1]$, then $\forall C, S \subseteq C$ where C is a convex set, $x_n \in C, n = 1, 2, \cdots, N$, so $x \in C$. Then x is contained in any convex set containing S, i.e., $x \in CH(S)$.

(2) $CH(S) \subseteq B$. $\forall x \in S, x \in A(x) \subseteq B$, so $S \subseteq B$. Then we show B is a convex set. $\forall x, y \in B$, we say $x = \sum_{m=1}^M a_m x_m, y = \sum_{n=1}^N b_n y_n$ where $\sum_{m=1}^M a_m = 1, \sum_{n=1}^N b_n = 1, a_m, b_n \in [0,1], x_m, y_n \in S \forall m, n,$. Then, $z = tx + (1-t)y, \forall t \in [0,1], z = t \sum_{m=1}^M a_m x_m + (1-t) \sum_{n=1}^N b_n y_n$ with $\sum_{m=1}^M t a_m + \sum_{n=1}^N (1-t) b_n = t + (1-t) = 1$, so $z \in A(x_1, x_2, \cdots, x_M, y_1, y_2, \cdots, y_N) \subseteq S$. Therefore, B is convex set containing S, so $CH(S) \subseteq B$. □

LEMMA 2.13. *Let $S \subseteq \mathbb{R}^L$ be convex. If $x \in CH(S)$, then $\exists x_1, x_2, \cdots, x_{L+1} \in S, t_1, t_2, \cdots, t_{L+1} \in [0,1]$ s.t. $\sum_{n=1}^{L+1} t_n = 1$ and $x = \sum_{n=1}^{L+1} t_n x_n$. In other words, any point belongs to a convex hull can be expressed by a linear composition of at most $L+1$ points belongs to the original set.*

Proof. If $x \in CH(S)$, then $\exists x_1, x_2, \cdots, x_N \in S, x_i \neq x_j, \forall i, j$, s.t. $x \in A(x_1, x_2, \cdots, x_N)$, we can just prove that if $N > L+1$, then $\exists x_1', x_2', \cdots, x_{N-1}' \in S$ s.t. $x \in A(x_1', x_2', \cdots, x_{N-1}')$. Let

$$x = \sum_{n=1}^N t_n x_n \tag{2.150}$$

where $t_n \in (0,1), \forall n, \sum_{n=1}^N t_n = 1$ (if $\exists n$ s.t. $t_n = 0$ or 1, we are done).

As $x_n \in \mathbb{R}^L, \forall n$, then let $X_{L,N} = [x_1, x_2, \cdots, x_N]$, $rank(X) \leq L < N - 1$. Thus, $\exists c_1, c_2, \cdots, c_N$ s.t. $\sum_{n=1}^N c_n^2 > 0$, and

$$\sum_{n=1}^N c_n x_n = \mathbf{0}, \sum_{n=1}^N c_n = 0 \tag{2.151}$$

Let $\epsilon = \min_{c_n<0} \frac{t_n}{|c_n|}$, so $\epsilon > 0$ and $t_n + \epsilon c_n \geq 0, \forall n$ and $\exists n_0$ s.t. $t_{n_0} + \epsilon c_{n_0} = 0$. Thus,

$$x = x + \epsilon \sum_{n=1}^{N} c_n x_n = \sum_{n=1}^{N} (t_n + \epsilon c_n) x_n = \sum_{n=1, n\neq n_0}^{N} (t_n + \epsilon c_n) x_n \tag{2.152}$$

where $\sum_{n=1,n\neq n_0}^{N} (t_n + \epsilon c_n) = 0$. So $x \in A(x_1, x_2, \cdots, x_{n_0-1}, x_{n_0+1}, \cdots, x_N)$ as desired. □

LEMMA 2.14. *Let $S_1, S_2 \subseteq \mathbb{R}^L, x \in CH(S_1), y \in S_2$, then $x + y \in CH(S_1 + S_2)$.*

Proof. As $x \in CH(S_1)$, then $\exists x_1, x_2, \cdots, x_N \in S, a_1, a_2, \cdots, a_N \in [0, 1]$ s.t. $x = \sum_{n=1}^{N} a_n x_n, \sum_{n=1}^{N} a_n = 1$. As $y \in S_2$, then $x_n + y \in S_1 + S_2, n = 1, 2, \cdots, L$. So $x + y = \sum_{n=1}^{N} a_n(x_n + y) \in CH(S_1 + S_2)$. □

THEOREM 2.139. *Let $S_1, S_2, \cdots, S_N \subseteq \mathbb{R}^L$, show that $CH(\sum_{n=1}^{N} S_n) = \sum_{n=1}^{N} CH(S_n)$.*

Proof. (1) Show $CH(\sum_{n=1}^{N} S_n) \subseteq \sum_{n=1}^{N} CH(S_n)$.

Let $\forall x \in CH(\sum_{n=1}^{N} S_n)$, we have $x = \sum_{m=1}^{M} a_m x_m, x_m = \sum_{n=1}^{N} x_m^n$ where $x_m^n \in S_n, \sum_{m=1}^{M} a_m = 1, a_m \in [0, 1], \forall m$. So we say $x = \sum_{m=1}^{M} (\sum_{n=1}^{N} x_m^n) = \sum_{m=1}^{M} (\sum_{n=1}^{N} a_m x_m^n) = \sum_{n=1}^{N} (\sum_{m=1}^{M} a_m x_m^n) \in \sum_{n=1}^{N} CH(S_n)$ as $\sum_{m=1}^{M} a_m x_m^n \in CH(S_n)$, so $x \in \sum_{n=1}^{N} CH(S_n)$.

(2) Next we show the $CH(\sum_{n=1}^{N} S_n) \supseteq \sum_{n=1}^{N} CH(S_n)$.

Let $x \in \sum_{n=1}^{N} CH(S_n)$, $x = \sum_{n=1}^{N} x^n = \sum_{n=1}^{N} (\sum_{m=1}^{M} a_{m,n} x_m^n) = \sum_{m=1}^{M} (\sum_{n=1}^{N} a_{m,n} x_m^n)$. If we can find, $x = \sum_{m=1}^{M} (\sum_{n=1}^{N} a_{m,n} x_m^n) = \sum_{k=1}^{K} b_k (\sum_{n=1}^{N} y_k^n)$ where $\sum_{k=1}^{K} b_k = 1, y_k^n \in S_n, b_k \in [0, 1], \forall k, n$. Then $x \in CH(\sum_{n=1}^{N} S_n)$.

Now we show how to find $\sum_{k=1}^{K} b_k (\sum_{n=1}^{N} y_k^n)$, then $\sum_{m=1}^{M} a_{m,n} x_m^n = \sum_{k=1}^{K} b_k y_k^n, \forall n = 1, 2, \cdots, N$. Let $K = M^N$, then we got $M^N - 1$ variables to solve $(M-1)N$ equations and, we see $M^N - 1 \geq (M-1)N$ when $M \geq N$, so the problem is solvable, i.e., such b_k, y_k^n is available. Moreover, let $y_k^n \in \{x_1^n, x_2^n, \cdots, x_M^n\}$, then we have $\sum_{k=1}^{K} b_k = \sum_{m=1}^{M} a_m = 1$. Thus, $x = \sum_{k=1}^{K} b_k (\sum_{n=1}^{N} y_k^n), \sum_{n=1}^{N} y_k^n \in S_n, \sum_{k=1}^{K} b_k = 1, b_k \in [0, 1], \forall k$. So $x \in CH(\sum_{n=1}^{N} S_n)$. We give an example to show how why such b_k, y_k^n exist in Example 2.1. □

Example 2.1. *Let $M = 3, N = 2$, $A = \{a_{m,n}\} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/4 & 1/4 \end{bmatrix}'$, and $X = \{x_m^n\} = \begin{bmatrix} x_1^1 & x_2^1 & x_3^1 \\ x_2^1 & x_2^2 & x_3^2 \end{bmatrix}'$. Let $B = \{b_k\} = [b_1, b_2, \cdots, b_9]', Y = \{y_k^n\} = \begin{bmatrix} x_1^1 & x_1^1 & x_1^1 & x_2^1 & x_2^1 & x_2^1 & x_3^1 & x_3^1 & x_3^1 \\ x_1^2 & x_1^2 & x_1^2 & x_2^2 & x_2^2 & x_2^2 & x_3^2 & x_3^2 & x_3^2 \end{bmatrix}'$. So we need to solve*

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \\ b_8 \\ b_9 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \\ 1/2 \\ 1/4 \\ 1/4 \end{bmatrix} \tag{2.153}$$

$$Ax = b \tag{2.154}$$

which is surely solvable as the $rank([A, b]) = rank(A) = 5 < 9$.

2.8 The Fixed Point Theorem

Consider a set S and a function f from S to S, i.e., a transformation of S into itself. Great interest is attached to the existence of an element x' s.t. $x' = f(x')$, i.e., which coincides with its image, or which does not move in the transformation. Such an element is called a fixed point of the transformation.

THEOREM 2.140. *(Brouwer's Theorem) If S is a non-empty, compact, convex subset of $\mathbb{R}^L$ and if f is a continuous function from S to S, then f has a fixed point.*

Proof. We first show the statement, demoted as (a), is equivalent to (b): Let $D = \{x \in \mathbb{R}^L|||x|| \leq 1\}$. If the function $h : D \to D$ is continuous, then h has a fixed point.

Define a homogeneous mapping $\phi : D \to S$.

(b)→ (a): Define $h = \phi^{-1} \cdot f \cdot \phi : D \to D$, then h is continuous and ha a fixed point, i.e., $h(x_0) = x_0$, or $\phi^{-1}(f(\phi(x_0))) = (\phi^{-1} \cdot f \cdot \phi)(x_0) = x_0$, then $f(\phi(x_0)) = \phi(x_0)$, then $\phi(x_0)$ is a fixed point of f.

(a)→(b): Define $f = \phi \cdot h \cdot \phi^{-1} : S \to S$, then f is continuous and has a fixed point x_0, that is, $f(x_0) = x_0$ or $\phi(f(\phi^{-1}(x_0))) = (\phi \cdot h \cdot \phi^{-1})(x_0) = x_0$, then $h(\phi^{-1}(x_0)) = \phi^{-1}(x_0)$, hence $\phi^{-1}(x_0)$ is a fixed point of h.

In the same way, we know (b) is equivalent to (c): Let $D = \{x \in \mathbb{R}|||x|| \leq 1\} = [-1, 1]$. If the function $h : D \to D$ is continuous, then h has a fixed point.

Now we show (c) hold. Define function $h' = h - Id : D \to D$, then h' is continuous with $h'(x) = h(x) - x$ and hence $h'(-1) = h(-1) + 1 \geq 0$ and $h'(1) = h(1) - 1 \leq 0$, thus h' has a solution to $h'(x) = 0$, that is h has a fixed point. □

The generalization of this result to correspondences from a set to itself will play an essential role later on. Consider now a set S and a correspondence ϕ from S to S. A fixed point of the correspondence ϕ is an element x' such that $x' \in \phi(x')$, i.e., which belongs to its image-set.

THEOREM 2.141. *(Kakutani's Theorem) If S is a nonempty, compact, convex subset of $\mathbb{R}^L$, and if φ is an u.s.c. correspondence from S to S s.t. for all $x \in S$ the set $\varphi(x)$ is convex (nonempty), then φ has a fixed point.*

3 General Equilibrium

We introduce the classic general equilibrium theory in this section. Suppose all economic activities take place at the same position and same time, i.e., we do not concern about time and place here.

3.1 Producers

Let l be the total amount of types of commodities, $i \in \mathbb{N}_{++}$. An action a, production or consumption, is a point of $\mathbb{R}^l$, the commodity space. A price system p is a point of $\mathbb{R}^l$. The value of an action a relative to a price system p is the inner product $p \cdot a$.

The number of n of producer is a given positive integer. Each producer is indicated by an index $j = 1, 2, \cdots, n$. The j-th producer chooses a point, his production or his supply y_j in a given nonempty subset of $\mathbb{R}^l$, his production Y_j, i,e, $y_j \in Y_j$. Given a production y_j for each producer, $y = \sum_{j=1}^n y_j$ is called the total production or the total supply, and the set $Y = \sum_{j=1}^n Y_j$ is called the total production set.

We now study the properties of Y_j and Y.

Assumption 3.1. *Y_j is closed.*

Assumption 3.2. $\mathbf{0} \in Y_j$ *(possibility of inaction)*

Assumption 3.3. $\mathbf{0} \in Y$ *(possibility of inaction)*

If $\mathbf{0} \in Y_j, j = 1, 2, \cdots, n$, then $\mathbf{0} = \sum_{i=1}^n \mathbf{0} \in \sum_{j=1}^n Y_j = Y$, i.e., $\mathbf{0} \in Y$.

Assumption 3.4. *$Y \cap \Omega \subseteq \{\mathbf{0}\}$ where $\Omega = \mathbb{R}_+^l = \{x = (x_1, x_2, \cdots, x_l) \in \mathbb{R}^l | x_i \geq 0, i = 1, 2, \cdots, l\}$ (impossibility of free production)*

Assumption 3.5. *$Y \cap (-Y) \subseteq \{\mathbf{0}\}$ (irreversibility)*

Assumption 3.6. *Return to scale,*

- *non-decreasing returns to scale (IRTS): if $y_j \in Y_j$, then $\forall t \geq 1$, we have $ty_j \in Y_j$;*
- *non-increasing returns to scale (DRTS): if $y_j \in Y_j$, then $\forall t \in [0, 1]$, we have $ty_j \in Y_j$;*
- *constant returns to scale (CRTS): if $y_j \in Y_j$, then $\forall t \geq 0$, we have $ty_j \in Y_j$.*

Assumption 3.7. *$Y_j + Y_j \subseteq Y_j$ (additivity)*

Assumption 3.8. *Y_j is convex. (convexity)*

LEMMA 3.1. *If Y_j is convex and $\mathbf{0} \in Y_j$, then it is DRTS.*

Proof. If $y_j \in Y_j$, then $\forall t \in [0,1]$, we have $ty_j = ty_j + (1-t)\mathbf{0} \in Y_j$ (convex), so $ty_j \in Y_j$, i.e., Y_j is DRTS. □

LEMMA 3.2. *If Y_j is convex and CRTS, then Y_j is additive.*

Proof. $\forall y_j^1, y_j^2 \in Y_j$, as Y_j is CRTS, so $2y_j^1, 2y_j^2 \in Y_j$. Then as Y_j is convex, $y_j^1 + y_j^2 = \frac{1}{2}(2y_j^1 + 2y_j^2) \in Y_j$, so Y_j is additive. □

LEMMA 3.3. *If Y_j is closed and convex for $j = 1, 2, \cdots, n$, and $Y \cap (-Y) = \{\mathbf{0}\}$, then Y is closed* [1].

Proof. With Theorem 2.132, to show Y is closed, we need to show $\mathbf{A}Y_1, \mathbf{A}Y_2, \cdots, \mathbf{A}Y_n$ are p.s.i.

We first show $\sum_{j=1}^n \mathbf{A}Y_j \subseteq Y$. Since $\mathbf{0} \in Y$, then $\mathbf{0} = \sum_{j=1}^n y_j, y_j \in Y_j, \forall j$. Then $\forall j$, we have $\mathbf{0} \in Y_j - \{y_j\}$, as Y_j is closed and convex, so is $Y_j - \{y_j\}$. Then $Y_j - \{y_j\}$ contains its asymmetric cone, i.e., $\mathbf{A}(Y_j - \{y_j\}) \subseteq Y_j$. As $\mathbf{A}Y_j = \mathbf{A}(Y_j - \{y_j\})$ (a-translation), so $\mathbf{A}Y_j \subseteq Y_j - \{y_j\}$. So $\sum_{j=1}^n \mathbf{A}Y_j \subseteq \sum_{j=1}^n (Y_j - \{y_j\}) = Y - \mathbf{0} = Y$.

Now we show $\forall y_j, j = 1, 2, \cdots, n$ s.t. $\sum_{j=1}^n y_j = \mathbf{0}$, we have $y_j = \mathbf{0}, \forall j$. Suppose not, let $y_k \neq \mathbf{0}$. Then $-y_k = \mathbf{0} + \sum_{j=1, j\neq k}^n y_k \subseteq Y$, i.e., $y_k \in -Y$. As $y_k = y_k + \sum_{j=1, j\neq k}^n y_j \subseteq Y$, so $y_k \in Y \cap (-Y) = \{\mathbf{0}\}$, contradicting to $y_k \neq \mathbf{0}$. So $\mathbf{A}Y_1, \mathbf{A}Y_2, \cdots, \mathbf{A}Y_n$ are p.s.i. □

Assumption 3.9. *Y is convex.*

If Y_j is convex $j = 1, 2, \cdots, n$, then $Y = \sum_{j=1}^n Y_j$ is convex. So "Y is convex" is weaker than Y_j is convex $j = 1, 2, \cdots, n$.

Assumption 3.10. *Y_j is a cone with vertex $\mathbf{0}$. (constant returns to scale)*

If every Y_j is a cone with vertex $\mathbf{0}$, so is Y

Assumption 3.11. *$-\Omega \subseteq Y$ (free of disposal)*

Assumption 3.12. $Y - \Omega \subseteq Y$

LEMMA 3.4. *If Y_j is convex, closed, and $-\Omega \subseteq Y$, then $Y - \Omega \subseteq Y$.*

Proof. Since $-\Omega \subseteq Y$, Y is nonempty. As Y is nonempty, convex and closed, with Theorem 2.136, we have $Y + Y_\infty = Y$.

$-\Omega \subseteq Y$ implies $-\Omega = (-\Omega)_\infty \subseteq Y_\infty$, so $Y - \Omega \subseteq Y + Y_\infty = Y$. □

LEMMA 3.5. *If $Y \cap (-Y) = \{\mathbf{0}\}$ and $-\Omega \subseteq Y$, then $Y \cap \Omega \subseteq \{\mathbf{0}\}$.*

Proof. $-\Omega \subseteq Y$ implies $\Omega \subseteq -Y$, then $Y \cap \Omega \subseteq Y \cap (-Y) = \{\mathbf{0}\}$. □

Given a price system p and a production y_j, the profit of the j-th producer is $p \cdot y_j$, the total profit is $p \cdot y$. All producers take prices as given.

Given the price system p, the j-th producer chooses in his production set Y_j so as to maximize his profit. The resulting action is called a equilibrium production of the j-th producer relative to p.

Define $T_j' \subseteq \mathbb{R}^l$ as

$$T_j' = \{p \in \mathbb{R}^l | pY_j \text{ has a maximum}\} \neq \varnothing \tag{3.1}$$

(T_j' depends on Y_j) Given $p \in T_j'$, let

$$\eta_j = \{y_j \in Y_j | p \cdot y_j = \max pY_j\} \tag{3.2}$$

i.e., $\eta_j : T_j' \rightrightarrows y_j$, the maximizer of profit maximization.η_j is named as the supply correspondence of the j-th producer,

Let $\pi_j(p)$ be the maximum profit when $p \in T_j'$, called the profit function of the j-th producer,

$$\pi_j : T_j' \to \mathbf{R}, \pi_j(p) = \max pY_j = py_j, \forall y_j \in \eta_j(p) \tag{3.3}$$

$\forall t > 0$, we have $\eta_j(tp) = \eta_j(p), \pi_j(tp) = t\pi_j(p)$.

[1] Note that based on Y_j is closed for $\forall j$, we do not have $Y = \sum_{j=1}^n$ is closed necessarily. E.g., let $A = Z$ and $B = \{n + \frac{1}{n}, n \in \mathbb{N}_{++}\}$, then $1 + \frac{1}{n} \in A + B, \forall n \in \mathbb{N}_{++}$, so $1 \in \overline{A+B}$, but $1 \notin A + B$.

For all producers, we define the total supply correspondence η and total profit function π as,

$$\eta : \bigcap_{j=1}^{n} T'_j \rightleftarrows Y, \eta(p) = \sum_{j=1}^{n} \eta_j(p) \tag{3.4}$$

$$\pi : \bigcap_{j=1}^{n} T'_j \rightleftarrows \mathbb{R}, \pi(p) = \sum_{j=1}^{n} \pi_j(p) \tag{3.5}$$

($\bigcap_{j=1}^{n} T'_j$ ensures the maximizer exists for all producers) Also, $\forall t > 0$, we have $\eta(tp) = \eta(p), \pi(tp) = t\pi(p)$.

LEMMA 3.6. *Let $y_j \in Y_j, j = 1, 2, \cdots, n$. Given p, $p \cdot y = \max p \cdot Y$ if and only if $p \cdot y_j = \max p \cdot Y_j$ where $y = \sum_{j=1}^{n} y_j$.*

Proof. (1) y maximizes $pY \Rightarrow y_j$ maximizes $pY_j, \forall j$. Suppose not, $\exists k$ and $y'_k \in Y_k$ s.t. $py'_k > py_k$, then $y' = \sum_{j=1, j\neq k}^{n} y_j + y'_k \in Y$ s.t. $py' - py = py'_k - py_k > 0$, i.e., y is not the maximizer of total profit, contradiction.

(2) y_j maximizes $pY_j, \forall j \Rightarrow y$ maximizes pY. Suppose not, then $\exists y' \in Y$ s.t. $py' > py$. Then $\exists k$ s.t. $py'_k > py_k, y'_k \in Y_k$ (otherwise, $py' \leq py$), so y_k is not the maximizer of k-th profit, contradiction. □

If $\mathbf{0} \in Y_j$, given $p \in T'_j$, $\mathbf{0}$ may be the maximizer, so the maximum must be non-negative.

If Y_j is additive, then given $p \in T'_j$, the maximum profit must be non-positive, otherwise infinite profit is possible.

If Y_j is convex, given $p \in T'_j$, if $p = \mathbf{0}$, then the maximum profit is zero, so $\eta_j(p) = Y_j$. If $p \neq \mathbf{0}$, then $\eta_j(p) = Y_j \cap H$(a hyperplane, convex, $py_j = \pi_j(p) = py_j, \forall y_j \in \eta_j(p)$), so $\eta_j(p)$ is convex. Thus, Y_j is convex implies $\eta_j(p), p \in T'_j$ is convex. Further, if Y_j is convex $\forall j$, then $\eta(p), p \in \bigcap_{j=1}^{n} T'_j$ is convex.

Y_j is a cone with vertex $\mathbf{0}$ (CRTS). Its polar set,

$$T'_j = Y_j^\circ = \{p \in \mathbb{R}^l | p \cdot y_j \leq 0, \forall y_j \in Y\} \tag{3.6}$$

so $T'_j = Y_j^\circ$ is a closed, convex cone with vertex $\mathbf{0}$. (CRTS does not permit positive profit, so if $py_j > 0$, there is not maximizer, i.e., $p \notin T'_j$)

LEMMA 3.7. *Let Y_j be a cone with vertex $\mathbf{0}$ (CRTS) and $T'_j = Y_j^\circ$ (polar). If $p \in int(T'_j) = int(Y_j^\circ)$ (int implies interior), then $\mathbf{0}$ is the unique maximizer.*

Moreover, if $p \in \partial Y_j^\circ$(boundary), and Y_j is closed (so Y_j° is closed), then the cone of maximizers is not reduced to the $\mathbf{0}$.

Proof. (1) As $p \in Y_j^\circ$, so $\forall y_j \in Y$, we have $p \cdot y_j \leq 0$. So $\mathbf{0}$ is the maximizer.

Suppose $\mathbf{0}$ is not the unique maximizer, then $\exists y_j^0 \neq 0$ s.t. $p \cdot y_j^0 = 0$. Assume $y_{ik}^0 \neq 0$, as $p \in int(Y_j^\circ)$, then $\exists \epsilon_k$and $\epsilon = (0, 0, \cdots, \epsilon_k, \cdots, 0)$ s.t. $\epsilon_k y_{jk}^0 > 0$ (same sign) and $p + \epsilon \in Y_j^\circ$, then

$$(p + \epsilon) \cdot y_j^0 = p \cdot y_j^0 + \epsilon_k y_{jk}^0 = \epsilon_k y_{jk}^0 > 0 \tag{3.7}$$

which is prohibited by CTRS.

(2) We need to show $\exists y_j^0 \in Y_j$ s.t. $y_j^0 \neq \mathbf{0}$ and $p \cdot y_j^0 = 0$.

$p \in \partial Y_j^\circ$ implies $\exists (p^k) \subseteq \mathbf{R}^l \backslash Y_j^\circ$ s.t. $p^k \to p$. $p^k \in \mathbf{R}^l \backslash Y_j^\circ$ implies $\exists y_j^k \in Y_j$ s.t. $p^k y_j^k > 0$. As Y_j is a closed cone with vertex $\mathbf{0}$, then we can let $||y_j^k|| = 1$ for $\forall k$. As $\{y_k^0\}$ is in the closed unit sphere $\{y \in \mathbb{R}^l | ||y|| = 1\}$. So with the Weierstrass Theorem, we say there is a subsequence (y^{k_q}) s.t. $y^{k_q} \to y_j^0$ and $y_j^0 \in Y_j, ||y_j^0|| = 1$. So

$$p \cdot y_j^0 = \lim_{q \to \infty} p^{k_q} y_j^{k_q} \geq 0 \tag{3.8}$$

As $p \in Y_j^\circ$, so $p \cdot y_j^0 \leq 0$, so $p \cdot y_j^0 = 0$,i.e., y_j^0 is also the maximizer. Further, $\forall t \geq 0$, we have $ty_j^0 \in Y_j$ and $p(ty_j^0) = t(py_j^0) = 0$, so ty_j^0 is the maximizer. Therefore the cone of maximizers is not reduced to the single point $\mathbf{0}$. □

If every Y_j is a cone with vertex $\mathbf{0}$, so is Y. Given p, there is a maximum of $p \cdot y_j$ on Y_j fpr every Y_j if and only if there is a maximum of $p \cdot y$ on Y. Therefore, $\bigcap_{j=1}^{n} Y_j^\circ = Y^\circ$, the polar of Y. Given $p \in Y^\circ$, then the Lemma 3.7 also holds for $\eta(p)$ and Y.

If $-\Omega \subseteq Y$ (free disposal), given $p = (p_1, p_2, \cdots, p_n) \in \bigcap_{j=1}^{n} T'_j$, if $p_i < 0$, we can make infinite profit by increasing the total input of the i-th commodity.

Suppose given p, the profit maximizer is y_j and given p, the profit maximizer is y'_j, and let $\Delta p = p' - p$, $\Delta y_j = y'_j - y_j$. Since $py \geq py'$ and $p'y' \geq p'y$,

$$p\Delta y_j \leq 0, p'\Delta y_j \geq 0 \Rightarrow \Delta p \Delta y_j = (p' - p)\Delta y_j = p'\Delta y_j - p\Delta y_j \geq 0 \tag{3.9}$$

If only p_i varies, we have $\Delta p_i \Delta y_{ji} \geq 0$. That is, if the price of commodities increase and other prices remain, the producer will increase or leaves unchanged his output of the commodities (resp., decrease input). Moreover, for total producers,

$$p\Delta y \leq 0, p'\Delta y \geq 0 \Rightarrow \Delta p \Delta y \geq 0 \tag{3.10}$$

Now we assume Y_j be nonempty and compact. Then $\forall p$, $p \cdot y_j$ defines a continuous function of y_j on Y_j, then $p \cdot Y_j$ has a maximum, i.e., $T'_j = \mathbb{R}^l$ (so we do not have to concern existence of $\eta_j(p)$).

THEOREM 3.1. *If Y_j is nonempty compact, then $\forall p \in \mathbb{R}^l$, $\eta_j(p)$ is u.s.c. and $\pi_j(p)$ is continuous.*

Proof. Let $f(p, y_j) = p \cdot y_j$, then f is a numerical function from $\mathbb{R}^l \times Y_j \to \mathbb{R}$. Let $\varphi(p) = Y_j$ be the correspondence from $\mathbb{R}^l \rightrightarrows Y_j$, and it is continuous and compact. From the Berge's Maximization Theorem, the maximization function π_j must be continuous and the maximizer correspondence η must be u.s.c. □

Further, if Y_j is compact $\forall j$, then Y is compact. From Theorem 3.1, we have η is u.s.c. and π is continuous.

3.2 Consumers

The number m of consumer is a given positive integer. Each consumer is indicated by an index $i = 1, 2, \cdots, m$. The i-th consumer chooses a point, his consumption or his demand x_i, in a given nonempty subset of $\mathbb{R}^l$, his consumption set X_t. Given a consumption x_i for each consumer, $x = \sum_{i=1}^{m} x_i$ is called the total consumption or the total demand: the set $X = \sum_{i=1}^{m} X_t$ is called the total consumption set.

Assumption 3.13. *X_i is closed.*

Assumption 3.14. *X is closed.*

Assumption 3.15. *X_i has a lower bound for $\leq$ (lower boundedness).*

Assumption 3.16. *X has a lower bound for $\leq$.*

LEMMA 3.8. *If $\tilde{x}_i$ is a lower bound for $X_i, \forall i$, then $\tilde{x} = \sum_{i=1}^{m} \tilde{x}_i$ is a lower bound for X. Conversely, if $\tilde{x}$ is a lower bound for X, then $\exists(\tilde{x}_i)$ s.t. $\tilde{x} = \sum_{i=1}^{m} \tilde{x}_i$ and $\tilde{x}_i$ is a lower bound for X_i.*

Proof. (1) $\forall x_i \in X_i, \tilde{x}_i \leq x_i$ hold for $\forall i$, then $\forall x \in X$, we have $\exists x_i \in X_i, \forall i$. So $\tilde{x} = \sum_{i=1}^{m} \tilde{x}_i \leq \sum_{i=1}^{m} x_i = x$, i.e., X has a lower bound $\tilde{x}$.

(2) If $\tilde{x}$ is a lower bound of X, then $\forall x \in X$, we have $\tilde{x} \leq x$. First we show X_i is lower bounded for $\forall i$. If X_k is not lower bounded, given M arbitrarily, $\exists x_k \in X_k$ s.t. and take $x_i \in X_i, \forall i \neq k$ arbitrarily, we have $\exists x_k \in X_k$ s.t. $x_k < M - \sum_{i=1, i\neq k}^{m} x_i$, so $x = \sum_{i=1}^{m} x_k < M$, i.e., X is not lower bounded, contradiction.

Now we claim $\exists(\tilde{x}_i)$ s.t. $\tilde{x} = \sum_{i=1}^{m} \tilde{x}_i$ and $\tilde{x}_i \leq x_i, \forall x_i \in X_i$ for $\forall i$. Suppose not, let $\tilde{x}_i = \inf\{x_i \in X_i\}, i = 1, 2, \cdots, m-1$ and $\tilde{x}_m = \tilde{x} - \sum_{i=1}^{m-1} \tilde{x}_i$, then $\tilde{x}_i$ is a lower bound of X_i for $\forall i$ and $\tilde{x}_m$ is not a lower bound of x_m, i.e., $\exists x_m \in X_m$ s.t. $x_m < \tilde{x}_m$. Then $\exists x_1, x_2, \cdots, x_{m-1}$ s.t. $\sum_{i=1}^{m-1} x_i + x_m < \tilde{x}_m + \sum_{i=1}^{m-1} \tilde{x}_i = \tilde{x}$. As $\sum_{i=1}^{m-1} x_i + x_m \in X$, so $\tilde{x}$ is not a lower bound of X, contradiction. □

LEMMA 3.9. *If X_i is closed and has a lower bound for $\leq$, then X is closed.*

Proof. With Theorem 2.132, to show X is closed, we just need to show $\mathbf{A}X_1, \mathbf{A}X_2, \cdots, \mathbf{A}X_m$ are p.s.i. Let $\tilde{x}_i$ be a lower bound of $X_i, \forall i$. Then $X_i \subseteq \Omega + \{\tilde{x}_i\}$, then $\mathbf{A}X_i \subseteq \mathbf{A}(\Omega + \{\tilde{x}_i\}) = \mathbf{A}\Omega = \Omega$ (a-translation) for $\forall i$. Thus $\mathbf{A}X_i \subseteq \Omega, \forall i$.

So, $\forall x_i \in \mathbf{A}X_i, i = 1, 2, \cdots, m$ s.t. $\sum_{i=1}^{m} x_i = \mathbf{0}$, if $x_k > \mathbf{0}$, then $\sum_{i=1, i\neq k}^{m} x_i = -x_k < \mathbf{0}$, i.e., $\exists k'$ s.t. $x_{k'} < \mathbf{0}$, contradicting to $x_{k'} \in \mathbf{A}X_{k'} \subseteq \Omega$.

(Further,$\sum_{i=1}^{m} \mathbf{A}X_i \subseteq \sum_{i=1}^{m} \Omega = \Omega$) □

Assumption 3.17. *X_i is connected.*

Assumption 3.18. *X_i is convex.*

If X_i is convex, then it is connected. (shown)

DEFINITION 3.1. *(Preference Preordering) Let $x_i^1, x_i^2 \in X_i$ and $\succeq_i$ be an binary relation, named preference preordering. If x_i^1 is preferred to x_i^2, we say $x_i^1 \succeq_i x_i^2$. We assume $\succeq_i$ is complete and transitive.*

Let $\sim_i$ be an binary relation, named indifference preordering, defined by $x_i^1 \sim_i x_i^2$ if $x_i^1 \succeq_i x_i^2$ and $x_i^2 \succeq_i x_i^1$.

Let $\succ_i$ be an binary relation, named strong preference preordering, defined by $x_i^1 \succ_i x_i^2$ if not $x_i^2 \succeq_i x_i^1$, i.e., $x_i^1 \succeq_i x_i^2$ and not $x_i^1 \sim_i x_i^2$.

DEFINITION 3.2. *(Indifference class) Given $x_i' \in X_i$, the set*

$$\{x_i \in X_i | x_i \sim_i x_i'\} \tag{3.11}$$

is called the indifference class of x_i'.

DEFINITION 3.3. *(Satiation) GIven $x_i' \in X_i$, it is a Satiation consumption if*

$$x_i' \succeq_i x_i, \forall x_i \in X_i \tag{3.12}$$

(A greatest element of X_i for $\succeq_i$ is called a satiation consumption.)

Assumption 3.19. *No satiation consumption exists for the i-th consumer.*

DEFINITION 3.4. *(Utility Function) Given $\succeq_i$ defined on X_i,if function u_i s.t. $u_i(x_i^1) \geq u_i(x_i^2)$ if and only if $x_i^1 \succeq_i x_i^2, \forall x_i^1, x_i^2 \in X_i$, then u_i is called the utility function representing $\succeq_i$.*

LEMMA 3.10. *If f is an increasing function and u_i represents $\succeq_i$, then $f(u_i)$ also represents $\succeq_i$.*

Proof. $\forall x_i^1, x_i^2 \in X_i$, we have $x_i^1 \succeq_i x_i^2$ iff $u_i(x_i^1) \geq u_i(x_i^2)$ iff $f(u_i(x_i^1)) \geq f(u_i(x_i^2))$, done. □

Assumption 3.20. *$\succeq_i$ is continuous, i.e., $\forall x_i' \in X_i$ the set $\{x_i \in X | x_i' \succeq_i x_i\}$ and $\{x_i \in X_i | x_i \succeq_i x_i'\}$ are closed.*

LEMMA 3.11. *The following statement are equivalent.*

(1) $\succeq_i$ is continuous;

(2) $\forall x_i' \in X_i$ the set $\{x_i \in X_i | x_i' \succeq_i x_i\}$ and $\{x_i \in X_i | x_i \succeq_i x_i'\}$ are closed.

(3) $\forall x_i' \in X_i$ the set $\{x_i \in X_i | x_i' \succ_i x_i\}$ and $\{x_i \in X_i | x_i \succ_i x_i'\}$ are closed.

Proof. From definition, $(1) \Leftrightarrow (2)$. We show $(2) \Leftrightarrow (3)$. Given $x_i' \in X_i$, $\succeq_i$ is complete, so $\forall x_i \in X$, we have either $x_i \succeq_i x_i'$ or $x_i' \succ_i x_i$, i.e., $\{x_i \in X_i | x_i' \succeq_i x_i\}$ is the complement of $\{x_i \in X_i | x_i' \succ_i x_i\}$. Thus, $\{x_i \in X_i | x_i' \succeq_i x_i\}$ is closed iff $\{x_i \in X_i | x_i' \succ_i x_i\}$ is open. So do $\{x_i \in X_i | x_i \succeq_i x_i'\}$ and $\{x_i \in X_i | x_i \succ_i x_i'\}$. Therefore, $(2) \Leftrightarrow (3)$. □

LEMMA 3.12. *If $\succeq$ is continuous on X and $x^2 \succ x^1, x^1, x^2 \in X$, then $\exists x \in [x^1, x^2]$ s.t. $x^2 \succ x \succ x^1$. Further, $\forall x' \in X$ s.t. $x^2 \succ x' \succ x^1$, we have $\exists x \in [x^1, x^2]$ s.t. $x \sim x'$.*

Proof. (1) Let $X_{x^1} = \{x \in [x^1, x^2] | x^1 \succeq x\}$ and $X^{x^2} = \{x \in [x^1, x^2] | x \succeq x^2\}$. Since $\succeq$ is continuous, X_{x^1}, X^{x^2} are closed. If not $\exists x \in [x^1, x^2]$ s.t. $x^2 \succ x \succ x^1$, then $\forall x \in [x^1, x^2]$, either $x \in X_{x^1}$ or $x \in X^{x^2}$, so X_{x^1}, X^{x^2} form a partition of $[x^1, x^2]$, and both closed, contradicting to $[x^1, x^2]$ is connected.

(2) Suppose not, then $\forall x \in [x^1, x^2]$ we have either $x \succ x'$ or $x' \succ x$, then let $Y_{x^1} = \{x \in [x^1, x^2] | x^1 \succ x\}$ and $Y^{x^2} = \{x \in [x^1, x^2] | x \succ x'\}$. So Y_{x^1}, Y^{x^2} form a partition of $[x^1, x^2]$. As $\succeq$ is continuous, so Y_{x^1}, Y^{x^2} are open and form a partition of $[x^1, x^2]$, contradicting to $[x^1, x^2]$ is connected. □

THEOREM 3.2. *(Debrue's Theorem) Let X be a connected subset of $\mathbb{R}^l$, and $\succeq$ is a complete preordering. If $\succeq$ is continuous, then there is a continuous utility function representing $\succeq$.*

Proof. $\forall X \subseteq \mathbb{R}^l$, we have $\exists D \subseteq X$ s.t. D is countable and D is dense in X, i.e., $X \subseteq \overline{D}$. (formally, let $D = X \cap \mathbb{Q}^l$(rational number), Theorem 2.19)

(1) If $x', x'' \in X$ s.t. $x'' \succ x'$, then $\exists x \in D$ s.t. $x'' \succ x \succ x'$. Let $X_{x'} = \{x \in X | x' \succeq x\}, X^{x''} = \{x \in X | x \succeq x''\}$, then $X_{x'}, X^{x''}$ are disjoint, nonempty and closed ($\succeq$ is continuous). As X is connected, $X_{x'}, X^{x''}$ can not be a partition of X, i.e.,

$$X_{x'} \cup X^{x''} \neq X \tag{3.13}$$

Suppose not $\exists x \in D$ s.t. $x'' \succ x \succ x'$, then $D \subseteq X_{x'} \cup X^{x''}$, so $\overline{D} \subseteq \overline{X_{x'} \cup X^{x''}} = X_{x'} \cup X^{x''}$. As $X \subseteq \overline{D}$, then

$$X \subseteq X_{x'} \cup X^{x''} \tag{3.14}$$

With $X_{x'} \cup X^{x''} \subseteq X$, then $X = X_{x'} \cup X^{x''}$, contradicting to Equation 3.13.

(2) Define utility function u' on D. Take two real numbers a, b arbitrarily s.t. $a < b$. If D has a least element x^α, take $u'(x^\alpha) = a$. If D has a greatest element x^β, take $u'(x^\beta) = b$. Remove from D all elements indifferent to x^α or x^β, and let D' denote the remaining set.

Then D' has no least and greatest element. Suppose not, WLOG, let x^γ be a least element of D'. Then $x^\gamma \succ x^\alpha$, from (1), we have $\exists x \in D$ s.t. $x^\gamma \succ x \succ x^\alpha$, so $x \in D'$ and $x^\gamma \succ x$, contradicting to x^γ is the least element of D'

(3) Assign value to $u'(x), \forall x \in D'$ now. Since D' is countable, all elements can be ranked as $x^1, x^2, \cdots$ arbitrarily . Also, $\mathbb{Q}' = \mathbb{Q} \cap (a, b)$ is countable and we rank its elements as $r^1, r^2, \cdots$ arbitrarily. Now we assign values to u' one by one,

- consider x^1, take $q_1 = 1$ and $u'(x^1) = r^{q_1}$;
- consider x^2,
  - if $x^2 \sim x^1$, take $q_2 = q_1$;
  - if $x^\beta \succ x^2 \succ x^1$, take an element q_2 from $(r^{q_1}, b) \cap Q'$ arbitrarily;
  - if $x^1 \succ x \succ x^\alpha$, take an element q_2 from $(a, r^{q_1}) \cap Q'$ arbitrarily;

  let $u'(x^2) = r^{q_2}$;
- consider x^n,
  - if $\exists m < n$ s.t. $x^n \sim x^m$, take $q_n = q_m$;
  - if $\exists m, m' < n$ s.t. $x^{m'} \succ x \succ x^m$, take an element q_n from $(q^m, q^{m'}) \cap Q'$ arbitrarily;
  - if $x^n \succ x^m$ where s.t. $m < n$ and $x^m \succeq x^{m'}, \forall m' < n$, then take an element q_n from $(q^m, b) \cap Q'$ arbitrarily;
  - if $x^m \succ x^n$ where s.t. $m < n$ and $x^{m'} \succeq x^m, \forall m' < n$, then take an element q_n from $(a, q^m) \cap Q'$ arbitrarily;

So we construct a function u' from D' to $\mathbf{Q}'$, which represents $\succeq$.

In this way, plus (2) (replace X with D), we have every element of $\mathbb{Q}'$ has been taken eventually.

(3) Extension from X to X. The utility function to be defined on X will be denoted u. If $x \in X$, let $D_x = \{x' \in D | x \succeq x'\}, D^x = \{x' \in D | x' \succeq x\}$. If x is a least element of X, take $u(x) = a$. If x is a greatest element of X, take $u(x) = b$. In other words, now we show $\sup u'(D_x) = \inf u'(D^x)$.

If $x' \in D_x$ and $x'' \in D^x$, then $x'' \succeq x'$, i.e.,$\forall r' \in u'(D_x), r'' \in u'(D^x)$ we have $r' \leq r''$.Thus, $\sup u'(D_x) \leq \inf u'(D^x)$.

We can not have $\sup u'(D_x) < \inf u'(D^x)$ as all elements have been taken out by u'. So take $u(x) = \sup u'(D_x) = \inf u'(D^x)$.

(4) Continuity of u. With Theorem , we just show $\forall c \in \mathbb{R}$, $u^{-1}([c, \infty))$ and $u^{-1}((-\infty, c])$ is closed in X. If $c \notin [a, b]$, then $u^{-1}([c, \infty)), u^{-1}((-\infty, c])$ are $\varnothing$, so closed. If $c \in \{a, b\}$, $u^{-1}([c, \infty))$ and $u^{-1}((-\infty, c])$ are the indifference class of a continuous preference, surely closed (intersection of two closed sets).

If $c \in (a, b)$, we first show $u^{-1}([c, \infty))$ is closed here. If $t \in \mathbb{R}$, define $X_t = \{x \in X | u(x) \leq t\}, X^t = \{x \in X | u(x) \geq t\}$. As

$$[c, \infty) = \bigcap_{r \leq c, r \in \mathbb{Q}'} [r, \infty) \tag{3.15}$$

So, $X^c = \bigcap_{r \leq c, r \in \mathbb{Q}'} X^r$. Let $x \in X$ s.t. $u(x) = r$, then $X^r = \{x' \in X | x' \succeq x\}$ is closed ($\succeq$ is closed). With Theorem 2.38, we have X^c is closed, which equals to $u^{-1}([c, \infty))$. We know $u^{-1}([c, \infty))$ is closed is the same way. $\square$

Now we study the convexity of preference and we assume X_i is convex here.

Assumption 3.21. *(Weak-convexity)*$\forall x_i^1, x_i^2 \in X_i$ *and* $x_i^2 \succeq_i x_i^1$*, then* $\forall t \in (0, 1)$*, we have* $tx_i^2 + (1-t)x_i^1 \succeq_i x_i^1$.

LEMMA 3.13. *If* $\succeq$ *is weak-convexity, then* $\forall x_i' \in X_i, X^{x_i'} = \{x_i \in X_i | x_i \succeq_i x_i'\}$ *and* $X_{x_i'} = \{x_i \in X_i | x_i' \succeq_i x_i\}$ *are convex.*

Proof. $\forall x_i^1, x_i^2 \in X^{x_i'}$, WOLG, let $x_i^2 \succeq x_i^1$, then $\forall t \in (0,1)$, from weak-convexity condition, we have $tx_i^2 + (1-t)x_i^1 \succeq_i x_i^1 \succeq_i x_i'$ (resp. $tx_i^2 + (1-t)x_i^1 \succeq_i x_i^1 \succ_i x_i'$), so $tx_i^2 + (1-t)x_i^1 \in X^{x_i'}$ (resp. $X_{x_i'}$), i.e., $X^{x_i'}$ resp. $X_{x_i'}$) is convex. □

Assumption 3.22. *(Convexity)* $\forall x_i^1, x_i^2 \in X_i$ *and* $x_i^2 \succ_i x_i^1$*, then* $\forall t \in (0,1)$*, we have* $tx_i^2 + (1-t)x_i^1 \succ_i x_i^1$.

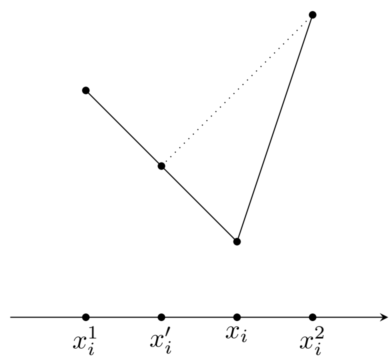


Figure 11: Continuous, convexity induce weak-convexity

LEMMA 3.14. *Given* X_i*, if* $\succeq_i$ *is continuous and convex, then it is weak-convex.*

Proof. $\forall x_i^1, x_i^2 \in X_i$ s.t. $x^2 \succeq_i x_i^1$, then $\forall t \in (0,1)$, we want to show $tx_i^2 + (1-t)x_i^1 \succeq_i x_i^1$. Suppose not, $\exists x_i \in (x_i^1, x_i^2)$ (open segment) s.t. $x_i^1 \succ_i x_i$, then from Lemma 3.12, $\exists x_i' \in (x_i^1, x_i)$ s.t. $x_i^1 \succ_i x_i' \succ_i x_i$. Then x_i must be in (x_i', x_i^2). Further, $x_i^2 \succeq_i x_i^1 \succ x_i'$ from convexity condition, we have $x_i \succ_i x_i'$, contradiction. See Figure 11. □

LEMMA 3.15. *If* $\succeq_i$ *is convex, then* $\forall x_i' \in X_i$*, if it is not a satiation point, then* x_i' *is adherent to* $\{x_i \in X_i | x_i \succ_i x_i'\}$.

Proof. Take a $x_i \in \{x_i \in X_i | x_i \succ_i x_i'\}$ arbitrarily, then $\forall \epsilon > 0$, we can let $t \in (0,1)$ s.t. $t < \frac{||x_i - x_i'||}{\epsilon}$. As $\succeq_i$ is convex, so $tx_i + (1-t)x_i' \succ_i x_i'$, i.e., $tx_i + (1-t)x_i' \in \{x_i \in X_i | x_i \succ_i x_i'\}$ and $||tx_i + (1-t)x_i' - x_i'|| = t||x_i - x_i'|| < \epsilon$. Thus, x_i' is adherent to $\{x_i \in X_i | x_i \succ_i x_i'\}$. □

Assumption 3.23. *(Strong-convexity)* $\forall x_i^1, x_i^2 \in X_i$ *and* $x_i^2 \sim_i x_i^1$*, then* $\forall t \in (0,1)$*, we have* $tx_i^2 + (1-t)x_i^1 \sim_i x_i^1$.

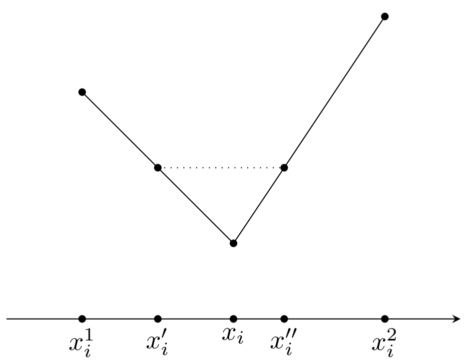


Figure 12: Continuous, strong convexity induce convexity

LEMMA 3.16. *Given* X_i*, if* $\succeq_i$ *is continuous and strong-convex, then it is convex.*

Proof. See Figure 12, similar to Lemma 3.14. Let $x_i^2 \succ_i x_i^1$, suppose the statement not hold, then $\exists x_i \in (x_i^1, x_i^2)$ s.t. $x_i^1 \succeq_i x_i$. If $x_i^1 \sim_i x_i$, from strong convexity condition, we say $\exists x_i' \in (x_i^1, x_i)$ s.t. $x_i^2 \succ_i x_i' \succ_i x_i$ (let $t \to 0$). If $x_i^1 \succ_i x_i$, then from Lemma 3.12, we also have $\exists x_i' \in (x_i^1, x_i)$ s.t. $x_i^1 \succ_i x_i' \succ_i x_i$.

Then $x_i^2 \succ_i x_i' \succ_i x_i$, so from Lemma 3.12, $\exists x_i'' \in (x_i, x_i^1)$ s.t. $x_i'' \sim_i x_i'$ and $x_i \in (x_i', x_i'')$. Thus, from strong convexity, we have $x_i \succ_i x_i'$, contradiction. □

There is another convexity, for distinct $x_i^1, x_i^2 \in X_i$, if $x_i^2 \succeq_i x_i^1$, then $\forall t \in (0,1)$, we have $tx_i^2 + (1-t)x_i^1 \succ_i x_i^1$. This implies strong-convexity, convexity and weak-convexity (trivially).

Given the price system p and his wealth w_i, a real number, the i-th consumer chooses his consumption x_i in his consumption set X_i so that his expenditure $p \cdot x_i$ satisfies the wealth constraint $p \cdot x_i \leq w_i$. The point $w = (w_i) \in \mathbb{R}^m$ is called the wealth distribution. The point $(p, w) \in \mathbb{R}^{l+m}$ is called the price-wealth pair.

When $p \neq \mathbf{0}$, the hyperplane

$$H_i = \{a \in \mathbb{R}^l | p \cdot a = w_i\} \tag{3.16}$$

is called the wealth hyperplane.

Define S_i

$$S_i = \{(p, w) \in \mathbb{R}^{l+1} | \exists x_i \in X_i \text{ s.t. } p_i \cdot x_i \leq w_i\} \tag{3.17}$$

i.e., it contains all (p, w) which ensure the choose set it not empty under wealth constraint.

Given $(p, w) \in S_i$, define

$$\gamma_i(p, w) = \{x_i \in X_i | p \cdot x_i \leq w_i\} \tag{3.18}$$

i.e., $\gamma_i : S_i \rightrightarrows X_i$, called budget constraint set.

Trivially, S_i is a cone with vertex $\mathbf{0}$ and $\forall t > 0, \gamma_i(tp, tw) = \gamma_i(p, w)$.

LEMMA 3.17. *Let $X = \mathbb{R}^N, \forall x^q \to x, x^q \in X, \forall q$, we have $\forall y \in X, \forall \delta > 0, \exists q'$ s.t. $\forall q > q', |x^q y - xy| < \delta$.*

Proof. If $y = \mathbf{0}$, the statement is evident. If $y \neq \mathbf{0}$, let $y = (y_1, y_2, \cdots, y_N)$, then $\sum_{i=1}^N |y_i| > 0$. $\forall \delta > 0$, let $0 < e < \frac{\delta}{\sum_{i=1}^N |y_i|}$. As $x^q \to x$, so $\exists q'$ s.t. $\forall q > q', |x^q - x| < e$, thus $|x_i^q - x_i| < e, \forall i, q$. So

$$|x^q y - xy| = |\sum_{i=1}^N (x_i^q - x_i) y_i| \leq \sum_{i=1}^N |(x_i^q - x_i) y_i| \leq \sum_{i=1}^N |(x_i^q - x_i)||y_i| \leq \sum_{i=1}^N e|y_i| = e \sum_{i=1}^N |y_i| < \delta \tag{3.19}$$

□

THEOREM 3.3. *If X_i is compact, convex and $(p^0, w^0) \in S_i$ s.t. $w_i^0 \neq \min p^0 \cdot X_i$, then γ_i is continuous at (p^0, w^0).*

Proof. We need to show γ_i is u.s.c. and l.s.c.

(1) u.s.c. The graph of γ_i, $\{(p, w, x_i) \in S_i \times X_i | p \cdot x_i \leq w_i\}$ is clearly closed in $S_i \times X_i$. Hence γ_i is u.s.c.

(2) l.s.c. (i.s.c.) $\forall (p^q, w^q) \to (p^0, w^0)$ and $x_i^0 \in \gamma_i(p^0, w^0)$, we want to show $\exists (x_i^q)$ s.t. $x_i^q \to x_i^0$ and $x_i^q \in \gamma_i(p^q, w^q)$, i.e., $p^q \cdot x_i^q \leq w_i^q$.

If $p^0 \cdot w_i^0 < w_i^0$, then from Lemma 3.17, we have $\exists q'$ s.t. $\forall q' > q, p^q \cdot x_i^0 < w_i^q$. Define (x_i^q) as

- take x_i^q from $\gamma_i(p^q, w^q)$ arbitrarily when $q \leq q'$;
- $x_i^q = x_i^0$ when $q > q'$.

If $p^0 x_i^0 = w_i^0$, as $w_i^0 \neq Min p^0 X_i$, so $\exists x_i'$ s.t. $p^0 x_i' \leq w_i^0$. With Lemma 3.17 and $\forall (p^q, w^q) \to (p^0, w^0)$, let $\delta = w^0 - p^0 x_i' > 0, \epsilon = p^0(x_i^0 - x_i') > 0$, we have

$$\exists q_1', s.t. \forall q > q_1', p^q x_i' < p^0 x_i' + \frac{\delta}{2} \tag{3.20}$$

$$\exists q_2', s.t. \forall q > q_2', w_i^q > w_i^0 - \frac{\delta}{2} \tag{3.21}$$

$$\exists q_3', s.t. \forall q > q_3', p^q(x_i^0 - x_i') > p^0(x_i^0 - x_i') - \frac{\epsilon}{2} > 0 \tag{3.22}$$

Therefore, let $q' = \max\{q_1', q_2', q_3'\}$, we have $\forall q > q'$,

$$p^q x_i' < w_i^q, p^q x_i' < p^q x_i^0 \tag{3.23}$$

As $a^q = t^q x_i^0 + (1 - t^q) x_i'$ and $p^q a^q = w_i^q$, so

$$p^q(t^q x_i^0 + (1 - t^q) x_i') = w_i^q \tag{3.24}$$

$$t^q p^q x_i^0 + (1 - t^q) p^q x_i' = w_i^q \tag{3.25}$$

$$t^q = \frac{w_i^q - p^q x_i'}{p^q x_i^0 - p^q x_i'} \tag{3.26}$$

So $t^q > 0, \forall q$ and $t^q \to 1$ as $p^q x_i^q \to w^0$. So $a^q \to x_i^0$.

Then we construct a sequence (x_i^q),

- take an arbitrary $x_i^q \in \gamma_i(p^q, w^q), q \leq q'$;
- take $x_i^q = a^q$ if $q > q'$ and $t^q \in [0,1]$; then $x_i^q = a^q \in \gamma_i(p^q, w^q)$
- take $x_i^q = x_i^0$ if $q > q'$ and $t^q > 1$; then $p^q x_i^q = p^q x_i^0 = p^q \frac{1}{t^q}(a^q + (t^q - 1)x_i') = \frac{1}{t^q}(p^q a^q + (t^q - 1)p^q x_i') < \frac{1}{t^q} t^q w_i^q = w_i^q$, so $x_i^q \in \gamma_i(p^q, w^q)$.

So $x_i^q \in x_i^0$ and $x_i^q \in \gamma_i(p^q, w^q), \forall q$, i.e., γ_i is i.s.c. at (p^0, w^0). $\square$

Given a price-wealth pair $(p, w) \in S_i$, the i-th consumer chooses, in the set $\gamma_i(p, w)$, a greatest element for his preference preordering $\succeq_i$. The result action is an equilibrium consumption of the i-th consumer relative to (p, w).

The action x_i' is called an equilibrium consumption of the i-th consumer relative to the price system p if it is a greatest elements of $\gamma_i(p, w)$ for $\succeq_i$.

Define

$$S_i' = \{(p, w) \in S_i | \gamma_i(p, w) \text{ has a greatest element for } \succeq_i\} \tag{3.27}$$

to ensure an equilibrium consumption exists.

Define

$$\xi_i(p, w) = \{x_i \in \gamma_i(p, w) | x_i \text{ is a greatest element of } \gamma_i(p, w) \text{ for } \succ_i\} \tag{3.28}$$

i.e., $\xi_i : S_i' \rightrightarrows X_i$, the demand correspondence of the i-th consumer.

If there is a utility function u_i on X_i, the maximum utility, when the price-wealth pair is (p, w) in S_i', is denoted by $v_i(p, w)$. The function v_i from S_i' to $\mathbb{R}$ is called the indirect utility function of the i-th consumer, i.e.,

$$v_i(p, w) = \max u_i(\gamma_i(p, w)) \tag{3.29}$$

Both ξ_i, v_i are positivity homogeneous of degree one. If $t > 0$, then

$$\xi_i(tp, tw) = \xi_i(p, w), v_i(tp, tw) = v_i(p, w) \tag{3.30}$$

Given a price-wealth pair (p, w), there is a greatest element of $\gamma_i(p, w)$ for $\forall i$ iff $(p, w) \in \bigcap_{i=1}^{m} S_i'$. (All consumers' decisions are irrelative)

The total demand correspondence is defined as

$$\xi : \bigcap_{i=1}^{m} S_i' \rightrightarrows X, \xi(p, w) = \sum_{i=1}^{m} \xi_i(p, w) \tag{3.31}$$

Then $\forall t > 0$, we have $\xi(p, w) = \xi(p, w)$.

We now study the duality of satiation consumption.

If $x_i' \in \xi_i(p, w)$, then $\forall x_i \in \gamma_i(p, w)$, we have $x_i' \succeq_i x_i$, i.e.,

$$p \cdot x_i \leq w_i \Rightarrow x_i' \succeq_i x_i \tag{3.32}$$

Conversely, if $x_i \succeq_i x_i'$, then $p \cdot x_i > w_i$.

We define the dual concept of ξ_i as (Hicksian demand)

$$\zeta_i(p, x_i^0) = \{x_i \in X_i | x_i \succeq_i x_i^0, p \cdot x_i \leq p \cdot x_i', \forall x_i' \succeq_i x_i^0\} \tag{3.33}$$

i.e., $\zeta_i : \mathbb{R}^l \times X_i \rightrightarrows X_i$, the minimizer of expenditure on the set $\{x_i \in X_i | x_i \succeq_i x_i^0\}$.

THEOREM 3.4. *Let X_i be a convex, and $\succeq_i$ is continuous and $w_i \neq \min p \cdot X_i$ is excluded. Given w_i, if $x_i' \in \zeta_i(p, x_i^0)$ s.t. $p \cdot x_i' = w_i$, then $x_i' \in \xi_i(p, w_i)$.*

Proof. We first show if $\succeq_i$ is continuous, then $x_i^0 \sim_i x_i'$. Suppose not, $x_i' \succ_i x_i^0$, let $X_{x_i^0} = \{x_i \in X_i | x_i \succeq_i x_i^0\}$. As $\succeq_i$ is continuous, then $X^{x_i^0}$ is closed. $x_i' \succ_i x_i^0$ implies $x_i' \in int(X^{x_i^0})$(otherwise x_i' is the adherent of $\{x_i \in X_i | x_i^0 \succeq_i x_i\}$, so $x_i^0 \sim_i x_i'$). As $\gamma_i(p, w_i)$ is closed, then $\gamma_i(p, w_i) \cap X^{x_i^0}$ is closed. As $x_i' \in int(X^{x_i^0})$, then $\exists x_i'' \in X^{x_i^0}$ s.t. $x_i'' \in int(\gamma_i(p, w_i))$, i.e., $p \cdot x_i'' < w_i$. But $x_i'' \succeq_i x_i^0$, so $x_i' \notin \zeta(p, x^0)$, contradiction.

Then we show $x_i' \in \xi_i(p, w_i)$, i.e., $\forall x_i^2 \in \gamma_i(p, w_i)$, we have $x_i' \succeq_i x_i^2$. Since $w_i \neq \min p \cdot X_i$, then $\exists x_i^1 \in X_i$ s.t. $p \cdot x_i^1 < w_i$. $\forall x_i^2$ s.t. $p \cdot x_i^2 = w_i$, then $\forall x_i \in (x_i^1, x_i^2)$, we have $p \cdot x_i < w_i$. Then $x_i' \sim_i x_i^0 \succ_i x_i$, so $x_i \in X_{x_i'} = \{x_i \in X | x_i' \succeq x_i\}$, which is closed. As x_i^2 is an accumulation point of (x_i^1, x_i^2), so x_i^2 is adherent to $X_{x_i'}$. Plus $X_{x_i'}$ is closed, we have $x_i^2 \in X_{x_i'}$, i.e., $x_i' \succeq_i x_i^2$. Thus, $x_i' \in \xi_i(p, w_i)$. $\square$

THEOREM 3.5. *If $\succeq_i$ is convex and continuous, and x_i' is not a satiation point, s.t. $p\cdot x_i' = w_i$. If $x_i' \in \xi_i(p, w_i)$, then $x_i' \in \zeta_i(p, x_i')$.*

Proof. To show $x_i' \in \zeta_i(p, x_i')$, we need to show, $\forall x_i^2 \in X_i$ s.t. $x_i^2 \succeq_i x_i'$, we have $p \cdot x_i^2 \geq px_i' = w_i$.

Since x_i' is not a satiation point, then $\exists x_i^1 \in X_i$ s.t. $x_i^1 \succ_i x_i'$. Given such $x_i^2, \forall x_i \in (x_i^2, x_i^1)$, with convexity, we have $x_i \succ_i x_i^2 \succeq_i x_i'$. Since $x_i' \in \xi_i(p, w_i)$, we then obtain $p \cdot x_i > w_i$. By continuity of $p \cdot x_i$, we have $p \cdot x_i^2 \geq w_i$ as desired. □

Let $\succeq_i$ be continuous then there is a continuous utility function representing $\succeq_i$. If X_i is nonempty compact subset of $\mathbb{R}^l$, then $\forall (p, w) \in S_i$, $\gamma_i(p, w)$ is nonempty compact. Thus the target function $u_i(x_i)$ is continuous and the domain $\gamma_i(p, w)$ is compact, so we say the maximizer is attainable with Theorem 2.75. Thus, $S_i' = S_i$. Hold these assumption in later chapter.

THEOREM 3.6. *Let u_i be a utility function defined on $S_i \times X_i$ and γ_i is the budget constraint (a correspondence from S_i to X_i). If u_i is continuous and γ_i is compact-valued and continuous, from Berge's Maximization Theorem, we have the indirect utility function (maximum function) $v_i(p, w), \forall (p, w) \in S_i$ is continuous and the Marshall demand (the maximizer) $\xi_i(p, w), \forall (p, w) \in S_i$ is compact-valued and upper semi-continuous.*

3.3 Equilibrium

In economy, the total resources are a given point ω in $\mathbb{R}^l$. An economy E is defined by : for each $i = 1, 2, \cdots, m$ a nonempty subset X_i of $\mathbb{R}^l$ completely preordered by $\succeq_i$; for each $j = 1, 2, \cdots, n$, a nonempty subset Y_j of $\mathbb{R}^l$; a point ω of $\mathbb{R}^l$.

A state of E is an $(m + n)$-tuple of points of $\mathbb{R}^l$.

Given a state $((x_i), (y_j))$ of E, the point $x - y$ is the net demand, the point $z = x - y - \omega$ is the excess demand. Z denotes the set $X - Y - \{\omega\}$.

A state $((x_i), (y_j))$ of E is a market equilibrium if $x - y = \omega$. The set of market equilibrium of E is denoted by $\mathbb{M}$ (a linear manifold), i.e.,

$$\mathbb{M} = \{((x_i), (y_j)) \in \mathbb{R}^{l(m+n)} | \sum_{i=1}^{m} x_i - \sum_{j=1}^{n} y_j = \omega\} \tag{3.34}$$

A state $((x_i), (y_j))$ of E is attainable if $x_i \in X_i$ for every i, $y_j \in Y_j$ for every j, $x - y = \omega$. The set of attainable states of E is denoted by $\mathbb{A}$, i.e.,

$$\mathbb{A} = \mathbb{M} \cap [(\prod_{i=1}^{m} x_i) \times (\prod_{j=1}^{n} y_j)] \tag{3.35}$$

Given an economy E, a consumption for the i-th consumer (resp. a production for the j-th producer) is attainable if it is the component corresponding to his of some attainable state. The set of his attainable consumption (resp. production) is his attainable consumption (resp. production) set, denoted by $\hat{X}_i$ (resp. $\hat{Y}_j$).

Now study properties of the attainable set $\mathbb{A}$.

It is clear that $\mathbb{A}$ is nonempty iff $\omega \in X - Y$, i.e., $\mathbf{0} \in Z$.

LEMMA 3.18. *Given an economy E, if every X_i and every Y_j are closed, then $\mathbb{A}$ is closed.*

Proof. X_i, Y_j are closed for $\forall i, j$, then $(\prod_{i=1}^{m} X_i) \times (\prod_{j=1}^{n} Y_j)$ is closed (product). A linear manifold $\mathbb{M}$ is closed, thus the intersection of two closed set $\mathbb{A}$, is closed. □

LEMMA 3.19. *Given an economy E, if every X_i and every Y_j are convex, then $\mathbb{A}$ is convex.*

Proof. If every X_i and every Y_j are convex, then with Theorem 2.119, we say $(\prod_{i=1}^{m} X_i) \times (\prod_{j=1}^{n} Y_j)$ is convex. The linear manifold $\mathbb{M}$ is convex, thus, from Theorem 2.118, we say $\mathbb{A}$ is convex.

Further, the set $X - Y$ and Z are convex from Theorem 2.119. □

LEMMA 3.20. *Let E be an economy such that X has a lower bound for $\leq$, Y is closed and convex and $Y \cap \Omega = \{\mathbf{0}\}$.*

(1) If $n = 1$ and / or $Y \cap (-Y) \subseteq \mathbf{0}$, then $\mathbb{A}$ is bounded.

(2) If X is closed, then $X - Y$ is closed.

Proof. (1) To show $\mathbb{A}$ is bounded, according to Theorem 2.132, we can just show the intersection of $\mathbf{A}(\prod_{i=1}^{m} X_i) \times (\prod_{j=1}^{n} Y_j)$ and $\mathbf{A}\mathbb{M}$ is $\mathbf{0}$.

According to Theorem 2.129, $\mathbf{A}(\prod_{i=1}^{m} X_i) \times (\prod_{j=1}^{n} Y_j)$ is contained in $(\prod_{i=1}^{m} \mathbf{A}X_i) \times (\prod_{j=1}^{n} \mathbf{A}Y_j)$, so it is sufficient to show

$$(\prod_{i=1}^{m} \mathbf{A}X_i) \times (\prod_{j=1}^{n} \mathbf{A}Y_j) \cap \mathbf{A}\mathbb{M} = \{\mathbf{0}\} \tag{3.36}$$

Define a linear manifold,

$$\mathbb{M}' = \{((x_i), (y_j)) \in \mathbb{R}^{l(m+n)} | \sum_{i=1}^{m} x_i + \sum_{j=1}^{n} y_j = \mathbf{0}\} \tag{3.37}$$

Since the $\mathbb{M}'$ is derived from $\mathbb{M}$ by a translation, hence has the same asymptotic cone as $\mathbb{M}$. $\mathbb{M}'$ is a closed cone with vertex $\mathbf{0}$, and hence coincides with its own asymptotic cone (Theorem 2.137), thus Equation 3.36 equals to

$$(\prod_{i=1}^{m} \mathbf{A}X_i) \times (\prod_{j=1}^{n} \mathbf{A}Y_j) \cap \mathbf{A}\mathbb{M}' = \{\mathbf{0}\} \tag{3.38}$$

We have $\mathbf{A}X_i \subseteq \mathbf{A}X$ (as $X = X_i + T$). Since X has a lower bound for $\leq$, then $\mathbf{A}X \subseteq \Omega$ (see details in Lemma 3.9). Thus, $\mathbf{A}X_i \subseteq \Omega$, consequently,

$$\sum_{i=1}^{M} \mathbf{A}X_i \subseteq \Omega \tag{3.39}$$

Similarly, $\mathbf{A}Y_j \subseteq \mathbf{A}Y$. Since Y is closed, convex and owns $\mathbf{0}$, then With Theorem,

$$\mathbb{A}Y \subseteq Y \tag{3.40}$$

Hence $\mathbb{A}Y_j \subseteq Y, \forall j$, consequently,

$$\sum_{j=1}^{n} \mathbf{A}Y_j \subseteq Y \tag{3.41}$$

Because of $Y \cap \Omega = \{\mathbf{0}\}$, the relation $\sum_{i=1}^{m} x_i + \sum_{j=1}^{n} y_j = \mathbf{0}, \forall x_i \in \mathbf{A}X_i, \forall y_j \in \mathbf{A}Y_j$ thus implies $\sum_{i=1}^{m} x_i = \mathbf{0} = \sum_{j=1}^{n} y_j$.

$\sum_{i=1}^{m} x_i = \mathbf{0}$ and $x_i \in \Omega, \forall i$ implies $x_i = \mathbf{0}, \forall i$.

If $n = 1$, $\sum_{j=1}^{n} y_j = \mathbf{0}$ implies $y_j = \mathbf{0}$.

If $Y \cap (-Y) \subseteq \mathbf{0}$, then from proof of Lemma 3.3, we have $\sum_{j=1}^{n} y_j = \mathbf{0}$ implies $y_j = \mathbf{0}, \forall j$. Done.

(2) To show $X - Y$ is closed, it suffices to show $\mathbf{A}X$ and $\mathbf{A}(-Y)$ are p.s.i., which is equivalent to show $\mathbf{A}X \cap \mathbf{A}Y = \mathbf{0}$, which directly follows $\mathbf{A}X \subseteq \Omega$ and $Y \cap \Omega = \{\mathbf{0}\}$. ☐

In a private economy, consumers own firms and the ownership is defined by $\theta = (\theta_{ij})$, where θ_{ij} denotes the share of i-th consumers to the j-th firm. We assume $\sum_{i=1}^{m} \theta_{ij} = 1, \theta_{ij} \geq 0, \forall i, j$. Then total wealth of the i-th consumer is

$$w_i = p \cdot \omega_i + \sum_{j=1}^{n} \theta_{ij} \pi_j(p) \tag{3.42}$$

where $\pi_j(p)$ is the j-th firm's profit.

DEFINITION 3.5. *(Economy) A private ownership economy $\mathscr{E}$ is defined by:*

(1) an economy $((X_i, \succeq_i), (Y_j), \omega)$;

(2) $\sum_{i=1}^{m} \omega_i = \omega$ where $\omega_i \in \mathbb{R}^l, \forall i$;

(3) $\sum_{i=1}^{m} \theta_{ij} = 1, \forall j$ where $\theta_{ij} \geq 0$.

DEFINITION 3.6. *(Equilibrium) An equilibrium of the private ownership economy $\mathscr{E}$ is an $(m+n+l)$-tuple $((x_i^*), (y_j^*), p^*)$ of points of $\mathbf{R}^l$ such that,* *(a) x_i^* is a greatest element of $\{x_i \in X_i | p^* \cdot x_i \le p^*\omega_i + \sum_{j=1}^{n} \theta_{ij} p^* \cdot y_j^*\}$ for $\succeq_i$ for $\forall i$;*

(b) y_j^* maximizes profit relative to p^* on Y_j for $\forall j$;

(c) $x^* - y^* = \omega$.

How to find the equilibrium? What is under control is the price system p. Given p, firms first choose product and then consumers choose consumption; we adjust p to make production and consumption are compatible with endowment.

Consider a private economy $\mathscr{E}$ and let C be the set of p in $\mathbb{R}^l$. Given a p, the j-th producer chooses y_j in the set $\eta_j(p)$ (best production plan) and the profit is $\pi_j(p) = p \cdot y_j$. Hence the i-th consumer's wealth becomes $p \cdot \omega_i + \sum_{j=1}^{n} \theta_{ij}\pi_j(p)$, so he chooses a consumption from

$$\xi_i(p, p \cdot \omega_i + \sum_{j=1}^{n} \theta_{ij}\pi_j(p)) \tag{3.43}$$

Thus, a p directly decide a consumption set and let $\eta_i'(p)$ denotes the correspondence. Let $\xi'(p) = \sum_{i=1}^{m} \xi_i'(p)$. So we can say, given p, the best x_i, y_j s.t. $x_i \in \xi_i'(p), \eta_j(p), \forall i, j$. The set of excess demand is

$$\zeta(p) = \xi'(p) - \eta(p) - \{\omega\} \tag{3.44}$$

$\zeta : C \rightrightarrows Z$ ($\zeta(p)$ is a subset of $Z = X - Y - \{\omega\}$). If $\mathbf{0} \in \zeta(p)$, we say we have find an equilibrium.

As $p \cdot x_i \le p \cdot \omega_i + \sum_{j=1}^{n} \theta_{ij}\pi_j(p)$ hold for $\forall i$, we sum them up and obtain,

$$p \cdot x \le p\omega + p \cdot y \Rightarrow p \cdot z \le 0 \tag{3.45}$$

Thus, $p \cdot z \le 0, \forall p, \forall z \in \zeta(p)$, written as $p \cdot \zeta(p) \le 0$.

Now we assume the production is free disposal, i.e., $-\Omega \subseteq Y_j, \forall j$. Then the equilibrium condition (market clear) becomes,

$$x^* - y^* \le \omega \tag{3.46}$$

equivalently, $z^* \le 0$. So now we just need to find a p s.t. $\zeta(p) \cap (-\Omega) \neq \varnothing$.

In the free disposal case, we have $p \in \Omega$ (else, firms profit may be infinite). In addition $\{\mathbf{0}\}$ must be excluded if some consumers are insatiable (and we do this rightly). Then $\sum_{h=1}^{l} p_j > 0$. Hence $\zeta(p) = \zeta(\frac{1}{\sum_{h=1}^{l} p_h})$ (price is nominal), so we can normalize the price system as

$$P = \{p \in \Omega | \sum_{h=1}^{l} p_h = 1\} \tag{3.47}$$

which is convex and compact.

Now the problem can be generalized as: a correspondence ζ from P to Z s.t. $p \cdot \zeta(p) \le 0$ whether there is a p s.t. $\zeta(p) \cap (-\Omega) \neq \varnothing$? (if this problem is solvable, the original problem has a solution) The answer is given by the following Lemma.

LEMMA 3.21. *(Debrue-Gale-Nikaido Lemma) Let Z be a compact subset of $\mathbb{R}^l$. If ζ is an u.s.c. correspondence from P to Z s.t. $p \cdot \zeta(p) \le 0$, and $\zeta(p)$ is convex for $\forall p$, then there is a $p \in P$ s.t. $\zeta(p) \cap (-\Omega) \neq \varnothing$.*

Proof. P is nonempty, compact and convex (Equation 3.47). Let $Z' = CH(Z)$ to construct a nonempty compact and convex set. Given $z \in Z'$, define

$$\mu(z) = \{p \in P | p \cdot z = \max P \cdot z\} \tag{3.48}$$

which is an u.s.c. correspondence from the Berge's Maximization Theorem (Z is compact, $\phi(z) = P$ is continuous, $p \cdot z$ is continuous). Since P is convex, then

(1) $z = \mathbf{0}$, $\mu(z) = P$ is convex;

(2) $z \neq \mathbf{0}$, $\mu(z) = P \cap \{p \in \mathbb{R}^l | p \cdot z = \max P \cdot z\}$ is the intersection of two convex set, thus is convex.

So $\mu(z)$ is convex for $\forall z \in Z'$.

Now consider the correspondence φ from $P \times Z'$ to itself by $\varphi(p, z) = \mu(z) \times \zeta(p)$. As P, Z' is nonempty, compact and convex, so is $P \times Z'$. As μ, ζ are u.s.c., so is φ. Further, $\mu(z), \zeta(p)$ are convex, so is $\varphi(p, z)$. Therefore all conditions of Kakutani's Theorem is satisfied, φ has a fixed point (p^*, z^*), s.t. $(p^*, z^*) \in \mu(z^*) \times \zeta(p^*)$, i.e.,

$$p^* \in \mu(z^*), z^* \in \zeta(p^*) \tag{3.49}$$

$p^* \in \mu(z^*)$ implies $p \cdot z^* \leq p^* \cdot z^*, \forall p \in P$. $z^* \in \zeta(p^*)$ implies $p^* \cdot z^* \leq 0$. Now we show $z^* \in -\Omega$.

Let $p = (1, 0, \cdots, 0)$, then $p \cdot z^* \leq 0$ implies $z_1^* \leq 0$. Similarly, for a p the k-th coordinate be 1 and other coordinates be 0, denoted as p_k. Then $p_k \cdot z^* \leq 0$ implies $z_k^* \leq 0$. So $z_k^* \leq 0, \forall k$, i.e., $z^* \in -\Omega$. Therefore, $z^* \in \zeta(p^*) \cap (-\Omega)$ and p^* is our target. Talent! □

THEOREM 3.7. *(Existence of General Equilibrium for Competitive Economics) The private ownership economy $\mathscr{E} = ((X_i, \succeq_i), (Y_j), (\omega_i), (\theta_{ij}))$ has an equilibrium if,*

(a) for every i,

(1) X_1 is closed, convex and has a lower bound for $\leq$;

(2) there is no satiation consumption in X_i;

(3) $\succeq_i$ is continuous;

(4) $\succeq_i$ is convex;

(b) for every j,

(1) $\mathbf{0} \in Y_j$;

(2) Y is closed and convex;

(3) $Y \cap (-Y) \subseteq \mathbf{0}$;

(4) $-\Omega \subseteq Y$.

Proof. We use Lemma 3.21 to show the existence of general equilibrium, but the Y_j may not be closed and convex, and X_i, Y_j may not be bounded. Thus, we would extend the original economy into more general cases, and show they share the same equilibrium.

(1) An $\mathscr{E}$-equilibrium if an $\overline{\dot{\mathscr{E}}}$-equilibrium.

Let $\dot{Y}_j$ denote the convex hull of Y_j and $\overline{\dot{Y}_j}$ denote the closed convex hull of Y_j, we show $\sum_{j=1}^n \overline{\dot{Y}_j} = Y$.

As $Y_j \subseteq \overline{\dot{Y}_j}, \forall j$, then $Y \subseteq \sum_{j=1}^n \overline{\dot{Y}_j}$. Theorem 2.139 tells $\sum_{j=1}^n \dot{Y}_j = \dot{Y}$, further, Theorem 2.112 tells $\sum_{j=1}^n \overline{\dot{Y}_j} \subseteq \overline{\sum_{j=1}^n \dot{Y}_j} = \overline{\dot{Y}} = Y$ (Y is closed and convex).

If $((x_i^*), (y_j^*), p^*)$ is an $\mathscr{E}$-equilibrium, then y_j^* maximizes $p^* \cdot y_j$ on Y_j, i.e., $Y_j \subseteq \{y_j \in \mathbb{R}^l | p^* \cdot y_j \leq p^* \cdot y_j^*\} = H_j$. It is trivial to check H_j is closed and convex (half space), then $\dot{Y}_j \subseteq \dot{H}_j = H_j$, then $\overline{\dot{Y}_j} \subseteq \overline{H_j} = H_j$, i.e., y_j^* also maximizes $p^* \cdot y_j$ on $\overline{\dot{Y}_j}$. Thus, $\eta_j(p) = \overline{\dot{\eta}}_j(p)$, then $\xi_j'(p) = \overline{\dot{\xi}}_j'(p)$, i.e., $((x_i^*), (y_j^*), p^*)$ is an $\overline{\dot{\mathscr{E}}}$-equilibrium.

(2) An $\overline{\dot{\mathscr{E}}}$-equilibrium is an $\tilde{\mathscr{E}}$-equilibrium.

In (1), we show $\sum_{j=1}^n \overline{\dot{Y}_j} = Y$. $Y \cap (-Y) \subseteq \{\mathbf{0}\}$ and $-\Omega \subseteq Y$ implies $Y \cap \Omega = \mathbf{0}$. Then from Lemma 3.20, the attainable set $\mathbb{A}$ is bounded for $\overline{\dot{\mathscr{E}}}$ (also for $\mathscr{E}$). Let K be a closed cube of $\mathbb{R}^l$ with center $\mathbf{0}$ containing these $m + n$ sets in its interior (containing the projection of $\mathbb{A}$). Define

$$\tilde{X}_i = X_i \cap K, \tilde{Y}_j = \overline{\dot{Y}}_j \cap K \tag{3.50}$$

Then $\tilde{X}_i$ is compact, convex and containing x_i^0 ($x_i^0 \ll \omega_i$ implies it is attainable) and $\tilde{Y}_j$ is compact, convex and owns $\mathbf{0}$.

If $((x_i^*), (y_j^*), p^*)$ is an $\overline{\dot{\mathscr{E}}}$-equilibrium, the state $((x_i^*), (y_j^*))$ is attainable, so $x_i^* \in \tilde{X}_i \subseteq X_i$ and $y_j^* \in \tilde{Y}_j \subseteq \overline{\dot{Y}}_j$, therefore $((x_i^*), (y_j^*), p^*)$ is an $\tilde{\mathscr{E}}$-equilibrium.

Summing up the conclusions of (1) and (2): an $\mathscr{E}$-equilibrium is an $\tilde{\mathscr{E}}$-equilibrium.

(3) An $\mathscr{E}$-equilibrium price system is > 0.

Let p^* be an $\mathscr{E}$-equilibrium price system. Because there is no satiation point in X_i, so $p^* \neq \mathbf{0}$ (otherwise $\xi(p, w)$ is empty). As $-\Omega \subseteq Y$, $p^* \geq 0$ (otherwise infinite profit), thus $p^* > 0$. Since p^* is nominal, we can normalize p^* in P (Equation 3.47).

(4) U.s.c. of $\tilde{\zeta}_j, \tilde{\xi}'_j$ on P.

Since $\tilde{Y}_j$ is compact, then supply correspondence $\tilde{\zeta}_j$ from P to $\tilde{Y}_j$ is u.s.c. and the profit function $\tilde{\pi}_j$ from P to $\mathbb{R}$ is continuous (Theorem 3.1).

Since $x_i^0 \ll \omega_i$, one has $p \cdot x_i^0 < p \cdot \omega_i, \forall p \in P$. Since $\mathbf{0} \in \tilde{Y}_j$, then $\tilde{\pi}_j(p) \geq 0$. So $p \cdot x_i^0 < p \cdot \omega_i + \sum_{j=1}^n \theta_{ij}\pi_j(p), \forall p \in P$. As $\tilde{\gamma}_i$ is continuous from Theorem 3.3, with Theorem 3.6, and Theorem 2.91, we have $\tilde{\xi}'_i$ is u.s.c. on P.

(5) There is $p^* \in P$ and $z \in -\Omega$ such that $z \in \tilde{\zeta}(p^*)$.

The set $\tilde{Z} = \sum_{i=1}^m \tilde{X}_i - \sum_{j=1}^n \tilde{Y}_i - \{\omega\}$ is compact as a sum of compact sets. Since $\tilde{\xi}'_i, \tilde{\zeta}_j$ are u.s.c. for $\forall i, j$ so $\tilde{\zeta}$ is u.s.c. from Theorem 2.98.

Since $\tilde{X}_i$ is convex, there is no satiation point in X_i and $\succeq_i$ is continuous, we have $\tilde{\xi}'_i$ is convex $\forall p \in P$ (Lemma 3.14). Similarly, from the convexity of $\tilde{Y}_j$, follows that $\tilde{\eta}_j(p)$ iw convex for $\forall p \in P$. Thus $\tilde{\zeta}(p)$ is convex, as a sum of convex sets $\forall p \in P$. Finally, $p \cdot \zeta(\tilde{p}) \leq 0$ from the proof of Lemma 3.21, thus, from Lemma 3.21, the assertion of the title is proved.

(6) Definition of the $\mathscr{E}$-equilibrium actions x_i^*, y_j^*.

Since $z \in \tilde{\zeta}(p^*)$, there is $x_i^* \in \tilde{\xi}'_i(p), \forall i$ and $y_j \in \tilde{\eta}_j(p^*), \forall j$, s.t.

$$\sum_{i=1}^m x_i^* - \sum_{j=1}^n y_j - \omega = z \tag{3.51}$$

Let $y = \sum_{j=1}^n y_j$, as $y_j \in \dot{\overline{Y}}_j$, then $y \in \sum_{j=1}^n \overline{Y}_j = Y$. The set Y is convex and closed, therefore $y \in Y, z \in -\Omega$ implies $y + z \in Y$ (Lemma 3.4) Hence, $\exists y_j^* \in Y_j, \forall j$ s.t.

$$\sum_{j=1}^n y_j^* = y + z \tag{3.52}$$

So

$$\sum_{i=1}^m x_i^* - \sum_{j=1}^n y_j^* - \omega = 0 \tag{3.53}$$

Thus, the statement $((x_i^*), (y_j^*), p^*)$ is attainable for $\tilde{\dot{\mathscr{E}}}$, hence $x_i^*, y_j^* \in int(K), \forall i, j$.

(7) Properties of x_i^*.

Define

$$w_i = p^* \cdot \omega_i + \sum_{j=1}^n \theta_{ij} p^* \cdot y_j \tag{3.54}$$

Since $x_i^* \in \tilde{\xi}'_i(p^*)$, the consumption set x_i^* is the greatest element of $\tilde{\gamma}_i(p^*, w)$ for $\succeq_i$. Then we claim x_i^* is the greatest element of $\gamma_i(p^*, w)$, i.e., $x_i^* \in \xi_i(p^*, w)$.

Suppose not, $\exists x'_i \in \gamma_i(p^*, w)$ s.t. $x'_i \succ_i x_i^*$. Let $x_i(t) = (1-t)x_i^* + tx'_i, t \in (0,1)$, then from the convexity of $\succeq_i$, $x_i(t) \succ_i x_i^*, \forall t \in (0,1)$. As $t \to 0$, $x_i(t) \in int(K)$ as $x_i^* \in int(K)$, hence in $\tilde{\gamma}_i(p^*, w) = K \cap \gamma_i(p^*, w)$, so x_i^* would therefore not be a greatest element of $\tilde{\gamma}_i(p^*, w)$ for $\succeq_i$, contradiction.

(8) Properties of y_j^*.

Since $x_i^* \in \xi_i(p^*, w)$ and $\succeq_i$ is convex, so $p^* \cdot x_i^* = w_i$. Summing up for $\forall i$, by $\sum_{i=1}^n \theta_{ij} = 1, \forall j$, we have $p^* \cdot z = 0$, hence from Equation 3.52,

$$p^* \cdot y^* = p^* \cdot y \tag{3.55}$$

Since $y_j \in \tilde{\eta}_i(p^*)$, therefore from Lemma 3.6, $y \in \zeta(p)$, i.e., y maximizes total profit on $\tilde{Y}$, and so is y^* as $p^* \cdot y^* = p^* \cdot y$.

In particular $p^* \cdot y_j^* = p^* \cdot y_j$, hence

$$w_i = p^* \cdot \omega_i + \sum_{j=1}^n \theta_{ij} p^* \cdot y_j^* \tag{3.56}$$

this with $x_i^* \in \xi_i(p^*, w)$ implies $x_i^* \in \xi_i(p^*, p^* \cdot \omega_i + \sum_{j=1}^n \theta_{ij} p^* \cdot y_j^*)$ (the (a) condition of equilibrium, Definition 3.6). Equation 3.53 is the (c) condition of equilibrium. Since y_j^* maximizes total profit on $\tilde{Y}$ and $y_j^* \in int(K)$, similar to proof of (7), we know y_j^* maximizes profit relative to p^* on $\dot{\overline{Y}}_j$, hence on Y_j. Thus, the (b) condition of equilibrium is satisfied. □

Part II
Advanced Macroeconomics

4 Preliminary Mathematics

4.1 The Kuhn-Tuck Condition and the Envelop Theorem

THEOREM 4.1. *(Kuhn-Tuck Conditions) For a nonlinear programming problem, Kuhn-Tuck condition is its solution's necessary condition. Precisely, the nonlinear programming problem and its Kuhn-Tuck condition are as follows:*

$$\max_{\mathbf{x}} f(\mathbf{x}) \tag{4.1}$$
$$s.t. \begin{cases} \mathbf{g}(\mathbf{x}) \leq \mathbf{0} \\ \mathbf{h}(\mathbf{x}) = \mathbf{0} \end{cases}$$

where bold expressions mean vectors. We construct the Langrage function:

$$L(\mathbf{x}, \lambda, \mu) = f(\mathbf{x}) - \lambda \mathbf{g}(\mathbf{x}) + \mu \mathbf{h}(\mathbf{x}) \tag{4.2}$$

The Kuhn-Tuck condition of the optimal solution $\mathbf{x}^*, \lambda^*, \mu^*$ satisfies (necessary condition):

$$\mathbf{g}(\mathbf{x}^*) \leq \mathbf{0} \tag{4.3}$$
$$\mathbf{h}(\mathbf{x}^*) = \mathbf{0} \tag{4.4}$$
$$\lambda^* \mathbf{g}(\mathbf{x}^*) = \mathbf{0}, \lambda^* \geq \mathbf{0} \tag{4.5}$$
$$\nabla_{\mathbf{x}} L(\mathbf{x}, \lambda, \mu) = \mathbf{0} \tag{4.6}$$

The condition (4.3, 4.4) are the constraints; the condition (4.5) is known as complementary slackness; the last condition is the F.O.C. of the corresponding linear programming problem (∇ denotes gradient).

The Kuhn-Tuck Condition is the workhorse method to solve nonlinear programming problem in macroeconomics.

THEOREM 4.2. *(The Envelop Theorem) For a nonlinear programming problem, given θ*

$$\max_{\mathbf{x}} f(\mathbf{x}, \theta) \tag{4.7}$$
$$s.t. \mathbf{g}(\mathbf{x}, \theta) \leq \mathbf{0} \tag{4.8}$$

Let $L(\mathbf{x}, \lambda, \theta) = f(\mathbf{x}, \theta) - \lambda \mathbf{g}(\mathbf{x}, \theta)$ be the Langrage function, $\mathbf{x}^*, \theta^*$ be the solution and $V(\theta) = f(\mathbf{x}^*(\theta), \theta)$ be the indirect target function, then

$$\frac{\partial V(\theta)}{\partial \theta} = \frac{\partial L(\mathbf{x}^*(\theta), \lambda^*(\theta), \theta)}{\partial \theta} \tag{4.9}$$

Proof. Since $\mathbf{x}^*, \lambda^*$ is the solution, the Kuhn-Tuck condition holds:

$$\frac{\partial L(\mathbf{x}(\theta), \lambda(\theta), \theta)}{\partial \mathbf{x}}|_{\mathbf{x}=\mathbf{x}^*, \lambda=\lambda^*} = \frac{\partial f(\mathbf{x}, \theta) - \lambda \mathbf{g}(\mathbf{x}, \theta)}{\partial \mathbf{x}}|_{\mathbf{x}=\mathbf{x}^*, \lambda=\lambda^*} = \frac{\partial f(\mathbf{x}, \theta)}{\partial \mathbf{x}}|_{\mathbf{x}=\mathbf{x}^*, \lambda=\lambda^*} = \mathbf{0} \tag{4.10}$$

Note that $\lambda^* \mathbf{g}(\mathbf{x}^*) = \mathbf{0}$. So

$$\frac{\partial V(\theta)}{\partial \theta} = \frac{\partial f(\mathbf{x}(\theta), \theta)}{\partial \theta}|_{\mathbf{x}=\mathbf{x}^*} = \frac{\partial f(\mathbf{x}(\theta), \theta)}{\partial \mathbf{x}(\theta)} \frac{\partial \mathbf{x}(\theta)}{\partial \theta}|_{\mathbf{x}=\mathbf{x}^*} + \frac{\partial f(\mathbf{x}(\theta), \theta)}{\partial \theta}|_{\mathbf{x}=\mathbf{x}^*} \tag{4.11}$$
$$= \frac{\partial f(\mathbf{x}(\theta), \theta)}{\partial \theta}|_{\mathbf{x}=\mathbf{x}^*} \tag{4.12}$$
$$= \frac{\partial f(\mathbf{x}(\theta), \theta) - \lambda \mathbf{g}(\mathbf{x}, \theta)}{\partial \theta}|_{\mathbf{x}=\mathbf{x}^*, \lambda=\lambda^*} \tag{4.13}$$
$$= \frac{\partial L(\mathbf{x}(\theta), \lambda(\theta), \theta)}{\partial \theta}|_{\mathbf{x}=\mathbf{x}^*, \lambda=\lambda^*} \tag{4.14}$$
$$= \frac{\partial L(\mathbf{x}^*(\theta), \lambda^*(\theta), \theta)}{\partial \theta} \tag{4.15}$$

□

The Envelop Theorem indicates that we can calculate derivatives of the value function (indirect target function) in recursive programming problems.

4.2 The Contraction Mapping Theorem and the Berge's Maximization Theorem

This Contraction Mapping Theorem and Berge's Maximization Theorem is for the dynamic programming in advanced macroeconomics. (Stocky and Lucas, 1989)

DEFINITION 4.1. *(Complete Metric Space) A metric space (S, ρ) is complete if every Cauchy sequence in S converges to an element in S.*

DEFINITION 4.2. *(Contraction Mapping) Let (S, ρ) be a metric space and $T : S \to S$ be a function mapping of S into itself. T is a contraction mapping (with modulus β) if for some $\beta \in (0, 1)$, $\rho(Tx, Ty) \leq \beta\rho(x, y), \forall x, y \in S$.*

THEOREM 4.3. *(**Contraction Mapping Theorem**) If (S, ρ) is a complete metric space and $T : S \to S$ is a contraction mapping with modulus β, then*

1. *$\exists! v \in S$ s.t. $Tv = v$ (only a fixed point);*
2. *$\forall v_0 \in S, \forall n \in \mathbb{N}, \rho(T^n v_0, v) \leq \beta^n \rho(v_0, v)$.*

Proof. (1) We first find such a candidate v and then prove it is unique. Given any $v_0 \in S$, let $v_1 = Tv_0, v_2 = Tv_1 = T^n v_0, \cdots$, we obtain a sequence $\{v_q\}_{q=0}^{\infty}$. As T is a contraction mapping with modulus β, $\rho(v_{n+1}, v_n) = \rho(Tv_n, Tv_{n-1}) \leq \beta\rho(v_n, v_{n-1})$, which implies the metric of two sequential elements are contractive. We say

$$\rho(v_{n+1}, v_n) \leq \beta\rho(v_n, v_{n-1}) \leq \beta^2\rho(v_{n-1}, v_{n-2}) \leq \cdots \leq \beta^n\rho(v_1, v_0) \tag{4.16}$$

We now verify the Cauchy criteria. $\forall \epsilon > 0$, let $N > \frac{ln(\frac{1-\beta}{\epsilon}\rho(v_1,v_0))}{ln\beta}$, so $\forall m > n > N$

$$\begin{aligned}
\rho(v_m, v_n) &\leq \rho(v_m, v_{m-1}) + \rho(v_{m-1}, v_{m-2}) + \cdots + \rho(v_{n+1}, v_n) && (4.17)\\
&\leq (\beta^{m-1} + \beta^{m-2} + \cdots + \beta^n)\rho(v_1, v_0) && (4.18)\\
&= \beta^n(1 + \beta + \cdots + \beta^{m-n-1})\rho(v_1, v_0) && (4.19)\\
&= \beta^n \frac{1 - \beta^{m-n}}{1 - \beta}\rho(v_1, v_0) && (4.20)\\
&< \beta^n\rho(v_1, v_0) && (4.21)\\
&< \epsilon && (4.22)
\end{aligned}$$

Thus, $\{v_q\}_{q=0}^{\infty}$ is a convergent sequence. As (S, ρ) is complete, so $\exists v \in S$ s.t. $v_q \to v$. To show $Tv = v$, we say

$$\rho(Tv, v) \leq \rho(Tv, T^n v_0) + \rho(T^n v_0, v) \leq \beta\rho(v, T^{n-1}v_0) + \rho(T^n v_0, v) = \beta\rho(v_{n-1}, v) + \rho(v_n, v) \tag{4.23}$$

As $v_q \to v$, so $\lim\limits_{q\to\infty} \rho(v_q, v) = 0$.

$$\rho(Tv, v) = \lim_{n\to 0}(Tv, v) \leq \lim_{n\to 0} \beta\rho(v_{n-1}, v) + \rho(v_n, v) = 0 \tag{4.24}$$

So $\rho(Tv, v) = 0$, i.e., $Tv = v$ from the definition of ρ.

Now we show v is unique. Suppose not, $\exists v, v', v \neq v'$ s.t. $Tv = v, Tv' = v'$, then $\rho(v, v') = \rho(Tv, Tv') \leq \beta\rho(v, v')$ which is impossible as $\beta \in (0, 1)$. So v is unique.

(2) from any start point v_0, we want to show every move makes the path nearer to the fixed point v. The proof is easy. $\rho(T^n v_0, v) = \rho(T^n v_0, T^n v) \leq \beta\rho(T^{n-1}v_0, v) = \beta\rho(T^{n-1}c_0, T^{n-1}v) \leq \beta^2\rho(T^{n-1}v_0, v) \leq \cdots \leq \beta^n\rho(v_0, v)$. □

To show a mapping is a contraction mapping may be difficult, but there is a sufficient condition which applies perfectly in most macroeconomics problems.

THEOREM 4.4. *(**Bloackwell's Sufficient Conditions for a Contraction**) Let $X \in \mathbb{R}^l$ and $B(X)$ be a space of bounded functions $f : X \to \mathbb{R}$, with the sup norm. Let $T : B(X) \to B(X)$ be an operator satisfying,*

1. *(monotonicity) $f, g \in B(X)$ and $f(x) \leq g(x)$, for all $x \in X$, implies $(Tf)(x) \leq (Tg)(x)$, for all $x \in X$;*
2. *(discounting) there exists $\beta \in (0, 1)$ s.t. $[T(f + a)](x) \leq (Tf)(x) + \beta a$, all $f \in B(X), a \geq 0, x \in X$.*

Then T is a contraction mapping with modulus β.

Proof. If $f(x) \leq g(x), \forall x \in X$, we say $f \leq g$. $\forall f, g \in B(X), f \leq g + ||f - g||$, then

$$Tf \leq T(g + ||f - g||) \leq Tg + \beta ||f - g|| \tag{4.25}$$

Reversing the roles of f, g gives by the same logic, we have

$$Tg \leq Tf + \beta ||f - g|| \tag{4.26}$$

So we have $||Tf - Tg|| \leq \beta ||f - g||$. $\square$

All recursive programming problems included in this notes all satisfy the Bloackwell's sufficient condition.

THEOREM 4.5. ***(Berge's Maximization Theorem)*** *If ϕ is a continuous numerical function in $X \times Y$ and Γ is a compact-valued continuous mapping of X into Y s.t. $\forall x, \Gamma x \neq \varnothing$. Then the numerical function M defined by*

$$M(x) = \max\{\phi(x, y) | y \in \Gamma x\} \tag{4.27}$$

is continuous. The mapping $\Phi : X \rightrightarrows Y$ defined by

$$\Phi x = \{y \in \Gamma x | \phi(x, y) = M(x)\} \tag{4.28}$$

is nonempty compact-valued u.s.c. (the maximizer is attainable as ϕ is continuous and the domain is compact).

5 Growth Models

5.1 The Solow Model

Let Y_t denote output, C_t denote consumption, I_t denote investment, K_t denote capital and L denote labor input (constant). Relations of macroeconomic variables are

$$C_t + I_t = F(K_t, L) = Y_t \tag{5.1}$$
$$K_{t+1} = (1 - \delta) K_t + I_t \tag{5.2}$$
$$I_t = sF(k_t, L) \tag{5.3}$$

where $F(\cdot)$ is the production function (always CRS, constant returns to scale, like $F(K, L) = AK^\alpha L^{1-\alpha}, \alpha \in (0, 1)$), δ is the depreciation rate (linear) and s is the saving rate. To reflect economy's character and make sure the model is solvable, four assumption are given as follows:

1. F is strictly increasing and concave and twice differentiable in K;
2. $F(0, L) = 0$;
3. $F'_K(0, L) \geq \frac{\delta}{s}$;
4. $\lim\limits_{R \to +\infty} \delta F'_K(K, L) + (1 - \delta) < 1$.

We show $\exists K^*, \forall K_0 > 0$, the economy will reach a stable status, $K = K^*$ now.

Let $H(K_t) = K_{t+1} - K_t = SF(K_t, L) - \delta K_t$. $H(K_0) = 0, H'(K_t) = SF'_{K_t}(K_t, L) - \delta, H''(K_t) = sF''(K_t, L) \leq 0$(marginal return decreasing). Then $H'(K_t)$ is strictly decreasing and $H'(K_0) > 0$. So $H(K_t)$ increases first and decreases then (gets 0 when $K_t = K^*$), which indicates that if $K_t < K^*$, K^* will increase and decrease otherwise.

There are two applications: (1) economic growth (s decides the stable status); (2) business cycle ($A_t \in \{A_H, A_L\}$ denotes two states of economy).

5.2 Models of Optimal Growth

In finite periods $T + 1$, we maximize a representative consumer's lifetime utility function, $u(c_0, c_1, \cdots, c_T) = \sum_{t=0}^{T} \beta^t u(c_t)$, where $u(\cdot)$ denotes the utility function satisfying $u' > 0$ and $\lim\limits_{c \to 0} u'(c) = +\infty$, and $\beta \in (0, 1)$

denotes the discount factor. The maximize problem is

$$\max_{\{c_t,k_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t u(c_t) \tag{5.4}$$
$$s.t. \begin{cases} c_t + k_{t+1} \leq f(k_t) = F(k_t, N) + (1-\delta)k_t, \forall t = 0, 1, \cdots, T \\ c_t \geq 0, \forall t = 0, 1, \cdots, T \\ k_{t+1} \geq 0, \forall t = 0, 1, \cdots, T \\ k_0 > 0 \text{ given} \end{cases}$$

In the problem above, (1) the first constraint is daily consumption and investment is limited by the output, and the inequality sign will become equality sigh in the maximize solution as $u(\cdot)$ is an strictly increasing function so that any left output should be consumed or invested; (2) as $\lim_{c\to 0} u'(c) = +\infty$ exists, c_t must be positive as it's little increase above zero will bring much increase to utility (that indicates that $k_t > 0, t = 0, 1, \cdots, T$). Therefore, the first two constraints are tight. The solution of that nonlinear programming problem is obtained with Kuhn-Tuck condition. Moreover, $k^{T+1} = 0$, i.e., the output in the last period must be consumed entirely.

According to the theorem 4.1, we construct the Langrage function as:

$$L = \sum_{t=0}^{T} \beta^t [u(c_t) + \mu_t k_{t+1}], c_t = f(k_t) - k_{t+1} \tag{5.5}$$

The first order condition is

$$\frac{\partial L}{\partial k_{t+1}} = -\beta^t u'(c_t) + \beta^t \mu_t + \beta^{t+1} u'(c_{t+1}) f'(k_{t+1}) = 0, t = 0, 1, \cdots, T-1 \tag{5.6}$$
$$\frac{\partial L}{\partial k_{T+1}} = -\beta^T u'(c_T) + \beta^T \mu_T = 0 \tag{5.7}$$

Kuhn-Tucker condition: $\mu_t k_{t+1} = 0, k_{t+1} \geq 0, \mu_t \geq 0, t = 0, 1, \cdots, T$. With $-\beta^T u'(c_T) + \beta^T \mu_T = 0$ and $u'(c_T) > 0$, we have $\mu_T > 0$ (so $k_{T+1} = 0$ as expected). As $k_t > 0$, we have $\mu_t = 0$ for $t = 1, 2, \cdots, T$.

The summary statement of the first order conditions is the **Euler equation**:

$$u'(c_t) = \beta u'(c_{t+1}) f'(k_{t+1}), t = 0, 1, \cdots, T-1 \tag{5.8}$$

The equation's meaning is marginal cost of investment $=$ discount $\times$ marginal benefit of investment $\times$ marginal product of investment.

Use the Euler equation in the following toy example.

Example 5.1. $u(c) = log(c), f(k) = Ak$.

$$\max_{\{c_t,k_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t u(c_t) \tag{5.9}$$
$$s.t.\ c_t + k_{t+1} = Ak_t, t = 0, 1, 2 \cdots, T$$

(period-by-period budget constraint)

We can eliminate k by preliminary change and get the lifetime budget constraint,

$$c_0 + \frac{c_1}{A} + \cdots + \frac{c_T}{A^T} + \frac{k_{T+1}}{A^{T+1}} = Ak_0 \tag{5.10}$$

Use $k_{T+1} = 0$ to eliminate k_{T+1}. With the Euler equation, for $t = 0, 1, \cdots, T-1$

$$u'(c_t) = \beta u'(c_{t+1}) f'(k_{t+1}) \tag{5.11}$$
$$\frac{1}{c_t} = \beta \frac{1}{c_{t+1}} A \tag{5.12}$$
$$c_{t+1} = \beta A c_t \tag{5.13}$$
$$\Rightarrow c_t = (\beta A)^t c_0 \tag{5.14}$$

Bring c_t to the lifetime budget constraint, we have

$$c_0 = \frac{1-\beta}{1-\beta^{T+1}} Ak_0, c_t = (\beta A)^t c_0, t = 1, 2, \cdots, T \tag{5.15}$$

$\frac{c_{t+1}}{c_t} = \beta A, \beta \in (0,1)$ *implies that consumption allocation is decided by impatience and technology. The more patient you are (bigger β), more consumption will be allocated to future.*

$A = f'(k_t)$ *denotes the real gross interest rate, as the derivative of output on capital show the efficiency of capital usage.*

Example 5.2. $u(c) = log(c), f(k) = Ak^\alpha, \alpha \in (0,1)$.

$$\max_{\{c_t,k_{t+1}\}_{t=0}^T} \sum_{t=0}^T \beta^t u(c_t) \tag{5.16}$$
$$s.t.\ c_t + k_{t+1} = Ak_t^\alpha, t = 0, 1, 2 \cdots, T$$

With Euler equation, for $t = 1, 2, \cdots, T$ we drop c_t and search path of k_t,

$$\frac{1}{c_t} = \beta \frac{1}{c_{t+1}} A\alpha k_{t+1}^{\alpha-1} \tag{5.17}$$
$$\frac{1}{Ak_t^\alpha - k_{t+1}} = \frac{\alpha\beta A k_{t+1}^{\alpha-1}}{Ak_{t+1}^\alpha - k_{t+2}} \tag{5.18}$$
$$Ak_{t+1}^\alpha - k_{t+2} = \alpha\beta A k_{t+1}^{\alpha-1}(Ak_t^\alpha - k_{t+1}) \tag{5.19}$$

Induce from $t = T - 1$,

$$\frac{1}{Ak_{T-1}^\alpha - k_T} = \frac{\alpha\beta A k_T^{\alpha-1}}{Ak_T^\alpha - 0} = \frac{\alpha\beta}{k_T} \tag{5.20}$$
$$k_T = \alpha\beta(Ak_{T-1}^\alpha - k_T) \tag{5.21}$$
$$k_T = \frac{\alpha\beta A k_{T-1}^\alpha}{1+\alpha\beta} \tag{5.22}$$

With start and final statuses and iteration equations, we can solve the paths. Moreover, $c_t = Ak_t^\alpha - k_{t+1}, t = 0, 1, \cdots, T$. Precisely, when $t = T - 2$, take equation (5.21,5.22) into equation (5.19):

$$\frac{k_T}{\alpha\beta} = \alpha\beta A k_{T-1}^{\alpha-1}(Ak_{T-2}^\alpha - k_{T-1}) \tag{5.23}$$
$$\frac{Ak_{T-1}^\alpha}{1+\alpha\beta} = \alpha\beta A k_{T-1}^{\alpha-1}(Ak_{T-2}^\alpha - k_{T-1}) \tag{5.24}$$
$$k_{T-1} = (\alpha\beta)(1+\alpha\beta)(Ak_{T-2}^\alpha - k_{T-1}) \tag{5.25}$$
$$\Rightarrow k_{T-1} = \frac{\alpha\beta + \alpha^2\beta^2}{1+\alpha\beta+\alpha^2\beta^2} Ak_{T-2}^\alpha \tag{5.26}$$

The difference of expressions between k_T and k_{T-1} is the coefficients, the latter multiply $\alpha\beta$ on the numerator and denominator. Apply the iteration process on $t = T-3, T-4, \cdots, 2$ and we get (by conjecture)

$$k_{t+1} = \frac{\alpha\beta + \alpha^2\beta^2 + \cdots + \alpha^{T-t}\beta^{T-t}}{1+\alpha\beta+\alpha^2\beta^2+\cdots+\alpha^{T-t}\beta^{T-t}}, Ak_t^\alpha \tag{5.27}$$

That all we need to obtain the path of k and c.

Next, we extend this optimal growth model to infinite horizon case.

$$\max_{\{c_t,k_{t+1}\}_{t=0}^T} \sum_{t=0}^T \beta^t u(c_t) \tag{5.28}$$
$$s.t. \begin{cases} c_t + k_{t+1} \le f(k_t), t = 0, 1, \cdots \\ c_t \ge 0, k_{t+1} \ge 0 k_0 \text{ given} \end{cases}$$

From the solution of finite Example 5.2, let $T \to \infty$, $\begin{cases} k_{t+1} = \alpha\beta A k_t^{\alpha} \\ c_t = (1-\alpha\beta)Ak_t^{\alpha} \end{cases}$. If they are the solution, there is a globally convergent steady state level of capital.

Let $H(k_t) = k_{t+1} - k_t = \alpha\beta A k_t^{\alpha} - k_t$, $H(0) = 0$, $H'(k_t) = \alpha^2\beta A k_t^{\alpha-1} - 1$, $\lim_{k\to 0} H'(k) = +\infty$, $\lim_{k\to\infty} H'(k) = -1$, $H^{''}(k_t) = \alpha^2(\alpha-1)\beta A k_t^{\alpha-2} < 0, \alpha \in (0,1)$. Then $\exists$ unique $k^* > 0 s.t. H(k^*) = 0$, i.e., $k_{t+1} = k_t = k^* \Rightarrow k^* = (\alpha^2\beta A)^{\frac{1}{1-\alpha}}$

A rise in β or A increases the long-run level of capital, meaning larger output and consumption is steady state.

Similar to solow's model, saving rate $s = \frac{y-c}{y} = \frac{Ak^{\alpha} - (1-\alpha\beta)Ak^{\alpha}}{Ak^{\alpha}} = \alpha\beta, f'(k) = \alpha A k^{\alpha-1}$. A rise in A does not change the saving rate. (balanced income effect and substitution effect)

Generally, we desire sufficient solution more than just necessary solution given by Kuhn-Tuck condition. That target need more delicate but elegant constraints on the nonlinear programming, named tranversality condition:

$$\lim_{t\to\infty} \beta^t u'(c_t) f'(k_t) k_t = 0 \tag{5.29}$$

The implication of this condition is "in the future, we all die". If the tranversality condition holds, we can solve the optimal growth problem, like $u(c) = log(c), f(k) = Ak^{\alpha}$.

$$\begin{aligned}
\lim_{t\to\infty} \beta^t u'(c_t) f'(k_t) k_t &= \lim_{t\to\infty} \beta^t \frac{1}{c_t} A\alpha k_t^{\alpha-1} k_t && (5.30)\\
&= \lim_{t\to\infty} \beta^t \frac{1}{(1-\alpha\beta)Ak_t^{\alpha}} (\alpha A k_t^{\alpha-1}) k_t && (5.31)\\
&= \lim_{t\to\infty} \beta^t \frac{\alpha}{1-\alpha\beta} = 0 && (5.32)
\end{aligned}$$

So we have $\begin{cases} k_{t+1} = \alpha\beta A k_t^{\alpha} \\ c_t = (1-\alpha\beta)Ak_t^{\alpha} \end{cases}$ (Markov Property). Note that all historical information accumulated in present, that is the key for recursive analysis.

5.3 Recursive Analysis (Dynamic Programming)

DEFINITION 5.1. *(Stationary) A problem is **stationary** if whenever the structure of the choice, problem that a decision maker faced is identical at every point in time.*

PROPOSITION 5.1. *If the problem is stationary, then for any two periods $t \neq s$. $k_t = k_s$ implies $k_{t+j} = k_{s+j}, \forall j > 0$. That is, he would not change his mind if he could decide all over again.*

Moreover, we define $g(k_t) = k_{t+1}$ the policy function. The stationary condition implies policy function does not vary with time. Finite horizon problem is not stationary as $g(k_{T+1}) = 0$.

The basic idea of recursive analysis is to solve equation with iteration, i.e, $\max_{x,y} f(x,y) = \max_x\{\max_y f(x,y)\}$. In the optimal growth model, we maximize over $\{k_{s+1}\}_{s=t}^{\infty}$ by choice of $\{k_{s+1}\}_{s=t+1}^{\infty}$ conditional on k_{t+1} and then choose k_{t+1}. In mathematics,

$$V(k_t) = \max \sum_{s=t}^{\infty} \beta^{s-t} F(k_s, k_{s+1}) \tag{5.33}$$
$$s.t. k_{s+1} \in P(k_s), \forall s \geq t.$$

where $P(k_s)$ is the feasible choice set for k_{s+1} given k_s.

$$\begin{aligned}
V(k_t) &= \max_{k_{t+1}\in P(k_t)} \{F(k_t, k_{t+1}) + \max_{\{k_{s+1}\}_{s=t}^{\infty}, k_{s+1}\in P(k_s), \forall s\geq t} \sum_{s=t+1}^{\infty} \beta^{s-t} F(k_s, k_{s+1})\} && (5.34)\\
&= \max_{k_{t+1}\in P(k_t)} \{F(k_t, k_{t+1}) + \beta \max_{\{k_{s+1}\}_{s=t}^{\infty}, k_{s+1}\in P(k_s), \forall s\geq t} \sum_{s=t+1}^{\infty} \beta^{s-(t+1)} F(k_s, k_{s+1})\}\\
&= \max_{k_{t+1}\in P(k_t)} \{F(k_t, k_{t+1}) + \beta V(k_{t+1})\}
\end{aligned}$$

where k_t is current capital / state variable and k_{t+1} is next period capital / control variable. We rewrite the problem in a general form,

$$V(k) = \max_{k' \in P(k)} \{F(k, k') + \beta V(k')\} \tag{5.35}$$

Equation 5.35 is the famous **Bellman Equation**. Our target is to find a suitable function, named value function and have $k'^* = g(k)$.

To find a suitable function, we need the Contraction Mapping Theorem.

THEOREM 5.1. *(Contraction Mapping Theorem) To solve Equation 5.35, we first give constraints to it:*

1. *F is continuously differentiable in its two arguments;*
2. *F is strictly increasing in its first argument and decreasing in its second argument (strictly not necessary);*
3. *F is strictly concave and bounded;*
4. *$\beta \in (0, 1)$;*
5. *Γ is a non-empty, compact-valued, and continuous, convex.*

Then:

1. *There exists a function $V(\cdot)$ that solves the Bellman equation and the solution is unique;*
2. *$V(\cdot)$ is strictly concave;*
3. *$V(\cdot)$ is strictly increasing;*
4. *$V(\cdot)$ is continuously differentiable;*
5. *It is possible to find $V(\cdot)$ by the following iteration process:*
   (a) *Pick any initial V_0 function, eg. $V_0(k) = 0, \forall k$;*
   (b) *Find:*

$$V_1(k) = \max_{k' \in P(k)} \{F(k, k') + \beta V_0(k')\} \tag{5.36}$$

$$V_2(k) = \max_{k' \in P(k)} \{F(k, k') + \beta V_2(k')\} \tag{5.37}$$

$$\dots$$

$$V_{n+1}(k) = \max_{k' \in P(k)} \{F(k, k') + \beta V_n(k')\} \tag{5.38}$$

$$\dots$$

   and $\{V_j\}_{j=0}^{\infty}$ converges to V.

6. *Optimal behavior can be characterized by a function g with $k' = g(k)$ that is increasing so long as F_2 ($\frac{\partial F}{\partial k'}$) is increasing in k.*

Remark: pros, theorem-based (developed VFI toolkit) and cons, converges slowly. To find more materials on this issue, see Dynamic Economics by adda & cooper or the teaching video of Zhigang Feng in Bilibili.

We apply the contraction mapping theorem to a special example.

Example 5.3. *Solving a parametric dynamic programming problem. $u(c) = log(c), f(k) = Ak^{\alpha}$, then $V(k) = \max_{c,k'}\{log(c) + \beta V(k')\}$ s.t. $c = Ak^{\alpha} - k'$, i.e.,*

$$V(k) = \max_{Ak^{\alpha} \geq k' \geq 0} \{log(Ak^{\alpha} - k') + \beta V(k')\} \tag{5.39}$$

Solution: With Contraction Mapping Theorem, we set $V_0(k) = 0$ and have (note that we do not write the constraints)

$$
\begin{aligned}
V_1(k) &= \max_{k'}\{log(Ak^\alpha - k') + \beta V_0(k)\} && (5.40)\\
&= \max_{k'}\{log(Ak^\alpha - k')\} && (5.41)\\
&\Rightarrow k^{'*} = 0, V_1(k) = log(A) + \alpha log(k') && (5.42)\\
V_2(k) &= \max_{k'}\{log(Ak^\alpha - k') + \beta V_1(k)\} && (5.43)\\
&= \max_{k'}\{log(Ak^\alpha - k') + \beta(logA + \alpha logk')\} && (5.44)\\
& f.o.c. \frac{1}{Ak^\alpha - k'} = \beta\alpha\frac{1}{k}, k' = \beta\alpha(Ak^\alpha - k'), k^{'*} = \frac{\beta\alpha Ak^\alpha}{1+\beta\alpha} && (5.45)\\
&\Rightarrow V_2(k) = log(Ak^\alpha - \frac{\alpha\beta Ak^\alpha}{1+\alpha\beta}) + \beta(logA + \alpha log(\frac{\alpha\beta Ak^\alpha}{1+\alpha\beta})) && (5.46)\\
&= (\alpha + \alpha^2\beta)logk + log(A - \frac{\alpha\beta A}{1+\alpha\beta}) + \beta logA + \alpha\beta log(\frac{\alpha\beta A}{1+\alpha\beta}) && (5.47)\\
conjecture: & V(k) = a + blogk && (5.48)\\
&\Rightarrow V(k) = a + blogk = \max_{k'}\{log(Ak^\alpha - k') + \beta(a + blogk')\} && (5.49)\\
& f.o.c.: \frac{1}{Ak^\alpha - k'} = \beta b\frac{1}{k'}, k^{'*} = \frac{\beta b}{1+\beta b}Ak^\alpha && (5.50)\\
V(k) &= a + blogk = log(Ak^\alpha - \frac{\beta b}{1+\beta b}Ak^\alpha) + \beta(a + blog(\frac{\beta b}{1+\beta b})Ak^\alpha) && (5.51)\\
&\Rightarrow \begin{cases} a = \cdots \\ b = \frac{\alpha}{1-\alpha\beta} \end{cases} && (5.52)
\end{aligned}
$$

Then $k^{'*} = \frac{\beta\alpha/(1-\alpha\beta)}{1+\beta\alpha/(1-\alpha\beta)}Ak^\alpha = \alpha\beta Ak^\alpha$, just the same solution of the finite period version.

The **Projection method** is more efficient than the contraction mapping theorem. What we use is called functional Euler equation. Let policy function be $k^{'*} = g(k)$, assume we have found the suitable $V(k)$ for Equation 5.35, i.e., $V(k) = F(k, g(k)) + \beta V(g(k))$. F.O.C of Equation 5.35 is $F_2 + \beta V'(k') = 0$, i.e., $F_2 + \beta V'(g(k)) = 0$.

With $V(k) = F(k, g(k)) + \beta V(g(k))$, we can derive that (the Envelop Theorem)

$$
\begin{aligned}
V'(k) &= F_1 + F_2 g'(k) + \beta V'(g(k))g'(k) && (5.53)\\
&= F_1 + g'(k)(F_2 + \beta V'(g(k))) && (5.54)\\
&= F_1 && (5.55)
\end{aligned}
$$

Then, we have

$$V'(k) = F_1(k, g(k)) \tag{5.56}$$

which is called **Benveniste-Scheinkman's condition**, and the idea behind that is we just need to decide k while k' is indirectly decided (also, we can replace k with k'). We are only interested in $g(k)$, and we can replace $V'(k')$ with $F_1(k', g(k'))$ in the f.o.c. and get

$$F_2(k, g(k)) + \beta F_1(g(k), g(g(k))) = 0 \tag{5.57}$$

which is called the **Euler equation**, derived from B-S condition and f.o.c.

Example 5.4. $F(k, k') = u(f(k) - g(k)), u(c) = log(c), F(k) = Ak^\alpha$.
Then we obtain the Euler equation:

$$
\begin{aligned}
& F_2(k, g(k)) + \ betaF_1(g(k), g(g(k))) = 0, \forall k && (5.58)\\
& \frac{1}{Ak^\alpha - g(k)} = \beta\frac{\alpha Ag^{\alpha-1}(k)}{Ag^\alpha(k) - g(g(k))}, \forall k && (5.59)
\end{aligned}
$$

According to contraction mapping theorem, the solution is unique. We try $g(k) = SAk^\alpha$ and get $S = \alpha\beta$, then $g^*(k) = \alpha\beta Ak^\alpha$.

6 Competitive Equilibrium in Dynamic Models

- planner's problem: no prices;
- competitive equilibrium: take prices as given, but these prices being set ensure market-clearing;
- representative households: own capital and labor;
- representative firms: tech $\rightarrow$ production;
- a competitive equilibrium is a set of prices and quantities $\{c_t, n_t, k_{t+1}, \cdots\}$ such that
  1. households choose quantities to maximize utility given wealth $\{k_t\}$, factor $\{l_t\}$ endowments evaluated at prices $\{p_t, w_t, r_t\}$;
  2. firms choose production to maximize profits at given prices;
  3. the quantities chosen by households and firms are feasible, the aggregate quantity of each commodity demanded is produced using the factors supplied (market clearing, 3 markets)

In dynamic economics, we have to describe how trade over time occurs, two choices:

- date-0 trade: all tradings take place at date 0, financial assets unnecessary (order we act);
- sequential trade: agents may borrow or save, one-period assets.

6.1 An Endowment Economy with Date-0 Trade

- No production, no firm;
- An infinitely-lived representative household has $\{w_t\}_{t=0}^{\infty}$ endowments;
- All households can do is trade.
- Utility $\sum_{t=0}^{\infty} \beta^t u(c_t)$.
- p_t, price of c_t and $p_0 \equiv 1$ (relative price of c_t compared to c_0);
- Value of endowments $\sum_{t=0}^{\infty} p_t w_t$;
- Budget constraints: $\sum_{t=0}^{\infty} p_t c_t \leq \sum_{t=0}^{\infty} p_t w_t$;
- Arror-Debrue economy;

A competitive equilibrium is a set of prices $\{p_t\}_{t=0}^{\infty}$ and quantities $\{c_t^*\}_{t=0}^{\infty}$, s.t.

$$\begin{cases} \{c_t^*\}_{t=0}^{\infty} \text{ solves } \max\limits_{\{c_t^*\}_{t=0}^{\infty}} \{\sum_{t=0}^{\infty} \beta^t u(c_t)\} \text{ s.t. } \sum_{t=0}^{\infty} p_t c_t \leq \sum_{t=0}^{\infty} p_t w_t, c_t \geq 0, \forall t \\ c_t^* = w_t, \forall t \end{cases}$$

Remark: In an endowment economy, prices must induce consumption to equal to endowment as there is no way to shift resources over time.

We solve it with the Langrage method.

$$L = \sum_{t=0}^{\infty} \beta^t u(c_t) + \lambda(\sum_{t=0}^{\infty} p_t w_t - \sum_{t=0}^{\infty} p_t c_t) \tag{6.1}$$

$$f.o.c.[c_t] : \beta^t u'(c_t) = \lambda p_t, \forall t \Rightarrow \beta^{t+1} u'(c_{t+1}) = \lambda p_{t+1}, \forall t \tag{6.2}$$

$$\Rightarrow \frac{p_t}{p_{t+1}} = \frac{u'(c_t^*)}{\beta u'(c_{t+1}^*)} \tag{6.3}$$

$$\Rightarrow \frac{p_0}{p_t} = \frac{u'(c_0^*)}{\beta^t u'(c_t^*)}, p_0 = 1 \tag{6.4}$$

$$\Rightarrow p_t = \frac{\beta^t u'(c_t^*)}{u'(c_0^*)} = \beta^t \frac{u'(w_t)}{u'(w_0)}, \forall t \tag{6.5}$$

the decided prices equal to the discount utility ration over time. As you see, everyone acts the same way, no sale, no buy.

6.2 Sequential Trade in the Endowment Economy

- Need assets, i.e., bonds denoted as a_t, the interest is $R_t = 1 + r_t$;
- Households can borrow or lend;
- Representative households, capital market clear, $a_t^* = 0, \forall t$;
- Budget constraints: $c_t + a_{t+1} = a_t R_t + w_t, \forall t$.
- No ponzi game condition: $\lim_{t\to\infty} (\prod_{s=0}^{t} R_{t+1})^{-1} a_{t+1} = 0$ (limitation on borrow new loans to pay matured loans);
- A competitive equilibrium is a set of sequences $\{c_t^*, a_{t+1}^*\}_{t=0}^{\infty}$ and $\{R_t\}_{t=0}^{\infty}$ s.t. $\{c_t^*, a_{t+1}^*\}_{t=0}^{\infty}$ solves

$$\max_{\{c_t^*\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{6.6}$$

$$s.t. \begin{cases} c_t + a_{t+1} = a_t R_t + w_t, \forall t \\ \lim_{t\to\infty} (\prod_{s=0}^{t} R_{t+1})^{-1} a_{t+1} = 0 \\ c_t \geq 0, \forall t a_0 = 0 \text{ given} \end{cases} \tag{6.7}$$

$$c_t^* = w_t, a_t^* = 0, \forall t \tag{6.8}$$

$$L = \sum_{t=0}^{\infty} \beta^t [u(c_t) + \lambda_t (a_t R_t + w_t - c_t - a_{t+1})] \tag{6.9}$$

$$f.o.c.[c_t] : u'(c_t) = \lambda_t \tag{6.10}$$

$$[a_{t+1}] : -\lambda_t + \beta \lambda_{t+1} R_{t+1} = 0 \tag{6.11}$$

$$\Rightarrow R_{t+1}^* = \frac{u'(c_t^*)}{\beta u'(c_{t+1}^*)} = \frac{u'(w_t)}{\beta u'(w_{t+1})} \tag{6.12}$$

Compared Equation 6.12 with Equation 6.5, we have

$$R_{t+1} = \frac{p_t}{p_{t+1}} \tag{6.13}$$

We reach the same end through two different trade ways.

6.3 The Neoclassical Growth Model with Date-0 Trade

1. Households: 1 unit of time;
2. $u(\{c_t, 1 - n_t\}_{t=0}^{\infty}) = \sum_{t=0}^{\infty} \beta^t u(c_t)$ (leisure $1 - n_t$ is not desired in this model)
3. Households own capital k_t, with rent rate r_t and depreciation rate δ;
4. w_t (wage);
5. Production function $F(k, n)$, is strictly increasing, concave and homogeneous of degree one (e.g., $f(kx) = kf(x)$).
6. p_t: relative price of c_t at t, compared to c_0, $p_0 = 1$; r_t, w_t are prices compared to c_t, e.g., 1 unit of capital at t can trade $r_t p_t$ unit of consumption at $t = 0$.
7. A date-0 competitive equilibrium is a set of quantities:
    - prices $\{p_t, r_t, w_t\}_{t=0}^{\infty}$;
    - quantities $\{c_t^*, n_t^*, R_{t+1}^*\}_{t=0}^{\infty}$ such that

– $\{c_t^*, n_t^*, R_{t+1}^*\}$ solves the households' problem

$$\max_{\{c_t, w_t, R_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{6.14}$$

$$s.t. \begin{cases} \sum_{t=0}^{\infty} p_t(c_t + k_{t+1}) \leq \sum_{t=0}^{\infty} p_t(r_t k_t + (1-\delta)k_t + w_t n_t) \\ c_t \geq 0, \forall t \\ k_0 \text{ given} \end{cases} \tag{6.15}$$

Here the first limitation means the resources to consume or invest come from rent, left capital and wage.

– $\{k_t^*, n_t^*\}_{t=0}^{\infty}$ solves the firms' problem:

$$\max_{\{k_t, n_t\}} p_t F(k_t, n_t) - p_t r_t k_t - p_t w_t n_t \tag{6.16}$$

Here firms make zero profit and fail to consider long run (actually, consideration of long run makes no difference).

– market clear: $c_t^* + k_{t+1}^* = F(k_t^*, n_t^*) + (1-\delta)k_t^*, \forall t$

Here all resources coming from new production and left capital must be consumed or invested.

8. If $w_t > 0, n_t^* = 1, \forall t$ (leisure is not valued);

First solve the firms' problem: f.o.c,

$$[k_t] : F_1(k_t^*, 1) = r_t \tag{6.17}$$

$$[n_t] : F_2(k_t^*, 1) = w_t \tag{6.18}$$

Note that $n_s = n_d$, i.e., labor market clear. We usually ignore labor market clear and capital rent market clear condition as they are trivial.

Next fix the consumers' problem:

$$L = \sum_{t=0}^{\infty} \beta^t u(c_t) + \lambda(\sum_{t=0}^{\infty} p_t(r_t k_t + (1-\delta)k_t + w_t n_t) - \sum_{t=0}^{\infty} p_t(c_t + k_{t+1})) \tag{6.19}$$

$$f.o.c.[c_t] : \beta^t u'(c_t) = \lambda p_t \Rightarrow \frac{p_t}{p_{t+1}} = \frac{u'(c_t^*)}{\beta u'(c_{t+1}^*)} \tag{6.20}$$

$$[k_{t+1}] : -\lambda p_t + \lambda p_{t+1}(r_{t+1} + 1 - \delta) = 0 \Rightarrow \frac{p_t}{p_{t+1}} = r_{t+1} + 1 - \delta \tag{6.21}$$

$$\Rightarrow \frac{u'(c_t^*)}{\beta u'(c_{t+1}^*)} = r_{t+1} + 1 - \delta = F_1(k_{t+1}^*, 1) + 1 - \delta \tag{6.22}$$

$$\Rightarrow u'(c_t^*) = \beta u'(c_{t+1}^*)[F_1(k_{t+1}^*, 1) + 1 - \delta] \tag{6.23}$$

Equation 6.23 is the **Euler Equation**. Here, $\frac{p_t}{p_{t+1}} = 1 + (r_{t+1} - \delta)$ is the gross real interest rate and $\frac{u'(c_t^*)}{\beta u'(c_{t+1}^*)}$ is the marginal rate of substitution of consumption goods between t and $t+1$.

6.4 The Neoclassical Growth Model with Sequential Trade

1. Two prices $R_t = r_t + 1 - \delta, w_t$ both in units of current consumption c_t.

2. k_t is the asset.

3. A competitive equilibrium is a sequence $\{c_t^*, k_{t+1}^*, n_t^*, R_t, w_t\}_{t=0}^{\infty}$ such that

   • $\{c_t^*, n_t^*, k_{t+1}^*\}_{n=0}^{\infty}$ solves

$$\max_{\{c_t, k_{t+1}, n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{6.24}$$

$$s.t. \begin{cases} c_t + k_{t+1} \leq R_t k_t + w_t n_t, \forall t \\ c_t \geq 0, \forall t \\ \lim\limits_{t\to\infty} (\prod_{t=0}^{\infty} R_{t+1})^{-1} k_{t+1} = 0(\text{ TVC }) \end{cases} \tag{6.25}$$

- $\{k_t^*, n_t^*\}_{t=0}^{\infty}$ solves $\max\limits_{\{k_t, n_t\}} F(k_t, n_t) - r_t k_t - w_t n_t$
- market clear: $c_t^* + k_{t+1}^* = F(k_t^*, n_t^*) + (1-\delta)k_t^*, \forall t$

Firms' problem, f.o.c: $r_t = F_1(k_t^*, 1) = R_t - (1-\delta) \Rightarrow R_t = F_1(k_t^*, 1) + (1-\delta)$.
Households' problem, f.o.c:

$$L = \sum_{t=0}^{\infty} \beta^t [u(c_t) + \lambda_t (R_t k_t + w_t n_t - c_t - k_{t+1})] \tag{6.26}$$

$$f.o.c.[c_t] : u'(c_t) = \lambda_t \tag{6.27}$$

$$[k_{t+1}] : \lambda_t = \beta \lambda_{t+1} R_{t+1} \tag{6.28}$$

$$\Rightarrow u'(c_t) = \beta u'(c_{t+1}) R_{t+1} \tag{6.29}$$

$$\Rightarrow u'(c_t^*) = \beta u'(c_{t+1}^*)[F_1(k_{t+1}^*, 1) + 1 - \delta] \tag{6.30}$$

We get the same Euler Equation with the last problem. Note that here L also can be $L = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} (R_t k_t + w_t n_t - c_t - k_{t+1})$, which won't change the Euler Equation.

6.5 A Date-0 Economy with N Households

For the household i, $i = 1, 2, \cdots, N$,

1. $l_t^i \in [0,1]$ time endowment;
2. k_0^i initial capital;
3. $u^i(\{c_t^i, 1-n_t^i\}_{t=0}^{\infty}) = \sum_{t=0}^{\infty} \beta^t u^i(c_t^i)$ (so $n_t^i = l_t^i$ in equilibrium as leisure is not valued);
4. A date-0 competitive equilibrium of the N-agents economy is a set of sequences
   - prices $\{p_t, r_t, w_t\}_{t=0}^{\infty}$;
   - quantities $\{c_t^*, n_t^*, k_{t+1}^*, \{c_t^{i*}, n_t^{i*}, k_{t+1}^{i*}\}_{i=1}^{N}\}_{t=0}^{\infty}$ (c_t^*, n_t^*, k_{t+1}^* are aggregate variables and the left are individual variables) such that
     - $\{c_t^{i*}, n_t^{i*}, k_{t+1}^{i*}\}_{t=0}^{\infty}$ solves households' problem for each i,

$$\max_{\{c_t^i, n_t^i, k_{t+1}^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta_i^t u^i(c_t^i) \tag{6.31}$$

$$s.t. \begin{cases} \sum_{t=0}^{\infty} p_t(c_t^i + k_{t+1}^i) \le \sum_{t=0}^{\infty} (r_t k_t^i + (1-\delta)k_t^i + w_t n_t^i) p_t \\ c_t^i \ge 0, 0 \le n_t^i \le l_t^i, \forall t \\ k_0^i \text{ given} \end{cases} \tag{6.32}$$

     - $\{k_t^*, n_t^*\}_{t=0}^{\infty}$ solves $\max\limits_{k_t, n_t} p_t F(k_t, n_t) - p_t r_t k_t - p_t w_t n_t$
     - market clear:
       * $n_t^* = \sum_{i=1}^{N} n_t^{i*}$;
       * $k_t^* = \sum_{i=1}^{N} k_t^{i*}$
       * $c_t^* = \sum_{i=1}^{N} c_t^{i*}$
       * $c_t^* + k_{t+1}^* = F(k_t^*, n_t^*) + (1-\delta)k_t^*, \forall t$

       the first three are aggregation variables.

Now we can solve it. From the f.o.c. $[k_t^i]$ of households' problem, we have $\frac{p_t}{p_{t+1}} = r_{t+1} + 1 - \delta, \forall t$ and take it to the first constraint:

$$\sum_{t=0}^{\infty} p_t(c_t^i + k_{t+1}^i) \le \sum_{t=0}^{\infty} p_t(\frac{p_{t-1}}{p_t} k_t^i + w_t l_t^i) \tag{6.33}$$

$$\sum_{t=0}^{\infty} p_t c_t^i + \sum_{t=0}^{\infty} p_t k_{t+1}^i \le \sum_{t=0}^{\infty} p_{t-1} k_t^i + \sum_{t=0}^{\infty} p_t w_t l_t^i \tag{6.34}$$

$$\sum_{t=0}^{\infty} p_t c_t^i \le p_{-1} k_0^i + \sum_{t=0}^{\infty} p_t w_t l_t^i \tag{6.35}$$

Let $t = -1, p_{-1} = p_0(r_0 + 1 - \delta) = r_0 + 1 - \delta$, so

$$\sum_{t=0}^{\infty} p_t c_t^i \leq (r_0 + 1 - \delta)k_0^i + \sum_{t=0}^{\infty} p_t w_t l_t^i \tag{6.36}$$

The left part is lifetime spending and the right part includes initial wealth ($(r_0 + 1 - \delta)k_0^i$) and permanent income ($\sum_{t=0}^{\infty} p_t w_t l_t^i$,independent with k_t).

If $u(\cdot)$ is strictly concave, then $\{c_t^i\}_{t=0}^{\infty}$ is unchanged as long as $(r_0 + 1 - \delta)k_0^i + \sum_{t=0}^{\infty} p_t w_t l_t^i$ holds, which implies only the change of permanent income and initial capital (usually given) can alter the consumption path. Summarized as **Permanent Income Hypothesis**: consumption is a function of permanent income rather that a function of current income, e.g., if the government offers you a subsidy, \$100 and raises the taxation by $\$100R$ (R is the discount rate) at next period, you should not change you consumption path as your permanent income does not change. Permanent income hypothesis holds in a non-frication world, but we face many constraints in the real world, e.g., credit constraints. If the government can eliminate this constraints and improve efficiency, the consumption path may change.

6.6 A Two Period Endowment Economy

$$\max_{c_0, c_1} u(c_0) + \beta u(c_1) \tag{6.37}$$

$$s.t. \begin{cases} c_0 + a_1 \leq R_0 a_0 + w_0 \\ c_1 + a_2 \leq R_1 a_1 + w_1 \\ a_0 = 0 \text{ given} \\ c_0, c_1 \geq 0 \end{cases} \tag{6.38}$$

This is a representative household model, so in equilibrium there is no asset trade, $a_t = 0, \forall t$ (implies $c_t = w_t, t = 0, 1$).

$$L = u(c_0) + \beta u(c_1) + \lambda_0(R_0 a_0 + w_0 - c_0 a_1) + \beta\lambda_1(R_1 a_1 + w_1 - c_1 - a_2) \tag{6.39}$$

$$f.o.c.[c_0] : u'(c_0) = \lambda_0 \tag{6.40}$$

$$[c_1] : u'(c_1) = \lambda_1 \tag{6.41}$$

$$[a_1] : -\lambda_0 + \beta\lambda_1 R_1 = 0 \tag{6.42}$$

$$\Rightarrow u'(c_0) = \beta u'(c_1) R_1 \tag{6.43}$$

$$\Rightarrow R_1 = \frac{u'(c_0)}{\beta u'(c_1)} = \frac{u'(w_0)}{\beta u'(w_1)} \tag{6.44}$$

Assume $w_0 = w_1$, then $R_1 = \frac{1}{\beta}$ (the equilibrium rate is equal to the compensation of impatience).

We then analyze this economy from data-0 trade setting. The constraint becomes $p_0 c_0 + p_1 c_1 \leq p_0 w_0 + p_1 w_1$ (note that we take $a_t = 0, t = 0, 1, 2$). The Langrage function is

$$L = u(c_0) + \beta u(c_1) + \lambda(p_0 w_0 + p_1 w_1 - p_0 c_0 - p_1 c_1) \tag{6.45}$$

$$f.o.c.[c_0] : u'(c_0) = \lambda p_0 \tag{6.46}$$

$$[c_1] : \beta u'(c_1) = \lambda p_1 \tag{6.47}$$

$$\Rightarrow \frac{p_1}{p_0} = \frac{\beta u'(c_1)}{u'(c_0)} = \frac{\beta u'(w_1)}{u'(w_0)} \tag{6.48}$$

$$\Rightarrow R_1 = \frac{p_0}{p_1} = \frac{u'(w_0)}{\beta u'(w_1)} \tag{6.49}$$

Note that here is a representative household model, there is no trade. But in the following example, we allow trade and offer a bond to invest, i.e., $a_1 \neq 0$. This extends the constraints to $c_0 + \frac{c_1}{R} \leq w_0 + \frac{w_1}{R} = w$, where R replaces the prices to transform consumption in different periods. Further, we take R as given, then we can analyze the consumption path's change.

Example 6.1. *$u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}, \sigma > 0, u'(c) = c^{-\sigma}$, let $R = R_1$.*
Then $u'(c_0) = \beta u'(c_1)R, c_0^{-\sigma} = (\beta R)c_1^{-\sigma} \Rightarrow c_1 = (\beta R)^{1/\sigma} c_0$. Implication:

1. *Under this utility, we have constant elasticity of intertemporal substitution of consumption.*

$$\eta = -\frac{dlog(c_1/c_0)}{dlog(p_1/p_0)} = \frac{1}{\sigma}\frac{dlog(\beta R)}{dlog(R)} == \frac{1}{\sigma}\frac{dlog(R)}{dlog(R)} = \frac{1}{\sigma} \tag{6.50}$$

Budget constraint: $c_0 + \frac{c_1}{R} = w_0 + \frac{w_1}{R}, c_0 + \frac{(\beta R)^{1/\sigma}c_0}{R} = w_0 + \frac{w_1}{R} = w \Rightarrow c_0 = \frac{w}{1+(\beta R^{1-\sigma})^{1/\sigma}}, c_1 = \frac{(\beta R)^{1/\sigma}w}{1+(\beta R^{1-\sigma})^{1/\sigma}}$

2. *If R does not change and w changes, the slope of the curve (budget constraints $c_0 + \frac{c_1}{R} \leq w$) does not change.*

3. *c_0, c_1 are independent with w_0, w_1, depending only on the sum w (permanent income hypothesis). Further, $\frac{dc_0}{dw} = \frac{1}{1+(\beta R^{1-\sigma})^{1/\sigma}} < 1$, consumers smooth consumption over his lifetime.*

4. $a_1 = w_0 - a_0 = w_0 - \frac{w}{1+(\beta R^{1-\sigma})^{1/\sigma}} = \frac{(\beta R^{1-\sigma})^{1/\sigma}w_0 - w_1/R}{1+(\beta R^{1-\sigma})^{1/\sigma}}$. *Let $w_1 = 0$ for simplicity, we get* $a_1 = \frac{(\beta R^{1-\sigma})^{1/\sigma}w_0}{1+(\beta R^{1-\sigma})^{1/\sigma}}$ *and* $\frac{da_1}{dR} = A\frac{1-\sigma}{\sigma} = \begin{cases} > 0, \sigma < 1 \\ = 0, \sigma = 1 \\ < 0, \sigma > 1 \end{cases}$, $A = \frac{v'}{(1+v)^2}t^{1/\sigma-1}\beta R^{-\sigma}/\sigma^2 > 0, t = \beta R^{1-\sigma}, v = t^{1/\sigma}$. *When R increases, it impose two effects, (a) income effect, $c_0 \nearrow, c_1 \searrow$; (b) substitution effect, $c_0 \searrow, c_1 \nearrow$. When $\sigma < 1$, substitution effect $>$ income effect then $c_0 \searrow, a_1 \nearrow$; when $\sigma > 1$, income effect $>$ substitution effect then $c_0 \nearrow, a_1 \searrow$. That is the bigger $\eta = 1/\sigma$ is, the stronger the substitution effect is.*

6.7 Government Debt and Tax

- $\{g_t\}$, government spending;
- $\{\tau_t\}$, lump-sum tax;
- $\{d_t\}$, government debt;
- Given $\{g_t, \tau_t\}$;
- A competitive equilibrium is a set of prices and quantities, $\{R_t, w_t\}_{t=0}^{\infty}$ and $\{c_t^*, k_{t+1}^*, n_t^*, d_{t+1}^*\}_{t=0}^{\infty}$ such that

    1. Households, $\{c_t^*, k_{t+1}^*, n_t^*, d_{t+1}^*\}_{t=0}^{\infty}$ solves

$$\max_{\{c_t, k_{t+1}, d_{t+1}, n_t\}_{n=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{6.51}$$

$$s.t. \begin{cases} c_t + k_{t+1} + d_{t+1} + \tau_t = R_t k_t + R_t d_t + w_t n_t, \forall t \\ \lim\limits_{t\to\infty} \frac{d_{t+1}}{\prod_{s=0}^{t} R_s} = 0, \lim\limits_{t\to\infty} \frac{k_{t+1}}{\prod_{s=0}^{t} R_s} = 0 \\ k_0, d_0 \text{ given} \end{cases} \tag{6.52}$$

    2. Firms, $\{k_t^*, n_t^*\}_{t=0}^{\infty}$ solves

$$\max_{k_t, n_t} F(k_t, n_t) - R_t k_t + (1-\delta)k_t - w_t n_t \tag{6.53}$$

    3. Government budget constraint holds, $g_t + R_t d_t = d_{t+1} + \tau_t, \forall t$ (balance, g_t produces nothing)
    4. Market clear, $c_t^* + k_{t+1}^* + g_t^* = F(k_t^*, n_t^*) + (1-\delta)k_t^*, \forall t$

Implication: government bounds are not net real wealth in economy. Take the government budget constraint into the households' constraint, we have

$$c_t = R_t k_t - k_{t+1} - w_t n_t - g_t, \forall t \tag{6.54}$$

Define $R_{0,t} = \prod_{s=0}^{t} R_s$, and the lifetime budget constraint of households,

$$\sum_{t=0}^{\infty} \frac{c_t}{R_{0,t}} = \sum_{t=0}^{\infty} \frac{k_t}{R_{0,t-1}} - \sum_{t=0}^{\infty} \frac{k_{t+1}}{R_{0,t}} + \sum_{t=0}^{\infty} \frac{w_t n_t}{R_{0,t}} - \sum_{t=0}^{\infty} \frac{g_t}{R_{0,t}} \tag{6.55}$$

$$= k_0 + \sum_{t=0}^{\infty} \frac{w_t n_t}{R_{0,t}} - \sum_{t=0}^{\infty} \frac{g_t}{R_{0,t}} \tag{6.56}$$

Clearly, the lifetime utility of households (controlled by the path of c_t) is decided by initial capital, lifetime wages and government spending, which implies if the total government spending holds still, the welfare of households does not change. This is called **Ricardian Equilibrium Theorem**.

Take any competitive equilibrium $\{R_t, w_t\}$ and $\{c_t^*, k_{t+1}^*, n_t^*\}$, given $\{g_t, \tau_t\}$. Now perturb the path of taxes to $\{\tau_t^a\}_{t=0}^{\infty}$ but do not change $\{g_t\}$. As the households budget constraints is not affected, there will be no change in prices or $\{c_t^*, k_{t+1}^*, n_t^*\}$.

6.8 Recursive Competitive Equilibrium

6.8.1 The Neoclassical Growth Model

Planner's problem,

$$V(k) = \max_{c,k'} u(c) + \beta V(k') \tag{6.57}$$

$$s.t. c + k' = F(k,1) + (1-\delta)k \tag{6.58}$$

Gross real interest rate, $R = F_1(k,1) + (1-\delta)$, real wage rate, $w = F_2(k,1)$.

- Aggregate state variables $\bar{k}$, individual state variables k (affected by market and self constraints);
- Households' problem,

$$v(k,\bar{k}) = \max_{c,k'} u(c) + \beta v(k',\bar{k}') \tag{6.59}$$

$$s.t. \begin{cases} c + k' = R(\bar{k})k + w(\bar{k}) \cdot 1(\text{ price taker}) \\ \bar{k}' = G(\bar{k}) \end{cases} \tag{6.60}$$

Then $k' = g(k,\bar{k})$ is the individual policy function.

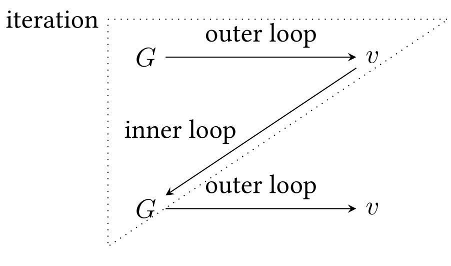


Figure 13: Double recursive methods

We solve the problem by double recursive methods, see Figure 13. A recursive competitive equilibrium is a set of functions: quantities $G(\bar{k}), g(k,\bar{k})$, value $V(k,\bar{k})$ and prices $R(\bar{k}), w(\bar{k})$ such that

1. $v(k,\bar{k})$ solves households' problem; $g(k,\bar{k})$ is the associated policy function;
2. prices are competitive determined, $R(\bar{k}) = F_1(\bar{k},1) + 1 - \delta, w(\bar{k}) = F_2(\bar{k},1)$;
3. Individual decisions are consistent with aggregates, $G(\bar{k}) = g(\bar{k},\bar{k}), \forall \bar{k}$.

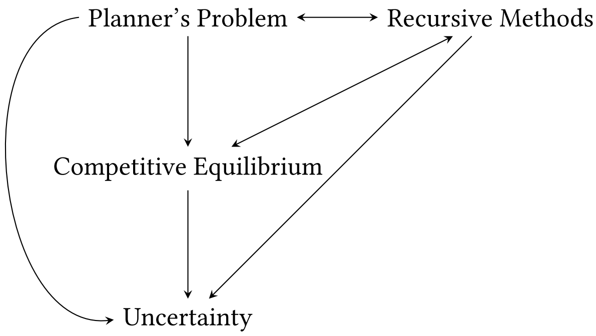


Figure 14: Framework of neoclassical growth models

6.8.2 An Endowment Economy with Two Agents

- Asset market equilibrium: $a_t^1 + a_t^2 = 0, \forall t(A_1 + A_2 = 0)$;
- A recursive competitive equilibrium of the two agents endowment economy is a set of functions: quantities $G(A_1), g(a_1, A_1), g(a_2, A_1)$, values $V_1(a_1, A_1), V_2(a_2, A_1)$ and prices $q(A_1)$ such that

  1.
  $$V_i(a_i, A_1) = \max_{c_1, a_i'} u_i(c_i) + \beta_i v_i(a_i', A_1') \tag{6.61}$$
  $$s.t. \begin{cases} c_i + a_i' q(A_1) = a_i + w_i, i = 1, 2 (q = \frac{1}{R}, \text{ price of bonds}) \\ a_i' \geq a, \text{ credit constraints} \\ A_1' = G(A_1) \end{cases} \tag{6.62}$$

  2. consistency, $g_1(A_1, A_1) = G(A_1), g_2(-A_1, A_1) = -G(A_1)$.

Now we want to give a framework of neoclassical growth models, see Figure 14.

7 Uncertainty and Neoclassical Growth Models

7.1 Maximization under Uncertainty

7.1.1 Incomplete Market

Example 7.1. • *2-period economy, $t = 0, 1$;*

- *A household in a large economy;*
- *Consumes in both periods;*
- *Faces earning risk in his second period when he works;*
- *In period 1, n possible states of the world, the real wage varies across these states $w \in \{\tilde{w}_1, \tilde{w}_2, \cdots, \tilde{w}_n\}$, and the probability of state i is $\pi_i = Pr(w = \tilde{w}_i), i = 1, 2, \cdots, n, \sum_{i=1}^n \pi_i = 1$.*
- The household maximizes expected lifetime utility, $u = \sum_{i=1}^n \pi_i u(c_0, c_{1i}, n_i)$ (Von Neumann-Morgenstan utility function);
- Incomplete market (the number of different assets is less than n). One extreme example is only a single risk-free asset, i.e., $a, q = \frac{1}{k}$.
  - period 0 budget constraint: $c_0 + qa = I$ (individual income);
  - there are n separate period budget constraints: $c_{1i} = a + \tilde{w}_i n_i, i = 1, 2, \cdots, n$, where a is the asset return.
  - market are incomplete, as we do not have as many as assets as state-of-nature.

- Assume $u(c_0, c_{1i}, n_i) = u(c_0) + \beta u(c_{1i}) + \beta v(n_i)$ where $v'(n_i) < 0$ (we desire leisure instead of work).

Then, the household solves,

$$\max_{c_0, a, \{c_{1i}, n_i\}_{i=1}^n} u(c_0) + \beta \sum_{i=1}^n \pi_i [u(c_{1i}) + v(n_i)] \tag{7.1}$$

$$s.t. \begin{cases} c_0 + qa = I \\ c_{1i} = a + \tilde{w}_i n_i, i = 1, 2, \cdots, n \end{cases} \tag{7.2}$$

We solve this by Langrage method. The Langrage function and f.o.c. are

$$L = u(c_0) + \beta \sum_{i=1}^n \pi_i [u(c_{1i}) + v(n_i)] + \lambda(I - c_0 - qa) + \sum_{i=1}^n \lambda_i (a + \tilde{w}_i n_i - c_{1i}) \tag{7.3}$$

$$f.o.c. [c_0] : u'(c_0) = \lambda \tag{7.4}$$

$$[a] : \lambda q = \sum_{i=1}^n \lambda_i \tag{7.5}$$

$$[c_{1i}] : \beta \pi_i u'(c_{1i}) = \lambda_i, i = 1, 2, \cdots, n \tag{7.6}$$

$$[n_i] : \beta \pi_i v'(n_i) = -\lambda_i \tilde{w}_i, i = 1, 2, \cdots, n \tag{7.7}$$

From f.o.c. of $[c_{1i}]$, $[n_i]$, we have

$$\tilde{w}_i u'(c_{1i}) = -v'(n_i), i = 1, 2, \cdots, n \tag{7.8}$$

which is called **labor-leisure condition**, implying the marginal value of another unit of time devoted to work equals to the marginal utility of leisure.

$[c_0]$, $[a]$, $[c_{1i}]$ implies

$$qu'(c_0) = \beta \sum_{i=1}^n \pi_i u'(c_{1i}) \tag{7.9}$$

which is the **Euler Equation**, and the right part means discounted expected marginal benefit. We give an example to study the model.

Example 7.2. *Assume $u(c) = \log(c), v(n) = \log(1 - n)$. Then the labor-leisure condition shows $\frac{\tilde{w}_i}{c_{1i}} = \frac{1}{1-n_i}$, so $c_{1i} = \tilde{w}_i(1 - n_i), i = 1, 2, \cdots, n$.*

Then period 1 budget constraint becomes $c_{1i} = a + \tilde{w}_i n_i = a + w_i - c_{1i}$, so $c_{1i} = \frac{a+\tilde{w}_i}{2}$. In this case, consumption ineach state i fluctuates with wage shock. Thus, not full insured against consumption risk. This is a consequence of incomplete market.

From Euler Equation, we have $\frac{q}{I-qa} = \beta \sum_{i=1}^n \pi_i \frac{2}{a+\tilde{w}_i}$ solve for a.

7.1.2 Complete Market

- Instead of a single-risk-free asset, there are state-contingent claims, n separate assets at date 0;
- Arrow securities: the i-th asset is bought at price $q_j, j = 1, 2, \cdots, n$ and pay

$$\begin{cases} 1, \text{ if in date 1, the sate of world is } j \\ 0, otherwise \end{cases}$$

- Period 0 budget constraint: $c_0 + \sum_{i=1}^n q_i a_i = I$;
- Period 1 budget constraint: $c_{1i} = a_i + \tilde{w}_i n_i, i = 1, 2, \cdots, n$;
- The risk-free asset price: $q = \sum_{i=1}^n q_i$;
- We can derive a single lifetime budget constraint (eliminate a_i): $c_0 + \sum_{i=1}^n q_i c_{1i} = I + \sum_{i=1}^n q_i \tilde{w}_i n_i$;
- The Langrage function: $L = u(c_0) + \beta \sum_{i=1}^n \pi_i [u(c_{1i}) + v(n_i)] + \lambda(I + \sum_{i=1}^n q_i \tilde{w}_i n_i - c_0 - \sum_{i=1}^n q_i c_{1i})$;

- F.o.c.:

$$[c_0] : u'(c_0) = \lambda \tag{7.10}$$
$$[c_{1i}] : \beta\pi_i u'(c_{1i}) = \lambda q_i, i = 1, 2, \cdots, n \tag{7.11}$$
$$[n_i] : \beta\pi_i v'(n_i) = -\lambda q_i \tilde{w}_i, i = 1, 2, \cdots, n \tag{7.12}$$

- Labor-leisure condition: $\tilde{w}_i u'(c_{1i}) = -v'(n_i)$;
- Euler equation: $q_i u'(c_0) = \beta\pi_i u'(c_{1i}), i = 1, 2, \cdots, n$;
- Assume actuarially for asset prices, $q_i = \pi_i q, i = 1, 2, \cdots, n$ (fair price). Then $u'(c_0)q = \beta u'(c_{1i}), i = 1, 2, \cdots, n$, $c_{1i} = c$, constant over states of nature.
- A risk-averse consumer with fair insurance will full insure himself;
- Complete market gives a marginal rate of substitution between c_0 and each c_{1i}, a, q_i. MRS between c_{1i} and c_{1j} is $\frac{u'(c_{1i})}{u'(c_{1j})} = \frac{q_i/\pi_i}{q_j/\pi_j}$;

Example 7.3. *Assume $u(c) = \log(c), v(n) = \log(1 - n)$.*
F.o.c. are

$$[c_0] : \frac{1}{c_0} - \lambda = 0 \tag{7.13}$$
$$[c_{1i}] : \beta\pi_i \frac{1}{c_{1i}} - \lambda q_i = 0, \forall i \tag{7.14}$$
$$[n_i] : -\beta\pi_i \frac{1}{1-n} - \lambda q_i \tilde{w}_i = 0 \tag{7.15}$$
$$\Rightarrow \beta\pi_i c_0 = q_i c_{1i} \tag{7.16}$$
$$\Rightarrow \beta\pi_i c_0 = q_i w_i (1 - n_i) \tag{7.17}$$

The lifetime budget constraint then becomes,

$$c_0 + \sum_{i=1}^{n} q_i c_{1i} = I + \sum_{i=1}^{n} q_i \tilde{w}_i n_i \tag{7.18}$$
$$c_0 + \sum_{i=1}^{n} \beta\pi_i c_0 = I + \sum_{i=1}^{n} (q_i \tilde{w}_i - \beta\pi_i c_0) \tag{7.19}$$
$$c_0(1 + 2\beta) = I + \sum_{i=1}^{n} q_i \tilde{w}_i \tag{7.20}$$

Note that $\sum_{i=1}^{n} \pi_i = 1$. We solved c_0. From the Euler Equation 7.16, $c_{1i} = \beta\frac{\pi_i}{q_i} c_0$. Hence complete markets are not sufficient to derive full insurance when $\frac{\pi_i}{q_i}$ is not constant.

7.2 Markov Chains

DEFINITION 7.1. *(Markov chain) A stochastic process that the probability distribution of the random variable next period depends only on it current value.*

DEFINITION 7.2. *(Stationary markov chain) Let $x_t \in X$, a stationary markov chain is a stochastic process $\{x_t\}_{t=0}^{\infty}$ defined by X, P, π_0 such that there exists a stationary (invariant) distribution $\pi = \pi P$, where X is $1 \times n$ set of values, $X = \{\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n\}$, P is a $n \times n$ transition matrix, π_0 is a $1 \times n$ initial probability distribution for X_0, and $P_{i,j} = Pr\{x_{t+1} = \tilde{x}_j | x_t = \tilde{x}_i\}$, surely, $\sum_i P_{i,j} = 1, \forall j$.*

Then, $Pr\{x_{t+2} = \tilde{x}_j | x_t = \tilde{x}_i\} = \sum_{k=1}^{n} P_{i,k} P_{k,j} = \{P^2\}_{i,j}$, and further, $Pr\{x_{t+q} = \tilde{x}_j | x_t = \tilde{x}_i\} = \{P^q\}_{i,j}$, i.e., given π_0, we say the probability of X_t is $\pi_t = \pi_0 P^t$.

A stationary distribution satisfies: $\pi = \pi P$, i.e., $\pi I = \pi P$ iff $\pi(I - P) = 0$, i.e., we can define π as an eigenvector of P associated with the eigenvalue $\lambda = 1$, e.g., $P = \begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{bmatrix}$, and $\pi = (2/3, 1/3)$.

7.3 The Neoclassical Growth Model with Uncertainty

- Total factor productivity (TFP) shock (uncertainty);
- z, exogenous productivity shock;
- $z \in Z$, Z is countable;
- $z_t \in Z, \forall t$ be a random variable following a stationary stochastic process;
- $Z^t = \prod_{i=1}^{t} Z$ (catersian product);
- $z^t = \{z_0, z_1, \cdots, z_t\} \in Z^{t+1}$, a history from period 0 to period t, let $Pr\{z^t\} \equiv \pi(z^t)$;
- Denote $\prec$ (be contained): $z^t \prec z^{t+s}$ if $z^{t+s} = \{z^t, z^{t+1}, \cdots, z^{t+s}\}$;
- If we assume $\{z_t\}$ is a markov chain, then $Pr\{z_{t+1}, z_t|z^t\} = Pr\{z_{t+1}|z^t\} = Pr\{z_{t+1}|z_t\}$.

7.3.1 Planners' Problem

- output: at $\{z^t\}, y_t(z^t) = z_t F(k_t(z^{t-1}), n(z^t))$ where z_t is the total productivity factor and k_t is the capital path;
- utility: $u(\{c_t(z^t), 1 - n_t(z^t)\}_{z^t \in Z^{t+1}}) = \sum_{t=0}^{\infty} \beta^t \sum_{z^t \in Z^{t+1}} \pi(z^t) u(c_t(z^t))$ (leisure is not preferred);
- $k_{t+1}(z^t) = (1-\delta)k_z(z^{t-1}) + i_t(z^t)$;
- $c_t(z^t) + i_t(z^t) \leq y_t(z^t)$;
- $n_t(z^t) \in [0,1] \Rightarrow n_t = 1, \forall z^t$ as n_t is not in the u;
- $u(\cdot,\cdot)$ is is strictly increasing, concave and twice continuously differentiable;
- $\beta \in (0,1), \delta \in (0,1)$;
- given k_0;
- planner chooses $c_t(z^t), n_t(z^t), k_{t+1}(z^t)$ and each $z^t \in Z^{t+1}, \forall t$;
- $L = \sum_{t=0}^{\infty} \beta^t \sum_{z^t \in Z^{t+1}} \pi(z^t)[u(c_t(z^t)) + \lambda_t(z^t)[z_t F(k_t(z^t), 1) + (1-\delta)k_t(k^{t-1}) - c_t(z^t) - k_{t+1}(z^t)]]$.

F.O.C. for some specific event $\bar{z}^t$ are:

$$[c_t(\bar{z}_t)] : u'(c_t(\bar{z}^t)) = \lambda_t(\bar{z}^t) \tag{7.21}$$

$$[k_{t+1}(\bar{z}^t)] : \pi(\bar{z}^t)\lambda_t(\bar{z}_t) = \beta \sum_{z_{t+1} \in Z} \pi(z_{t+1}, \bar{z}^t)\lambda_{t+1}(z_{t+1}, \bar{z}^t)(z_{t+1}F_1(k_{t+1}(\bar{z}^t), 1) + 1 - \delta) \tag{7.22}$$

Define

$$\pi(z_{t+1}|\bar{z}^t) = \frac{\pi(z_{t+1}, \bar{z}^t)}{\pi(\bar{z}^t)} \tag{7.23}$$

then,

$$\lambda_t(\bar{z}^t) = \beta \sum_{z_{t+1} \in Z} \pi(z_{t+1}|\bar{z}^t)\lambda_{t+1}(z_{t+1}, \bar{z})(z_{t+1}F_1(k_{t+1}|\bar{z}^t, 1) + 1 - \delta) \tag{7.24}$$

Market clear,

$$\bar{z}_t F(k_t(\bar{z}^{t-1}), 1) + (1-\delta)k_t(\bar{z}^{t-1}) - k_{t+1}(\bar{z}^t) - c_t(\bar{z}^t) = 0 \tag{7.25}$$

With Equation 7.21, 7.24, we obtain the **Euler Equation**,

$$u'(c_t(\bar{z}^t)) = \beta \sum_{z_{t+1} \in Z} \pi(z_{t+1}|\bar{z}^t)u'(c_{t+1}(z_{t+1}, \bar{z}^t))(z_{t+1}F_1(k_{t+1}(\bar{z}^t), 1) + 1 - \delta) \tag{7.26}$$

7.3.2 Recursive Formulation

- The planner's problem: we have to assume a first order markov process for $\{z_t\}_{t=0}^{\infty}$;
- The value function,

$$V(k,z) = \max_{k'}\{u(zf(k) - k' + (1-\delta)k) + \beta \sum_{z' \in Z} \pi(z'|z)V(k', z')\} \tag{7.27}$$

  and the associated policy function is $k' = g(k,z)$ (capital and world state are state variables);
- We divide domain into two sets:
  1. transient set: a set of values of capital, which cannot occur in the long terms;
  2. ergodic set: a set stat the capital will never leave once it is there.

  e.g., $Z = \{z_l, z_h\}$, see Figure 15

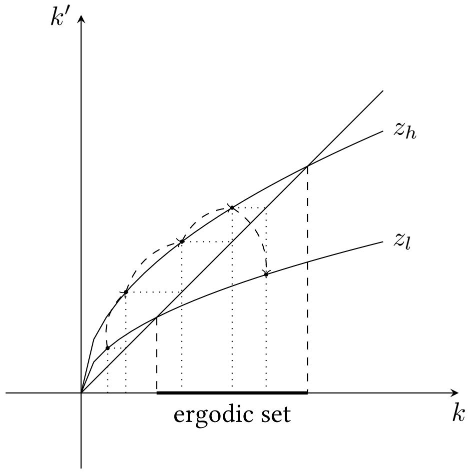


Figure 15: Example of transient sets and ergodic sets

- Stationary stochastic process for (k,z): $P_h(k)(z=h), P_l(k)(z=l)(P(k,z)$ joint density), $P(k,z)$ s.t.
  1. $\int(P_h(k) + P_l(k))dk = 1$;
  2. $\int P_h(k)dk = \pi_h, \int P_l(k)dk = \pi_l$ where π_h, π_l are invariant probability of z_h, z_l;
  3. $Pr\{k \leq \bar{k}, z = z_h\} = \int_{k\leq\bar{k}} P_h(k)dk = [\int_{k=g_h(k)\leq\bar{k}} P_h(k)dk]\pi_{hh} + [\int_{k=g_l(k)\leq\bar{k}} P_l(k)dk]\pi_{lh}$; $Pr\{k \leq \bar{k}, z = z_l\} = \int_{k\leq\bar{k}} P_h(k)dk = [\int_{k=g_h(k)\leq\bar{k}} P_h(k)dk]\pi_{hl} + [\int_{k=g_l(k)\leq\bar{k}} P_l(k)dk]\pi_{ll}$;

  Now we derive B-S condition and the Euler Equation. The F.O.C. of Equation 7.27,

$$[k'] : -u'(zf(k) - k' + (1-\delta)k) + \beta \sum_{z'\in Z} \pi(z'|z)v_1(k', z') = 0 \tag{7.28}$$

  Let $g(k) = k'$ be the policy function. With the Envelop Theorem, the value function's derivative w.r.t. k is,

$$V_1(k,z) = u'(zf(k) - g(k) + (1-\delta)k)(zf'(k) - g'(k) + 1 - \delta) + \beta \sum_{z'\in Z} \pi(z'|z)V_1(g(k), z')g'(k) \tag{7.29}$$

  Take Equation 7.28 into 7.29, we obtain the **B-S condition**,

$$V_1(k,z) = u'(zf(k) - k' + (1-\delta)k)(zf'(k) + 1 - \delta) \tag{7.30}$$

  With the B-S condition and Equation 7.28, we can eliminate V, obtaining the **Euler Equation**,

$$u'(zf(k) - k' + (1-\delta)k) = \beta \sum_{z'\in Z} \pi(z'|z)[u'(z'f(k')) - k'' + (1-\delta)k](z'f'(k') + 1 - \delta) \tag{7.31}$$

  where $k' = g(k), k'' = g(g(k))$. Then we can solve this model with linearization of Euler equation.

Example 7.4. *$\{z_t\}_{t=0}^{\infty}$ follows an $AR(1)$ process, $z_{t+1} = \rho z_t + (1-\rho)\bar{z} + \varepsilon_{t+1}$ where $|\rho| < 1, E(\varepsilon_t) = 0, E(\varepsilon_t^2) = \sigma^2 < \infty, E(\varepsilon_t \varepsilon_{t+j}) = 0, \forall j \geq 1$ and $\bar{z}$ is the long run value of z_t. Given $\bar{z}$ and associated steady state level of caption $\bar{k}$, we have*

$$u'(\bar{c}) = \beta u'(\bar{c})[\bar{z} f(\bar{k}) + 1 - \delta] \tag{7.32}$$
$$\Rightarrow \bar{k} = f^-1(\frac{\beta^{-1} - 1 + \delta}{z}) \tag{7.33}$$
$$\Rightarrow \bar{c} = \bar{z} f(\bar{k}) - \delta \bar{k} \tag{7.34}$$

Define $\hat{k} = k - \bar{k}, \hat{z} = z - \bar{z}$, derivatives from steady state. Then we rewrite the Euler equation 7.31 as

$$LHS \approx a_L \hat{z} + b_L \hat{k} + c_L \hat{k}' + d_L \tag{7.35}$$
$$RHS \approx E_z[a_R \hat{z}' + b_R \hat{k}' + c_R \hat{k}''] + d_R \tag{7.36}$$

e.g., $a_L = u''(\bar{c}) f(\bar{k})$. We have LHS = RHS if $\hat{z} = \hat{z}' = \hat{k} = \hat{k}' = \hat{k}''$ (the steady state, so $d_L = d_R$).

Then $\approx a_L \hat{z} + b_L \hat{k} + c_L \hat{k}' = E_z[a_R \hat{z}' + b_R \hat{k}' + c_R \hat{k}'']$ (difference equation, 2rd order). We can solve this by guess and verify. Gauss $\hat{k}' = g_k \hat{k} + g_z \hat{z}$, then $\hat{k}'' = g_k \hat{k}' + g_z \hat{z}' = g_k^2 \hat{k} + g_k g_z \hat{z} + g_z \hat{z}'$, then we have

$$LHS \approx a_L \hat{z} + b_L \hat{k} + c_L(g_k \hat{k} + g_z \hat{z}) + d_L = (a_L + c_L g_z)\hat{z} + (b_L + c_L g_k)\hat{k} + d_L \tag{7.37}$$
$$RHS \approx a_R E_z \hat{z}' + b_R g_k \hat{k} + b_R g_z \hat{z} + c_R g_k^2 \hat{k} + c_R g_k g_z \hat{z} + c_R g_z E_z \hat{z}' + d_R \tag{7.38}$$

Then we get $A\hat{z} + BE_z \hat{z}' + C\hat{k} = 0$ where

$$\begin{cases} A = a_L + c_L g_z - b_R g_z - c_R g_k g_z \\ B = -a_R - c_R g_z \\ C = b_L + c_L g_k - b_R g_k - c_R g_k^2 \end{cases}$$

. Since $z_{t+1} = \rho z_t + (1-\rho)\bar{z} + \varepsilon_{t+1}$, we have $\hat{z}' = z' - \bar{z} = \rho + (1-\rho)\bar{z} + \varepsilon' - \bar{z} = \rho(z - \bar{z}) + \varepsilon' = \rho \hat{z} + \varepsilon', E_z \hat{z}' = \rho \hat{z}$ Then we replace $E_z \hat{z}'$ with $\hat{k}$ and get $A'\hat{z} + B'\hat{k} = 0$, where

$$\begin{cases} A' = a_L + c_L g_z - b_R g_z - c_R g_k g_z - c_R g_z \rho \\ B' = b_L + c_L g_k - b_R g_k - c_R g_k^2 \end{cases}$$

Let $A' = B' = 0$, we can solve g_k, g_z, which verifies our gauss.

Moreover, we may want to do simulation and impulse responses.

- simulation: given $\hat{k}_0$, we simulate $\{z_0, z_1, \cdots, z_T\}$ and calculate $\{\hat{z_0}, \hat{z_1}, \cdots, \hat{z_T}\}$. Then

$$\{\hat{k}_0, \hat{z}_0\} \Rightarrow \hat{k}_1 = g_k \hat{k}_0 + g_z \hat{z}_0 \tag{7.39}$$
$$\{\hat{k}_1, \hat{z}_1\} \Rightarrow \hat{k}_2 \tag{7.40}$$
$$\cdots \tag{7.41}$$
$$\{\hat{k}_T, \hat{z}_T\} \Rightarrow \hat{k}_{T+1} \tag{7.42}$$
$$\tag{7.43}$$

- impulse response: one time shock, $\hat{z}_0 = \Delta, \hat{z}_1 = \rho \hat{z}_0, \hat{z}_2 = \rho^2 \hat{z}_0, \cdots$.

$$\hat{k}_0 = 0 \tag{7.44}$$
$$\hat{k}_1 = g_z \Delta \tag{7.45}$$
$$\hat{k}_2 = g_k \hat{k}_1 + g_z \hat{z}_1 = g_k g_z \Delta + g_z \rho \Delta \tag{7.46}$$
$$\cdots \tag{7.47}$$
$$\hat{k}_t = (g_k^{t-1} + g_k^{t-2}\rho + \cdots + g_k \rho^{t-2} + \rho^{t-1}) g_z \Delta \tag{7.48}$$

and $|g_z| < 1$ or otherwise the system is not convergent (ensure $\lim_{t \to \infty} \hat{k}_t = 0$). A typical impulse response function is like Figure 16.

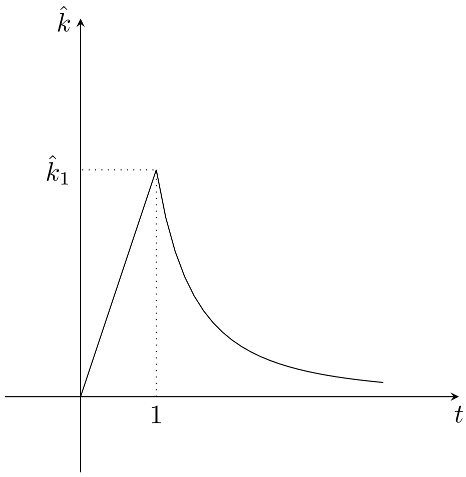


Figure 16: Impulse response function

The limitation of linearization is it is only for small shock under error control and certainly not for Chinese economy as it is not convergent.

Finally, we provide an example to show that we can eliminate state variables sometimes to simplify the formulation.

Example 7.5.

$$\max_{\{c_t(z^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{z^t \in Z^{t+1}} \beta^t \pi(z^t) u(c_t(z^t)) \tag{7.49}$$

$$s.t. z_t \in \{z_h, z_l\}, \text{ follows a first order Markov process} \tag{7.50}$$

$$\begin{cases} z^t = (z_l, z^{t-1}) : c_t(z^t) + q_{h,t}(z^t) a_{h,t+1}(z^t) a_{l,t+1}(z^t) = w_t(z^t) + a_{l,t}(z^{t-1}) \\ z^t = (z_h, z^{t-1}) : c_t(z^t) + q_{h,t}(z^t) a_{h,t+1}(z^t) a_{l,t+1}(z^t) = w_t(z^t) + a_{h,t}(z^{t-1}) \\ \text{no Ponzi-game condition} \end{cases} \tag{7.51}$$

To simplify, let

$$z_t = z_l : w(z^t) = w_l \tag{7.52}$$

$$z_t = z_h : w(z^t) = w_h \tag{7.53}$$

We define

$$z_t = z_l : x_t(z^t) = w_l + a_{l,t}(z^{t-1}) \tag{7.54}$$

$$z_t = z_h : x_t(z^t) = w_l + a_{h,t}(z^{t-1}) \tag{7.55}$$

The recursive formulation becomes

$$V(x, z_i) = \max_{a'_l, a'_h} u(x - q_{i,h} a'_h - q_{i,l} q'_l) + \beta[\pi_{i,h} V(x'_h, z_h) + \pi_{i,l} V(x'_l, z_l)] \tag{7.56}$$

$$\text{where} \begin{cases} x'_h = w_h + a'_h \\ x'_l = w_l + a'_l \end{cases} \tag{7.57}$$

$$\text{policy function} \begin{cases} a'_h = g_{i,h}(x), i = l, h \\ a'_l = g_{i,l}(x), i = l, h \end{cases} \tag{7.58}$$

7.4 Competitive Equilibrium under Uncertainty

7.4.1 The Neoclassical Growth Model with Complete Markets

Arrow-Debrue date-0 trading competitive equilibrium is $\{c_t(z^t), k_{t+1}(z^t), l_t(z^t), p_t(z^t), r_t(z^t), w_t(z^t)\}_{t=0}^{\infty}$ s.t.

- $\{c_t(z^t), k_{t+1}(z^t), l_t(z^t)\}_{t=0}^{\infty}$ solves household's problem,

$$\max_{\{c_t(z^t),k_{t+1}(z^t),l_t(z^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t \pi(z^t) u(c_t(z^t), 1 - l_t(z^t)) \tag{7.59}$$

$$s.t. \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} p_t(z^t)[c_t(z^t) + k_{t+1}(z^t)] \leq \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} p_t(z^t)[(r_t(z^t) + 1 - \delta)k_t(z^{t-1}) + w_t(z^t)l_t(z^t)] \tag{7.60}$$

where $p_t(z^t)$ contains information of $\pi(z^t)$.

- F.O.C. of firm's problem

$$r_t(z^t) = z_t F_k(k_t(z^{t-1}), l_t(z^t)) \tag{7.61}$$
$$w_t(z^t) = z_t F_l(k_t(z^{t-1}), l_t(z^t)) \tag{7.62}$$

- Market clear

$$c_t(z^t) + k_{t+1}(z^t) = (1 - \delta)k_t(z^{t-1}) + z_t F(k_t(z^{t-1}, l_t(z^t)) + (1 - \delta)), \forall t, \forall z^t \tag{7.63}$$

From the budget constraint and market clear condition, we obtain two terms in the left hand side and right hand side, get

$$k^z_{t+1}[-p_t(z^t) + \sum_{z_{t+1} \in Z} p_{t+1}(z_{t+1}, z^t)[r_{t+1}(z_{t+1,z^t}) + 1 - \delta]] \tag{7.64}$$

if Equation 7.64 is not equal to zero, we can obtain unbounded wealth by setting

$$k_{t+1}(z^t) = \begin{cases} +\infty, (7.64) > 0 \\ -\infty, (7.64) < 0 \end{cases} \tag{7.65}$$

So we have the so called No-arbitrage condition

$$p_t(z^t) = \sum_{z_{t+1} \in Z} p_{t+1}(z_{t+1}, z^t)[r_{t+1}(z_{t+1,z^t}) + 1 - \delta] \tag{7.66}$$

We had better rewrite it as

$$k_{t+1}(z^t)[p_t(z^t) - \sum_{z_{t+1} \in Z} p_{t+1}(z_{t+1}, z^t)[r_{t+1}(z_{t+1,z^t}) + 1 - \delta]] = 0 \tag{7.67}$$

to stress the underlying asset.

Sequential trade, assume state is finite, $Z = \{z_1, z_2, \cdots, z_n\}$, we say z_{t+1} has n possible shock values,

$$z_{t+1} = \begin{cases} = z_1 \\ = z_2 \\ \cdots \\ = z_n \end{cases} \tag{7.68}$$

- assume there are q assets with asset j paying off r_{ij} consumption units in $t + 1$ if the realized state is z_i. The payoff matrix is thus

$$R \equiv \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1q} \\ r_{21} & r_{22} & \cdots & r_{2q} \\ & & \cdots & \\ r_{n1} & r_{n2} & \cdots & r_{nq} \end{bmatrix} \tag{7.69}$$

where r_{ij} is the payoff of a_j under state z_i.

- then the portfolio $a = [a_1, a_2, \cdots, a_q]'$ pays $p = Ra$, where $p_i = \sum_{j=1}^{q} r_{ij} a_j$.

- if $Rank(R) = n$, then the market structure is complete; arrow securities with $q < n$ is not complete,

$$R_{n\times q} \equiv \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & 0 \end{bmatrix} \tag{7.70}$$

7.4.2 General Equilibrium under Uncertainty: the Case of Two Agent Types in a Two Period Setting

Assumption:

- random shock $z \in \{z_1, z_2, \cdots, z_n\}, \pi_j = Pr(z = z_j), \bar{z} = \sum_{j=1}^{n} \pi_j z_j$(the expected value of z)
- preference $u_i = u_i(c_0^i) + \beta \sum_{j=1}^{n} \pi_j u_i(c_j^i), i = 1, 2$ where $u_1(x) = x$ (risk neutral) and $u_2(x)$ is strictly concave and $\lim_{x\to 0} u_2'(x) = +\infty$ (risk aversion)
- endowments w_0 consumption goods in period 0 and 1 unit of labor in period 1.
- technology $y : y_j = z_j k^\alpha (\frac{n}{2})^{1-\alpha}$ and $y_j = z_j k^\alpha$ when $n = 2$ (two units of labor), so the F.O.C. of firm's problem is $r_j = z_j \alpha k^{\alpha-1}, w_j = z_j \frac{1-\alpha}{2} k^\alpha$

Structure 1: one asset (incomplete market).

Capital is the only asset. Let k be aggregate capital, a_i be capital stock hold by agent i (under market clear condition, we have $a_1 + a_2 = k$), the budget constraint for each agent is

$$c_0^i + a_i = w_0 \tag{7.71}$$
$$c_j^i = a_i r_j + w_j, i = 1, 2, \cdots, n \tag{7.72}$$

- agent 1: no arbitrage condition tells

$$a_i[-1 + \beta \sum_{j=1}^{n} \pi_j r_j] = 0 \tag{7.73}$$
$$\Rightarrow 1 = \beta \sum_{j=1}^{n} \pi_j r_j \tag{7.74}$$
$$1 = \alpha\beta k^{\alpha-1} \sum_{j=1}^{n} \pi_j z_j \tag{7.75}$$
$$\Rightarrow k^* = (\bar{z}\alpha\beta)^{\frac{1}{1-\alpha}} \tag{7.76}$$

(note that $\bar{z} = \sum_{j=1}^{n} \pi_j z_j$)

- agent 2: Euler equation 7.31 tells

$$u_2'(w_0 - a_2) = \beta \sum_{j=1}^{n} \pi_j u_2'(a_2 r_j + w_j) r_j \tag{7.77}$$

From k^*, we know w_j^*, r_j^*, also a_2^* (from $r_j = z_j \alpha k^{\alpha-1}, w_j = z_j \frac{1-\alpha}{2} k^\alpha, a_1 + a_2 = k$). Then $c_j^2 = a_2^* r_j^* + w_j^*$, where r_j^*, w_j^* are stochastic, then c_j^2 is stochastic. Therefore, agent 2 is not full insured.

Structure 2: arrow securities (complete market)

- n different arrow securities $\{a_j\}_{j=1}^n, \{q_j\}_{j=1}^n$
- total saving $s = \sum_{j=1}^{n} q_j(a_{1,j} + a_{2,j})$, equals to total investment $k = \sum_{j=1}^{n} q_j(a_{1,j} + a_{2,j})$
- total return in state j is $r_j k$, so $a_{1,j} + a_{2,j} = r_j k$. Further, we have $\sum_{j=1}^{n} q_j(a_{1,j} + a_{2,j}) = \sum_{j=1}^{n} q_j r_j k = k$, so $\sum_{j=1}^{n} q_j r_j = 1$ which is a no-arbitrage condition.

- the budget constraint of agent i is

$$c_0^i + \sum_{j=1}^{n} q_j a_{i,j} = w_0 \tag{7.78}$$

$$c_j^i = a_{i,j} + w_j, i = 1, 2, \cdots, n \tag{7.79}$$

The Langrage function and F.O.C for agent i (induce lifetime budget constraint),

$$L = u_i(c_0^i) + \beta \sum_{j=1}^{n} \pi_j u_i(c_j^i) + \lambda(\omega_0 + \sum_{j=1}^{n} q_j w_j - c_0^i - \sum_{j=1}^{n} q_j c_j^i) \tag{7.80}$$

$F.O.C.$

$$[c_0^i] : u_i'(c_0^i) - \lambda = 0 \tag{7.81}$$

$$[c_j^i] : \beta \pi_j u_i'(c_j^i) - \lambda q_j = 0 \tag{7.82}$$

$$\Rightarrow q_j = \beta \pi_j \frac{u_i'(c_j^i)}{u_i'(c_0^i)} \tag{7.83}$$

(Euler Equation)

- agent 1: $u_1(c) = c$, so

$$q_j = \beta \pi_j \tag{7.84}$$

The no-arbitrage condition $\sum_{j=1}^{n} q_j r_j = 1$ and $r_j = z_j \alpha k^{\alpha-1}$ tell,

$$1 = \sum_{j=1}^{n} q_j z_j \alpha k^{\alpha-1} = \bar{z} \alpha k^{\alpha-1} \Rightarrow k^* = (\bar{z}\alpha)^{\frac{1}{1-\alpha}} \tag{7.85}$$

- agent 2: With Equation 7.84 we rewrite the Euler Equation,

$$u_2'(c_0^2) = u_2'(c_j^2), j = 1, 2, \cdots, n \tag{7.86}$$

So we say agent 2 is able to obtain full insurance. (agent 1 is risk neutral, we do not need to ensure him)

7.4.3 General Equilibrium under Uncertainty: Multi-periods Model tieh Two Agent Types

Follow the setting of "the case of two agent types in a two period" model. Now extend it to multiply periods.

Structure 1: one asset (incomplete market)

- agent 1's problem

$$\max \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t \pi(z^t) c_{1,t}(z^t) \tag{7.87}$$

$$s.t. c_{1,t}(z^t) + a_{1,t+1}(z^t) = r_t(z^t) + a_{1,t}(z^{t-1}) + w_t(z^t) \tag{7.88}$$

- firm's problem, F.O.C.

$$r_t(z^t) = z_t \alpha k_t^{\alpha-1}(z^{t-1}) + 1 - \delta \tag{7.89}$$

$$w_t(z^t) = z_t(\frac{1-\alpha}{2}) k_t^{\alpha}(z^{t-1}) \tag{7.90}$$

where $k_t = a_{1,t}(z^{t-1}) + a_{2,t}(z^{t-1})$, aggregate capital.

- The Langrage function and F.O.C.,

$$L = \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t \pi(z^t)[c_{1,t}(z^t) + \lambda_t(z^t)[r_t(^t) a_{1,t}(z^{t-1}) + w_t(z^t) - c_{1,t}(z^t) - a_{1,t+1}(z^t)]] \tag{7.91}$$

$F.O.C.$

$$[a_{1,t}(z^t)] : \lambda_t(z^t)\pi(z^t) - \beta \sum_{z_{t+1} \in Z} \pi(z_{t+1}, z^t) \lambda_{t+1}(z_{t+1}, z^t) - r_{t+1}(z_{t+1}, z^t) = 0 \tag{7.92}$$

$$[c_{1,t}(^t)] : 1 - \lambda_t(z^t) = 0 \tag{7.93}$$

Then,

$$1 = \beta \sum_{z_{t+1}} \frac{\pi(z_{t+1}, z^t)}{\pi(z^t)} r_{t+1}(z_{t+1}, z^t) \tag{7.94}$$

$$= \beta E_{z_{t+1}|z^t}[r_{t+1}] \tag{7.95}$$

$$\Rightarrow 1 = \beta E_{z_{t+1}|z^t}[z_{t+1}\alpha k_{t+1}^{\alpha-1}(z^t) + 1 - \delta] \tag{7.96}$$

$$= \alpha\beta k_{t+1}^{\alpha-1}(z^t) E_{z_{t+1}|z^t}[z_{t+1}] + \beta(1-\delta) \tag{7.97}$$

$$\Rightarrow k_{t+1}(z^t) = [\frac{1/\beta - 1 + \delta}{\alpha E(z^{t+1}|z^t)}]^{\frac{1}{\alpha-1}} \tag{7.98}$$

- agent 2's Euler Equation

$$u'(c_{2,t}(z^t)) = \beta E_{z_{t+1}|z^t}[u'(c_{2,t+1}(z^{t+1}))(1 - \delta + \alpha z_{t+1} k_{t+1}^{\alpha-1}(z^t))] \tag{7.99}$$

We can solve $c_{2,t}(z^t)$ from the Euler Equation, not full insured.

Structure 2: Arrow securities (complete market)

- budget constraint:

$$c_t^i(z^t) + \sum_{j=1}^{n} q_j(z^t) a_{j,t+1}^i(z^t) = a_{s,t}^i(z^{t-1}) + w_t(z^t) \tag{7.100}$$

where s is the world state at t. Aggregate savings equal to aggregate investments implies

$$\sum_{j=1}^{n} q_j(z^t)(a_{j,t+1}^1(z^t) + a_{j,t+1}^2(z^t)) = k_{t+1}(z^t) \tag{7.101}$$

When the state is j, returns of securities and capital must equal,

$$a_{j,t+1}^1(z^t) + a_{j,t+1}^2(z^t) = [1 - \delta + z_j \alpha z_j \alpha k_{t+1}^{\alpha-1}(z^t)] k_{t+1}(z^t) \tag{7.102}$$

So we obtain the no-arbitrage condition,

$$1 = \sum_{j=1}^{n} q_j(z^t)[1 - \delta + z_j \alpha k_{t+1}^{\alpha-1}(z^t)] \tag{7.103}$$

- the Langrage function and F.O.C.,

$$L = \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t \pi(z^t)[u^i(c_t^i(z^t)) + \lambda_t^i(z^t)[a_{z^t,t}^i(z^{t-1}) + w_t(z^t) \tag{7.104}$$

$$- c_t^j(z^t) - \sum_{j=1}^{n} q_{j,q}(z^t) a_{j,t+1}^i(z^t)]] \tag{7.105}$$

$$F.O.C.$$

$$[c_t^i(z^t)] : u^{i'}(c_t^i(z^t)) - \lambda_t^i(z^t) = 0 \tag{7.106}$$

$$[a_{j,t+1}^i(z^t)] : -\pi(z^t)\lambda_t^i(z^t) q_{j,t}(z^t) + \beta\pi(z_{t+1}, z^t)\lambda_{t+1}^i(z_{t+1}, z^t) = 0 \tag{7.107}$$

- agent 1's F.O.C. implies ($u'(c) = 1$),

$$q_{j,t}(z^t) = \beta \frac{\pi(z_j, z^t)}{\pi(z^t)} = \beta\pi(z_j|z^t) \tag{7.108}$$

- agent 2's F.O.C. and Equation 7.108 implies,

$$\pi(z^t)u'(c_t(z^t))q_{j,t}(z^t) = \beta\pi(z_j, z^t)u'(c_{t+1}(z_j, z^t)) \tag{7.109}$$

$$\Rightarrow u'(c_t^2(z^t)) = u'(c_{t+1}^2(z_j, z^t)), \forall j \tag{7.110}$$

which implies agent 2 is full insured.

8 Applications

8.1 Asset Pricing

8.1.1 The Lucas Tree Model

- A countable set of agents, with identical preferences and initial endowments;
- Each agent initially owns a fruit tree, yields uncertain dividends, which agents consume but not store;
- z_t is the fruit per tree at z, $z_t \in Z = \{z_1, z_2, \cdots, z_N\}$, $Pr\{z_{t+1} = z_j | z_t = z_i\} = \pi_{ij} \geq 0, \sum_{j=1}^N \pi_{ij} = 1, \forall i = 1, 2, \cdots, N$;
- s_t: shares in fruit trees of typical agent at the start of period t, p_t: ex-dividend prices;
- Budget constraint: $c_t + p_t s_{t+1} \leq (z_t + p_t)s_t$;
- Equilibrium: $c_z^* = z_t, s_t^* = 1$ (no body buys or sells trees);
- Recursive formulation:

$$V(s, z_i) = \max_{s' \in p(s,z)} u((z_i + p(z_i))s - p(z_i)s') + \beta \sum_{j=1}^{N} \pi_{ij} V(s', z_j) \tag{8.1}$$

  where s denote individual state and z_i denote aggregate state, $p(s, z) = [0, \frac{z+p(z)}{p(z)} s]$;

  F.O.C. $-p(z_i)u'(c_i) + \beta \sum_{j=1}^N \pi_{ij} V_1(s', z_j) = 0$;

  Derivative w.r.t. s of V, $V_1(s, z_i) = u'(c)[(z_i + p(z_i)) - p(z_i)g'(s)] + \beta \sum_{j=1}^n \pi_{ij} C_1(g(s), z_j)g'(s)$ where $g(s) = s'$ denotes the policy function.

  Combine these two equations, we have the B-S condition, $V_1(s, z_i) = u'(c)(z_i + p(z_i))$;

  Take the B-S condition to the F.O.C., giving the Euler equations, $p(z_i)u'(c) = \beta \sum_{j=1}^N \pi_{ij} u'(c_j)(z_j + p(z_j))$.

  In equilibrium, we have $c_i = z_i, \forall i$, so the Euler equation becomes $p(z_i)u'(c_i) = \beta \sum_{j=1}^N \pi_{ij} u'(z_j)(z_j + p(z_j))$, then

$$p(z_i) = \beta \sum_{j=1}^{N} \pi_{ij} \frac{u'(z_j)}{u'(z_i)} (z_j + p(z_j)) \tag{8.2}$$

$$\Rightarrow p(z) = \beta E_z[\frac{u'(z')}{u'(z)} (z' + p(z'))] \tag{8.3}$$

$$\Rightarrow 1 = \beta E_z[\frac{u'(z')}{u'(z)} \frac{z' + p(z')}{p(z)}] \tag{8.4}$$

  Now we get the pricing equation, where $\frac{u'(z')}{u'(z)}$ is called the pricing kernel, $\frac{z'+p(z')}{p(z)}$ denotes the return and sometimes; rewrite the pricing equations, $1 = E_z[\beta \frac{u'(z')}{u'(z)} \frac{z'+p(z')}{p(z)}]$ we call $\beta \frac{u'(z')}{u'(z)}$ the stochastic discount factor.

8.1.2 Risk Premia

- Let $Q(z, z')$ denote the probability that z_{t+1} is z' after z_t takes on the value of z;
- Allow for continuous dividend preocess, $z \in Z$;
- Assume $Q(z, \cdot)$ satisfies for any continuous function $f : Z \to \mathbb{R}$, $g(z) = \int_z f'(z)Q(z, z')dz'$ is continuous in Z.
- Lucas tree, s, price $p(z)$; risk free asset / bond, a (within period, give 1, price $q(z)$);

- Recursive formulation

$$V(a,s,z) = \max_{(a',s')\in P(a,s,z)} u((z+p(z))s + a - p(z)s' - q(z)a') + \beta \int_{z'} V(a',s',z')Q(z,z')dz' \quad (8.5)$$

$$F.O.C. \quad (8.6)$$

$$[s'] : u'(c)p(z) = \beta \int_{z'} V_2(a',s',z')Q(z,z')dz' \quad (8.7)$$

$$[a'] : u'(c)q(z) = \beta \int_{z'} V_1(a',s',z')Q(z,z')dz' \quad (8.8)$$

Derivatives of the value function,

$$[s] : V_2(a,s,z) = u'(c)[z + p(z) - p(z)g'(s)] + \beta \int_{z'} V_2(a',s',z')g'(s)Q(z,z')dz' \quad (8.9)$$

$$[a] : V_1(a,s,z) = u'(c)[1 - q(z)h'(a)] + \beta \int_{z'} V_1(a',s',z')Q(z,z')dz' \quad (8.10)$$

where $g(s) = s', h(a) = a'$ denote policy functions. Combine these four equations, giving

$$V_2(a,s,z) = u'(c)(z + p(z)) \quad (8.11)$$

$$V_1(a,s,z) = u'(c) \quad (8.12)$$

In equilibrium: $a = 0, s = s' = 1$(then $c = z, c; = z'$) then take B-S conditions back to F.O.C., we say

$$q(z)u'(z) = \beta \int_{z'} u'(z')Q(z,z')dz' \quad (8.13)$$

$$p(z)u'(z) = \beta \int_{z'} u'(z')(z' + p(z'))Q(z,z')dz' \quad (8.14)$$

- let $e(z,z') = \frac{z'+p(z')}{p(z)}$, then gross return on equity;
- let $R(z) = \frac{1}{q(z)}$, the risk-free real interest rate;
- Define $E_z(f(z,z')) = \int_{z'} f(z,z')Q(z,z')dz'$ for any function f;
- So Equation 8.14 implies

$$1 = \int_{z'} \beta \frac{u'(z')}{u'(z)} \frac{z'+p(z')}{p(z)} Q(z,z')dz' = E_z[\beta \frac{u'(z')}{u'(z)} e(z,z')] \quad (8.15)$$

$$= E_z[\beta \frac{u'(z')}{u'(z)}]E_z[e(z,z')] + cov_z[\beta \frac{u'(z')}{u'(z)}, e(z,z')] \quad (8.16)$$

$$1 = \frac{E_z[e(z,z')]}{R(z)} + cov_z[\beta \frac{u'(z')}{u'(z)}, e(z,z')] \quad (8.17)$$

Note that Equation 8.13 and $R(z) = \frac{1}{q(z)}$ implies, $\frac{1}{R(z)} = E_z[\beta \frac{u'(z')}{u'(z)}]$, so

$$\frac{E_z[e(z,z')] - R(z)}{R(z)} = -cov_z[\beta \frac{u'(z')}{u'(z)}, e(z,z')] \quad (8.18)$$

where $\frac{E_z[e(z,z')]-R(z)}{R(z)}$ is the risk premia. As z' increases, $\frac{u'(z')}{u'(z)}$ decreases: if $e(z,z')$ increases, then $cov_z < 0$, indicating risk Premia is positive; if $e(z,z')$ decreases, then $cov_z > 0$, indicating risk Premia is negative, which is an insurance, helping smoothing consumption;

8.2 The Equity Premium Puzzle

- SP500 return 6% higher than U.S. government bonds;
- The above fact can not be explained by the neoclassical growth model, hence there exists a puzzle call the **equity premium puzzle**.

8.2.1 The Model

- Key ingredients: an endowment economy, a representative agent and complete market;
- Preference $u = E_0[\sum_{t=0}^{\infty} \beta^t u(c_t)]$ where $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$(CRRA), where
  - $\frac{1}{\sigma}$, intertemporal elasticity of substitution;
  - coefficient of relative risk aversion, $-\frac{u''(c)}{u'(c)}c = -\frac{-\sigma c^{-1-\sigma}c}{c^{-\sigma}} = \sigma$
- endowment process, y_t. $y_{t+1} = x_{t+1}y_t$ where x_{t+1} is a random variable that take n values $x_{t+1} \in \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ and x_{t+1} is a first order Markov process, $p_{ij} = P(x_{t+1} = \lambda_j | x_t = \lambda_i)$;
- Asset prices: Lucas tree, dividend $d_t = y_t = z_t$, with the pricing equation 8.3 we derive the price of the lucas tree by iteration,

$$p_t^e = E_t[\beta \frac{u'(z_{t+1})}{u'(z_t)}(z_{t+1} + p(z_{t+1}))] \tag{8.19}$$

$$= E_t[\beta \frac{u'(z_{t+1})}{u'(z_t)}(z_{t+1} + E_{t+1}[\beta \frac{u'(z_{t+2})}{u'(z_{t+1})}(z_{t+2} + p(z_{t+2}))])] \tag{8.20}$$

$$= E_t[\beta \frac{u'(z_{t+1})}{u'(z_t)} z_{t+1} + \beta^2 \frac{u'(z_{t+2})}{u'(z_{t+1})}[z_{t+2} + p_{z_{t+2}}]] \tag{8.21}$$

$$\cdots \tag{8.22}$$

$$= E_t[\sum_{s=t+1}^{\infty} \beta^{s-t} \frac{u'(z_s)}{u'(z_t)} z_s] \tag{8.23}$$

Then

$$p_t^e = E_t[\sum_{s=t+1}^{\infty} \beta^{s-t} \frac{u'(y_s)}{u'(y_t)} ds] \tag{8.24}$$

$$p_t^e(x_t, y_t) = E[\sum_{s=t+1}^{\infty} \beta^{s-t} (\frac{y_t}{y_s})^{\sigma} y_s | x_t, y_t] \tag{8.25}$$

- For each $x_i, i = 1, 2, \cdots, n$, let $\Phi_{ij} = Q(z_i, z_j)$. $p^e(x_i, y) \equiv p_i^e(y) = \beta \sum_{j=1}^n \Phi_{ij} (\frac{y}{y\lambda_j})^{\sigma} [y\lambda_j + p_j^e(y\lambda_j)], \forall y, \forall i$.
- Guess and verify $p_i^e(y) = p_i^e y$, so

$$p_i^e y = \beta \sum_{j=1}^n \Phi_{ij} \lambda^{-\sigma} [y\lambda_j + p_j^e y \lambda_j] \tag{8.26}$$

$$p_i^e = \beta \sum_{j=1}^n \Phi_{ij} \lambda_j^{1-\sigma} (1 + p_j^e) \tag{8.27}$$

- price of risk-free asset

$$p_i^{rf}(y) = \beta \sum_{j=1}^n \Phi_{ij} \lambda_j^{-\sigma} \tag{8.28}$$

- Gross return $R_{ij}^e = \frac{\lambda_j y + p_j^e(\lambda_j y)}{p_i^e(y)} = \frac{\lambda_j y + p_j^e \lambda_j y}{p_i^e}$;
- Net return $r^e = R_{ij}^e - 1 = \frac{(1+(p_j^e))\lambda_j}{p_i^e} - 1$;
- Conditional expected return: $r^e = E_i(r_{ij}^e) = \sum_{j=1}^n \Phi_{ij} r_{ij}^e$ where π_i is the invariant (long run) distribution of the markov process. r^e: long-run market return.
- Risk-free return $r_i^{rf} = \frac{1}{p_i^{rf}} - 1$; long-run risk-free return: $r^{rf} = \sum_{i=1}^n \pi_i r_i^{rf}$;
- Equity premium $r^e - r^{rf} = 6\%$.

8.2.2 Calibration

- Mehra and Prescoff assume two states $n = 2$, $\lambda_1 = 1 + \mu + \delta, \lambda_2 = 1 + \mu - \delta$ where μ is the average consumption growth $\frac{c_{t+1}-c_t}{c_t}$;
- δ is the variation in the growth rate;
- $\mu = 0.018, \delta = 0.036$;
- Transition matrix $\Phi = \begin{bmatrix} \phi & 1-\phi \\ 1-\phi & \phi \end{bmatrix}$, $\phi = 0.43$ to match first order serial correlation of $\frac{c_{t+1}-c_t}{c_t}$.
- If (β, σ) is normal, to match 6%, we need r^{rf} very high; if other elements are normal, we need β, σ extreme risk aversion, impossible.
- Solutions:
  - Epstein-Zin preferences;
  - Habit persistence (John Cochrane, 1990s);
  - Disaster risk (Bansel and Yaron)

8.3 Optimal Taxation with Commitment: Ramsey Problem

We study the optimal taxation ratio on capital τ_t^k and on income τ_t^n under the condition: the government maximize the total lifetime utility of households. Assume there is a risk-free bond, gives 1 within period and is priced at R_t.

Set up:

- households: $\{c_t^*, l_t^*\}_{t=0}^{\infty}$ solves households' problem,

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \tag{8.29}$$

$$s.t. \begin{cases} n_t + l_t = 1 \\ c_t + k_{t+1} + \frac{b_{t+1}}{R_t} = (1-\tau_t^n) w_t n_t + (1-\tau_t^k) r_t k_t + (1-\delta) k_t + b_t \\ R_t = (1-\tau_{t+1}^k) r_{t+1} + 1 - \delta (\text{no arbitrage condition}) \end{cases} \tag{8.30}$$

  There are two assets, so we need no arbitrage condition. (bond returns equal to capital return)

- firms: $\{k_t^*, n_t^*\}_{t=0}^{\infty}$ to maximize $\pi = F(k_t, n_t) - r_t k_t - w_t n_t$
- government: budget balance, $g_t = \tau_t^k r_t k_t + \tau_t^n w_t n_t + \frac{b_{t+1}}{R_t} - b_t$
- market clear: $c_t + g_t + k_{t+1} = F(k_t, n_t) + (1-\delta) k_t$

A primal approach to solve the Ramsey problem, comprised of five steps,

step (1) obtain the F.O.C. of the households and firms, as well as any arbitrage pricing conditions; solve these conditions for $\{q_t^0 = \prod_{i=0}^{t} R_i^{-1}, r_t, w_t, \tau_n^k, \tau_t\}_{t=0}^{\infty}$ as functions of the condition $\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}$;

step (2) substitute these expressions for taxes and prices in terms of the allocation into the households' present-value budget constraint. This is an intertemporal constraint involving only the allocation;

step (3) solve for the Ramsey allocation by maximizing households lifetime utility, subject to the resource constraint (market clear) and "implementability condition" derived in step (2);

step (4) after the Ramsey allocation is solved, use the formulas from step (1) to find taxes and prices.

The present-value budget constraint by eliminating b_t, k_t,

$$\sum_{t=0}^{\infty} q_t^0 c_t = \sum_{t=0}^{\infty} [q_t^0 (1-\tau_t^n) w_t n_t] + [(1-\tau_0^k) r_0 + 1 - \delta] k_0 + b_0 \tag{8.31}$$

where $q_t^0 = \prod_{i=0}^{t} R_i^{-1}$.

Now construct the Ramsey plan.

Step 1, the Langrage function,

$$L \sum_{t=0}^{\infty} \beta^t u(c_t, 1-n_t) + \lambda[\sum_{t=0}^{\infty}[q_t^0(1-\tau_t^n)w_t n_t] + [(1-\tau_0^k)r_0 + 1 - \delta]k_0 + b_0 - \sum_{t=0}^{\infty} q_t^0 c_t] \tag{8.32}$$

F.O.C., let $f_a(t)$ denote $f'(a_t)$ for u, F.

$$[c_t] : \beta^t u_c(t) - \lambda q_t^0 = 0 \tag{8.33}$$

$$[n_t] : -\beta^t u_l(t) + \lambda q_t^0(1-\tau_t^n)w_t = 0 \tag{8.34}$$

$$q_0^0 = 1 \Rightarrow q_t^0 = \beta^t \frac{u_c(t)}{u_c(0)} \tag{8.35}$$

$$(1-\tau_t^n)w_t = \frac{u_l(t)}{u_c(t)} \tag{8.36}$$

from the no arbitrage condition,

$$\frac{q_{t-1}^0}{q_t^0} = (1-\tau_{t+1}^k)r_{t+1} + 1 - \delta \tag{8.37}$$

Firm's F.O.C. gives $r_t = F_k(t), w_t = F_n(t)$.

Step 2, substitute equations 8.35, 8.36 and $r_0 = F_k(0)$ into the present-value budget constraint,

$$\sum_{t=0}^{\infty} \beta^t [u_c(t)c_t - u_l(t)n_t] - A = 0 \tag{8.38}$$

where $A = A(c_0, n_0, \tau_0^k) = u_c(0)[[(1-\tau_0^k)F_k(0) + (1-\delta)]k_0 + b_0]$

Step 3, the Ramsey problem is to maximize households' lifetime utility s.t. Equation 8.38, and the resource constraint (market clear). Let Φ be the Langrage multiplier of Equation 8.38 and define

$$V(c_t, n_t, \Phi) = u(c_t, 1-n_t) + \Phi[u_c(t)c_t - u_l(t)n_t] \tag{8.39}$$

$$L = \sum_{t=0}^{\infty}\{V(c_t, n_t, \Phi) + \theta_t[F(k_t, n_t) + (1-\delta)k_t - c_t - g_t - k_{t+1}]\} - \Phi A \tag{8.40}$$

where $\{\theta_t\}_{t=0}^{\infty}$ is a sequence of Langrage multipliers. For given k_0, b_0 fix τ_0^k F.O.C. w.r.t. $\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}$,

$$[c_t] : V_c(t) = \theta_t, t \geq 1 \tag{8.41}$$

$$[n_t] : V_n(t) = -\theta_t F_n(t), t \geq 1 \tag{8.42}$$

$$[k_{t+1}] : \theta_t = \beta\theta_{t+1}[F_k(t+1) + 1 - \delta], t \geq 0 \tag{8.43}$$

$$[c_0] : V_c(0) = \theta_0 + \Phi A_c \tag{8.44}$$

$$[n_0] : V_n(0) = -\theta_0 F_n(0) + \Phi A_n \tag{8.45}$$

$$V_n(0) = [\Phi A_v - V_c(0)]F_n(0) + \Phi A_n \tag{8.46}$$

so, we have the equilibrium condition,

$$V_c(t) = \beta V_c(t+1)[F_k(t+1) + 1 - \delta], t \geq 1 \tag{8.47}$$

$$V_n(t) = -V_c(t)F_n(t), t \geq 1 \tag{8.48}$$

Plus the market clear and budget constraint,

$$c_t + g_t + k_{t+1} = F(k_t, n_t) + (1-\delta)k_t \tag{8.49}$$

$$\sum_{t=0}^{\infty} \beta^t [u_c(t)c_t - u_l(t)n_t] - A = 0 \tag{8.50}$$

We can seek an allocation $\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}$ and a multiplier Φ.

Step 4, after an allocation is found, obtain $q_t^0, r_t, w_t, \tau_t^n, \tau_t^k$, then the optimal tax rate: assume $g_t = g$ for all $t \geq T$ and there exists a solution to the Ramsey problem and that it converges to a time invariant allocation. Then because $V_c(t)$ converges to a constant, then Equation 8.47 implies ($V_c(t) = V_c(t+1)$)

$$1 = \beta(F_k + 1 - \delta) \tag{8.51}$$

Equation 8.33 $\beta^t u_c(t) - \lambda q_t^0 = 0 \Rightarrow \frac{q_t^0}{q_{t+1}^0} = \frac{1}{\beta}\frac{u_c(t)}{u_c(t+1)} \to 1$, then as $t \to \infty$, $\frac{q_t^0}{q_{t+1}^0} \to \frac{1}{\beta}$, so Equation 8.37 becomes,

$$1 = \beta[(1-\tau^k)F_k + 1 - \delta], t \to \infty \tag{8.52}$$

With Equation 8.51 and 8.52, we can solve τ^k, and the optimal capital tax is zero in the long run.

The limitation of this model,

1. tax is a proportion to capital / income (linear);
2. research on representative households (homogeneous).

The improved version is called New Dynamic Public Finance (still limited in step (3), in reality, the government has other aims)

9 Models of Money

9.1 Dynamic General Equilibrium Cash-Credit Goods Market

Assume households (homogeneous, representative) consume and product goods (there is no firm) but the household can only consume goods produced by others, instead of his production (must exchange, s.t. CIA constraint). There are two goods, cash goods and credit goods, the former must by purchased by cashes, but the latter can be purchased by credit and must repay at the end of the period.

The model is set as follows.

- household chooses $\{C_{1t}, C_{1t}\}_{t=0}^{\infty}$ to maximize lifetime utility,

$$\sum_{t=0}^{\infty} \beta^t E_t[\frac{1}{\alpha}C_{1t}^{\alpha} + \frac{\varphi}{\alpha}C_{2t}^{\alpha}] \tag{9.1}$$

  where C_{1t} is the consumption of cash good and C_{1t} is the consumption of credit good at t; leisure is not valued so the production containt will be tight.

- production,

$$C_{1t}^s + C_{2t}^s \leq 1 \tag{9.2}$$

- cash-in-advance constraint (CIA),

$$P_t C_{1t} \leq M_t \tag{9.3}$$

  where P_t is the nominal price of consumption (one price, no arbitrage) and M_t denotes the money holding.

- law of motion of money

$$M_{t+1} = M_t + P_t(C_{1t}^s + C_{2t}^s - C_{1t} - C_{2t}) + (g_{t+1} - 1)M_t^s \tag{9.4}$$

  where g_{t+1} is given by the government policy and M_t^s is the money supply. Here M_0 is given and $M_t \geq 0, \forall t$. (view $(g_{t+1} - 1)M_t^s$ as government transfer)

- Money market clear, $M_t^s = M_t, \forall t$, so Equation 9.4 implies $C_{1t}^s + C_{2t}^s = C_{1t} + C_{2t}$, i.e., goods market clear.

- Transversality conditions (TVC, no arbitrage condition)

$$\lim_{T\to\infty} E[\beta^T u'(C_{1T})\frac{M_T}{P_T}] = 0 \tag{9.5}$$

  which implies the discounted value of money at infinite period is zero. (money is memory, goods are real)

The Langrage function and F.O.C.,

$$L = \max_{\{C_{1t},C_{2t},M_{t+1}\}} \sum_{t=0}^{\infty} \beta^t E_t[\frac{1}{\alpha}C_{1t}^{\alpha} + \frac{\varphi}{\alpha}C_{1t}^{\alpha} + \mu_t(M_t - P_tC_{1t}) \tag{9.6}$$

$$+ \lambda_t[M_t + P_t(1 - C_{1t} - C_{1t}) + (g_{t+1} - 1)M_t^s - M_{t+1}]] \tag{9.7}$$

$$F.O.C. \tag{9.8}$$

$$[C_{1t}] : C_{1t}^{\alpha-1} - \mu_tP_t - \lambda_tP_t = 0 \tag{9.9}$$

$$[C_{2t}] : \varphi c_{2t}^{\alpha-1} - \lambda_tP_t = 0 \tag{9.10}$$

$$[M_{t+1}] : -\lambda_t + \beta E_t[\mu_{t+1} + \lambda_{t+1}] = 0 \tag{9.11}$$

Why do we take E_t out at $[C_{1t}], [C_{2t}]$?[2]

Then we say

$$\lambda_t + \mu_t = \frac{C_{1t}^{\alpha-1}}{P_t} \tag{9.12}$$

$$\lambda_t = \frac{\varphi_{2t}^{\alpha-1}}{P_t} \tag{9.13}$$

$$\Rightarrow \varphi\frac{C_{2t}^{\alpha-1}}{P_t} = \beta E_t[\frac{C_{1(t+1)}^{\alpha-1}}{P_{t+1}}] \tag{9.14}$$

which implies the cost of giving up a unit C_{2t} equals the discounted utility of but an unit of C_{1t} at $t + 1$. (cash holding gives no interest, only to buy cash goods)

Government set the money supply rule,

$$M_{t+1}^s = M_t^s + T_t, T_t = (g_{t+1} - 1)M_t^s \tag{9.15}$$

where $g_{t+1} = \rho g_t + (1 - \rho)\bar{g} + \epsilon_t$ where ϵ_t is a white noise process ($AR(1)$).

DEFINITION 9.1. *A competitive monetary equilibrium is a stochastic process*

$$\{C_{1t}^s, C_{2t}^s, C_{1t}, C_{2t}, P_t, M_t, M_t^s, g_{t+1}\}_{t=0}^{\infty} \tag{9.16}$$

s.t.

(a) households maximize utility subject to budget constraint, CIA constraint, non-negativity constraint, TVC by choosing $\{C_{1t}, C_{2t}, C_{1t}^s, C_{2t}^s, M_{t+1}\}_{t=0}^{\infty}$ *taking* $M_0, \{P_t, M_t^s, g_{t+1}\}_{t=0}^{\infty}$ *as given;*

(b) the government set the money supply rule, $M_{t+1}^s = g_{t+1}M_t^s$*;*

(c) market clear, $M_t^s = M_t, C_{1t}^s = C_{1t}, C_{2t}^s = C_{2t}$.

Imposing market clear, households budget constraint becomes,

$$M_{t+1}^s = M_t^s + (g_{t+1} - 1)M_t^s \Rightarrow M_{t+1}^s = g_{t+1}M_t^s \tag{9.17}$$

Collecting equilibrium equations,

$$M_{t+1}^s = g_{t+1}M_t^s \tag{9.18}$$

$$1 = C_{1t} + C_{2t} \tag{9.19}$$

$$M_t^s = P_tC_{1t} \tag{9.20}$$

$$\frac{\varphi C_{2t}^{\alpha-1}}{P_t} = \beta E_t[\frac{C_{1(t+1)}^{\alpha-1}}{P_{t+1}}] \tag{9.21}$$

and two boundary condition, M_0^s given and TVC.

[2] When the household make decisions at t, the past information is known, i.e., the household chooses consumption based on Φ_t (information set). Actually, $[C_{1t}] : E_t[C_{1t}^{\alpha-1} - \mu_tP_t - \lambda_tP_t = 0|\Phi_t] = C_{1t}^{\alpha-1} - \mu_tP_t - \lambda_tP_t = 0$ (known can be take out), and so does $[C_{2t}]$. As for $[M_{t+1}]$, we have $E_t[-\lambda_t + \beta E_{t+1}[\mu_{t+1} + \lambda_{t+1}]|\Phi_t] = -\lambda_t + \beta E_t[\mu_{t+1} + \lambda_{t+1}] = 0$ (the law of ierated expectations).

There are four unknowns, $\{M_t^s, C_{1t}, C_{2t}, P_t\}$. Substitute out M_t^s with Equation 9.20 and C_{2t} using Equation 9.19, we obtain

$$P_{t+1}C_{1(t+1)} = g_{t+1}P_tC_{1t} \tag{9.22}$$

$$(1-C_{1t})^{\alpha-1} = \frac{\beta}{\varphi}E_t[\frac{P_t}{P_{t+1}}C_{1(t+1)}^{\alpha-1}] \tag{9.23}$$

From Equation 9.22, we see $\frac{P_t}{P_{t+1}} = \frac{C_{1(t+1)}}{g_{t+1}C_{1t}}$, then

$$(1-C_{1t})^{\alpha-1} = \gamma E_t[\frac{1}{g_{t+1}}\frac{C_{1(t+1)}^{\alpha}}{C_{1t}}] \tag{9.24}$$

where $\gamma = \frac{\beta}{\varphi}$. In steady state, we have $C_{1t} = C_1, g_{t+1} = \bar{g}$ and so Equation 9.24 becomes

$$1 = \frac{\gamma}{\bar{g}}(\frac{C_1}{1-C_1})^{\alpha-1} \tag{9.25}$$

Example 9.1. *A constant money growth, $g_{t+1} = \bar{g} > \beta$ (use lowercase for simplicity, e.g., c_1 replaces C_1). In steady state, $c_{1t} = c_{1(t+1)} = c_1$, then*

$$(1-c_1)^{\alpha-1} = \gamma\frac{1}{\bar{g}}\frac{c_1^{\alpha}}{c_1} \tag{9.26}$$

so

$$c_1 = [(\frac{\gamma}{\bar{g}})^{\frac{1}{\alpha-1}} + 1]^{-1} \tag{9.27}$$

TVC,

$$\lim_{T\to\infty} E_t[\beta^T(c_{1T})^{\alpha-1}c_{1T}] = \lim_{T\to\infty}\beta^T c_{1T}^{\alpha} = 0 \tag{9.28}$$

The price level, from Equation 9.22 and $c_{1(t+1)} = c_{1t}$ and $g_{t+1} = \bar{g}$,

$$\frac{p_{t+1}}{p_t} = \bar{g} \tag{9.29}$$

which implies the inflation increases with $\bar{g}$.

From Equation 9.27, we have

$$\frac{\partial c_1}{\partial \bar{g}} < 0 \tag{9.30}$$

i.e., inflation causes households to "economize" on activities that requires money.

If there is no CIA constraint, the solution is a Pareto optimal allocation, as

$$L = \sum_{t=0}^{\infty}\beta^t[\frac{c_{1t}^{\alpha}}{\alpha} + \frac{\varphi}{\alpha}c_{2t}^{\alpha} + \lambda_t(1-c_{1t}-c_{2t})] \tag{9.31}$$

$$F.O.C. \Rightarrow 1 = \frac{1}{\varphi}(\frac{c_1^*}{c_2^*})^{\alpha-1} \tag{9.32}$$

The left hand side is the cocial MRT (marginal rate of transition) and the right hand is the socaial MRS (marginal rate of substitution). ($c_1^* + c_2^* = 1$)

Whether is there ia government policy $\{g_{t+1}\}_{t=0}^{\infty}$ where the allocation from the monetary equilibrium equals the Pareto optimal? YES, (plus Equation 9.25, 9.32 and $c_1^* + c_2^* = 1$)

$$1 = \frac{1}{\varphi}(\frac{c_1^*}{1-c_1^*})^{\alpha-1} \tag{9.33}$$

$$\bar{g} = \gamma(\frac{c_1}{1-c_1})^{\alpha} \tag{9.34}$$

$$c_1 = c_1^* \tag{9.35}$$

$$\Rightarrow \bar{g} = \beta = \varphi\gamma \tag{9.36}$$

which is called the "Fredman Rule": optimal deflation ($\bar{g} < 1$).

Example 9.2. *Random policy,*

$$\hat{g}_{t+1} = \rho\hat{g}_t + \hat{\epsilon}_{t+1}, \hat{\epsilon}_{t+1} \sim (0, \sigma^2), |\rho| < 1 \tag{9.37}$$

where $\hat{g}_t$ is log-derivation of money growth from the steady state, i.e.,

$$g_t = \bar{g}e^{\hat{g}_t} \Rightarrow \hat{g}_t = \log g_t - \log \bar{g} \tag{9.38}$$

The log-linearization : $X_t = \bar{X}e^{\hat{X}_t} \simeq \bar{X}(1 + \hat{X}_t)$. Now log-linearization Equation 9.24,

$$(1 - c_{1t})^{\alpha-1} = \gamma E_t[\frac{1}{g_{t+1}}\frac{c_{1(t+1)}^{\alpha}}{c_{1t}}] \tag{9.39}$$

$$LHS = c_{2t}^{\alpha-1} \tag{9.40}$$

$$= (\bar{c}_2 e^{\hat{c}_{2t}})^{\alpha-1} \tag{9.41}$$

$$= \bar{c}_2^{\alpha-1}e^{(\alpha-1)\hat{c}_{2t}} \tag{9.42}$$

$$= \bar{c}_2^{\alpha-1}(1 + (\alpha - 1)\hat{c}_{2t}) \tag{9.43}$$

$$c_{2t} = 1 - c_{1t} \tag{9.44}$$

$$\Rightarrow \bar{c}_2 e^{\hat{c}_{2t}} = 1 - \bar{c}_1 e^{\hat{e}_{1t}} \tag{9.45}$$

$$\Rightarrow \bar{c}_2(1 + \hat{c}_{2t}) = 1 - \bar{c}_1(1 + \hat{c}_{1t}) \tag{9.46}$$

$$\bar{c}_1 + \bar{c}_2 = 1 \Rightarrow \bar{c}_2\hat{c}_{2t} = -\bar{c}_1\hat{c}_{1t} \tag{9.47}$$

$$\Rightarrow \hat{c}_{2t} = -\frac{\bar{c}_1}{\bar{c}_2}\hat{c}_{1t} \tag{9.48}$$

$$= -\frac{\bar{c}_1}{1 - \bar{c}_1}\hat{c}_{1t} \tag{9.49}$$

So the left hand side equals

$$LHS = (1 - \bar{c}_1)^{\alpha-1}[1 + (\alpha - 1)(-\frac{\bar{c}_1}{1 - \bar{c}_1})\hat{c}_{1t}] \tag{9.50}$$

The right hand side,

$$RHS = \gamma E_t[\frac{c_{1(t+1)}^{\alpha}}{g_{t+1}c_{1t}}] \tag{9.51}$$

$$= \gamma E_t[\frac{(\bar{c}_1 e^{\hat{c}_{1(1+t)}})^{\alpha}}{\bar{g}e^{\hat{g}_{t+1}}\bar{c}_1 e^{\hat{c}_{1t}}}] \tag{9.52}$$

$$= \gamma E_t[\frac{\bar{c}_1^{\alpha}}{\bar{g}\bar{c}_1}e^{\alpha\hat{c}_{1(t+1)} - \hat{g}_{t+1} - \hat{c}_{1t}}] \tag{9.53}$$

$$= \gamma E_t[\frac{\bar{c}_1^{\alpha-1}}{\bar{g}}e^{\alpha\hat{c}_{1(t+1)} - \hat{g}_{t+1} - \hat{c}_{1t}}] \tag{9.54}$$

$$\simeq \gamma E_t[\frac{\bar{c}_1^{\alpha-1}}{\bar{g}}(1 + \alpha\hat{c}_{1(t+1)} - \hat{g}_{t+1} - \hat{c}_{1t})] \tag{9.55}$$

$$\tag{9.56}$$

Then, plus Equation 9.25 (to drop one term on each sides), we have

$$LHS = RHS \tag{9.57}$$

$$(1 - \bar{c}_1)^{\alpha-1}[1 + (\alpha - 1)(-\frac{\bar{c}_1}{1 - \bar{c}_1})\hat{c}_{1t}] = \gamma E_t[\frac{\bar{c}_1^{\alpha-1}}{\bar{g}}(1 + \alpha\hat{c}_{1(t+1)} - \hat{g}_{t+1} - \hat{c}_{1t})] \tag{9.58}$$

$$(1 - \bar{c}_1)^{\alpha-2}(1 - \alpha)\bar{c}_1\hat{c}_{1t} = \gamma E_t[\frac{\bar{c}_1^{\alpha-1}}{\bar{g}}(\alpha\hat{c}_{1(t+1)} - \hat{g}_{t+1} - \hat{c}_{1t})] \tag{9.59}$$

Conjecture $\hat{c}_{1t} = \phi\hat{g}_t$, plus Equation 9.25 (to drop multiplier on each sides), Equation 9.59 becomes

$$\varphi\hat{g}_t = \frac{1}{1 - \alpha}\frac{1 - \hat{c}_1}{\hat{c}_1}E_t[\alpha\phi\hat{g}_{t+1} - \hat{g}_{t+1} - \phi\hat{g}_t] \tag{9.60}$$

$$\phi\hat{g}_t = \frac{1}{1 - \alpha}\frac{1 - \hat{c}_1}{\hat{c}_1}E_t[(\alpha\phi - 1)(\rho\hat{g}_t + \hat{\epsilon}_{t+1}) - \phi\hat{g}_t] \tag{9.61}$$

$$= \frac{1}{1 - \alpha}\frac{1 - \hat{c}_1}{\hat{c}_1}[(\alpha\phi - 1)\rho\hat{g}_t - \phi\hat{g}_t] \tag{9.62}$$

Case A: $\rho = 0$, *then* $\phi = -\frac{1}{1-\alpha}\frac{1-\bar{c}_1}{\bar{c}_1}\phi\hat{g}_t$, *then* $\phi = 0$, *i.e., if* $\hat{g}_t$ *is i.i.d., then* $\hat{c}_{1t} = 0$, c_{1t} *is a constant(unanticipated money growth has no real effects on quantities)*

Case B: $\rho \neq 0$, *then* $\phi = \frac{-\rho}{1+(1-\alpha)(1-\alpha\rho)(\frac{1-\bar{c}_1}{\bar{c}_1})}$. *If* $\rho > 0$, *then* $\phi < 0$, *and* $\hat{c}_{1t} = \phi\hat{g}_t$ *implies if* $\hat{g}_t$ *increases, then* $\hat{c}_{1t}$ *decreases (anticipated inflation increases, then decreases* c_{1t} *and increases* c_{2t} $(c_{1t} + c_{2t} = 1)$ *to transfer less cash to the next period).*

9.2 The Cash-Credit Goods Model with Bond Market

Assume one period nominal bond, denoted as B_{t+1} (within period, less state variables, better). The household lend one dollar at t, receives R_t at the end of t, so the new constraint becomes

$$P_tC_{1t} \leq M_t - \frac{B_{t+1}}{R_t} \tag{9.63}$$

and the budget constraint,

$$M_{t+1} = M_t + B_{t+1} - \frac{B_{t+1}}{R_t} + p_t(1 - c_{1t} - c_{2t}) + (g_{t+1})M_t^s \tag{9.64}$$

The timeline, see Figure 17.

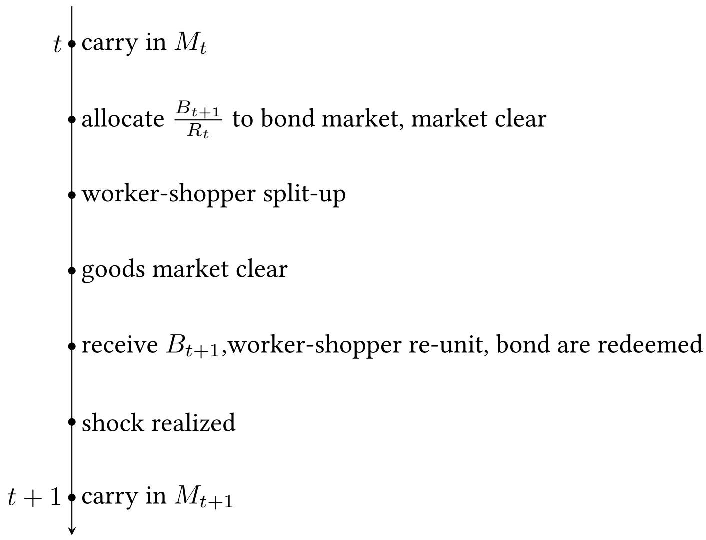


Figure 17: Timeline of cash-credit goods market with bond

Let $u(C_{1t}, C_{2t}) = \frac{1}{\alpha}C_{1t}^{\alpha} + \frac{\varphi}{\alpha}C_{2t}^{\alpha}$, then the Langrage function and F.O.C.,

$$\begin{aligned} L = \sum_{t=0}^{\infty}\beta^t E_t[u(C_{1t}, C_{2t}) + \lambda_t[B_{t+1} - \frac{B_{t+1}}{R_t} + P_t(1 - C_{1t} - C_{2t}) + (g_{t+1} - 1)M_t^s + M_t - M_{t+1}] \\ + \mu_t[M_t - \frac{B_{t+1}}{R_t} - P_tC_{1t}]] \end{aligned} \tag{9.65}$$

$$F.O.C. \tag{9.66}$$

$$[C_{1t}] : C_{1t}^{\alpha-1} - \lambda_tP_t + \mu_tP_t = 0 \tag{9.67}$$

$$[C_{2t}] : \varphi C_{2t}^{\alpha-1} = \lambda_tP_t \tag{9.68}$$

$$[M_{t+1}] : \lambda_t = \beta E_t[\lambda_{t+1} + \mu_{t+1}] \tag{9.69}$$

$$[B_{t+1}] : \lambda(1 - \frac{1}{R_t}) - \mu_t\frac{1}{R_t} = 0 \tag{9.70}$$

Equation 9.67 implies $\lambda_t + \mu_t = \frac{C_{1t}^{\alpha-1}}{P_t}$, plus Equation 9.69, we have $\lambda_t = \beta E_t[\frac{C_{1(t+1)}^{\alpha-1}}{P_{t+1}}]$. Equation 9.70 implies

$\lambda_t R_t = \mu_t + \lambda_t$. Summing up,

$$\varphi \frac{C_{2t}^{\alpha-1}}{P_t} R_t = \frac{C_{1t}^{\alpha-1}}{P_t} \tag{9.71}$$

$$\Rightarrow R_t = \frac{C_{1t}^{\alpha-1}}{\varphi C_{2t}^{\alpha-1}} = \frac{u_{1t}}{u_{2t}} \tag{9.72}$$

where the left hand side is the private MRT between cash and credit goods (save money to buy bond) and the right hand is the MRS.

Further, Equation 9.68 implies

$$R_t = \frac{c_{1t}^{\alpha-1}/p_t}{\beta E_t[c_{1(t+1)}^{\alpha-1}/p_{t+1}]} = \frac{c_{1(t+1)}^{\alpha-1}}{\beta E_t[c_{1t}^{\alpha-1}/\pi_{t+1}]}, \pi_{t+1} = \frac{p_{t+1}}{p_t} \tag{9.73}$$

Then, in steady state, $\bar{R} = \frac{\bar{\pi}}{\beta}$. Now log-linearization,

$$LHS = \bar{R}e^{\hat{R}_t} \simeq \bar{R}(1 + \hat{R}_t) \tag{9.74}$$

$$RHS = \frac{1}{\beta} \frac{\bar{c}_1^{\alpha-1} e^{(\alpha-1)\hat{c}_{1t}}}{E_t[\frac{\bar{c}_1^{\alpha-1}}{\bar{\pi}} e^{(\alpha-1)\hat{c}_{1(t+1)} - \hat{\pi}_{t+1}}]} \tag{9.75}$$

$$= \frac{\bar{\pi}}{\beta} \frac{1}{E_t[e^{(\alpha-1)\hat{c}_{1(t+1)} - \hat{\pi}_{t+1} - (\alpha-1)\hat{c}_{1t}}]} \tag{9.76}$$

$$LHS = RHS \Rightarrow \hat{R}_t = E_{[}\hat{\pi}_{t+1} + (\alpha - 1)(\hat{c}_{1(t+1) - \hat{c}_{1t}})] \tag{9.77}$$

which is exactly the Fisher Equation: nominal interest rate = real interest rate + inflation.

9.3 Dixit-Stiglitz Monopolistic Competition

Household preferences: $u(C, N)$ where c is an aggregator over $i \in [0, 1]$ type of goods,

$$C = [\int_0^1 C(i)^{\frac{\theta-1}{\theta}} di]^{\frac{\theta}{\theta-1}} \tag{9.78}$$

(view $\int_0^1$ as summation) where $C(i)$ is the consumption of the good i and the parameter θ governs the price elasticity of the individual goods.

Cost minimization by households,

$$\min_{[C(i)]_{i=0}^1} \int_0^1 P(i)C(i)di \tag{9.79}$$

$$s.t. C \leq [\int_0^1 C(i)^{\frac{\theta-1}{\theta}} di]^{\frac{\theta}{\theta-1}} \tag{9.80}$$

The Langrage function and F.O.C. tell

$$L = \int_0^1 P(i)C(i)di - \lambda([\int_0^1 C(i)^{\frac{\theta-1}{\theta}} di]^{\frac{\theta}{\theta-1}} - C) \tag{9.81}$$

$$[C(i)] : P(i) = \lambda \frac{\theta}{\theta - 1} [\int_0^1 C(i)^{\frac{\theta-1}{\theta}} di]^{\frac{\theta}{\theta-1}-1} \frac{\theta - 1}{\theta} C(i)^{-\frac{1}{\theta}} \tag{9.82}$$

$$\tag{9.83}$$

Then

$$P(i) = \lambda [\int_0^1 C(i)^{\frac{\theta-1}{\theta}} di]^{\frac{\theta}{\theta-1}-1} C(i)^{-\frac{1}{\theta}} \tag{9.84}$$

$$\Rightarrow P(i)^{1-\theta} = \lambda^{1-\theta} [\int_0^1 C(i)^{\frac{\theta-1}{\theta}} di]^{-1} C(i)^{\frac{\theta-1}{\theta}} \tag{9.85}$$

$$\Rightarrow [\int_0^1 P(i)^{1-\theta} di]^{\frac{1}{1-\theta}} = \lambda [\int_0^1 C(i)^{\frac{1-\theta}{\theta}} di]^{\frac{1}{\theta-1}} [\int_0^1 C(i)^{\frac{\theta-1}{\theta}} di]^{\frac{1}{1-\theta}} \tag{9.86}$$

$$\Rightarrow [\int_0^1 P(i)^{1-\theta} di]^{\frac{1}{1-\theta}} = \lambda \tag{9.87}$$

Define

$$P \equiv [\int_0^1 P(i)^{1-\theta} di]^{\frac{1}{1-\theta}} = \lambda \tag{9.88}$$

Then, Equation 9.84 implies

$$\begin{aligned} P(i) &= P[\int_0^1 C(i)^{\frac{\theta-1}{\theta}} di]^{\frac{1}{\theta-1}} C(i)^{-\frac{1}{\theta}} && (9.89)\\ &= PC^{\frac{1}{\theta}} C(i)^{-\frac{1}{\theta}} && (9.90)\\ \Rightarrow \frac{P(i)}{P} &= (\frac{C(i)}{C})^{-\frac{1}{\theta}} && (9.91)\\ \Rightarrow \frac{C(i)}{C} &= (\frac{P(i)}{P})^{-\theta} && (9.92)\\ \Rightarrow C(i) &= C[\frac{P(i)}{P}]^{-\theta} && (9.93) \end{aligned}$$

which is called **the demand curve**. Under this curve, the total cost of consumption is

$$\begin{aligned} \int_0^1 P(i)C(i)di &= \int_0^1 P(i)C[\frac{P(i)}{P}]^{-\theta} di && (9.94)\\ &= CP^{\theta} \int_0^1 P(i)^{1-\theta} di && (9.95)\\ &= CP^{\theta} P^{1-\theta} && (9.96)\\ &= PC && (9.97) \end{aligned}$$

Formally,

$$\int_0^1 P(i)C(i)di = PC \tag{9.98}$$

Firms' problem, production function is $Y(i) = N(i)$, and in equilibrium, $Y(i) = C(i)$. Firm i' aims to maximize profit,

$$\max_{C(i),P(i)} \Pi(i) = P(i)C(i) - WN(i) \tag{9.99}$$

$$s.t. \begin{cases} C(i) = [\frac{P(i)}{P}]^{-\theta} C \\ Y(i) = N(i) = C(i) \end{cases} \tag{9.100}$$

constant W implies the wage is the same in all firms (one price, no arbitrage) and we normalize it at 1, i.e., $w = 1$. Then the problem becomes,

$$\max_{[c(i)]} [\frac{C(i)}{C}]^{-\frac{1}{\theta}} PC(i) - C(i) \tag{9.101}$$

$$F.O.C.[C(i)] : (1 - \frac{1}{\theta})(\frac{C(i)}{C})^{-\frac{1}{\theta}} P = 1 \tag{9.102}$$

In a symmetric equilibrium, $C(i) = C$ and $P = P(i) = \frac{\theta}{\theta-1}$.

Gross mark-up (price plus):

$$\frac{P}{W} = \frac{\frac{\theta}{\theta-1}}{1} = \frac{\theta}{\theta-1} \tag{9.103}$$

where p is the price and w is the marginal cost.

- if $\theta = +\infty$, then $P = W$, the market is in perfect competition;
- if $\theta \to 1^+$, then the mark-up becomes infinite. (monopoly)

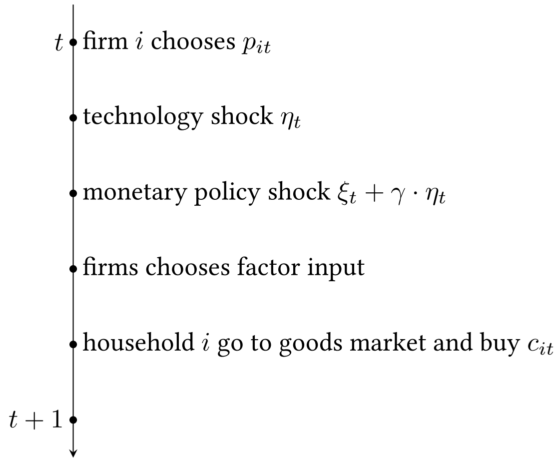


Figure 18: Timeline of the MIU model

9.4 Monty in Utility, Gali (1999), AER

Assume households own firms. Firms decide the goods price at the start of a period, then the technology shock happens, the government use monetary policy as response, giving a monetary shock, then firms choose input factors to meet households' consumption demand. See Figure 18.

- household, MIU, money in utility,

$$\max_{M_t,N_t,U_t,C_t} \sum_{t=0}^{\infty} \beta^t E_t[\log C_t + \lambda_m \log(\frac{M_t}{P_t}) - \frac{\lambda_N}{1+\sigma_N} N_t^{1+\sigma_N} - \frac{\lambda_U}{1+\sigma_U} U_t] \tag{9.104}$$

$$s.t. \begin{cases} \int_0^1 P_{it} C_{it} di + M_t = W_t \cdot N_t + V_t \cdot U_t + M_{t-1} + T_t + \Pi_t \\ C_t = [\int_0^1 C_{it}^{\frac{\epsilon-1}{\epsilon}} di]^{\frac{\epsilon}{\epsilon-1}} \end{cases} \tag{9.105}$$

where N_t denotes labor hours, W_t denotes wages of labor houers, U_t denotes labor efforts, V_t denotes wages of labor efforts, T_t denotes tax / transfer from the government, Π_t denotes firms' profit and ϵ denotes the elasticity of consumption.

Note that, the household first choose total consumption C_t, then C_{it} is decided by the demand curve, Equation 9.93.

With Equation 9.98, the Langrage function and F.O.C.,

$$L = \sum_{t=0}^{\infty} \beta^t E_t[\log C_t + \lambda_m \log(\frac{M_t}{P_t}) - \frac{\lambda_N}{1+\sigma_N} N_t^{1+\sigma_N} - \frac{\lambda_U}{1+\sigma_U} U_t \tag{9.106}$$

$$+ \lambda_t(W_t N_t + V_t U_t + M_{t-1} + T_t + \Pi_t - P_t C_t - M_t)] \tag{9.107}$$

$F.O.C.$

$$[C_t] : \frac{1}{C_t} - \lambda_t P_t = 0 \tag{9.108}$$

$$[N_t] : -\lambda_N N_t^{\sigma_N} + \lambda_t W_t = 0 \tag{9.109}$$

$$[U_t] : -\lambda_t U_t^{\sigma_U} + \lambda_t N_t = 0 \tag{9.110}$$

$$[M_t] : \lambda_m \frac{1}{M_t} - \lambda_t + \beta E_t[\lambda_{t+1}] = 0 \tag{9.111}$$

Then we obtain equilibrium conditions for households,

$$\frac{W_t}{P_t} = \lambda_N \cdot C_t \cdot N_t^{\sigma_N} \text{ hours-consumption} \tag{9.112}$$

$$\frac{V_t}{P_t} = \lambda_U \cdot C_t \cdot U_t^{\sigma_U} \text{ efforts-consumption} \tag{9.113}$$

$$\frac{1}{C_t} = \lambda_m \cdot \frac{P_t}{M_t} + \beta E_t[\frac{1}{C_{t+1}} \cdot \frac{P_t}{P_{t+1}}] \text{ consumption Euler Equation} \tag{9.114}$$

- the firm i want to maximize profit, but have to meet demand at their pre-set price, can still look for cost minimizing way to meet their demand, giving, (given Y_{it})

$$\min_{N_{it},U_{it}} N_{it}W_t + U_{it}N_t \tag{9.115}$$
$$s.t. Y_{it} = Z_t[N_{it}^{\theta}U_{it}^{1-\theta}]^{\alpha} \tag{9.116}$$

the F.O.C. gives,

$$\frac{U_{it}}{N_{it}} = \frac{1-\theta}{\theta}\frac{W_t}{V_t} \tag{9.117}$$

The the firm i can choose the price,

$$\max_{(N_{it}),(U_{it})} E_{t-1}[\frac{1}{C_t}(P_{it}Y_{it} - W_tN_{it} - V_tU_{it})] \tag{9.118}$$
$$s.t. \begin{cases} Y_{it} = (\frac{P_{it}}{P_t})^{-\epsilon}Y_t \text{ demand curve} \\ Y_{it} = Z_t[N_{it}^{\theta}U_{it}^{1-\theta}]^{\alpha} \text{ production function} \end{cases} \tag{9.119}$$

where $\frac{1}{C_t}$ denotes the marginal utility of consumption, Z_t denotes the technology shock, common to all firms.

Take Equation 9.117 into the production function, we see

$$Y_{it} = Z_tN_{it}^{\alpha}(\frac{1-\theta}{\theta}\frac{W_t}{V_t})^{\alpha(1-\theta)} \tag{9.120}$$
$$= N_{it}^{\alpha}\kappa_t \tag{9.121}$$

where $\kappa_t = Z_t(\frac{1-\theta}{\theta}\frac{W_t}{V_t})^{\alpha(1-\theta)}$. Then with the demand curve, we know N_{it},

$$N_{it} = \kappa^{\frac{1}{-\alpha}}Y_{it}^{\frac{1}{\alpha}} \tag{9.122}$$
$$= \kappa^{-\frac{1}{\alpha}}(Y_t(\frac{P_{it}}{P_t})^{-\epsilon})^{\frac{1}{\alpha}} \tag{9.123}$$
$$= (\kappa^{-1}Y_tP_t^{\epsilon})^{\frac{1}{\alpha}}P_{it}^{-\frac{\epsilon}{\alpha}} \tag{9.124}$$

Then take constraints, Equation 9.117, 9.124 into the target function, we maximize

$$L = E_{t-1}[\frac{1}{C_t}(p_{it}Y_t(\frac{P_{it}}{P_t})^{-\epsilon} - \frac{1}{\theta}W_t(\kappa^{-1}Y_tP_t^{\epsilon})^{\frac{1}{\alpha}}P_{it}^{-\frac{\epsilon}{\alpha}})] \tag{9.125}$$

$F.O.C.$

$$[P_{it}] : E_{t-1}[\frac{1}{C_t}(Y_tP_t^{\epsilon}(1-\epsilon)P_{it}^{-\epsilon} - \frac{1}{\theta}W_t(\kappa^-1Y_tP_t^{\epsilon})^{\frac{1}{\alpha}}(-\frac{\epsilon}{\alpha})P_{it}^{-\frac{\epsilon}{\alpha}-1})] = 0 \tag{9.126}$$
$$\Rightarrow E_{t-1}[\frac{1}{C_t}(\alpha\theta Y_tP_t^{\epsilon}P_{it}^{-\epsilon} - W_t(\kappa^-1Y_tP_t^{\epsilon})^{\frac{1}{\alpha}}\frac{\epsilon}{\epsilon-1}P_{it}^{-\frac{\epsilon}{\alpha}-1})] = 0 \tag{9.127}$$
$$E_{t-1}[\frac{1}{C_t}(\alpha\theta Y_t(\frac{P_{it}}{P_t})^{-\epsilon} - W_tN_{it}Y_{it}^{-\frac{1}{\alpha}}Y_t^{\frac{1}{\alpha}}P_t^{\frac{\epsilon}{\alpha}}\frac{\epsilon}{\epsilon-1}P_{it}^{-\frac{\epsilon}{\alpha}-1})] = 0 \tag{9.128}$$

Note that $(\frac{Y_{it}}{Y_t})^{-\frac{1}{\alpha}}(\frac{P_{it}}{P_t})^{-\frac{\epsilon}{\alpha}} = (\frac{P_{it}}{P_t})^{\frac{\epsilon}{\alpha}}(\frac{P_{it}}{P_t})^{-\frac{\epsilon}{\alpha}} = 1$, then we obtain the pricing Euler Equation,

$$E_{t-1}[\frac{1}{C_t}(\alpha\theta P_{it}Y_{it} - \frac{\epsilon}{\epsilon-1}W_tN_{it})] = 0 \tag{9.129}$$

- shocks
  - $\frac{Z_t}{Z_{t-1}} = \exp(\eta_t)$, permanent technology shocks;
  - $\frac{M_t^s}{M_{t-1}^s} = \exp\{\xi_t + \gamma \cdot \eta_t\}$, where ξ_t denotes purely random shock from money supply, γ denotes response of money supply to a change in technology, and constant γ implies symmetric response of government to changes in technology level. If $\gamma > 0$, central banks "accommodates" technology increases. ξ_t, η_t are set as white noise, normally distributed,

$$\xi_t \sim N(0, s_m^2), \eta_t \sim N(0, s_Z^2) \tag{9.130}$$

then $\xi_t + \gamma\eta_t \sim N(0, s_m^2 + \gamma^2 s_Z^2)$.

- symmetric equilibrium,

$$P_{it} = P_t, C_{it} = C_t, Y_{it} = Y_t, U_{it} = U_t, N_{it} = N_t \tag{9.131}$$

- equilibrium in the monetary market $M_t^s = M_t, \forall t$ implies

$$\frac{M_t}{M_{t-1}} = \exp\{\xi_t + \gamma \cdot \eta_t\} \tag{9.132}$$

From Equation 9.114, (ignore $\cdot$)

$$\frac{1}{C_t} = \lambda_m \frac{P_t}{M_t} + \beta E_t[\frac{1}{c_{t+1}}\frac{P_t}{P_{t+1}}] \tag{9.133}$$
$$\Rightarrow \frac{1}{C_t} = \lambda_m \frac{P_t}{M_t} + \beta E_t[\frac{P_t}{P_{t+1}}\lambda_m \frac{P_{t+1}}{M_{t+1}} + \beta E_{t+1}[\frac{1}{c_{t+2}}\frac{P_{t+1}}{P_{t+2}}]] \tag{9.134}$$
$$= \lambda_m \frac{P_t}{M_t} + \beta E_t[\frac{P_t}{P_{t+1}}\lambda_m \frac{P_{t+1}}{M_{t+1}} + \beta[\frac{1}{C_{t+2}}\frac{P_{t+1}}{P_{t+2}}]] \tag{9.135}$$
$$= \lambda_m \frac{P_t}{M_t} + \lambda_m \beta E_t[\frac{P_t}{P_{t+1}}\frac{P_{t+1}}{M_{t+1}}] + \beta^2 E_[\frac{P_t}{P_{t+1}}\frac{1}{C_{t+1}}\frac{P_{t+1}}{P_{t+1}}] \tag{9.136}$$
$$= \lambda_m \frac{P_t}{M_t} + \lambda_m \beta E_t[\frac{P_t}{M_{t+1}}] + \beta^2 E_[\frac{P_t}{p_{t+2}}\frac{1}{C_{t+1}}] \tag{9.137}$$
$$\cdots \tag{9.138}$$
$$\Rightarrow \frac{1}{C_t} = \lambda_m \frac{P_t}{M_t} + \lambda_m \beta E_t[\frac{P_t}{M_{t+1}}] + \lambda_m \beta^2 E_t[\frac{P_t}{M_{t+2}}] + \cdots \tag{9.139}$$
$$= \lambda_m \frac{P_t}{M_t}[1 + \beta E_t[\frac{M_t}{M_{t+1}}] + \beta^2 E_t[\frac{M_t}{M_{t+1}}\frac{M_{t+1}}{M_{t+1}}] + \cdots] \tag{9.140}$$
$$= \lambda_m \frac{P_t}{M_t}[1 + \beta E_t[e^{-(\xi_{t+1}+\gamma\eta_{t+1})}] + \beta^2 E_t[e^{-(\xi_{t+1}+\gamma\eta_{t+1})}e^{-(\xi_{t+2}+\gamma\eta_{t+2})}] + \cdots] \tag{9.141}$$

As $\xi_t + \gamma\eta_t \sim N(0, s_m^2 + \gamma^2 s_Z^2)$, so $E[e^{-(\xi_t+\gamma\eta_t)}] = e^{\frac{1}{2}(s_m^2+\gamma^2 s_Z^2)}, \forall t$ [3]. Thus, let $\kappa = e^{\frac{1}{2}(s_m^2+\gamma^2 s_Z^2)}$ and assume $\beta\kappa < 1$, we have

$$\frac{1}{C_t} = \lambda_m \frac{P_t}{M_t}[1 + \beta E_t[e^{-(\xi_{t+1}+\gamma\eta_{t+1})}] + \beta^2 E_t[e^{-(\xi_{t+1}+\gamma\eta_{t+1})}e^{-(\xi_{t+2}+\gamma\eta_{t+2})}] + \cdots] \tag{9.142}$$
$$= \lambda_m \frac{P_t}{M_t}[1 + \beta\kappa + \beta^2\kappa^2 + \cdots] \tag{9.143}$$
$$= \lambda_m \frac{P_t}{M+t}\frac{1}{1-\beta\kappa} \tag{9.144}$$
$$\Rightarrow C_t = \frac{1}{\lambda_m}\frac{M_t}{P_t}(1-\beta\kappa) = \Phi\frac{M_t}{P_t} \tag{9.145}$$

where $\Phi = \frac{1}{\lambda_m}[1-\beta\kappa] = \frac{1}{\lambda}[1 - \beta e^{\frac{1}{2}}(s_m^2 + \gamma^2 s_Z^2)]$.

From Equation 9.112, 9.113, 9.117, we have

$$\frac{W_t}{V_t} = \frac{\lambda_N}{\lambda_U}\frac{N_t^{\sigma_N}}{U_t^{\sigma_I}} = \frac{\theta}{1-\theta}\frac{U_t}{V_t} \tag{9.146}$$

so

$$U_t = A^{\frac{1}{\alpha(1-\theta)}} N_t^{(1+\sigma_N)/(1+\sigma_U)} \tag{9.147}$$

where $A = (\frac{\lambda_N(1-\theta)}{\lambda_U\theta})^{\frac{\alpha(1-\theta)}{1+\sigma_U}}$ (add $\alpha(1-\theta)$ for latter simplicity).

From production function, we have

$$Y_t = Z_t[N_t^\theta U_t^{1-\theta}]^\alpha \tag{9.148}$$
$$= Z_t N_t^{\alpha\theta} U_t^{\alpha(1-\theta)} \tag{9.149}$$
$$= Z_t N_t^{\alpha\theta} A N_t^{\alpha(1-\theta)\frac{1+\sigma_N}{1+\sigma_U}} \tag{9.150}$$
$$= A Z_t N_t^\varphi \tag{9.151}$$

[3] If $X \sim N(\mu, \sigma^2)$, then $E(e^{uX}) = e^{\mu u + \frac{1}{2}u^2\sigma^2}$ (monment generating function)

i.e.,

$$Y_t = AZ_t N_t^{\varphi} \tag{9.152}$$

where $\varphi = \alpha\theta + \alpha(1-\theta)(\frac{1+\sigma_N}{1+\sigma_U})$.

Finally, evaluating Equation 9.112 and 9.129 at the symmetric equilibrium and combine with Equation 9.145 and 9.152, we obtain a set of expressions for the equilibrium levels of prices, output, employment and productivity in terms of the exogenous deriving variables.[4]

We are not interested in coefficients, so denoted as ζ for all coefficients here. With Equation 9.145 and 9.145, we have $N_t^{\varphi} = \zeta \frac{M_t}{Z_t}\frac{1}{P_t}$. Use lowercase for the log of original variables, e.g., $n_t = \log(N_t)$. With Equation 9.112, we have $P_t = \zeta \frac{W_t N_t}{C_t} N_t^{\sigma_N - 1}$. Combine these equations and Equation 9.129, we have[5]

$$P_t = E_{t-1}[\zeta \frac{M_t}{Z_t}] = \zeta \frac{M_{t-1}}{Z_{t-1}} \tag{9.153}$$

Thus, $p_t = \log(\zeta) + (m_{t-1} - z_{t-1}), \Delta p_t = (\xi_{t-1} + \gamma\eta_{t-1} - \eta_{t-1})$.

Equation 9.145 implies $\Delta y_t + \Delta p_t = \xi_t + \gamma\eta_t$, so $\Delta y_t = \gamma\eta_t + (1-\gamma)\eta_{t-1}$. Equation 9.152 implies $n_t = \frac{1}{\varphi}(y_t - z_t)$, so $\Delta n_t = \frac{1}{\varphi}\Delta\xi_t - \frac{1-\gamma}{\varphi}\Delta\eta_t$. Summing up,

$$\Delta p_t = \xi_{t-1} - (1-\gamma)\eta_{t-1} \tag{9.154}$$

$$\Delta y_t = \Delta\xi_{t-1} + \gamma\eta_t + (1-\gamma)\eta_{t-1} \tag{9.155}$$

$$\Delta x_t = \Delta y_t - \Delta n_t = (1 - \frac{1}{\varphi})\Delta\xi_t + (\frac{1-\gamma}{\varphi} + \gamma)\eta_t + (1-\gamma)(1-\frac{1}{\varphi})\eta_{t-1} \tag{9.156}$$

where Δp_t denotes the inflation, Δy_t denotes output and x_t denotes the log of labor productivity. (p_t is chosen before the realization of shocks) Rewrite Equation 9.155 with the lag operator, we have

$$y_t = \xi_t + \frac{1}{1-L}(\gamma\eta_t + (1-\gamma)\eta_t) \tag{9.157}$$

which implies ξ_t has temporary effects but η_t has long-lasting effects.

The unconditional covariances among the growth rates of output, labor productivity, employment, are

(a) $Cov(\Delta y_t, \Delta n_t) = \frac{1}{\varphi}(2s_m^2 + (1-\gamma)(1-2\gamma)s_Z^2)$.

If $\gamma \in [0, \frac{1}{2})$, then $Cov(\Delta y_t, \Delta n_t) > 0$, the employment growth is procychical which means the employment growth varies in the same direction with output.

(b) $Cov(\Delta y_t, \Delta x_t) = \frac{1}{\varphi}(2(\varphi-1)s_m^2 + (\gamma+\varphi-1)s_Z^2)$.

If $\varphi > 1$ (sufficient condition), the measured labor productivity is procychical.

(c) $Cov(\Delta n_t, \Delta x_t) = \frac{1}{\varphi^2}[2(\varphi-1)s_m^2 - (1-\gamma)[(2-\varphi) + 2\gamma(\varphi-1)]s_Z^2]$.

- conditional on the technology being the only source of fluctuations, $Cov(\Delta n_t, \Delta x_t|Z) = -\frac{1-\gamma}{\varphi^2}[(2-\varphi) + 2\gamma(\varphi-1)]s_Z^2$. If $\gamma \in [0,1]$ and $\varphi \in (1,2)$, then $Cov(\Delta n_t, \Delta x_t|Z) < 0$, i.e., technology shocks generate a negative co-movement between employment and productivity production.
- conditional on monetary shock being the source of fluctuations, $Cov(\Delta n_t, \Delta x_t|M) = \frac{2}{\varphi^2}(\varphi-1)s_m^2$. If $\gamma \in [0,1)$ and $\varphi \in (1,2)$, then $Cov(\Delta n_t, \Delta x_t|M) > 0$, i.e., monetary shocks generate a positive co-movement between employment and productivity growth.

Thus, the model predicts

- technology shocks $\Rightarrow$ negative correlation;
- monetary shocks $\Rightarrow$ positive correlation.

This is in contrast with the prediction of RBC models (e.g., Christiano and Eichenbanm(1992)),

- technology shocks $\Rightarrow$ positive correlation;

[4] We restate the story to show how variables interact: firms set the price P_t, then technology and monetary shocks realize, the money supply M_t directly decide C_t, hence Y_t, then firms choose N_t to meet C_t, wages are thus decided by P_t, C_t, N_t and the gap in the budget constraint is filled by government transfers.

[5] Shocks are white noise, so firms can only expect tomorrow will be the same as today.

- monetary shocks $\Rightarrow$ negative correlation.

We need to test it, using a structural VAR (vector autocorrelation regression) model.
Let q_t and Δq_t be

$$q_t = \begin{bmatrix} x_t \\ n_t \end{bmatrix} \tag{9.158}$$

$$\Delta q_t = A(L)\Delta q_t + v_t \tag{9.159}$$

where L is the lag operator. If $L = 1$, then $A(L)\Delta q_t = A_1 \cdots \Delta q_{t-1}$; if $L = p$, then $A(L)\Delta q_t = A_1\Delta q_{t-1} + A_1\Delta a_{t-2} + \cdots + A_p\Delta q_{t-p}$.

Example 9.3. *$L = 1$. $\Delta q_t = A(L)\Delta q_t + v_t$ implies*

$$\Delta q_t = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \Delta x_{t-1} \\ \Delta n_{t-1} \end{bmatrix} + \begin{bmatrix} v_{12} \\ v_{2t} \end{bmatrix} \tag{9.160}$$

$$\Delta x_t = a_{11}\Delta x_{t-1} + a_{12}\Delta n_{t-1} + v_{1t} \tag{9.161}$$

$$\Delta n_t = a_{21}\Delta x_{t-1} + a_{22}\Delta n_{t-1} + v_{2t} \tag{9.162}$$

We can estimate $A(L)$ and v_t by OLS. v_t is statistical innovation rather than structural / fundamental innovation. We need to identify structural shocks,

$$v_t = C\varepsilon_t \tag{9.163}$$

where ε_t is a structural shocks. We want to find C.
We know $E[v_t v_t'] = \Sigma$ (can be estimated by OLS) and assume $E[\varepsilon_t\varepsilon_t'] = I$ (identical matrix), then

$$E[v_t v_t'] = E[C\varepsilon_t\varepsilon_t' C'] = CE[\varepsilon_t\varepsilon_t']C' = CC' \tag{9.164}$$

So $CC' = \Sigma$, where Σ is symmetric, so we need another condition to obtain 4 unknowns in C.
Make identification assumption: only technology shock has long run effects on labor productivity Δx_t, then

$$v_t = C\varepsilon_t = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix} \tag{9.165}$$

where ε_{1t} denotes the technology shock.

$$\Delta q_t = A(L)q_t + C\varepsilon_t \tag{9.166}$$

$$(I - A(L))\Delta q_t = C\varepsilon_t \tag{9.167}$$

$$\Delta q_t = [I - A(L)]^{-1}C\varepsilon_t \tag{9.168}$$

Let $B = [I - A(L)]^{-1}C = (b_{ij})$, then

$$\Delta q_t = B\varepsilon_t = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix} \tag{9.169}$$

The identification assumption implies $b_{12} = 0$. Now C is solvable.
Let $\hat{C} = (\hat{c}_{ij})$, then

$$\begin{bmatrix} \Delta x_t \\ \Delta n_t \end{bmatrix} = \begin{bmatrix} \hat{a}_{11} & \hat{a}_{12} \\ \hat{a}_{21} & \hat{a}_{22} \end{bmatrix} \begin{bmatrix} \Delta x_{t-1} \\ \Delta n_{t-1} \end{bmatrix} + \begin{bmatrix} \hat{c}_{11} & \hat{c}_{12} \\ \hat{c}_{21} & \hat{c}_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix} \tag{9.170}$$

Then the technology shock is given by $\varepsilon_{it} = u_t$ and the response are

$$\Delta x_t = \hat{c}_{11}u_t \tag{9.171}$$

$$\Delta n_t = \hat{c}_{21}u_t \tag{9.172}$$

$$\Delta x_{t+1} = \hat{a}_{11}\Delta x_t + \hat{a}_{12}\Delta n_t \tag{9.173}$$

$$= \hat{a}_{11}\Delta\hat{c}_{11}u_t + \hat{a}_{12}\hat{c}_{21}u_t \tag{9.174}$$

$$\Delta n_{t+1} = \hat{a}_{21}\hat{c}_{11}u_t + \hat{a}_{22}\Delta n_t \tag{9.175}$$

$$= \hat{a}_{21}\hat{c}_{11}u_t + \hat{a}_{22}\hat{c}_{21}u_t \tag{9.176}$$

$$\cdots$$

9.5 A Standard New Keynesian Model for Monetary Analysis

Households,

$$E_t \sum_{i=0}^{\infty} \beta^i [\frac{c_{t+i}^{1-\sigma}}{1-\sigma} + \frac{\gamma}{1-b}(\frac{M_{t+i}}{p_{t+i}})^{1-b} - \chi \frac{N_{t+i}}{1+\eta}^{1+\eta}] \tag{9.177}$$

$$c_t = [\int_0^1 c_{jt}^{\frac{\theta-1}{\theta}} dj]^{\frac{\theta}{\theta-1}}, \theta > 1 \tag{9.178}$$

cost minimization (Dixit-Stiglitz Monopolistic Competition setting),

$$p_t \equiv [\int_0^q p_{jt}^{1-\theta} dj]^{\frac{1}{1-\theta}} \text{ price aggregator} \tag{9.179}$$

$$c_{it} = (\frac{p_{jt}}{p_t})^{-\theta} c_t \tag{9.180}$$

and budget constraint,

$$c_t + \frac{M_t}{p_t} + \frac{B_t}{p_t} = (\frac{W_t}{p_t})N_t + \frac{M_{t-1}}{p_t} + (1+i_{t-1})\frac{B_{t-1}}{p_t} + \Pi_t \tag{9.181}$$

where B_t is the nominal bond, i_{t-1} is the nominal interest rate, $(1+i_{t-1})\frac{B_{t-1}}{p_t}$ is the bond return and Π_t is the real profits (valued on consumption).

The Langrage function and F.O.C.,

$$L = E_t \sum_{i=0}^{\infty} \beta^i [\frac{c_{t+i}^{1-\sigma}}{1-\sigma} + \frac{\gamma}{1-b}(\frac{M_{t+i}}{p_{t+i}})^{1-b} - \chi \frac{N_{t+i}}{1+\eta}^{1+\eta} \tag{9.182}$$

$$-\lambda_{t+i}((\frac{W_t}{p_t})N_t + \frac{M_{t-1}}{p_t} + (1+i_{t-1})\frac{B_{t-1}}{p_t} + \Pi_t - c_t \frac{M_t}{p_t}\frac{B_t}{p_t})] \tag{9.183}$$

$$F.O.C.(\text{ given } \Phi_t, i=0)$$

$$[c_t] : c_t^{-\sigma} - \lambda_t = 0 \tag{9.184}$$

$$[M_t] : \gamma(\frac{M_t}{p_t})^{-b}\frac{1}{p_t} - \frac{1}{p_t}\lambda_t + \beta E_t[\lambda_{t+1}\frac{1}{p_{t+1}}] = 0 \tag{9.185}$$

$$[N_t] : -\chi N_t^{\eta} + \lambda_t \frac{W_t}{p_t} = 0 \tag{9.186}$$

$$[B_t] : -\frac{1}{p_t}\lambda_t + \beta E_t[\lambda_{t+1}\frac{p_t}{p_{t+1}}] = 0 \tag{9.187}$$

Equation 9.184 and 9.186 implies $\frac{W_t}{p_t} = \frac{\chi N_t^{\eta}}{c_t^{-\sigma}}$. Take Equation 9.184 into Equation 9.185, we have $c_t^{-\sigma} = \gamma(\frac{M_t}{p_t})^{-b} + \beta E_t[c_{t+1}^{-\sigma}\frac{p_t}{p_{t+1}}]$; combine this equation with Equation 9.187, we eliminate $\frac{p_t}{p_{t+1}}$, giving $\frac{\gamma(\frac{M_t}{p_t})^{-b}}{c_t^{-\sigma}} = \frac{i_t}{1+i_t}$. Take this equation into $c_t^{-\sigma} = \gamma(\frac{M_t}{p_t})^{-b} + \beta E_t[c_{t+1}^{-\sigma}\frac{p_t}{p_{t+1}}]$, we clear $\frac{M_t}{p_t}$, and get $c_t^{-\sigma} = \beta(1+i_t)E_t[\frac{p_t}{p_{t+1}}c_{t+1}^{-\sigma}]$.

Summing up,

$$c_t^{-\sigma} = \beta(1+i_t)E_t(\frac{p_t}{p_{t+1}})c_{t+1}^{-\sigma} \tag{9.188}$$

$$\frac{\gamma(\frac{M_t}{p_t})^{-b}}{c_t^{-\sigma}} = \frac{i_t}{1+i_t} \tag{9.189}$$

$$\frac{\chi N_t^{\eta}}{c_t^{-\sigma}} = \frac{W_t}{p_t} \tag{9.190}$$

Firms, production function $c_{ji} = Y_{it} = Z_t N_{jt}, E[Z_t] = 1$.

Galvo (1983) type price stickiness: each period, the firms that adjust the price are randomly selected and a fraction $1-\omega$ of all firms adjust while the remaining ω fraction firms do not adjust. The larger ω implies the expected time between price changes is longer.

Before analyzing the firms' pricing decision, consider its cost minimization, [6]

$$\min_{N_t}(\frac{W_t}{p_t})N_t + \varphi_t(c_{jt} - Z_t N_{jt}) \tag{9.191}$$

$$F.O.C.\varphi_t = \frac{W_t/P-t}{Z_t} \tag{9.192}$$

which is the real marginal cost. Firms' pricing decisions problems then involve picking p_{jt} to maximize

$$\max_{p_{jt}} E_t \sum_{i=1}^{\infty} \omega^i \Delta_{i,t+i}[(\frac{p_{jt}}{p_{t+i}})c_{j(t+i)} - \varphi_{t+i}c_{j(t+i)}] \tag{9.193}$$

where $(\frac{p_{jt}}{p_{t+i}})c_{j(t+i)}$ is the income and $\varphi_{t+i}c_{j(t+1)}$ is the real cost and $\Delta_{i,t+i} = \beta^i(\frac{c_{t+i}}{c_t})^{-\sigma}$ is the discount factor. (ω^i implies it has no chance to adjust prices from t to $t+i$)

Using the demand curve to eliminate $c_{j(t+i)}, c_{j(t+i)} = c_{t+i}(\frac{p_{j(t+i)}}{p_{t+i}})^{-\theta} = c_{t+i}(\frac{p_{jt}}{p_{t+i}})^{-\theta}$ ($p_{jt} = p_{j(t+i)}$ because the firm is not allowed to adjust price from t to $t+i$).

Then, firms maximize

$$\max_{p_{jt}} E_t \sum_{i=1}^{\infty} \omega^i \Delta_{i,t+i}[(\frac{p_{jt}}{p_{t+i}})^{1-\theta} - \varphi_{t+i}(\frac{p_{jt}}{p_{t+i}})^{-\theta}]c_{t+i} \tag{9.194}$$

All firms are homogeneous. All firms adjusting in period t facing the same problem, so all adjusting firms will set the same price, let p_t^* denote the optimal price.

F.O.C.

$$[p_{jt}] : E_t \sum_{i=0}^{\infty} \omega^i \Delta_{i,t+i}[(1-\theta)\frac{p_t^*}{p_{t+i}} + \theta\varphi_{t+i}](\frac{1}{p_t^*})(\frac{p_t^*}{p_{t+i}})^{-\theta}c_{t+i} = 0 \tag{9.195}$$

$$\Rightarrow \frac{p_t^*}{p_t} = \frac{\theta}{\theta-1}\frac{E_t\sum_{i=0}^{\infty}\omega^i\beta^i c_{t+i}^{1-\sigma}\varphi_{t+i}(\frac{p_{t+i}}{p_t})^{\theta}}{E_t\sum_{i=0}^{\infty}\omega^i\beta^i c_{t+i}^{1-\sigma}(\frac{p_{t+i}}{p_t})^{\theta-1}} \tag{9.196}$$

Recall $p_t \equiv [\int_0^1 p_{jt}^{1-\theta}dj]^{\frac{1}{1-\theta}}$, then

$$p_t^{1-\theta} = [(1-\omega)p_t^{*(1-\theta)} + \omega(1-\omega)p_{t-1}^{*(1-\theta)} + \omega^2(1-\omega)p_{t-2}^{*(1-\theta)} + \cdots] \tag{9.197}$$

$$= (1-\omega)p_t^{*(1-\theta)} + \omega[(1-\omega)p_{t-1}*(1-\theta) + \omega(1-\omega)p_{t-2}*(1-\theta) + \cdots] \tag{9.198}$$

$$\Rightarrow p_t^{1-\theta} = (1-\omega)p_t^{*(1-\theta)} + \omega p_{t-1}^{1-\theta} \tag{9.199}$$

Equation 9.196 and 9.199 can be approximated around a zero average inflation steady state equilibrium (see details in Walsh(2017), ch8 appendix).

$$\hat{\pi}_t = \beta E_t\hat{\pi}_{t+1} + \kappa\hat{\varphi}_t \tag{9.200}$$

where $\pi_t = \frac{p_t}{p_{t-1}}$, $\hat{x}_t$ is log derivation from the steady state, x is any variable and $\kappa = \frac{(1-\omega)(1-\beta\omega)}{\omega}$

New Keynesian Pillps Curve (NKPC),

$$\hat{\pi}_t = \beta E_t\hat{\pi}_{t+1} + \kappa\hat{\varphi}_t \tag{9.201}$$

where $\hat{\varphi}_t$ denotes the real marginal cost.

From labor-leisure condition and firm's cost minimization, we have

$$\hat{\varphi} = (\hat{W}_t - \hat{p}_t) - (\hat{y}_t - \hat{n}_t) = (\sigma+\eta)[\hat{y}_t - \frac{1+\eta}{\sigma+\eta}\hat{Z}_t] \tag{9.202}$$

where $\hat{c}_t = \hat{y}_t$.

Now we obtain the General Equilibrium,

$$\begin{cases} \hat{y}_t = E_t\hat{y}_{t+1} - \frac{1}{\sigma}(\hat{i}_t - E_t\hat{\pi}_{t+1}) \text{ Euler Equation} \\ \hat{\pi}_t = \beta E_t\hat{\pi}_{t+1} + \kappa(\sigma+\eta)(\hat{y}_t - \frac{1+\eta}{\sigma+\eta}\hat{Z}_t) \text{ NKPC} \\ \hat{i}_t = \delta_\pi\hat{\pi} + \delta_y\hat{ty}_t + v_t \text{ monetary policy} \end{cases} \tag{9.203}$$

Here the monetary policy is given by the government. Taylor Rule: we need $\delta_\pi > 1$ to insure an unique equilibrium.

[6] $\max_{N_t} \frac{W_t}{p_t}N_t$, s.t. $c_{jt} = Z_t N_{jt}$ (production function)

Part III
Advanced Microeconomics

10 Choice, Preference and Utility

10.1 Fundamental Theories of Choice

This section introduces the basis of consumption theory, and choice, preference, and utility the same method to analyze choice.

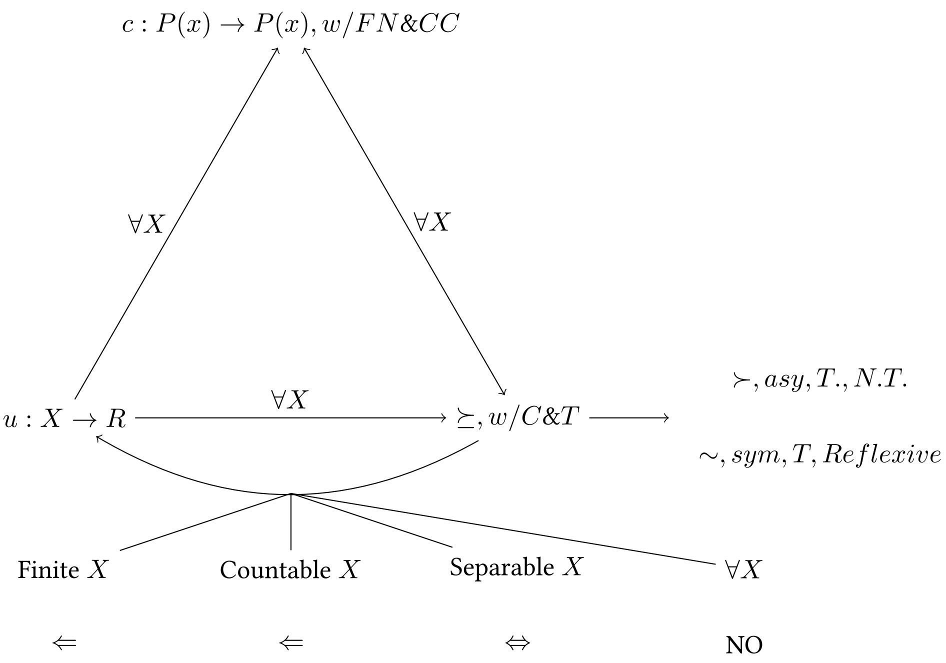


Figure 19: Frame of choice, preference and utility

The first work of this section is in Figure 19, i.e., to define choice function, preference and utility precisely, prove that they are equivalent under specific conditions. Then we will introduce how to describe the choice behavior of people and give properties of reference.

DEFINITION 10.1. *(Choice Set) A choice set X is a set of possible objects that an individual might choose.*

DEFINITION 10.2. *(Choice Function) A choice function $c : P(x) \to P(x)$ where $P(x) = \{A \subseteq X\}$ s.t. $c(A) \subseteq A$.*

We have two assumptions for choice function:

Assumption 10.1. *Finite Non-emptiness, FN A choice function c is FN if $\forall A \subseteq X, A \neq \varnothing, ||A|| < \infty \Rightarrow c(A) \neq \varnothing$.*

Assumption 10.2. *(Choice Coherence, CC) A choice function c is CC if $x \in c(A), y \notin c(A), x, y \in B \cap A \Rightarrow y \notin c(B)$.*

DEFINITION 10.3. *(Utility Function) $u : X \to R, \forall x \subseteq X, c(A) \equiv \{x \in A | u(x) \geq u(y), \forall y \in A\}$.*

DEFINITION 10.4. *(Weak Reference Relation) A preference $\succeq$ is a binary relation on X s.t. $\forall x, y \in X, x \succeq y$ means that x is at least as good as y. Then $\forall A \subseteq X, c(A) \equiv \{x \in A | x \succeq y, \forall y \in A\}$. Note that a binary relation is a subset of $X \times X$.*

We have two assumptions for weak preference relation:

Assumption 10.3. *(Completeness, C) A preference $\succeq$ on X is complete if $\forall x, y \in X$, we have $x \succeq y$ or $y \succeq x$ (or both).*

Assumption 10.4. *(Transitivity, T) A preference on X is transitive if $\forall x \succeq y, y \succeq z \Rightarrow x \succeq z$.*

Now we start to prove Figure 19 step by step.

LEMMA 10.1. *$\forall X$, if $\exists u : X \to R$, then (a) define $\succeq_u$ by $x \succ_u y$ if $u(x) \geq u(y)$, then $\succeq_u$ is complete and transitive. (b) define $c_u(A) \equiv \{x \in A | u(x) \geq u(y), \forall y \in A\}$, then c_u is FN and CC.*

Proof. (a) $\forall x, y \in X$,$u(x), u(y) \in \mathbb{R}$, we have $u(x) \geq u(y)$ or $u(y) \geq u(x)$ or both, so $\succeq_c$ is complete.

$\forall x, y, z \in X$, $x \succeq_c y, y \succeq_c z$ implies $u(x) \geq u(y), u(y) \geq u(z)$, so $u(x) \geq u(z)$, i.e., $x \succeq_c z$. Then $\succeq$ is transitive. Note that we use the complete and transitive properties of real numbers.

(b) (Induction) If $||A|| = 1, A = \{x\}$, then $c_u(A) = \{x\}$. Suppose $c_u(A) \neq \varnothing, \forall ||A|| \leq k$, then when $||A|| = k + 1$, without loss of generality (WLOG), let $A = A' \cup \{x\}, ||A'|| = k$, so $c_u(A') \neq \varnothing$. WLOG, let $g \in c_u(A')$, then consider $u(y), u(x)$. If $u(y) \geq u(x)$, we have $y \in c_u(A)$; if $u(y) \leq u(x)$, we have $x \in c_u(A)$ (or both). So $c_u(A) \neq \varnothing$, that is FN.

$\forall x, y \in A \cap B, x \in c_u(A), y \notin c_u(A)$, we need to show (NTS) $y \notin c_u(B)$. As $u(x) \geq u(z), \forall z \in A$. As $y \notin c_u(A)$, $\exists z^* \in A$ such that (s.t.) $u(z^*) > u(y)$. Then $u(x) \geq u(z^*) > u(y)$, so $u(x) \geq u(y)$, implying $y \notin c_u(B)$, that is CC. (Note that we use transitive property of $\mathbb{R}$). $\square$

LEMMA 10.2. *If $\succeq$ on X is C and T, define $c_\succeq A \equiv \{x \in A | x \succeq y, \forall y \in A\}$, then $c_\succeq$ is FN and CC.*

Proof. $\forall x, y \in A \cap B, x \in c_u(A), y \notin c_u(A)$, then $\forall z \in A, x \succeq z, \exists z^*, \urcorner(y \succeq z^*)$. NTS $\urcorner(y \succeq x)$, suppose not, $y \succeq x$, then $y \succeq x \succeq z^*$, then $y \succeq z^*$, contradicting to $\urcorner(y \succeq^*)$. So we have $\urcorner(y \succeq x)$, that is CC.

(Induction) When $||A|| = 1, A = \{x\}$, we have $c_\succeq(A) = \{x\}$ as $x \succeq x$ (complete). Suppose $\forall ||A|| \leq k, c_\succeq(A) \neq \varnothing$, we NTS $\forall ||A|| = k+1, c_\succeq(A) \neq \varnothing$. WLOG, let $A = A' \cup \{x\}, ||A'|| = k$, then $c_\succeq(A') \neq \varnothing$. Let $y \in c_\succeq(A')$. Then we have either $x \succeq y$ (then $x \in c_\succeq(A)$) or $y \succeq x$ (then $y \in c_\succeq(A)$) or both. Then $c_\succeq(A) \neq \varnothing$, that is FN. $\square$

LEMMA 10.3. *$\forall X, \exists c : P(X) \to P(X)$ is FN and CC. Define $\succeq_c$ by $x \succeq_c y$ if $x \in c(\{x, y\})$, then $\succeq_c$ is C and T. Define $c_\succeq(A) \equiv \{x \in A | x \succeq_c y, \forall y \in A\}$, then $c_\succeq(\cdot)$ is FN and CC. and $\forall A \subseteq X, c(A) = \varnothing$ or $c(A) = c_\succeq(A)$.*

Proof. First show $\succeq_c$ is complete. $\forall x, y \in X$, $\{x, y\} \in X$, then $c(\{x, y\}) \neq \varnothing$ as c is FN. So we have either $x \in c(\{x, y\})$ (then $x \succeq_c y$) or $y \in c(\{x, y\})$ (then $y \succeq x$) or both. Then $\succeq_c$ is complete.

Next show $\succeq_c$ is transitive. $\forall x, y, z \in X, x \succeq_c y, y \succeq_c z$, we NTS $x \succeq_c z$. Suppose not, $x \notin c(\{x, z\})$, i.e., $z \in c(\{x, z\})$. As c is CC, so $x \notin c(\{x, y, z\})$. If $y \in c(\{x, y, z\})$, then with $x \notin c(\{x, y, z\})$ and c is CC, we have $x \notin c(\{x, y\})$, contradicting to $x \succeq_c y$. So $y \notin c(\{x, y, z\})$. Then $z \in c(\{x, y, z\})$. With $y \notin c(\{x, y, z\})$, we have $y \notin c(\{y, z\})$, contradicting to $y \succeq_c z$. Therefore $x \in c(\{x, z\})$, i.e., $x \succeq_c z$.

Finally we show $\forall A \subseteq X, c(A) = \varnothing$ or $c(A) = c_\succeq(A)$. $\forall A \subseteq X$, if $c(A) = \varnothing$, then done. Then we discuss what happens if $c(A) \neq \varnothing$.

First show $c(A) \subseteq c_\succeq(A)$. We know $x \in c_\succeq(A)$ iff $x \succeq_c y, \forall y \in A$. Suppose $x \notin c_\succeq(A)$, then $\exists y \in A, \urcorner(x \succeq_c y)$, then $x \notin c(\{x, y\}), y \in c(\{x, y\})$, then $x \notin c(A)$ (CC), contradicting to $x \in c(A)$. So $x \in c_\succeq(A)$.

Then we show $c_\succeq(A) \subseteq c(A)$. Let $x \in c_\succeq(A)$ and suppose $x \notin c(A), c(A) \neq \varnothing$. Then $\exists y \in c(A)$, so $x \notin c(\{x, y\})$, further, $\urcorner(x \succeq y)$, then $x \notin c_\succeq(A)$, contradicting to $x \in c_\succeq(A)$. So $x \in c(A)$. $\square$

DEFINITION 10.5. *(No-Worse-Than and No-Better-Than) For $\succeq$ on X, we define $\forall x \in X$,*

$$NWT \equiv \{y \in X | y \succeq x\} \tag{10.1}$$

$$NBT \equiv \{y \in X | x \succeq y\} \tag{10.2}$$

Note that if we find the a utility function representation of $\succeq$, NWT and NBT are the contour set of the utility function. Remembering this saving your time latter.

LEMMA 10.4. *For $\succeq$ on X, let $\succeq$ be complete and transitive. We have*

1. *$\forall x \in X, NBT(x), NWT(x) \neq \varnothing$;*
2. *$\forall x, y \in X, x \succeq y$ iff $NBT(y) \subseteq NBT(x)$;*
3. *In addition, $\urcorner(y \succeq x)$ iff $NBT(y) \subsetneq NBT(x)$.*

Proof.

1. $\forall x \in X$ we have $x \succeq x$ as $\succeq$ is complete. So $x \in NBT(x), x \in NWT(x)$.
2. $\forall x, y \in X, x \succeq y$, then $y \in NBT(x)$. $\forall z \in NBT(y)$, i.e., $y \succeq z$, as $\succeq$ is transitive, we have $x \succeq z$, i.e., $z \in NBT(x)$. So $NBT(y) \subseteq NBT(x)$.

   $\forall x, y \in X, NBT(y) \subseteq NBT(x)$, with $y \in NBT(y)$, so $y \in NBT(x)$, i.e., $x \succeq y$.

3. $\neg(y \succeq x)$, then $x \notin NBT(y)$ while $x \in NBT(x)$ and so $NBT(y) \subsetneq NBT(x)$ with the second conclusion.

This lemma tells that NBT is nested. □

LEMMA 10.5. *For finite X, if $\exists \succeq$ on X is complete and transitive, then $\exists u : X \to \mathbb{R}$ s.t. $x \succeq y$ iff $u(x) \geq u(y)$.*

Proof. To prove a function u is a representation of a preference $\succeq$, we NTS $\forall x, y \in X, x \succeq y$ iff $u(x) \geq u(y)$.

Let $u(x) \equiv ||NBT(x)||, \forall x \in X$. Then we NTS $x \succeq y$ iff $u(x) \geq u(y)$. That is intuitive as NBT is nested: $x \succeq y$ iff $NBT(y) \subseteq NBT(x)$ iff $||NBT(x)|| \geq ||NBT(y)||$ iff $u(x) \geq u(y)$. □

DEFINITION 10.6. *(Strict Preference and Indifference Relation) For a given weak preference $\succeq$ on X, we define a binary relation $\succ$ on X by $\forall x, y \in X, x \succ y$ if $x \succeq y$ and $\neg(y \succeq x)$. We define a binary relation $\sim$ on X by $\forall x, y \in X$, $x \sim y$ if $x \succeq y$ and $y \succeq x$.*

DEFINITION 10.7. *(Strict-Better-Than (SBT) and Strict-Worse-Than (SWT)) For a $\succ$ defined on X, we define $\forall x \in X, SBT(x) \equiv \{y \in X | y \succ x\}$ and $SWT(x) \equiv \{y \in X | x \succ y\}$.*

We say $\succeq$ and $\succ, \sim$ are equivalent by following two lemmas.

LEMMA 10.6. *If $\succeq$ on X is complete and transitive, then we have*

1. *$\forall x \succ y$ iff $\neg(y \succeq x)$;*
2. *$\succ$ is asymmetric (if $x \succ y$, then $\neg(y \succ x)$);*
3. *$\sim$ is symmetric (if $x \sim y$, then $y \sim x$);*
4. *$\succ$ and $\sim$ are transitive;*
5. *If $x \succeq y, y \succ z$ or $x \succ y, y \succeq z$, then $x \succ z$;*
6. *$\succ$ is negatively transitive (if $x \succ y$, then $\forall z \in X$, either $x \succ z$ or $z \succ y$);*
7. *$\sim$ if reflexive ($\forall x \in X, x \sim x$);*

Proof.
1. $x \succ y$ iff $x \succeq y, \neg(y \succeq z)$ iff $\neg(y \succeq z)$ ($\succeq$ is complete);
2. $x \succ y$ iff $x \succeq y, \neg(y \succeq x)$ iff $\neg(\neg(x \succeq y))$ iff $\neg(y \succeq x)$;
3. $x \sim y$ iff $x \succeq y, y \succeq x$ iff $y \succeq x, x \succeq y$ iff $y \sim x$;
4. (1) $\forall x \succ y, y \succ z$, then $x \succeq y, \neg(z \succeq y)$. If $z \succeq x$, we have $z \succeq x \succeq y$, contradicting to $\neg(z \succeq y)$, so $\neg(z \succeq x)$, i.e., $x \succ z$. (2) $x \sim y, y \sim z$, then $x \succeq y, y \succeq z, z \succeq y, y \succeq x$ then $x \sim z, z \sim x$, so $x \sim z$.
5. $x \succeq y, y \succ z$, then $x \succeq y, y \succeq z, \neg(z \succeq y)$, then $x \succeq z, \neg(z \succeq x)$ (if not, $z \succeq x \succeq y$, contradicting to $\neg(z \succeq y)$), then $x \succ z$. The other situation is the same.
6. $x \succ y$, suppose $\neg(x \succ z)$ and $\neg(z \succ y)$, then $z \succeq x, y \succeq x$, then $y \succeq x$, i.e., $\neg(x \succ y)$, contradicting to $x \succ y$.
7. $\forall x \in X, x \succeq x$, so $x \sim x$.

□

LEMMA 10.7. *Suppose $\succ$ is asymmetric and negatively transitive, we define $\succeq$ on X by $x \succeq y$ if $\neg(y \succ x)$ and define $\sim$ on X by $x \sim y$ if $\neg(x \succ y)$ and $\neg(y \succ x)$, then $\succeq$ is complete and transitive. (Further, if we define $\succ', \sim'$ on X from $\succeq$, then $\succ', \succ$ are identical, $\sim', \sim$ are identical.)*

Proof. First consider $\forall x, y \in X$, we have $x \succeq y$ or $\neg(x \succeq y)$. If $\neg(x \succeq y)$, then $\neg(\neg(y \succ x))$, then $y \succ x$, then $\neg(x \succ y)$, i.e., $y \succeq x$. So $\succeq$ is complete.

Second consider $x \succeq y, y \succeq z$, suppose $\neg(x \succeq z)$, then $z \succ x$. Then $\forall y$, we have either $z \succ y$ or $y \succ z$ (negatively transitive), contradicting to $\neg(z \succ y)$ and $\neg(y \succ x)$ as $y \succeq z$ and $x \succeq y$ show. So $x \succeq z$. □

Then we want to find what a preference need to find a utility function representation.

LEMMA 10.8. *For a countable X and $\succeq$ on X which is complete and transitive, then $\exists u : X \to \mathbb{R}$ s.t. $x \succeq y$ iff $u(x) \geq u(y)$.*

Proof. Let $X = \{x_1, x_2, \cdots\}$, we define $d : X \to \mathbb{R}$ by $d(x_i) = (\frac{1}{2})^i$. Then we define $u(x) = \sum\limits_{x_i \in NBT(x)} x_i$. We claim that u is a representation of $\succeq$. First u is well defined as d is positive and X is countable, so it is positive and finite. As NBT is nested, so $x \succeq y$ iff $NBT(y) \succeq NBT(x)$ iff $u(y) \leq u(x)$. $\square$

LEMMA 10.9. *Suppose $\succeq$ on X is complete and transitive, then $\succeq$ can be represented by $u : X \to \mathbb{R}$ iff $\exists X^* \subseteq X, X^*$ is countable and $\forall x, y \in X, x \succ y$, we have $\exists x^* \in X^*, x \succeq x^* > y$ (X is separable).*

Proof. First show the if part. We know $\exists X^* \subseteq X$ s.t. $\forall x \succ y, \exists x^* \in X^*, x \succeq x^* \succ y$. Then we have to find an utility function. Let $X^* = \{x_1^*, x_2^*, \cdots\}$, and $d : X^* \to \mathbb{R}, d(x_i^*) = (\frac{1}{2})^i$. Now define $u : X \to \mathbb{R}$ by $u(x) = \sum\limits_{z \in X^* \cap NBT(x)} d(z)$. We claim $u(x)$ is what we want, i.e., NTS $x \succeq y$ iff $u(x) \geq u(y)$.

If $x \succeq y$, then $NBT(y) \subseteq NBT(x)$, so $X^* \cap NBT(y) \subseteq X^* \cap NBT(x)$, then $u(x) \geq u(y)$.

If $\urcorner(x \succeq y)$, then $y \succ x$, then $NBT(y) \supsetneq NBT(x)$, then $\exists x^* \in X, y \succeq x^* \succ x$, then $x^* \in X^* \cap NBT(y), x^* \notin X^* \cap NBT(x)$, so $NBT(y) \cap X^* \supsetneq NBT(x) \cap X^*$, then $u(y) > u(x)$, i.e., $\urcorner(u(x) \geq u(y))$.

Then we show the only if part. We know $\exists u$ represents $\succeq$, we need to construct X^*, which is somewhat technical. As we know $\mathbb{Q}$ is countable, so maybe $X^* \subseteq \mathbb{Q}$ is feasible. As $\mathbb{Q}$ is dense in $\mathbb{R}$, we always can find a rational number near any value of u. Now follow me closely.

Consider $\mathbb{Q} = \{q_1, q_2, \cdots\}$, $I_n = \{q_n, q_n'\}$, then I_n is countable as $\mathbb{Q} \times \mathbb{Q}$ is. Then we define $\tilde{u} \equiv \{r \in \mathbb{R} | \exists x \in X, u(x) = r\}$, i.e., the range of u. Now consider $\forall I_n \in \{I_n\}$, we have one of situations hold (see Figure 20):

1. $\tilde{u} \cap I_n = \varnothing$, then $\exists x_n \in X, u(x_n) \in [q_n, q_n']$;

2. $\tilde{u} \cap I_n = \varnothing$, then $u(x_n) = \inf\{r \in \tilde{u} | r > q_n'\}$;

3. $\tilde{u} \cap I_n = \varnothing, \urcorner(\exists X \in X) u(x) = \inf\{r \in \tilde{u} | r > q_n'\}$.

Then $\{x_n\}$ is countable certainly. Now we define $X^* = \{x_n\}$ and claim X^* is want we attempt to find.

Consider $\forall x, y \in X, x \succ y$, then $u(x) > u(y)$, we NTS $\exists x^* \in X^*$ s.t. $x \succeq x^* \succ y$, i.e., $u(x) \geq u(x^*) > u(y)$. As we know $\mathbb{Q}$ is dense in $\mathbb{R}$, so $\exists q \in \mathbb{Q}$ s.t. $u(x) > q > u(y)$.

We define $\bar{r} = \inf\{r \in \tilde{u} | r > q\}$. Then we have $\bar{r} = u(x)$ or $\bar{r} < u(x)$ (no $\bar{r} > u(x)$).

If $\bar{r} = u(x)$, then $u(x) = \bar{r} > q > u(y)$, then $\exists q' \in \mathbb{Q}$ s.t. $u(x) = \bar{r} > q > q' > u(y)$ and $I_n = [q', q] \in \{I_n\}$, so $\exists x_n \in X^*$ s.t. either $x_n \in [q', q]$ or $u(x_n) = \bar{r}$, then $x \succeq x_n \succ y, x_n \in X^*$.

If $\bar{r} < u(x)$, then $u(x) > \bar{r} > q > u(y)$, then $\exists q' \in \mathbb{Q}$ s.t. $I_n = [q, q'], u(x) > \bar{r} > q' > q > u(y)$, then $\exists x_n \in X^*, u(x_n) = \bar{r}$ s.t. $u(x) > \bar{r} = u(x_n) > u(y)$, i.e., $x \succ x_n \succ y$, i.e., $x \succeq x_n \succ y$.

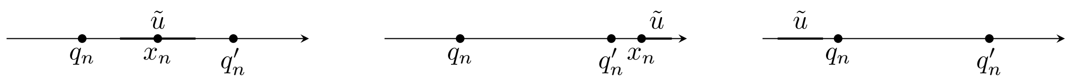


Figure 20: Construct a countable X^* with rational intervals

$\square$

Now we reach the furthest hill of possible X which can define a utility function, but separable X is not desired for the lack of economic implication while continuous X is common in economics. So we study continuous X now.

Assumption 10.5. *(Continuity) A complete and transitive preference on $X = \mathbb{R}^K$ is continuous if $\forall x, y \in X, x \succ y$, then $\exists \varepsilon > 0$ s.t. $\forall x', y' \in X, ||x' - x|| < \varepsilon, ||y' - y|| < \varepsilon$, we have $x' \succ y'$.*

LEMMA 10.10. *The following statements are equivalent:*

1. *$\succeq$ is continuous;*

2. *$\{x_n\} \subseteq X, x_n \succeq y(y \succeq x_n), \forall n$ and $\lim\limits_{n \to \infty} x_n = x$, then $x \succeq y(y \succeq x)$;*

3. *$\{x_n\} \subseteq X, \lim\limits_{n \to \infty} x_n = x, x \succ y(y \succ x)$, then $\exists N \in \mathbb{N}, \forall n > N$ we have $x_n \succ y(y \succ x_n)$;*

4. *$\forall x \in X, NBT(x), NWT(x)$ are closed;*

5. *$\forall x \in X, SBT(x), SWT(x)$ are open.*

$$\begin{tikzcd}
(1) \arrow[r] & (2) \arrow[r] & (3) \arrow[dl] \\
& (5) \arrow[ul] \arrow[r] & (4)
\end{tikzcd}$$

Figure 21: Proof logic of equivalent statements of continuity

Proof. We prove this according to the way of Figure 21.

$(4) \Leftrightarrow (5)$: Since $NBT(x) \cap SBT(x) = \varnothing, NBT(x) \cup SBT(x) = X$, so $NBT(x) = \mathbf{C}_X SBT(x)$, and we have $NWT(x) = \mathbf{C}_X SWT(x)$ analogously. As the complement of an open set is a closed set and vice versa, we reach the conclusion.

$(4), (5) \Rightarrow (1)$: This proof needs some skills. We want to show that $\exists w, x \succ w \succ y$. Since $x \in SBT(w), y \in SWT(w)$ and SBT, SWT are open ,so $\exists \varepsilon > 0, \forall ||x' - x|| < \varepsilon, ||y' - y|| < \varepsilon$, we have $x' \in SBT(w), y' \in SWT(w)$, so $x' \succ z \succ y'$, i.e., $x' \succ y'$. Then we claim such w exists and can be find with the following approach. (Like the immediate value theorem, $\forall f(x_1) > f(x_2), \exists x$ between$x_1, x_2, f(x_1) > f(x) > f(x_2)$)

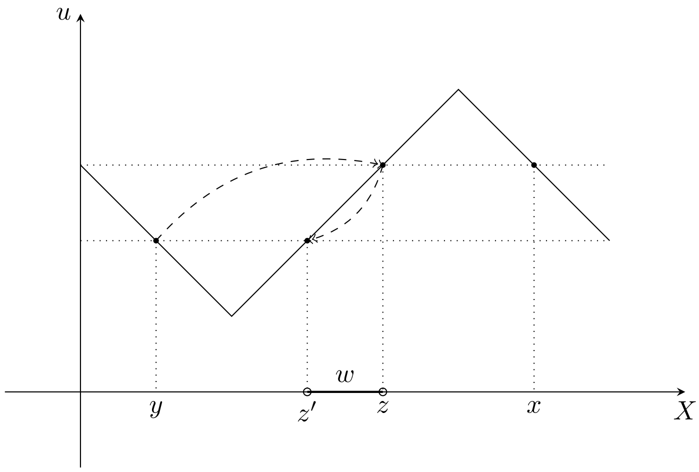


Figure 22: Quasi-immediate value theorem

The existence of w is obvious from Figure 22. Let $A = \{\alpha x + (1-\alpha)y | \alpha \in [0,1]\}$, which is closed. Then $A \cap NWT(x)$ is closed and non-empty ($x \in A \cap NWT(x)$). Since $y \notin NWT(x)$, we search $z \sim x$ by increasing α, i.e., moving from y to x until touching $NWT(x)$. Let $\alpha^* = \inf\{\alpha \in [0,1] | \alpha x + (1-\alpha)y \in NWT(x)\}$, clearly, α^* is available. And $\forall \alpha \in [0, \alpha^*), \alpha x + (1-\alpha)y \in NBT(x)$, then $z \in NBT(x)$ as $NBT(x)$ is closed and z is a limit of such elements. So $z \sim x$.

Using the same trick, let $B = \{\beta z + (1-\beta)y | \beta \in [0,1]\}$, moving from z to y and we can find $z' = \beta^* z + (1-\beta^*)y$ such that $z' \sim y$ and $\forall \beta \in (\beta^*, 1], \beta z + (1-\beta)y \succ y$.

Together we have $x \sim z = \alpha^* x + (1-\alpha^*)y$ and $y \sim z' = \alpha^*\beta^* x + (1-\alpha^*\beta^*)y$. For $\forall \gamma \in (\alpha^*\beta^*, \alpha^*), x \succ \gamma x + (1-\gamma)y \succ y$. Pick one such γ and let $w = \gamma x + (1-\gamma)y$. Then $x \succ w \succ y$.

$(1) \Rightarrow (2)$: Suppose $\exists \{x_n\} \subseteq X, x_n \to x, x_n \succeq y$ but $y \succ x$. Since $y \succ x$, then $\exists \varepsilon > 0, \forall ||x - x'|| < \varepsilon, ||y - y|| = 0 < \varepsilon$, we have $x' < y$, contradicting to $x_n \to x, x_n \succeq y$.

$(2) \Rightarrow (3)$: Suppose $\exists \{x_n\} \subseteq X, x_n \to x, x \succ y$ but $\forall N \in \mathbf{N}, \exists n > N$ s.t. $y \succeq x_n$. Then we collect such x_n and denote as x_k, s.t. $\{x_k\} \subseteq \{x_n\}, x_k \to x$ and $y \succeq x_k, \forall k$, then $y \succeq x$, contradicting to $x \succ y$.

$(3) \Rightarrow (5)$: Suppose $SWT(x)$ is not open, then $\exists y \in SWT(x)$, s.t. $\forall \varepsilon > 0, \exists y'$ s.t. $||y' - y|| < \varepsilon$ and $y' \notin SWT(x)$ (means y is in the boundary of $SWT(x)$ and this is immediate from the definition of open sets). Let $\varepsilon_1 = 1, \exists y_1$ such that $||y_1 - y|| < \varepsilon$ but $y_1 \succeq x$. Then let $\varepsilon_2 = \min\{1/2, ||y_1 - y||\}$, then $\exists y_2$ s.t. $||y_2 - y|| < \varepsilon_2$ and $y_2 \succeq x$, surely, $y_2 \neq y_1$. Repeat this step and we construct a sequence $\{y_n\}$ s.t. $y_n \succeq y$ and $y_n \to x$, i.e., $\urcorner(x \succeq y_n)$, violating (3). $\square$

THEOREM 10.1. *If $X = \mathbb{R}_+^K$ and $\succeq$ on X is complete, transitive and continuous, then $\exists u : X \to \mathbb{R}$ s.t. $u(x) \succeq u(y)$ iff $x \succeq y$.*

Proof. We just need to show $\mathbb{R}_+^K$ is separable. As $\mathbb{Q}_+^K \subseteq \mathbb{R}_+^K$ and $\mathbb{Q}_+^K$ is countable, we NTS $\forall x, y \in \mathbb{R}_+^K, \exists x^* \in \mathbb{Q}_+^K$ s.t. $x \succeq x^* \succ y$.

Since $\succeq$ is continuous, then from the last lemma, we have $\exists w \in X, x \succ w \succ y$. Then $w \in SWT(x) \cap SBT(y)$, which is open, so $\exists \varepsilon > 0, \forall w' \in X, ||w' - w|| < \varepsilon$, we have $w' \in SWT(x) \cap SBT(y)$. As $\mathbb{Q}^K$ is dense on $\mathbb{R}^K$, so $\exists x^* \in \mathbb{Q}^K$ s.t. $||x^* - w|| < \varepsilon$, so $x^* \in SWT(x) \cap SBT(y)$, i.e., $x \succ x^* \succ y$, loosely, $x \succeq x^* \succ y$. □

DEFINITION 10.8. *(Maximal and Greatest Elements) For $\forall X, \succeq$ on X. For $\forall A \subseteq X$, we call $x \in A$ a $\succeq$-maximal element of A if $A \cap NWT(x) = \{x\}$. We call $x \in A$ a $\succeq$-greatest element if $A \subseteq NBT(x)$. ($\forall A \subseteq X$, if $\exists x \in A$, x is a $\succeq$-greatest element, then $c(A) \neq \varnothing$)*

LEMMA 10.11. *Let X be an arbitrary set, $\succeq$ on X is complete, transitive and continuous, then $\forall$ non-empty compact $A \subseteq X$, we have A has a $\succeq$-greatest element. (A set X on an arbitrary space is compact if any open cover of it has a finite subcovering. In particular, $X \subseteq \mathbb{R}^K$ is compact iff it is closed and bounded)*

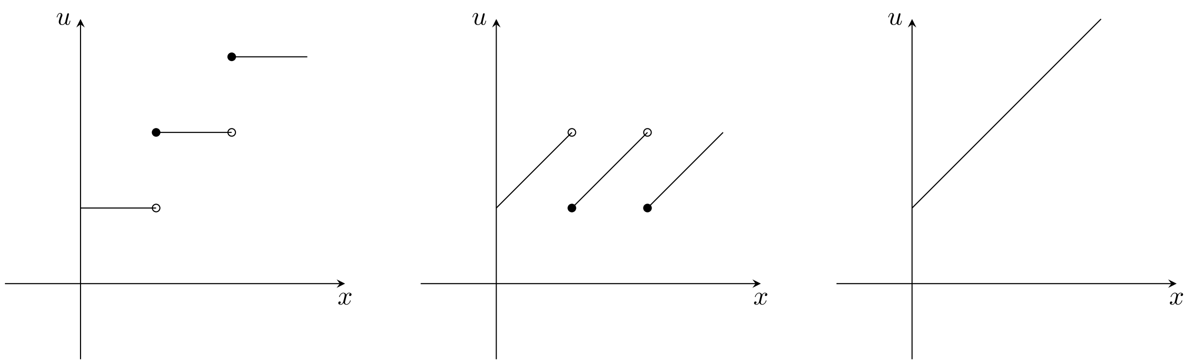
Proof. Suppose not, for $\forall y \in A, \exists x \in A$ s.t. $y \in SWT(x)$. Then $\{SWT(x) | y \in SWT(x), \forall y \in A\}$ is an open cover of A, as A is compact, so $\exists \{x_N\} = \{x_1, x_2, \cdots, x_N\} \in A$ s.t. $\{SWT(x_1), SWT(x_2), \cdots, SWT(x_N)\} \supseteq$ covering A, i.e.,$A \subseteq \bigcup_{x \in \{x_N\}} SWT(x)$. Then $c(\{x_N\}) \neq \varnothing$ (FN). WLOG, Let $x_i \in c(\{x_N\})$, then $x_1 \succeq x_i, \forall x_i \in \{x_N\}$, i.e., $x \notin SWT(x_i), \forall x_i \in \{x_N\}$. That contradicts to $x \in A, A \subseteq \bigcup_{x \in \{x_N\}} SWT(x)$. So A has a $\succeq$-greatest element. □

LEMMA 10.12. *If u is a utility function, representing $\succeq$. If $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing, then let $v = f(u), v : \mathbb{R} \to \mathbb{R}$ is another utility function representing $\succeq$. (utility function is ordinal)*

Proof. $\forall x, y \in X, x \succeq y$ iff $u(x) \succeq u(y)$ iff $v(x) = f(u(x)) \geq f(u(y)) = v(y)$. □

10.2 Properties of Preference

Illustration of properties of preference is in Figure 23.



(a) monotonicity (a) insatiability (a) convexity

Figure 23: Properties of preference

Assumption 10.6. *(Monotonicity) Let $X = \mathbb{R}^K, \succeq$ on X is monotone, if $\forall x, y \in X, x \geq y$ ($x_i \geq y_i, i = 1, 2, \cdots, K$) implies $x \succeq y$. $\succeq$ on X is strictly monotone if $\forall x, y \in X, x \geq y, x \neq y$ implies $x \succ y$.*

LEMMA 10.13. *Let $X = \mathbb{R}^K, \succeq$ on X and u represent $\succeq$. Then $\succeq$ is monotone iff u is non decreasing; $\succeq$ is strictly monotone iff u is increasing.*

Proof. $\succeq$ is monotone iff $\forall x, y \in X, x \geq y$ implies $x \succeq y$ iff $\forall x, y \in X, x \geq y$ implies $u(x) \geq u(y)$ iff u is non decreasing.

$\succeq$ is strictly monotone iff $\forall x, y \in X, x \geq y, x \neq y$ implies $x \succ y$ iff $\forall x, y \in X, x \geq y, x \neq y$ implies $u(x) > u(y)$ iff u is increasing. □

Assumption 10.7. *(Insatiability) Let $X = \mathbb{R}^K, \succeq$ on X is locally insatiable if $\forall x \in X, \forall \varepsilon > 0$ we have $\exists x'$ s.t. $||x' - x|| < \varepsilon$ and $x' \succ x$. $\succeq$ is globally insatiable if $\forall x \in X, \exists y \in X$ s.t. $y \succ x$.*

LEMMA 10.14. *If $\succeq$ is strictly monotone, then it is monotone and locally insatiable. If $\succeq$ is locally insatiable, it is globally insatiable.*

Proof. This is immediate from the definition. We just explain if $\succeq$ is strictly monotone, then it is locally insatiable. $\forall x \in X \subseteq \mathbb{R}^K, \forall \varepsilon > 0$, let $y = x + (\varepsilon/(2K))d, d = (1, 1, \cdots, 1)$, then $||y - x|| = \varepsilon/\sqrt{4K} < \varepsilon$, so $y \succ x$ as $\succeq$ is strictly monotone. Also, we see y near x and $y \succ x$, so $\succeq$ is locally insatiable. $\square$

DEFINITION 10.9. *(Convexity) Consider $X = \mathbb{R}^K$, $\succeq$ on X is convex if $\forall x, y \in X, x \succeq y$, then $\forall \lambda \in [0, 1]$, we have $\lambda x + (1 - \lambda)y \succeq y$. $\succeq$ on X is strictly convex if $\forall x, y \in X, x \neq y$, we have $\forall \lambda \in (0, 1), \lambda x + (1 - \lambda)y \succ y$. (Convexity implies marginal utility decreasing)*

LEMMA 10.15. *$\succeq$ is convex iff $NWT(x)$ is convex for $\forall x \in X$. (A set A is convex if $\forall x, y \in A, \forall \lambda \in [0, 1]$, we have $\lambda x + (1 - \lambda)y \in A$).*

Proof. $\succeq$ is convex iff $\forall x, y \in X, x \succeq y, \forall \lambda \in [0, 1]$ we have $\lambda x + (1 - \lambda)y \succeq y$ iff $\forall x, y \in X, x, y \in NWT(y)$, then $\forall \lambda \in [0, 1], \lambda x + (1 - \lambda)y \in NWT(y)$ iff $\forall y \in X, NWT(y)$ is convex. $\square$

LEMMA 10.16. *$\succeq$ on X is locally insatiable if it is globally insatiable and strictly convex. $\succeq$ on X is strictly monotone if it is monotone and strictly convex.*

Proof. This follows intuition. For $\forall x \in X$, we have $\exists y \in X, y \succ x$ as $\succeq$ is globally insatiable. Then $\forall \lambda \in (0, 1)$, we have $\lambda y + (1 - \lambda)x \succ x$ as $\succeq$ is strictly convex. Let $d = ||y - x||$, $\forall \varepsilon > 0$, let $0 < \lambda < \min\{\varepsilon/d, 1\}$, then $||\lambda y + (1 - \lambda)x \succ x - x|| = ||\lambda(y - x)|| = \lambda d < \varepsilon$ and $\lambda y + (1 - \lambda)x \succ x$, that implies $\succeq$ is locally insatiable.

If $\succeq$ is monotone and strictly convex, $\forall x, y \in X, y \geq x, y \neq x$, we have $y \succeq x$ and $\frac{1}{2}y + \frac{1}{2}x \succ x$ from strict convexity of $\succeq$. But $y \geq \frac{1}{2}y + \frac{1}{2}x \geq$, so $y \succeq \frac{1}{2}y + \frac{1}{2}x$ from the monotonicity of $\succeq$. So $y \succ x$, which implies $\succeq$ is strictly monotone. $\square$

We summarize this lemma in Figure 24

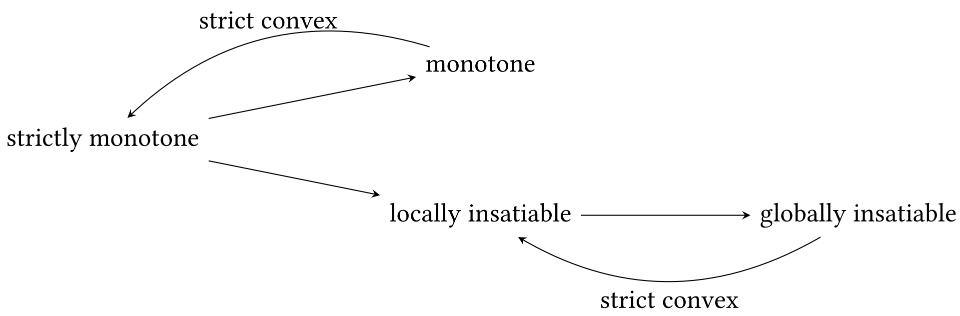


Figure 24: Equivalent properties of preference

LEMMA 10.17. *Let u represent $\succeq$ on X. Then we have:*

1. *$\succeq$ is convex iff u is quasi-concave (looser than concave);*
2. *$\succeq$ is strictly convex if u is strictly quasi-concave (looser than strictly concave);*

We say:

- *A function f is concave if $\forall x, y, \forall \lambda \in [0, 1]$, we have $f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)$;*
- *A function f is strictly concave if $\forall x \neq y, \forall \lambda \in (0, 1)$, we have $f(\lambda x + (1 - \lambda)y) > \lambda f(x) + (1 - \lambda)f(y)$;*
- *A function f is quasi-concave if $\forall x, y, \forall \lambda \in [0, 1]$, we have $f(\lambda x + (1 - \lambda)y) \geq \min\{f(x), f(y)\}$;*
- *A function f is strictly quasi-concave if $\forall x \neq y, \forall \lambda \in (0, 1)$, we have $f(\lambda x + (1 - \lambda)y) > \min\{f(x), f(y)\}$;*

Certainly, $\lambda f(x) + (1 - \lambda)f(y) \geq \min\{f(x), f(y)\}$, so if a function f is (strictly) concave, we have f is (strictly) quasi-concave but not vice versa.

Proof. $\succeq$ is convex iff $\forall x, y, x \succeq y$, then $\forall \lambda \in [0,1], \lambda x + \lambda y \succeq y$ iff $\forall x, y, u(x) \geq u(y)$, then $\forall \lambda \in [0,1], u(\lambda x + (1-\lambda)y) \geq y$ iff u is quasi-concave.

$\succeq$ is strictly convex iff $\forall x \neq y, x \succeq y$, then $\forall \lambda \in (0,1), \lambda x + \lambda y \succ y$ iff $\forall x \neq y, u(x) \geq u(y)$, then $\forall \lambda \in (0,1), u(\lambda x + (1-\lambda)y) > u(y)$ iff u is strictly quasi-concave. □

LEMMA 10.18. *If $\succeq$ on $X = \mathbb{R}_+^K$ is continuous and strictly monotone, it has a continuous utility representation.*

Proof. We construct such a utility function with Figure 25.

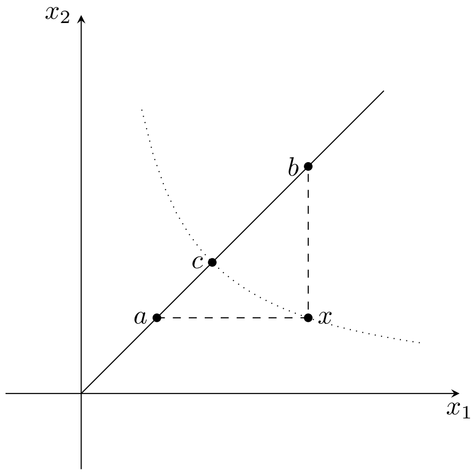


Figure 25: Equivalent point of x on the diagonal line

We see if $\forall x \in X, \exists r \in \mathbb{R}$ s.t. $x \sim rd, d = (1, 1, \cdots)$, then let $u(x) = r$, we obtain a continuous utility function.We first show such r exists. Let $\forall x \in X, x = (x_1, x_2, \cdots, x_K), \overline{x} = \max\{x_1, x_2, \cdots, x_K\}, \underline{x} = \min\{x_1, x_2, \cdots, x_K\}$. If $\overline{x} = \underline{x}$, we have $r = \overline{x}$. If not, then we have $\overline{x}d > x > \underline{x}d$, since $\succeq$ is strictly monotone, so $\overline{x}d \succ \underline{x}d$. Let $a = \overline{x}d, b = \underline{x}d$, and $A = \{y = \lambda a + (1-\lambda)b | \lambda \in [0,1]\}$, which is closed. Since $\succeq$ is continuous, $NBT(x)$ is closed, so $A \cap NBT(x)$ is closed and non-empty (a belongs to it). Pick $r = \sup\{r | rd \in A \cap NBT(x)\}$, then we have $x \sim rd$. Further by the monotonicity of $\succeq$, we have r is unique.

Then we have $\forall x \in X, \exists! r \in R$ (! means only) s.t. $x \sim rd$. Let $u(x) = r$, we claim $u(\cdot)$ is a continuous utility function. First $\forall x \in X, r$ is non-negative, so u is well defined. Then $\forall x \succeq y$ iff $x \sim u(x)d \succeq u(y)d \sim y$ iff $u(x) \geq u(y)$ by monotonicity.

Then we show $\succeq$ is continuous. Suppose not, then $\exists x \in X, u$ is not continuous on x, so $\exists \{x_n\} \subseteq X, x_n \to x$ but $\urcorner(u(x_n))$, i.e., $\exists \delta > 0$ s.t. $u(x_n) > u(x) + \delta$ or $u(x_n) < u(x) - \delta, \forall n$. Since $\{x_n\}$ is infinite, so we have infinite $u(x_n) > u(x) + \delta$ or infinite $u(x_n) < u(x) - \delta$. WLOG, let $\{x_{n_k}\} \subseteq \{x_n\}$ s.t. $u(x_{n_k}) > u(x) + \delta, \forall k$. So $x_n \succ [u(x) + \delta]d \succ u(x)d$. Let $r^* = u(x) + \delta$, we have $r^*d \succ u(x)d$. As $\succeq$ is continuous and $x_{n_k} \to x$, we have $x \succeq r^*d$, contradicting to $r^*d \succ x$. So $u(\cdot)$ is continuous. □

THEOREM 10.2. *(Debreu's Theorem) If $\succeq$ on $X = \mathbb{R}_+^K$ is continuous, then it has a continuous utility representation. (it holds conversely)*

Proof. We need to construct such a continuous utility function. Let $\overline{X} \equiv \{x \in X | x \succeq y, \forall y \in X\}, \underline{X} = \{x \in X | y \succeq x, \forall y \in X\}, Y \equiv \mathbb{Q}_+^K, X^* = X \backslash (\overline{X} \cup \underline{X}), Y^* = X^* \cap Y$.

We know Y^* is countable as $\mathbb{Q}_+^K$ is countable. Then we claim $\urcorner(\exists y \in Y^*)$ s.t. $y \succeq y' \forall y' \in Y^*$, $\urcorner(\exists y \in Y^*)$ s.t. $y' \succeq y \forall y' \in Y^*$.

Suppose $\exists y \in Y^*$ s.t. $y \succeq y', \forall y' \in Y^*$. If $\overline{X} \neq \varnothing, \forall x^* \in \overline{X}$, we have $x^* \succ y$. Then $\exists z \in X^*$ s.t. $x^* \succ z \succ y$. However, we can insert a $y'' \in Y$ near z as $\mathbb{Q}_+^K$ is dense in $\mathbb{R}_+^K$ and obtain $x^* \succ y'' \succ y$ for $\succeq$ is continuous. So $y'' \in Y^*$ (for $x^* \succ y''$), contradicting to $y \succeq y', \forall y' \in Y^*$.

Suppose $\overline{X} = \varnothing$, then $\forall x \in X, \exists x' \in X$ s.t. $x' \succ x$. Then $\forall y \in Y^*, \exists z \in X^*$ s.t. $z \succ y$. Then we can find a $y'' \in Y$ near z and $y'' \succ y$ from the continuity of $\succeq$, contradicting to $y \succeq y' \forall y' \in Y^*$. We can prove $\urcorner(\exists y \in Y^*)$ s.t. $y' \succeq y \forall y' \in Y^*$ analogously.

Then we can construct the utility function. Let $Y^* = \{y_1, y_2, \cdots\}$. Let $u(x) = 1, \forall x \in \overline{X}, u(x) = 0, \forall x \in \underline{X}$. Let $u(y_1) = \frac{1}{2}$. Then moving from y_2 to infinite elements, for y_n we have four cases:

1. If $y_n \sim y_m$ for $m \leq n-1$, let $u(y_n) = u(y_m)$;
2. If $y_n \succ y_m, \forall m = 1, 2, \cdots, n-1$, let $u(y_n) = \frac{1}{2}(1 + u(y_m))$ where $y_m \succeq y_i, i = 1, 2, \cdots, n-1$;

3. If $y_m \succ y_n, \forall m = 1, 2, \cdots, n-1$, let $u(y_n) = \frac{1}{2}u(y_m)$ where $y_i \succeq y_m, i = 1, 2, \cdots, n-1$;
4. If $\exists m, m' \in \{1, 2, \cdots, n-1\}$ s.t. $y_m \succ y_n \succ y_{m'}$ and $y_i \succeq y_m$ or $y_{m'} \succeq y_i$ for $\forall i \in \{1, 2, \cdots, n-1\}$, let $u(y_n) = \frac{1}{2}(u(y_m) + u(y_{m'}))$.

By the construction of Y^*, the range of $u : Y^* \to (0, 1)$ is exactly all dyadic numbers in $(0, 1)$, i.e., expressed by $\frac{k}{2^l}, k, l \in \mathbb{N}_{++}$ and $k < 2^l$. First we say$\{0, 1\}$ are the limits of $u(Y^*)$, that hols because $\forall y \in Y^*, \exists z^* \in Y^*$ s.t. $z^* \succ y$ and $\exists z^{'*} \in Y^*$ s.t. $y \succ z^{'*}$.

Then we want to show the range of $u(Y^*)$ is dense on $(0, 1)$, i.e., $\forall y_i \succ y_j \in Y^*, \exists z^* \in Y^*$ s.t. $y_i \succ z^* \succ y_j$. That holds as $\succeq$ is continuous and Y^* is dense on X.

Now we define left points. $\forall x \in X^* \backslash Y^*$, let $u(x) \equiv \sup\{u(y)|y \in Y^*, x \succeq y\} \equiv \sup\{u(y)|y \in Y^* \cap NBT(x)\}$.

We show u is a utility function representing $\succeq$ by $x \sim y$ iff $u(x) = u(y)$ (a. $x, y \in Y^*$, then $u(x) = u(y)$; b. $x \in X^* \backslash Y^*, y \in Y^*$, then $u(x) = u(y)$; c. $x, y \in X^* \backslash Y^*$, then they get the same supremum, so $u(x) = u(y)$) and $x \succ y$ iff $\exists z_1, z_2 \in Y^*$ s.t. $u(x) \geq u(z_1) > u(z_2) \geq u(y)$.

Lastly we show u is continuous. Suppose not, $\exists \{x_n\} \subseteq X$ s.t. $x_n \to x$ but $\lim_{n\to\infty} u(x_n) \neq x_n$. Then $\exists \delta > 0$ we can find a subsequence $\{x'_n\}$ of $\{x_n\}$ such that $x'_n \to x$ and either $u(x'_n) > u(x) + \delta$ or $u(x'_n) < u(x) - \delta$. WLOG, let $u(x'_n) > u(x) + \delta, \forall n$. So $\lim_{n\to\infty} u(x'_n) > u(x) + \delta$. Then, as $u(Y^*)$ is dense on $(0, 1)$, so can find $z^* \in Y^*$ such that $u(x_n) + \frac{1}{2}\delta > u(z^*) > u(x)$. As $\lim_{n\to\infty} u(x'_n) > u(x) + \delta$, then $\exists N \in \mathbb{N}$ s.t. $\forall n > N, u(x'_n) > u(x) + \delta - \frac{1}{2}\delta = u(x) + \frac{1}{2}\delta$. So $x'_n \succ z^*, \forall n > N$. Then from the continuity of $\succeq$, we have $x \succeq z^*$. But from $u(z^*) > u(x)$ we have $z^* \succ x$, contradiction immediately. So u is continuous. (Note that the existence of z^* comes from $u(Y^*)$ is dense on $(0, 1)$) □

Finally, we introduce a specific utility function, quasi-linear utility function.

Assumption 10.8. *(Quasi-linearity) A preference $\succeq$ is quasi-linear in the K-th commodity if it can be represented by an utility function of the form $U(x, m) = u(x) + m$ for some sub-utility function $u : \mathbb{R}^K_+ \to \mathbb{R}$.*

LEMMA 10.19. *$\succeq$ on X is quasi-linear iff*

(1) $\forall x \in \mathbb{R}^K_+, \forall m, m' \in \mathbb{R}$ we have $m \geq m'$ iff $(x, m) \succeq (x, m')$;

(2) $\forall x, x' \in \mathbb{R}^K_+, \forall m, m', m'' \in \mathbb{R}$, we have $(x, m) \succeq (x', m')$ iff $(x, m + m'') \succeq (x', m' + m'')$;

(3) $\forall x, x' \in \mathbb{R}^{K-1}_+$ we have $\exists m, m' \in \mathbb{R}_+$ s.t. $(x, m) \sim (x', m')$.

Proof. We first show the only if part. Let u represent $\succeq$ which is quasi-linear, then $U(x, m) = u(x) + m$. So we have

(1) $\forall x, m, m' \in \mathbb{R}, m \geq m'$ iff $u(x) + m \geq u(x) + m'$ iff $U(x, m) \geq U(x, m')$ iff $(x, m) \succeq (x, m')$;

(2) $\forall x, x', m, m', (x, m) \succeq (x', m')$ iff $U(x, m) \geq U(x', m')$ iff $u(x) + m \geq u(x') + m'$ iff $u(x) + m + m'' \geq u(x') + m' + m''$ iff $U(x, m + m'') \geq U(x, m' + m'')$ iff $(x, m + m'') \succeq (x', m' + m'')$;

(3) $\forall x, x'$, WLOG, let $u(x) \geq u(x')$, then let $u(x) = u(x') + u(x) - u(x')$, i.e., $m = 0, m' = u(x) - u(x')$.

Then we prove the tough if part and we need to construct such a utility function.

From (1), we have if $\exists x, (x, m) \succeq (x, m')$, then $m \geq m'$. Further, we say if $\exists x, (x, m) \sim (x, n)$, then $m = n$, immediately from $(x, m) \sim (x, n)$ equals to $(x, m) \succeq (x, n)$ and $(x, n) \succeq (x, m)$, then $m \geq n$ and $n \geq m$, i.e., $m = n$.

Let $\mathbf{0} = (0, 0, \cdots, 0) \in \mathbb{R}^K_+$, then $\forall x \in \mathbb{R}^K_+$, we have $\exists m, m' \in \mathbb{R}$ s.t. $(x, m) \sim (\mathbf{0}, m')$ from (3). Then we let $u(x) = m' - m$ and we claim $u(x)$ is a function of x, i.e., $m' - m$ is unique for $\forall x$. Suppose not, $\exists n, n', n - n' \neq m - m'$ s.t. $(x, n) \sim (\mathbf{0}, n')$, then $(x, n + m') \sim (\mathbf{0}, n' + m') \sim (x, m + n')$, so $n + m' = m + n'$, i.e., $m' - m = n' - n$, contradicting to $u(x)$ is not unique. So $u(x)$ is unique, and let $m = 0, u(x) = m'$, so $(x, 0) \sim (\mathbf{0}, u(x))$.

We claim such $U(x, m) = u(x) + m$ represents $\succeq$. We NTS $\forall (x, m) \succeq (x', m')$ iff $U(x, m) \geq U(x', m')$. Since $(x, m) \sim (x, 0 + m) \sim (\mathbf{0}, u(x) + m)$ and $(x', m') \sim (\mathbf{0}, u(x') + m')$ analogously. So $(x, m) \succeq (x', m')$ iff $(\mathbf{0}, u(x) + m) \succeq (\mathbf{0}, u(x') + m')$ iff $u(x) + m \geq u(x') + m'$ iff $U(x, m) \geq U(x', m')$. Therefore, we obtain the u. □

11 Consumer's Problem

11.1 Consumer's Demand and Duality

We study how consumers choose goods under constrains and what the choice gives out. Commonly, let $X = \mathbb{R}_+^K$ denote a choice set (assumed good), $p \in R_+^K$ denote a price vector and w denote the total endowment, and consumers are price takers. We then define the core conceptions of this section.

DEFINITION 11.1. *(Budget Set) Let $B(p, w) \equiv \{x \in \mathbb{R}_+^K | px \leq w\}$. This defines the choice set.*

DEFINITION 11.2. *(Marshallian Demand) Let $D(p, w) \equiv \{x \in B(p, w) | x \succeq y, \forall y \in B(p, w)\}$. Marshallian demand tells the best choice available under wealth constrains.*

DEFINITION 11.3. *(Hicksian Demand) Let $H(p, x^0) \equiv \{x \succeq x^0 | px \leq py, \forall y \succeq x^0\}$. Hicksian demand tells the best choice of the least wealth you need to reach the same utility level of x^0.*

We can rewrite Marshallian demand and Hicksian demand with sets and show them in Figure 26.

LEMMA 11.1.

$$D(p, w) = B(p, w) \cap [\bigcap_{x \in B(p,w)} NWT(x)] \tag{11.1}$$

$$H(p, x^0) = NWT(x^0) \cap [\bigcap_{x \in NWT(x^0)} B(p, px)] \tag{11.2}$$

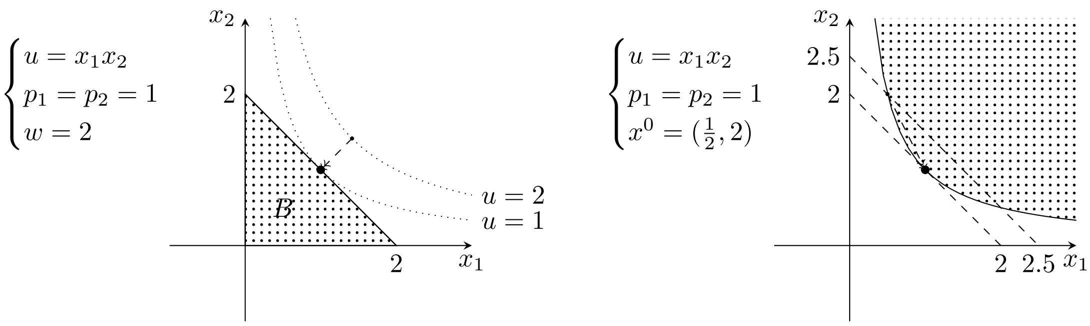


Figure 26: How Marshallian demand and Hicksian demand evolve

LEMMA 11.2. *(1) If $\succeq$ is continuous, both $D(p, w), H(p, x^0)$ are closed.*
(2) If in addition ((1) holds), $p \in \mathbb{R}_{++}^K$, both $D(p, w), H(p, x^0)$ are non-empty and compact and $D(p, w) = D(\lambda p, \lambda w), H(p, x^0) = H(\lambda p, x^0), \forall \lambda > 0$.
(3) If in addition ((1), (2) hold), $\succeq$ is convex, both $D(p, w), H(p, x^0)$ are convex; $\succeq$ is strictly convex, both $D(p, w), H(p, x^0)$ are singletons.

Proof. (1) If $\succeq$ are continuous, then $\forall x, NWT(x)$ are closed. Then $\bigcap_{x \in B(p,w)} NWT(x)$ are closed, and also $B(p, w)$ is closed, so $D(p, w)$ is closed.

As $\bigcap_{x \in NWT(x^0)} B(p, px)$ are closed, so the intersection of that and $NWT(x^0)$ is closed, i.e., $H(p, x^0)$ is closed. (Note that we use $B(p, w)$ is closed and an arbitrary intersection of closed sets is closed.)

(2) With $p \in \mathbb{R}_{++}$, we have $B(p, w)$ is bounded and closed, i.e., it is compact. We rewrite $D(p, w) = \underset{x \in B(p,w)}{argmax}\ u(x)$, since $\succeq$ is continuous, so u is continuous. As $B(p, w)$ is compact, so the range of u is compact, so we can obtain the maximum, also the maximizer. That is $D(p, w)$ is non-empty and further the maximizer set is compact.

We rewrite $H(p, x^0) = \underset{x \in NWT(x^0) \cap B(p, px^0)}{argmax} -px$. As $NWT(x^0)$ is closed and $B(p, px^0)$ is compact, so the domain is compact. With $-px$ is continuous, we know the range of it is compact, so the maximum and maximizer are available. That is $H(p, x^0)$ is non-empty and further the maximizer set is compact. (Note that we use the maximizer set of a continuous function on a compact domain is non-empty and compact).

Moreover, $D(p, w)$ and $D(\lambda p, \lambda w)$ ask for the same domain $B(p, w)$, so the maximizer set remains, i.e, $D(p, w) = D(\lambda p, \lambda w)$. With the same reason, we have $H(p, x^0) = H(\lambda p, x^0)$.

(3) $\succeq$ is convex (with (1), (2) stay). $\forall x, y \in D(p,w)$, we have $x \sim y$. Then $\forall \lambda \in [0,1]$, we have $\lambda x + (1-\lambda)y \sim x \succeq x$, and such $\lambda x + (1-\lambda)y$ is available as $p(\lambda x + (1-\lambda)y) = \lambda px + (1-\lambda)py \leq \lambda w + (1-\lambda)w = w$. (Note that $x, y \in B(p,w)$).

Further, if $\succeq$ is strictly convex, we want $D(p,w)$ is a singleton. Suppose not, let $x, y \in D(p,w)$, then $x \sim y$, implying $x \succeq y$. As $\succeq$ is strictly convex, $\forall \lambda \in (0,1)$, we have $\lambda x + (1-\lambda)y \succ y$ while $\lambda x + (1-\lambda)y \succ y$ belongs to $B(p,w)$ as we proved. So $y \notin D(p,w)$. Therefore, $D(p,w)$ is a singleton.

Then we show $H(p,x^0)$ is convex. $\forall x, y \in H(p, px^0)$, we have $px = py$. Without loss of generality, let $x \succeq y$, then $\forall \lambda \in [0,1]$, (since $\succeq$ is convex) we have $\lambda x + (1-\lambda)y \succeq y \succeq x^0$. Moreover, $p(\lambda x + (1-\lambda)y) = \lambda px + (1-\lambda)py = px = py$. Then $\lambda x + (1-\lambda)y \in H(p,x^0)$.

Further, if $\succeq$ is strictly convex, we want to show $H(p,x^0)$ is a singleton. Suppose not, let $x, y \in H(p,x^0), x \neq y$, then $px = py, x \succeq x^0, y \succeq x^0$. Without loss of generality, let $x \succeq y$, then as $\succeq$ is strictly convex, $\forall \lambda \in (0,1)$, we have $\lambda x + (1-\lambda)y \succ y \succeq x^0$. Then we want to find z near $\lambda x + (1-\lambda)y$ which costs less. Since $\succeq$ is continuous, then $\exists \varepsilon > 0, \forall z$ such that $||\lambda x + (1-\lambda)y - z|| < \varepsilon$, we have $z \succ x^0$ (as $\lambda x + (1-\lambda)y \succ x^0$). Let $z = \lambda x + (1-\lambda)y - (\varepsilon/n)d$ ($d = (1,1,\cdots)$, n is the dimension of choice sets), we have $||\lambda x + (1-\lambda)y - z|| = \varepsilon/\sqrt{n} \leq \varepsilon$ and $pz = p(\lambda x + (1-\lambda)y - (\varepsilon/n)d) = py - (\varepsilon/n)pd < py$. So $y \notin H(p,x^0)$ (as we find a better choice, costing less but happier). ☐

LEMMA 11.3. *(Duality I) If $\succeq$ is locally insatiable, $\forall x^* \in D(p,w)$, we have $D(p,w) \subseteq H(p,x^*)$ and $px^* = w$.*

Proof. $\forall x' \in D(p,w), x^* \in D(p,w)$, we have $x' \sim x^*$, implying $x' \succeq x^*$. Suppose $x' \notin H(p,x^*)$, then $\exists \tilde{x} \in H(p,x^*), \tilde{x} \succeq x^*$ and $p\tilde{x} < px' \leq w$ (as $x' \sim x^*, x' \notin H(p,x^*)$). Since $\succeq$ is locally insatiable, we say $\forall \varepsilon > 0, \exists x''$ such that $||x'' - \tilde{x}|| < \varepsilon$ and $x'' \succ \tilde{x}$. Since $p\tilde{x} < w$, we can let $\varepsilon < w - p\tilde{x}$, then $px'' < w$, i.e., $x'' \in B(p,w)$. Since $x'' \succ \tilde{x} \succeq x^*$, then $x^* \notin D(p,w)$ as x'' costs less but brings more happiness. Therefore, $x' \in H(p,x^0)$ as we desire.

If $px^* < w$, we can find $x''' \succ x^*$ but $px''' < w$ as $\succeq$ is locally insatiable as we do before. Then $px^* = w$. ☐

LEMMA 11.4. *(Duality II) If $\succeq$ is continuous, $\forall x^* \in H(p,x^0)$ such that $px^* > 0$, we have $H(p,x^0) \subseteq D(p,px^*)$ and $x^* \sim x^0$.*

Proof. Let $w = px^* > 0$. $x^* \in H(p,x^0)$ implies $x^* \succeq x^0$. Suppose $x^* \succ x^0$, with continuity of $\succeq$, we can find $x^* - d/n$ ($d = (1,1,\cdots)$) near x^* (as long as n large enough) such that $x^* - d/n \succ x^0$ and $p(x^* - d/n) = px^* - pd/n < px^* = w$. So we find a choice costing less but bringing more happiness than x^*, then $x^* \notin H(p,x^0)$. Therefore $x^* \sim x^0$.

Now we want to show $H(p,w) \subseteq D(p,px^*)$. $\forall x' \in H(p,x^0)$, we have $px' = px^* = w$ and $x' \succeq x^0$. Then $x' \in B(p,w) = B(p,px^*)$. As $x' \succeq x^0 \sim x^*$, so $x' \succeq x^*$. Suppose $x' \notin D(p,px^*)$, then $\exists \tilde{x} \in B(p,px^*)$ such that $\tilde{x} \succ x' \succeq x^0$, so $\tilde{x} \succ x'$. With continuity of $\succeq$, we can find $\tilde{x} - d/n$ near $\tilde{x}$, that is $\tilde{x} - d/n \succ x^0$ and $p(\tilde{x} - d/n) < p\tilde{x} \leq w$. So $x^* \notin H(p,x^0)$, contradicting to $x^* \in H(p,x^0)$. Hence, $x' \in D(p,px^*)$, i.e., $H(p,x^0) \subseteq D(p,px^*)$. ☐

DEFINITION 11.4. *(Indirect Utility) For a given $u(\cdot)$, we define $v: \mathbb{R}_+^K \times \mathbb{R}_+ \to \mathbb{R}$ s.t. $v(p,w) \equiv u(x^*)$ for $\forall x^* \in D(p,w)$. (We refuse to discuss when it is non-empty). More precisely, let $v(p,w) \equiv \sup\{u(x)|px \leq w\}$.*

DEFINITION 11.5. *(Expenditure Function) For some $p \in \mathbb{R}_+^K$, let $e(p,x^0) = \inf\{px|x \succeq x^0\}$ for $\forall x^0 \in \mathbb{R}_+^K$. Note that $e(p,\cdot): \mathbb{R}_+^K \to \mathbb{R}_+$.*

LEMMA 11.5.

1. *If $p \in \mathbb{R}_{++}^K$, $u(\cdot)$ is continuous, then $v(p,w)$ is continuous;*
2. *$v(p,w) = v(\lambda p, \lambda w)$;*
3. *$v(p,w)$ is weakly increasing in w, weakly decreasing in p. If $\succeq$ is strictly monotone, the relation is strict;*
4. *$\forall t \in [0,1], v(tp + (1-t)p', tw + (1-t)w') \leq \max\{v(p,w), v(p',w')\}$.*

Proof.

1. From Berge's Maximization Theorem, let $f = u(\cdot), \Gamma(p,w) = B(p,w), \theta = (p,w), h(\theta) = v(\theta)$, then with $B(p,w)$ compact-valued and continuous, $u(\cdot)$ continuous, we have $v(p,w)$ is continuous.
2. Since $D(\lambda p, \lambda w) = D(p,w)$, we have the same best choice x^*, so $u(x^*)$ remains.
3. This is easy. As w increases, we get $B(p,w)$ looser, so $v(p,w)$ increases. As p increases, we get $B(p,w)$ tighter, so $v(p,w)$ decreases.

4. $\forall t \in [0,1]$, let $p^t = tp + (1-t)p'$, $w^t = tw + (1-t)w'$, we show $v(p^t, w^t) \leq \max\{v(p,w), w(p', w')\}$ as follows. Let $x^t \in D(p^t, w^t)$, then $v(p^t, w^t) = u(x^t)$ and $p^t x^t \leq w^t$. That is $[tp + (1-t)p']x^t \leq tw + (1-t)w'$, so we have $px^t \leq w$ or $p'x^t \leq w'$ (or both), i.e., either $x^t \in B(p,w)$ or $x^t \in B(p', w')$ (or both). Without loss of generality, let $x^t \in B(p,w)$. If $x^t \in B(p,w)$, $x^* \in D(p,w)$, we have $x^* \succeq x^t$, so $v(p,w) = u(x^*) \geq u(x^t) = v(p^t, w^t)$. So $v(p^t, w^t) \leq v(p,w)$ or $v(p^t, w^t) \leq v(p', w')$, that is $v(p^t, w^t) \leq \max\{v(p,w), v(p', w')\}$.

□

LEMMA 11.6. *If $\succeq$ is continuous, $p \in \mathbb{R}^K_{++}$, then we have*

1. *$px = e(p, x^0)$ for $\forall x \in H(p, x^0)$;*
2. *If in addition, $x \succeq \mathbf{0}$ for $\forall x \in X$, we have $e(p, \cdot) : X \to \mathbb{R}$ is a continuous utility representation;*
3. *$\lambda e(p, x^0) = e(\lambda p, x^0), \forall \lambda > 0$;*
4. *$\forall t \in [0,1], te(p, x^0) + (1-t)e(p', x^0) \leq e(tp + (1-t)p', x^0)$.*

Proof. 1. Since $\succeq$ is continuous and $B(p, px^0) \cap NWT(x^0)$ is compact, so the minimizer is available, then $H(p, x^0)$ is non-empty. Then we have this statement immediately from the definition.

2. First we show $e(p, \cdot)$ represents $\succeq$, i.e., $\forall x, x', x \succeq x'$ iff $e(p,x) \geq e(p,x')$.

   If $x \succeq x'$, then $NWT(x) \subseteq NWT(x')$, thus $e(p,x) \geq e(p,x')$ (the former's domain is tighter than the latter).

   From Lemma 11.2, we know $\forall x, H(p,x)$ is non-empty and compact. Let $x_a \in H(p,x), x_b \in H(p,x')$, then $e(p,x) = px_a, e(p,x') = px_b, px_a \geq px_b$ and $x_a \sim x, x_b \sim x'$ from Duality II. Further, $x_a \in D(p, px_a) = D(p, e(p,x)), x_b \in D(p, px_b) = D(p, e(p,x'))$ from Duality II. Then, from Lemma 11.5, and $e(p,x) \geq e(p,x')$, we have $v(p, e(p,x)) \geq v(p, e(p,x'))$, i.e., $x_a \succeq x_b$, i.e., $x_a \sim x \succeq x' \sim x_b$.

   Next we show $e(p, \cdot)$ is continuous. Since $e(p,x) = -\max_{y \in NWT(x) \cap B(p,px)} -py$, then from the Berge's Maximization Theorem, we immediately have $e(p, \cdot)$ is continuous (the domain is compact-valued and continuous, the target function is continuous; analogously, $e(\cdot, x)$ and the minimizer set $H(p,x)$ is upper semi- continuous).

3. Immediately from Lemma 11.2.

4. Note that $\forall p_1, p_2, x$ and $x^* \in H(p_1, x)$ we have $p_2 x^* \geq e(p_2, x)$. Suppose not, then $p_2 x^* < e(p_2, x)$, which implies $x^* \notin H(p_2, p_2 x)$. Since $H(p_2, x)$ is not-empty ($e(p_2, x)$ exists), so $\exists x' \in H(p_2, x)$, and $x' \succeq x$. Since $x^* \in H(p_1, x)$, we say $x^* \succeq x$. Thus, with $p_2 x^* < e(p_2, x) = p_2 x'$, we have $x' \notin H(p_2, x)$ (x^* costs less and brings more happiness). Therefore we ensure that the statement holds. (we assume $p \in \mathbb{R}^K_{++}$ to ensure $H(p,x)$ non-empty , do not assume $\succeq$ is continuous)

   Then let $\bar{p} = tp + (1-t)p'$, and $x^* \in H(\bar{p}, x^0)$. Then from the last statement, we have $px^* \geq e(p, x^0)$ and $p'x^* \geq e(p', x^0)$. So $\forall t \in [0,1]$, we have $tpx^* + (1-t)p'x^* \geq te(p, x^0) + (1-t)e(p', x^0)$, i.e., $e(\bar{p}, x^0) = \bar{p}x^* \geq te(p, x^0) + (1-t)e(p', x^0)$.

   This statement show that variant prices better consumers as they provide opportunity to reallocate wealth on different goods for consumers, i.e., bringing substitution effects.

□

We have show how to find $v(p,w)$ and $e(p, \cdot)$ and their properties, next we study the transformation conversely, see Figure 27.

LEMMA 11.7. *(Roy's Identity) Let $\succeq$ be continuous, locally insatiable, strictly convex, and $u(\cdot)$ represents $\succeq$ which is continuous. Then for every (p,w) such that $v(p,w)$ is differentiable, $D(p,w) = -\frac{\partial v(p,w)/\partial p}{\partial v(p,w)/\partial w}$ (assume $p \in \mathbb{R}^K_{++}$ to ensure $D(p,w)$ non-empty).*

Proof. First we need to show $D(p,w)$ is a singleton as it is a function constructed by $v(p,w)$. Since $\succeq$ is continuous and strictly convex, so $D(p,w)$ is a singleton from Lemma 11.2.

$$\begin{tikzcd}
D(p,w) \arrow[r, leftrightarrow, "\text{duality}"] \arrow[d, "\text{definition}"] & H(p,x^0) \arrow[d] \\
v(p,w) \arrow[u, bend left, "\text{Roy identity}"] & e(p,x^0) \arrow[u, bend right, "\text{Shephard' lemma}"']
\end{tikzcd}$$

Figure 27: Transformation between demand and utility function, expenditure function

Then let $D(p,w) = \{x^*\}$, we have $px^* = w$ with $\succeq$ being locally insatiable from Duality I. Since $v(p,w)$ is the maximum of $u(x)$ under a constrain $px \le w$, so the corresponding Language function is $L = u(x) + \lambda(w - px)$, we say f.o.c. ($v = L$ when $x = x^*$ (refer to Kuhn-Tuck condition), see Envelope Theorem) implies

$$\frac{\partial v}{\partial p}|_{x=x^*} = \frac{\partial L}{\partial p}|_{x=x^*} = -\lambda x^* = 0 \tag{11.3}$$

$$\frac{\partial v}{\partial w}|_{x=x^*} = \frac{\partial L}{\partial w}|_{x=x^*} = -\lambda = 0 \tag{11.4}$$

$$\Rightarrow x^* = -\frac{\partial v / \partial p}{\partial v / \partial w} \tag{11.5}$$

□

LEMMA 11.8. *(Shephard's lemma) For* $\forall x \in X = \mathbb{R}^K_+$, *if* $\succeq$ *is continuous, strictly convex, and* $x \succeq \mathbf{0}$ *for* $\forall x$ *and* $p \in \mathbb{R}^K_{++}$, *then* $e(\cdot, x)$ *is differentiable and* $\frac{\partial e(p,x)}{\partial p} = H(p,x)$.

Proof. First, since $\succeq$ is continuous and strictly convex, $p \in \mathbb{R}^K_{++}$, we have $H(p,x)$ is a singleton. Then we need to show, $\forall p_0 \in \mathbb{R}^K_{++}, \forall \{p_n\} \subseteq \mathbb{R}^K_{++}, p_n \to p_0$ and $p_n \neq p_0, \forall n$, we have

$$\lim_{n\to\infty} \frac{|e(p_n,x) - e(p_0,x)|}{||p_n - p_0||} = H(p_0,x) = x^* \tag{11.6}$$

$$\Leftrightarrow \lim_{n\to\infty} \frac{|e(p_n,x) - e(p_0,x) - (p_n - p_0)x^*|}{||p_n - p_0||} = 0 \tag{11.7}$$

Let $\{x_n\} = H(p_n, x)$, then $e(p_n,x) = p_n x_n$ from definition. As $e(p_0,x) = p_0 x^*$, we have $e(p_n,x) - e(p_0,x) - (p_n - p_0)x^* = p_n x_x - p_n x^* \le 0$ ($e(p_n,x) \le p_n x^*, x^* \succeq x$). Also, $e(p_0,x) \le p_0 x_n$, so $0 \ge e(p_n,x) - e(p_0,x) - (p_n - p_0)x^* \ge e(p_n,x) - p_0 x_n - (p_n - p_0)x^* = p_n x_n - p_0 x_n - (p_n - p_0)x^* = (p_n - p_0)(x_n - x^*)$, so

$$\lim_{n\to\infty} \frac{|e(p_n,x) - e(p_0,x) - (p_n - p_0)x^*|}{||p_n - p_0||} \le \lim_{n\to\infty} \frac{|(p_n - p_0)(x_n - x^*)|}{||p_n - p_0||} \le \lim_{n\to\infty} ||x_n - x^*|| \tag{11.8}$$

(Cauchy Inequality: $|xy| \le ||x|| \cdot ||y||$) The remaining task is to prove $\lim\limits_{n\to\infty} ||x_n - x^*|| = 0$, i.e., $H(p,x)$ is continuous with respect to p. According to Berge's Maximization Theorem, we know $H(p,x)$ is u.s.c., with it is a singleton correspondence, we know $H(p,x)$ is also l.s.c., so it is continuous. □

THEOREM 11.1. *(Law of Demand) Let* $p, p' \in \mathbb{R}^K_+, w \ge 0$, $\succeq$ *be locally insatiable. For* $\forall x \in D(p,w)$, $x' \in H(p',x)$, *we have* $(x' - x)(p' - p) \le 0$. *For* $\forall x'' \in D(p', p'x)$, *we have* $(x'' - x)(p' - p) \le 0$.

Proof. Since $\succeq$ is locally insatiable, so $x \in D(p,w) \subseteq H(p,x)$ (Duality I), then $px \le px'$. $x' \in H(p',w)$, so $p'x' \le p'x$. Then $(x' - x)(p' - p) = (p'x' - p'x) + (px - px') \le 0$.

Since $x'' \in D(p', p'x)$, then $p'x'' = p'x$ (Duality I). As $x \in H(p,x)$, so $px \le px''$. Then $(x'' - x)(p' - p) = (p'x'' - p'x) + (px - px'') = px - px'' \le 0$. □

11.2 Revealed Preference

In reality, we can observe wealth, prices and choices only, and from these we want to judge whether the choice theory is right.

LEMMA 11.9. *If $\succeq$ is complete, transitive and locally insatiable, and $x^* \in D(p,w)$, for some $p \in \mathbb{R}_+^K, w \in \mathbb{R}_+$, we can say*

1. *$x^* \succeq x, \forall x \in B(p,w)$;*
2. *$\forall x^* \succ x, \forall px \leq w$ (Duality I, , $x^* \in H(p,x^*)px* = w$, if $x \succeq x^*, px < w$, we have $x^* \notin H(p,x^*)$ as x better)*

DEFINITION 11.6. *(Revealed Reference, GARP) Let data: $\{(p_i, w_i, x_i), i = 1, 2, \cdots, J\}$, we have*

1. *$x_i \succeq_d x_j$ if $p_i x_j \leq w_i = p_i x_i$;*
2. *$x_i \succ_d x_j$ if $p_i x_j < w_i$;*
3. *$x_i \succeq_r x_j$ if $x_i \succeq_d x_2 \succeq_d \cdots \succeq_d x_j$ (indirect);*
4. *$x_i \succ_r x_j$ if $x_i \succeq_d x_2 \succeq_d \cdots \succ_d x_k \succeq \cdots \succeq_d x_j$ (indirect);*

Assumption 11.1. *(GARP) The data satisfies GARP if $\neg(\exists x_i)$ s.t. $x_i \succ_r x_i$.*

THEOREM 11.2. *Afriat's Therefore If a finite set of data violates GARP, it is not a choice according to $\succeq$ which is complete, transitive and locally insatiable.*

If the data set satisfies GARP, then $\exists \succeq$ which is complete, transitive, strictly monotone, continuous, convex that generate the data (and locally insatiable surely).

Proof. (1) Immediately from Lemma 11.6.

(2) Let $\mathscr{J} = \{1, 2, \cdots, J\}$. We need to construct such a utility function which satisfies if $x_i \succeq_r x_j$, then $u(x_i) \geq u(x_j)$ but we cannot say if $u(x_i) \geq u(x_j)$, we can find the inequality chian, as in data, the revealed preference is not complete necessarily.

First, we claim $p_i x_i = w_i, i = 1, 2, \cdots, J$. Suppose not, we have $\exists i, p_i x_i < w_i$, then $x_i \succ_d x_i$, violating GARP.

Second, let $n(i) = ||\{x_j | x_i \succ_r x_j\}, \forall j||$. Then we have

(1) $n(i) < n(j)$ iff $p_i x_i < p_i x_j$;

(2) $n(i) = n(j)$ iff $p_i x_i \leq p_i x_j$;

(3) $\exists i, n(i) = 0$.

If (1) not hold, then $n(i) < n(j)$ but $w_i = p_i x_i \geq p_i x_j q$. Then $\forall x_k$ s.t. $x_j \succ_r x_k$, we have $x_i \succ_r$, then $n(i) \geq n(j)$, violating $n(i) < n(j)$. We can obtain (2) analogously. If (3) not hold, then $\forall i, \exists j, x_i \succeq x_j$. Then let $i = 1$ we get j_1 and $x_1 \succ_r x_{j_1}$, let $i = j_1$, we get j_2 and $x_{j_1} \succ_r x_{j_2}$. In this way, we can obtain a infinite sequence $\{1, j_1, j_2, \cdots\}$ and they are different, violating finite sample size.

Next, we claim $\exists v_i, \alpha_i \in \mathbb{R}, i = 1, 2, \cdots, J$ s.t. $\alpha_i > 0$ and $v_i \leq v_j + \alpha_j(p_j x_i - p_j x_j)$ for all $i, j = 1, 2, \cdots, J$.

First see a specific example: suppose $n(i) = 0, i = 1, 2, \cdots, J$, then $n(j) \leq n(i), p_j x_j \leq p_j x_i, \forall i, j = 1, 2, \cdots, J$. Then let $v_i = v^*, \alpha_i = \alpha^*, \forall i$ which satisfies the condition.

Now we reach the conclusion by mathematical induction. Let $n = 1$, then $n(1) = 0$, then any v_1, α_1 is acceptable. Let $n = 2$, then WLOG, let $n(1) = 0$, if $n(2) = 0$, we are done; if $n(2) = 1$, we can let $\alpha_1 = 1, v_1 = 0, v_2 = p_1 x_2 - p_1 x_2, \alpha_2 = -v_2/(p_2 x_1 - p_2 x_2)$, which satisfies conditions.

Now suppose when $n = J - 1$, we can find $\{v_i, \alpha_i\}_{i=1}^{J-1}$. When $n = J$, we say either $n(i) = 0$ or $n(i) \geq 0$ for $\forall i \in \mathscr{J}$. We put all observation into a partition $\mathscr{A} = \{i \in 1, 2, \cdots, J | n(i) = 0\}$ and $\mathscr{B} = \{i \in 1, 2, \cdots, J | n(i) > 0\}$. According to (3), we know $\mathcal{A}$ in non-empty. Now we first let $\alpha_i = \alpha^* > 0, v_i = v^*, \forall i \in \mathscr{A}$. And for $\{x_j, j \in \mathscr{B}\}$, we know it contains $J - 1$ elements at most. So we ignore $\mathscr{A}$ and directly decide the $\alpha_j, v_j, \forall j \in \mathscr{B}$ with the assumption hold for $n = J - 1$. It should be noticed that we only get $v_i \leq v_j + \alpha_j(p_j x_i - p_j x_j), \forall i, j \in \mathscr{B}$ or $i, j \in \mathscr{A}$, not $i \in \mathscr{A}, j \in \mathscr{B}$ or conversely!

Then we adjust α^*, v^* to extend that. First, let $i \in \mathscr{A}, j \in \mathscr{B}$, and we want $v_i \leq v_j + \alpha_j(p_j x_i - p_j x_j)$. Although we do not know whether $p_j x_i - p_j x_j \geq 0$, we know the value, and we can just let

$$v^* = v_i \leq \min_{i \in \mathscr{A}, j \in \mathscr{B}} v_j + \alpha_j(p_j x_i - p_j x_j) \tag{11.9}$$

solving the first part. Next, let $j \in \mathscr{A}, i \in \mathscr{B}$, we want $v^* = v_i \leq v_j + \alpha_j(p_j x_i - p_j x_j)$. As $n(j) = 0$, so we know $p_j x_i - p_j x_j > 0$, then we just need α_j big enough, i.e.,

$$a^* = a_j \geq \max_{i \in \mathscr{B}, j \in \mathscr{A}} \frac{v^* - v_i}{p_j x_i - p_j x_j} \tag{11.10}$$

(what if $p_j x_i - p_j x_j = 0$?). In this way, we bridges the gap and have

$$v_i \leq v_j + \alpha_j(p_j x_i - p_j x_j), \forall i, j \in \mathscr{J} \tag{11.11}$$

Now we give the utility function,

$$u(x_j) = \min_{i \in \mathscr{J}} v_i + \alpha_i(p_i x_j - p_i x_i), \forall j \in \mathscr{J} \tag{11.12}$$

and we know $u(x_j) \geq v_j, \forall j \in \mathscr{J}$. We claim $u(\cdot)$ is strictly monotone, continuous and concave.

Generally, we can show that if f_i is strictly monotone, continuous and concave for $\forall i \in \mathscr{J}$, then $f(x) = \min_{i \in \mathscr{J}} f_i(x)$ is strictly monotone, continuous and concave.

First show f is strictly monotone, let $\forall x_1 < x_2$, then $f_i(x_1) < f_i(x_2), \forall i \in \mathscr{J}$. Then $f(x_2) = \min_{i \in \mathscr{J}} f_i(x_2) > f_i(x_1), \exists i \in \mathscr{J}$ and then $f_i(x_1) \geq \min_{i \in \mathscr{J}} f_i(x_1) = f(x_1)$. So $f(x_2) > f(x_1)$.

Second we show f is continuous, let $\forall x_q \to x$, then $\lim_{q \to \infty} f_i(x_q) = f_i(x)$. WLOG, let $f_1(x) \leq f_i(x), \forall i \in \mathscr{J}$, then $f(x) = f_1(x)$. So $\forall \varepsilon > 0, \exists N, \forall q > N$, we have $f_i(x_q) > f_i(x) - \varepsilon \geq f_1(x) - \varepsilon$ and $f_1(x_q) < f_1(x) + \varepsilon$, then $f(x_q) = \min_{i \in \mathscr{J}} f_i(x_q) > f_1(x) - \epsilon$. Moreover $f(x_q) = \min_{i \in \mathscr{J}} f_i(x_q) \leq f_1(x_q) < f_1(x) + \epsilon$. then $|f(x_q) - f_1(x)| < \epsilon$, so $\lim_{q \to \infty} f(x_q) = f_1(x) = f(x)$.

Finally we show f is concave, let $\forall t \in [0,1], x, y$, we have $f_i(tx + (1-t)y) \geq t f_i(x) + (1-t) f_i(y)$. We have $f(tx + (1-t)y) = \min_{i \in \mathscr{J}} f_i(tx + (1-t)y) \geq \min_{i \in \mathscr{J}} t f_i(x) + (1-t) f_i(y) \geq \min_{i \in \mathscr{J}} t f_i(x) + \min_{i \in \mathscr{J}} (1-t) f_i(y) = t f(x) + (1-t) f(y)$.

Now since $v_i + \alpha_i(p_i x_j - p_i x_i)$ is strictly monotone, continuous and concave $\forall j \in \mathscr{J}$, we know u is strictly monotone, continuous and concave.

The remaining task is to show that u can generate the data, i.e., $x_i \in D(p_i, w_i)$, i.e., $\forall x_j$, if $p_i x_i - p_i x_j \geq 0$, we have $u(x_i) \geq u(x_j)$.

As $p_i x_i - p_i x_j \geq 0$, we have $u(x_j) = \min_{i \in \mathscr{J}} v_i + \alpha_i(p_i x_j - p_i x_i) \leq v_i + \alpha_i(p_i x_j - p_i x_i) \leq v_i \leq u(x_i)$ (please refer to the definition of $u(x_i)$). Hence we reach the conclusion. □

12 Choice under Uncertainty

12.1 Six Assumptions

In previous chapters, we establish a utility theory to analyze choices under certainty. Now we move to people's behavior under uncertainty. This is rather difficulty if we directly set the probability space as a choice set, which is a $\mathbb{R}^\infty$ space generally. To make it easy to handle, we need the expected utility theory.

DEFINITION 12.1. *Simple Lottery Let $Z = \{z_1, z_2, \cdots, z_N\}$ denote a finite possible consequences. A simple lottery over Z is a vector $L = (p_1, p_2, \cdots, p_N)$ such that $p_i \geq 0, \forall i$ and $\sum_{i=1}^N p_i = 1$, where p_i represents the probability that the outcome z_i occurs in the lottery Z. The set of all simple lottery is denoted as $\Delta(L)$ or Δ simply. We use $p_1 z_1 + p_2 z_2 + \cdots + p_N z_N$ to represent the lottery $(p_1, p_2, \cdots, p_N)$ over Z.*

For a preference defined on $\Delta(L)$, we have three assumptions beyond transitive and complete.

Assumption 12.1. *(Continuity) If $x \succ y \succ z$, then $\exists \lambda, \mu \in (0,1)$ s.t. $\lambda x + (1-\lambda) z \succ y \succ \mu x + (1-\mu) z$.*

Assumption 12.2. *(Independence) If $x \succ y$, then $\forall \lambda \in (0,1], \forall z$, we have $\lambda x + (1-\lambda) z \succ \lambda y + (1-\lambda) z$.*

Assumption 12.3. *(Consequentialism) If $L_j = \sum_{n=1}^N p_n^j z_n, j = 1, 2, \cdots, J$, and $\tilde{L} = \sum_{j=1}^J q_j L_j$. Let $r_n = \sum_{j=1}^J q_j p_n^j, n = 1, 2, \cdots, N$, so we have $\tilde{L} \sim L = \sum_{n=1}^N r_n z_n$.*

Assumption 12.4. *(Monotonicity) Let $x \succ z$, and $\forall \lambda, r \in [0,1]$. If $\lambda > r$, then $\lambda x + (1-\lambda) z \succ r x + (1-r) z$.*

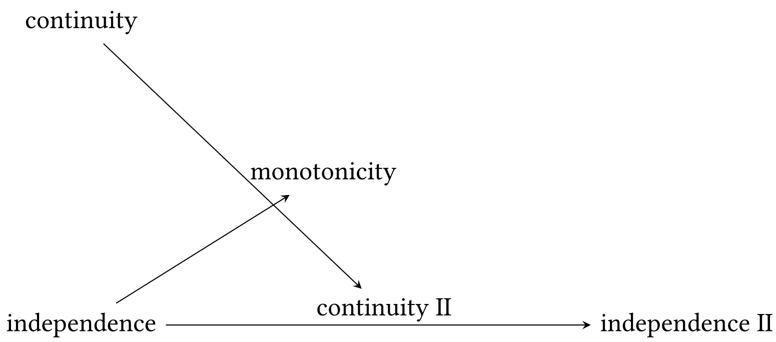


Figure 28: Proof of equivalent assumptions for choice under uncertainty

Assumption 12.5. *(Continuity II)* $\forall x \succ y \succ z$, *we have* $\exists! \beta \in (0,1)$ *s.t.* $\beta x + (1-\beta) z \sim y$.

Assumption 12.6. *(Independence II) If* $x \sim y$, *then* $\forall \lambda \in [0,1], \forall z$, *we have* $\lambda x + (1-\lambda) z \sim \lambda y + (1-\lambda) z$.

We now show that under consequentialism, the assumption of continuity, independence is equivalent to the assumption of monotonicity, continuity II and independence II, see Figure 28.

LEMMA 12.1. *Independence* $\Rightarrow$ *monotonicity.*

Proof. Let $x \succ z$, we want to show that $\forall \lambda, r \in [0,1], \lambda > r$, then $\lambda x + (1-\lambda) z \succ r x + (1-r) z$.

We first show monotonicity hold for $\lambda = 1$ or $r = 0$. Let $\lambda = 1$, we NTS $x \succ rx + (1-r)z$, i.e., $rx + (1-r)x \succ rx + (1-r)z$, which is obvious if $x \succ z$ and $\succ$ is independent. Let $r = 0$, we NTS $\lambda x + (1-\lambda) z \succ z$, i.e., $\lambda x + (1-\lambda) z \succ \lambda z + (1-\lambda) z$, which is obvious if $x \succ z$ and $\succ$ is independent. The idea is, if $x \succ z$, with independence, $\forall \alpha \in (0,1)$, we have $x \succ \alpha x + (1-\alpha) z \succ z$.

Now we show $\lambda x + (1-\lambda) z \succ rx + (1-r) z$ if $\lambda > r, \lambda, r \in (0,1)$. If we can find $t \in (0,1)$ s.t. $\lambda x + (1-\lambda) z = tx + (1-t)[rx + (1-r)z]$, then the conclusion is self evident. Solve t, $\lambda = t + (1-t) r$, $t = \frac{\lambda - r}{1-r}$. So we can say $\lambda x + (1-\lambda) z \succ rx + (1-r) z$ if $\lambda > r, \lambda, r \in [0,1]$. □

LEMMA 12.2. *Continuity, monotonicity* $\Rightarrow$ *continuity II.*

Proof. Let $x \succ y \succ z$, then We NTS $\exists! \beta \in (0,1)$ s.t. $\beta x + (1-\beta) z \sim y$. With continuity, we know $\exists \lambda, \mu \in (0,1)$ s.t. $\lambda x + (1-\lambda) z \succ y \succ \mu x + (1-\mu) z$. With monotonicity, we know $\lambda > \mu$. Further, $\forall \lambda' \in (\lambda, 1)$, we have $\lambda' x + (1-\lambda') z \succ y$, and $\forall \mu' \in (0, \mu)$, we have $y \succ \mu' x + (1-\mu')$. Let $\lambda_1 = \lambda, \mu_1 = \mu$, then according to continuity, we have $\exists \lambda_2, \mu_2 \in (\lambda_1, \mu_1)$ s.t. $\lambda_2 x + (1-\lambda_2) z \succ y \succ \mu_2 x + (1-\mu_2) z$. Repeat the step, we get an decreasing and bounded (μ_1) sequence (λ_q) and an increasing and bounded (λ_1) sequence (μ_q). So with Weierstrass Theorem we know $\exists \lambda^*, \mu^*$ s.t. $\lambda_q \to \lambda^*, \mu_q \to \mu^*$. We claim $\lambda^* x + (1-\lambda^*) z \sim y$. Suppose not, if $\lambda^* x + (1-\lambda^*) z \succ y$, then $\exists \lambda' \in (\mu^*, \lambda^*)$ s.t. $\lambda^* x + (1-\lambda^*) z \succ \lambda' x + (1-\lambda') z \succ y$, contradicting to $\lambda_q \to \lambda^*$. If $y \succ \lambda^* x + (1-\lambda^*) z$, violating $\lambda_q x + (1-\lambda_q) z \succ y, \forall q$ and $\succ$ is continuous. Therefore, $\lambda^* x + (1-\lambda^*) z \sim y$. In the same way, we know $\mu^* x + (1-\mu^*) z \sim y$. Now we claim $\lambda^* = \mu^*$, if not, $\lambda^* > \mu^*$, then $y \sim \lambda^* x + (1-\lambda^*) z \succ \mu^* x + (1-\mu^*) z \sim y$ with monotonicity, contradiction. □

LEMMA 12.3. *Independence, continuity II* $\Rightarrow$ *independence II.*

Proof. Let $x \sim y$, then $\forall \lambda \in [0,1], \forall z$, we have $\lambda x + (1-\lambda) z \sim \lambda y + (1-\lambda) z$.

If $z = x$ or $z = y$, we show the statement hold. WLOG, let $z = x$, so NTS $x \sim \lambda y + (1-\lambda) x, \forall \lambda \in [0,1]$. Suppose not, WLOG, let $x \succ \lambda y + (1-\lambda) x$, then $y \succ \lambda y + (1-\lambda x)$, so $(1-\lambda) x + \lambda y \succ \lambda y + (1-\lambda) x$, contradiction. Note that we use, if $x \succ z, y \succ z$, then with independence, we have $\lambda x + (1-\lambda) y \succ \lambda z + (1-\lambda) y$ and $(1-\lambda) y + \lambda z \succ (1-\lambda) y + \lambda z = z$. So $\lambda x + (1-\lambda) y \succ z$.

If $z \sim x \sim y$, then $\lambda x + (1-\lambda) z \sim z, \lambda y + (1-\lambda) z \sim z$, so $\lambda x + (1-\lambda) z \sim \lambda y + (1-\lambda) z$.

If $z \succ x \sim y$, then NTS $\lambda x + (1-\lambda) z \sim \lambda y + (1-\lambda) z$. As $z \succ x$, with independence, we have $z \succ \lambda x + (1-\lambda) z \succ x$. Suppose not, WLOG, let $\lambda x + (1-\lambda) z \succ \lambda y + (1-\lambda) z$, then $z \succ \lambda x + (1-\lambda) z \succ \lambda y + (1-\lambda) z$. With continuity II, we have $\exists! \beta \in (0,1)$ s.t. $\lambda x + (1-\lambda) z \sim \beta(\lambda y + (1-\lambda) z) + (1-\beta) z = \lambda(\beta y + (1-\beta) z) + (1-\lambda) z$. Since $z \succ y$, $\beta y + (1-\beta) z \succ y \sim x$, so $\lambda(\beta y + (1-\beta) z) + (1-\lambda) z \succ \lambda x + (1-\lambda) z$, contradiction.

If $x \sim y \succ z$, the proof is similar, we just need to replace $\succ$ with $\prec$. □

The remaining task is to show continuity and independence under monotonicity, continuity II and independence II.

12.2 vNM Expected Utility Theorem

THEOREM 12.1. *(vNM Expected Utility) If $\succeq$ on Δ is complete, transitive and satisfies continuity, independence and consequentialism, then we have $\exists f : \Delta \to R$ represents $\succeq$, where f is linear with respect to (w.r.t.) probability.*

Proof. Consider $Z = \{z_1, z_2, \cdots, z_N\}$, WLOG, $\exists z_w, z_B \in Z$ s.t. $z_B \succ z \succ z_W, \forall z \in Z, z \neq z_W, z \neq z_B$. Now according to continuity II, we have $\exists! f(z) \in [0, 1]$ s.t. $z \sim f(z)z_B + (1 - f(z))z_W$ where $f : Z \to [0, 1]$. We claim f is what we need.

First, we show f represents $\succeq$. $\forall p, q \in \Delta(L)$, we NTS $p \succeq q$ iff $f(p) \geq f(q)$. With monotonicity, $f(p) \geq f(q)$ iff $f(p)z_B + (1 - f(p))z_W \succeq f(q)z_B + (1 - f(q))z_W$ iff $p \succeq q$ as $f(p)z_B + (1 - f(p))z_W \sim p, f(q)z_B + (1 - f(q))z_W \sim q$.

Next, we show f is linear w.r.t. probability, i.e., $\forall \lambda \in (0, 1)$, we NTS $f(\lambda p+(1-\lambda)q) = \lambda f(p)+(1-\lambda)f(q)$. $p \sim f(p)z_B+(1-f(p))z_W$, and $q \sim f(q)z_B+(1-f(q))z_W$, then according to independence II, $\lambda p+(1-\lambda)q \sim \lambda(f(p)z_B + (1 - f(p))z_W) + (1 - \lambda)f(q)z_B + (1 - f(q))z_W = [\lambda f(p) + (1 - \lambda)f(q)]z_B + [1 - \lambda f(p) - (1 - \lambda)f(q)]z_W$. With continuity II, for $\lambda p + (1 - \lambda)q$, the correspondence $f(\lambda p + (1 - \lambda)q)$ is unique, so $\lambda f(p) + (1 - \lambda)f(q) = f(\lambda p + (1 - \lambda)q)$. □

LEMMA 12.4. *(Positive Affine Transformation) Suppose u is a expected utility function representing $\succeq$. Then v is another expected utility function representing $\succeq$ iff v is a positive affine transformation of u. (positive affine transformation implies $\exists A, B, A > 0$ s.t. $v = Au + B$)*

Proof. First, $f(x) = Ax + B, A > 0$ is a strictly monotone function, so v represents $\succeq$.

Next, we want to show such A, B exist. Suppose such A, B do not exist, then $\exists z_1 \succ z_2 \succ z_3$ s.t. if we define a positive affine transformation $\tilde{u}$ of u by

$$\tilde{u}(z_i) = \frac{v(z_3)u(z_1) - v(z_1)u(z_3)}{u(z_1) - u(z_3)} + \frac{v(z_1) - v(z_3)}{u(z_1) - u(z_3)}u(z_i) \tag{12.1}$$

we have $\tilde{u}(z_1) = v(z_1), \tilde{u}(z_3) = v(z_3)$ but $\tilde{u}(z_2) \neq v(z_2)$. Thus,

$$v(z_2) = \frac{v(z_2) - v(z_3)}{v(z_1) - v(z_3)}v(z_1) + \frac{v(z_1) - v(z_2)}{v(z_1) - v(z_3)}v(z_3) \tag{12.2}$$

but

$$\frac{v(z_2) - v(z_3)}{v(z_1) - v(z_3)}\tilde{u}(z_1) + \frac{v(z_1) - v(z_2)}{v(z_1) - v(z_3)}\tilde{u}(z_3) = v(z_2) \neq \tilde{u}(z_2) \tag{12.3}$$

The ranking between $\frac{v(z_2)-v(z_3)}{v(z_1)-v(z_3)}z_1 + \frac{v(z_1)-v(z_2)}{v(z_1)-v(z_3)}z_3$ and z_2 depends on whether $\succeq$ is represented by v or $\tilde{u}$, which generates a contradiction. □

13 Firm's Problem

We assume firms are price taker. Firm's behavior is similar to an individual.

13.1 Single Output Firm

DEFINITION 13.1. *(Production Function) Let $q = f(x)$ where x is the quantity vector of inputs and q is the quantity of output. We call f a production function. Note that q is a scalar, not a vector here.*

Assumption 13.1. *Let $f : \mathbb{R}_+^{k-1} \to \mathbb{R}_+$ s.t.*

1. *$f(0) = 0$;*
2. *$\sup\limits_{x\in\mathbb{R}_+^{k-1}} f(x) = +\infty$;*
3. *f is non-decreasing;*
4. *f is continuous and quasi-concave.*

DEFINITION 13.2. *(Isoquant) For $\forall q \in \mathbb{R}_+$, we define $\{x \in \mathbb{R}_+^{k-1} = q\}$ be the isoquant of the output q. Further, we define $V(q) \equiv \{x \in \mathbb{R}_+^{k-1} | f(x) \geq q\}$.*

DEFINITION 13.3. *(Cost Function) For* $\forall w \in \mathbb{R}_+^{k-1}$*(a price vector), let* $c(w,q) \equiv \inf_{x \in V(q)} \{wx\}$.

LEMMA 13.1. *(Revealed Cost) Define* $V^* \equiv \bigcap_{w \in \mathbb{R}_+^{k-1}} \{x \in \mathbb{R}_+^{k-1} | wx \geq c(w,q)\}$. *Let* $c^*(w,q) \equiv \inf_{x \in V^*(q)} wx$. *Then* $V^*(q) = V(q)$ *if assumption 13.1 holds. Further, if* $w \in \mathbb{R}_{++}^{k-1}$*, then* $c(w,q) = \min_{x \in V(q)} wx$. *(For a set* X*, let* $co(X)$ *denote the convex hull of* X *and* $cl(X)$ *denote the closure of* X*)(See Figure 29)*

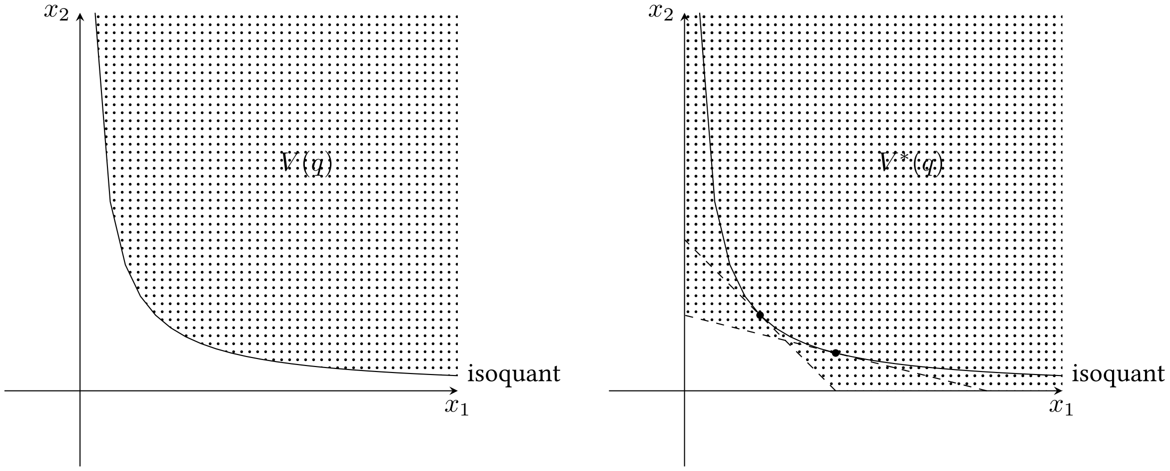


Figure 29: Equivalent of $V(q)$ and $V^*(q)$

Proof. We prove the statement by three steps.

step(1) $V(q) = clco(V(q)+\mathbb{R}_+^{k-1})$. $\forall x \in V(q), y \in \mathbb{R}_+^{k-1}$, as f is non decreasing, we have $f(x+y) \geq f(x) \geq q$, so $x+y \in C(q)$, i.e., $V(q)+\mathbb{R}_+^{k-1} \subseteq V(q)$. Next, for $\forall x \in V(q), \mathbf{0} \in \mathbb{R}_+^{k-1}$, so $x = x+\mathbf{0} \in V(q)+\mathbb{R}_+^{k-1}$, i.e. $V(q) \subseteq V(q) + \mathbb{R}_+^{k-1}$. Now we have $V(q) = V(q) + \mathbb{R}_+^{k-1}$ from f is non decreasing.

Now we show $clco(V(q)) = V(q)$. Since f is quasi concave, we have $V(q)$ is convex (like $NWT(x)$). Since f is continuous, we have $V(q)$ is closed (also like $NWT(x)$). So $co(V(q)) = V(q), cl(V(q)) = cl(V(q))$, i.e., $clco(V(q)) = cl(V(q)) = V(q)$.

step(2) $V^*(q) = clco(V^*(q) + \mathbb{R}_+^{k-1})$. For $\forall x \in V^*(q), \forall y \in \mathbb{R}_+^{k-1}$, we say $w(x+y) \geq w(x) \geq c(w,q), \forall w \in \mathbb{R}_+^{k-1}$, so $x+y \in V^*(q)$, i.e., $V^*(q) + \mathbb{R}_+^{k-1} \subseteq V^*(q)$. Next, $\forall x \in V^*(q), \mathbf{0} \in \mathbb{R}_+^{k-1}$, we have $x = x + \mathbf{0} \in V^*(q) + \mathbb{R}_+^{k-1}$, so $V^*(q) \subseteq V^*(q) + \mathbb{R}_+^{k-1}$. Thus, $V^*(q) = V^*(q) + \mathbb{R}_+^{k-1}$.

Now we need to show $V^*(q)$ is closed and convex. Since any intersection of closed (or convex) set is also closed (or convex), we just need to show that given w, the set $V_w(q) = \{x \in \mathbb{R}_+^{k-1} | wx \geq c(w,q)\}$ is closed and convex. Note that $c(w,q)$ is fixed, so $V_w(q)$ is a half space of $\mathbb{R}^{k-1}$ containing the boundary, so it is closed and convex.

step(3)
- $V(q) \subseteq V^*(q)$. $\forall x \in V(q)$, then $f(x) \geq q$. $\forall w \in \mathbb{R}_+^{k-1}$, we have $wx \geq c(w,q)$ from the definition. Then $x \in V^*(q)$. Thus, $V(q) \subseteq V^*(q)$.
- $\forall x^0 \notin V(q) \Rightarrow x^0 \notin V^*(q)$. If $x^0 \notin V(q) = clco(V(q) + \mathbb{R}_+^{k-1})$, $\exists \varepsilon > 0$ s.t. $\forall N(x^0, \epsilon)$, we have $N(x^0, \varepsilon) \cap V(q) = \varnothing$. So we can construct a convex set. Let $\delta \in [0, \varepsilon/\sqrt{k-1})$, so $x^\delta = x + \delta I$ where $I = [1, 1, \cdots, 1]'$, so $x^\delta \in N(x^0, \varepsilon)$, then $x^\delta \notin V(q)$. Now we obtain a convex set $X^\delta = \{x^\delta = x + \delta I | \delta \in [0, \varepsilon/\sqrt{k-1})\}$ s.t. $X^\delta \cap V(q) = \varnothing$. From the Separation of Convex Set Theorem, we know $\exists w^0 \in \mathbb{R}_+^{k-1} \backslash \{0\}$ s.t. $\sup_{x^\delta \in X^\delta} w^0 x^\delta \leq \inf_{x \in V(q)} w^0 x$. Moreover, $w^0 \in \mathbb{R}_{++}^{k-1}$ and if not, let $w_i^0 < 0$, and $x_i = +\infty$, we obtain an $-\infty$ cost which is not possible. Then $w^0 x^0 < \sup_{x^\delta \in X^\delta} w^0 x^\delta \leq \inf_{x \in V(q)} w^0 x$. So $\exists w^0 \in \mathbb{R}_+^{k-1}$ s.t. $w^0 x^0 < \inf_{x \in V(q)} w^0 x = c(w^0, q)$, contradicting to the definition of $c(w^0, q)$. Thus, $x^0 \notin V^*(q)$.

□

LEMMA 13.2. *If assumptions 13.1 hold and* $w \in \mathbb{R}_{++}^{k-1}$*, then* $c(\cdot, q)$ *is non decreasing, homogenous of degree* 1*, concave and continuous. (analogously from the expenditure function* $e(\cdot, q)$*)*

DEFINITION 13.4. *(Conditional Input Demand Function) Define $x(w,q) \equiv \{x \in V(q) | wx' \geq wx, \forall x' \in V(q)\}$ as a conditional input demand function.*

LEMMA 13.3. *If assumptions 13.1 hold and $x(w,q)$ is well defined (may not exist but we assume it exists not) and $w \in \mathbb{R}_+^{k-1}$, then*

1. *$x(\cdot,q)$ is continuous;*
2. *$c(w,q) = wx(w,q)$;*
3. *(Shephard's lemma) $\frac{\partial c(w,q)}{\partial w} = x(w,q)$.*

LEMMA 13.4. *If a function $c(\cdot,q) : \mathbb{R}_{++}^{k-1} \to \mathbb{R}_+$ for some $q \in \mathbb{R}_+$ and $c(\cdot,q)$ satisfies non decreasing, homogenous of degree 1, concave and differentiable (stronger than continuous). Then $V^*(q) \equiv \bigcap_{w \in \mathbb{R}_{++}^{k-1}} \{x \in \mathbb{R}_+^{k-1} | wx \geq c(w,q)\}$, i.e., $c(w,q) = \inf_{x \in V^*(q)} wx$ ($c(\cdot,q)$ is a cost function).*

Proof. Let $x(w,q) = \frac{\partial c(w,q)}{\partial w}, \forall w \in \mathbb{R}_{++}^{k-1}$, then (1) $x(w,q) \in \mathbb{R}_+^{k-1}$ as $c(w,q)$ is non decreasing; (2) $c(w,q) = wx(w,q)$ because $c(w,q)$ is homogenous of degree 1 (Euler Theorem, if $f(tx) = tf(x), \forall t > 0$, then $\frac{\partial tf(x)}{\partial t} = f(x) = \frac{\partial f(tx)}{\partial t} = \sum_i \frac{\partial f(tx)}{\partial tx_i} \overset{t=1}{\to} \sum_i \frac{\partial f(x)}{\partial x_i}$).

Next, as $c(\cdot,q)$ is concave, $\forall w' \in \mathbb{R}_{++}^{k-1}$, we have $(w'-w)x(w,q) \geq c(w',q) - c(w,q) = c(w',q) - wx(w,q)$, so $w'x(w,q) \geq c(w',q)$, i.e., $x(w,q) \in V^*(q)$.

With $c(w,q) = wx(w,q)$ and $x(w,q) \in V^*(q) = V(q)$, we have $c(w,q) = wx(w,q) = \inf_{x \in V(q)} wx$, i.e., $c(\cdot,q)$ is a cost function. □

DEFINITION 13.5. *(Profit Maximization) Given a price vector, $p = (p_q, w) \in \mathbb{R}_+^k$, the profit function is $\pi(p) \equiv \max_{q \in \mathbb{R}_+, x \in \mathbb{R}_+^{k-1}} p_q q - wx$ and the corresponding maximizer $q(p)$ and $x(p)$ are called the output supply function and the input demand function. The profit maximization can be decomposed into two steps, first, given q, calculate $c(w,q)$, second, optimize the profit, i.e., $\max_{q \in \mathbb{R}_+} p_q q - c(w,q)$. (Note that $pi, , q, x$ may not exist but here we assume they exist)*

LEMMA 13.5. *If assumptions 13.1 hold and π, q, x are well defined, then we have (analogously with the indirect utility function $v(p,w)$)*

- *$\pi(p)$ is increasing in p_q, non increasing in w;*
- *$\pi(p)$ is homogenous of degree 1;*
- *$\pi(p)$ is convex. Let $p, p' \in \mathbb{R}_+^k, \forall \lambda \in (0,1)$, let $p^\lambda = \lambda p + (1-\lambda)p'$. Then $p_q q(p^\lambda) - wx(p^\lambda) \leq \pi(p), p'_q q(p^\lambda) - w'x(p^\lambda) \leq \pi(p')$, so $p_q^\lambda q(p^\lambda) - w^\lambda x(p^\lambda) \leq \lambda\pi(p) + (1-\lambda)\pi(p')$, then $\pi(p)$ is convex.*
- *(Hotelling Lemma) If $\pi(\cdot)$ is differentiable, then $\frac{\partial \pi}{\partial p_q} = q(p)$, $\frac{\partial \pi}{\partial w} = -x(p)$ frm the Envelope Theorem.*

13.2 Generalized Firm

Now we extend single output firms to multi-output firms. Let $-x$ denote a quantity vector of input and q denote a quantity vector of output, so $(q, -x) \in \mathbb{R}^k$ is a production plan. If $y \geq x$, we say $(q, -y)$ is also a production plan. If $q' > q$, we say $(q', -x)$ is not a production plan.

DEFINITION 13.6. *(Production Possibility Set) A firm can transfer vectors of goods into other vectors of goods, described by $y \in \mathbb{R}^k$. The set of all net put vectors of the firm is given by $Y \subseteq \mathbb{R}^k$.*

Now we can rewrite the profit maximization problem as $\pi(p) = \max_{q, x \in V(q)} p_q q - wx = \max_{(q,-x) \in Y} (p_q, w) \cdot (q, -x) = \max_{y \in Y} py$.

LEMMA 13.6. *(Law of Supply) Let Y be a production possibility set, y is the profit maximizer at p and y' is the profit maximizer at p', then $(p-p')(y-y') \geq 0$.*

Proof. Since $py \geq py'$ and $p'y' \geq p'y$, $(p-p')(y-y') = py - py' + p'y' - p'y \geq 0$. (Note that y, y' is the profit maximizer) □

DEFINITION 13.7. *(Technical Effiency) A production plan $y^0 \in Y$ is technically efficient if $\forall y \geq y^0, y \neq y^0$, we have $y \notin Y$.*

LEMMA 13.7. *If $y \in Y$ is a profit maximizer at some $p \in \mathbb{R}^k_{++}$, then y is technically efficient.*

Proof. Suppose not, $\exists y' \geq y, y' \neq y, y' \in Y$, then $py' > py$, so y is not the profit maximizer at p, contradiction. □

LEMMA 13.8. *Let Y be convex, $y^0 \in Y$ is technically efficient, then $\exists p \in \mathbb{R}^k_+\backslash\{0\}$ s.t. y^0 is the profit maximizer at p^0. See Figure 30*

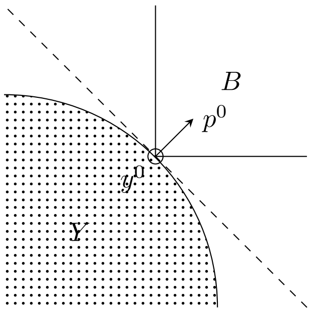


Figure 30: The corresponding price vector of a profit maximizer

Proof. $y^0 \in Y$, so $Y \neq \varnothing$, moreover, Y is convex. Let $B = \{y^0\} + \mathbb{R}^k_+\backslash\{0\} \neq$, so $B\varnothing$ and is convex. As $y^0 \notin B$ and y^0 is technically efficient, so $B \cap Y = \varnothing$. From the Separation of Convex Sets Theorem, we know $\exists p^0 \in \mathbb{R}^k\backslash\{0\}$ s.t. $\sup\limits_{y\in Y} p^0 y \leq \inf\limits_{y \in B} p^0 y$. First I claim $p^0 \in \mathbb{R}^k_+\backslash\{0\}$. Suppose not, let $p^0_i < 0$, then we can let $y_i = -\infty$ so get an infinite profit, which is not possible. So $p^0 \in \mathbb{R}^k_+\backslash\{0\}$. Next, $p^0 y^0 = \inf\limits_{y\in B} p^0 y = \sup\limits_{y \in Y} p^0 y$, so y^0 is the profit maximizer at p^0. □

The production possibility set Y may satisfies the four following assumptions.

Assumption 13.2. *(No Free Production, NFP) $Y \cap \mathbb{R}^k_+ \subseteq \{0\}$.*

Assumption 13.3. *(Free Disposal, FD) If $y \in Y, y' \leq y$, then $y' \in Y$.*

Assumption 13.4. *(Ability to Shut Down) $\mathbf{0} \subseteq Y$.*

Assumption 13.5. *(Return to Scale, RTS) Increasing Return to Scale (IRTS): $\forall y \in Y, \forall \alpha > 1$, then $\alpha y \in Y$;*
Decreasing Return to Scale (DRTS): $\forall y \in Y, \forall \alpha \in (0, 1)$, then $\alpha y \in Y$;
Constant Return to Scale (CRTS): $\forall y \in Y, \forall \alpha > 0$, then $\alpha y \in Y$;
See Figure 31

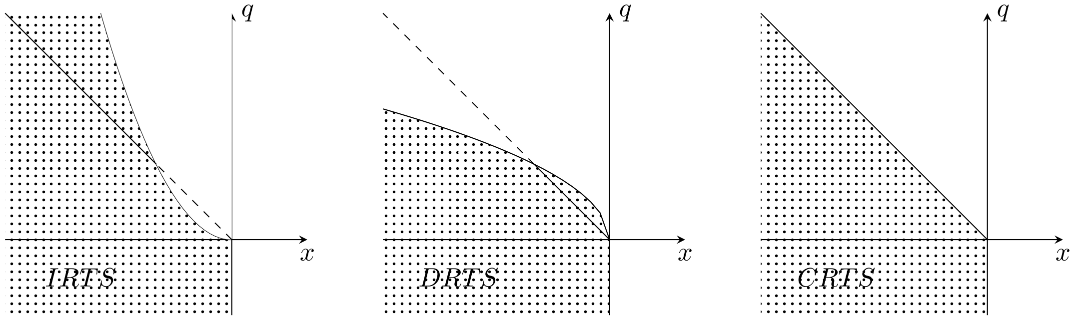


Figure 31: Return to scale

LEMMA 13.9. *Suppose Y is IRTS, then $\forall p \in \mathbb{R}^k_{++}$, if $\exists y \in Y$ s.t. $py > 0$. Then $\sup\limits_{y \in Y} py = +\infty$. That is Y is IRTS globally and the firm is a price taker can not hold simultaneously.*

Part IV
Advanced Econometrics

14 Preliminary Mathematics

We introduce essential mathematics for modern econometrics, including basic probability, matrix algebra and asymptotic theory. Let $X, Y, \cdots$ denote random variable and $x, y, \cdots$ denote realizations.

14.1 Conditional Expectation and Conditional Variance

Let $X, Y, \cdots$ denote random variables and $E, Var, |$ denote expectation, variance and conditioning.

Basic properties of conditional expectation.[7]

THEOREM 14.1. *(Law of Iteraed Expectation)*

$$E[E[X|Y]] = E[X] \tag{14.1}$$

which is a weak form of $E[E[X|I_t]I_s] = E[X|I_s]$ where I_t, I_s are two σ-algebra and $I_s \subset I_t$.

To guess X, we can first guess X under all states of Y and take expectations.

LEMMA 14.1. *(Known can be Taken out) If $h(\cdot)$ is a deterministic function, then*

$$E[Xh(Y)|Y] = h(Y)E[X|Y] \tag{14.2}$$

(what is known can be taken out; under Y, since we know $h(Y)$, we do not have to guess it) In particular, if $E[X|Y] = 0$, then

$$E[Xh(Y)] = 0 \tag{14.3}$$

as $E[Xh(Y)] = E[E[Xh(Y)|Y]] = E[h(Y)E[X|Y]] = E[h(Y)0] = 0$.

DEFINITION 14.1. *(Statistically Independent) If X, Y are statistically independent, then $E[XY] = E[X]E[Y]$, so $E[X|Y] = E[X]$.*

LEMMA 14.2. *If $E[X] = 0$, then $Cov(X, Y) = E[XY]$.*

Proof. $Cov(X, Y) = E[(X - EX)(Y - EY)] = E[XY - XEY] = E[XY] - E[XE[Y]] = E[XY] - E[Y]E[X] = E[XY]4$ ($E[Y]$ known, take out) □

LEMMA 14.3. *$\forall X, Y$, the covariance of $E(X|Y)$ and $X - E[X|Y]$ is zero.*

Proof. As $E[X - E[X|Y]|Y] = E[X|Y] - E[E[X|Y]|Y] = E[X|Y] - E[X|Y] = 0$, so $Cov(E[X|Y], X - E[X|Y]|Y) = E[E[X|Y](X - E[X|Y])|Y] = E[X|Y]E[X - E[X|Y]|Y] = 0$ ($E[X|Y]$ is a random variable depending on Y, and is known conditioning on Y, then take it out; take expectation to obtain unconditional outcome, still 0) □

LEMMA 14.4.

$$\begin{aligned} Var[X] &= E[Var[X|Y]] + Var[E[X|Y]] && (14.4)\\ &= Var[X - E[X|Y]] + Var[E[X|Y]] && (14.5)\\ &\geq Var[X - E[X|Y]] && (14.6) \end{aligned}$$

(take equality when X, Y are statistically independent)

Proof. Let $E[X] = 0$, then

$$\begin{aligned} Var[X] = E[X^2] &= E[(X - E[X|Y] + E[X|Y])^2] && (14.7)\\ &= E[(X - E[X|Y])^2] + 2E[(X - E[X|Y])E[X|Y]] + E[(E[X|Y])^2] && (14.8)\\ &= E[(X - E[X|Y])^2] + E[(E[X|Y])^2] && (14.9)\\ &\geq E[(X - E[X|Y])^2] && (14.10) \end{aligned}$$

(take equality when X, Y are statistically independent, and $E[(X - E[X|Y])E[X|Y]] = 0$ has shown). Further,

$$Var[X - E[X|Y]] = E[(X - E[X|Y])^2] = E[E[(X - E[X|Y])^2|Y]] = E[Var[X|Y]] \tag{14.11}$$

□

[7]The continuous time finance course gives clear proof for them.

14.2 Kronecker Product and Vce-operator

DEFINITION 14.2. *(Kronecker product) Let $A_{m\times n}, B_{p\times q}$ be two matrices, and*

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \cdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \tag{14.12}$$

*then the **Kronecker product** of A and B are*

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ & & \cdots & \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix} \tag{14.13}$$

The dimension of $A \otimes B$ is $mp \times nq$ and we can view $A \otimes B$ as the combination of $m \times n$ sub-matrices $a_{ij}B$, denoted as $Conb(a_{ij}B)$ (only for me).

With the definition of Kronecker product, we have the following properties (suppose all operation are possible)

LEMMA 14.5. *Let $A, B, C, D, A_1, A_2, B_1, B_2$ be matrices, then (distribution law, condensation law)*

1. $A \otimes (B_1 + B_2) = A \otimes B_1 + A \otimes B_2$
2. $(A_1 + A_2) \otimes B = A_1 \otimes B + A_2 \otimes B$
3. $(A \otimes B)(C \otimes D) = AC \otimes BD$

Proof. (1) Let $A = \{a_{ij}\}$, then $A \otimes (B_1 + B_2)$ is $Conb(a_{ij}(B_1 + B_2))$. As $\forall i, j, a_{ij}(B_1 + B_2) = a_{ij}B_1 + a_{ij}B_2$, so $A \otimes (B_1 + B_2) = Conb(a_{ij}(B_1 + B_2)) = Conb(a_{ij}B_1) + Conb(a_{ij}B_2) = A \otimes B_1 + A \otimes B_2$.

(2) Let $A_k = \{a_{ij}^k\}, k = 1, 2$, then $(A_1 + A_2) \otimes B$ is $Conb((a_{ij}^1 + a_{ij}^2)B)$. As $\forall i, j, (a_{ij}^1 + a_{ij}^2)B = a_{ij}^1 B + a_{ij}^2 B$, so $(A_1 + A_2) \otimes B = Conb((a_{ij}^1 + a_{ij}^2)B) = Conb(a_{ij}^1 B + a_{ij}^2 B) = Conb(a_{ij}^1 B) + Conb(a_{ij}^2 B) = A_1 \otimes B + A_2 \otimes B$.

(3) Let $A_{m\times n}, B_{m'\times n'}, C_{n\times p}, D_{n'\times p'}$, then $A \otimes B = Conb(a_{ij}B), C \otimes D = COnb(c_{jk}D), (A \otimes B)(C \otimes D) = Conb(\sum_{j=1}^{n} a_{ij}c_{ji}BD) = Conb((AC)_{ji}BD) = AC \otimes BD$. □

DEFINITION 14.3. *(vce-operator) For any matrix (or vector) A, **vce-operator** rearranges all elements into a column vector column by column, like*

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \cdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, vce(A) = \begin{bmatrix} a_{11} \\ a_{21} \\ \cdots \\ a_{m1} \\ a_{12} \\ a_{22} \\ \cdots \\ a_{m2} \\ \cdots \\ a_{1n} \\ a_{2n} \\ \cdots \\ a_{mn} \end{bmatrix} \tag{14.14}$$

i,e, we have $a_{ij} = (vec(A))_{(j-1)m+i}, i = 1, 2, \cdots, m, j = 1, 2, \cdots, n$.

Also, we have the following basic properties of vce-operator.

LEMMA 14.6. *Let A, B be two matrices and a be a scalar. Then,*

(1) $vce(A + B) = vce(A) + vce(B)$

(2) $vce(aA) = a \cdot vce(A)$

Proof. Let $A_{m\times n}, B_{m\times n}$.

(1) $\forall i,j, (vec(A)+vce(B))_{(j-1)m+i} = (A+B)_{ij} = a_{ij}+b_{ij} = (vec(A))_{(j-1)m+i} + (vec(B))_{(j-1)m+i}$, so $vce(A+B) = vce(A)+vce(B)$.

(2) $\forall i,j, (vce(aA))_{(j-1)m+i} = (aA)_{ij} = aA_{ij} = a(vce(A))_{(j-1)m+i}$, so $vce(aA) = a\cdot vce(A)$. □

For a row vector a, $vce(a) = a^\top$; for a column vector b, $vce(b) = b$. We always assume a vector is a column vector if there is no explanation.

Vce-operator and Kronecker product have close relation, we can prove some important equations of them.

LEMMA 14.7. *For any column vectors a, b, we have $vce(ab^\top) = b\otimes a$.*

Proof. Let $b = [b_1, b_2, \cdots, b_n]^\top$, then

$$\begin{aligned} vce(ab^\top) &= vce(a[b_1, b_2, \cdots, b_n]) && (14.15)\\ &= vce([ab_1, ab_2, \cdots, ab_n]) && (14.16)\\ &= \begin{bmatrix} ab_1 \\ ab_2 \\ \cdots \\ ab_n \end{bmatrix} && (14.17)\\ &= b\otimes a && (14.18) \end{aligned}$$

□

LEMMA 14.8. *For any matrices A, B, C, we have $vce(ABC) = (C^\top \otimes A)vce(B)$*

Proof. Let $B_{m\times n} = [b_1, b_2, \cdots, b_n], I_n = [e_1, e_2, \cdots, e_n]$ (an identity matrix), then

$$b_i e_i^\top = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ & & \cdots & \\ b_{i1} & b_{i2} & \cdots & b_{in} \\ & & \cdots & \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \forall i \quad (14.19)$$

$$\Rightarrow B = \sum_{i=1}^n b_i e_i^\top \quad (14.20)$$

So,

$$\begin{aligned} vce(ABC) &= vce(A\sum_{i=1}^n b_i e_i^\top C) = \sum_{i=1}^n vce(Ab_i e_i^\top C) && (14.21)\\ &= \sum_{i=1}^n vce(Ab_i(C^\top e_i)^\top) = \sum_{i=1}^n (C^\top e_i)\otimes(Ab_i) && (14.22)\\ &= \sum_{i=1}^n (C^\top \otimes A)(e_i\otimes b_i) = (C^\top\otimes A)\sum_{i=1}^n (e_i\otimes b_i) && (14.23)\\ &= (C^\top\otimes A)\sum_{i=1}^n vce(b_i e_i^\top) = (C^\top \otimes A)vce(B) && (14.24) \end{aligned}$$

Note that there are $A_2 \otimes B(A\otimes B)(C\otimes D) = AC\otimes BD$ used reversibly and $vce(ab^\top) = b\otimes a$ used once. In particular, when $A = I$, we have $vce(BC) = (C^\top \otimes I)vce(B)$. This lemma is of great importance, Remember it! □

DEFINITION 14.4. *(Trace) The summation of all diagonal elements of matrix A is named **trace**, written as $trace(A)$ or $tr(A)$. Note that A must be a square matrix. Let $A_n = (a_{ij})$, then $tr(A) = \sum_{i=1}^n a_{ii}$.*

LEMMA 14.9. *Let $A_{m\times n}, B_{n\times m}$ be two matrices and C_n be a square, then $tr(AB) = tr(BA), tr(C) = tr(C^\top)$. (if $tr(AB)$ exists, i.e., AB is a square, then BA exists and is a square, i.e., $tr(BA)$ exists)*

Proof. (1) $tr(AB) = \sum_{i=1}^{m}(AB)_{ii} = \sum_{i=1}^{m}(\sum_{j=1}^{n} a_{ij}b_{ji}) = \sum_{j=1}^{m}(\sum_{i=1}^{n} b_{ji}a_{ij}) = \sum_{j=1}^{m}(BA)_{jj} = tr(BA)$.
(2) $tr(C) = \sum_{i=1}^{n} C_{ii} \sum_{i=1}^{n} c_{ii} = \sum_{i=1}^{n} C_{ii}^{\top} = tr(C^{\top})$. □

LEMMA 14.10. *Let A_m, B_n be two squares, then $tr(A \otimes B) = tr(A)tr(B)$ (the trace of a Kronecker product equals to multiplication of traces of each matrix).*

Proof.

$$tr(A \otimes B) = tr(\begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ & & \cdots & \\ a_{m1}B & a_{m2}B & \cdots & a_{mm}B \end{bmatrix}) \tag{14.25}$$

$$= tr(a_{11}B) + tr(a_{22}B) + \cdots + tr(a_{mm}B) \tag{14.26}$$

$$= (a_{11} + a_{22} + a_{mm})tr(B) \tag{14.27}$$

$$= (trA)(trB) \tag{14.28}$$

□

LEMMA 14.11. *Let $A_{m\times n}, B_{m\times n}$ be two matrices, then $(vceA)^{\top}vce(B) = tr(A^{\top}B)$.*

Proof. $(vceA)^{\top}vce(B) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}b_{ij} = \sum_{j=1}^{n}(\sum_{i=1}^{m} a_{ij}b_{ij}) = \sum_{j=1}^{n}(A^{\top}B)_{jj} = tr(A^{\top}B)$. □

LEMMA 14.12. *Let A, B, C, D be four matrices and $ABCD$ exists, then $tr(ABCD) = (vceD^{\top})^{\top}(C^{\top} \otimes A)vce(B)$.*

Proof.

$$tr(ABCD) = tr(D(ABC)) \tag{14.29}$$

$$= (vceD^{\top})^{\top}vce(ABC) \tag{14.30}$$

$$= (vceD^{\top})^{\top}(C^{\top} \otimes A)vce(B) \tag{14.31}$$

Further, with $\forall A_{m\times n}, B_{n\times m}, C_{n\times n}$ we have $tr(AB) = tr(BA)$ and $tr(C) = tr(C^{\top})$, we have other ways to calculate $tr(ABCD)$, e.g.,

$$tr(ABCD) = tr(DABC) \tag{14.32}$$

$$= tr(C(DAB)) \tag{14.33}$$

$$= vce(C^{\top})^{\top}vce(DAB) \tag{14.34}$$

$$= vce(C^{\top})^{\top}(B^{\top} \otimes D)vec(A) \tag{14.35}$$

$$tr(ABCD) = tr((ABCD)^{\top}) \tag{14.36}$$

$$= tr(D^{\top}C^{\top}B^{\top}A^{\top}) \tag{14.37}$$

$$= vce(D)^{\top}vce(C^{\top}B^{\top}A^{\top}) \tag{14.38}$$

$$= vce(D)^{\top}(A \otimes C^{\top})vce(B^{\top}) \tag{14.39}$$

□

14.3 Matrix Calculus

Now introduce matrix calculus and its expression with vce-operator and Kronecker product.

DEFINITION 14.5. *(Derivatives of Matrix) Let $f_{m\times 1}$ be a vector equation and $x_{n\times 1}$ be a variable vector. Then the derivative (or Jacobian matrix) of f is defined as:*

$$Df(x)_{m\times n} \equiv \frac{\partial f(x)}{\partial x^{\top}} = \begin{bmatrix} \frac{\partial f_1(x)}{\partial x_1} & \frac{\partial f_1(x)}{\partial x_2} & \cdots & \frac{\partial f_1(x)}{\partial x_n} \\ \frac{\partial f_2(x)}{\partial x_1} & \frac{\partial f_2(x)}{\partial x_2} & \cdots & \frac{\partial f_2(x)}{\partial x_n} \\ & & \cdots & \\ \frac{\partial f_m(x)}{\partial x_1} & \frac{\partial f_m(x)}{\partial x_2} & \cdots & \frac{\partial f_m(x)}{\partial x_n} \end{bmatrix} \tag{14.40}$$

Let $F_{m\times p}$ be a matrix equation and $X_{n\times q}$ be a variable matrix. Then the derivative of F includes $mnpq$ partial derivatives (span them to vectors first) , i.e.,

$$DF(X)_{mp\times nq} \equiv \frac{\partial vce(F(X))}{\partial vce(X)} = \begin{bmatrix} \frac{\partial vce(F(X))}{\partial x_{11}} & \cdots & \frac{\partial vce(F(X))}{\partial x_{n1}} & \cdots & \frac{\partial vce(F(X))}{\partial x_{1q}} & \cdots \frac{\partial vce(F(X))}{\partial x_{nq}} \end{bmatrix} \tag{14.41}$$

Note that every row of DF is comprised of one element of F's derivative w.r.t. all elements of X.

We assume α be a constant, a, b be constant vectors ,A be a constant matrix, and F, G are matrix equations with the same dimensions. $I = [e_1, e_2, \cdots, e_n]$ be an identity matrix in later chapters.

LEMMA 14.13. *Let x be a n-vector (column), $A_{m\times n}$ be a matrix, and $y = Ax$, then $\frac{\partial y}{\partial x^\top} = A$.*

Proof. Let $y = [y_1, y_2, \cdots, y_m]$, $y_i = \sum_{j=1}^n a_{ij}x_j$. So $\frac{\partial y_i}{\partial x_j} = a_{ij}$, then

$$Df = \frac{\partial f(x)}{\partial x} = \begin{bmatrix} \frac{\partial y_1(x)}{\partial x_1} & \frac{\partial y_1(x)}{\partial x_2} & \cdots & \frac{\partial y_1(x)}{\partial x_n} \\ \frac{\partial y_2(x)}{\partial x_1} & \frac{\partial y_2(x)}{\partial x_2} & \cdots & \frac{\partial y_2(x)}{\partial x_n} \\ & & \cdots & \\ \frac{\partial y_m(x)}{\partial x_1} & \frac{\partial y_m(x)}{\partial x_2} & \cdots & \frac{\partial y_m(x)}{\partial x_n} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \cdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = A \tag{14.42}$$

□

LEMMA 14.14. *Let X_n be a square and $\varphi(X) = tr(X)$,then $D\varphi(X) = (vceI_n)^\top$*

Proof. $\varphi(X) = \sum_{i=1}^n x_{ii}$, then $\frac{\partial\varphi(X)}{x_{ii}} = 1, \frac{\partial\varphi(X)}{x_{ij}} = 0, i \neq j, \forall i, j$. So

$$\begin{aligned} D\varphi(X) &= \begin{bmatrix} \frac{\partial vce(\varphi(X))}{\partial x_{11}} & \cdots & \frac{\partial vce(\varphi(X))}{\partial x_{n1}} & \cdots & \frac{\partial vce(\varphi(X))}{\partial x_{1q}} & \cdots \frac{\partial vce(\varphi(X))}{\partial x_{nq}} \end{bmatrix} & (14.43) \\ &= \begin{bmatrix} \frac{\partial\varphi(X)}{\partial x_{11}} & \cdots & \frac{\partial\varphi(X)}{\partial x_{n1}} & \cdots & \frac{\partial\varphi(X)}{\partial x_{1q}} & \cdots \frac{\partial\varphi(X)}{\partial x_{nq}} \end{bmatrix} & (14.44) \\ &= \begin{bmatrix} e_1, e_2, \cdots, e_n \end{bmatrix} & (14.45) \\ &= vce(I_n)^\top & (14.46) \end{aligned}$$

□

Derivatives give a vector outcome of the derivative information of the equation matrix. But we are interested in the total effects of X on every element of F, i.e., $\sum_{s,t} \frac{f_{ij}}{X_{s,t}}$. Differentials satisfy that appetite with dense derivative matrix with the same dimension of F, that simplifies our calculation.

DEFINITION 14.6. *Differential Let $f_{m\times 1}, F_{m\times p}$ be equation matrices and $x_{n\times 1}, X_{n\times q}$ be a variable vector / matrix. If $Df, DvceF$ exists, the differentials of f, F, denoted as df, dF is given by*

$$df(x) = A(x)dx \Leftrightarrow Df(x) = A(x) \tag{14.47}$$

$$dvecF(X) = A(X)dvecX \Leftrightarrow DF(X) = A(X) \tag{14.48}$$

Then we obtain dF by the "inverse" operation of vce-operator.

$$vce(dF) = dvce(F), vce(dX) = dvce(X) \tag{14.49}$$

We use dF in practice frequently, but remember how it is constructed. But to operate dF, we have to go back to $dvce(F)$.

To interpret df, let $f(x) = [f_1(x), f_2(x), \cdots, f_m(x)]$, then

$$(df)_i = df_i(x) = \sum_{j=1}^n \frac{\partial f_i(x)}{\partial x_j} = \sum_{j=1}^n (A(x))_{ij}dx_j, \forall i = 1, 2, \cdots, m \tag{14.50}$$

so $\frac{\partial f(x)}{\partial x^\top} = Adx$ where $dx = [dx_1, dx_2, \cdots, dx_n]^\top$.

Moreover, we see, $\forall i, j, (dF)_{ij} = (dvce(F))_{(j-1)p+i} = \sum_{s=1}^n \sum_{t=1}^q \frac{\partial(vce(F))_{j-1}p+i}{X_{st}} = \sum_{s=1}^n \sum_{t=1}^q \frac{\partial f_{ij}}{X_{st}}$, i.e.,

$$(dF)_{ij} = \sum_{s=1}^n \sum_{t=1}^q \frac{\partial f_{ij}}{X_{st}} \tag{14.51}$$

i.e., $(dF)_{ij}$ equals to the summation of f_{ij}'s partial derivatives w.r.t. all elements of X. That's way we say differentials are condense.

LEMMA 14.15. *Differentials of general functions :*

(1) $dA = \mathbf{0}$*;*

(2) $d(\alpha F) = \alpha dF$*;*

(3) $d(F + G) = dF + dG; d(F - G) = dF - dG$*;*

(4) $dtr(F) = tr(dF), \forall F_n$ *(square);*

(5) $d(F^\top) = (dF)^\top$*;*

(6) $d(vceF) = vce(dF)$*;*

(7) $d(FG) = (dF)G + F(dG)$*;*

(8) $d(F \otimes G) = (dF) \otimes G + F \otimes (dG)$*.*

Proof. Note that differentials do not change the arrangement of elements, i.e., the differentials of matrix is equivalent to the matrix of differentials on every elements.

Let $X_{n\times q}$ and $\sum_{s,t}$ denote $\sum_{s=1}^n \sum_{t=1}^q$.

(1) $\forall i, j, (dA)_{ij} = \sum_{s,t} \frac{\partial a_{ij}}{\partial X_{s,t}} = 0$, so $dA = \mathbf{0}$.

(2) $\forall i, j, (d(\alpha F))_{ij} = \sum_{s,t} \frac{\partial \alpha f_{ij}}{\partial X_{s,t}} = \alpha \sum_{s,t} \frac{\partial f_{ij}}{\partial X_{s,t}} = \alpha(dF)_{ij}, d(\alpha F) = \alpha(dF)$.

(3) $\forall i, j, (d(F + G))_{ij} = \sum_{s,t} \frac{\partial f_{ij}+g_{ij}}{\partial X_{s,t}} = \sum_{s,t} \frac{\partial f_{ij}}{\partial X_{s,t}} + \sum_{s,t} \frac{\partial g_{ij}}{\partial X_{s,t}} = (dF)_{ij} + (dG)_{ij}$, so $d(F + G) = dF + dG$.

(4) $tr(dF) = \sum_{i=1}^n \sum_{s,t} \frac{\partial f_{ii}}{\partial X_{s,t}} = \sum_{s,t} \frac{\partial \sum_{i=1}^n f_{ii}}{\partial X_{s,t}} = \sum_{s,t} \frac{\partial tr(F)}{X_{s,t}} = dtr(F)$.

(5) $\forall i, j, (dF^\top)_{ij} = \sum_{s,t} \frac{\partial f_{ji}}{\partial X_{s,t}} = ((dF)^\top)_{ij}$, so $(dF^\top) = (dF)^\top$.

(6) Let $F_{m\times p}$ be a matrix, then $\forall i, j, (d(vceF))_{(j-1)p+i} = d((vce(F))_{(j-1)p+i}) = d(F_{ij}) = (dF)_{i,j} = vce(dF)_{(j-1)p+i}$.

(7) $\forall i, j,$

$$
\begin{aligned}
(d(FG))_{ij} &= d(FG)_{ij} && (14.52)\\
&= d\sum_k f_{ik}g_{kj} && (14.53)\\
&= \sum_k d(f_{ik}g_{kj}) && (14.54)\\
&= \sum_k (df_{ik})g_{kj} + f_{ik}dkj && (14.55)\\
&= \sum_k (df_{ik})g_{kj} + \sum_k f_{ik}dkj && (14.56)\\
&= ((dF)G)_{ij} + (FdG)_{ij} && (14.57)\\
\Rightarrow d(FG) &= (dF)G + F(dG) && (14.58)
\end{aligned}
$$

(8) $\forall i, j, s, t$, for an element $f_{ij}g_{st}$ of sub-matrix $f_{ij}G$, we have

$$d(f_{ij}g_{st}) = (df_{ij})g_{st} + f_{ij}dg_{st} \tag{14.59}$$

where $(df_{ij})g_{st}$ belongs to $(dF) \otimes G$ and $f_{ij}dg_{st}$ belongs to $F \otimes (dG)$ and they are at the same position, so

$$(F \otimes G) = (dF) \otimes G + F \otimes (dG) \tag{14.60}$$

□

In particular, let F be constant, then $dF = \mathbf{0}$, so $\forall G, d(FG) = (dF)G + F(dG) = FdG$ and $d(GF) = (dF)G + G(dF) = (dF)G$. Further, let AC be constant, then $d(AFC) = A(d(FC)) = A(dF)C$.

LEMMA 14.16. *Differentials of scalar equations:*

1. $\varphi(X) = a^\top Xb, D\varphi = (b \otimes a)^\top$*;*
2. $\varphi(X) = a^\top XX^\top a, D\varphi = 2(X^\top a \otimes a)^\top$

3. $\varphi(X) = a^\top X^\top X a, D\varphi = 2(a \otimes Xa)^\top$

Proof. (1) Note that $dvce(F) = vce(dF), dvce(X) = vce(dX)$, $d(AF) = AdF$ and $A^\top \otimes B^\top = (A \otimes B)^\top$[8]. $\forall A$, if A is a scalar, $vce(A) = A, dA = dvce(A)$.

$$d\varphi = dvce(\varphi) = dvce(a^\top Xb) = d((b^\top \otimes a^\top)vce(X)) = (b^\top \otimes a^\top)dvce(X) = (b \otimes a)^\top dvceX \tag{14.61}$$

$$\Rightarrow D\varphi(X) = \frac{\partial \varphi}{\partial (vceX)^\top} = (b \otimes a)^\top \tag{14.62}$$

(2)

$$d\varphi = d(a^\top XX^\top a) = a^\top d(XX^\top)a = a^\top((dX)X^\top + XdX^\top)a \tag{14.63}$$
$$= a^\top((dX)X^\top + X(dX)^\top)a = a^\top(dX)X^\top a + (a^\top(dX)X^\top a)^\top \tag{14.64}$$

Let $Y = (a^\top(dX)X^\top a)$, then Y is a variable scalar, so $Y = vce(Y^\top) = vce(Y) = vce(a^\top(dX)X^\top a) = (a^\top X \otimes a^\top)vce(dX) = (a^\top X \otimes a^\top)dvce(X) = (X^\top a \otimes a)^\top dvce(X)$. So,

$$dvce(\varphi) = vce(d\varphi) = vce(Y + Y^\top) = 2vce(Y) = 2(X^\top a \otimes a)^\top dvce(X) \tag{14.65}$$

i.e., $D\varphi(X) = \frac{\partial\varphi}{\partial(vceX)^\top} = 2(X^\top a \otimes a)^\top$.

(3) Similar to (2),

$$d\varphi = d(a^\top X^\top Xa) = a^\top d(X^\top X)a = a^\top(d(X^\top)X + X^\top dX)a \tag{14.66}$$
$$= a^\top((dX)^\top X + X^\top dX)a = (a^\top X^\top dXa)^\top + (a^\top X^\top dXa) \tag{14.67}$$
$$Y = a^\top X^\top dX, Y = vce(Y^\top) = vce(Y) = vce(a^\top X^\top dXa) \tag{14.68}$$
$$= (a^\top \otimes a^\top X^\top)^\top vce(dX) = (a \otimes Xa)^\top dvce(X) \tag{14.69}$$
$$\Rightarrow dvce(\varphi) = vce(d\varphi) = vce(Y + Y^\top) = 2vce(Y) = 2(a \otimes Xa)^\top dvce(X) \tag{14.70}$$
$$\Rightarrow D\varphi(X) = \frac{\partial\varphi}{\partial(vceX)^\top} = 2(a \otimes Xa)^\top \tag{14.71}$$

□

LEMMA 14.17. *Differentials of trace:*

1. $dtr(F) = tr(dF)$;
2. $d\varphi = tr(A^\top dX) \Rightarrow D\varphi(X) = (vceA)^\top$;
3. $\varphi(X) = tr(X^p), D\varphi = p(vce(A^\top)^{p-1})^\top$;
4. $\varphi(X) = tr(X^\top X), D\varphi = 2(vceX)^\top$.

Proof. (1) $dtr(F) = d\sum_i f_{ii} = \sum_i df_{ii} = tr(dF)$;
(2) $d\varphi = tr(A^\top dX) = (vceA)^\top vce(dX) = (vceA)^\top dvceX$, so $D\varphi = (vceA)^\top$;
(3) Note that $dtr(F) = tr(dF)$ and $tr(AB) = tr(BA)$.

$$d\varphi = dtr(X^p) = tr(dX^p) = tr[(dX)X^{p-1}] + tr[X(dX)X^{p-2}] + \cdots + tr[(X^{p-1}dX)] \tag{14.72}$$
$$= tr[X^{p-1}dX] + tr[X^{p-2}XdX] + \cdots + tr[X^{p-1}dX] \tag{14.73}$$
$$= ptr(X^{p-1}dX) = p[vce(X^{p-1})^\top]^\top dvceX = p(vce(X^\top)^{p-1})^\top dvceX \tag{14.74}$$
$$\Rightarrow D\varphi = p(vce(X^\top)^{p-1})^\top \tag{14.75}$$

(3) $d\varphi = dtr(X^\top X) = tr(d(X^\top X)) = tr((dX^\top)X + X^\top(dX)) = tr((dX)^\top X + X^\top(dX)) = tr((dX)^\top X) + tr(X^\top(dX)) = tr(X^\top dX) + tr(X^\top dX) = 2tr(X^\top dX) \Rightarrow D\varphi = 2(vceX)^\top$. □

LEMMA 14.18. *Differentials of vector equations:*

1. $f(x) = Ax, Df(x) = A$;

[8] $A^\top \otimes B^\top = Conb(a_{ji}B^\top) = (Conb(Ba_{ij}))^\top = (A \otimes B)^\top$.

2. $f(x) = A(x)x, Df(x) = (x^\top \otimes I)\frac{\partial vceA}{\partial x^\top} + A;$

3. $f(X) = Xa, Df = a^\top \otimes I;$

4. $f(X) = X^\top a, Df = I \otimes a^\top.$

Proof. (1) $dvcef = vcedf = vce(d(Ax)) = vce((dA)x + Adx) = vce(Adx) = (I_1 \otimes A)vcedx = Advcex$, so $Df = A$;

(2) $dvcef = vcedf = vced(A(x)x) = vce((dA(x))x + A(x)dx) = vce(dA(x)x) + vce(A(x)dx) = (x^\top \otimes I)dvceA + (I_1 \otimes A(x))vcedx = ((x^\top \otimes I)\frac{\partial vceA}{\partial x^\top} + A)dvcex, Df(x) = (x^\top \otimes I)\frac{\partial vceA}{\partial x^\top} + A.$

(3) $dvcef = vcedf = vce(d(Xa)) = vce((dX)a) = (a^\top \otimes I)vcedX = (a^\top \otimes I)dvceX, Df = a^\top \otimes I.$

(4) $dvcef = vcedf = vce(d(X^\top a)) = vce((dX)^\top a) = vce(a^\top dX) = (I \otimes a^\top)vcedX = (I \otimes a^\top)dvceX, Df = I \otimes a^\top.$ ☐

LEMMA 14.19. *Differentials of matrix equations:*

1. $F(x) = xx^\top, DF = x \otimes I + I \otimes x;$

2. $F(X) = AXB, DF = B^\top \otimes A;$

3. $F(X) = X^2, DF = X^\top \otimes I_n + I_n \otimes X;$

4. $F(X) = X^p, DF = \sum_{ii=0}^{p-1}(X^\top)^{ii} \otimes X^{p-1-ii};$

5. $F(X) = X^\top, DF = K_{mn};$

6. $F(X) = X^\top X, DF = (I_{n^2} + K_n)(I_n \otimes X^\top)$

Proof. (1) $dvceF = vcedF = vce(d(xx^\top)) = vce((dx)x^\top) + vce(x(dx^\top)) = (x \otimes I)vcedX + (I \otimes x)vcedx^\top = (x \otimes I)dvcex + (I \otimes x)dvcex^\top = (x \otimes I)dvcex + (I \otimes x)dvcex$, so $DF = x \otimes I + I \otimes x$.

(2) $dvceF = vcedF = vce(d(AXB)) = vce(A(dX)B) = (B^\top \otimes A)dvceX, DF = B^\top \otimes A.$

(3) $dvceF = vcedF = vce(d(X^2)) = vce[(dX)X + X(dX)] = vce[(dX)X] + vce[X(dX)] = (X^\top \otimes I_n)vcedX + (I_n \otimes X)vcedX = (X^\top \otimes I_n)dvceX + (I_n \otimes X)dvceX, DF = X^\top \otimes I_n + I_n \otimes X.$

(4) $dvceF = vcedF = vce(dX^p) = vce[d(X)X^{p-1} + Xd(X)X^{p-2} + \cdots + X^{p-1}dX] = vce[d(X)X^{p-1}] + vce[Xd(X)X^{p-2}] + \cdots + vce[X^{p-1}dX] = \sum_{i=1}^p vce[X^{i-1}(dX)X^{p-i}] = \sum_{i=1}^p[(X^{p-i})^\top \otimes X^{i-1}]vcedX = \sum_{i=1}^p[(X^\top)^{p-i} \otimes X^{i-1}]dvceX$, so $DF = \sum_{i=1}^p(X^\top)^{p-i} \otimes X^{i-1}$.

(5) $dvceF = vcedF = vce(dX^\top) = dvceX^\top = K_{mn}dvceX, DF = K_{mn}$, where K_{mn} is named after the commutation matrix, s.t. $vceX^\top = K_{mn}vceX$. When $m = 3, n = 2$, $K_{mn} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

Moreover, K_{mn} is not symmetric, but $|K_{mn}| \neq 0$ and $K_{mn}^2 = I_{mn}$. With K_{mn}, we can rewrite the Kronecker product with $(A \otimes B)K_{nq} = K_{mp}(B \otimes A)$ for A_{mn}, B_{pq}. Note that (1)if $\forall x, Ax = Bx$, then $A = B$ (we can set $x = e_1, e_2, \cdots, e_n$, so $A = AE = A[e_1, e_2, \cdots, e_n] = [Ae_1, Ae_2, \cdots, Ae_n] = [Be_1, Be_2, \cdots, Be_n] = BE = B$) and (2)$vce(C^\top) = K_{d_1d_2}vce(C)$. As $\forall C_{n\times q}$, we have $(A \otimes B)K_{nq}vce(C) = (A \otimes B)vce(C^\top) = vce(BC^\top A^\top) = K_{mp}vce(BCA^\top)^\top = K_{mp}vce(ACB^\top) = K_{mp}(B \otimes A)vce(C)$.

(6) we have $(A \otimes B)K_{nq} = K_{mp}(B \otimes A)$ for A_{mn}, B_{pq}. $dvceF = vcedF = vce(d(X^\top X)) = vce((dX^\top)X + X^\top dX) = (X^\top \otimes I)dvceX^\top + (I \otimes X^\top)dvceX = (X^\top \otimes I)K_{n^2}vcedX + (I \otimes X^\top)vcedX = K_{n^2}(I \otimes X^\top)vcedX + (I \otimes X^\top)vcedX = (K_{n^2} + I_{n^2})(I \otimes X^\top)dvceX, DF = (K_{n^2} + I_{n^2})(I \otimes X^\top).$ ☐

LEMMA 14.20. *Differentials of inverse matrix equations:*

- $F(X^{-1}), DF = -X^{-1}(dX)X^{-1};$
- $\varphi(X) = tr(AX^{-1}), D\varphi = -(vce(X^{-1}AX^{-1})^\top)^\top.$

Proof. (1) $X^{-1}X = I, d(X^{-1}X) = dX^{-1}X + X^{-1}dX = \mathbf{0}, dX^{-1} = -X^{-1}(dX)X^{-1}$. When $|dX| \neq 0, dX^{-1}$ is not singular.

(2) $d\varphi = trA(dX^{-1}) = -trAX^{-1}(dX)X^{-1} = -trX^{-1}AX^{-1}dX, D\varphi(X) = -(vce(X^{-1}AX^{-1})^\top)^\top.$ ☐

LEMMA 14.21. *Differentials of idempotent matrix equations:* $M = I_n - X(X^\top X)^{-1}X^\top, X_{n\times k}, dM?dvceM?$

$$\begin{aligned}
dM &= -[(dX)(X^\top X)^{-1}X^\top + X(d(X^\top X)^{-1})X^\top + X(X^\top X)^{-1}(dX)^\top] && (14.76)\\
&= -(dX)(X^\top X)^{-1}X^\top + X(X^\top X)^{-1}(d(X^\top X))(X^\top X)^{-1}X^\top - X(X^\top X)^{-1}(dX)^\top && (14.77)\\
&= -(dX)(X^\top X)^{-1}X^\top + X(X^\top X)^{-1}[(d(X^\top))X + X^\top dX](X^\top X)^{-1}X^\top - X(X^\top X)^{-1}(dX)^\top && (14.78)\\
&= -(dX)(X^\top X)^{-1}X^\top + X(X^\top X)^{-1}d(X^\top)X(X^\top X)^{-1}X^\top + X(X^\top X)^{-1}X^\top dX(X^\top X)^{-1}X^\top && (14.79)\\
&- X(X^\top X)^{-1}(dX)^\top && (14.80)\\
&= -M(dX)(X^\top X)^{-1}X^\top - X(X^\top X)^{-1}(dX)^\top M && (14.81)\\
dvceM &= -(X(X^\top X)^{-1} \otimes M)dvceX - (M \otimes X(X^\top X)^{-1})dvceX^\top && (14.82)\\
&= -((X(X^\top X)^{-1} \otimes M) + (M \otimes X(X^\top X)^{-1})K_{nk})dvceX && (14.83)\\
&= -(I_{n^2} + K_{nk})(X(X^\top X)^{-1} \otimes M)dvceX && (14.84)\\
DM &= -(I_{n^2} + K_n)(X(X^\top X)^{-1} \otimes M) && (14.85)
\end{aligned}$$

Note that $M^\top = M^{-1} = M, M^2 = I$.

Define

$$exp(X) = \sum_{k=0}^{\infty} \frac{1}{k!}X^k \tag{14.86}$$

LEMMA 14.22. *Differentials of exponential matrix equations:*

1. $dexp(xA) = Aexp(xA)dx;$
2. $dexp(X) = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \sum_{j=0}^{k} X^j(dX)X^{k-j};$
3. $tr(dexp(X)) = tr(exp(X)dX).$

Proof. (1) First we know, $exp(X) = \sum_{k=0}^{\infty} \frac{1}{k!}X^k$. $dexp(xA) = \sum_{k=0}^{\infty} \frac{1}{k!}(dx^k)A^k = \sum_{k=0}^{\infty} \frac{1}{k!}kx^{k-1}A^k dx = \sum_{k=1}^{\infty} \frac{1}{(k-1)!}x^{k-1}A^k dx = A\sum_{k=0}^{\infty} \frac{1}{k!}(xA)^k dx = Aexp(xA)dx$.

(2) $dexp(X) = \sum_{k=0}^{\infty} \frac{1}{k!}dX^k = \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j=0}^{k-1} X^j(dX)X^{k-j-1} = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \sum_{j=0}^{k} X^j(dX)X^{k-j}$

(3) $tr(dexp(X)) = \sum_{k=0}^{\infty} \frac{1}{(k+1)!}(k+1)tr(X^k dX) = tr(\sum_{k=0}^{\infty} \frac{1}{k!}X^k dX) = tr(exp(X)dX)$ □

LEMMA 14.23. *Differentials of log-matrix equations: Let*

$$|x| \cdot \rho(A) < 1, log(I_n - X) = -\sum_{k=1}^{\infty} \frac{1}{k}X^k, dlog(I_n - xA) = -A(I_n - xA)^{-1}dx \tag{14.87}$$

Proof. $dlog(I_n - xA) = -\sum_{k=1}^{\infty} \frac{1}{k}(dx^k)A^k = -\sum_{k=1}^{\infty} \frac{1}{k}x^{k-1}A^k dx = -A\sum_{k=0}^{\infty}(xA)^k dx = -A(I_n - xA)^{-1}dx$. □

LEMMA 14.24. *Differentials of matrix's Determinant: We have* $|X| = \sum_{ii=1}^{n} c_{ij}x_{ij}$*,, where* c_{ij} *is named as algebraic complement and* $c_{ij} = (-1)^{i+j}|M_{ij}|$*. Prove* $d|X| = |X|trX^{-1}dX$.

Proof. Define $C = (c_{ij})$, we have $C^\top X = |X|I_n$ and $C^\top = |X|X^{-1}$. $\frac{\partial |X|}{\partial x_{ij}} = \frac{\partial}{\partial x_{ij}}(c_{1j}x_{1j} + c_{2j}x_{2j} + \cdots + c_{nj}x_{nj}) = c_{ij}$, $d|X| = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}dx_{ij} = tr[(C^\top)dX] = |X|tr[X^{-1}dX]$, $\frac{\partial |X|}{\partial (vceX)^\top} = |X|((vceX^\top)^{-1})^\top$. □

With $d|X| = |X|tr[X^{-1}dX]$*, we can calculate more differentials:*

1. $d|AXB| = |AXB|trB(AXB)^{-1}AdX;$
2. $d|XX^\top| = 2|XX^\top|trX^\top(XX^\top)^{-1}dX;$
3. $|X| > 0, dlog|X| = trX^{-1}dX;$
4. $|X^\top AX| > 0, dlog|X^\top AX| = tr[(X^\top A^\top X)^{-1}X^\top A^\top + (X^\top AX)^{-1}X^\top A]dX.$

Proof. (1) $d|AXB| = |AXB|tr(AXB)^{-1}d(AXB) = |AXB|tr(AXB)^{-1} = |AXB|tr(AXB)^{-1}A(dX)B = |AXB|trB(AXB)^{-1}AdX$;

(2) Note $tr(A^\top B) = vce(A)^\top vce(B), vce(B + B^\top) = vce(B)$,

$$d|XX^\top| = |XX^\top|tr(XX^\top)^{-1}d(XX^\top) = |XX^\top|tr[(XX^\top)^{-1}((dX)X^\top + X(dX)^\top)] \tag{14.88}$$
$$= 2|XX^\top|tr(XX^\top)^{-1}(dX)X^\top = 2|XX^\top|trX^\top(XX^\top)^{-1}dX \tag{14.89}$$

(3) $dlog|X| = \frac{1}{|X|}d|X| = \frac{1}{|X|}|X|trX^{-1}dX = trX^{-1}dX$.

(4) Note that $trA(B + C) = tr(AB) + tr(AC) = tr(B^\top A^\top) + tr(AC) = tr(B^\top A^\top + AC)$.

$$dlog|X^\top AX| = \frac{1}{|X^\top AX|}d|X^\top AX| = \frac{1}{|X^\top AX|}|X^\top AX|tr(X^\top AX)^{-1}d(X^\top AX) \tag{14.90}$$
$$= tr(X^\top AX)^{-1}((dX)^\top AX + X^\top AdX) \tag{14.91}$$
$$= tr[(X^\top A^\top X)^{-1}X^\top A^\top(dX) + (X^\top AX)^{-1}X^\top AdX] \tag{14.92}$$
$$= tr[(X^\top A^\top X)^{-1}X^\top A^\top + (X^\top AX)^{-1}X^\top A]dX \tag{14.93}$$

□

14.4 Law of Large Numbers

Asymptotics is the most important property of estimator in econometrics, and we depend on it to ensure the estimator stable and easy to make inference. The core conceptions we concerned are consistency and asymptotic normality.

DEFINITION 14.7. ***Consistency*** *implies that the more data we get, the closer we are to the truth, i.e., as long as we have enough data, the gauss will be right.*

DEFINITION 14.8. ***Asymptotic Normality*** *implies that as we get more data, the mean of variables behaves in the way following the normal distribution more likely.*

A classic problem implemented consistency and asymptotic normality is Example 14.1.

Example 14.1. *Let $X_1, X_2, \cdots, X_n$ be a IID random variable series, and $E(X_i) = \mu, Var(X_i) = \sigma^2$. σ^2 is known. The questions are*

1. *estimate μ;*
2. *calculate the confidence interval of μ in 99%;*
3. *test $H_0 : \mu = \mu_0, H_1 : \mu \neq \mu_0$.*

To answer these questions, we need Law of Large Numbers (LLNs) and Central Limit Theorems (CLTs).

Let $X_1, X_2, \cdots, X_n$ be IID random samples with the probability density function f. Given a equation g, we can define

$$Y_1 = g(X_1) \tag{14.94}$$
$$Y_2 = g(X_1, X_2) \tag{14.95}$$
$$\cdots \tag{14.96}$$
$$Y_n = g(X_1, X_2, \cdots, X_n) \tag{14.97}$$
$$\epsilon_i|x_i \sim i,i.d.N(0, \sigma^2) \tag{14.98}$$

The statistics $Y_1, Y_2, \cdots, Y_n$ is a random variable series.

DEFINITION 14.9. *(Convergence in Probability) Let $Y_1, Y_2, \cdots, Y_n$ be a random variable series, c be a constant or variable. If $\forall \varepsilon > 0$,*

$$\lim_{n\to\infty} P(|Y_n - c| > \varepsilon) = 0 \tag{14.99}$$

Then, we say Y_n converges to c in probability, written as $Y_n \to^p c$ or $\underset{n\to\infty}{plim}\, Y_n = c$ [9] . If $Y_n \to^p c$, we have $Y_n - c \to^p 0$. Moreover, for a vector process $Y_n = (Y_{n1}, Y_{n2}, \cdots, Y_{nk})^\top$, if $Y_{ni} \to^p c, i = 1, 2, \cdots, k$, then $Y_n \to^p c = (c_1, c_2, \cdots, c_k)^\top$.

[9] $plim$ is the bbr. of probability limit

DEFINITION 14.10. *(Consistent Estimator) Let $\hat{\theta}$ be an estimator of the scalar value θ. If*

$$\hat{\theta} \to^p \theta \tag{14.100}$$

Then we say $\hat{\theta}$ is consistent.

Now we are ready to introduce LLNs.

THEOREM 14.2. *(Chebychev's LLN) Let $X_1, X_2, \cdots, X_n$ be a IID random series with $E(X_i) = \mu < \infty, Var(X_i) = \sigma^2 < \infty$, then*

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \to^p E(X_i) = \mu \tag{14.101}$$

Proof. Let X be a random variable with $E(X) = \mu < \infty, Var(X) = \sigma^2 < \infty$, then $\forall \varepsilon > 0$,

$$P(|X - c| > \varepsilon) \leq \frac{Var(X)}{\varepsilon^2} = \frac{\sigma^2}{\varepsilon^2} \tag{14.102}$$

That is the famous Chebychev's inequality. Thus,

$$P(|\bar{X}_n - \mu| \geq \varepsilon) \leq \frac{Var(\bar{X}_n)}{\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2} \tag{14.103}$$

$$\Rightarrow \lim_{n\to\infty} P(|\bar{X}_n - \mu| \geq \varepsilon) \leq \lim_{n\to\infty} \frac{\sigma^2}{n\varepsilon^2} = 0 \tag{14.104}$$

Note that this proof relies on convergence in mean square, i.e., $MSE(\bar{X}_n, \mu) = E[(\bar{X}_n - \mu)^2] = \frac{\sigma^2}{n} \to 0$. We can decompose MSE with

$$MSE(\bar{X}_n, \mu) = E[(\bar{X}_n - \mu)^2] = (E(\bar{X}_n) - \mu)^2 + E(\bar{X}_n - E(\bar{X}_n))^2 = bias(\bar{X}_n, \mu)^2 + Var(\bar{X}_n) \tag{14.105}$$

Check that manually! Remember $E(\bar{X}_n)$ and μ are constants. If $MSE \to 0$, we have $X_n \to^p \mu$. Because when $MSE \to 0, bias, Var \to 0$, but the inverse form does not hold. □

THEOREM 14.3. *(Kolmogrov's and Khinchine's LLN) Let $X_1, X_2, \cdots, X_n$ be a IID random variable series with $E(X_i) = \mu, E(|X_i|) < \infty$, then*

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \to^p E(X_i) = \mu \tag{14.106}$$

Note that Kolmogrov's and Khinchine's LLN can be applied for fat-tailed distribution without variance.

THEOREM 14.4. *(Markov's LLN) Let $X_1, X_2, \cdots, X_n$ be an uniformly unbounded and unrelated random series with $E(X_i) = \mu_i < \infty, Var(X_i) = \sigma_i^2 \leq M < \infty$, then*

$$\bar{X}_n - \bar{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i - \frac{1}{n}\sum_{i=1}^{n} \mu_i = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu_i) \to^p 0 \tag{14.107}$$

Markov's LLN can be proved with Chebychev's inequality. We can rewrite $Var(X_i) = \sigma_i^2 \leq M < \infty$ as $sup_i \sigma_i^2 < \infty$.

For LLN, a general rule is if you loose some assumptions, you must tight some other conditions.

THEOREM 14.5. *(Series Relation's LLN) Let $X_1, X_2, \cdots, X_n$ be a random variable series, and if $\sigma_{ij} = Cov(X_i, X_j)$ exist for all i, j and $\lim_{|i-j|\to\infty} \sigma_{ij} \to 0$, then*

$$\lim_{n\to\infty} P(|\bar{X}_n - \bar{\mu}| > \varepsilon) \leq \lim_{n\to\infty} \frac{M}{n\varepsilon^2} = 0 \tag{14.108}$$

We summarize all LLNs in Table 3.

Then Slutsky's Theorem ensures some simple transformations do not change the convergence in probability.

THEOREM 14.6. *(Slutsky's Theorem 1) $\{Y_n\}, \{Z_n\}$ are random variable series and b, c, d are constants.*

1. *if $Y_n \to^p c$, $bY_n \to^p c$;*

LLN	statistics	series relation
Chebychev's LLN	$E(X_i) = \mu < \infty, Var(X_i) = \sigma^2 < \infty$	IID
Kolmogorov's and Khinchine's LLN	$E(X_i) = \mu, E(\lvert X_i \rvert) < \infty$	IID
Markov's LLN	$E(X_i) = \mu_i < \infty, Var(X_i) = \sigma_i^2 \leq M < \infty$	series unrelated
Series Relation's LLN		$\lim\limits_{\lvert i-j \rvert \to \infty} \sigma_{ij} \to 0$

Table 3: Summary of LLNs

2. *if* $Y_n \to^p c, Z_n \to^p d, Y_n + Z_n \to^p c + d$*;*
3. *if* $Y_n \to^p c, Z_n \to^p d, d \neq 0, Y_n/Z_n \to^p c/d, Y_n Z_n \to^p cd$*;*
4. *if* $Y_n \to^p c$*,* $h(\cdot)$ *are continuous at* c*, thus,* $h(Y_n) \to^p h(c)$*.*

See the application of Slutsky's Theorem in lemma 14.25.

LEMMA 14.25. *Let* $X_1, X_2, \cdots, X_n$ *be a IID random variable series with* $E(X_i) = \mu, Var(X_i) = \sigma^2 < \infty$*. Then the variance of sample* $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X}_n)^2$ *is a consistent estimator of* σ^2*.*

Solution: First we know $\bar{X}_n$ *is a consistent estimator of* μ*. Then we write* $\hat{\sigma}^2$ *as*

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X}_n)^2 \tag{14.109}$$

$$= \frac{1}{n}\sum_{i=1}^n (X_i - \mu)^2 - (\mu - \bar{X}_n)^2 \tag{14.110}$$

$$= \frac{1}{n}\sum_{i=1}^n (X_i - \mu)^2 + o_p(1) \tag{14.111}$$

where $Y_n = -(\mu - \bar{X}_n)^2 \to^p 0$ *because* $-x^2$ *is continuous at* 0*. Moreover,* $\sqrt{x}$ *is also continuous, so* $\hat{\sigma}^2 \to^p \sigma^2$ *implies* $\hat{\sigma} \to \sigma$*.*

14.5 Central Limit Theorems

Then, we introduce convergence in distribution, which is important for hypothesis test.

DEFINITION 14.11. *(Convergence in Distribution) Let* $X_1, X_2, \cdots, X_n$ *be a IID random variable series, with* $E(X_i) = \mu, Var(X_i) = \sigma^2$*. We define a random variable series,* $Y_n = \sqrt{n}\frac{\bar{X}_n - \mu}{\sigma}$*. If* $n \to \infty$

$$F_{Y_n}(y) = Pr(Y_n \leq Y) \to F_W(y) = Pr(W \leq y) \tag{14.112}$$

holds for every continuous point of CDF_W*, then we say* Y_n *convergences in distribution to* W*, i.e.,*

$$Y_n \to^d W \tag{14.113}$$

In application, we want $W \sim N(\mu_w, \sigma_w)$ *or* χ_m^2*. CLT is an essential method to prove the convergence in distribution. If* n *is big enough, we agree* $Pr(Y_n \in A) \simeq Pr(W \in A), \forall A \subseteq R$*.*

Let $Y_n = (Y_{n_1}, Y_{n_2}, \cdots, Y_{n_k})^\top$ *be a multi random variable series, we have:* $Y_n \to^d W$ *iff* $\forall \lambda \in R^k, \lambda^\top Y_n \to^d \lambda^\top W$*.*

The relation between convergence in distribution and convergence in probability is:

- $Y_n \to^p Y \Rightarrow Y_n \to^d Y$*;*
- $Y_n \to^d c$*,* c *is a constant* $\Rightarrow Y_n \to^p c$*.*

THEOREM 14.7. *(Lindeberg-Levy CLT) Let* $X_1, X_2, \cdots, X_n$ *be an IID random variable with* $E(X_i) = \mu, Var(X_i) = \sigma^2 < \infty$*, then* $n \to \infty$*,*

$$Y_n = \sqrt{n}\frac{\bar{X}_n - \mu}{\sigma} = \frac{1}{\sqrt{n}}\sum_{i=1}^n \frac{X_i - \mu}{\sigma} \to^d Z \sim N(0,1) \tag{14.114}$$

that is, $\forall y \in R, \lim_{n\to\infty} Pr(Y_n \le y) \to \Phi(y)$. *We can use normal distribution to approach* Y_n*, i.e.,* $Y_n \sim^a N(0,1)$*, so with slutsky theorem, we have* $\bar{X}_n \sim^a N(\mu, \frac{\sigma^2}{n})$ *which mean the consistent estimator of* $Var(\bar{X}_n)$*:* $a\hat{var}(\bar{X}_n) = \frac{\hat{\sigma}^2}{n}$ [10].

We extend the LL-CLT to multivariate version.

THEOREM 14.8. *(Multivariate Lindeberg-Levy CLT) Let* $X_1, X_2, \cdots, X_n$ *be an IID random vector series with* $E(X_i) = \mu, Var(X_i) = E[(X_i-\mu)(X_i-\mu)^\top] = \Sigma$ *(not singular). Let* $\Sigma = \Sigma^{1/2}\Sigma^{1/2T}$*, then*

$$\sqrt{n}\Sigma^{-1/2}(\bar{X}_n - \mu) \to^d Z \sim N(0, I) \tag{14.115}$$

then, we have $\sqrt{n}(\bar{X}_n - \mu) \sim^a N(0,\Sigma), \sqrt{n}\bar{X}_n \sim^a N(\mu,\Sigma), \bar{X}_n \sim^a N(\mu, n^{-1}\Sigma)$*, and then* $avar(\bar{X}_n) = n^{-1}\Sigma$*. If* Σ *is unknown, but* $\hat{\Sigma} \to^p \Sigma$*, then the consistent estimator of* $a\hat{var}(\bar{X}_n) = n^{-1}\hat{\Sigma}$*, e.g.,* $\hat{\Sigma} = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X}_n)(X_i - \bar{X}_n)^\top$.

THEOREM 14.9. *(Lindeberg-Feller CLT) Let* $X_1, X_2, \cdots, X_n$ *ne an independent random variable with* $E(X_i) = \mu_i, Var(X_i) = \sigma_i^2 < \infty$*. We define* $\bar{\mu}_n = \frac{1}{n}\sum_{i=1}^n \mu_i, \bar{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^n \sigma_i^2$*. If* $\lim_{n\to\infty} \max_i \frac{\sigma_i^2}{n\bar{\sigma}_n^2} = 0, \lim_{n\to\infty} \bar{\sigma}_n^2 = \bar{\sigma}^2 < \infty$*, then*

$$\sqrt{n}\frac{\bar{X}_n - \bar{\mu}_n}{\bar{\sigma}_n} \to^d Z \sim N(0,1) \tag{14.116}$$

$$\sqrt{n}(\bar{X} - \bar{\mu}_n) \to^d \bar{\sigma} Z \sim N(0, \bar{\sigma}^2) \tag{14.117}$$

We have an equivalent CLT of Lindeberg-Feller CLT as follows.

THEOREM 14.10. *(Liapounov's CLT) Let* $X_1, X_2, \cdots, X_n$ *be an independent random variable with* $E(X_i) = \mu_i, Var(X_i) = \sigma_i^2 < \infty$*. We define* $\bar{\mu}_n = \frac{1}{n}\sum_{i=1}^n \mu_i, \bar{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^n \sigma_i^2$*. If* $\exists \delta > 0, E[|X_i - \mu_i|^{2+\delta}] \le M < \infty$ *and* $\exists N \in \mathbb{N}, \forall n > N, \bar{\sigma^2}_n$ *is positive and bounded, then*

$$\sqrt{n}\frac{\bar{X}_n - \bar{\mu}_n}{\bar{\sigma}_n} \to^d Z \sim N(0,1) \tag{14.118}$$

$$\sqrt{n}(\bar{X} - \bar{\mu}_n) \to^d \bar{\sigma} Z \sim N(0, \bar{\sigma^2}) \tag{14.119}$$

$$\lim_{n\to\infty} \bar{\sigma}_n^2 = \bar{\sigma^2} < \infty \tag{14.120}$$

DEFINITION 14.12. *(Asymptotic Normality) A consistent estimator* $\hat{\theta}$ *follows asymptotic normality iff* $\sqrt{n}(\hat{\theta} - \theta) \to^d N(0,\Sigma)$ *iff* $\hat{\theta} \sim^a N(\theta, n^{-1}\Sigma), avar(\hat{\theta}) = n^{-1}\Sigma$*. If* $\hat{\Sigma} \to^p \Sigma$*, then* $\hat{\theta} \sim^a N(\theta, n^{-1}\hat{\Sigma}), a\hat{var}(\hat{\theta}) = n^{-1}\hat{\Sigma}$.

We then extend the Slutsky theorem to convergence in distribution.

THEOREM 14.11. *Let* $\{Y_n\}, \{Z_n\}$ *be random variable series,* W *be a random variable and* c *be a constant with* $Y_n \to^d W, Z_n \to^p c$*. Then*

- $Z_n Y_n \to^d cW$;
- $c \ne 0, Y_n/Z_n \to^d W/c$;
- $Y_n + Z_n \to^d W + c$.

Note that $Y_n \to^d W, Z_n \to^d Z$ *cannot induce* $Y_n + Z_n \to^d W + Z$ *as they maybe related.*

Beyond CLTs, two fancy methods are available to induce asymptotic distribution, CMT and the Delta Method.

THEOREM 14.12. *(Continuous Mapping Theorem, CMT) Assume* $h : R \to R$ *is continuous everywhere and* $Y_n \to^d W$*. Then* $\lim_{n\to\infty} h(Y_n) \to^d h(W)$

This is one of the most popular theorems in statistics.

THEOREM 14.13. *(The Delta Method) Assume* $\hat{\theta}$ *is a consistent estimator of* θ*,* $\sqrt{n}(\hat{\theta} - \theta) \to^d W \sim N(0,\sigma^2)$*. For a given function* g*,* $\eta = g(\theta)$*, if* $g : R \to R$ *is continuously differentiable at* θ *and* $g' = \frac{dg}{d\theta}$ *is continuous, then*

$$\sqrt{n}(\hat{\eta} - \eta) = \sqrt{n}(g(\hat{\theta}) - g(\theta)) \to^d W^* \sim N(0, g'(\theta)^2\sigma^2) \tag{14.121}$$

[10] $avar$ is the abbr. of asymptotic variance, we also use covariance denote the variance of a vector variable

or equivalently,

$$g(\hat{\theta}) \sim^a N(g(\theta), \frac{g'(\theta)^2\sigma^2}{n}) \tag{14.122}$$

Note that $avar(\hat{\eta}) = \frac{g'(\theta)^2\sigma^2}{n}$ *and the consistent estimator is* $a\hat{v}ar(\hat{\eta}) = a\hat{v}ar(g(\hat{\theta})) = \frac{g'(\hat{\theta})^2\hat{\sigma}^2}{n}$ *where* $\hat{\theta} \to^p \theta, \hat{\sigma}^2 \to^p \sigma^2$.

The Delta Method originates from Taylor expansion: $g(\hat{\theta}) = g(\theta) + g'(\tilde{\theta})(\hat{\theta}-\theta), \tilde{\theta} = \lambda\hat{\theta} + (1-\lambda)\theta, 0 \le \lambda \le 1$. *Then,* $\sqrt{n}[g(\hat{\theta}) - g(\theta)] = g'(\tilde{\theta})\sqrt{n}(\hat{\theta} - \theta)$. *As* $\tilde{\theta}$ *is between* $\hat{\theta}$ *and* θ, $\hat{\theta} \to^p \theta$, *then* $\tilde{\theta} \to^p \theta$. *With Slutsky theorem, we have* $g'(\tilde{\theta}) \to^p g'(\theta)$. *Therefore,*

$$\sqrt{n}(g(\tilde{\theta}) - g(\theta)) \to^d g'(\theta)N(0, \sigma^2) \sim N(0, g'(\theta)\sigma^2) \tag{14.123}$$

We can extend the delta method to multivariate. $\theta \in \mathbb{R}^k$ *and its consistent estimator is* $\hat{\theta}$, $\sqrt{n}(\hat{\theta} - \theta) \to^d N(0, \Sigma), \hat{\theta} \sim^a N(\theta, n^{-1}\Sigma)$. *Let* $\eta = g(\theta) : \mathbb{R}^k \to \mathbb{R}^j, j \le k$. *If* $g(\theta)$ *is continuous and its first order derivatives are continuous, we define* $\frac{\partial g(\theta)}{\partial \theta^\top}{}_{j \times k}$. *Then,*

$$\sqrt{n}(\hat{\eta} - \eta) = \sqrt{n}(g(\hat{\theta}) - g(\theta)) \to^d N(0, (\frac{\partial g(\theta)}{\partial \theta^\top})\Sigma(\frac{\partial g(\theta)}{\partial \theta^\top})^\top) \tag{14.124}$$

With $\hat{\Sigma} \to^p \Sigma$, *we can replace* Σ *to the consistent estimator, i.e.,*

$$g(\hat{\theta}) \sim^a N(g(\theta), \frac{1}{n}(\frac{\partial g(\hat{\theta})}{\partial \theta^\top})\hat{\Sigma}(\frac{\partial g(\hat{\theta})}{\partial \theta^\top})^\top) \tag{14.125}$$

$$a\hat{v}ar(g(\hat{\theta})) = \frac{1}{n}(\frac{\partial g(\hat{\theta})}{\partial \theta^\top})\hat{\Sigma}(\frac{\partial g(\hat{\theta})}{\partial \theta^\top}) \tag{14.126}$$

Much data is time series, which is surely not independent with each other. Luckily, we have developed tools to analyze it.

DEFINITION 14.13. *(Stochastic Process)* $\{Y_n\}_{n=1}^{\infty}$ *is a random variable series. Its realizations are observed data series* $\{y_t\}_{t=1}^{\infty} = \{\cdots, Y_1 = y_1, Y_2 = y_2, \cdots, Y_n = y_n, \cdots\}$.

If $\forall t_0$ *as* $N, T \to \infty$, $\frac{1}{N}\sum_{k=1}^{N} Y_{t_0}^{(k)} \to \frac{1}{T}\sum_{t=1}^{T} Y_t$, *we say the stochastic process is* ***ergodic***. *Thus, we can view the ergodic stochastic process as random sample with* N *big enough.*

DEFINITION 14.14. *(Strict Stationary)* $\forall r \in \mathbb{N}$ *and* $\{t_1, t_2, \cdots, t_r\}$, *the joint distribution* $(Y_t, Y_{t_1}, \cdots, Y_{t_r})$ *depends on* $t_1 - t, t_2 - t, \cdots, t_r - t$, *instead of* t. *Then the stochastic process* $\{Y_t\}_{t=1}^{\infty}$ *is* ***strict stationary***.

$\{Y_t\}_{t=1}^{\infty}$ *is strict stationary implying:(1)* $\forall t, E(Y_t), Var(Y_t)$ *does not change; (2)* $\forall g(\cdot), g(Y_t)$ *is also strict stationary; (3) If* $\{Y_t\}_{t=1}^{\infty}$ *is IID, and independent with* $X \sim N(0,1)$, *then* $\{Z_t\} = \{Y_t + X\}$ *is strict stationary.*

DEFINITION 14.15. *(Covariance or Weak Stationary) For the stochastic process* $\{Y_t\}_{t=1}^{\infty}$, *if* $E(Y_t) = \mu$ *independent with* t *and* $Cov(Y_t, Y_{t-j}) = \gamma_j$ *exists, bounded and only dependent on* $j = 0, 1, 2, \cdots$ *instead of* t, *then* $\{Y_t\}_{t=1}^{\infty}$ *is covariance or weak stationary.*

If the strict stationary process $\{Y_t\}_{t=1}^{\infty}$ *with existed mean and bounded variance for all* t, *then* $\{Y_t\}_{t=1}^{\infty}$ *is covariance stationary, which is decided by mean, variance and covariance only and uniquely.*

Then we connect covariance stationary and ergodic by the following proposition.

LEMMA 14.26. *(Ergodic and Stationary process' LLN) Let* $\{Y_t\}$ *be a covariance stationary process with* $E(Y_t) = \mu, Cov(Y_t, Y_{t-j}) = \gamma_j$. *If*

$$\sum_{j=0}^{\infty} |\gamma_j| < \infty \tag{14.127}$$

then $\{Y_t\}$ *is ergodic for the mean, i.e.,* $\bar{Y}_t \to^p E(Y_t) = \mu$.

Example 14.2. *(MA(1) is ergodic for the mean)* $MA(1) : Y_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1}, |\theta| < 1, \varepsilon_t \sim IID(0, \sigma^2)$. *Then,* $E(Y_t) = \mu, \gamma_0 = \sigma^2(1+\theta^2), \gamma_1 = \sigma^2\theta, \gamma_k = 0, k > 1$. *So,* $\sum_{j=0}^{\infty}|\gamma_j| = \sigma^2(1 + \theta^2 + |\theta|) < \infty$, $\{Y_t\}$ *is variance stationary and ergodic for the mean.*

THEOREM 14.14. *(Ergodic Theorem) Let* $\{Y_t\}$ *be an ergodic and covariance stationary stochastic process with* $E(Y_t) = \mu$, *then* $\bar{Y} = \frac{1}{T}\sum_{t=1}^{T} Y_t \to^p E(Y_t) = \mu$.

Ergodic theorem is an extension of Kolmogorov's LLN for stochastic process, with series relation allowed. Any transformation $\{g(Y_t)\}$ *is ergodic and covariance stationary. Thus, if* $E(g(Y_t))$ *exists, we have* $\bar{g} = \frac{1}{T}\sum_{t=1}^{T} g(Y_t) \to^p E[g(Y_t)]$, *which have wide applications, e.g., proving* $\hat{\gamma}_j \to \gamma_j$.

Let $\{Y_t\}$ be a random variable series, $\{I_t\}$ be the information set series. $\forall t, I_t \subseteq I$, where I is the full information set. The conditional expectation is defined as: $E(Y_t|I_s) = \int_{-\infty}^{+\infty} y_t f(y_t|I_s) dy_t$ where $s < t$ and Y_t. The law of iterated expectation tells us: if $I_1 \subseteq I_2$, Y is a random variable and $E[Y|I_1], E[Y|I_2]$ exist, then $E[Y|I_1] = E[E[Y|I_2]|I_1]$. If $I_1 = \varnothing$, $E[Y|I_1] = E[Y] = E[E[Y|I_2]]$.

DEFINITION 14.16. *(Martingales) (Y_t, I_t) is martingales if*

- $I_t \subseteq I_{t+1}, \forall t$ *(information accumulation);*
- $Y_t \subseteq I_t$ *(Y_t is adapted to I_t);*
- $E[|Y_t|] < \infty$*;*
- $E[Y_t|I_{t-1}] = Y_{t-1}$ *(key).*

If (Y_t, I_t) is a martingale, then $\forall m > 0, E[Y_{t+m}|I_t] = E[E[Y_{t+m}|I_{t+m-1}]|I_t] = E[Y_{t+m-1}|I_t] = \cdots = E[Y_{t+1}|I_t] = Y_t$.

Example 14.3. *(Random Walk) Let $Y_t = Y_{t-1} + u_t$, where u_t is an IID series with $E(u_t) = 0, Var(u_t) = \sigma^2$ and $I_t = \{Y_1, Y_2, \cdots, Y_t\}$. Then, $E(Y_t|I_{t-1}) = Y_{t-1}$, that means (Y_t, I_t) is is a martingale. Moreover, we can replace u_t with $v_t = u_t/t$, which induces a heteroscedastic random walk, which is also a martingale as expectation does not change.*

DEFINITION 14.17. *(Martingales Difference Sequence, MDS) (u_t, I_t) is a martingale difference sequence if (u_t, I_t) is adapted sequence and $E(u_t|I_{t-1}) = 0$. Easily to induce, $E[u_{t+m}|I_t] = 0, \forall m$. We can transform a martingale (Y_t, I_t) to MDS (u_t, I_t) by defining $u_t = Y_t - E[Y_t|I_{t-1}] = Y_t - Y_{t-1}$.*
Further, let $t \leq s$ $E[u_t u_s] = E[u_t E[u_s|I_t]] = E[u_t 0] = 0$.

THEOREM 14.15. *(Multivariate CLT for stationary and ergodic MDS) If (u_t, I_t) is stationary and ergodic MDS with $E[u_t u_t^\top] = \Sigma_{k\times k}$, let $\bar{u} = \frac{1}{T}\sum_{t=1}^{\top} u_t$. Then, $\sqrt{T}\bar{u} = \frac{1}{\sqrt{T}}\sum_{t=1}^{\top} u_t \to^d N(0, \Sigma)$ or $\bar{u} \sim^a N(0, T^{-1}\Sigma)$.*

How to choose between LLN and CLT for Y_n ? As $n \to \infty$, if the covariance of Y_n converges to O, then, LLN works; if the covariance of Y_n converges to a deterministic matrix, which is always assumed beforehand, then CLT works.

15 Ordinary Least Squares

LEMMA 15.1. *(OLS Estimation) Let $rank(X_{n\times k}) = k, y_{n\times 1}$, the Ordinary Least Square estimate $\hat{\beta}$ minimize*

$$\varphi(\beta) = (y - X\beta)^\top (y - X\beta) \tag{15.1}$$

Show $\hat{\beta} = (X^\top X)^{-1} X^\top y$.

Proof. F.O.C.,

$$\begin{aligned} d\varphi &= 2(y - X\beta)^\top d(y - X\beta) && (15.2)\\ &= -2(y - X\beta)^\top X d\beta && (15.3)\\ &= 0 && (15.4)\\ \Rightarrow X^\top (y - X\beta) &= 0 && (15.5) \end{aligned}$$

Notice, $X^\top u = 0$ is the F.O.C. of that $\hat{\beta}$ minimizes $\varphi(\beta)$.
As $rank(X) = k$, so $X^\top X$ is invertible, then

$$\hat{\beta} = (X^\top X)^{-1} X^\top y \tag{15.6}$$

Furthermore, assume Ω is symmetric and full rank, let $\hat{\beta}$ minimize a weighted summation of squared residuals,

$$\varphi(\beta) = (y - X\beta)^\top \Omega^{-1} (y - X\beta) \tag{15.7}$$

F.O.C.,

$$d\varphi = (d(y - X\beta)^\top)\Omega^{-1}(y - X|\beta) + (y - X\beta)^\top\Omega^{-1}d(y - X\beta) \tag{15.8}$$
$$= -(d\beta)^\top X^\top\Omega^{-1}(y - X\beta) + (y - X\beta)^\top\Omega^{-1}(-X)d\beta \tag{15.9}$$
$$= -2(d\beta)^\top X^\top\Omega^{-1}(y - X\beta) \tag{15.10}$$
$$= 0 \tag{15.11}$$
$$\Rightarrow X^\top\Omega^{-1}(y - X\beta) = 0 \tag{15.12}$$
$$\hat{\beta} = (X^\top\Omega^{-1}X)^{-1}X^\top\Omega^{-1}y \tag{15.13}$$

□

Note that we used $d\varphi = a^\top b + b^\top a = 2a^\top b$ as $a^\top b, b^\top a$ are both scalars.

LEMMA 15.2. *(Constrained OLS) Let Ω be symmetric and full rank, and $\tilde{\beta}$ solve*

$$\min_{\beta}\varphi(\beta) = (y - X\beta)^\top\Omega^{-1}(y - X\beta) \tag{15.14}$$
$$s.t. R^\top\beta = c \tag{15.15}$$

where R satisfies $rank(R_{k\times r}) = r$. Show $\tilde{\beta}$ with $\hat{\beta}$ (the unconstrained ols estimate).

Proof. The Lagrange function

$$\psi(\beta) = \frac{1}{2}(y - X\beta)^\top\Omega^{-1}(y - X\beta) - I^\top(R^\top\beta - c) \tag{15.16}$$

where $I_{r\times 1}$ is the Lagrange multiplier vector. Thus

$$d\psi = (y - X\beta)^\top\Omega^{-1}d(y - X\beta) - I^\top R^\top d\beta \tag{15.17}$$
$$= -[(y - X\beta)^\top\Omega^{-1}X + I^\top R^\top]d\beta \tag{15.18}$$

so, F.O.C.,

$$X^\top\Omega^{-1}X\beta - RI = X^\top\Omega^{-1}y \tag{15.19}$$
$$R^\top\beta = c \tag{15.20}$$

First we solve I out and note that $(X^\top\Omega^{-1}X)^{-1}$ is not singular.

$$X^\top\Omega^{-1}X\beta - RI = X^\top\Omega^{-1}y \tag{15.21}$$
$$R^\top(X^\top\Omega^{-1}X)^{-1}X^\top\Omega^{-1}X\beta - R^\top(X^\top\Omega^{-1}X)^{-1}RI = R^\top(X^\top\Omega^{-1}X)^{-1}X^\top\Omega^{-1}y \tag{15.22}$$
$$R^\top\beta - R^\top(X^\top\Omega^{-1}X)^{-1}RI = R^\top(X^\top\Omega^{-1}X)^{-1}X^\top\Omega^{-1}y \tag{15.23}$$

note that $R^\top\beta = c, (X^\top\Omega^{-1}X)^{-1}X^\top\Omega^{-1}y = \hat{\beta}$(an unconstrained ols estimate).

$$c - R^\top(X^\top\Omega^{-1}X)^{-1}RI = R^\top\hat{\beta} \tag{15.24}$$
$$(X^\top\Omega^{-1}X)^{-1}RI = c - R^\top\hat{\beta} \tag{15.25}$$
$$\tilde{I} = (R^\top(X^\top\Omega^{-1}X)^{-1}R)^{-1}(c - R^\top\hat{\beta}) \tag{15.26}$$

with Equation 15.19, we say

$$X^\top\Omega^{-1}X\beta = RI + X^\top\Omega^{-1}y \tag{15.27}$$
$$\beta = (X^\top\Omega^{-1}X)^{-1}(RI + X^\top\Omega^{-1}y) \tag{15.28}$$
$$\tilde{\beta} = (X^\top\Omega^{-1}X)^{-1}(R\tilde{I} + X^\top\Omega^{-1}y) \tag{15.29}$$
$$= \hat{\beta} + (X^\top\Omega^{-1}X)^{-1}R\tilde{I} \tag{15.30}$$
$$= \hat{\beta} + (X^\top\Omega^{-1}X)^{-1}R(R^\top(X^\top\Omega^{-1}X)^{-1}R)^{-1}(c - R^\top\hat{\beta}) \tag{15.31}$$

Lastly, the constrained ols estimators are

$$\begin{cases} \tilde{I} = (R^\top(X^\top\Omega^{-1}X)^{-1}R)^{-1}(c - R^\top\hat{\beta}) \\ \tilde{\beta} = \hat{\beta} + (X^\top\Omega^{-1}X)^{-1}R\tilde{I} \end{cases} \tag{15.32}$$

where $\hat{\beta} = (X^\top\Omega^{-1}X)^{-1}X^\top\Omega^{-1}y$. □

15.1 The Geometry of Linear Regression

DEFINITION 15.1. *(Column Space or Span) Let $X_{n\times k} = [x_1, x_2, \cdots, x_k]$, then the column space (span) of X is defined as*

$$\mathscr{S}(X) \equiv \mathscr{S}(x_1, \cdots, x_k) \equiv \{z \in E^n | z = \sum_{i=1}^{k} b_i x_i, b_i \in \mathbb{R}\} = Xb \tag{15.33}$$

that is, every vector in $\mathscr{S}$ is a linear combination of $x_1, x_2, \cdots, x_k$.

DEFINITION 15.2. *(Orthogonal Complement) The orthogonal complement of $\mathscr{S}(X)$ in E^n, denoted as $\mathscr{S}^{\perp}(X)$, is*

$$\mathscr{S}^{\perp}(X) \equiv \{w \in E^n | w^{\top} z = 0, \forall z \in \mathscr{S}(X)\} \tag{15.34}$$

If the dimension of $\mathscr{S}(X)$ is k, then the dimension of $\mathscr{S}^{\perp}(X)$ is $n-k$.

DEFINITION 15.3. *(Linearly Dependent) A set of vectors $x_1, x_2, \cdots, x_k$ is linearly dependent if there exists coefficients $b_i, i = 1, 2, \cdots, k$ such that*

$$\sum_{i=1}^{k} b_i x_i = \mathbf{0} \tag{15.35}$$

and $\sum_{i=1}^{k} b_i^2 \neq 0$. Let $X = [x_1, x_2, \cdots, x_k], b = [b_1, b_2, \cdots, b_n]'$, then (15.35) *becomes*

$$Xb = \mathbf{0} \tag{15.36}$$

and $b \neq \mathbf{0}$.

LEMMA 15.3. *If the columns of X are linearly independent if and only if the matrix $X^{\top}X$ is invertible.*

Proof. (1) Only if part: if the columns of X are linearly dependent, then the matrix $X^{\top}X$ is not invertible.

Columns of X are linearly dependent imply $\exists b$ s.t. $Xb = \mathbf{0}$ and $b \neq \mathbf{0}$. Then

$$X^{\top}Xb = \mathbf{0} \tag{15.37}$$

Suppose $(X^{\top}X)^{-1}$ exists, so

$$b = Ib = (X^{\top}X)^{-1}(X^{\top}X)b = \mathbf{0} \tag{15.38}$$

violating $b \neq \mathbf{0}$.

(2) If part: if the matrix $X^{\top}X$ is not invertible, then the columns of X are linearly dependent.

Let b minimize

$$\varphi(b) = (Xb)^{\top}Xb \tag{15.39}$$

So $d\varphi = d(Xb)^{\top}(Xb) = (Xb)^{\top}d(Xb) + d(Xb)^{\top}Xb = (Xb)^{\top}Xdb + (Xdb)^{\top}Xb = 2(Xb)^{\top}Xdb$, F.O.C.,

$$(Xb)^{\top}X = \mathbf{0} \tag{15.40}$$

$$(X^{\top}X)b = \mathbf{0} \tag{15.41}$$

As $X^{\top}X$ is not invertible, so $b = \mathbf{0}$. As $(Xb)^{\top}Xb \geq 0$, so only when $b = \mathbf{0}$, we have $Xb = \mathbf{0}$, i.e., columns of X are linearly dependent.

In other words, if columns of X are linearly independent, then $X^{\top}X$ is invertible. □

Further, if Ω is a symmetric and full rank square, then can be decomposed as $\Omega = DD^{\top}$, where D is full rank square, so columns of $D^{\top}X$ are linearly independent, $(D^{\top}X)^{\top}D^{\top}X = X^{\top}\Omega X$ is invertible. (Need CHECK)

Let $X = [X_1, X_2]$, if $Rank(X) = Rank(X_1) = k_1$, then $\mathscr{S}(X) = \mathscr{S}(X_1)$.

Now we always assume the columns of any regressor matrix X are linearly independent (so $X^{\top}X$ is invertible).

For a regression model,

$$y = X\beta + u \tag{15.42}$$

The moment condition is

$$X^{\top}(y - X\beta) = \mathbf{0} \tag{15.43}$$

and the OLS estimate is

$$\hat{\beta} = (X^{\top}X)^{-1}X^{\top}y \tag{15.44}$$

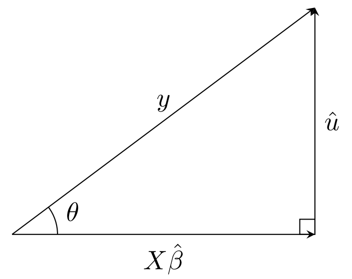


Figure 32: Orthogonal Decomposition of y

The fitted value is

$$\hat{y} = X\hat{\beta} = X(X^\top X)^{-1}X^\top y \tag{15.45}$$

and the residual is

$$\hat{u} = y - \hat{y} = (I - X(X^\top X)^{-1}X^\top)y \tag{15.46}$$

$\hat{y}$ is a linear combination of columns of X ($(X^\top X)^{-1}X^\top y$ is a vector), so $\hat{y} \in \mathscr{S}(X)$. We define the projection matrix

$$P_X = X(X^\top X)^{-1}X^\top \tag{15.47}$$

to show this relation, then $\hat{y} = P_X y$. Also, $\hat{u} \in \mathscr{S}^\perp(X)$, then we define then external projection matrix

$$M_X = I - P_X = I - X(X^\top X)^{-1}X^\top \tag{15.48}$$

Thus we can divide y into two orthogonal parts

$$y = P_X y + M_X y \tag{15.49}$$

We say $\mathscr{S}(X)$ is the image of P_X and $\mathscr{S}^\perp(X)$ is the image of M_X respectively.

LEMMA 15.4. *For any X with full rank, P_X and M_X are idempotent, symmetric and annihilate with each other (they are orthogonal).*

Proof. (1) idempotent. $P_X P_X = X(X^\top X)^{-1}X^\top X(X^\top X)^{-1}X^\top = X(X^\top X)^{-1}(X^\top X)(X^\top X)^{-1}X^\top = X(X^\top X)^{-1}X^\top = P_X, M_X M_X = (I - P_X)(I - P_X) = I - P_X - P_X + P_X P_X = I - P_X = M_X$.

(2) symmetric. $P_X^\top = (X(X^\top X)^{-1}X^\top)^\top = X(X^\top X)^{-1}X^\top = P_X, M_X^\top = (I - P_X)^\top = I - P_X = M_X$.

(3) annihilate. $P_X M_X = P_X(I - P_X) = P_X - P_X = \mathbf{O}$ which is a zero matrix. □

Since $P_X y$ and $M_X y$ are in two orthogonal spaces, the square of (15.49) becomes

$$||y||^2 = ||P_X y||^2 + ||M_X y||^2 \tag{15.50}$$

which implies

$$TSS = ESS + SSR \tag{15.51}$$

i.e., the total sum of squares (TSS) is equal to the explained sum of squares (ESS), plus the sum of squared residuals (SSR).

$P_X y$ is the projection of y respect to $\mathscr{S}(X)$ plane, it is naturally to take the cosine of the projection angle as the goodness of fit, see Figure 32,

$$R_u^2 = cos(\theta) = \frac{||P_X y||^2}{||y||^2} = \frac{ESS}{TSS} = 1 - \frac{SSR}{TSS} \tag{15.52}$$

R_u^2 implies the how much percent of variation of y can be explained by X! But surprisingly, R_u^2 can be manipulated! See Example 15.1.

Example 15.1. *(Manipulation of R_u^2) Let $y = [0, 2]^\top, X = [1, 1]^\top$, with ols estimation, we have $\hat{\beta} = 1, \hat{u} = [-1, 1]^\top$, i.e.,*

$$\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{15.53}$$

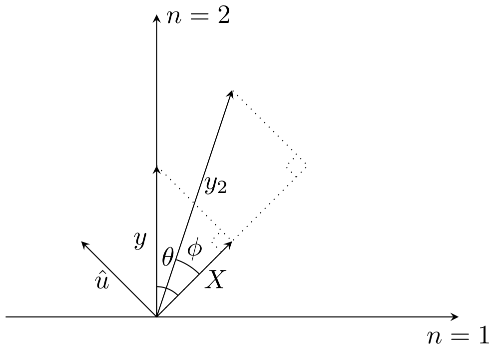


Figure 33: Manipulation of R_u^2

The $R^2 = cos^2\theta = \frac{\sqrt{2}}{2\cdot\sqrt{2}} = 1/2$*. Then, we add* $[1,1]^\top$ *to* y *(named* y_2*) and get*

$$\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}\cdot 2 + \begin{bmatrix}-1\\1\end{bmatrix} \tag{15.54}$$

Then, $R_u^2 = cos^2\phi = \frac{1+3}{\sqrt{10}\cdot\sqrt{2}} = \frac{2}{\sqrt{5}} > \frac{1}{2}$*. What happened? We improve the* R_u^2 *by just add the same number to every element of* y*! See Figure 33. Note that as we set a constant vector as* X *to plot* $X, y, \hat{u}$ *on a xoy space, the specific change of* y *just change the coefficient of constant, not* $\hat{u}$*. We cannot observe whether that transformation change coefficients of dependent variables, actually not!*

Let $\iota = [1, 1, \cdots, 1]^\top, c \in \mathbb{R}$, formally, the initial model is

$$\mathrm{y} = X\beta + \epsilon \tag{15.55}$$

where $X = [\iota, Z]$. The manipulation process is

$$y = \mathrm{y} + c\iota \tag{15.56}$$

$$y = X\beta + \varepsilon \tag{15.57}$$

$$\Rightarrow \mathrm{y} + c\iota = X\beta + \varepsilon \tag{15.58}$$

The fitted value is

$$P_X(\mathrm{y} + c\iota) = P_X\mathrm{y} + cP_XP_\iota\iota = P_X\mathrm{y} + cP_\iota\iota = P_X\mathrm{y} + c\iota \tag{15.59}$$

So the uncentered R_u^2 is

$$R_u^2 = \frac{||P_X\mathrm{y} + c\iota||^2}{||\mathrm{y} + c\iota||^2} \Rightarrow 1 \text{ as } c \to \infty \tag{15.60}$$

To avoid that manipulation, we center y before ols, i.e., $\mathrm{y} = M_\iota y = y - mean(y)$. The centered R square is

$$R_c^2 = \frac{||P_XM_\iota y||^2}{||M_\iota y||^2} \tag{15.61}$$

R_c^2 is what STATA gives in ols! If we also center X, i.e., the regressor becomes $M_\iota X$, we have to drop the constant term as $M_\iota\iota = 0$ to ensure $M_\iota X$ is not singular.

If ι is not a term of X, then $P_X(\mathrm{y} + c\iota) = P_X\mathrm{y} + cP_X\iota$, thus as long as $P_X\iota \neq 0$, the conclusion still holds.

Before proceed, we introduce a useful property of the projection matrix.

LEMMA 15.5. *Let* $X = [X_1, X_2]$*, then* $\mathscr{S}(X_i) \subseteq \mathscr{S}(X), i = 1, 2$*. Let*

$$P_1 \equiv P_{X_1} = X_1(X_1^\top X_1)^{-1}X_1^\top \tag{15.62}$$

Show $P_1P_X = P_XP_1 = P_1$ *and* $M_1M_X = M_XM_1 = M_X$*. Note that* X_1 *and* X_2 *are not orthogonal necessarily.*

Proof. Since $X_1 \in \mathscr{S}(X_1) \subseteq \mathscr{S}(X)$, $P_XX_1 = X_1$. Then

$$P_XP_1 = P_XX_1(X_1^\top X_1)^{-1}X_1^\top = X_1(X_1^\top X_1)^{-1}X_1^\top = P_1 \tag{15.63}$$

As P_1 and P_X are systemic, so $P_XP_1 = P_1$ implies $P_1 = (P_XP_1)^\top = P_1^\top P_X^\top = P_1P_X$, finished the first part.

$M_1M_X = (I - P_1)(I - P_X) = I - P_1 - P_X + P_1P_X = I - P_X = M_X$, and similarly $M_XM_1 = M_X$. Finished the proof. □

LEMMA 15.6. *Let $X = [X_1, X_2]$. Show $P_X - P_1 = P_{M_1 X_2}$.*

Proof. We show $\forall x \in \mathscr{S}(M_1X_2)$, $(P_X - P_1)x = x$ and $\forall w \in \mathscr{S}^{\perp}(M_1X_2)$, $(P_X - P_1)w = \mathbf{0}$.

(1) $x \in \mathscr{S}(M_1X_2)$ implies $\exists b$ s.t. $x = M_1X_2b$, so $(P_X - P_1)x = (P_X - P_1)M_1X_2b = (P_XM_1X_2 - P_1M_1X_2)b = P_XM_1X_2b = P_X(I - P_1)X_2b = (P_X - P_1)X_2b = (X_2 - P_1X_2)b = M_1X_2b$.

(2) $w \in \mathscr{S}^{\perp}(M_1X_2)$ implies $X_2^\top M_1 w = \mathbf{0}$ We want to show $(P_X - P_1)w = \mathbf{0}$.

$$
\begin{aligned}
X^\top M_1 w &= \begin{bmatrix} X_1^\top \\ X_2^\top \end{bmatrix} M_1 w = \mathbf{0} && (15.64)\\
X^\top (I - P_1)w &= \mathbf{0} && (15.65)\\
(P_X X)^\top (I - P_1)w &= \mathbf{0} && (15.66)\\
X^\top (P_X - P_1)w &= \mathbf{0} && (15.67)\\
X(X^\top X)^{-1}X^\top (P_X - P_1)w &= \mathbf{0} && (15.68)\\
P_X(P_X - P_1)w &= \mathbf{0} && (15.69)\\
(P_X - P_1)w &= \mathbf{0} && (15.70)
\end{aligned}
$$

Done. □

LEMMA 15.7. *(Linear Transformations of Regressors) Given $X_{n\times k}$ and $Rank(X) = k$. Let A be a nonsingular $k \times k$ matrix. A nonsingular linear transformation of X is*

$$
XA = [Xa_1, Xa_2, \cdots, Xa_k] \tag{15.71}
$$

Show that $\mathscr{S}(X) = \mathscr{S}(XA)$ and $P_{XA} = P_X, M_{XA} = M_X$.

Proof. (1) $\mathscr{S}(XA) \subseteq \mathscr{S}(X)$. $\forall z \in \mathscr{S}(XA), \exists b$ s.t. $z = XAb$, so $z = X(Ab) = Xc$, then $z \in \mathscr{S}(X)$. Note that $\mathscr{S}(XA) \subseteq \mathscr{S}(X)$ holds whether A is nonsingular.

$\mathscr{S}(X) \subseteq \mathscr{S}(XA)$. $\forall z \in \mathscr{S}(X)$, $\exists b$ s.t. $z = Xb$. As A is nonsingular, so $\exists A^{-1}$ s.t. $AA^{-1} = I$. Thus $z = Xb = XAA^{-1}b = (XA)(A^{-1}b) = (XA)c$, i.e., $z \in \mathscr{S}(XA)$.

Further, if $\mathscr{S}(X) = \mathscr{S}(Y), Rank(X) = Rank(Y)$, then $\exists A$ (nonsingular) s.t. $Y = XA$ (trivial from identical rank).

(2) Since $\mathscr{S}(X) = \mathscr{S}(XA)$, it is intuitive to see $P_{XA} = P_X$. Formally,

$$
\begin{aligned}
P_{XA} &= (XA)((XA)^\top XA)^{-1}(XA)^\top && (15.72)\\
&= XA(A^\top X^\top XA)^{-1}A^\top X^\top && (15.73)\\
&= XAA^{-1}(X\top X)^{-1}A^{\top -1}A^\top X^\top && (15.74)\\
&= X(X^\top X)^{-1}X^\top && (15.75)\\
&= P_X && (15.76)
\end{aligned}
$$

(3) $M_{XA} = I - P_{XA} = I - P_X = M_X$.

This lemma implies if two span is identical, then the corresponding projection matrix (external projection matrix) is also identical (also hold conversely trivially). □

Lemma 15.7 implies the following two regressions give the same fitted values and residuals:

$$
\begin{aligned}
y &= X\beta + u && (15.77)\\
y &= XA\gamma + u && (15.78)
\end{aligned}
$$

and $\hat{\beta} = A\hat{\gamma}$.

THEOREM 15.1. *(Frisch-Waugh-Lovell Theorem, FWL) Given two regression process,*

$$
\begin{aligned}
y &= X_1\beta_1 + X_2\beta_2 + u && (15.79)\\
M_1 y &= M_1X_2\beta_2 + M_1 u && (15.80)
\end{aligned}
$$

We have

1. *The estimator of β_2 in the both ols is the same in value.*

2. The residual of the both ols are the same in value.

Note that $M_1 y$(resp. $M_1 X_2$) is the residual of y(resp. X_2) regresses on X_1.

Proof. First we give the OLS estimator of Equation 15.80, named $\hat{\beta}^{(2)}$that

$$\begin{aligned}
\hat{\beta}^{(2)} &= ((M_1X_2)^\top M_1X_2)^{-1}(M_1X_2)^\top M_1y && (15.81)\\
&= (X_2^\top M_1X_2)^{-1}(M_1X_2)^\top M_1y && (15.82)\\
&= (X_2^\top M_1X_2)^{-1}X_2^\top M_1y && (15.83)
\end{aligned}$$

Now we say $y = P_Xy + M_Xy = X_1\hat{\beta}_1 + X_2\hat{\beta}_2 + M_Xy$,

$$\begin{aligned}
X_2^\top M_1y &= X_2^\top M_1X_1\hat{\beta}_1 + X_2^\top M_1X_2\hat{\beta}_2 + X_2^\top M_1M_Xy && (15.84)\\
&= X_2^\top M_1X_2\hat{\beta}_2 && (15.85)\\
\Rightarrow \hat{\beta}_2 &= (X_2^\top M_1X_2)^{-1}X_2^\top M_1y = \hat{\beta}^{(2)} && (15.86)
\end{aligned}$$

Note that we use $M_1X_1 = \mathbf{0}, X_2^\top M_1M_Xy = X_2^\top M_Xy = \mathbf{0}$. The former is the external projection of X_1 on itself, zero certainly! The later one hols as $X_2^\top$ is orthogonal with M_Xy. (note that residual is orthogonal with dependent variables)

Lastly, we prove the residuals do not change. With $y = P_Xy + M_Xy = X_1\hat{\beta}_1 + X_2\hat{\beta}_2 + M_Xy$, we have

$$\begin{aligned}
M_1y &= M_1X_1\hat{\beta}_1 + M_1X_2\hat{\beta}_2 + M_1M_Xy && (15.87)\\
&= M_1X_2\hat{\beta}_2 + M_Xy && (15.88)
\end{aligned}$$

Note that M_Xy is lucky to be the residuals of (15.80). Formally, residuals of (15.80) is

$$\begin{aligned}
M_{M_1X_2}M_1y &= (I - P_{M_1X_2})(I - P_1)y && (15.89)\\
&= (I - P_X + P_1)(I - P_1)y && (15.90)\\
&= (I - P_X)y && (15.91)\\
&= M_Xy && (15.92)
\end{aligned}$$

just the residuals of (15.79). □

Two step OLS with FWL theorem can help us to pretreat the independent variable, like centering, dropping seasonality and detrending.

LEMMA 15.8. *Consider the linear regression*

$$y = \beta_1\iota + X_2\beta_2 + u \qquad (15.93)$$

where ι is an n-vector of 1s, and X_2 is an $n \times (k-1)$ matrix of observations on the remaining regressors. Show using the FWL Theorem, that the OLS estimators of β_1 and β_2 can be written as

$$\begin{bmatrix}\hat{\beta}_1\\ \hat{\beta}_2\end{bmatrix} = \begin{bmatrix} n & \iota^\top X_2\\ \mathbf{0} & X_2^\top M_\iota X_2\end{bmatrix}^{-1}\begin{bmatrix}\iota^\top y\\ X_2^\top M_\iota y\end{bmatrix} \qquad (15.94)$$

where M_ι is the projection matrix derived from ι.

Proof. With FWL Theorem, we have

$$M_\iota y = M_\iota X_2\beta_2 + M_\iota u \qquad (15.95)$$

gives the same ols estimator of β_2

$$\hat{\beta}_2 = (X_2^\top M_\iota X_2)^{-1}X_2^\top M_\iota y \qquad (15.96)$$

The exogeneity condition implies $E[X^\top u] = 0$. So $\begin{cases}\iota^\top\hat{u} = 0\\ X_2^\top\hat{u} = 0\end{cases}$. Thus

$$\begin{aligned}
\iota^\top\hat{u} &= 0 && (15.97)\\
\iota^\top y - \iota^\top\hat{u} &= \iota^\top y && (15.98)\\
\iota^\top\hat{y} &= \iota^\top y && (15.99)\\
\iota^\top(\iota\hat{\beta}_1 + X_2^\top\hat{\beta}_2) &= \iota^\top y && (15.100)\\
\iota^\top\iota\hat{\beta}_1 + \iota^\top X_2\hat{\beta}_2 &= \iota^\top y && (15.101)
\end{aligned}$$

Combine (15.96) and (15.101), we obtain

$$\begin{cases} \iota^\top \iota \hat{\beta}_1 + \iota^\top X_2 \hat{\beta}_2 = \iota^\top y \\ X_2^\top M_\iota X_2 \hat{\beta}_2 = X_2^\top M_\iota y \end{cases} \tag{15.102}$$

$$\Rightarrow \begin{bmatrix} n & \iota^\top X_2 \\ \mathbf{0} & X_2^\top M_\iota X_2 \end{bmatrix} \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} \iota^\top y \\ X_2^\top M_\iota y \end{bmatrix} \tag{15.103}$$

$$\Rightarrow \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} n & \iota^\top X_2 \\ \mathbf{0} & X_2^\top M_\iota X_2 \end{bmatrix}^{-1} \begin{bmatrix} \iota^\top y \\ X_2^\top M_\iota y \end{bmatrix} \tag{15.104}$$

Note that we used $\iota^\top \iota = n$ here. □

LEMMA 15.9. *The bias caused by omitting variables. Let the full regression be*

$$y = X\beta + Z\gamma + u \tag{15.105}$$

and the ols estimator of β is $\tilde{\beta}$. Let the omitting variables regression be

$$y = X\beta + e \tag{15.106}$$

and the ols estimator of β is $\hat{\beta}$. Then the bias of β is given by

$$\tilde{\beta} - \hat{\beta} = (X^\top M_Z X)^{-1} X^\top M_Z M_X y \tag{15.107}$$

Proof. This is an useful and interesting problem. We can calculate the ols estimator of β directly with the FWL Theorem. That is

$$y = X\beta + Z\gamma + u \tag{15.108}$$
$$\Rightarrow M_Z y = M_Z X\beta + M_Z u \tag{15.109}$$
$$\Rightarrow \tilde{\beta} = ((M_Z X)^\top M_Z X)^{-1} (M_Z X)^\top M_Z y \tag{15.110}$$
$$= (X^\top M_Z X)^{-1} X^\top M_Z y \tag{15.111}$$
$$y = X\beta + e \tag{15.112}$$
$$\Rightarrow \hat{\beta} = (X^\top X)^{-1} X^\top y \tag{15.113}$$
$$\Rightarrow \tilde{\beta} - \hat{\beta} = (X^\top M_Z X)^{-1} X^\top M_Z y - (X^\top X)^{-1} X^\top y \tag{15.114}$$
$$(X^\top M_Z X)(\tilde{\beta} - \hat{\beta}) = X^\top M_Z y - X^\top M_Z X (X^\top X)^{-1} X^\top y \tag{15.115}$$
$$= X^\top M_Z [I - X(X^\top X)^{-1} X^\top] y \tag{15.116}$$
$$= X^\top M_Z [I - P_X] y \tag{15.117}$$
$$= X^\top M_Z M_X y \tag{15.118}$$
$$\Rightarrow \tilde{\beta} - \hat{\beta} = (X^\top M_Z X)^{-1} X^\top M_Z M_X y \tag{15.119}$$

Interestingly, this is just the ols estimator of

$$M_X y = M_Z X\beta + v \tag{15.120}$$

If $M_Z X = X$, u.e., $X \in \mathscr{S}^\perp(Z)$, then $\hat{\beta}_{bias} = 0$. However if $X \in \mathscr{S}(Z)$ (X and Z are multicollinear), then $M_Z X = 0$, so $\hat{\beta}_{bias}$ is arbitrary. □

Now we study influential observations and leverage. We are interested in the change of estimated coefficients if we drop an observation. Let

$$y = X\beta + \alpha e_t + u \tag{15.121}$$

where e_t is a unit basis vector with t-th element as 1 and other elements as 0. By the FWL Theorem, this regression is equivalent to

$$M_t y = M_t X\beta + residuals \tag{15.122}$$

where $M_t = M_{e_t} = I - e_t(e_t^\top e_t)^{-1} e_t^\top = I - e_t e_t^\top$. Here

$$M_t y = (I - e_t e_t^\top) y = y - y_t e_t \tag{15.123}$$

that is, t-th element of $M_t y$ is replaced by 0, so is $M_t X$. Let $\hat{\beta}^{(t)}$ denote the OLS estimate of (15.121). Let $Z = (X, e_t)$, then (15.121) implies

$$y = P_Z y + M_Z y = X\hat{\beta}^{(t)} + \hat{\alpha} e_t + M_Z y \tag{15.124}$$

Then premultiply P_X on both sides,

$$P_X y = X\hat{\beta}^{(t)} + \hat{\alpha} P_X e_t \tag{15.125}$$

where $P_X M_Z = \mathbf{O}$ as M_Z annihilates X and e_t. As $P_X y = X\hat{\beta}$, so

$$X(\hat{\beta}^{(t)} - \hat{\beta}) = -\hat{\alpha} P_X e_t \tag{15.126}$$

Now we derive $\hat{\alpha}$. By tht FWL Theorem, (15.121) is equivalent to

$$M_X y = \alpha M_X e_t + residuals \tag{15.127}$$

so

$$\hat{\alpha} = \frac{e_t^\top M_X y}{e_t^\top M_X e_t^\top} \tag{15.128}$$

where $e_t^\top M_X y$ is the t-th element of $M_X y$ and $e_t^\top M_X e_t^\top$ is the t-th diagonal element of M_X,

$$\hat{\alpha} = \frac{\hat{u}_t}{1 - h_t} \tag{15.129}$$

where $h_t = e_t^\top P_X e_t = ||P_X e_t||^2 \leq ||e_t||^2 = 1$, so $h_t \in [0, 1]$. Therefore (15.126) becomes

$$\hat{\beta}^{(t)} - \hat{\beta} = -\hat{\alpha}(X^\top X)^{-1} X^\top P_X e_t = -\frac{1}{1 - h_t}(X^\top X)^{-1} X_t^\top \hat{u}_t \tag{15.130}$$

The influence of observation t is decided by $h_t, \hat{u}_t$ and X_t itself. Observations with high h_t is called a high leverage point, but not influential necessarily. An important outcome of h_t is

$$\sum_{t=1}^{n} h_t = tr(P_X) = tr(X(X^\top X)^{-1} X^\top) \tag{15.131}$$

$$= tr(X^\top X(X^\top X)^{-1}) = tr(I_k) = k \tag{15.132}$$

15.2 Statistical Properties of Ordinary Least Squares

Assume the DGP,

$$y = X\beta_0 + u, u \sim IID(\mathbf{0}, \sigma_0^2 I) \tag{15.133}$$

and its OLS estimate

$$\hat{\beta} = (X^\top X)^{-1} X^\top y \tag{15.134}$$

replace y with β,

$$\hat{\beta} = (X^\top X)^{-1} X^\top (X\beta_0 + u) = \beta_0 + (X^\top X)^{-1} X^\top u \tag{15.135}$$

We now study the statistical properties of $\hat{\beta}$.

15.2.1 Unbias

Consider

$$y_t = X_t^\top \beta + u_t \tag{15.136}$$

where X_t is the transposed t-th row of X, then a column vector of $k \times 1$ dimension.

DEFINITION 15.4. *(Unbias) If $E[\hat{\beta}] = \beta_0$, then $\hat{\beta}$ is an unbiased estimate of β_0.*

The expectation of $\hat{\beta} - \beta_0$ is

$$E[\hat{\beta} - \beta_0] = E[\beta_0 + (X^\top X)^{-1} X^\top u - \beta_0] \tag{15.137}$$

$$= E[(X^\top X)^{-1} X^\top u] \tag{15.138}$$

To induce $E[(X^\top X)^{-1} X^\top u] = E[h(X)u] = 0$, we assume

$$E[u|X] = \mathbf{0} \tag{15.139}$$

which is sufficient (correct model specification), implying X is exogenous.

The construction of $\hat{\beta}$ implies the F.O.C.

$$X^\top u = 0 \tag{15.140}$$

must hold, which implies $E[X_t u_t] = 0$, i.e., X_t is predetermined (this is an outcome, not an assumption).

15.2.2 Consistency

DEFINITION 15.5. *(Consistency) If* $\underset{n\to\infty}{plim}\,\hat\beta = \beta_0$*, or equivalently,* $\hat\beta \to^p \beta_0$*, then* $\hat\beta$ *is consistency.*

$\hat\beta \to^p \beta_0$ is equivalent to $(X^\top X)^{-1}X^\top u \to^p \mathbf{0}$. Assume $\frac{1}{n}X^\top X \to^p E[X_t^\top X_t] = S$ and $E[u_t|X_t] = 0$, then

$$\underset{n\to\infty}{plim}\,\frac{1}{n}X^\top u = \underset{n\to\infty}{plim}\,\frac{1}{n}\sum_{t=1}^{n} X_t u_t = E[X_t u_t] = \mathbf{0} \tag{15.141}$$

$$\underset{n\to\infty}{plim}(X^\top X)^{-1}X^\top u = (\underset{n\to\infty}{plim}\,\frac{1}{n}(X^\top X))^{-1}(\underset{n\to\infty}{plim}\,\frac{1}{n}X^\top u) \tag{15.142}$$

$$= S^{-1}\mathbf{0} = \mathbf{0} \tag{15.143}$$

15.2.3 Efficiency

DEFINITION 15.6. *(Positive Semidefinite) Let* A *be a square. If* $\forall x \neq \mathbf{0}$*, we have* $x^\top Ax \geq 0$*, then* A *is positive semidefinite, further, if* $x^\top Ax > 0$*, then* A *is positive definite.*

LEMMA 15.10. *Any covariance matrix is positive semidefinite.*

Proof. Let b be any random vector with existing variance. $\forall x \neq \mathbf{0}$, $Var(x^\top b) = x^\top Var(b)x$. Since $x^\top b$ is a scalar, so $Var(x^\top b) \geq 0$, then $x^\top Var(b)x$. The equality holds if $x^\top b$ is invariant. □

LEMMA 15.11. *Any matrix* $B^\top B$ *is positive semidefinite.*

Proof. $\forall x \neq \mathbf{0}$, $x^\top B^\top x = (Bx)^\top(Bx) = ||Bx||^2 \geq 0$. The equality holds iff $Bx = \mathbf{0}$, i.e., columns of B are linear dependent.

Further, if A is positive definite, then $B^\top AB$ is positive semidefinite and further positive definite if B has full rank. □

LEMMA 15.12. *A matrix is positive* M *definite if and only if it is nonsingular.*

Proof. As preliminary transformation does not change the property of positive definite. (1) If M is nonsingular, then I is positive definite implies M is positive definite. (2) If M is singular, then $\exists b \neq 0$, s.t. $Mb = 0$, so $b^\top Mb = 0$, i.e., M is not positive definite. Done. □

LEMMA 15.13. *If* A *is symmetric and positive definite, then* $\exists B_{k\times k}$ *s.t.* $Rank(B) = k$ *and* $A = B^\top B$*. In particular,* B *can be chosen to be symmetric or upper / lower triangular.*

Proof. Omitted as it is out of scope. □

LEMMA 15.14. *Any external projection matrix is positive semidefinite.*

Proof. Let M_X be an external projection matrix. $\forall y \neq \mathbf{0}$, we have

$$y^\top M_X y = y^\top y - y^\top P_X y = y^\top y - (P_X y)^\top P_X y = ||y||^2 - ||P_X y||^2 \geq 0 \tag{15.144}$$

Thus M_X is positive semidefinite. □

LEMMA 15.15. *Lemma 15.14 can be extend to* $\forall Z$*, if* P *is an (external) projection matrix, then* $Z^\top PZ$ *is positive semidefinite.*

$$Z^\top PZ = Z^\top PP^\top Z = (P^\top Z)^\top(P^\top Z) \tag{15.145}$$

be positive semidefinite with Lemma 15.11.

LEMMA 15.16. *If* A *is a symmetric, positive definite matrix, then* A^{-1} *is also positive definite.*

Proof. $\forall x \neq \mathbf{0}$, $x^\top Ax = x^\top A^{-1}AA^{-1}x = (A^{-1}x)^{-1}A(A^{-1}x) > 0$ as A is positive definite. □

LEMMA 15.17. *If* $A_{k\times k}$ *is symmetric positive definite, then* $I - A$ *is positive definite iff* $A^{-1} - I$ *is positive definite.*

If A *and* B *are symmetric positive definite, then* $A - B$ *is positive definite iff* $B^{-1} - A^{-1}$ *is positive definite.*

Proof. A is symmetric positive definite, so $\exists R$ (symmetric) s.t. $A = R^2$, then $(R^{-1})^2 = A^{-1}, R^{-1}AR^{-1} = I$. $\forall x, x = R^{-1}z$,

$$x^\top(I-A)x = z^\top R^{-1}(I-A)R^{-1}z \tag{15.146}$$
$$= z^\top(R^{-1}R^{-1} - R^{-1}AR^{-1})z \tag{15.147}$$
$$= z^\top(A^{-1} - I)z \tag{15.148}$$

So $I - A$ is positive definite iff $A^{-1} - I$ is positive definite.

Let $A \in \mathscr{P}$ denote that A is positive definite, then

$$A - B \in \mathscr{P} \tag{15.149}$$
$$\Leftrightarrow R^{-1}(A-B)R^{-1} \in \mathscr{P} \tag{15.150}$$
$$\Leftrightarrow I - R^{-1}BR^{-1} \in \mathscr{P} \tag{15.151}$$
$$\Leftrightarrow RB^{-1}R - I \in \mathscr{P} \tag{15.152}$$
$$\Leftrightarrow B^{-1} - A^{-1} \in \mathscr{P} \tag{15.153}$$
$$\tag{15.154}$$

□

Assume X is exogenous. Since $u \sim IID(\mathbf{0}, \sigma_0^2 I)$, then $Var(u) = E[uu^\top] = \sigma_0^2 I$, then the variance of $\hat{\beta}$ is

$$Var[\hat{\beta}] = E[(\hat{\beta} - \beta_0)(\hat{\beta} - \beta_0)^\top] \tag{15.155}$$
$$= E[(X^\top X)^{-1}X^\top uu^\top X(X^\top X)^{-1}] \tag{15.156}$$
$$= E[E[(X^\top X)^{-1}X^\top uu^\top X(X^\top X)^{-1}|X]] \tag{15.157}$$
$$= E[(X^\top X)^{-1}X^\top E[uu^\top]X(X^\top X)^{-1}] \tag{15.158}$$
$$= \sigma_0^2 E[(X^\top X)^{-1}X^\top X(X^\top X)^{-1}] \tag{15.159}$$
$$= \sigma_0^2(X^\top X)^{-1} \tag{15.160}$$

Further,

$$Var(\hat{\beta}) = (\frac{1}{n}\sigma_0^2)(\frac{1}{n}X^\top X)^{-1} \tag{15.161}$$

We have assumed $\frac{1}{n}X^\top X \to^p S$, so $Var(\hat{\beta})$ is proportional to $\frac{1}{n}$, i.e., when the sample size increases, the variance of estimator decreases in the speed $\frac{1}{n}$.

THEOREM 15.2. *(Gauss-Markov Theorem) If $E[u|X] = \mathbf{0}$ and $E[uu^\top|X] = \sigma^2 I$, then the OLS estimator is more efficient than any unbiased linear estimator $\tilde{\beta}$, in the sense that $Var(\tilde{\beta}) - Var(\hat{\beta})$ is a positive semidefinite matrix.*

Proof. The DGP is given by (15.133). A linear estimator $\tilde{\beta}$ can be written as Ay, then

$$\tilde{\beta} = A(X\beta_0 + u) = AX\beta_0 + Au \tag{15.162}$$

$\tilde{\beta}$ is unbiased, so $E[AX\beta_0] = \beta_0$, i.e., $AX = I$ (take X as given). Let $C = A - (X^\top X)^{-1}X^\top$, then $\tilde{\beta} = Ay = (C + (X^\top X)^{-1}X^\top)y = Cy + \hat{\beta}$, i.e., $Cy = \tilde{\beta} - \hat{\beta}$. Thus $AX = I$ implies $CX = \mathbf{O}$.

$$E[(\hat{\beta} - \beta_0)(\tilde{\beta} - \hat{\beta})^\top] = E[(X^\top X)^{-1}X^\top uu^\top C^\top] \tag{15.163}$$
$$= (X^\top X)^{-1}X^\top \sigma_0^2 I C^\top \tag{15.164}$$
$$= \sigma_0^2(X^\top X)^{-1}X^\top C^\top \tag{15.165}$$
$$= \mathbf{O} \tag{15.166}$$

So

$$Var(\tilde{\beta}) = Var(\hat{\beta} + (\tilde{\beta} - \hat{\beta})) \tag{15.167}$$
$$= Var(\hat{\beta} + Cy) \tag{15.168}$$
$$= Var(\hat{\beta}) + Var(Cy) \tag{15.169}$$

Since a variance matrix is positive semidefinite, so $Var(\tilde{\beta}) - Var(\hat{\beta})$ is positive semidefinite. □

15.2.4 Asymptotic Distribution

Consider $y_t = X_t^\top \beta + \epsilon_t, t = 1, 2, \cdots, T, \beta = [\beta_1, \cdots, \beta_k]^\top$, where X_t is the transposed t-th row of X, thus a $k \times 1$ vector. We assume:

(a) $\{X_t, \epsilon_t\}$ is stationary and ergodic;

(b) $E[X_t X_t^\top] = \Sigma_{XX}$ is nonsingular, i.e., $Rand(\Sigma_{XX}) = k$;

(c) $E[X_{jt}\epsilon_t] = 0, \forall t, \forall j = 1, 2, \cdots, k$ (predetermined regressors);

(d) $\{g_t\} = \{X_t \epsilon_t\}$ is a MDS with $E[g_t g_t^\top] = E[X_t X_t^\top \epsilon_t^2] = S$ is not singular.

Assumption a implies $\{\epsilon_t\}$ is stationary, then $E(\epsilon_t^2) = \sigma^2$ is unconditional variance. Assumption b indicates non strict multicollinearity. Assumption d allows general conditional heteroscedasticity, e.g., $Var(\epsilon_t|X_t) = f(X_t)$, then we have to estimate S as a whole,

$$
\begin{aligned}
E[X_t X_t^\top \epsilon_t^2] &= E[E[X_t X_t^\top \epsilon_t^2 | X_t]] && (15.170)\\
&= E[X_t X_t^\top E[\epsilon_t^2 | X_t]] && (15.171)\\
&= E[X_t X_t^\top f(X_t)] && (15.172)\\
&= S && (15.173)
\end{aligned}
$$

If it is hemoscedasticity, then $E[X_t X_t^\top \epsilon_t^2] = E[E[X_t X_t^\top \epsilon_t^2 | X_t]] = \sigma^2 E[X_t X_t^\top] = \sigma^2 \Sigma_{XX} = S$.

The OLS estimator,

$$
\begin{aligned}
\hat{\beta} &= (\sum_{t=1}^{T} X_t X_t^\top)^{-1} \sum_{t=1}^{T} X_t y_t && (15.174)\\
&= (\sum_{t=1}^{T} X_t X_t^\top)^{-1} \sum_{t=1}^{T} X_t (X_t^\top \beta + \epsilon_t) && (15.175)\\
&= \beta + (\sum_{t=1}^{T} X_t X_t^\top)^{-1} \sum_{t=1}^{T} X_t \epsilon_t && (15.176)\\
\Rightarrow \hat{\beta} - \beta &= (\sum_{t=1}^{T} X_t X_t^\top)^{-1} \sum_{t=1}^{T} X_t \epsilon_t && (15.177)\\
&= (\sum_{t=1}^{T} X_t X_t^\top)^{-1} \sum_{t=1}^{T} g_t && (15.178)
\end{aligned}
$$

The expectation of $\sqrt{T}(\hat{\beta} - \beta)$ is

$$
E[\sqrt{T}(\hat{\beta} - \beta)] = E[\sqrt{T}(\sum_{t=1}^{T} X_t X_t^\top)^{-1} \sum_{t=1}^{T} X_t E[\epsilon_t | X_t]] = \mathbf{0} \quad (15.179)
$$

The variance of $\sqrt{T}(\hat{\beta} - \beta)$ is

$$
\begin{aligned}
Var[\sqrt{T}(\hat{\beta} - \beta)] &= Var[(\frac{1}{T}\sum_{t=1}^{T} X_t X_t^\top)^{-1} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} g_t] && (15.180)\\
&= E[(\frac{1}{T}\sum_{t=1}^{T} X_t X_t^\top)^{-1} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} g_t (\frac{1}{\sqrt{T}} \sum_{t=1}^{T} g_t)^\top [(\frac{1}{T}\sum_{t=1}^{T} X_t X_t^\top)^{-1}]^\top] && (15.181)\\
&= \Sigma_{XX}^{-1} E[\frac{1}{T}(\sum_{t=1}^{T} g_t)(\sum_{t=1}^{T} g_t)^\top](\Sigma_{XX}^{-1})^\top && (15.182)\\
&= \Sigma_{XX}^{-1} \frac{1}{T} \sum_{t=1}^{T} (E[g_t g_t^\top]) \Sigma_{XX}^{-1} && (15.183)\\
&= \Sigma_{XX}^{-1} S \Sigma_{XX}^{-1} && (15.184)
\end{aligned}
$$

Note that $E[g_t g_s] = \mathbf{O}, \forall t \neq s$ as g_t is MDS.

With assumption d, as $T \to \infty$, we have $\hat{\beta} \to^p \beta$ and $\sqrt{T}(\hat{\beta} - \beta) \to^d N(0, \Sigma_{XX}^{-1} S \Sigma_{XX}^{-1})$, which implies

$$\hat{\beta} \sim^a N(\beta, T^{-1}\Sigma_{XX}^{-1} S \Sigma_{XX}^{-1}) \tag{15.185}$$

Assume $\hat{\Sigma}_{XX} \to^p \Sigma_{XX}, \hat{S} \to^p S$, then

$$\hat{\beta} \sim^a N(\beta, T^{-1}\hat{\Sigma}_{XX}^{-1} \hat{S} \hat{\Sigma}_{XX}^{-1}) \tag{15.186}$$

where $\hat{\Sigma}_{XX}, \hat{S}$ can be estimated with sample covariance. That is famous White Heteroscedasticity Coherent estimator (White-HC) and we have

$$\hat{SE}_{HC}(\hat{\beta}_i) = \sqrt{[T(X^\top X)^{-1}\hat{S}(X^\top X)^{-1}]_{ii}}, i = 1, 2, \cdots, k \tag{15.187}$$

which just comes from the sample covariance of $\hat{\beta}$ by replacing $\hat{\Sigma}_{XX}$ with the sample covariance $\frac{1}{n}X^\top X$.

If errors are hemoscedasticity, i.e., $Var(\varepsilon_t|X_t) = \sigma^2, S = \sigma^2\Sigma_{XX}$ then

$$a\hat{v}ar(\hat{\beta}) = T^{-1}\Sigma_{XX}^{-1} S \Sigma_{XX}^{-1} = T^{-1}\sigma^2\Sigma_{XX}^{-1} \tag{15.188}$$

We just need to estimate $\sigma^2, \Sigma_{XX}^{-1}$. As $S_{XX} = T^{-1}X^\top X \to^p \Sigma_{XX}$ and $\hat{\sigma}^2 = T^{-1}\hat{\epsilon}^\top\hat{\epsilon} \to^p \sigma^2$ ($\hat{\epsilon} = y - X\hat{\beta}$ sample covariance and variance), we obtain $a\hat{v}ar(\hat{\beta}) = \hat{\sigma}^2(X^\top X)^{-1}$ (consistent).

Finally, we need to explain the consistent estimator of $S = E[g_t g_t^\top] = E[\varepsilon_t^2 X_t X_t^\top]$. If $E[(X_{ik}X_{ij})^2], \forall t, j = 1, 2, \cdots, k$ exist and bounded, then as $T \to \infty$

$$\hat{S} = \frac{1}{T}\sum_{t=1}^{T} \hat{\epsilon}_t^2 X_t X_t^\top \to^p S \tag{15.189}$$

where $\hat{\epsilon}_t = y_t - X_t\hat{\beta}$. Davidon and MacKinnon (1993) suggest replace T with $T - k$ to fix the degree of freedom, which has better characteristics for finite sample.

16 Hypothesis Test and Confidence Interval

16.1 A Brief Review of Hypothesis Test

We are interested in the real β_0, not available but can be estimated. The estimator $\hat{\beta}$ is a random variable, and two methods are capable of inferring, hypothesis test and confidence intervals (the same in nature). Basic conceptions include:

- Null hypothesis H_0, alternative hypothesis H_1;
- Hypothesis statistics: a random variable with distribution known if H_0 is true;
- Test types: unilateral test and bilateral test;
- Rejection region;
- Test;
- Precise test: with precise distribution of hypothesis statistics;
- Type I error: reject the true H_0 (significance level α);
- Type II error: fail to reject the wrong H_0;
- Size of hypothesis: the supremum of reject probability of all data generating process fulfilling H_0. For the precise test, it is the significance level α;
- Power of hypothesis: the probability of rejecting H_0. For the precise test, it is the significance level α;
- P value (marginal significance level): the minimum of α to reject H_0.

Table 4: Four popular distributions

	construction	note
Multivariate Normal Distribution	$x \sim N(\mu, \Omega)$	$a^\top x \sim N(a^\top \mu, a^\top \Omega a)$ if a is a constant vector
Chi-square Distribution	$x \sim N(0, \Omega), x^\top \Omega^{-1} x \sim \chi^2(m)$	If P is a projection matrix with rank r, $z \sim N(0, I_n)$, then $z^\top P z \sim \chi^2(r)$
Student-t Distribution	independent $z \sim N(0,1), y \sim \chi^2(m)$ then $t \equiv \frac{z}{\sqrt{y/m}} \sim t(m)$	
F Distribution	independent $y_1 \sim \chi^2(m_1), y_2 \sim \chi^2(m_2)$ then $F \equiv \frac{y_1/m_1}{y_2/m_2} \sim F(m_1, m_2)$	

Four popular distributions are summarized in Table 4.

Here we give a brief proof to the construction of $\chi^2(m)$. The standard construction of $\chi^2(m)$ is $\sum_{i=1}^m x_i, x_i \sim N(0,1), \forall i$ and x_i, x_j are independent for all $i \neq j$.

LEMMA 16.1. *If the m-vector x is distributed as $N(0, \Omega)$, then the quadratic form $x^\top \Omega^{-1} x$ is distributed as $\chi^2(m)$.*

If P is a projection matrix with rank r and z is an n-vector that is distributed as $N(0, I)$, then the quadratic form $z^\top P z$ is distributed as $\chi^2(r)$.

Proof. (1) Ω can be decomposed as $\Omega = AA^\top$ where A is invertible. Since $E(x) = 0$, so $E[A^{-1}x] = 0$, then the covariance of $A^{-1}x$ equals

$$E[A^{-1} x x^\top (A^\top)^{-1}] = A^{-1} \Omega (A^\top)^{-1} = A^{-1} A A^t op (A^\top)^{-1} = I_m \tag{16.1}$$

So $x^\top \Omega^{-1} x = x^\top (A^\top)^{-1} A^{-1} x = (A^{-1}x)^\top A^{-1} x$, where $A^{-1}x \sim N(0, I_m)$, so $x^\top \Omega^{-1} x \sim \chi^2(m)$.

(2) Let $P = Z(Z^\top Z)^{-1} Z^\top$ where $Z_{n \times r}$ and $Rank(Z) = r$. Let $z \sim N(0, I_n) x = Z^\top z$, then $E(x) = 0, Var(x) = Z^\top Z$, so

$$z^\top P z = z^\top Z (Z^\top Z)^{-1} Z^\top z \tag{16.2}$$

is just the form of $x^\top \Omega^{-1} x$, so $z^\top P z \sim \chi^2(r)$.

If $M = I - P$, then

$$z^\top M z = z^\top z - z^\top P z = \chi^2(n) - \chi^2(k) = \chi^2(n-k) \tag{16.3}$$

which implies this also statement holds for external projection matrix.[11] □

16.2 Hypothesis Test

16.2.1 T-test (one constrain)

Consider

$$y = X_1 \beta_1 + \beta_2 x_2 + u, u \sim N(0, \sigma^2 I) \tag{16.4}$$

and we run the OLS regression of it, giving $\hat{\beta}$ and we calculate $\hat{u}$. Here $M_X y = M_X(X\beta + u) = M_X u$ always holds.

We handle two cases:

(1) σ^2 is known;

(2) σ^2 is unknown.

If σ^2 is known, let $y = X_1 \beta_1 + \beta_2 x_2 + u, u \sim N(0, \sigma^2 I)$ where x_2 is a single variable and we wonder whether $\beta_2 = 0$. We need to construct a known statistics of β_2.

Form FWL theorem, we have the OLS estimators of the equation equals to the following equation's estimators:

$$M_1 y = \beta_2 M_1 x_2 + M_1 u, u \sim N(0, \sigma^2 I) \tag{16.5}$$

[11] P is singular as $tr(P) = k < n$ usually, so we can not apply (1).

Then

$$\hat{\beta}_2 = \frac{x_2^\top M_1 y}{x_2^\top M_1 x_2} = \beta_2 + \frac{x_2^\top M_1 u}{x_2^\top M_1 x_2} \tag{16.6}$$

Thus

$$Var(\hat{\beta}_2) = \sigma^2 (x_2^\top M_1 x_2)^{-1} \tag{16.7}$$

When $\beta_2 = 0$, then $M_1 y = M_1 u$ and

$$z_{\beta_2} = \frac{\hat{\beta}_2 - \beta_2}{\sqrt{Var(\hat{\beta}_2)}} = \frac{x_2^\top M_1 u}{\sigma (x_2^\top M_1 x_2)^{1/2}} \sim N(0,1) \tag{16.8}$$

As $\hat{\beta}_2$ is a linear combination of normally distributed r.v.s., so it follows a normal distribution. We normalize it to obtain a standard normal distribution.

If σ^2 is unknown, we need to find an estimator of it, and the sample standard error is great. Since $\sigma I = E(uu^\top) = Var(u)$, we know $\hat{u}^\top \hat{u}/n$ is a consistent estimator of σ^2 but it is biased.

As $\hat{u} = M_X y$, then $\hat{u}^\top \hat{u} = (M_X y)^\top (M_X y) = y^\top M_X y$. First we see $tr(P_X) = tr(X(X^\top X)^{-1} X^\top) = tr(X^\top X (X^\top X)^{-1})$. As $x^\top x$ is a $k \times k$ matrix, so $X^\top X (X^\top X)^{-1} = I_k$, i.e., $tr(P_X) = tr(I_k) = k$. We know $M_X = I - P_X$, so $tr(M_X) = tr(I) - tr(P_X) = n - k$.

As $\hat{u} = M_X y = M_X u, Var(\hat{u}) = E(\hat{u}\hat{u}^\top) = E(M_X u u^\top M_X) = \sigma^2 M_X$, so $Var(\hat{u}_i) = \hat{\sigma}_i^2 = Var(\hat{u})_{ii}$, then $\hat{u}^\top \hat{u} = \sum_{i=1}^n \hat{u}_i^2 = tr(Var(\hat{u})) = tr(\sigma^2 M_X) = \sigma^2 (n-k)$. Therefore, we have $E(\hat{u}^\top \hat{u}/(n-k)) = \sigma^2$. Therefore

$$s^2 = \frac{y^\top M_X y}{n-k} = \frac{\hat{u}^\top \hat{u}}{n-k} \tag{16.9}$$

is unbiased. We claim

$$t_{\beta_2} = \frac{x_2^\top M_1 y}{s (x_2^\top M_1 x_2)^{1/2}} \tag{16.10}$$

$$= (\frac{y^\top M_X y}{n-k})^{-1/2} \frac{x_2^\top M_1 y}{(x_2^\top M_1 x_2)^{1/2}} \tag{16.11}$$

$$= \frac{z_{\beta_2}}{\sqrt{y^\top M_X y/\sigma^2}/\sqrt{n-k}} \sim t(n-k) \tag{16.12}$$

We need to show three statements:

- $z_{\beta_2} \sim N(0,1)$ (known);
- $y^\top M_X y/\sigma^2 \sim \chi^2(n-k)$;
- z_{β_2} and $y^\top M_X y/\sigma^2 \sim \chi^2(n-k)$ are independent.

First, let $\epsilon = u/\sigma \sim N(0, I)$, then $y^\top M_X y/\sigma = u^\top M_X u/\sigma = \epsilon^\top M_X \epsilon$. Since $tr(M_X) = n-k$, so $\epsilon^\top M_X \epsilon \sim \chi^2(n-k)$ (Note that M_X is a projection matrix, see Table 4).

Second, z_{β_2} depends on y through $P_X y$ as

$$x_2^\top M_1 y = (P_X x_2)^\top M_1 y = x_2^\top P_X M_1 y = x_2^\top M_1 P_X y \tag{16.13}$$

As M_X and P_X are orthogonal, so $P_X y$ and $M_X y$ are independent, i.e., z_{β_2} and $y^\top M_X y/\sigma^2 \sim \chi^2(n-k)$ are independent.

16.2.2 F-test (multi-constrains)

Let the unconstrained model be

$$H_1 : y = X_1 \beta_1 + X_2 \beta_2 + u, u \sim N(0, \sigma^2 I) \tag{16.14}$$

and the constrained model $\beta_2 = \mathbf{0}_r$ be

$$H_0 : y = X_1 \beta_1 + u, u \sim N(0, \sigma^2 I) \tag{16.15}$$

Idea of F-test is if the constrain is right, extra variables bring a little fitness improvement model, i.e., SSE (sum of squared residuals) or R^2 remains. Let

$$F_{\beta_2} = \frac{(SSE_r - SSE_u)/r}{SSE_u/(n-k)} \sim F(r, n-k) \tag{16.16}$$

Intuitively, $SSE_r > SSE_u$ and $R_r^2 < R_u^2$, i.e., extra variables (extra information) never decrease the explaining ability of models. We need to show

- $(SSE_r - SSE_u)/\sigma^2 \sim \chi^2(r)$;
- $SSE_u/\sigma^2 \sim \chi^2(n-k)$;
- $SSE_r - SSE_u$ and SSE_u are independent.

First,

$$\begin{aligned} SSE_u &= (M_X y)^\top (M_X y) && (16.17)\\ &= (M_X(X\beta + u))^\top (M_X(X\beta + u)) && (16.18)\\ &= (M_X u)^\top (M_X u) && (16.19)\\ &= u^\top M_X u && (16.20) \end{aligned}$$

then let $\epsilon = u/\sigma \sim N(0, I)$, we have $SSE_u = u^\top M_X u = \sigma^2 \epsilon^\top M_X \epsilon$ and immediately, $SSE_u/\sigma^2 \sim \chi^2(tr(M_X)) \sim \chi^2(n-k)$.

Then we prove $(SSE_r - SSE_u)/\sigma^2 \sim \chi^2(r)$ (using a different expression of SSE_u).

The SSE for constrained model

$$SSE_r = (M_1 y)^\top (M_1 y) = y^\top M_1 y \tag{16.21}$$

Consider H_1 model, from the FWL theorem, we know

$$M_1 y = M_1 X_2 \beta_2 + M_1 u \tag{16.22}$$

so

$$SSE_u = SST - SSR = y^\top M_1 y - y^\top M_1 X_2 (X_2^\top M_1 X_2)^{-1} X_2^\top M_1 y \tag{16.23}$$

where $SSR = (P_{M_1X_2} M_1 y)^\top P_{M_1X_2} M_1 y = y^\top M_1 P_{M_1X_2} M_1 y$. Then

$$SSE_r - SSE_u = y^\top M_1 y - SSE_u = y^\top M_1 X_2 (X_2^\top M_1 X_2)^{-1} X_2^\top M_1 y \tag{16.24}$$

Under $H_0 : \beta_2 = 0$, we have $y^\top M_1 = (M_1 y)^\top = (M_1(X_1\beta_1 + u))^\top = (M_1 u)^\top = u^\top M_1 = \sigma\epsilon^\top M_1$, and also $M_1 y = M_1 \epsilon\sigma$. Then

$$SSE_r - SSE_u = \epsilon^\top A (A^\top A)^{-1} A^\top \epsilon \sigma^2 = \epsilon^\top P_A \epsilon \sigma^2 \tag{16.25}$$

where $A = M_1 X_2$, so $(SSE_r - SSE_u)/\sigma^2 \sim \chi^2(tr(P_A)) \sim \chi^2(r)$ as $tr(P_A) = tr(P_{M_1X_2}) = tr(P_X - P_1) = r$.

Lastly, M_X and $M_1 X_2$ are orthogonal as $M_X M_1 X_2 = M_X X_2 = \mathbf{O}$ (clear their link to y), so $SSE_r - SSE_u$ and SSR_u are independent.

16.2.3 Asymptotic T-test and F-test

We loose the assumption about u letting $u \sim IID(0, \sigma_0^2 I)$. Then the model becomes

$$y = X_1\beta_1 + \beta_2 x_2 + u, u \sim IID(0, \sigma_0^2 I) \tag{16.26}$$

We construct t_{β_2} and claim $t_{\beta_2} \sim^a N(0, 1)$. Then

$$\begin{aligned} t_{\beta_2} &= (\frac{y^\top M_X y}{n-k})^{-1/2} \frac{x_2^\top M_1 y}{(x_2^\top M_1 x_2)^{1/2}} && (16.27)\\ &= (\frac{y^\top M_X y}{n-k})^{-1/2} \frac{n^{-1/2} x_2^\top M_1 y}{(n^{-1} x_2^\top M_1 x_2)^{1/2}} && (16.28)\\ & && (16.29) \end{aligned}$$

With LLNs, we have $s^2 \equiv \frac{y^\top M_X y}{n-k} \to^p \sigma_0^2$, so $(\frac{y^\top M_X y}{n-k})^{-1/2} \to^p \frac{1}{\sigma_0}$. Under $H_0, \beta_2 = 0, M_1 y = M_1 u$, so

$$E(x_2^\top M_1 y y^\top M_1 x_2 | X) = E(x_2^\top M_1 u u^\top M_1 x_2 | X) = \sigma_0^2 x_2^\top M_1 x_2 \tag{16.30}$$

Then,

$$E((\frac{y^\top M_X y}{n-k})^{-1/2} \frac{n^{-1/2} x_2^\top M_1 y}{(n^{-1} x_2^\top M_1 x_2)^{1/2}} | X) = 0, Var((\frac{y^\top M_X y}{n-k})^{-1/2} \frac{n^{-1/2} x_2^\top M_1 y}{(n^{-1} x_2^\top M_1 x_2)^{1/2}} | X) = \sigma_0^2 \tag{16.31}$$

(conditional), further more,

$$E((\frac{y^\top M_X y}{n-k})^{-1/2} \frac{n^{-1/2} x_2^\top M_1 y}{(n^{-1} x_2^\top M_1 x_2)^{1/2}}) = 0, Var((\frac{y^\top M_X y}{n-k})^{-1/2} \frac{n^{-1/2} x_2^\top M_1 y}{(n^{-1} x_2^\top M_1 x_2)^{1/2}}) = \sigma_0^2 \tag{16.32}$$

(unconditional). As we have shown, M_X and P_X are independent, so with CLT, we have

$$t_{\beta_2} \sim^a N(0,1) \tag{16.33}$$

Similarly, we let

$$F_{\beta_2} = \frac{n^{-1/2} \epsilon^\top M_1 X_2 (n^{-1} X_2^\top M_1 X_2)^{-1} n^{-1/2} X_2^\top M_1 \epsilon / r}{\epsilon^\top M_X \epsilon / (n-k)} \tag{16.34}$$

$$E[n^{-1/2} \epsilon^\top M_1 X_2] = E[n^{-1/2} E[\epsilon | X]^\top M_1 X_2] = \mathbf{0} \tag{16.35}$$

$$Var[n^{-1/2} \epsilon^\top M_1 X_2] = E[n^{-1} X_2^\top M_1 \epsilon \epsilon^\top M_1 X_2] \tag{16.36}$$

$$= n^{-1} X_2^\top M_1 X_2 \tag{16.37}$$

So $n^{-1/2} \epsilon^\top M_1 X_2 (n^{-1} X_2^\top M_1 X_2)^{-1} n^{-1/2} X_2^\top M_1 \epsilon \sim^a \chi^2(r)$. Similarly, $\epsilon^\top M_X \epsilon \sim^a \chi^2(n-k)$.

Thus,

$$F_{\beta_2} \sim^a F(r, n-k) \tag{16.38}$$

Further, $\lim_{n\to\infty} F_{\beta_2} \sim^a F(r, +\infty)$ and $\lim_{n\to\infty} r F_{\beta_2} \sim^a \chi^2(r)$.

In conclusion, when we loose the assumption of u, we shold take asymptotic T-test and F-test. Standard normal distribution and t distribution are feasible to calculate p-value of t statistics. χ^2 and F distribution are feasible to calculate p-value of F statistics.

16.3 A Brief Introduction to Confidence Interval

A few conceptions:

- Test family: a family of many simple hypothesis tests;
- Covering probability: the probability of a random interval covering true values;
- Confidence interval: an interval comprised of not rejected θ;
- Confidence region: a region comprised of not joint rejected parameters, the shape is an ellipse;
- Precise confidence interval: construct the confidence interval with precise statistics;
- Asymptotic confidence interval: construct the confidence interval with asymptotic statistics;

An usual approach to construct a confidence interval follows: let $H_0 : \theta = \theta_0$ and $\tau(y, \theta_0)$ follows a known distribution, then

- y is a random sample;
- $\tau(\cdot)$ is a deterministic function;
- usually, let c_α be the tipping point, then no rejection probability is $Pr(\tau(y, \theta_0) \leq c_\alpha) = 1 - \alpha$;
- θ_0 belongs to a confidence interval iff $\tau(y, \theta_0) \leq c_\alpha$;
- by solving $\tau(y, \theta_0) = c_\alpha$, we construct a confidence interval $[\theta_l, \theta_u]$;
- c_α is the tipping point of a precise (asymptotic) distribution, then $[\theta_l, \theta_u]$ is a precise (asymptotic) confidence interval.

Confidence intervals are the opposite aspect of hypothesis test. We use the same statistics for hypothesis test and confidence interval, if a value is rejected, then it is not in the confidence interval and vice versa.

16.3.1 Confidence Interval and the Delta Method

Example 16.1. *Let $y = X_1\beta_1 + \beta_2 x_2 + u, u \sim N(0, \sigma^2 I)$, we test $H_0 : \beta_2 = \beta_{2,0}$ by $\frac{\hat{\beta}_2 - \beta_{2,0}}{s_2} \sim t(n-k)$. Then we have*

$$Pr(t_{\alpha/2} \leq \frac{\hat{\beta}_2 - \beta_{2,0}}{s_2} \leq t_{1-\alpha/2}) = 1 - \alpha \tag{16.39}$$

$$\Rightarrow Pr(\hat{\beta}_2 - s_2 t_{1-\alpha/2} \leq \beta_{2,0} \leq \hat{\beta}_2 - s_2 t_{\alpha/2}) = 1 - \alpha \tag{16.40}$$

then, we obtain a precise confidence interval: $[\hat{\beta}_2 - s_2 t_{1-\alpha/2}, \hat{\beta}_2 + s_2 t_{\alpha/2}]$.

Now we give an lemma of asymptotic confidence interval.

Example 16.2. $\tau(y, \theta_0) = (\frac{\hat{\theta} - \theta_0}{s_\theta})^2 \sim \chi^2(1)$ *under null hypothesis. Let c_α s.t. $Pr(x < c_\alpha) = 1 - \alpha$, e.g., $\alpha = 0.05, c_\alpha^{1/2} = 1.96$. Solve $\tau(y, \theta_0) = c_\alpha$, then $|\hat{\theta} - \theta_0| = s_\theta c_\alpha^{1/2}$, then*

$$\theta_l = \hat{\theta} - s_\theta c_\alpha^{1/2}, \theta_u = \hat{\theta} + s_\theta c_\alpha^{1/2} \tag{16.41}$$

So an asymptotic confidence interval is $[\theta_l, \theta_u]$.

Then we introduce the Delta Method. Assuming we have an estimator $\hat{\theta}$ s.t. $\sqrt{n}(\hat{\theta} - \theta) \rightarrow^d W \sim N(0, \sigma^2)$, then we can construct an confidence interval for a function of $\hat{\theta}$ if it is differentiable at θ. Let $\eta = g(\theta)$ and $g(\cdot)$ is differentiable everywhere and $g'(\cdot)$ is continuous. Then

$$\sqrt{n}(\hat{\eta} - \eta) = \sqrt{n}(g(\hat{\theta}) - g(\theta)) \rightarrow^d W^* \sim N(0, g'(\theta)^2 \sigma^2) \tag{16.42}$$

If θ, σ^2 is unknown, then $\hat{\eta} - \eta \sim^a N(0, s_\eta^2)$ where $\hat{\theta} \rightarrow^p \theta, \hat{\sigma}^2 \rightarrow^p \sigma^2, s_\eta^2 = g'(\hat{\theta})^2 \frac{\hat{\sigma}^2}{n}$. Then we see $[\hat{\eta} - s_\eta z_{1-\alpha/2}, \hat{\eta} + s_\eta z_{\alpha/2}]$. The problem is twofold: (1)s_η^2 and $g(\hat{\theta})$ are dependent; (2) $g(\hat{\theta})$ convergences slowly.

An alternative approach is compositing $g(\cdot)$ with the confidence interval of θ, i.e., $[g(\hat{\beta} - s_\theta z_{1-\alpha/2}), g(\hat{\theta} + s_\theta z_{\alpha/2})]$ (when $g' > 0$, and exchange the boundary when $g' < 0$). The disadvantage of this method is the confidence interval may not be symmetric.

16.3.2 Confidence Region

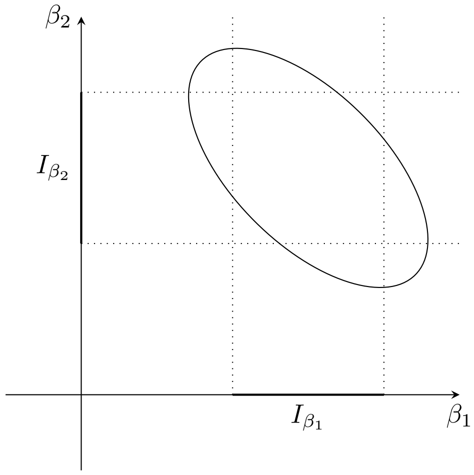


Figure 34: Confidence regions

To establish a joint statistics, we need the wald statistics: to test $H_0 : \beta = \beta_0$, let $\hat{\beta} \rightarrow^p \beta_0$, then we have

$$Wald \equiv n^{1/2}(\hat{\beta} - \beta_0)^\top (n\hat{Var}(\hat{\beta}))^{-1} n^{1/2}(\hat{\beta} - \beta_0) \tag{16.43}$$

If $n^{1/2}(\hat{\beta} - \beta_0) \sim^a N(0, nVar(\hat{\beta}))$, then $Wald \sim^a \chi^2(k)$ where k is the dimension of vector β.

For a model

$$y = X\beta_0 + u, u \sim IID(0, \sigma^2 I) \tag{16.44}$$

we have $\hat{\beta} - \beta = (X^\top X)^{-1} X^\top u$, then $n^{1/2}(\hat{\beta} - \beta) = (n^{-1}X^\top X)^{-1} n^{-1/2} X^\top u$. And $avar(n^{1/2}(\hat{\beta} - \beta_0)) = \Sigma_{XX}^{-1}\sigma^2\Sigma_{XX}\Sigma_{XX}^{-1} = \sigma^2\Sigma_{XX}$ where $\Sigma_{XX} = \underset{n\to\infty}{plim}\, n^{-1}X^\top X$. Therefore,

$$n^{1/2}(\hat{\beta} - \beta_0) \sim^a N(0, \sigma^2\Sigma_{XX}^{-1}), \hat{Var}(\hat{\beta}) = s^2(X^\top X)^{-1} \tag{16.45}$$

then

$$n^{1/2}(\hat{\beta} - \beta_0)^\top (n\hat{Var}(\hat{\beta}))^{-1} n^{1/2}(\hat{\beta} - \beta_0) \sim^a \chi^2(k) \tag{16.46}$$

Let c_α be the $1-\alpha$ upper quantile of $\chi^2(r)$, so an asymptotic confidence region satisfying

$$n^{1/2}(\hat{\beta} - \beta_0)^\top (n\hat{Var}(\hat{\beta}))^{-1} n^{1/2}(\hat{\beta} - \beta_0) \le c_\alpha \tag{16.47}$$

If $u \sim N(0, \sigma^2 I)$, we can find a precise confidence region.

Let $y = X_1\beta_1 + X_2\beta_2 + u, u \sim N(0, \sigma^2 I)$ where $X_{n\times k} = [X_1, X_2]$ we want to test $H_0 : \beta_2 = \beta_{2,0}$. Equivalently,

$$y^* \equiv y - X_2\beta_{2,0} = X_1\gamma_1 + X_2\gamma_2 + u, u \sim N(0, \sigma^2 I) \tag{16.48}$$

we need to test $\gamma_2 = 0$.

$$SSE_r = y^{*\top} M_1 y^*, SSE_u = y^{*\top} M_X y^* \tag{16.49}$$

$$SSE_r - SSE_u = y^{*\top} M_1 X_2 (X_2^\top M_1 X_2)^{-1} X_2^\top M_1 y^* \tag{16.50}$$

$$= (y - X_2\beta_{2,0}) M_1 X_2 (X_2^\top M_1 X_2)^{-1} X_2^\top M_1 (y - X_2\beta_{2,0}) \tag{16.51}$$

$$(X_2^\top M_1 X_2)^{-1} X_2^\top M_1 (y - X_2\beta_{2,0}) = (X_2^\top M_1 X_2)^{-1} X_2^\top M_1 y - (X_2^\top M_1 X_2)^{-1} X_2^\top M_1 X_2 \beta_{2,0} \tag{16.52}$$

$$= \hat{\beta}_2 - \beta_{2,0} \tag{16.53}$$

$$\Rightarrow SSE_r - SSE_u = (\hat{\beta}_2 - \beta_{2,0})^\top X_2^\top M_1 X_2 (\hat{\beta}_2 - \beta_{2,0}) \tag{16.54}$$

As $SSE_u = (y - X_2\beta_{2,0})^\top M_X (y - X_2\beta_{2,0}) = y^\top M_X y$, so

$$F_{\gamma_2} = \frac{(\hat{\beta}_2 - \beta_{2,0})^\top X_2^\top M_1 X_2 (\hat{\beta}_2 - \beta_{2,0})/r}{y^\top M_X y/(n-k)} \sim F(r, n-k) \tag{16.55}$$

Then let c_α be the $1-\alpha$ upper quantile, then the confidence region satisfying

$$(\hat{\beta}_2 - \beta_{2,0})^\top X_2^\top M_1 X_2 (\hat{\beta}_2 - \beta_{2,0}) \le c_\alpha r s^2 \tag{16.56}$$

where $s^2 = \frac{y^\top M_X y}{n-k}$.

Now show rF_{γ_2} is a wald statistics, with FWL theorem, we have

$$M_1 y = M_1 X_2 \beta_2 + M_1 u \tag{16.57}$$

$$Var(\hat{\beta}_2) = \sigma^2 (X_2^\top M_1 X_2)^{-1} \tag{16.58}$$

$$\hat{Var}(\hat{\beta}_2) = s^2 (X_2^\top M_1 X_2)^{-1} = \frac{y^\top M_X y}{n-k} (X_2^\top M_1 X_2)^{-1} \tag{16.59}$$

so $rF_{\gamma_2} \sim \chi^2(r)$.

LEMMA 16.2. *Let x_1, x_2 centered, $y = \beta_1 x_2 + \beta_2 x_2 + u, u \sim IID(0, \sigma^2 I)$, and $\hat{\rho} = \frac{x_1^\top x_2}{(x_1^\top x_1)^{1/2}(x_2^\top x_2)^{1/2}}$, prove the equation: $corr(\hat{\beta}_2, \hat{\beta}_2) = -\hat{\rho}$.*

Proof. We first evaluate $corr(\hat{\beta}_2, \hat{\beta}_2)$ with x_1, x_2, then substitute it with $\hat{\rho}$.

With the FWL Theorem, $M_1 y = M_1 x_2 \beta_2 + u$, so

$$\hat{\beta}_2 = \beta_2 + (x_2^\top M_1 x_2)^{-1} x_2^\top M_1 u \tag{16.60}$$

Then $Var(\hat{\beta}_2) = \sigma^2 (x_2^\top M_1 x_2)^{-1}$. Similarly, $\hat{\beta}_1 = \beta^1 + (x_1^\top M_2 x_1)^{-1} x_1^\top M_2 u, Var(\hat{\beta}_1) = \sigma^2 (x_1^\top M_2 x_1)^{-1}$.

$$Cov(\hat{\beta}_1, \hat{\beta}_2) = E[(\hat{\beta}_1 - \beta_1)(\hat{\beta}_2 - \beta_2)^\top] \tag{16.61}$$

$$= E[(x_2^\top M_1 x_2)^{-1} x_2^\top M_1 u u^\top M_2 x_1 (x_1^\top M_2 x_1)^{-1}] \tag{16.62}$$

$$= \sigma^2 (x_2^\top M_1 x_2)^{-1} x_2^\top M_1 M_2 x_1 (x_1^\top M_2 x_1)^{-1} \tag{16.63}$$

So

$$corr(\hat{\beta}_1, \hat{\beta}_2) = \frac{Cov(\hat{\beta}_1, \hat{\beta}_2)}{\sqrt{Var(\hat{\beta}_1)}\sqrt{Var(\hat{\beta}_2)}} \tag{16.64}$$
$$= \frac{\sigma^2(x_2^\top M_1 x_2)^{-1} x_2^\top M_1 M_2 x_1 (x_1^\top M_2 x_1)^{-1}}{\sqrt{\sigma^2(x_2^\top M_1 x_2)^{-1}}\sqrt{\sigma^2(x_1^\top M_2 x_1)^{-1}}} \tag{16.65}$$
$$= \frac{x_2^\top M_1 M_2 x_1}{\sqrt{x_2^\top M_1 x_2}\sqrt{x_1^\top M_2 x_1}} \tag{16.66}$$

Now we calculate these terms with $\hat{\rho}$.

$$x_2^\top M_1 x_2 = x_2^\top (I - P_1) x_2 \tag{16.67}$$
$$= x_2^\top (I - x_1(x_1^\top x_1)^{-1} x_1^\top) x_2 \tag{16.68}$$
$$= x_2^\top x_2 - \frac{x_2^\top x_1 x_1^\top x_2}{x_1^\top x_1} \tag{16.69}$$
$$= x_2^\top x_2 (1 - \hat{\rho}^2) \tag{16.70}$$

Similarly, $x_1^\top M_2 x_1 = x_1^\top x_1 (1 - \hat{\rho}^2)$.

$$x_2^\top M_1 M_2 x_1 = x_2^\top (I - x_1(x_1^\top x_1)^{-1} x_1^\top)(I - x_2(x_2^\top x_2)^{-1} x_2^\top) x_1 \tag{16.71}$$
$$= x_2^\top x_1 - \frac{x_2^\top x_1 x_1^\top x_2}{x_1^\top x_1} - \frac{x_2^\top x_2 x_2^\top x_1}{x_2^\top x_2} + \frac{x_2^\top x_1 x_1^\top x_2 x_2^\top x_1}{x_1^\top x_1 x_2^\top x_2} \tag{16.72}$$
$$= (\hat{\rho}^2 - 1) x_2^\top x_1 \tag{16.73}$$

Summing up,

$$corr(\hat{\beta}_1, \hat{\beta}_2) = \frac{x_2^\top M_1 M_2 x_1}{\sqrt{x_2^\top M_1 x_2}\sqrt{x_1^\top M_2 x_1}} \tag{16.74}$$
$$= \frac{x_2^\top x_1 (\hat{\rho}^2 - 1)}{\sqrt{x_2^\top x_2}\sqrt{x_1^\top x_1}(1 - \hat{\rho}^2)} = -\hat{\rho} \tag{16.75}$$

□

17 Generalized Least Squares and Related Topics

Consider the model

$$y = X\beta + u, E[u|X] = \mathbf{0}, E[uu^\top] = \Omega \tag{17.1}$$

if $E[u|X] = \mathbf{0}, E[uu^\top|X] = \sigma^2 I$, then the OLS estimator $\hat{\beta}$ is the best linear unbiased estimator from thr Gauss-Markov Theorem.

However, if $E[uu^\top] = \Omega \neq \sigma^2 I$, then $\hat{\beta}$ is not BLUE,

$$Var[\hat{\beta}|X] = (X^\top X)^{-1} X^\top \Omega X (X^\top X)^{-1} \neq \sigma^2 (X^\top X)^{-1} \tag{17.2}$$

implying we need to update the estimating method for the covariance matrix.

17.1 The GLS Estimator

Suppose Ω is known. As Ω is a covariance matrix, so can be decomposed

$$\Omega^{-1} = \Psi\Psi^\top \tag{17.3}$$

where Ψ is called the transformation matrix and nonsingular. (17.1) is equivalent to

$$\Psi^\top y = \Psi^\top X\beta + \Psi^\top u \tag{17.4}$$

Now $E[\Psi^\top u|X] = \mathbf{0}, Var[\Psi^\top u|X] = E[\Psi^\top u|X] = E[\Psi^\top uu^\top \Psi|X] = \Psi^\top E[E[uu^\top|X]]\Psi = \Psi^\top(\Psi\Psi^\top)^{-1}\Psi = I$, satisfying the Gauss-Markov condition. Then

$$\begin{aligned}
\hat{\beta}_{GLS} &= (X^\top\Omega^{-1}X)^{-1}X^\top\Omega^{-1}(X\beta_0 + u) && (17.5)\\
&= \beta_0 + (X^\top\Omega^{-1}X)^{-1}X^\top\Omega^{-1}u && (17.6)\\
E[\hat{\beta}_{GLS} - \beta_0] &= E[(X^\top\Omega^{-1}X)^{-1}X^\top\Omega^{-1}u] && (17.7)\\
&= E[(X^\top\Omega^{-1}X)^{-1}X^\top\Omega^{-1}E[u|X]] \qquad = 0 && (17.8)\\
& && (17.9)
\end{aligned}$$

and

$$\begin{aligned}
Var[\hat{\beta}_{GLS}] &= Vae[\hat{\beta}_{GLS} - \beta_0] && (17.10)\\
&= E[(X^\top\Omega^{-1}X)^{-1}X^\top\Omega^{-1}uu^\top[(X^\top\Omega^{-1}X)^{-1}X^\top\Omega^{-1}]^\top] && (17.11)\\
&= (X^\top\Omega^{-1}X)^{-1}X^\top\Omega^{-1}\Omega\Omega^{-1}X(X^\top\Omega^{-1}X)^{-1} && (17.12)\\
&= (X^\top\Omega^{-1}X)^{-1} && (17.13)
\end{aligned}$$

The OLS estimator of (17.4) is

$$\hat{\beta}_{GLS} = (X^\top\Psi\Psi^\top X)^{-1}X^\top\Psi\Psi^\top y = (X^\top\Omega^{-1}X)^{-1}X^\top\Omega^{-1}y \tag{17.14}$$

which is call the generalized least squares (GLS) estimator of β.

$\hat{\beta}_{GLS}$ can also be obtained by minimizing the GLS criterion function

$$(y - X\beta)^\top\Omega^{-1}(y - X\beta) \tag{17.15}$$

or solve the moment condition,

$$X^\top\Omega^{-1}(y - X\beta) = \mathbf{0} \tag{17.16}$$

which is also the F.O.C. of (17.15).

LEMMA 17.1. *The GLS estimator is the most efficient unbiased method of moments (MM) estimator.*

Proof. A MM estimator of the linear regression (17.1) is defined in terms of an $n\times k$ matrix of exogenous variances $W_{n\times k}$, where k is the dimension of β, by the equations

$$W^\top(y - X\beta) = \mathbf{0} \tag{17.17}$$

By solving it, we obtain

$$\hat{\beta}_W = (W^\top X)^{-1}W^\top y \tag{17.18}$$

The GLS (resp. OLS) estimator is a special case of MM estimator, $W = \Omega^{-1}X$ (resp. $W = X$).

$$\begin{aligned}
\hat{\beta}_W &= (W^\top X)^{-1}W^\top y = (W^\top X)^{-1}W^\top(X\beta_0 + u) = \beta_0 + (W^\top X)^{-1}W^\top u && (17.19)\\
E[\hat{\beta}_W|X, W] &= \beta_0 + E[(W^\top X)^{-1}W^\top u|X, W] = \beta_0 + (W^\top X)^{-1}W^\top E[u|X, W] = \beta && (17.20)\\
Var[\hat{\beta}_W] &= E[(\hat{\beta}_W - \beta_0)(\hat{\beta}_W - \beta_0)^\top] && (17.21)\\
&= E[(W^\top X)^{-1}W^\top uu^\top W(X^\top W)^{-1}] && (17.22)\\
&= (W^\top X)^{-1}W^\top\Omega W(X^\top W)^{-1} && (17.23)
\end{aligned}$$

To show $\hat{\beta}_{GLS}$ is more efficient than $\hat{\beta}_W$, we need to show $Var[\hat{\beta}_W] - Var[\hat{\beta}_{GLS}]$ is positive semidefinite, which is equivalent to show $Var[\hat{\beta}_{GLS}]^{-1} - Var[\hat{\beta}_W]^{-1}$ is positive semidefinite, i.e.,

$$X^\top\Omega^{-1}X - X^\top W(W^\top\Omega W)^{-1}W^\top X \tag{17.24}$$

is positive semidefinite.

Since Ω is symmetric positive semidefinite, then also is Ω^{-1}, so Ω^{-1} can be decomposed as $\Omega^{-1} = \Psi\Psi^\top$. Then

$$X^\top\Omega^{-1}X = X^\top\Psi\Psi^\top X = (\Psi^\top X)^\top\Psi^\top X \tag{17.25}$$

and

$$
\begin{aligned}
& X^\top W(W^\top \Omega W)^{-1} W^\top X && (17.26)\\
=& X^\top W(W^\top (\Psi\Psi^\top)^{-1} W)^{-1} W^\top X && (17.27)\\
=& (\Psi^\top X)^\top \Psi^{-1} W((\Psi^{-1}W)^\top \Psi^{-1} W)^{-1} (\Psi^{-1}W)^\top \Psi^\top X && (17.28)\\
=& (\Psi^\top X)^\top P_{\Psi^{-1}W} \Psi^\top X && (17.29)
\end{aligned}
$$

So

$$
\begin{aligned}
& X^\top \Omega^{-1} X - X^\top W(W^\top \Omega W)^{-1} W^\top X && (17.30)\\
=& (\Psi^\top X)^\top \Psi^\top X - (\Psi^\top X)^\top P_{\Psi^{-1}W} \Psi^\top X && (17.31)\\
=& (\Psi^\top X)^\top M_{\Psi^{-1}W} \Psi^\top X && (17.32)
\end{aligned}
$$

which is positive semidefinite with Lemma 15.15. □

In most cases, the GLS estimator $\hat{\beta}_{GLS}$ is more efficient, and is never less efficient, than the OLS estimator $\hat{\beta}$.

In practice, Ω is unknown usually. Now we study the structure of Ω.

A simple case of Ω is $\Omega = \sigma^2\Delta$ where Δ is known, then GLS estimator is given by

$$\hat{\beta}_{GLS} = (X^\top \Omega^{-1} X)^{-1} X^\top \Omega^{-1} y = (X^\top \Delta^{-1} X)^{-1} X^\top \Delta^{-1} y \tag{17.33}$$

If Ω is a diagonal matrix, which implies the error terms are heteroscedastic but uncorrelated, then Ψ can be set as

$$\Psi = \begin{bmatrix} \sigma_1^{-1} & 0 & \cdots & \\ 0 & \sigma_2^{-1} & \cdots & \\ & & \cdots & \\ & & \cdots & \sigma_n^{-1} \end{bmatrix} \tag{17.34}$$

where σ_t is the t-th diagonal element of $\Omega^{1/2}$. Then (17.4) becomes

$$\frac{1}{\sigma_t} y_t = \frac{1}{\sigma_t} X_t \beta + \frac{1}{\sigma_t} u_t, E[uu^\top] = \Omega = diag(\sigma_1^2, \cdots, \sigma_n^2) \tag{17.35}$$

and the criterion function becomes

$$\min_\beta \sum_{t=1}^n \frac{1}{\sigma_t^2}(y_t - X_t\beta)^2 \tag{17.36}$$

But here we still are ignorant to $\sigma_1, \sigma_2, \cdots, \sigma_t$.

17.2 Feasible Generalized Least Squares

Although Ω is unknown, it is reasonable to assume Ω depends on known exogenous variables Z, i.e.,

$$\Omega = \Omega(Z, \gamma) \tag{17.37}$$

where $\Omega(\cdot)$ is called the skedastic function (assume explicitly). We need to estimate γ to obtain Ω.

For example, let

$$y_t = X_t^\top \beta + u_t, E[u_t^2] = exp(Z_t^\top \gamma) \tag{17.38}$$

We can run OLS regression and obtain $\hat{u}_t$, which is an consistent estimator of u_t. Then run the auxiliary linear regression

$$log(\hat{u}_t^2) = Z_t^\top \gamma + v_t \tag{17.39}$$

to find $\hat{\gamma}$, which is an consistent estimator of γ. Then we can compute $\hat{\sigma}_t = \sqrt{exp(Z_t^\top \hat{\gamma})}, \forall t$, then we can run (17.36). This is an example of feasible weighted least squares.

Summing up, to obtain a feasible GLS estimator,

Step (1) assume the skedastic function $\Omega(Z_t, \gamma)$;

Step (2) obtain an consistent estimator of γ;

Step (3) calculate estimated $\hat{\Omega} = \Omega(Z_t, \hat{\gamma})$;

Step (4) replace Ω with $\hat{\Omega}$ and run (17.4), obtain the feasible GLS estimator $\hat{\beta}_{FGLS}$.

We can iterate Step (2)-(4) to update the outcome. Under certain regular conditions, $\hat{\beta}_{FGLS}$ is consistent with β and asymptotically equivalent to $\hat{\beta}_{GLS}$. Therefore $\hat{\beta}_{FGLS}$ is asymptotically efficient and asymptotically normal with big sample

$$\hat{\beta}_{FGLS} - \beta \sim^a N(0, (X^\top \Omega(\hat{\gamma})^{-1} X)^{-1}) \tag{17.40}$$

But with finite sample, $\hat{\beta}_{FGLS}$ is biased usually, and $\hat{\gamma} \neq \gamma$, the transformation matrix $\Psi(\hat{\gamma})$ is wrong and $E[\Psi(\hat{\gamma})^\top u | \Psi(\hat{\gamma})^\top X] \neq \mathbf{0}$, i.e., $\Psi(\hat{\gamma})^\top X$ is not exogenous.

The structure of Ω is of central interest, so we analyze some common cases below.

17.3 Heteroscedasticity

First test whether the error terms are heteroscedastic.

$$H_0 : E[u_t^2 | X] = \sigma^2, H_1 : E[u_t^2 | X] = \sigma^2(X_t) \tag{17.41}$$

Note that we assume the hemoscedasticity is related to regressors. The White test:

Step (1) run (17.1) and obtain an consistent estimator of error terms $\hat{u}_t$;

Step (2) run the auxiliary regression:

$$\hat{u}_t^2 = [X_{t1}^2, \cdots, X_{tk}^2, X_{t1}X_{t2}, \cdots, X_{t(k-1)}X_{tk}]\gamma + v_t \tag{17.42}$$

where $X_{t1} = \iota$;

Step (3) Test whether all slop coefficients are joint significant with F-test. (exclude the intercept)

The idea is, only $X_{t1}^2, \cdots, X_{tk}^2, X_{t1}X_{t2}, \cdots, X_{t(k-1)}X_{tk}$ affect the efficiency of the covariance matrix of $\hat{\beta}$, so we must eliminate this type of heterscedasticity at least.

Assume the DGP of hemoscedasticity is

$$u_t = exp(\delta + Z_t \gamma) v_t \tag{17.43}$$

where Z_t is exogenous ot determined variables (may contain X_t), $v_t \sim IID(0, \sigma^2)$ and independent with Z_t, γ is a parameter vector and δ is a constant. Replace u_t with its consistent estimator $\hat{u}_t$ and take square-log, giving

$$log(\hat{u}_t^2) = \delta^* + Z_t \gamma^* + log(v_t^2) = \delta^* + Z_t \gamma^* + \epsilon_t \tag{17.44}$$

where $\delta^* = 2\delta, \gamma^* = 2\gamma$.

Based on (17.44), we can obtain a consistent estimator $\hat{\gamma}$ of γ and then be capable to derive the FGLS estimator

$$\hat{\beta}_{FGLS} = (X^\top \Omega(\hat{\gamma})^{-1} X)^{-1} X^\top \Omega(\hat{\gamma})^{-1} y \tag{17.45}$$

Also, we can test heteroscedasticity:

$$H_0 : \gamma = 0, H_1 : \gamma \neq 0 \tag{17.46}$$

with F-test, similar to the White test.

17.4 Serial Corelation

Consider the simplest serial corelation process $AR(1)$:

$$y_t = X_t \beta + u_t, u_t = \rho u_{t-1} + \epsilon_t \tag{17.47}$$

where $|\rho| < 1$ (stationary condition). Assume $u_1 = \frac{1}{\sqrt{1-\rho^2}}\epsilon_1, \epsilon_t \sim IID(0, \sigma_\epsilon^2), \forall t$ and $E[\epsilon|X] = \mathbf{0}$. Then u_t is a covariance stationary process, implying

$$
\begin{aligned}
E[u_t] &= \rho E[u_{t-1}] + E[\epsilon_t] = \rho E[u_{t-1}] && (17.48)\\
\Rightarrow E[u_t] &= 0, \forall t && (17.49)\\
Var[u_t] &= \rho^2 Var[u_{t-1}] + Var[\epsilon_t] = \rho^2 Var[u_{t-1}] + \sigma_\epsilon^2 && (17.50)\\
\Rightarrow Var[u_t] &= \frac{\sigma_\epsilon^2}{1-\rho^2}, \forall t && (17.51)\\
Cov(u_t, u_{t-j}) &= Cov(\rho u_{t-1} + \epsilon_t, u_{t-j}) && (17.52)\\
&= Cov(\rho^2 u_{t-2} + \rho\epsilon_{t-1} + \epsilon_t, u_{t-j}) && (17.53)\\
&\cdots && (17.54)\\
&= Cov(\rho^j u_{t-j} + \rho^{j-1}\epsilon_{t-j+1} + \cdots + \epsilon_t, u_{t-j}) && (17.55)\\
&= \rho^j \frac{\sigma_\epsilon^2}{1-\rho^2}, \forall t, \forall j > 0 && (17.56)
\end{aligned}
$$

With precise structure of Ω, we can estimate it now.

Step (1) run OLS regression and obtain a consistent estimator $\hat{u}_t$ of u_t;

Step (2) run an auxiliary regression:

$$\hat{u}_t = \rho\hat{u}_{t-1} + \epsilon_t \tag{17.57}$$

and obtain a consistent estimator of ρ.

Step (3) the autocovariance matrix,

$$\Omega(\rho) = \frac{\sigma_\epsilon^2}{1-\rho^2}\begin{bmatrix} 1 & \rho & \cdots & \rho^{n-1} \\ \rho & 1 & \cdots & \rho^{n-2} \\ & & \cdots & \\ \rho^{n-1} & \rho^{n-2} & \cdots & 1 \end{bmatrix} \tag{17.58}$$

Decompose Ω: let $\Psi^\top u = \epsilon, Var(\epsilon) = \sigma_\epsilon^2 I$, with

$$\epsilon_t = u_t - \rho u_{t-1}, t > 1, \epsilon_t = u_1\sqrt{1-\rho^2}, t = 1 \tag{17.59}$$

we have $\Omega = \sigma_\epsilon^2\Delta(\rho) = (\Psi(\rho)\Psi(\rho)^\top)^{-1}$.

Step (4) obtain the FGLS estimator

$$\hat{\beta}_{FGLS} = (X^\top\Delta(\hat{\rho})^{-1}X)^{-1}X^\top\Delta(\hat{\rho})^{-1}y \tag{17.60}$$

We can iterate these procedures to improve efficiency.

To test a first-order serial corelation,

$$H_0 : \rho = 0, H_1 : \rho \neq 0 \tag{17.61}$$

we can

Step (1) run OLS to obtain $\hat{u}_t$;

Step (2) run an auxiliary regression: $\hat{u}_t = \rho\hat{u}_{t-1} + \epsilon_t$;

Step (3) test $\rho = 0$ by t-test.

To test p-th-order serial corelation, let the error terms follow $AR(p)$ process,

$$u_t = \rho_1 u_{t-1} + \rho_2 u_{t-2} + \cdots + \rho_p u_{t-p} + \epsilon_t, \epsilon \sim IID(0, \sigma_\epsilon^2) \tag{17.62}$$

and $E[\epsilon|X] = 0$. Then need to test

$$H_0 : \rho_1 = \rho_2 = \cdots = \rho_p = 0 \tag{17.63}$$

which is equivalent to $E[uu^\top|X] = \sigma_\epsilon^2 I$. Similar to the first-order serial corelation test, we can

Step (1) run OLS regression and obtain $\hat{u}_t$;

Step (2) run an auxiliary regression

$$\hat{u}_t = \rho_1 \hat{u}_{t-1} + \rho_2 \hat{u}_{t-2} + \cdots + \rho_p \hat{u}_{t-p} + \epsilon_t \tag{17.64}$$

Step (3) then test $\rho_1 = \rho_2 = \cdots = \rho_p = 0$ by F-test.

The serial corelation may be finite, a property of $MA(1)$, so we introduce $MA(1)$ briefly.
Let u_t follows the $MA(1)$ process,

$$u_t = \epsilon_t + \lambda \epsilon_{t-1}, \epsilon \sim IID(0, \sigma_\epsilon^2 I) \tag{17.65}$$

and $E[\epsilon|X] = 0$. It is trivial to check

$$Var[u_t] = (1 + \lambda^2)\sigma_\epsilon^2 \tag{17.66}$$

$$Cov(u_t, u_{t-1}) = \lambda \sigma_\epsilon^2 \tag{17.67}$$

$$Cov(u_t, u_{t-j}) = 0, \forall j > 1 \tag{17.68}$$

The correlation efficient of u_t and u_{t-1} is

$$|corr(u_t, u_{t-1})| = |\frac{\lambda}{1 + \lambda^2}| \leq \frac{1}{2} \tag{17.69}$$

Non-adjacent residuals have no relation. Further, $MA(q)$ is

$$u_t = \epsilon_t + \lambda_1 \epsilon_{t-1} + \cdots + \lambda_q \epsilon_{t-q}, \epsilon \sim IID(0, \sigma_\epsilon^2 I) \tag{17.70}$$

and $E[\epsilon|X] = 0$.

17.5 Random-Effects Estimation

Consider a balanced panel with m individuals and T periods and assume m is large but T is small. Fix T, then $mT \to \infty$ implies $m \to \infty$.

A random effects model is (only consider individual random effect)

$$y_{it} = X_{it}\beta + u_{it}, u_{it} = v_i + \epsilon_{it} \tag{17.71}$$

where $\epsilon_{it} \sim IID(0, \sigma_\epsilon^2), v_t \sim IID(0, \sigma_v^2)$ and ϵ, v are independent with X, ϵ is independent wih v. So

$$E[v_i + \epsilon_{it}|X] = 0 \tag{17.72}$$

and

$$Var[u_{it}] = \sigma_v^2 + \sigma_\epsilon^2 \tag{17.73}$$

$$Cov(u_{it}, u_{is}) = \sigma_v^2, \forall t \neq s \tag{17.74}$$

$$Cov(u_{it}, u_{js}) = 0, \forall t, s, i \neq j \tag{17.75}$$

Then $\Omega = E[uu^\top|X]$ can be written as

$$\Omega = \begin{bmatrix} \Sigma & 0 & \cdots & 0 \\ 0 & \Sigma & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & \Sigma \end{bmatrix} \tag{17.76}$$

where

$$\Sigma = \sigma_\epsilon^2 I_T + \sigma_v^2 \iota\iota^\top \tag{17.77}$$

Exogenous condition $E[u|X] = 0$ ensures $\hat{\beta}_{OLS}$ is unbiased and consistent. But $E[uu^\top|X] = \Omega \neq \sigma^2 I$, so $\hat{\beta}_{OLS}$ is not efficient. We can implement $\hat{\beta}_{FGLS}$ with the known structure of Ω:

$$\hat{\beta}_{FGLS} = (X^\top \Omega(\hat{\gamma})^{-1} X)^{-1} X^\top \Omega(\hat{\gamma})^{-1} y \tag{17.78}$$

where $\hat{\gamma} = (\hat{\sigma}_v^2, \hat{\sigma}_\epsilon^2)$ is a consistent estimator of γ. Now to find $\hat{\gamma}$ by the following procedures:

Step (1) run OLS regression and obtain $\hat{u}_t$;

Step (2) estimate γ with sample covariance with (17.73) and (17.74)

$$\hat{\sigma}_v^2 = \frac{1}{m}\sum_{i=1}^{m}[\frac{1}{T(T-1)}\sum_{t,s=1,t\neq s}^{T}\hat{u}_{it}\hat{u}_{is}] \to^p \hat{\sigma}_v^2 \tag{17.79}$$

$$\hat{\sigma}_\epsilon^2 = \frac{1}{m}\sum_{i=1}^{m}[\frac{1}{T}\sum_{t=1}^{T}\hat{u}_{it}^2] - \hat{\sigma}_v^2 \to^p \sigma_\epsilon^2 \tag{17.80}$$

Step (3) obtain $\Omega(\hat{\gamma})$ and $\hat{\beta}_{FGLS}$.

For a general panel model,

$$y_{it} = X_{it}\beta + u_{it}, E[u|X] = 0, E[uu^\top|X] = \Omega \tag{17.81}$$

where

$$\Omega_{mT\times mT} = \begin{bmatrix} \Sigma & 0 & \cdots & 0 \\ 0 & \Sigma & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & \Sigma \end{bmatrix}, \Sigma_{T\times T} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1T} \\ \gamma_{12} & \gamma_{22} & \cdots & \gamma_{2T} \\ & & \cdots & \\ \gamma_{1T} & \gamma_{2T} & \cdots & \gamma_{TT} \end{bmatrix} \tag{17.82}$$

An estimator of Σ is

$$\hat{\Sigma} = \frac{1}{m}\sum_{i=1}^{m}\hat{u}_i\hat{u}_i^\top \tag{17.83}$$

where $\hat{u}_i = [\hat{u}_{i1}, \cdots, \hat{u}_{iT}]^\top$ is the OLS regression residual of (17.81). Σ only depends on $\frac{1}{2}T(T+1)$ parameters, no matter how large the sample size is. FGLS is optional in this model.

17.6 Summary

Theoretically, GLS is the BLUE, while FGLS is neither linear or unbiased.

$\Omega(\hat{\gamma})$ is a nonlinear function of data, so $\hat{\beta}_{FGLS}$ is a nonlinear function of y, so that it is biased at most cases.

In a big sample, the FGLS estimator is consistent and more efficient than OLS but faced with mis-specification risk of the conditional covariance function $E(uu^\top|X)$.

In most cases, OLS estimator with rubost covariance matrix is more robust than the FGLS estimator, so suggest to use OLS + robust.

18 Instrumental Variables

Consider a model

$$y = X\beta + u, E[uu^\top] = \sigma^2 I \tag{18.1}$$

If $E[u_t|X_t] = 0$ does not hold, then $\hat{\beta}_{OLS}$ is not unbiased and not consistent. That happens under many scenarios, e.g., omitted variables, reverse causality, errors in variables.

Example 18.1. *(Errors in Variables) Consider a model*

$$y_t^\circ = \beta_1 + \beta_2 x_t^\circ + u_t^\circ, u_t^\circ \sim IID(0,\sigma^2) \tag{18.2}$$

where y_t° and x_t° are not observed. Instead, we observe

$$x_t = x_t^\circ + v_{1t}, v_{1t} \sim IID(0,\omega_1^2) \tag{18.3}$$

$$y_t = y_t^\circ + v_{2t}, v_{2t} \sim IID(0,\omega_2^2) \tag{18.4}$$

where v_{1t}, v_{2t} are independent with y_t°, x_t° and u_t°; v_{1t} and v_{2t} are independent with each other; x_t° and t° are independent with each other.

Collect all observed variables and form a model

$$y_t = \beta_1 + \beta_2 x_t + u_t^\circ + v_{2t} - \beta_2 v_{1t} = \beta_1 + \beta_2 x_t + u_t \tag{18.5}$$

Then

$$E[u_t|x_t] = E[u_t^\circ + v_{2t} - \beta_2 v_{1t}|v_{1t}] \quad (18.6)$$
$$= -\beta_2 v_{1t} \quad (18.7)$$
$$Cov(x_t, u_t) = E[x_t u_t] \quad (18.8)$$
$$= E[x_t E[u_t|x_t]] \quad (18.9)$$
$$= -E[(x_t^\circ + v_{1t})\beta_2 v_{1t}] \quad (18.10)$$
$$= -\beta_2 \omega_1^2 \quad (18.11)$$

Thus $E[u_t|x_t] \neq 0$.

18.1 The Simple IV Estimator

Consider (18.1), assume $E[u_t|X_t] \neq 0$, so some variables of $X_{n\times k}$ are endogenous. Given a matrix $W_{n\times k}$, such that

(a) instrument exogeneity: $E[u_t|W_t] = 0$, i.e., W is predetermined;

(b) instrument relevance: $W^\top X$ is nonsingular and $S_{W^\top X} = \underset{n\to\infty}{plim} \frac{1}{n} W^\top X$ is deterministic and nonsingular (to show consistency).

The moment conditions of (18.1) with W is

$$W^\top (y - X\beta) = 0_{k\times 1} \quad (18.12)$$

Then we obtain the simple IV estimator

$$\hat{\beta}_{IV} = (W^\top X)^{-1} W^\top y \quad (18.13)$$

Take $y = X\beta + u$ to $\hat{\beta}_{IV}$, obtain

$$\hat{\beta}_{IV} = (W^\top X)^{-1} W^\top (X\beta + u) \quad (18.14)$$
$$= \beta + (W^\top X)^{-1} W^\top u \quad (18.15)$$
$$\Rightarrow E[\hat{\beta}_{IV} - \beta] = E[(W^\top X)^{-1} W^\top u] \quad (18.16)$$
$$= (W^\top X)^{-1} E[W^\top u] \quad (18.17)$$

Since $E[u_t|W_t] = 0$, then $E[W^\top u] = 0$, so $\hat{\beta}_{IV}$ is consistent.

The variance of $\underset{n\to\infty}{plim}\, n^{1/2} \hat{\beta}_{IV}$ is

$$Var[\underset{n\to\infty}{plim}\, n^{1/2} \hat{\beta}_{IV}] = Var[\underset{n\to\infty}{plim}\, n^{1/2}(\hat{\beta}_{IV} - \beta)] \quad (18.18)$$
$$= Var[\underset{n\to\infty}{plim}(\frac{1}{n} W^\top X)^{-1} \frac{1}{\sqrt{n}} W^\top u] \quad (18.19)$$
$$= Var[\underset{n\to\infty}{plim}(S_{W^\top X})^{-1} \frac{1}{\sqrt{n}} W^\top u] \quad (18.20)$$
$$= (S_{W^\top X})^{-1} E[\frac{1}{\sqrt{n}} W^\top u u^\top W](S_{W^\top X}^\top)^{-1} \quad (18.21)$$
$$= \sigma^2 (S_{W^\top X})^{-1} S_{W^\top W} (S_{W^\top X}^\top)^{-1} \quad (18.22)$$

where $S_{W^\top W} \equiv \underset{n\to\infty}{plim}\, n^{-1} W^\top W$. Further,

$$\sigma^2 (S_{W^\top X})^{-1} S_{W^\top W} (S_{W^\top X}^\top)^{-1} \quad (18.23)$$
$$= \sigma^2 \underset{n\to\infty}{plim}\, n (W^\top X)^{-1} (W^\top W)(X^\top W)^{-1} \quad (18.24)$$
$$= \sigma^2 \underset{n\to\infty}{plim}(n^{-1} X^\top W (W^\top W)^{-1} W^\top X)^{-1} \quad (18.25)$$
$$= \sigma^2 \underset{n\to\infty}{plim}(n^{-1} X^\top P_W X)^{-1} \quad (18.26)$$

Note that $S_{W^\top X} = \underset{n\to\infty}{plim} \frac{1}{n} W^\top X$ is deterministic and nonsingular.

Assume $E[uu^\top|W] = \sigma^2 I$, then $\sqrt{n}(\hat{\beta}_{IV} - \beta)$ is asymptotically normal distributed,

$$\sqrt{n}(\hat{\beta}_{IV} - \beta) \sim^a N(0, \sigma^2[E[X_t^\top W_t]E[W_t^\top W_t]^{-1}E[W_t^\top X_t]]^{-1}) \tag{18.27}$$

where the asymptotic covariance matrix is

$$\sigma^2[E[X_t^\top W_t]E[W_t^\top W_t]^{-1}E[W_t^\top X_t]]^{-1} = \sigma^2 \underset{n\to\infty}{plim}(n^{-1}X^\top P_W X)^{-1} \tag{18.28}$$

If W is optional, the best choice is to minimize the asymptotic covariance matrix. Here σ^2 can be estimated with $\hat{\sigma}^2 = \frac{1}{n}\sum_{t=1}^n \hat{u}_{IV}$ where $\hat{u}_{IV} = y - X\hat{\beta}_{IV}$.

18.2 The Generalized IV Estimator

Consider $X_{n\times k}$ and a general iv $W_{n\times l}$, and the moment conditions

$$W^\top(y - X\beta) = 0_{l\times 1} \tag{18.29}$$

where all exogenous variables of X is contained in W. There are three cases:

(a) $l = k$, exactly identified, the generalized IV degenerates to the simple IV;

(b) $l < k$, underidentified, there is no unique solution unless we add new IV to ensure $l \geq k$ or impose $k - l$ constrains on β in addition;

(c) $l > k$, overidentified, study below

Assume $l > k$, we can select k linear combination of l columns of W to make it exactly identified. That is, to find a matrix $J_{l\times k}$ to construct $(MJ)_{n\times k}$; such J must satisfy

(1) full column rank, $Rank(J) = k$;

(2) deterministic or asymptotically deterministic at least;

(3) minimize the asymptotic covariance matrix

$$\sigma^2 \underset{n\to\infty}{plim}(\frac{1}{n}X^\top P_{WJ}X)^{-1} \tag{18.30}$$

Such J is

$$J = (W^\top W)^{-1}W^\top X \tag{18.31}$$

then

$$WJ = W(W^\top W)^{-1}W^\top X = P_W X \tag{18.32}$$

So the moment conditions become

$$X^\top P_W(y - X\beta) = 0 \tag{18.33}$$

($P_W X$ is a $n \times k$ matrix) which indicates the generalized IV estimator (GIV) is

$$\hat{\beta}_{GIV} = (X^\top P_W X)^{-1}X^\top P_W y \tag{18.34}$$

$\hat{\beta}_{GIV}$ is also the solution of

$$\min Q(\beta) = (y - X\beta)^\top P_W(y - X\beta) \tag{18.35}$$

Now we derive the distribution of $\hat{\beta}_{GIV}$, assuming

(1) $y_t = X_t^\top\beta + u_t, X_t = (x_{t1}, \cdots, x_{tk})^\top$;

(2) (y_t, X_t, W_t) is a random realization, where $W_t = [w_{t1}, \cdots, w_{tl}]^\top$ and $l > k$;

(3) instrument exogeneity: $E[u_t|W_t] = 0$;

(4) instrument relation: $\underset{n\to\infty}{plim} \frac{1}{n}W^\top X = E[W_t X_t^\top]_{l\times k}$ with full columns rank k;

(5) $E[u_t^2|W_t] = \sigma^2$;

(6) $W^\top W$ and $E[W_t W_t^\top]$ are nonsingular, otherwise, we can drop one IV and obtain the same $\hat{\beta}_{GIV}$.

The $\hat{\beta}_{IV}$ can be written as

$$\hat{\beta}_{GIV} = (X^\top P_W X)^{-1} X^\top P_W y = \beta_0 + (X^\top P_W X)^{-1} X^\top P_W u \tag{18.36}$$

then $P_W = W(W^\top W)^{-1} W^\top$,

$$\sqrt{n}(\hat{\beta}_{GIV} - \beta_0) \tag{18.37}$$
$$= (\frac{1}{n} X^\top P_W X)^{-1} \sqrt{n} \frac{1}{n} X^\top P_W u \tag{18.38}$$
$$= (\frac{1}{n} X^\top P_W X)^{-1} \frac{1}{n} X^\top W (\frac{1}{n} W^\top W)^{-1} \frac{1}{\sqrt{n}} W^\top u \tag{18.39}$$

Let $Q^* = [[E(W_t X_t^\top)]^\top E[W_t W_t^\top]^{-1} E[W_t X_t^\top]]^{-1}$

$$(\frac{1}{n} X^\top P_W X)^{-1} \to^p Q^* \tag{18.40}$$
$$\frac{1}{n} X^\top W (\frac{1}{n} W^\top W)^{-1} \to^p E[W_t X_t^\top]^\top E[W_t W_t^\top]^{-1} \tag{18.41}$$

and

$$E[\frac{1}{\sqrt{n}} W u] = 0 \tag{18.42}$$
$$Var[\frac{1}{\sqrt{n}} W^\top u] = E[\frac{1}{n} W^\top u u^\top W] \tag{18.43}$$
$$= \frac{1}{n} W^\top E[u u^\top] W \tag{18.44}$$
$$= \sigma^2 \frac{1}{n} W^\top W \tag{18.45}$$
$$\underset{n\to\infty}{plim}\, Var[\frac{1}{\sqrt{n}} W^\top u] = \sigma^2 E[W_t W_t^\top] \tag{18.46}$$
$$\Rightarrow \frac{1}{\sqrt{n}} W^\top u \to^a N(0, \sigma^2 E[W_t W_t^\top]) \tag{18.47}$$

Thus, the asymptotic variance of $\sqrt{n}(\hat{\beta}_{GIV} - \beta_0)$ is

$$Q = Q^* E[X_t W_t^\top] E[W_t W_t^\top]^{-1} \sigma^2 E[W_t W_t^\top] (E[X_t W_t^\top] E[W_t W_t^\top]^{-1} \sigma^2)^\top (Q^*)^\top \tag{18.48}$$
$$= Q^* E[X_t W_t^\top] E[W_t W_t^\top]^{-1} \sigma^2 E[W_t W_t^\top] E[W_t W_t^\top]^{-1} E[X_t W_t^\top] \sigma^2 Q^* \tag{18.49}$$
$$= \sigma^2 Q^* E[X_t W_t^\top] E[W_t W_t^\top]^{-1} E[X_t W_t^\top] \sigma^2 Q^* \tag{18.50}$$
$$= \sigma^2 Q^* (Q^*)^{-1} Q^* \tag{18.51}$$
$$= \sigma^2 Q^* \tag{18.52}$$

Therefore

$$\sqrt{n}(\hat{\beta}_{GIV} - \beta_0) \sim^a N(0, Q), Q = \sigma^2 Q^* \tag{18.53}$$

Now we prove the $WJ = (W^\top W)^{-1} W^\top X$ is the most efficient IV.

THEOREM 18.1. *For any linear combination $Z = W\Lambda^\top$ where Λ is asymptotically deterministic at least. The asymptotic covariance of the $\sqrt{n}(\hat{\beta}_Z - \beta_0)$ is*

$$\sigma^2 [E[X_t Z_t^\top] E[Z_t Z_t^\top]^{-1} E[Z_t X_t^\top]]^{-1} \tag{18.54}$$

Show that

$$\sigma^2 [E[X_t Z_t^\top] E[Z_t Z_t^\top]^{-1} E[Z_t X_t^\top]]^{-1} - \sigma^2 [E[X_t W_t^\top] E[W_t W_t^\top]^{-1} E[W_t X_t^\top]]^{-1} \tag{18.55}$$

is positive semidefinite, equivalently, to show

$$E[X_t W_t^\top] E[W_t W_t^\top]^{-1} E[W_t X_t^\top] - E[X_t Z_t^\top] E[Z_t Z_t^\top]^{-1} E[Z_t X_t^\top] \tag{18.56}$$

is positive semidefinite.

Proof. $Z = W\Lambda^\top$, so $Z_t = W_t\Lambda^\top$; replace Z_t, then (18.56) becomes

$$E[X_tW_t^\top]E[W_tW_t^\top]^{-1}E[W_tX_t^\top] - E[X_tW_t^\top]\Lambda^\top E[\Lambda W_tW_t^\top\Lambda^\top]^{-1}\Lambda E[W_tX_t^\top] \tag{18.57}$$
$$=E[X_tW_t^\top](E[W_tW_t^\top]^{-1} - \Lambda^\top E[\Lambda W_tW_t^\top\Lambda^\top]^{-1}\Lambda)E[W_tX_t^\top] \tag{18.58}$$
$$\tag{18.59}$$

Since $E[W_tW_t^\top]^{-1}$ is symmetric and positive semidefinite, then can be decomposed as $E[W_tW_t^\top]^{-1} = DD^\top$, then

$$E[X_tW_t^\top](E[W_tW_t^\top]^{-1} - \Lambda^\top E[\Lambda W_tW_t^\top\Lambda^\top]^{-1}\Lambda)E[W_tX_t^\top] \tag{18.60}$$
$$=E[X_tW_t^\top](DD^\top - \Lambda^\top(\Lambda(DD^\top)^{-1}\Lambda^\top)\Lambda)E[W_tX_t^\top] \tag{18.61}$$
$$=E[X_tW_t^\top]D(I - D^{-1}\Lambda^\top((D^{-1}\Lambda^\top)^\top D^{-1}\Lambda^\top)^{-1}(D^{-1}\Lambda^\top)^\top)D^\top E[W_tX_t^\top] \tag{18.62}$$
$$=(D^\top E[W_tX_t^\top])^\top M_{D^{-1}\Lambda^\top}D^\top E[W_tX_t^\top] \tag{18.63}$$

which is positive semidefinite from Lemma 15.15. □

The GIV estimator is commonly known as the two-stage least-squares, and the two-stage regression is

Step (1) run an auxiliary regression:

$$X_j = W\pi_j + residuals \tag{18.64}$$

for every variables in X. If X_j is exogenous, then it should be contained in W, i.e., $P_WX_j = X_j$. Obtain the optimal IV:

$$P_WX = W\hat{\pi} \tag{18.65}$$

Step (2) run the main regression

$$y = P_WX\beta + residuals \tag{18.66}$$

and obtain

$$\hat{\beta}_{2sls} = (X^\top P_WX)^{-1}X^\top P_Wy = \hat{\beta}_{GIV} \tag{18.67}$$

where $P_WX = \hat{X}$ is the fitted value from Step (1).

Notice that the estimator for σ^2

$$s^2 = \frac{||y - P_WX\hat{\beta}_{GIV}||^2}{n-k} \tag{18.68}$$

is not consistent. We usually take

$$\hat{\sigma}^2 = \frac{||y - X\hat{\beta}_{GIV}||^2}{n} \tag{18.69}$$

or other asymptotically equivalent estimator to estimate σ^2.

LEMMA 18.1. *Extra IV can improve the efficiency of $\hat{\beta}_{GIV}$.*

18.3 Testing Overidentified Restrictions

Consider

$$y = X\beta + u, u \sim IID(0, \sigma^2 I) \tag{18.70}$$

where $X_{n\times k}$ and $W_{n\times l}$ be IV s.t. $l > k$. To calculate $\hat{\beta}_{GIV}$, only k efficient IV are used. To satisfy the consistent condition, we just need the k iV to satisfy instrument exogeneity condition. Let W^* denote extra instrument s.t. $\mathscr{S}(W) = \mathscr{S}(P_WX, W^*)$. We test

$$H_0 : E[W_tu_t] = 0, H_1 : E[W_tu_t] \neq 0 \tag{18.71}$$

Step (1) obtain $\hat{u}_{GIV}$ by

$$\hat{u}_{t,GIV} = y_t - X_t^\top\hat{\beta}_{GIV} \tag{18.72}$$

Step (2) run the main regression

$$\hat{u}_{GIV} = Wb + residuals \tag{18.73}$$

(alternative regression: $\hat{u}_{GIV} = P_WXb_1 + W^*b_2 + residuals$, giving the same SSE)

Step (3) test statistics, under H_0,

$$T = nR_0^2 \sim^a \chi^2_{l-k} \tag{18.74}$$

where

$$nR_0^2 = \frac{\hat{u}_{GIV}^\top P_W \hat{u}_{GIV}}{\hat{\sigma}^2_{GIV}} \tag{18.75}$$

Another test is called the Sargan test. The criterion function $Q(\beta) = (y - X\beta)^\top P_W (y - X\beta)$ take $\hat{u}_{GIV}^\top P_W \hat{u}_{GIV} = 0$ at $\hat{\beta}_{GIV}$. A test statistics is

$$S = \frac{Q(\hat{\beta}_{GIV})}{\hat{\sigma}^2} \tag{18.76}$$

which equals to

$$nR_0^2 = \frac{\hat{u}_{GIV}^\top P_W \hat{u}_{GIV}}{\hat{\sigma}^2_{GIV}} = \frac{1}{\hat{\sigma}^2}(y - X\hat{\beta}_{GIV})^\top P_W (y - X\hat{\beta}_{GIV}) \tag{18.77}$$

18.4 Durbin-Wu-Hausman Test

In many cases, we have no idea whether IV is need. We can test whether there are endogenous variables by the Durbin-Wu-Hausman test.

Consider

$$y = X\beta + u, u \sim IID(0, \sigma^2 I) \tag{18.78}$$

Assume $X_{n\times k} = [Z_{n\times k_1}, Y_{n\times k_2}]$ where Z is exogenous but Y may be endogenous. Assume $W_{n\times l}, l > k$ is an efficient IV, containing Z. We test whether Y is endogenous by

$$H_0 : E[X_t u_t] = 0, H_1 : E[X_t u_t] \neq 0 \tag{18.79}$$

Under H_0, $\hat{\beta}_{GIV}$ and $\hat{\beta}_{OLS}$ are both consistent, but under H_1, $\hat{\beta}_{GIV}$ is consistent while $\hat{\beta}_{OLS}$ is not consistent. That is equivalent to test

$$y = X\beta + P_W Y\delta + u \tag{18.80}$$

whether $\delta_{k_2\times 1} = 0$ by F-test.

18.5 Summary

- As long as both instrument exogeneity and instrument relation conditions hold, the IV estimator is consistent. But if the IV is weak, that standard error may be large, also is the asymptotic covariance;
- When X is exogenous, IV and OLS are both consistent but OLS is more efficient;
- When endogeneity is not a serve problem or there is no good IV, you had better use OLS;
- When endogeneity is a serve problem you have to solve it, OLS and bad IV both not acceptable.

19 Generalized Method of Moments

Consider a linear regression model:

$$y_t = X_t^\top \beta_0 + u_t \tag{19.1}$$

where β_0 is a unknown $k \times 1$ vector and some regressors are endogenous. Let W_t be a $l \times 1$ IV. Assume $\{\omega_t\} = \{y_t, X_t, W_t\}$ is a stationary and ergodic stochastic process.

Define

$$g_t(\omega_t, \beta_0) = W_t u_t = W_t(y_t - X_t^\top \beta_0) \tag{19.2}$$

and W_t satisfies the k orthogonal conditions

$$\begin{aligned} E[g_t(\omega_t, \beta_0)] = E[W_t u_t] = E[W_t(y_t - X_t^\top \beta_0)] = 0 && (19.3)\\ \Rightarrow E[W_t y_t] - E[W_t X_t^\top]\beta_0 = 0 && (19.4)\\ \Rightarrow \Sigma_{Wy} = \Sigma_{WX}\beta_0 && (19.5) \end{aligned}$$

where $\Sigma_{Wy} = E[W_t y_t], \Sigma_{WX} = E[W_t X_t^\top]$.

Example 19.1. *Consider*

$$lnWage_i = \beta_1 + \beta_2 S_i + \beta_3 EXPR_i + \beta_4 IQ_i + u_i \tag{19.6}$$

where S_i denotes year of schooling, $EXPR_i$ denotes years of experience, IQ_i denotes score on IQ test and β_2 denotes rate of return to schooling. Assume $E[S_i u_i] = E[EXPR_i u_i] = 0$.

Since IQ_i is the proxy of unobservable ability, there is errors in variable:

$$IQ_i = ABILITY_i + error_i \Rightarrow E[IQ_i u_i] \neq 0 \tag{19.7}$$

There are some IV: AGE_i, age in years, MED_i, years of mother's education, and $E[AGE_i u_i] = 0, E[MED_i u_i] = 0$. Now restate the model:

$$y_i = X_i^\top \beta_0 + u_t \tag{19.8}$$

where

$$y_i = lnWage_i \tag{19.9}$$

$$X_i = (1, S_i, EXPR_i, IQ_i)^\top \tag{19.10}$$

$$W_i = (1, S_i, EXPR_i, AGE_i, MED_i)^\top \tag{19.11}$$

$$\omega_i = (lnWage_i, S_i, EXPR_i, IQ_i, AGE_i, MED_i)^\top \tag{19.12}$$

$$k = 4, l = 5 \tag{19.13}$$

Here $S_i, EXPR_i$ are called included exogenous variables and AGE_i, MED_i are called excluded exogenous variables.

Let $l = k$, just identification of β_0 implies that β_0 is the only solution of

$$E[g_t(\omega_t, \beta_0)] = 0 \tag{19.14}$$

thus, $\forall \beta \neq \beta_0$, we have $E[g_t(\omega_t, \beta)] \neq 0$. The rank condition,

$$Rank(\Sigma_{WX}) = Rank(E[W_t X_t^\top]) = k \tag{19.15}$$

which ensures β_0 is the only solution of $E[g_t(\omega_t, \beta_0)] = 0$. Under this condition,

$$\beta_0 = \Sigma_{WX}^{-1} \Sigma_{Wy} \tag{19.16}$$

To identify β_0, it is necessary to have

$$l \geq k \tag{19.17}$$

With the first order condition, if $l = k$, β_0 is just identified; if $l > k$, β_0 is overidentified; if $l < k$, β_0 is not identified.

Consider conditional heteroscedasticity, assume $\{g_t\} = \{W_t u_t\}$ is a stationary and ergodic martingale difference sequence (MDS), satisfying

$$E[g_t g_t^\top] = E[W_t W_t^\top u_t^2] = S \tag{19.18}$$

where $S_{l\times l}$ is nonsingular. If $Var[u_t|W_t] = \sigma^2$, then $S = \sigma^2 \Sigma_{WW}$.

S is the asymptotically covariance matrix of $\sqrt{n}\bar{g} = \sqrt{n}\frac{1}{n}\sum_{t=1}^n g_t(\omega_t, \beta_0)$, (CLT)

$$\sqrt{n}\bar{g} = \frac{1}{\sqrt{n}} \sum_{t=1}^n W_t u_t \to^d N(0, S) \tag{19.19}$$

$$S = E[g_t g_t^\top] = avar[\sqrt{n}\bar{g}] \tag{19.20}$$

19.1 Definition of Generalized Method of Moment

The Generalized Method of Moment (GMM) estimator is constructed by the moment conditions; the idea is to construct a function of β_0 to ensure the sample moments match with population moments.

$$g_n(\beta) = \frac{1}{n} \sum_{t=1}^n g_t(\omega_t, \beta) \tag{19.21}$$

$$= \frac{1}{n} \sum_{t=1}^n W_t (y_t - X_t^\top \beta) \tag{19.22}$$

$$= \begin{bmatrix} \frac{1}{n}\sum_{t=1}^n W_{1t}(y_t - X_t^\top \beta) \\ \cdots \\ \frac{1}{n}\sum_{t=1}^n W_{lt}(y_t - X_t^\top \beta) \end{bmatrix} \tag{19.23}$$

containing l linear equations. The population moment condition implies

$$\frac{1}{n}\sum_{t=1}^{n} W_t(y_t - X_t^\top\beta) = S_{Wy} - S_{WX}\beta = 0 \tag{19.24}$$

where

$$S_{Wy} = \frac{1}{n}\sum_{t=1}^{n} W_t y_t, S_{WX} = \frac{1}{n}\sum_{t=1}^{n} W_t X_t^\top \tag{19.25}$$

If $l = k$ (just identification), and S_{WX} is nonsingular, then the GMM estimator of β_0 is

$$\hat{\beta} = S_{WX}^{-1}S_{Wy} = (\frac{1}{n}\sum_{t=1}^{n} W_t X_t^\top)^{-1}\frac{1}{n}\sum_{t=1}^{n} W_t y_t \tag{19.26}$$

Here $\hat{\beta}$ is also called the indirect least squares (ILS) or instrumental variables (IV). Since

$$S_{Wy} = \frac{1}{n}\sum_{t=1}^{n} W_t y_t \tag{19.27}$$

$$= \frac{1}{n}\sum_{t=1}^{n} W_t(X_t^\top\beta_0 + u_t) \tag{19.28}$$

$$= S_{WX}\beta_0 + S_{Wu} \tag{19.29}$$

$$\Rightarrow \hat{\beta}_0 - \beta_0 = S_{WX}^{-1}S_{Wu} \tag{19.30}$$

The ergodic theorem and Slutsky's theorem indicate ($Rank(\Sigma_{WX}) = k$)

$$S_{WX} = \frac{1}{n}\sum_{t=1}^{n} W_t X_t^\top \to^p E[W_t X_t^\top] = \Sigma_{WX} \tag{19.31}$$

$$\Rightarrow S_{WX}^{-1} \to^p \Sigma_{WX}^{-1} \tag{19.32}$$

$$S_{Wu} = \frac{1}{n}\sum_{t=1}^{n} W_t u_t \to^p E[W_t u_t] = 0 \tag{19.33}$$

Thus

$$\hat{\beta} - \beta_0 = S_{WX}^{-1}S_{Wu} \to^p 0 \tag{19.34}$$

which implies $\hat{\beta}$ is a consistent estimator of β_0.

However, if $l > k$, then there is no solution for

$$S_{Wy} - S_{WX}\beta = 0 \tag{19.35}$$

since there is k unknown parameters and l equations. Alternatively, the GMM minimizes chi-square, i.e., to make $S_{Wy} - S_{WX}\beta$ close to 0. Let $\Lambda_{l\times l}$ be a symmetric positive semidefinite (may depend data) and $\Lambda \to^p \Lambda_0$ where Λ_0 is a symmetric positive semidefinite. The target function is defined as

$$Q(\beta, \Lambda) = n g_n(\beta)^\top \Lambda g_n(\beta) \tag{19.36}$$

$$= n(S_{Wy} - S_{WX}\beta)^\top \Lambda (S_{Wy} - S_{WX}\beta) \tag{19.37}$$

The F.O.C.,

$$Q(\beta, \Lambda) = n[S_{Wy}^\top \Lambda S_{Wy} - 2S_{Wy}^\top \Lambda S_{WX}\beta + \beta^\top S_{WX}^\top \Lambda S_{WX}\beta] \tag{19.38}$$

$$\frac{\partial Q(\beta, \Lambda)}{\partial \beta} = -2S_{WX}^\top \Lambda S_{Wy} + 2S_{WX}^\top \Lambda S_{WX}\beta \tag{19.39}$$

$$\Rightarrow \hat{\beta}(\Lambda) = (S_{WX}^\top \Lambda S_{WX})^{-1} S_{WX}^\top \Lambda S_{Wy} \tag{19.40}$$

$$= \beta_0 + (S_{WX}^\top \Lambda S_{WX})^{-1} S_{WX}^\top \Lambda S_{Wu} \tag{19.41}$$

where $S_{WX}^\top \Lambda S_{WX}$ is nonlinear (need $Rank(S_{WX}) = k$).

Under regular conditions,

$$\hat{\beta}(\Lambda) = (S_{WX}^{\top}\Lambda S_{WX})^{-1}S_{WX}^{\top}\Lambda S_{Wy} \to^p \beta_0 \tag{19.42}$$
$$\sqrt{n}(\hat{\beta}(\Lambda)) \to^d N(0, avar(\sqrt{n}\hat{\beta}(\Lambda))) \tag{19.43}$$

where
$$avar(\sqrt{n}\hat{\beta}(\Lambda)) = (\Sigma_{WX}^{\top}\Lambda_0\Sigma_{WX})^{-1}\Sigma_{WX}^{\top}\Lambda_0 S\Lambda_0\Sigma_{WX}(\Sigma_{WX}^{\top}\Lambda_0\Sigma_{WX})^{-1} \tag{19.44}$$

A consistent estimator of $avar(\sqrt{n}\hat{\beta}(\Lambda))$ is

$$a\hat{v}ar(\sqrt{n}\hat{\beta}(\Lambda)) = (S_{WX}^{\top}\Lambda S_{WX})^{-1}S_{WX}^{\top}\Lambda\hat{S}\Lambda S_{WX}(S_{WX}^{\top}\Lambda S_{WX})^{-1} \tag{19.45}$$

where S can be estimated with the Whit or hemoscedasticity consistent (HC), as $\hat{\beta}(\Lambda) \to^p \beta_0$, then

$$\hat{S}_{HC} = \frac{1}{n}\sum_{t=1}^{n} W_t W_t^{\top}\hat{u}_t^2 \to^p S \tag{19.46}$$

where $\hat{u}_t = y_t - X_t\hat{\beta}(\Lambda)$ (Note that here we assume $\{W_t u_t\}$ is MDS, so there is no serial relation, i.e., $E[W_t u_t u_s^{\top} W_s^{\top}] = 0, \forall t \neq s$).

Now prove the consistency and asymptotic covariance.

$$\hat{\beta}(\Lambda) = (S_{WX}^{\top}\Lambda S_{WX})^{-1}S_{WX}^{\top}\Lambda S_{Wy} \tag{19.47}$$
$$= (S_{WX}^{\top}\Lambda S_{WX})^{-1}S_{WX}^{\top}\Lambda(S_{WX}\beta_0 + S_{Wu}) \tag{19.48}$$
$$= \beta_0 + (S_{WX}^{\top}\Lambda S_{WX})^{-1}S_{WX}^{\top}\Lambda S_{Wu} \tag{19.49}$$

With the ergodic theorem and Slutsky's theorem, we have

$$S_{WX} = \frac{1}{n}\sum_{t=1}^{n} W_t X_t^{\top} \to^p E[W_t X_t^{\top}] = \Sigma_{WX} \tag{19.50}$$
$$S_{WX}^{\top}\Lambda \to^p \Sigma_{WX}^{\top}\Lambda_0 \tag{19.51}$$
$$(S_{WX}^{\top}\Lambda S_{WX})^{-1} \to^p (\Sigma_{WX}^{\top}\Lambda_0\Sigma_{WX})^{-1} \tag{19.52}$$
$$S_{Wu} = \frac{1}{n}\sum_{t=1}^{n} W_t u_t \to^p E[W_t u_t] = 0 \tag{19.53}$$
$$\Rightarrow \hat{\beta}(\Lambda) - \beta_0 \to^p 0 \tag{19.54}$$

thus, $\hat{\beta}(\Lambda)$ is a consistent estimator of β_0.

Moreover, with CLT of MDS, $\{g_t\} = \{W_t u_t\}$, we have

$$S = E[g_t g_t^{\top}] = E[W_t W_t^{\top} u_t^2] \tag{19.55}$$
$$\sqrt{n}S_{Wu} = \frac{1}{\sqrt{n}}\sum_{t=1}^{n} W_t u_t \to^d N(0, S) \tag{19.56}$$

With the Slutsky's theorem,
$$S_{WX}^{\top}\Lambda\sqrt{n}S_{Wu} \to^d \Sigma_{WX}^{\top}\Lambda_0 N(0, S) \tag{19.57}$$

So

$$\sqrt{n}(\hat{\beta}(\Lambda) - \beta_0) \to^d (\Sigma_{WX}^{\top}\Lambda_0\Sigma_{WX})^{-1}\Sigma_{WX}^{\top}\Lambda N(0, S) \tag{19.58}$$
$$= N(0, (\Sigma_{WX}^{\top}\Lambda_0\Sigma_{WX})^{-1}\Sigma_{WX}^{\top}\Lambda_0 S\Lambda_0\Sigma_{WX}(\Sigma_{WX}^{\top}\Lambda_0\Sigma_{WX})^{-1}) \tag{19.59}$$

There are eleven terms.

19.2 Efficient Generalized Method of Moment

The GMM estimator $\hat{\beta}(\Lambda)$ depends on a weighted matrix Λ, so it is naturally to choose a Λ to minimize $avar(\hat{\beta}(\Lambda))$. Hansen (1982) shows the best Λ is

$$\Lambda = \hat{S}^{-1} \tag{19.60}$$

where $\hat{S} \to^p S = E[W_t W_t^\top u_t^2] = avar(\sqrt{n} g_n(\beta_0))$. Then

$$avar(\sqrt{n}\hat{\beta}(\Lambda)) = (\Sigma_{WX}^\top \Lambda_0 \Sigma_{WX})^{-1} \tag{19.61}$$

$$a\hat{v}ar(\sqrt{n}\hat{\beta}(\Lambda)) = (\Sigma_{WX}^\top \hat{S}^{-1} \Sigma_{WX})^{-1} \tag{19.62}$$

$$\tag{19.63}$$

which is a consistent estimator of $avar(\sqrt{n}\hat{\beta}(\Lambda))$.

A two-step efficient GMM estimator is constructed by following procedures: let $\Lambda = I$ or $\Lambda = S_{WW}^{-1} = (\frac{1}{n} W^\top W)^{-1}$

Step (1) obtain $\hat{S}$:

$$\hat{S}(\Lambda) = \frac{1}{n} \sum_{t=1}^{n} W_t W_t^\top (y_t - X_t^\top \hat{\beta}(\Lambda))^2 \tag{19.64}$$

Step (2) calculate $\hat{\beta}$:

$$\hat{\beta}(\hat{S}(\Lambda)^{-1}) = \underset{\beta}{argmin}\, n g_n(\beta)^\top \hat{S}(\Lambda)^{-1} g_n(\beta) \tag{19.65}$$

$$\Rightarrow \hat{\beta}(\hat{S}(\Lambda)^{-1}) = (S_{WX}^\top \hat{S}(\Lambda) S_{WX})^{-1} S_{WX}^\top \hat{S}(\Lambda)^{-1} S_{Wy} \tag{19.66}$$

The disadvantage is the numerical value of $\hat{\beta}(\hat{S}(\Lambda)^{-1})$ depends on Λ. So we can repeat Step (1)-(2) to update $\hat{S}$, which is called the iterated efficient estimator (with the same asymptotic distribution with the two-step estimator).

Further, it is feasible to estimate S and β simultaneously, the continuous updating efficient GMM estimator is

$$\hat{\beta}(\hat{S}_{CU}^{-1}) = \underset{\beta}{argmax}\, Q(\beta, \hat{S}(\beta)^{-1}) \tag{19.67}$$

$$= \underset{\beta}{argmax}\, n g_n(\beta) \hat{S}(\beta)^{-1} g_n(\beta) \tag{19.68}$$

where $\hat{S}(\beta) = \frac{1}{n}\sum_{t=1}^{n} W_t W_t^\top (y_t - X_t^\top \beta)^2$.

19.3 2SLS and GMM

Under the hemoscedasticity assumption,

$$S = E[W_t W_t^\top u_t^2] = \sigma^2 \Sigma_{WW} \tag{19.69}$$

and a consistent estimator of it is $\hat{S} = \hat{\sigma}^2 S_{WW}$ where $\hat{\sigma}^2 \to^p \sigma^2, \hat{\sigma}^2 = \frac{1}{n}\sum_{t=1}^{n}(y_t - X_t^\top \hat{\beta})^2, \hat{\beta} \to^p \beta_0$.

The efficient GMM estimator is

$$\hat{\beta}(\hat{\sigma}^{-2} S_{WW}^{-1}) = (S_{WX}^\top \hat{\sigma}^2 S_{WW}^{-1} S_{WX})^{-1} S_{WX}^\top \hat{\sigma}^{-2} S_{WW}^{-1} S_{Wy} \tag{19.70}$$

$$= (S_{WX}^\top S_{WW}^{-1} S_{WX})^{-1} S_{WX}^\top S_{WW}^{-1} S_{Wy} \tag{19.71}$$

$$= \hat{\beta}(S_{WW}^{-1}) \tag{19.72}$$

which is independent with $\hat{\sigma}^2$.

In fact,

$$\hat{\beta}(S_{WW}^{-1}) = (S_{WX}^\top S_{WW}^{-1} S_{WX})^{-1} S_{WX}^\top S_{WW}^{-1} S_{Wy} \tag{19.73}$$

$$= (X^\top P_W X)^{-1} X^\top P_W y \tag{19.74}$$

$$= (\hat{X}^\top \hat{X})^{-1} \hat{X}^\top y \tag{19.75}$$

$$= \hat{\beta}_{2sls} \tag{19.76}$$

where $P_W = W(W^\top W)^{-1}W^\top, \hat{X} = P_W X$. The asymptotic covariance of $\hat{\beta}_{2sls}$ is

$$avar(\hat{\beta}_{2sls}) = (\Sigma_{WX}^\top S^{-1}\Sigma_{WX})^{-1} = \sigma^2(\Sigma_{WX}^\top \Sigma_{WW}^{-1}\Sigma_{WX})^{-1} \tag{19.77}$$

not dependent with $\hat{\sigma}^2$, but the consistent estimator

$$av\hat{a}r(\hat{\beta}_{2sls}) = \hat{\sigma}_{2sls}^2(S_{WX}^\top S_{WW}^{-1}S_{WX})^{-1}\hat{\sigma}_{2sls}^2 \tag{19.78}$$

$$\hat{\sigma}_{2sls}^2 = \frac{1}{n}\sum_{t=1}^{n}(y_t - X_t^\top \hat{\beta}_{2sls})^2 \tag{19.79}$$

dependent with $\hat{\sigma}_{2sls}^2$.

19.4 General Definition of Generalized Method of Moment

Assumption:

(1) random realization $\{\omega_t\}$ is IID;

(2) parameter $\beta_{k\times 1}$ is unknown and the parameter space is a open interval containing β_0;

(3) $g(\omega_t, \beta)_{l\times 1}$ are moment equations and $l > k$;

(4) $E[g(\omega_t, \beta_0)] = 0$ and $E[g(\omega_t, \beta)] \neq 0, \forall\beta \neq \beta_0$;

(5) $\forall\omega_t, \beta$, the $l \times k$ partial derivative matrix $\frac{\partial g(\omega_t,\beta)}{\partial\beta}$ is continuous and full column rank k; $\forall\beta$, the $E[\frac{\partial g(\omega_t,\beta)}{\partial\beta}]$ has full column rank k.

Let $\Lambda_{l\times l}$ be a symmetric positive semidefinite weighted matrix, and $\Lambda \to^p \Lambda_0$. The GMM estimator of β is defined as

$$\hat{\beta}(\Lambda) = \underset{\beta}{argmin}\, Q(\beta, \Lambda) = \underset{\beta}{argmin}\, ng_n(\beta)^\top \Lambda g_n(\beta) \tag{19.80}$$

where $g_n(\beta) = \frac{1}{n}\sum_{t=1}^{n} g(\omega_t, \beta)$. Under assumption (1)-(5), $\hat{\beta}(\Lambda)$ is consistent with β_0.

The F.O.C. of (19.80) is

$$n\frac{\partial g_n(\hat{\beta})^\top}{\partial\beta}\Lambda g_n(\hat{\beta}(\Lambda)) = 0 \tag{19.81}$$

The Taylor expansion of $\sqrt{n}g_n(\hat{\beta}(\Lambda))$ at β_0 is

$$\sqrt{n}(g_n(\hat{\beta}(\Lambda)) - g_n(\beta_0)) \simeq \frac{\partial g_n(\beta_0)}{\partial\beta^\top}\sqrt{n}(\hat{\beta}(\Lambda) - \beta_0) \tag{19.82}$$

So the F.O.C. becomes

$$\frac{\partial g_n(\hat{\beta})^\top}{\partial\beta}\Lambda\frac{\partial g_n(\beta_0)}{\partial\beta^\top}\sqrt{n}(\hat{\beta}(\Lambda) - \beta_0) \simeq -\sqrt{n}\frac{\partial g_n(\hat{\beta}(\Lambda))^\top}{\partial\beta}\Lambda g_n(\beta_0) \tag{19.83}$$

With assumption (5) and Λ is positive semidefinite,

$$\sqrt{n}(\hat{\beta}(\Lambda) - \beta_0) \simeq -[\frac{\partial g_n(\hat{\beta}(\Lambda))^\top}{\partial\beta}\Lambda\frac{\partial g_n(\beta_0)}{\partial\beta^\top}]^{-1}\frac{\partial g_n(\hat{\beta}(\Lambda))^\top}{\partial\beta}\Lambda\sqrt{n}(g_n(\beta_0) - 0) \tag{19.84}$$

CLT implies

$$\sqrt{n}(g_n(\beta_0) - 0) \sim^d N(0, E[g(\omega_t, \beta_0)g(\omega_t, \beta_0)^\top]) \tag{19.85}$$

From the continuity of partial derivatives and LLN, Slutsky's theorem, we have

$$\sqrt{n}(\hat{\beta}(\Lambda) - \beta_0) \sim^d N(0, Q) \tag{19.86}$$

where

$$Q = \{E[\frac{\partial g(\omega_t, \beta_0)^\top}{\partial\beta}]\Lambda_0 E[\frac{\partial g(\omega_t, \beta_0)}{\partial\beta^\top}]\}^{-1} \tag{19.87}$$

$$E[\frac{\partial g(\omega_t, \beta_0)^\top}{\partial\beta}]\Lambda_0 E[g(\omega_t, \beta_0)g(\omega_t, \beta_0)^\top]\Lambda_0 E[\frac{\partial g(\omega_t, \beta_0)}{\partial\beta^\top}] \tag{19.88}$$

$$\{E[\frac{\partial g(\omega_t, \beta_0)^\top}{\partial\beta}]\Lambda_0 E[\frac{\partial g(\omega_t, \beta_0)}{\partial\beta^\top}]\}^{-1} \tag{19.89}$$

The optimal weighted matrix is $\Lambda^{opt} = E[g(\omega_t, \beta_0)g(\omega_t, \beta_0)^\top]^{-1}$, thus $\sqrt{n}(\hat{\beta}(\Lambda)^{opt} - \beta_0) \sim^d N(0, Q^{opt})$

$$Q^{opt} = \{E[\frac{\partial g(\omega_t, \beta_0)^\top}{\partial \beta}]E[g(\omega_t, \beta_0)g(\omega_t, \beta_0)^\top]^{-1}E[\frac{\partial g(\omega_t, \beta_0)}{\partial \beta^\top}]\}^{-1} \tag{19.90}$$

Overidentified test: Sargan test for GMM,

$$S = ng_n(\hat{\beta}^{opt}_{GMM})\Lambda^{opt}g_n(\hat{\beta}^{opt}_{GMM}) \sim^a \chi^2_{l-k} \tag{19.91}$$

S is just the target function, $Q(\beta, \Lambda)$. Under the hemoscedasticity and MDS condition, the Sargan test statistics equals to Hansen J test statistics.

19.5 Application: Spatial Autoregression Model

Consider a spatial autoregression model

$$y = \rho Wy + X\beta + \epsilon \tag{19.92}$$

where X is exogenous but y is endogenous. [12]

Rewrite it

$$y = [Wy, X]\begin{bmatrix}\rho\beta\end{bmatrix} + \epsilon = Q\theta + \epsilon \tag{19.93}$$

where $Q = [Wy, X], \theta = [\rho, \beta^\top]^\top$. Assume there is a IV Z ($WX, W^2X, \cdots$ are optional) satisfying

(1) $E[Z^\top \epsilon] = 0$;

(2) $E[Z^\top Q]$ is nonsingular.

Then we can implement GMM estimation. Let $\Lambda_{l\times l}$ be a symmetric positive semidefinite matrix and $\Lambda \to^p \Lambda_0$. The GMM estimator is defined as

$$\hat{\theta}(\Lambda) = \underset{\theta}{argmin}\, ng_n(\theta)^\top \Lambda g_n(\theta) \tag{19.94}$$

$$= \underset{\theta}{argmin}\, n(S_{Zy} - S_{Zq}\theta)^\top \Lambda(S_{Zy} - S_{Zq}\theta) \tag{19.95}$$

where $g_n(\theta) = \frac{1}{n}\sum_{t=1}^n g(\omega_t, \theta) = \frac{1}{n}\sum_{t=1}^n z_t(y_t - q_t^\top\theta)$, z_i is the transposed i-the row of Z and q_i is the transposed j-th column of Q,

$$S_{Zy} = \frac{1}{n}\sum_{i=1}^n z_i y_i, S_{zq} = \frac{1}{n}\sum_{i=1}^n z_i q_i^\top \tag{19.96}$$

F.O.C.

$$\hat{\theta}(\Lambda) = (S_{Zq}^\top \Lambda S_{zq})^{-1}S_{zq}^\top \Lambda S_{Zy} \tag{19.97}$$

Let $\Lambda = \hat{S}^{-1}$ and $\hat{S} = \frac{1}{n}z_i z_i^\top(y_i q_i^\top \hat{\theta})^2$, the efficient GMM estimator is

$$\hat{\theta}_{GMM} = (S_{zq}^\top \hat{S}^{-1}S_{zq})^{-1}S_{zq}^\top \hat{S}^{-1}S_{zq} \tag{19.98}$$

with covariance

$$\hat{Var}(\hat{\theta}_{GMM}) = (S_{zq}^\top \hat{S}^{-1}S_{zq})^{-1} \tag{19.99}$$

20 Maximum Likelihood Estimation

20.1 The Maximum Likelihood Estimator

Suppose we have a set of observed data $x_1, x_2, \cdots, x_n$, given a fully parametric model (we can calculate the pdf as long as we get all parameters), how to estimate the parametric model? If we assume there is a real parameter vector θ for the model, then we can take all observations as produced by a model with a sample of θ. We observe it, so the data is equipped with the largest probability of realization.

Let $X_1, X_2, \cdots, X_n$ be an IID sample from the same distribution with pdf $f(x_i;\theta)$ where f is deterministic and θ is the parametric vector. Given IID observations $x_1, x_2, \cdots, x_n$, its joint density given $\theta \in \mathbf{R}^k$ is

$$f(x_1, x_2, \cdots, x_n; \theta) = \prod_{i=1}^n f(x_i; \theta) \tag{20.1}$$

[12](we can transform it to $y = (I - \rho W)^{-1}(X\beta + \epsilon)$ and then use MLE)

and generally, we have $f(x_1, x_2, \cdots, x_n; \theta) \geq 0$ and

$$\int \cdots \int f(x_1, x_2, \cdots, x_n; \theta) dx_1 \cdots dx_n = 1 \tag{20.2}$$

Now we take θ as a variable, so the joint density becomes the likelihood function,

$$L(\theta | x_1, x_2, \cdots, x_n) = f(x_1, x_2, \cdots, x_n; \theta) = \prod_{i=1}^{n} f(x_i; \theta) \tag{20.3}$$

where $L : \mathbb{R}^k \to \mathbb{R}_+$ is just a function, not any density. Let $x = (x_1, x_2, \cdots, x_n)$ denote the sample, we rewrite the joint density and the likelihood function as $f(x; \theta), L(\theta|x)$.

We believe the real θ is the maximizer of $L(\theta|x)$,

$$\hat{\theta}_{MLE} = \underset{\theta}{argmax}\, L(\theta|x) \tag{20.4}$$

Usually, we use the log-likelihood function, the problem is the same as $\ln(\cdot)$ is strictly monotone

$$\begin{aligned}\hat{\theta}_{MLE} &= \underset{\theta}{argmax} \ln L(\theta|x)\, \underset{\theta}{argmax} \prod_{i=1}^{n} f(x_i; \theta) && (20.5)\\ &= \underset{\theta}{argmax} \sum_{i=1}^{n} lnf(x_i; \theta) && (20.6)\\ & && (20.7)\end{aligned}$$

Summation is easier to handle than product. Now what we faced is a maximum problem. Recall the Berge's Maximization Theorem. If the target function and parameter space are "good", the maximizer is endowed with some great properties. Formally, we want the $f(x; \theta)$ is regular, i.e.,

1. the support region $S_X = \{x : f(x; \theta) > 0\}$ in independent with θ;
2. $f(x; \theta) > 0$ is at least three times differentiable;
3. the real $\theta \in \Theta$, which is compact.

If $f(x; \theta)$ is regular, we can derive the F.O.C. of the problem,

$$\frac{\partial lnL(\theta|x)}{\partial \theta} = \begin{bmatrix} \frac{\partial lnL(\theta|x)}{\partial \theta_1} \\ \frac{\partial lnL(\theta|x)}{\partial \theta_2} \\ \cdots \\ \frac{\partial lnL(\theta|x)}{\partial \theta_k} \end{bmatrix} = \mathbf{0} \tag{20.8}$$

The left hand side is called the score vector,

$$S(\theta|x) = \frac{\partial lnL(\theta|x)}{\partial \theta} \tag{20.9}$$

Then, the MLE satisfies $S(\hat{\theta}_{MLE}|x) = 0$. And

$$S(\theta|x) = \frac{\partial lnL(\theta|x)}{\partial \theta} = \frac{\partial lnf(x; \theta)}{\partial \theta} = \sum_{i=1}^{n} \frac{\partial lnf(x_i; \theta)}{\partial \theta} = \sum_{i=1}^{n} S(\theta|x_i) \tag{20.10}$$

where $S(\theta|x_i)$ is called x_i's contribution to the score vector.

THEOREM 20.1. *Let $f(x; \theta)$ be regular, then*

- $E(S(\theta|x_i)) = \int S(\theta|x_i) f(x_i; \theta) dx_i = 0$;
- $Var[S(\theta|x_i)] = I(\theta|x_i)$.
  - *θ is a scalar,*

$$Var[S(\theta|x_i)] = E(S(\theta|x_i)^2) = \int S^2(\theta|x_i) f(x_i; \theta) dx_i = I(\theta|x_i) \tag{20.11}$$

– *θ is a vector,*

$$Var[S(\theta|x_i)] = E(S(\theta|x_i)S(\theta|x_i)^\top) = \int S(\theta|x_i)S(\theta|x_i)^\top f(x_i;\theta)dx_i = I(\theta|x_i) \tag{20.12}$$

Proof.

$$E(S(\theta|x_i)) = \int S(\theta|x_i)f(x_i;\theta)dx_i = \int \frac{\partial lnf(x_i;\theta)}{\partial\theta} f(x_i;\theta)dx_i \tag{20.13}$$

$$= \int \frac{1}{f(x_i;\theta)}\frac{\partial f(x_i;\theta)}{\partial\theta} f(x_i;\theta)dx_i = \int \frac{\partial f(x_i;\theta)}{\partial\theta}dx_i \tag{20.14}$$

$$= \frac{\partial}{\partial\theta}\int f(x_i;\theta)dx_i = \frac{\partial 1}{\partial\theta} = 0 \tag{20.15}$$

□

If $f(x;\theta)$ is not regular, a numerical method call Newton-Raphson iteration can be used to estimate the maximizer. Given a start point $\hat{\theta}_1$, the question is which point is a better solution nearby? We approximate the $lnL(\theta|x)$ by Taylor Series Expansion at $\hat{\theta}_1$,

$$lnL(\theta|x) = lnL(\hat{\theta}_1|x) + \frac{\partial L(\hat{\theta}_1|x)}{\partial\theta^\top}(\theta-\hat{\theta}_1) \tag{20.16}$$

$$+\frac{1}{2}(\theta-\hat{\theta}_1)^\top\frac{\partial^2 L(\hat{\theta}_1|x)}{\partial\theta\partial\theta^\top}(\theta-\hat{\theta}_1) + o((\theta-\hat{\theta}_1)^\top(\theta-\hat{\theta}_1)) \tag{20.17}$$

Now maximize $lnL(\theta|x)$, the F.O.C. is

$$\frac{lnL(\hat{\theta}_1|x)}{\partial\theta} + \frac{\partial^2 lnL(\hat{\theta}_1|x)}{\partial\theta\partial\theta^\top}(\theta-\hat{\theta}_1) = \mathbf{0} \tag{20.18}$$

$$\Rightarrow \hat{\theta}_2 = \hat{\theta}_1 - [\frac{\partial^2 lnL(\hat{\theta}_1|x)}{\partial\theta\partial\theta^\top}]^{-1}\frac{lnL(\hat{\theta}_1|x)}{\partial\theta} \tag{20.19}$$

$$\Rightarrow \hat{\theta}_2 = \hat{\theta}_1 - H(\hat{\theta}_1|x)^{-1}S(\hat{\theta}_1|x) \tag{20.20}$$

So we can repeat

$$\hat{\theta}_{n+1} = \hat{\theta}_n - H(\hat{\theta}_n|x)^{-1}S(\hat{\theta}_n|x) \tag{20.21}$$

until $S(\hat{\theta}_n|x) \simeq 0$.

The Newton-Raphson method implies that the precision of MLE is decided by the curvature of $\hat{\theta}_{MLE}$.

- If $lnL(\theta|x)$ is very curved or steep around $\hat{\theta}_{MLE}$, the MLE is high-precision; we say x contains much information;
- If $lnL(\theta|x)$ is not curved or flat around $\hat{\theta}_{MLE}$, the MLE is low-precision; we say x contains little information;
- If $lnL(\theta|x)$ is completely flat around $\hat{\theta}_{MLE}$, the MLE is not identified; we say x contains no information;

Curvature is measured by the Hessian Matrix,

$$H(\theta|x) = \frac{\partial^2 lnL(\theta|x)}{\partial\theta\partial\theta^\top} \tag{20.22}$$

which is negative semi-finite. We use $-H(\theta|x)$ to measure the information contained in x and its expectation is the information matrix,

$$I(\theta|x) = E(-H(\theta|x)) = -E(H(\theta|x)) \tag{20.23}$$

which decides the precision of MLE.

20.2 Properties of MLE

THEOREM 20.2. *(Cramer-Rao Inequality) Let $X_1, X_2, \cdots, X_n$ be IID samples with pdf $f(x;\theta)$. $\hat{\theta}$ is an unbiased estimator of θ. If $f(x;\theta)$ is regular, then we say*

$$Var(\hat{\theta}) \geq I(\theta|x)^{-1} = CRLB \tag{20.24}$$

where $I(\theta|x) = -E(H(\theta|x))$ is the information matrix and CRLB is the abbr. of Cramer-Rao Lower Bound. If $E(\hat{\theta}) = \theta, Var(\hat{\theta}) = I(\theta|x)^{-1}$, then $\hat{\theta}$ is the Best Unbiased Estimator (BUE) of θ.

We introduce the invariance, consistency and asymptotic normality of MLE here.

If $\hat{\theta}$ is the MLE of θ and $\alpha = h(\theta)$ is a one-to-one function, then $\hat{\alpha}_{MLE} = h(\hat{\theta})$ is the MLE of α. (why?)

If $X_1, X_2, \cdots, X_n$ is IID samples from the same distribution, then under the general regularity conditions, MLE of θ s.t.

- $\hat{\theta}_{MLE} \rightarrow^p \theta$;
- $\sqrt{n}(\hat{\theta}_{MLE} - \theta) \rightarrow^d N(0, I(\theta|x_i)^{-1})$, so we have $avar(\sqrt{n}(\hat{\theta}_{MLE} - \theta)) = I(\theta|x_i)^{-1}$, $\hat{\theta}_{MLE} \sim^a N(\theta, \frac{1}{n}I(\theta|x_i)^{-1}) = N(\theta, I(\theta|x)^{-1})$.

MLE is the extremum estimator of

$$Q_n(\theta) = \frac{1}{n}\sum_{i=1}^{n} m(x_i,\theta), m(x_i,\theta) = lnf(x_i,\theta) \tag{20.25}$$

The consistency of MLE needs

1. $Q_n(\theta), Q_0(\theta)$ is continuous;
2. $Q_n(\theta) = \frac{1}{n}\sum_{i=1}^{n} lnf(x_i,\theta) \rightarrow^p E[lnf(x_i,\theta)] = Q_0(\theta)$ uniformly in Θ;
3. Θ is compact;
4. $Q_0(\theta)$ obtains the maximum only at θ_0;

When $f(x_i;\theta)$ is continuous, $Q_n(\theta), Q_0(\theta)$ are continuous. The second condition needs $E[\sup_{\theta} |lnf(x_i,\theta)|] < \infty$. If $Pr(f(x_i|\theta) \neq f(x_i|\theta_0)) > 0$ for all $\theta \neq \theta_0$, we have the fourth condition.

Assume the real pdf is $f(x_i|\theta_0)$, $E(lnf(x_i|\theta))$ exists and is finite for $\forall\theta$. Then $\forall x_i$ and $f(x_i|\theta)$, we want to show $P(f(x_i|\theta)) \neq f(x_i|\theta_0) > 0$. Define the likelihood ratio $\alpha(x_i) = \frac{f(x_i|\theta)}{f(x_i|\theta_0)}$, then need to show

$$P(\alpha(x_i) \neq 1) = P(\frac{f(x_i|\theta)}{f(x_i|\theta)} \neq 1) > 0 \tag{20.26}$$

THEOREM 20.3. *(Jensen's Inequality) Let $c(x)$ be strictly concave and x is a random variable, not a scalar, so we have*

$$E(c(x)) < cE(x) \tag{20.27}$$

Since $ln(x)$ is strictly concave and α is a random variable, not a constant, so from Jensen's inequality, we have

$$E[ln\alpha(x_i)] = E[ln\frac{f(x_i|\theta)}{f(x_i|\theta_0)}] < lnE[\frac{f(x_i|\theta)}{f(x_i|\theta_0)}] \tag{20.28}$$

$$E[\frac{f(x_i|\theta)}{f(x_i|\theta_0)}] = \int \frac{f(x_i|\theta)}{f(x_i|\theta_0)} f(x_i|\theta_0)dx_i = \int f(x_i|\theta)dx_i = 1 \tag{20.29}$$

$$\Rightarrow E[ln\frac{f(x_i|\theta)}{f(x_i|\theta_0)}] < ln1 = 0 \tag{20.30}$$

Thus, for $\theta \neq \theta_0$, we have

$$E[lnf(x_i|\theta) - lnf(x_i|\theta_0)] < 0 \Rightarrow E[lnf(x_i|\theta)] < E[lnf(x_i|\theta_0)] \tag{20.31}$$

Therefore $Q_0(\theta) = E[lnf(x_i;\theta)]$ takes the maximum only at θ_0.

The asymptotic normality of $\hat{\theta}_{MLE}$ is obtained by the Taylor expansion of $S(\hat{\theta}_{MLE}|x)$ near θ_0,

$$\mathbf{0} = S(\hat{\theta}_{MLE}|x) = S(\theta_0|x) + H(\bar{\theta}|x)(\hat{\theta}_{MLE} - \theta_0) \tag{20.32}$$

where $\bar{\theta} = \lambda\hat{\theta}_{MLE} + \lambda\theta_0, \lambda \in (0,1)$. So

$$H(\bar{\theta}|x)(\hat{\theta}_{MLE} - \theta_0) = -S(\theta_0|x) \tag{20.33}$$

$$\sqrt{n}(\hat{\theta}_{MLE} - \theta_0) = -(\frac{1}{n}H(\bar{\theta}|x))^{-1}(\frac{1}{\sqrt{n}}S(\theta_0|x)) \tag{20.34}$$

Assume $\frac{1}{n}H(\bar{\theta}|x)$ uniformly converges to $E(H(\theta_0|x_i))$, then

$$\frac{1}{n}H(\bar{\theta}|x) = \frac{1}{n}\sum_{i=1}^{n} H(\bar{\theta}|x_0) \to^p E[H(\theta_0|x_i)] = -I(\theta_0|x_i) \tag{20.35}$$

Further, if $\{S(\theta_0|x_i)\}$ is an ergodic-stationary process, and $E(S(\theta_0|x_i)S(\theta_0|x_i)^\top) = I(\theta_0|x_i)$, so

$$\frac{1}{\sqrt{n}}S(\theta_0|x) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} S(\theta_0|x_i) \to^d N(0, I(\theta_0|x_i)) \tag{20.36}$$

Thus,

$$\sqrt{n}(\hat{\theta}_{MLE} - \theta_0) \to^d I(\theta_0|x_i)^{-1}N(\mathbf{0}, I(\theta_0|x_i)) = N(0, I(\theta_0|x_i)^{-1}) \tag{20.37}$$

$$\hat{\theta}_{MLE} \sim^a N(\theta_0, I(\theta_0|x)^{-1}) \tag{20.38}$$

where $I(\theta_0|x) = nI(\theta_0|x_i)$.

In practice, $I(\theta|x_i) = -E[H(\theta|x_i)] = Var(S(\theta_0|x_i))$ is unknown, so we have to estimate it, like,

$$\hat{I}_1(\hat{\theta}_{MLE}|x_i) = -\frac{1}{n}\sum_{i=1}^{n} H(\hat{\theta}_{MLE}|x_i) \to^p -E(H(\theta_0|x_i)) = I(\theta_0|x_i) \tag{20.39}$$

$$\hat{I}_2(\hat{\theta}_{MLE}|x_i) = \frac{1}{n}\sum_{i=1}^{n} S(\hat{\theta}_{MLE}|x_i)S(\hat{\theta}_{MLE}|x_i)^\top \to^p E[S(\theta_0|x_i)S(\theta_0|x_i)^\top] = I(\theta_0|x_i) \tag{20.40}$$

So,

$$\hat{I}_1(\hat{\theta}_{MLE}|x) = n\hat{I}_1(\hat{\theta}_{MLE}|x_i) = -H(\hat{\theta}_{MLE}|x) \tag{20.41}$$

$$\hat{I}_2(\hat{\theta}_{MLE}|x) = n\hat{I}_2(\hat{\theta}_{MLE}|x_i) = \sum_{i=1}^{n} S(\hat{\theta}_{MLE}|x_i)S(\hat{\theta}_{MLE}|x_i)^\top \tag{20.42}$$

20.3 Hypothesis Test of MLE

To test $H_0 : \theta = \theta_0, H_1 : \theta \neq \theta_0$, we have three approaches:

1. $Wald = \frac{(\hat{\theta}_{MLE}-\theta_0)^2}{\hat{I}(\hat{\theta}_{MLE})^{-1}} = (\hat{\theta}_{MLE} - \theta_0)\hat{I}(\hat{\theta}_{MLE})(\hat{\theta}_{MLE} - \theta_0)^\top$;
2. $LM = \frac{S(\theta_0|x)^2}{I(\theta_0|x)} = S(\theta_0|x)I(\theta_0|x)^{-1}S(\theta_0|x)$;
3. $LR = 2[lnL(\hat{\theta}_{MLE}|x) - lnL(\theta_0|x)]$

Under H_0, we have $Wald, LM, LR \to^d \chi^2(1)$.

Wald statistics is constructed by the asymptotic covariance of $\hat{\theta}_{MLE}$, t-value

$$t = \frac{\hat{\theta}_{MLE} - \theta_0}{\hat{SE}(\hat{\theta}_{MLE})} = \frac{\hat{\theta}_{MLE} - \theta_0}{\sqrt{\hat{I}(\hat{\theta}_{MLE}|x)^{-1}}} = (\hat{\theta}_{MLE} - \theta_0)\sqrt{\hat{I}(\hat{\theta}_{MLE}|x)} \sim^a N(0,1) \tag{20.43}$$

With continuous mapping theorem

$$Wald = t^2 = (\hat{\theta}_{MLE} - \theta_0)^2\hat{I}(\hat{\theta}_{MLE}|x) \sim^a \chi^2(1) \tag{20.44}$$

$\hat{\theta}_{MLE}$ is the solution of

$$\frac{dlnL(\hat{\theta}_{MLE}|x)}{d\theta} = S(\hat{\theta}_{MLE}|x) = 0 \tag{20.45}$$

If H_0 is true, $\frac{dlnL(\theta_{MLE}|x)}{d\theta} = S(\theta_{MLE}|x)$ equals to 0 and $\neq 0$ otherwise. The LM statistics is based on the difference between $S(\theta_{MLE}|x)$ and 0,

$$S(\theta_0|x) \sim^a N(0, nI(\theta_0|x_i)) = N(0, I(\theta_0|x)) \tag{20.46}$$

If H_0 holds,

$$LM = \frac{S(\theta_0|x)^2}{I(\theta_0|x)} = S(\theta_0|x)^2 I(\theta_0|x)^{-1} \tag{20.47}$$

$$= [\frac{1}{\sqrt{n}} S(\theta_0|x)]^2 [\frac{1}{n} I(\theta_0|x)]^{-1} \tag{20.48}$$

$$\rightarrow^d (N(0, I(\theta_0|x_i)))^2 I(\theta_0|x_i)^{-1} = \chi^2(1) \tag{20.49}$$

In practice, $I(\theta_0|x)$ is unknown, but can be estimated consistently

$$\tilde{I}(\theta_0|x) = \frac{1}{n}\sum_{i=1}^{n} S(\theta_0|x_i)^2 \rightarrow^d E[S(\theta_0|x_i)^2] = I(\theta_0|x_i) \tag{20.50}$$

Consider $maxlnL(\theta|x), s.t. \theta = \theta_0$, the Lagrange function

$$L = lnL(\theta|x) + \lambda(\theta - \theta_0) \tag{20.51}$$

F.O.C.

$$0 = \frac{\partial L}{\partial \theta} = S(\tilde{\theta}|x) + \tilde{\lambda} \Rightarrow \tilde{\lambda} = -S(\tilde{\theta}|x) \tag{20.52}$$

and $\tilde{\theta} = \theta_0$, then $\tilde{\lambda} = -S(\tilde{\theta}|x) = -S(\theta_0|x)$ and

$$LM = S(\theta_0|x)^2 I(\theta_0|x)^{-1} = \tilde{\lambda}^2 I(\theta_0|x)^{-1} \tag{20.53}$$

Consider the likelihood ratio

$$\lambda = \frac{L(\theta|x)}{L(\hat{\theta}_{MLE}|x)} = \frac{L(\theta_0|x)}{maxL(\theta|x)} \tag{20.54}$$

then $\lambda \in (0, 1]$. If $H_0 : \theta = \theta_0$, then $\lambda \simeq 1$ and $\lambda < 1$ otherwise.

$$LR = -2ln\lambda = -2ln\frac{L(\theta_0|x)}{L(\hat{\theta}_{MLE}|x)} \tag{20.55}$$

$$= -2ln[lnL(\theta_0|x) - lnL(\hat{\theta}_{MLE}|x)] \tag{20.56}$$

$$= 2ln[lnL(\hat{\theta}_{MLE}|x) - lnL(\theta_0|x)] \tag{20.57}$$

$$\tag{20.58}$$

Under certain regular condition, $LR \rightarrow^d \chi^2(1)$.

20.4 MLE and GMM

Let $X_1, \cdots, X_n$ be IID samples from the same model. Given pdf $f(x_i|\theta)$, to maximize the log-likelihood function

$$lnL(\theta|x) = \sum_{i=1}^{n} lnf(x_i|\theta) \tag{20.59}$$

obtain F.O.C.

$$\frac{\partial lnL(\hat{\theta}_{MLE}|x)}{\partial \theta} = S(\hat{\theta}_{MLE}|x) = 0 \tag{20.60}$$

Under regular conditions,

$$\hat{\theta}_{MLE} \sim^a N(\theta, \frac{1}{n} I(\theta|x_i)^{-1}) \tag{20.61}$$

where $I(\theta|x_i) = -E[H(\theta|x_i)] = E[S(\theta|x_i)S(\theta|x_i)^\top]$.

Given $l \geq k$ moments conditions,

$$E[g(x_i, \theta)] = 0 \tag{20.62}$$

calculate the sample moments

$$g_n(\theta) = \frac{1}{n}\sum_{i=1}^{n} g(x_i, \theta) \tag{20.63}$$

If $l > k$, the efficient GMM estimator is the argument of

$$\min Q(\theta, \hat{S}^{-1}) = n g_n(\theta)^\top \hat{S}^{-1} g_n(\theta) \tag{20.64}$$

where $S = E[g(x_i, \theta)g(x_i, \theta)^\top]$. The F.O.C.

$$\frac{Q(\hat{\theta}_{GMM}, \hat{S}^{-1})}{\partial\theta} = G_n(\hat{\theta}_{GMM})^\top \hat{S}^{-1} g_n(\hat{\theta}_{GMM}) = 0 \tag{20.65}$$

Under regular conditions

$$\hat{\theta}_{GMM} \sim^a N(\theta, \frac{1}{n}(G^\top S^{-1} G)^{-1}), G = E[\frac{\partial g(x_i, \theta)}{\theta^\top}] \tag{20.66}$$

As the MLE estimator's covariance reaches the Cramer-Rao lower bound, so

$$avar(\hat{\theta}_{GMM}) - avar(\hat{\theta}_{MLE}) \tag{20.67}$$

is positive semidefinite, i.e., $\hat{\theta}_{MLE}$ is more efficient than $\hat{\theta}_{GMM}$. If the moment conditions equal to the score vector,

$$g(x_i, \theta) = S(\theta|x_i) \tag{20.68}$$

then $\hat{\theta}_{MLE}$ is equivalent to $\hat{\theta}_{GMM}$ (k moment conditions, just identified), as

$$g_n(\hat{\theta}_{GMM}) = \frac{1}{n} S(\hat{\theta}_{GMM}) = 0 \Rightarrow \hat{\theta}_{GMM} = \hat{\theta}_{MLE} \tag{20.69}$$

$$G = E[\frac{\partial S(\theta|x_i)}{\partial\theta^\top}] = E[H(\theta|x_i)] = -I(\theta|x_i) \tag{20.70}$$

$$S = E[S(\theta|x_i)S(\theta|x_i)^\top] = I(\theta|x_i) \tag{20.71}$$

So the asymptotic covariance of $\hat{\theta}_{GMM}$ and $\hat{\theta}_{MLE}$ equal to each other:

$$(G^\top S^{-1} S)^{-1} = I(\theta|x_i)^{-1} \tag{20.72}$$

Part V
English

21 Academic English

English is a simple and elegant language, while it is still out of my range. With generous help from Mrs. Cao, my teacher, it comes to me that I am not born a bad English speaker, I was prevent by my culture values and inner scars. That is bad. I decide to acquire enormous English knowledge and culture to become a great English speaker and find myself.

This note covers topics including academic presentation, reading, writing, listening and etc.

21.1 Presentation

In class, an academic presentation is limited in four minutes and Q&A is limited in one minute. Oral English speed is between 125 and 150 words per minute and so 400 to 500 words are allowed in an academic presentation, which implies contents must be clear and compressed in a logic manner, or otherwise audiences is not capable to catch up on you. Remember that you are selling your paper! A typical presentation framework is Figure 35.

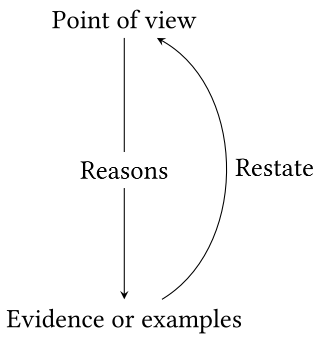


Figure 35: Framework of presentation

Apart from the logic of presentation, body language, full preparation and acquaintance of audiences' background also make a great difference on the presentation, but you can not learn it from books, please, learn by doing and keep awareness of these in any presentation.

21.2 Read

Reading and listening are processes of receiving information, to make they efficient, we have to be acquaint with English grammar because English conveys information by clear and logic sentence, instead of context like Chinese. In other words, sentences are all the author want to deliver and you can understand it just by reading them separately. So, we summarize English grammars in this section, including sentence structures and general vocabulary grammars.

Master the main idea of a paper by subject, thesis statement and outline (logic steps):

- subject: a phrase or a simple sentence, which tells the main field or main idea;
- thesis statement:

    1. main topic
    2. structure

  Thesis statement is the most important sentence in you paper which not only reveal the main idea (only one, using phrase usually), but also show the structure of following contents, usually located in the introduction of the paper. A thesis statement must cover the main topic and the structure of the whole paper.

  For example:

This paper discusses science as a vocation, including the external and internal conditions, science as the courses of progress, and the meaning of science.

- outline:

  1. coordination: relations of the same level items, MECE principle[13]. Items of the same level must be mutually exclusive and collectively exhaustive. In other words, items of the same level must cover different parts of the article (of part of it) totally but not cover each other.
  2. subordination: relations of the mother-son items, son items division must follow the same standard, e.g., Word (mother) - positive features, negative features (two sons) (sons also the MECE principle, two items at least!).
  3. division: son items must be mutually exclusive;
  4. parallelism: the sentence pattens and word class at the same position must be identical.

  The leading letter and an example,

  I. Types of programs
     A. Word processing
     B. Desktop publishing
  II. Evaluation of programs
     A. Word processing
        1. Word
        2. Word Perfect
     B. Desktop publishing
        1. Page Maker
        2. Quark Express

  In a short paper, two levels are at best; in a long paper, four levels are at most. (the fourth level Roman numeral uses lowercase of alphabet, e.g., a as the leading letter; we can also use decimal pattern of leading letter, e.g., 1., 1.1., 1.1.1., 1.1.1.1)

21.3 Write

Basic writing methods include seven types:

1. comparison and contrast;
2. exemplification;
3. classification;
4. definition;
5. cause-and-effect;
6. description;
7. narration.

Forbid any explicitly logical terms, e.g., firstly, used in a short essay.

[13] Mutually exclusive collectively exhaustive

Part VI
Continuous Time Finance: Mathematical Tools and Applications

22 General Probability Theory

22.1 Probability Space and Expectation

In this section, we give a quick introduction to the probability theory. "Probability" is defined as the frequencies that certain events occur, and we can view it as a mapping from event sets into $[0, 1]$. We use σ-algebra, $(\Omega, \mathcal{F}, P)$ to describe a probability space. Suppose we are doing a random experiment, let a set Ω include all possible outcomes of the experiment. Ω is called the sample space, $\omega \in \Omega$ is called a sample point.

DEFINITION 22.1. *(σ-algebra) Given a set Ω, let $\mathcal{F}$ be a collection of subsets of Ω. $\mathcal{F}$ is called a σ-algebra if it satisfies the following properties:*

(1) $\varnothing \in \mathcal{F}$;

(2) if $A \in F$, then $A^c \in F$;

(3) if $A_i \in F, \forall i \in I$, then $\underset{i\in I}{\cap} A_i \in F$.

σ-algebra is closed for common set operations, such as set intersection and set difference.

Example 22.1. *Given a collection of subsets of Ω, we can construct a smallest σ-algebra that included this collection (generated by this collection). Let $\Omega_1, \Omega_2 \subseteq \Omega$, then the σ-algebra generated by $\{\Omega_1, \Omega_2\}$ is*

$$\varnothing, \Omega, \Omega_1, \Omega_2, \Omega_1^c, \Omega_2^c, \Omega_1 \cup \Omega_2, (\Omega_1 \cup \Omega_2)^c \tag{22.1}$$

E.g., $\Omega = \{1, 2, 3\}, \Omega_1 = \{1\}, \Omega_2 = \{2\}$, then the smallest σ-algebra is

$$\varnothing, \{1, 2, 3\}, \{1\}, \{2\}, \{2, 3\}, \{1, 3\}, \{1, 2\}, \{3\} \tag{22.2}$$

DEFINITION 22.2. *(Probability Measure) Given a set Ω, let $\mathcal{F}$ be a σ-algebra of the subsets of Ω. A probability measure P is a mapping from $\mathcal{F}$ to $[0, 1]$, satisfying the following properties*

- *$P(\Omega) = 1$;*
- *accountable additivity: for a countable collection of disjoint sets $A_i \in \mathcal{F}$, then $P(\overset{\infty}{\underset{i=1}{\cap}} A_i) = \sum_{i=1}^{\infty} P(A_i)$. ($p(\Omega) = p(\Omega \cup \varnothing) = P(\Omega) + P(\varnothing) \Rightarrow p(\varnothing) = 0$)*

DEFINITION 22.3. *(Probability Space) If above requirements are satisfied, the triplet $(\Omega, \mathcal{F}, P)$ is call a probability space.*

DEFINITION 22.4. *(Borel Set) Let $\Omega = \mathbb{R}$, then the σ-algebra generated by $\{[a, b], a, b \in \mathbb{R}\}$ is called the Borel σ-algebra. Every set A belongs to the Borel σ-algebra is called a Borel set. Any set defined on $\mathbb{R}$ is a Borel set.*

DEFINITION 22.5. *(Random Variable) Given a probability space $(\Omega, \mathcal{F}, P)$, a random variable X is a mapping from Ω to the set of real numbers $\mathbb{R}$, $X : \Omega \to \mathbb{R}$, satisfying $\forall B \in \mathbb{R}, X^{-1}(B) \in \mathbb{F}$.*

A probability measure maps sets in $\mathcal{F}$ to $\mathbb{R}$ while a random variable maps Ω to $\mathbb{R}$.

DEFINITION 22.6. *(Distribution) Every random variable X includes a measure on $\mathbb{R}$, denoted by $\mu_X(\cdot)$, satisfying $\mu_X(B) = P(\omega : X(\omega) \in B)$ for every Borel set B of $\mathbb{R}$.*

The distribution is a weighting scheme of the Borel set of $\mathbb{R}$. If $\exists f(x)$ (non-negative) s.t. $\forall B \in B(\mathbb{R})$ we have $\mu_X(B) = \int_B f(x)dx$, then $f(x)$ is called the density function of random variable X.

DEFINITION 22.7. *(Expectation) Given a probability space $(\Omega, \mathcal{F}, P)$, and a random variable. The expectation of X is defined as the Lebesgue integral $\int_\Omega X(\omega)dP(\omega)$.*

The Lebesgue integral is constructed by

step(1) assume $X(\omega) \geq 0, \forall \omega$;

step(2) let $0 = y_0 < y_1 < \cdots < y_k < \cdots$ be a partition of $\mathbb{R}_+$ and $\Pi = \max\{y_{k+1} - y_k, k \in \mathbb{N}\}$;

step(3) Let $A_k = \{\omega : y_k \leq X(\omega) \leq y_{k+1}\}$, then $A_k \in \mathcal{F}$. Therefore, $P(A_k)$ is defined, so the lower Lebesgue sum

$$LS_\Pi^- = \sum_{k=1}^{\infty} y_k P(A_k) \tag{22.3}$$

step(4)

$$\text{Lebesgue integral} = \lim_{\Pi \to 0} LS_\Pi^- = \lim_{\Pi \to 0} \sum_{k=1}^{\infty} y_k P(A_k) \tag{22.4}$$

step(5) if $X(\omega)$ can be either positive or positive, we can construct two non-negative random variables $X^+ = \max\{X, 0\}, X^- = \min\{-X, 0\}$, then $X = X^+ - X^-$. So

$$\int_\Omega X(\omega) dP(\omega) = \int_\Omega X^+(\omega) dP(\omega) - \int_\Omega X^-(\omega) dP(\omega) \tag{22.5}$$

We can also define $LS_\Pi^+ = \sum_{k=1}^{\infty} y_{k+1} P(A_k)$ and take the same limit.
Let a, b be constants and X, Y be random variables, the properties of expectation includes

- linearity, $E(aX + bY) = aE(X) + bE(y)$;
- comparability, if $X \leq Y$ a.s., then $E(X) \leq E(Y)$;
- Jensen's inequality, if $\varphi(\cdot)$ is a convex function, and $|E[X]| < \infty$, then $\varphi(E[X]) \leq E[\varphi(X)]$;
- Cauchy's inequality, $(E[XY])^2 \leq E[X^2]E[Y^2]$;
- if the distribution of X permits a density $f(x)$, then the expectation can be computes through the Riemann integral $\int_\mathbb{R} x f(x) dx$.

22.2 Convergence of Integrals

Let $\{f_n(x)\}_{n=1}^{\infty}$ be a sequence of functions defined on a domain D. Take $\tilde{x} \in D$, if the sequence $\{f_n(\tilde{x})\}$ converges, then we say $\{f_n(x)\}_{n=1}^{\infty}$ converges at $\tilde{x}$. Define $f(\tilde{x}) = \lim_{n \to \infty} f_n(\tilde{x})$. If $\exists f(x)$ s.t. $\forall x \in D$, we have $\lim_{n \to \infty} f_n(x) = f(x)$, we say $\{f_n(x)\}_{n=1}^{\infty}$ converges point by point. If $\exists A = \{x \in D | \{f_n(x)\} \text{ is not convergent}\}$, and $m(A) = 0$ (the measure of A is zero, i.e., A is countable), we say $\{f_n\}$ converges almost surely (a.s.). If $\{f_n(x)\}$ converges to $f(x)$ a.s., we do not have $\lim_{n \to \infty} \int_D f_n(x) dx = \lim \int_D f(x) dx$ necessarily, see Example 22.2.

Example 22.2. *Let $\sigma_n^2 = \frac{1}{n}, f_n = \frac{1}{\sqrt{2\pi/n}} e^{-\frac{x^2}{2/n}}, x \in \mathbb{R}$, which is the density function of $N(0, \sigma_n^2)$. Let $f(x) \equiv 0, \forall x \in \mathbb{R}$.*

As $n \to \infty$, we have $f_n(x) \to 0, \forall x \in \mathbb{R} \backslash \{0\}$, so $f_n(x) \to f(x) \equiv 0, \forall x \in \mathbb{R} \backslash \{0\}$. $m(\{0\}) = 0$, so $f_n \to f$ a.s.

But $\int_\mathbb{R} f_n(x) dx = 1, \forall n \in \mathbb{N}_+$, so $\lim_{n \to \infty} \int_\mathbb{R} f_n(x) dx = 1$. However, $\int_\mathbb{R} f(x) dx = 0$.

With sufficient conditions, we can obtain $\lim_{n \to \infty} \int_D f_n(x) dx = \lim \int_D f(x) dx$ surely, two types.

THEOREM 22.1. *(Monotone Convergence Theorem) If $\{f_n(x)\}$ converges to $f(x)$ in a monotone way, then we have $\int_\Omega f_n(x) dx$ converges to $\int_\Omega f(x) dx$. (This is from $\{\int_D f_n(x) dx\}$ is a infinite monotone sequence, must obtain a limit, just the $\int_D f(x) dx$)*

THEOREM 22.2. *(Domainated Convergence Theorem) If there is an integrable function $g(x)$ s.t. $|f_n(x)| \leq g(x)$ almost everywhere, then $f_n(x) \to f(x)$ implies $\int_\Omega f_n(x) dx \to \int_\Omega f(x) dx$.*

22.3 Change of Measure

Probability measure is just a mapping of $\mathcal{F}$ into $[0,1]$, so for the same event, we can assign a different probability for it in our interest, which is a change of measure. We do that with a non-negative random variable Z s.t. $E[Z] = \int_\Omega Z(\omega)d\omega = 1$ by

$$\tilde{P}(A) = \int_A Z(\omega)dP(\omega) = \int_\Omega Z(\omega)I_A(\omega)dP(\omega) \tag{22.6}$$

where $I_A(\omega) = \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases}$ (an indicator function). $\tilde{P}(A)$ can be interpreted as the average value of Z on the set A, under probability measure P.

If $Z > 0$ a.s., then $\tilde{P}$ and P are equivalent probability measures in the sense that $\tilde{P}(A) = 0$ iff $P(A) = 0$.

Now we verify $\tilde{P}(A)$ is a probability measure indeed, i.e., $\tilde{P}(\Omega) = 1$ and countable additivity. First, $\tilde{P}(\Omega) = \int_\Omega Z(\omega)dP(\omega) = E[Z] = 1$.

Next, let $\{\{A_i\}_{i=1}^\infty\}$ be pairwise disjoint, we want to show $\tilde{P}(\bigcup_{n=1}^{\infty} A_i) = \sum_{i=1}^\infty \tilde{P}(A_i)$.

Let $B_n = \cup_{i=1}^n A_i, B_\infty = \bigcup_{n=1}^{\infty} A_i$. Then $I_{B_1} \le I_{B_2} \le \cdots$ and $\lim_{n\to\infty} I_{B_n} = B_\infty$, so $\lim_{n\to\infty} I_{B_n}Z(\omega) = B_\infty Z(\omega), \forall\omega \in \Omega$. With the Monotone Convergence Theorem, we have

$$\lim_{n\to\infty} \tilde{P}(B_n) = \lim_{n\to\infty} \int_\Omega I_{B_n}Z(\omega)dP(\omega) = \int_\Omega I_{B_\infty}Z(\omega)dP(\omega) = \tilde{P}(B_\infty) \tag{22.7}$$

Moreover,

$$\tilde{P}(B_n) = \int_\Omega Z(\omega)I_{B_n}dP(\omega) \tag{22.8}$$

$$= \int_\Omega Z(\omega)(\sum_{i=1}^n I_{A_i}(\omega))dP(\omega) \tag{22.9}$$

$$= \sum_{i=1}^n \int_\Omega Z(\omega)I_{A_i}(\omega)dP(\omega) \tag{22.10}$$

$$= \sum_{i=1}^n \tilde{P}(A_i) \tag{22.11}$$

(finite sum and integral can exchange positions) So $\tilde{P}(B_n) = \sum_{i=1}^n \tilde{P}(A_i)$. Then

$$\sum_{i=1}^\infty \tilde{P}(A_i) = \lim_{n\to\infty} \sum_{i=1}^n \tilde{P}(A_i) = \lim_{n\to\infty} \tilde{P}(B_n) = \tilde{P}(B_\infty) \tag{22.12}$$

Now given a random variable X, its expectation is $E(X) = \int_\Omega X(\omega)dP(\omega)$ under P, we are interested in $\tilde{E}(X) = \int_\Omega X(\omega)d\tilde{P}(\omega)$. Since $\forall A \in \mathcal{F}$, we have

$$\tilde{P}(A) = \tilde{E}(I_A) = \int_\Omega I_A(\omega)d\tilde{P}(\omega) \tag{22.13}$$

$$\text{in an other way} = \int_\Omega I_A(\omega)Z(\omega)dP(\omega) \tag{22.14}$$

So $d\tilde{P}(\omega) = Z(\omega)dP(\omega)$, then

$$\tilde{E}(X) = \int_\Omega X(\omega)d\tilde{P}(\omega) = \int_\Omega X(\omega)Z(\omega)dP(\omega) = E(XZ) \tag{22.15}$$

DEFINITION 22.8. *(Randon-Nikodym Derivative) If we define a new probability measure $\tilde{P}$ by $\tilde{P}(A) = \int_A Z(\omega)dP(\omega)$, then Z is called the Randon-Nikodym derivative of $\tilde{P}$ respect to P, formally written As*

$$Z = \frac{d\tilde{P}}{dP} \tag{22.16}$$

THEOREM 22.3. *(Randon-Nikodym Theorem) Let P and $\tilde{P}$ be equivalent probability measures defined on a probability space $(\Omega, \mathcal{F})$. Then there exists an almost surely positively variable Z s.t. $E(Z) = 1$ and*

$$\tilde{P}(A) = \int_A Z(\omega) dP(\omega), \forall A \in \mathcal{F} \tag{22.17}$$

See Example 22.3

Example 22.3. *Given a probability space $(\Omega, \mathcal{F}, P)$ and a random variable $X \sim N(0,1)$. Let $Y = X + \theta, \theta > 0$(constant). Show that $Z = e^{\theta X - \frac{1}{2}\theta^2}$ is the Randon-Nikodym derivative which makes $Y \sim N(0,1)$ under $\tilde{P}$ derived by Z from P.*

Proof. Define $\tilde{P}(A) = \int_A Z(\omega) dP(\omega)$. The c.d.f. of Y under $\tilde{P}$ is given by

$$\tilde{F}(y) = \tilde{P}(Y \leq y) = \tilde{E}(I_{Y \leq y}) \tag{22.18}$$

$$= E(I_{Y \leq y} e^{-\theta X - \frac{1}{2}\theta^2}) = E(I_{X \leq y - \theta} e^{-\theta X - \frac{1}{2}\theta^2}) \tag{22.19}$$

$$= \int_{\mathbb{R}} f(x) I_{x \leq y-\theta} e^{-\theta x - \frac{1}{2}\theta^2} dx \tag{22.20}$$

$$= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} I_{x \leq y-\theta} e^{-\theta x - \frac{1}{2}\theta^2} dx \tag{22.21}$$

$$= \int_{-\infty}^{y-\theta} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2} - \theta x - \frac{1}{2}\theta^2} dx \tag{22.22}$$

$$= \int_{-\infty}^{y-\Theta} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x+\theta)^2}{2}} dx \tag{22.23}$$

$$u = x + \theta \Rightarrow = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du \tag{22.24}$$

So $y \in N(0,1)$ under $\tilde{P}$. $\square$

The remaining question is how to find such a Z? If $X \sim N(0,1)$, we have $E(e^{tX}) = e^{\frac{1}{2}t^2}$, and $\tilde{E}(e^{tY})$ need to be $e^{\frac{1}{2}t^2}$, i.e.,

$$\tilde{E}(e^{tY}) = E(e^{t(X+\theta)} Z) = e^{t\theta} E(e^{tX} Z) \tag{22.25}$$

Guess $Z = e^{aX+b}$,

$$e^{t\theta} E(e^{tX} Z) = e^{t\theta} E(e^{tX} e^{aX+b}) = e^{t\theta+b} E(e^{(t+a)X}) = e^{t\theta+b} e^{\frac{1}{2}(t+a)^2} = e^{\frac{1}{2}t^2} \tag{22.26}$$

So $t\theta + b + \frac{1}{2}(t+a)^2 = \frac{1}{2}t^2$, then $a = -\theta, b = -\frac{1}{2}\theta^2$.

23 Information and Conditioning

23.1 Filtration and Measurability

We develop a model to describe information in this section.

Example 23.1. *Consider an experiment of tossing a coin twice. If a head appears, record an "H"; if a tail appears, record a "T". Then the sample space is*

$$\Omega = \{HH, HT, TH, TT\} \tag{23.1}$$

If you know the first tossing is "H", then we can partition the sample space into two subsets,

$$\Omega_1 = \{HH, HT\}, \Omega_2 = \{TH, TT\} \tag{23.2}$$

and we say Ω_1 will happen but Ω_2 does not. The σ-algebra generated by the result of the first tossing is

$$\{\{HH, HT\}, \{TH, TT\}, \varnothing, \Omega\} \tag{23.3}$$

where $\varnothing, \Omega$ contains no information while the biggest σ-algebra (generated by the result of all tossing) contains full information.

DEFINITION 23.1. *(Filtration) Let Ω be a nonempty set. Let T be a positive number (representing the final time), and assume that for each $t \in [0, T]$, there exists a σ-algebra $\mathcal{F}_t$ s.t. if $s \leq t$, then $\mathcal{F}_s \subseteq \mathcal{F}_t$. Then the collection of σ-algebra $\mathcal{F}_t, t \in [0, T]$ is call a filtration.*

$\mathcal{F}_t$ is increasing as t increases, implying information accumulating over time (no loss, infinite).

Example 23.2. *The filtration of Example 23.1 is*

$$\begin{aligned}
\mathcal{F}_0 =& \{\varnothing, \Omega\} && (23.4)\\
\mathcal{F}_1 =& \{\{HH, HT\}, \{TH, TT\}, \varnothing, \Omega\} && (23.5)\\
\mathcal{F}_2 =& \{\{HH\}, \{HT\}, \{TH\}, \{TT\}, && (23.6)\\
& \{HH, HT\}, \{TH, TT\}, && (23.7)\\
& \{HT, TH, TT\}, \{HH, TH, TT\}, \{HH, HT, TT\}, \{HH, HT, TH\}, && (23.8)\\
& \varnothing, \Omega\} && (23.9)\\
& && (23.10)
\end{aligned}$$

DEFINITION 23.2. *(σ-algebra Generated by a Random Variable) Let X be a random variable on a nonempty sample space Ω. The σ-algebra generated by X, denoted by $\sigma(X)$, is the collection of all subsets of Ω of the form $\{\omega \in \Omega : X(\omega) \in B\}$ where B runs over all the Borel subsets of $\mathbb{R}$. ($\sigma(X)$ is the smallest σ-algebra that can reveal X)*

Example 23.3. *Follow Example 23.1, we define $X_i, i = 1, 2$ as $X_i = 1$ if getting head in the ith tossing and $X_i = -1$ otherwise. So the σ-algebra generated by $X_1, X_2, X_1 + X_2$ is*

$$X_1^{-1}(B) = \begin{cases} \{HH, HT\}, \text{ if} 1 \in B, -1 \notin B \\ \{TH, TT\}, \text{ if} 1 \notin B, -1 \in B \\ \Omega, \text{ if} 1 \in B, -1 \in B \\ \varnothing, \text{ if} 1 \notin B, -1 \notin B \end{cases} \tag{23.11}$$

$$X_2^{-1}(B) = \begin{cases} \{HH, TH\}, \text{ if} 1 \in B, -1 \notin B \\ \{HT, TT\}, \text{ if} 1 \notin B, -1 \in B \\ \Omega, \text{ if} 1 \in B, -1 \in B \\ \varnothing, \text{ if} 1 \notin B, -1 \notin B \end{cases} \tag{23.12}$$

$$(X_1 + X_2)^{-1}(B) = \begin{cases} \{HH\}, \text{ if} 2 \in B, 0, -2, \notin B \\ \{TT\}, \text{ if} -2 \in B, 0, 2, \notin B \\ \{HT, TH\}, \text{ if} 0 \in B, 2, -2, \notin B \\ \{HH, HT, TH\}, \text{ if} 0, 2 \in B, -2 \notin B \\ \{TT, HT, TH\}, \text{ if} 0, -2 \in B, 2 \notin B \\ \{HH, TT\}, \text{ if} 2, -2 \in B, 0 \notin B \\ \Omega, \text{ if} 2, 0, -2 \in B \\ \varnothing, \text{ if} 2, 0, -2 \notin B \end{cases} \tag{23.13}$$

DEFINITION 23.3. *(Measurability) Let X be a random variable defined on a nonempty sample space Ω. Let $\mathcal{G}$ be a σ-algebra of subsets of Ω. If every set in $\sigma(X)$ in also in $\mathcal{G}$, then we say X is $\mathcal{G}$-measurable.*

A random variable X is $\mathcal{G}$-measurable iff the information in $\mathcal{G}$ is enough to determine the value of X ($\sigma(X) \subseteq \mathcal{G}$). In Example 23.3, X_1 is $\mathcal{F}_1$-measurable but X_2 is not $\mathcal{F}_1$, and X_1, X_2 is $\mathcal{F}$-measurable.

DEFINITION 23.4. *(Adaptability of a Stochastic Process) Let Ω be a nonempty sample space with a filtration $\mathcal{F}_t, t \in [0, T]$. Let $X(t)$ be a collection of random variables indexed by $t \in [0, T]$ (a stochastic process). If for $\forall t, X(t)$ is $\mathcal{F}_t$-measurable, then we say $X(t)$ is adapted.*

DEFINITION 23.5. *Independence between Two Sets Let $(\Omega, \mathcal{F}, P)$ be a probability space, two sets in $\mathcal{F}$ are Independent if*

$$P(A \cap B) = P(A)P(B) \tag{23.14}$$

DEFINITION 23.6. *(Independence between σ-algebras) Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $\mathcal{G}, \mathcal{H}$ be two sub-σ-algebras of $\mathcal{F}$, then we say $\mathcal{G}, \mathcal{H}$ are Independent if*

$$P(A \cap B) = P(A)P(B), \forall A \in \mathcal{G}, \forall B \in \mathcal{H} \tag{23.15}$$

DEFINITION 23.7. *(Independence between r.v.s) Two random variables X and Y defined on the same probability space $\{\Omega, \mathcal{F}, P\}$ are independent if $\sigma(X), \sigma(Y)$ are independent. Similarly, we say X is independent with $\mathcal{G}$ (a σ-algebra) if $\sigma(X)$ and $\mathcal{G}$ are independent.*

Independence depends on the probability measure we are using.

THEOREM 23.1. *(Preservation of Independence) If X and Y are independent random variables, and f, g are measurable functions on $\mathbb{R}$, then $f(X), g(Y)$ are also independent.*

THEOREM 23.2. *(Ways to Check for Independence)*

- *the joint distribution measure factors;*
- *the joint c.d.f. function factors;*
- *the joint density function factors;*
- *the moment generating function factors.*

Keep in mind: non-correlation implies independence only for normal random variables.

23.2 Conditional Expectation

DEFINITION 23.8. *(Conditional Expectation) Let $(\Omega, \mathcal{F}, P)$ be a probability space, let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$, and let X be an integrable random variable. The conditional expectation of X given $\mathcal{G}$, denoted bt $E[X|\mathcal{G}]$, is any random variable that satisfies the following properties*

- *measurability with respect to $\mathcal{G}$: $E[X|\mathcal{G}]$ is $\mathcal{G}$-measurable;*
- *Free of bias: for any $A \in \mathcal{G}$, we have*

$$\int_A E[X|\mathcal{G}](\omega)dP(\omega) = \int_A X(\omega)dP(\omega) \tag{23.16}$$

The conditional expectation and the original random variable have the same average value on every set in $\mathcal{G}$. The random variable $[X|\mathcal{G}]$ is unique a.s..

Conditional expectations have some properties which we will use frequently in practice.

(1) linearity: $E[c_1X + c_2X|\mathcal{G}] = c_1E[X|\mathcal{G}] + c_2E[Y|\mathcal{G}]$;

(2) take out known: $E[XY|\mathcal{G}] = XE[Y|\mathcal{G}]$ if X is G-measurable;

(3) iterated conditioning (tower property): $E[E[X|\mathcal{G}]|\mathcal{H}] = E[X|\mathcal{H}]$ if $\mathcal{H} \subseteq \mathcal{G}$;

(4) Jensen's inequality: for a concave function φ, $E[\varphi(X)|\mathcal{G}] \leq \varphi(E[X|\mathcal{G}])$;

(5) Cauchy's inequality: $(E[XY|\mathcal{G}])^2 \leq E^2[X|\mathcal{G}]E^2[Y|\mathcal{G}]$.

Now we prove them one by one.
(1) $E[c_1X + c_2X|\mathcal{G}] = c_1E[X|\mathcal{G}] + c_2E[Y|\mathcal{G}]$. Let $Z_X = [X|\mathcal{G}], Z_Y = [Y|\mathcal{G}]$, we need to show $c_1Z_X + c_2Z_Y = E[c_1X + c_2X|\mathcal{G}]$.

a. $c_1Z_X + c_2Z_Y$ is $\mathcal{G}$-measurable;

b. free of bias: $\forall A \in \mathcal{G}$, we need to show $\int_A c_1Z_X + c_2Z_YdP = \int_A c_1X + c_2YdP$.

$$LHS = c_1\int_A Z_XdP + c_2\int_A Z_YdP \tag{23.17}$$

$$= c_1\int_A XdP + c_2\int_A YdP \tag{23.18}$$

$$= \int_A c_1X + c_2YdP = RHS \tag{23.19}$$

(we use $\int_A Z_XdP = \int_A XdP, \int_A Z_YdP = \int_A YdP$ as $Z_X = E[X|\mathcal{G}], Z_Y = E[Y|\mathcal{G}]$)

(2) $E[XY|\mathcal{G}] = XE[Y|\mathcal{G}]$ if X is $\mathcal{G}$-measurable.

a. suppose X is an indicator variable, $I_B(\omega) = \begin{cases} 1, \omega \in B \\ 0, \omega \notin B \end{cases} \quad B \in \mathcal{G}$,

so $E[XY|\mathcal{G}] = E[I_B Y|\mathcal{G}]$. Let $Z_Y = E[Y|\mathcal{G}]$, then we need to show $I_B Z_Y$ is the expectation of $I_B Y$.

a) $I_B Z_Y$ is $\mathcal{G}$-measurable;

b) free of bias, $\forall A \in \mathcal{G}, \int_A I_B Z_Y dP = \int_A I_B Y dP$.

$$LHS = \int_A I_B Z_Y dP = \int_{A\cap B} Z_Y dP = \int_{A\cap B} Y dP = \int_A I_B Y dP = RHS \tag{23.20}$$

b. suppose $X = \sum_{i=1}^n c_i I_{B_i}(\omega)$, then we want to show $E[\sum_{i=1}^n c_i I_{B_i}(\omega) Y|\mathcal{G}] = \sum_{i=1}^n c_i I_{B_i}(\omega) E[Y|\mathcal{G}]$.

$$LHS = E[\sum_{i=1}^n c_i I_{B_i}(\omega) Y|\mathcal{G}] = \sum_{i=1}^n c_i E[I_{b_i} Y|\mathcal{G}] = \sum_{i=1}^n c_i I_{B_i}(\omega) E[Y|\mathcal{G}] = RHS \tag{23.21}$$

c. $\forall X$ is $\mathcal{G}$-measurable, we say $\exists X_n = \sum_{i=1}^n I_{B_i}, \forall n$ s.t. $\lim_{n\to\infty} X_n = X$ (the Monotone Convergence Theorem), then we need to show $\forall A \in \mathcal{G}, \int_A X Z_Y dP = \int_A XY dP$.

$$LHS = \int_A \lim_{n\to\infty} X_n Z_Y dP = \lim_{n\to\infty} \int_A X_n Z_Y dP = \lim_{n\to\infty} \int_A X_n Y dP = \int_A \lim_{n\to\infty} X_n Y dP = \int_A XY dP \tag{23.22}$$

(3) $E[E[X|\mathcal{G}]|\mathcal{H}], H \subseteq G$ (or $E[E[X|\mathcal{F}_s]|\mathcal{F}_t] = E[X|\mathcal{F}_t], s < t$). Let $Z_X^G = E[X|\mathcal{G}], Z_X^H = E[X|\mathcal{H}]$, then we need to show $Z_X^H = E[Z_X^G|\mathcal{H}]$.

a. Z_X^H is $\mathcal{H}$-measurable;

b. free of bias, $\forall A \in \mathcal{H} \subseteq \mathcal{G}, \int_A Z_X^H dP = \int_A Z_X^G dP$.

$$LHS = \int_A Z_X^H dP = \int_A X dP, RHS = \int_{A\subseteq \mathcal{H}} Z_X^G dP = \int_{A\subseteq \mathcal{G}} Z_X^G dP = \int_A X dP = LHS \tag{23.23}$$

THEOREM 23.3. *An Independence Theorem Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. Suppose that the random variables $X_1, X_2 \cdots, X_K$ are $\mathcal{G}$-measurable and the random variables $Y_1, Y_2, \cdots, Y_L$ are $\mathcal{G}$-independent. Let $f(x_1, x_2, \cdots, x_K, y_1, y_2, \cdots, y_L)$ be a function of variables $x_1, x_2, \cdots, x_K$ and $y_1, y_2, \cdots, y_L$ and define*

$$g(x_1, x_2, \cdots, x_K) = E[f(x_1, x_2, \cdots, x_K, Y_1, Y_2, \cdots, Y_L)] \tag{23.24}$$

(given $\mathcal{G}$, $X_1, X_2, \cdots, X_K$ are known) then we have

$$E[f(X_1, X_2, \cdots, X_K, Y_1, Y_2, \cdots, Y_L)|\mathcal{G}] = g(X_1, X_2, \cdots, X_K) \tag{23.25}$$

Example 23.4. *Suppose $X, Y, Z \sim N(0,1)$ and Y, Z are $\sigma(X)$-independent, so*

$$E[x^2Y^2 + Y^2Z^2 + x^2Z^2|\sigma(X)] = E[x^2Y^2 + Y^2Z^2 + x^2Z^2] \tag{23.26}$$
$$= x^2E[Y^2] + E[Y^2Z^2] + x^2E[Z^2] \tag{23.27}$$
$$= x^2 + 1 + x^2 \tag{23.28}$$
$$= 2x^2 = g(x) \tag{23.29}$$
$$\Rightarrow E[X^2Y^2 + Y^2Z^2 + X^2Z^2|\sigma(X)] = g(X) = 2X^2 + 1 \tag{23.30}$$

DEFINITION 23.9. *(Martingale (MG), sub-MG, super-MG) Let $(\Omega, \mathcal{F}, P)$ be a probability space, T be a positive constant and $\mathcal{F}_t, t \in [0,T]$ be a filtration. Consider an adapted stochastic process M_t defined on $[0,T]$. In a.s. sense,*

- *if $E[M_t|\mathcal{F}_s] = M_s, \forall 0 \le s \le t \le T$, we say M_t is a martingale;*
- *if $E[M_t|\mathcal{F}_s] \le M_s, \forall 0 \le s \le t \le T$, we say M_t is a super-martingale;*
- *if $E[M_t|\mathcal{F}_s] \ge M_s, \forall 0 \le s \le t \le T$, we say M_t is a sub-martingale;*

Whether a stochastic process M_t is a MG depends on the probability measure. Risk-free money market account $M_t = e^{rt}$ in general is not a MG, but we can discount it by $D_t = e^{-rt}$ to obtain a MG, i.e., $M_t D_t = 1$.

Example 23.5. *Consider a bet: tossing a fair coin, for every dollar of stake you put in, if head appears you get* $\$1$*, otherwise you lose* $\$1$*.*

Doubling strategy: putting in a stake of $\$1$ *in the first round; in the* i*th round, if head appears, then stop betting, and otherwise, double the stake in the* $i+1$*th round. If head appears in the* $k+1$*th round, he P&L is*

$$PL = -1 - 2 - \cdots - 2^{k-1} + 2^k = 1 \tag{23.31}$$

and never win's probability

$$\lim_{k\to\infty} (\frac{1}{2})^k = 0 \tag{23.32}$$

The problem is the loss is not bounded. This is an example of aritrage, which is strictly prohibited in financial model.

DEFINITION 23.10. *(Markov Process) Let* $(\Omega, \mathcal{F}, P)$ *be a probability space,* T *be a positive constant, and* $\mathcal{F}_t, t \in [0, T]$ *be a filtration. Consider an adapted stochastic process* X_t *defined on* $[0, T]$*. Assume that for all* $0 \le s \le t \le T$ *and for every non-negative, Borel-measurable function* f*, there is a Borel-measurable function* g *s.t.*

$$E[f(X_t)|\mathcal{F}_s] = g(X_s) \tag{23.33}$$

then we say X_t *is a Markov process.*

At any point, the future distribution of a Markov process only depends on its current value, not on how it arrives this value, i.e., path independent, e.g., a random walk.

24 Brownian Motion

DEFINITION 24.1. *(Random Walk) Consider a random experiment of tossing a fair coin infinitly many times, and assume different tosses are independent. The sample space is* $\omega = \omega_1\omega_2\cdots$ *where each* ω_j *equas to* H *or* T*. Define a random variable on this sample spaces as follows*

$$X_j = \begin{cases} 1, \omega_j = H \\ -1, \omega_j = T \end{cases} \tag{24.1}$$

and $M_0 = 0$*,*

$$M_k = \sum_{j=1}^{k} X_j, k = 1, 2, \cdots \tag{24.2}$$

Then the process M_k *is called a random walk.*

The increment of a random walk M_k is

$$I(k_2, k_1) \equiv M_{k_2} - M_{k_1}, k_2 > k_1, k_1, k_2 \in \mathbb{N}_+ \tag{24.3}$$

The random walk have four properties,

(1) the random walk has independent increment over non-overlappin intervals;

(2) the mean of $I(k_2, k_1)$ is zero, the variance of $I(k_2, k_1)$ is $k_2 - k_1$, i.e.,

$$E[I(k_2, k_1)] = \sum_{j=k_1}^{k_2} E[X_j] = \sum_{j=k_1}^{k_2} 0 = 0 \tag{24.4}$$

$$Var[I(k_2, k_1)] = Var[\sum_{j=k_1}^{k_2} X_j] = \sum_{j=k_1}^{k_2} Var[X_j] = \sum_{j=k_1}^{k_2} E[X_j^2] = \sum_{j=k_1}^{k_2} 1 = k_2 - k_1 \tag{24.5}$$

(3) the random walk is a martingale, but not stationary (variant variance), $\forall k_2 > k_1$

$$
\begin{aligned}
M_{k_1} &= E[M_{k_2}|\mathcal{F}_{k_1}] && (24.6)\\
\Leftrightarrow & E[M_{k_2} - M_{k_1}|\mathcal{F}_{k_1}] = 0 && (24.7)\\
\Leftrightarrow & E[\sum_{j=k_1+1}^{k_2} X_j|\mathcal{F}_{k_1}] = 0 && (24.8)\\
\Leftrightarrow & X_j \text{ independent with } \mathcal{F}_{k_1}, \forall j > k_1 && (24.9)
\end{aligned}
$$

$$\Leftrightarrow E[\sum_{j=k_1+1}^{k_2} X_j] = 0 \tag{24.10}$$

(4) quadratic variation,

$$[M, M]_n = \sum_{j=1}^{n}(M_j - M_{j-1})^2 = \sum_{j=1}^{n} X_j^2 = n \tag{24.11}$$

which is calculated path by path while, the variance is calculated by averaging over all paths.

DEFINITION 24.2. *(Scaled Random Walk) Let M_k be a random walk. Given $n \in \mathbb{N}$ and $\forall t > 0$, we define a stochastic process as follows*

$$W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{nt} \tag{24.12}$$

The process is called a scaled random walk. Firstly We define the process on $t = \frac{m}{n}, W^{(n)}(\frac{m}{n}) = \frac{1}{\sqrt{n}} M_m$. Next if $nt \notin \mathbb{N}$, then M_{nt} is calculated by linearly interpolating the nearest points.

The limiting process of the scaled random walk is called a Brownian Motion.

DEFINITION 24.3. *(Brownian Motion, BM) Let $(\Omega, \mathcal{F}, P)$ be a probability space. $\forall \omega \in \Omega$, suppose there is a continuous function $W_t, t \geq 0$ s.t. $W_0 = 0$. Then W_t is called a Brownian Motion if,*

(1) $W_0 = 0$;

(2) $\forall 0 = t_0 < t_1 < \cdots < t_m$, the increments $W_{t_1} - W_{t_0}, W_{t_2} - W_{t_1}, \cdots, W_{t_m} - W_{t_{m-1}}$ are independent;

(3) each of these increments is normally distributed with $E[W_{t_i} - W_{t_{i-1}}] = 0, Var[W_{t_i} - W_{t_{i-1}}] = t_i - t_{i-1}$. Thus, with centering limiting theorem (CLT), we have $W_s - W_t \sim N(0, s-t), \forall s > t$.

Brownian Motions (BM) do exist. The sample space can be thought as all the possible paths of the Brownian Motion.

We can simulate a sample pathe of a BM ina given discret points $t_1 < t_2 < \cdots < t_m$,

step (1). set $t_0 - 0$ and $W_{t_0} = 0$;

step (2). draw m independent standrad normal random variables $z_1, z_2, \cdots, z_m$;

step (3). for $i = 1, 2, \cdots, m$, set $W_{t_i} = W_{t_{i-1}} + z_i\sqrt{t_i - t_{i-1}}$.

Properties of a BM,

(1) $\forall t_1 < t_2 < \cdots < t_m$, the random vector $(W_{t_1}, W_{t_2}, \cdots, W_{t_m})$ is normally distributed;

$$\begin{bmatrix} w_{t_1} \\ w_{t_2} \\ \cdots \\ w_{t_m} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ & & \cdots & \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} W_{t_1} \\ W_{t_2} - W_{t_1} \\ \cdots \\ W_{t_m} - W_{t_{m-1}} \end{bmatrix} \tag{24.13}$$

denoted as $\mathbf{W}_m = A\Delta\mathbf{W}_m$. A is not sigular ($|A| \neq 0$), so $\mathbf{W}_m$ is normally distributed. Moreover,

$$A^{-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & \cdots & 0 & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & -1 & 0 \end{bmatrix} \tag{24.14}$$

(2) $\forall s \le t, Cov(W_s, W_t) = s$.

$$\begin{aligned} Cov(W_s, W_t) &= E[W_sW_t] - E[W_s]W[W_t] && (24.15)\\ &= E[W_sW_t] && (24.16)\\ &= E[W_s(W_t - W_s) + W_s^2] && (24.17)\\ &= E[W_s(W_t - W_s)] + E[W_s^2] && (24.18)\\ &= E[W_s]EW_t - W_s + E[W_s^2] && (24.19)\\ &= 0 + Var[W_s] + [E[W_s]]^2 && (24.20)\\ &= s && (24.21)\\ \Rightarrow Cov(\mathbf{W}_m) &= \begin{bmatrix} t_1 & t_1 & \cdots & t_1 \\ t_1 & t_2 & \cdots & t_2 \\ & & \cdots & \\ t_1 & t_2 & \cdots & t_m \end{bmatrix} && (24.22) \end{aligned}$$

where $\mathbf{W}_m = [W_1, W_2, \cdots, W_m]'$.

(3) moment generating function,

$$\begin{aligned} \varphi(u_1, u_2, \cdots, u_m) &= E[e^{\sum_{i=1}^m u_i W_{t_i}}] && (24.23)\\ u = [u_1, u_2, \cdots, u_m] W &= [W_{t_1}, W_{t_2}, \cdots, W_{t_m}] && (24.24)\\ \varphi &= E[e^{u'W}] && (24.25)\\ &= E[e^{u'A\Delta W}] && (24.26)\\ v = A'u \rightarrow &= E[e^{v'\Delta W}] && (24.27)\\ &= E[e^{\sum_{i=1}^m v_i \Delta W_i}] && (24.28)\\ &= \prod_{i=1}^m E[e^{v_i \Delta W_i}] && (24.29)\\ &= e^{\frac{1}{2}v_1^2 t_1} \prod_{i=2}^m e^{\frac{1}{2}v_i^2(t_i - t_{i-1})} && (24.30)\\ v = A'u &= \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ & & \cdots & \\ 0 & 0 & \cdots & 1 \end{bmatrix}, v_i = u_1 + u_2 + \cdots + u_i, \forall i && (24.31)\\ \Rightarrow \varphi(u_1, u_2, \cdots, u_m) &= exp\{\frac{1}{2}(u_1 + u_2 + \cdots t_m)^2 t_1 && (24.32)\\ &+ \frac{1}{2}(u_2 + u_2 + \cdots t_m)^2(t_2 - t_1) && (24.33)\\ &+ \cdots && (24.34)\\ &+ \frac{1}{2}u_m^2(t_m - t_{m-1})\} && (24.35) \end{aligned}$$

Note that if $X \sim N(0,1)$, then $E[e^{vX}] = e^{\frac{1}{2}v^2}, \forall v$; if $Y \sim N(0, \sigma^2)$, then $E[e^{vY}] = E[e^{v\sigma X}] = e^{\frac{1}{2}v^2\sigma^2}$. If a process satisfies all three properties, then it is a BM.

DEFINITION 24.4. *(A Filtration for a BM) Let $(\Omega, \mathcal{F}, P)$ be a probability space on which is defined a Brownian Motion $W_t, \forall t \ge 0$. A filtration for the Brownian Motion is a collection of σ-algebra $\mathcal{F}_t$ s.t.*

(1) information accumulation, $\forall 0 \le s < t$, we have $\mathcal{F}_s \subseteq \mathbf{F}_t$;

(2) additivity, $\forall s \ge 0$, W_s is $\mathcal{F}_s$-measurable;

(3) independence of future increments, $\forall 0 \le t < u, W_u - W_t \perp \mathcal{F}_t$ (independent with $\mathcal{F}_t$).

Assume W_t is a Brownian Motion, then each of the following processes is a martingale,

(1) the BM W_t;

(2) the seqaured BM $W_t^2 - t$;

(3) the exponential process $e^{\theta W_t - \frac{1}{2}\theta^2 t}$.

We show that ont by one.

(1) W_t. $\forall 0 \le t < s$, we want $W_t = E[W_s|\mathcal{F}_t]$.

$$\begin{aligned} E[W_s|\mathcal{F}_t] &= E[W_s - W_t + W_t|\mathcal{F}_t] && (24.36)\\ &= E[W_s - W_t|\mathcal{F}_t] + E[W_t|\mathcal{F}_t] && (24.37)\\ &= W_t && (24.38) \end{aligned}$$

(2) $W_t^2 - t$. $\forall 0 \le t < s$, we want $E[W_s^2 - s|\mathcal{F}_t] = W_t^2 - t$.

$$\begin{aligned} E[W_s^2|\mathcal{F}_t] &= E[(W_s - W_t)^2 + W_t^2 + 2W_t(W_s - W_t)|\mathcal{F}_t] && (24.39)\\ &= E[(W_s - W_t)^2|\mathcal{F}_t] + E[W_t^2|\mathcal{F}_t] + 2E[W_t(W_s - W_t)|\mathcal{F}_t] && (24.40)\\ &= E[(W_s - W_t)^2] + E[W_t^2] + 2W_t E[W_s - W_t|\mathcal{F}_t] && (24.41)\\ &= s - t + W_t^2 + 0 + 0 && (24.42)\\ &= W_t^2 + s - t && (24.43)\\ \Rightarrow E[W_s^2 - s|\mathcal{F}_t] &= W_t^2 - t && (24.44) \end{aligned}$$

(3) $e^{\theta W_t - \frac{1}{2}\theta^2 t}$. $\forall 0 \le t < s$, we want $E[e^{\theta W_s - \frac{1}{2}\theta^2 s}|\mathcal{F}_t] = e^{\theta W_t - \frac{1}{2}\theta^2 t}$.

$$\begin{aligned} E[e^{\theta W_s - \frac{1}{2}\theta^2 s}|\mathcal{F}_t] &= E[e^{\theta(W_s - w_t) - W_t \frac{1}{2}\theta^2 s}|\mathcal{F}_t] && (24.45)\\ &= e^{\theta W_t - \frac{1}{2}\theta^2 s} E[e^{\theta(W_s - W_t)}|\mathcal{F}_t] && (24.46)\\ &= e^{\theta W_t - \frac{1}{2}\theta^2 s} e^{\frac{1}{2}\theta^2 (s-t)} && (24.47)\\ &= e^{\theta W_t - \frac{1}{2}\theta^2 t} && (24.48) \end{aligned}$$

If M_t is an MG, φ is a convex function, then $\forall 0 \le t < s$,

$$E[\varphi(M_s)|\mathcal{F}_t] \ge \varphi(E[M_s|\mathcal{F}_t]) = \varphi(M_t) \Rightarrow \varphi(M_t) \le E[\varphi(M_s)|\mathcal{F}_t] \tag{24.49}$$

i.e., M_t is a super-MG.

THEOREM 24.1. *(A BM is a Markov Process) Let $W_t, t \ge 0$ be a Brownian Motion and $\mathcal{F}_t$ be a filtration for it. Then W_t is a Markov Process.*

Proof. $\forall f, \forall 0 \le t \le s$, we need to find a g s.t $E[f(W_s)|\mathcal{F}_t] = g(W_t)$. Let

$$g(x) = \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{+\infty} f(w+x) e^{-\frac{w^2}{2(t-s)}} dw \tag{24.50}$$

$$\begin{aligned} E[f(W_s)|\mathcal{F}_t] &= E[f(W_s - W_t + W_t)|\mathcal{F}_t] && (24.51)\\ &= E[H(W_s - W_t, W_t)|\mathcal{F}_t] && (24.52)\\ &= E[H(\Delta W, \Delta W_t)|\mathcal{F}_t] && (24.53)\\ &= E[H(\Delta W, x)|\mathcal{F}_t] && (24.54)\\ &= E[H(\Delta W, x)] && (24.55)\\ &= E[f(\Delta W + x)] && (24.56)\\ &= \int_{-\infty}^{+\infty} f(y+x) \frac{1}{\sqrt{2\pi(s-t)}} e^{-\frac{y^2}{2(s-t)}} dy && (24.57)\\ &\equiv g(x) && (24.58)\\ &= g(W_t) && (24.59) \end{aligned}$$

□

DEFINITION 24.5. *(First-order Variation) Given a function f defined on $[0, T]$, let $\Pi : 0 = t_0 < t_1 < \cdots < t_n = T$ be a partition of the interval $[0, T]$ and let $|\Pi| = \max(t_k - t_{k-1})$ be the module of the partition. The quadratic variation is defined as the limit*

$$FV_T(f) = \lim_{|\Pi|\to 0} \sum_{k=1}^{n} |f(t_k) - f(t_{k-1})| \tag{24.60}$$

provided it exists.

If f is differentiable, then

$$FV_T(f) = \int_0^T |f'(t)|dt \tag{24.61}$$

as

$$FV_T(f) = \lim_{|\Pi|\to 0} \sum_{k=1}^{n} |f(t_k) - f(t_{k-1})| \tag{24.62}$$

$$= \lim_{|\Pi|\to 0} \sum_{k=1}^{n} |f'(\epsilon_k)|(t_k - t_{k-1}), \epsilon_k \in (t_{k-1}, t_k), \forall k \tag{24.63}$$

$$= \int_0^T |f'(t)|dt \tag{24.64}$$

DEFINITION 24.6. *(Quadratic Variation) Given a function f defined on $[0, T]$, let $\Pi : 0 = t_0 < t_1 < \cdots < t_n = T$ be a partition of the interval $[0, T]$ and let $|\Pi| = \max(t_k - t_{k-1})$ be the module of the partition. The quadratic variation is defined as the limit*

$$[f, f](T) = \lim_{|\Pi|\to 0} \sum_{k=1}^{n} |f(t_k) - f(t_{k-1})|^2 \tag{24.65}$$

provided it exists.

If f has continuous derivatives, then $[f, f](T) = 0$, as

$$[f, f](T) = \lim_{|\Pi|\to 0} \sum_{k=1}^{n} |f(t_k) - f(t_{k-1})|^2 \tag{24.66}$$

$$= \lim_{|\Pi|\to 0} \sum_{k=1}^{n} |f(t_k) - f(t_{k-1})|^2 \tag{24.67}$$

$$= \lim_{|\Pi|\to 0} \sum_{k=1}^{n} |f(\epsilon_k)|^2 (t_k - t_{k-1})^2, \epsilon_k \in (t_{k-1}, t_k), \forall k \tag{24.68}$$

$$\leq \lim_{|\Pi|\to 0} |\Pi| \sum_{k=1}^{n} |f(\epsilon_k)|^2 |t_k - t_{k-1}| \tag{24.69}$$

$$= (\lim_{|\Pi|\to 0} |\Pi|) \int_0^T |f'(t)|dt \tag{24.70}$$

$$= 0 \tag{24.71}$$

$$\Rightarrow [f, f](T) = 0 \tag{24.72}$$

THEOREM 24.2. *Let W_t be a Brownian Motion, then*

$$[W, W](T) = T, \forall T \geq 0, a.s. \tag{24.73}$$

Proof. Let $\Pi : 0 = t_0 < t_1 < \cdots < t_n = T$ be a partition of the interval $[0, T]$ and $W_k = W_{t_k}, S_\Pi =$

$\sum_{k=1}^{n}(W_k - W_{k-1})^2$. Then

$$
\begin{aligned}
E(S_\Pi) &= E[\sum_{k=1}^{n}(W_k - W_{k-1})^2] && (24.74)\\
&= \sum_{k=1}^{n} E[(W_k - W_{k-1})^2] && (24.75)\\
&= \sum_{k=1}^{n} Var[W_k - W_{k-1}] && (24.76)\\
&= \sum_{k=1}^{n}(t_k - t_{k-1}) && (24.77)\\
&= t_n - t_0 && (24.78)\\
&= T \Rightarrow \lim_{|\Pi|\to 0} E[S_\Pi] \qquad = T && (24.79)\\
Var[S_\Pi] &= Var[\sum_{k=1}^{n}(W_k - W_{k-1})^2] && (24.80)\\
&= \sum_{k=1}^{n} Var[(W_k - W_{k-1})^2] && (24.81)\\
&= \sum_{k=1}^{n} E[(W_k - W_{k-1})^4] - (E[(W_k - W_{k-1})^2])^2 && (24.82)\\
&= \sum_{k=1}^{n} 3(t_k - t_{k-1})^2 - (t_k - t_{k-1})^2 && (24.83)\\
&= \sum_{k=1}^{n} 2(t_k - t_{k-1})^2 && (24.84)\\
\Rightarrow & 0 \le Var[S_\Pi] \le 2\sum_{k=1}^{n}|t_k - t_{k-1}||\Pi| = T|\Pi| && (24.85)\\
\Rightarrow \lim_{|\Pi|\to 0} & Var[S_\Pi] \le \lim_{|\Pi|\to 0} T|\Pi| = 0 && (24.86)\\
& && (24.87)
\end{aligned}
$$

Thus, with $\lim_{|\Pi|\to 0} E[S_\Pi] = T$, $\lim_{|\Pi|\to 0} Var[S_\Pi] = 0$, we have

$$[W, W](T) = T, \forall T \ge 0, a.s. \tag{24.88}$$

□

The remaining task is to show that if $X \in N(0, \sigma^2)$, then $E[X^4] = 3\sigma^4$. We derive the moment generating function step by step. Let $X \sim N(\mu, \sigma^2)$, then

$$
\begin{aligned}
\varphi(u) &= E[e^{uX}] = e^{u\mu + \frac{1}{2}u^2\sigma^2} && (24.89)\\
\varphi'(u) &= E[Xe^{uX}] = (\mu + \sigma^2 u)e^{u\mu + \frac{1}{2}u^2\sigma^2} && (24.90)\\
\varphi''(u) &= E[X^2 e^{uX}] = e^{u\mu + \frac{1}{2}u^2\sigma^2}(\mu + \sigma^2 u^2 + \sigma^2) && (24.91)\\
\varphi'''(u) &= E[X^3 e^{uX}] = e^{u\mu + \frac{1}{2}u^2\sigma^2}[((\mu + \sigma^2 u)^2 + \sigma^2)(\mu + \sigma^2 u) + 2(\mu + \sigma^2 u)\sigma^2] && (24.92)\\
\varphi''''(u) &= E[X^4 e^{uX}] = e^{u\mu + \frac{1}{2}u^2\sigma^2} && (24.93)\\
& [A(\mu + \sigma^2 u) + 2(\mu + \sigma^2 u)\sigma^2(u + \sigma^2 u) + ((\mu + \sigma^2 u)^2) + \sigma^2]\sigma^2 + 2\sigma^2(Check) && (24.94)\\
\mu = 0 \Rightarrow & E[X^4 e^{uX}] = u^4 + 6\sigma^2 u^2 + 4\sigma^4 && (24.95)\\
u = 0 \Rightarrow & E[X^4] = 3\sigma^4 && (24.96)
\end{aligned}
$$

The first order variation of W_t (BM) is infinite. Suppose not, then there is a path W_t s.t.

$$\lim_{|\Pi|\to 0}\sum_{k=1}^{n}|W_k - W_{k-1}| = A < \infty \tag{24.97}$$

So, $0 \leq \lim_{|\Pi|\to 0} \sum_{k=1}^n |W_k - W_{k-1}|^2 \leq \max_k |W_k - W_{k-1}| \lim_{|\Pi|\to 0} \sum_{k=1}^n |W_k - W_{k-1}| = 0$, vioalting $[W,W](T) = T$.

Moreover, for higher order variation ($p \geq 2$), we have

$$0 \leq \lim_{|\Pi|\to 0} \sum_{k=1}^{n} |W_k - W_{k-1}|^p \leq \lim_k \max_k |W_k - W_{k-1}|^{p-1} (\sum_{k=1}^{n} |W_k - W_{k-1}|^2) = 0 \tag{24.98}$$

Further, $\lim_{|\Pi|\to 0} \sum_{k=1}^n |W_k - W_{k-1}||t_k - t_{k-1}| \leq \lim_{|\Pi|\to 0} |W_k - W_{k-1}| T = 0$.

The **Box** algebra refines all the outcomes As

$$dW_t dW_t = dt, dW_t dt = 0, dtdt = 0 \tag{24.99}$$

which is of great importance!

DEFINITION 24.7. *(Geometric Brownian Motion,GBM) Let W_t be a Brownian Motion, and μ, σ be constants. The process*

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t} \tag{24.100}$$

is called a Geometric Brownian Motion process (GBM), or a log-normal process. μ and σ are called the drift and volatility of the process.

Example 24.1. *Interpreting S_t as the price process of an asset, if we are given thck-by-tick data $S_{t_i}, i = 1, 2, \cdots, n$, how to estimate σ?*

Proof. Let $S_i = S_{t_i}, W_i = W_{t_i}, \forall i$ (denote) and $T = t_n$.

$$S_i = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t_i + \sigma W_i} \tag{24.101}$$

$$\log S_i \log S_0 + (\mu - \frac{1}{2}\sigma^2) t_i + \sigma W_i \tag{24.102}$$

$$\log S_{i-1} \log S_0 + (\mu - \frac{1}{2}\sigma^2) t_{i-1} + \sigma W_{i-1} \tag{24.103}$$

$$\Rightarrow \log \frac{S_i}{S_{i-1}} = (\mu - \frac{1}{2}\sigma^2)(t_i - t_{i-1}) + \sigma(W_i - W_{i-1}) \tag{24.104}$$

$$(\log \frac{S_i}{S_{i-1}})^2 = (\mu - \frac{1}{2}\sigma^2)^2 (t_i - t_{i-1})^2 + \sigma^2 (W_i - W_{i-1})^2 \tag{24.105}$$

$$+ 2(\mu - \frac{1}{2}\sigma^2)(t_i - t_{i-1})\sigma(W_i - W_{i-1}) \tag{24.106}$$

$$\Rightarrow \sum_{i=1}^{n} (\log \frac{S_i}{S_{i-1}})^2 = (\mu - \frac{1}{2}\sigma^2)^2 \sum_{i=1}^{n} (t_i - t_{i-1})^2 + \sigma^2 \sum_{i=1}^{n} (W_i - W_{i-1})^2 \tag{24.107}$$

$$+ 2(\mu - \frac{1}{2}\sigma^2)\sigma \sum_{i=1}^{n} (t_i - t_{i-1})(W_i - W_{i-1}) \tag{24.108}$$

$$\simeq (\mu - \frac{1}{2}\sigma^2)^2 \int_0^T (d_t)^2 + \sigma^2 \sum_{i=1}^{n} \int_0^T (dW_t)^2 + 2(\mu - \frac{1}{2}\sigma^2)\sigma \int_0^T dW_t dt \tag{24.109}$$

$$= \sigma^2 T \tag{24.110}$$

$$\Rightarrow \hat{\sigma}^2 = \frac{1}{T} \sum_{i=1}^{n} (\log \frac{S_i}{S_{i-1}})^2 \tag{24.111}$$

□

25 Ito's Calculus

25.1 Ito's Formula

We are interested in a P&L function:

$$\sum_{i=0}^{n-1} f(t_i)(W(t_{i+1}) - W(t_i)) \tag{25.1}$$

where $f(t_i)$ is the position of stocks at t_i and $W(t_i)$ is the price of stocks at t_i. In generally, we want to calculate

$$I(t) = \int_0^T \Delta(t) dW(t) \tag{25.2}$$

where $\Delta(t)$ is $\mathcal{F}$-measurable. We start from a simple process, let $0 = t_0 < t_1 < \cdots < t_n = T$ be a partition of T, and $\Delta(t)$ be a constant in each interval. So, let $t \in [t_k, t_{k+1}]$

$$\begin{aligned}
I(t) &= \int_0^t \Delta(s) dW(s) &(25.3)\\
&= \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \Delta(s) dW(s) + \int_{t_k}^{t} \Delta(s) dW(s) &(25.4)\\
&= \sum_{i=0}^{k-1} \Delta(t_i) \int_{t_i}^{t_{i+1}} dW(s) + \Delta(t_k) \int_{t_k}^{t} dW(s) &(25.5)\\
&= \sum_{i=0}^{k-1} \Delta(t_i)(W(t_{i+1}) - W_{t_i}) + \Delta(t_k)(W(t) - W(t_k)) &(25.6)
\end{aligned}$$

LEMMA 25.1. *$I(t)$ has four properties if $\Delta(s)$ is a deterministic and simple function of s, not varies at t:*

1. *$I(t)$ is $\mathcal{F}$-measurable;*
2. *$I(t)$ is a martingale process, and $E(I(t)) = 0$;*
3. *(Ito' isometry): $Var(I(t)) = E(I(t)^2) = E\int_0^t \Delta(s)^2 ds$;*
4. *(Quadratic variation): $[I, I](t) = \int_0^t \Delta(s)^2 ds$.*

Proof. Let $I(t) = \sum_{i=0}^{k-1} \Delta(t_i)(W(t_{i+1}) - W_{t_i}) + \Delta(t_k)(W(t) - W(t_k))$.

1. we see all $W(t_i), \Delta(t_i), t_i \le t$ is $\mathcal{F}$-measurable, so $I(t)$ is $\mathcal{F}$-measurable.
2. $\forall s > t$, we need to show $E[I(s)|\mathcal{F}(t)] = I(t)$. There are two cases:
   - $s, t \in [t_j, t_{j+1}]$. $I(s) - I(t) = \Delta(t_j)(W(s) - W(t))$, then $E[I(s)|\mathcal{F}(t)] - I(t) = E[I(s) - I(t)|\mathcal{F}(t)] = E[\Delta(t_j)(W(s) - W(t))|\mathcal{F}(t)] = \Delta(t_j)E[(W(s) - W(t))|\mathcal{F}(t)] = 0$.
   - $t \in [t_k, t_{k+1}], s \in [t_j, t_{j+1}], k < j$. Then,

$$\begin{aligned}
I(s) - I(t) = \Delta(t_k)(W(t_{k+1}) - W(t_k)) + &\qquad (25.7)\\
\sum_{i=k+1}^{j-1} \Delta(t_i)(W(t_{i+1}) - W(t_i)) + \Delta(t_j)(W(s) - W(t_j)) &\qquad (25.8)
\end{aligned}$$

$$\begin{aligned}
E[\Delta(t_k)(W(t_{k+1}) - W(t_k))|\mathcal{F}(t)] &= \Delta(t_k)E[W(t_{k+1}) - W(t_k)|\mathcal{F}(t)] = 0 &(25.9)\\
E[\Delta(t_j)(W(s) - W(t_j))|\mathcal{F}(t)] &= E[E[\Delta(t_j)(W(s) - W(t_j))|\mathcal{F}(t_j)]|\mathcal{F}(t)] &(25.10)\\
&= E[\Delta(t_j)0|\mathcal{F}(t)] = 0 &(25.11)
\end{aligned}$$

$$E[\sum_{i=k+1}^{j-1} \Delta(t_i)(W(t_{i+1}) - W(t_i))|\mathcal{F}(t)] = 0 \tag{25.12}$$

$$E[I(t) - I(s)|\mathcal{F}(t)] = 0 \tag{25.13}$$

$$E[I(t) - I(0)|\mathcal{F}(0)] = I(0) = \Delta(t_0)(W(t_0) - W(t_0)) = 0, i.e., E[I(0)] = 0 \tag{25.14}$$

3. First, $Var(I(t)) = E[I(t)^2] + E(I(t))^2 = E(I(t)^2)$.

$$I(t)^2 = \sum_{i=0}^{k-1} \Delta_i^2 (W_{i+1} - W_i)^2 + \Delta_k^2 (W_t - W_k)^2 + \tag{25.15}$$

$$2 \sum_{0 \le i < j \le k-1} \Delta_i \Delta_j (W_{i+1} - W_i)(W_{j+1} - W_j) \tag{25.16}$$

$$E[\Delta_i (W_{i+1} - W_i)^2] = E[E[\Delta_i^2 (W_{i+1} - W_i)^2 | \mathcal{F}_i]] \tag{25.17}$$

$$= E[\Delta_i^2 E[(W_{i+1} - W_i)^2 | \mathcal{F}_i]] \tag{25.18}$$

$$= E(\Delta_i^2 (t_{i+1} - t_i)), i = 0, 1, \cdots, k-1 \tag{25.19}$$

$$E[\Delta_k^2 (W_t - W_k)^2] = E[\Delta_k^2 (t - t_k)] \tag{25.20}$$

$$E[\Delta_i \Delta_j (W_{i+1} - W_i)(W_{j+1} - W_j)] = E[E[\Delta_i \Delta_j (W_{i+1} - W_i)(W_{j+1} - W_j) | \mathcal{F}_j]] \tag{25.21}$$

$$= E[\Delta_i \Delta_j (W_{i+1} - W_i) E[(W_{j+1} - W_j) | \mathcal{F}_j]] \tag{25.22}$$

$$= E[\Delta_i \Delta_j 0] = 0 \tag{25.23}$$

$$\Rightarrow E[I(t)^2] = \sum_{i=0}^{k-1} E[\Delta_i^2 (t_{i+1} - t_i)] + E[\Delta_k^2 (t - t_k)] \tag{25.24}$$

$$= E[\sum_{i=0}^{k-1} \Delta_i^2 (t_{i+1} - t_i) + \Delta_k^2 (t - t_k)] = E[\int_0^t \Delta(s)^2 ds] \tag{25.25}$$

4. Let $t_i = s_0^i < s_1^i < \cdots < s_{m_i}^i = t_{i+1}$ partition $[t_i, t_{i+1}]$, then

$$[I, I]([t_i, t_{i+1}]) = \sum_{k=1}^{m_i} (I(s_k^i) - I(s_{k-1}^i))^2 \tag{25.26}$$

$$= \sum_{k=1}^{m_i} \Delta(t_i)^2 (W(s_k^i) - W(s_{k-1}^i))^2 \tag{25.27}$$

$$= \Delta(t_i)^2 \sum_{k=1}^{m_i} (W(s_k^i) - W(s_{k-1}^i))^2 \tag{25.28}$$

$$(|\pi_i| \to 0) = \Delta(t_i)^2 (t_{i+1} - t_i) \tag{25.29}$$

$$\Rightarrow [I, I](t) = \sum_{i=0}^{k-1} \Delta(t_i)^2 (t_{i+1} - t_i) + \Delta(t_k)^2 (t - t_k) \tag{25.30}$$

$$= \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \Delta(s)^2 ds + \int_{t_k}^{k} \Delta(s)^2 ds \tag{25.31}$$

$$= \int_0^{t_k} \Delta(s)^2 ds + \int_{t_k}^{t} \Delta(s)^2 ds \tag{25.32}$$

$$= \int_0^t \Delta(s)^2 ds \tag{25.33}$$

□

If $\Delta(s)$ is a stochastic function of s, e.g., $W(s)$, then we we can construct a sequence of deterministic simple function $\Delta_n(s)$, obtain the integral and take its limit. See Example 25.1

Example 25.1. *Show* $I(t) = \int_0^T W_t dW_t = \frac{W_T - T^2}{2}$.

Proof. Let $0 = t_0 < t_1 < \cdots < t_n = T$ partition $[0, T]$, then $\Delta_n(t) = W(t_i), t \in [t_i, t_{i+1}]$.

$$\begin{aligned}
I_n(t) &= \int_0^T \Delta_n(t) dW_t && (25.34)\\
&= \sum_{i=0}^{n-1} W_i(W_{i+1} - W_i) && (25.35)\\
&= \sum_{i=0}^{n-1} (W_i - W_{i+1} + W_{i+1})(W_{i+1} - W_i) && (25.36)\\
&= \sum_{i=0}^{n-1} -(W_{i+1} - W_i)^2 + W_{i+1}(W_{i+1} - W_i) && (25.37)\\
&= \sum_{i=0}^{n-1} -(W_{i+1} - W_i)^2 + \sum_{i=0}^{n-1} W_{i+1}^2 - \sum_{i=0}^{n-1} W_{t+1}W_i && (25.38)\\
&= \sum_{i=0}^{n-1} -(W_{i+1} - W_i)^2 + \sum_{i=1}^{n} W_i^2 - \sum_{i=0}^{n-1} W_{t+1}W_i && (25.39)\\
&= \sum_{i=0}^{n-1} -(W_{i+1} - W_i)^2 + \sum_{i=0}^{n-1} W_i^2 - \sum_{i=0}^{n-1} W_{t+1}W_i + W_n^2 - W_0^2 && (25.40)\\
&= \sum_{i=0}^{n-1} -(W_{i+1} - W_i)^2 - \sum_{i=0}^{n-1} W_i(W_{i+1} - W_i) + W_n^2 && (25.41)\\
\Rightarrow I_n(t) &= -\sum_{i=0}^{n-1} (W_{i+1} - W_i)^2 - I_n(t) + W_n^2 && (25.42)\\
\Rightarrow I_n(t) &= \frac{-\sum_{i=0}^{n-1}(W_{i+1} - W_i)^2 + W_n^2}{2} && (25.43)\\
\Rightarrow I(t) &= \lim_{n\to\infty} I_n(t) && (25.44)\\
&= \lim_{n\to\infty} \frac{-\sum_{i=0}^{n-1}(W_{i+1} - W_i)^2 + W_n^2}{2} && (25.45)\\
&= \frac{W_T^2 - T^2}{2} && (25.46)
\end{aligned}$$

□

THEOREM 25.1. *(Ito's Fomula) Let $f(t, x)$ be a function with continuous partial derivatives $f_t(t, x)$, $f_x(t, x)$ and $f_{xx}(t, x)$ and let $W(t)$ be a Brownian motion. Then $\forall T_2 > T_1 \geq 0$, we have*

$$\begin{aligned}
&f(T_2, W(T_2)) - f(T_1, W(T_1)) && (25.47)\\
&= \int_{T_1}^{T_2} f_t(t, W(t))dt + \int_{T_1}^{T_2} f_x(t, W(t))dW(t) + \frac{1}{2}\int_{T_1}^{T_2} f_{xx}(t, W(t))dt && (25.48)
\end{aligned}$$

Its differential form is

$$df(t, W(t)) = f_t(t, W(t))dt + f_x(t, W(t))dW(t) + \frac{1}{2} f_{xx}(t, W(t))dt \quad (25.49)$$

Proof. We prove the differential form with taylor expansion and Box algebra. In a simple case, $f(t, W_t) = f(W_t)$

$$\begin{aligned}
df(W_t) &\simeq f(W_t + dW_t) - f(W_t) && (25.50)\\
&= f(W_t) + f'(W_t)dW(t) + \frac{1}{2} f''(W_t)(dW_t)^2 + \cdots - f(W_t) && (25.51)\\
&= f'(W_t)dW_t + \frac{1}{2} f''(W_t)dt && (25.52)
\end{aligned}$$

Note that $dW_t dW_t = dt, dW_t dt = 0, dtdt = 0$. Then, for $f(t, W_t)$, we have

$$df(t, W_t) = f(t + dt, W_t + dW_t) - f(t, W_t) \tag{25.53}$$
$$= f(t, W_t) + f_t(t, W_t)dt + f_x(t, W_t)dW_t \tag{25.54}$$
$$+ \frac{1}{2}f_{tt}(t, W_t)d^2t + \frac{1}{2}f_{xx}(t, w_t)d^2W_t \tag{25.55}$$
$$+ f_{tx}(t, w_t)dtdW_t + \cdots - f(t, w_t) \tag{25.56}$$
$$= f_t(t, W_t)dt + f_x(t, W_t)dW_t + \frac{1}{2}f_{xx}(t, w_t)dt \tag{25.57}$$

□

Example 25.2. *Verify* $\int_0^T W_t dW_t = \frac{W_T - T^2}{2}$.

Proof. Let $f(x) = \frac{1}{2}x^2$, then $d(f(W_t)) = W_t dW_t + \frac{1}{2}dt$, so $\int_0^T df(W_t) = \int_0^T W_t dW_t + \int_0^T \frac{1}{2}dt$, $\frac{1}{2}W_T^2 - \frac{1}{2}W_0^2 = \int_0^T W_t dW_t + \frac{1}{2}T$, then $\int_0^T W_t dW_t = \frac{W_T^2 - T}{2}$. □

DEFINITION 25.1. *(Ito's Process) Let* $X_t = X_0 + \int_0^t \Delta(u)dW_u + \int_0^t \Theta(u)du$*, where* $\Delta(u), \Theta(u)$ *are adapted stochastic process, i.e.,* $\Delta(t), \Theta(t)$ *is* $\mathcal{F}(t)$*-measurable. Then* X_t *is a Ito's process.*

LEMMA 25.2. *(Quadratic Variation of Ito's Process) Let* $[X, X](t) = \int_0^t dX_u dX_u, dX_u = \Delta(u)dW_u + \Theta(u)du$*. Then,*

$$(dX_u)^2 = \Delta^2(u)d^2W_u + \Theta^2(u)d^2u + 2\Delta(u)\Theta(u)dW_t du \tag{25.58}$$
$$= \Delta^2(u)du \tag{25.59}$$
$$\Rightarrow \int_0^t d^2X_u = \int_0^t \Delta^2(u)du \tag{25.60}$$

Only Ito's integral contributes to the Q.V.

THEOREM 25.2. *(Ito's Formula for Ito's Process) Let* $f(t, x)$ *be a continuous function with continuous partial derivatives, and* X_t *be an Ito's process, we have*

$$df(t, X_t) = f_t(t, X_t)dt + f_x(t, X_t)dX_t + \frac{1}{2}f_{xx}(t, w_t)\Delta^2(t)dt \tag{25.61}$$

.

Proof. As $dX_t dX_t = \Delta^2(t)dt, dX_t dt = \Delta(t)dW_t dt + \Theta_t dtdt = 0, dtdt = 0$, so we just need to substitute W_t, d^2W_t with $X_t, \Delta^2(t)dt$ in Ito's formula and get that differential form.

Moreover, the integral form is given by

$$f(T_2, X_{T_2}) - f(T_1, X_{T_1}) = \int_{T_1}^{T_2} df(t, X_t) \tag{25.62}$$
$$= \int_{T_1}^{T_2} f_t(t, X_t)dt + f_x(t, X_t)dX_t + \frac{1}{2}f_{xx}(t, X_t)\Delta^2(t)dt \tag{25.63}$$
$$= \int_{T_1}^{T_2} f_t(t, X_t)dt + \int_{T_1}^{T_2} f_x(t, X_t)dX_t + \int_{T_1}^{T_2} \frac{1}{2}f_{xx}(t, X_t)\Delta^2(t)dt \tag{25.64}$$
$$= \int_{T_1}^{T_2} f_t(t, X_t)dt + \int_{T_1}^{T_2} f_x(t, X_t)(\Delta(t)dW_t + \Theta_t dt) \tag{25.65}$$
$$+ \int_{T_1}^{T_2} \frac{1}{2}f_{xx}(t, X_t)\Delta^2(t)dt \tag{25.66}$$
$$= \int_{T_1}^{T_2} [f_t(t, X_t) + f_x(t, X_t)\Theta_t + \frac{1}{2}f_{xx}(t, X_t)\Delta^2(t)]dt \tag{25.67}$$
$$+ \int_{T_1}^{T_2} f_x(t, X_t)\Delta(t)dW_t \tag{25.68}$$

□

DEFINITION 25.2. *(Generalized GBM) We parameterize the α, σ with s in GBM, obtaining*

$$S_t = S_0 \exp\{\int_0^t (\alpha(s) - \frac{1}{2}\sigma^2(s))ds + \int_0^t \sigma(s)dW_s\} \tag{25.69}$$

where $\alpha(t), \sigma(t)$ is $(\mathcal{F})(t)$-measurable.

Let $X_t = \int_0^t (\alpha(s) - \frac{1}{2}\sigma^2(s))ds + \int_0^t \sigma(s)dW_s$ be a Ito's process, then $S_t = S_0 e^{X_t}, dS_t = S_0 de^{X_t}$. Let $f(x) = e^x$, then $f'(x) = e^x, f''(x) = e^x$. According to Ito's formula for Ito's process, we say,

$$\begin{aligned} de^{X_t} &= e^{X_t}dX_t + \frac{1}{2}e^{X_t}\sigma^2(t)dt & (25.70)\\ &= e^{X_t}[(\alpha(t) - \frac{1}{2}\sigma^2(t))dt + \sigma(t)dW_t] + \frac{1}{2}e^{X_t}\sigma^2(t)dt & (25.71)\\ &= e^{X_t}[(\alpha(t) - \frac{1}{2}\sigma^2(t)) + \frac{1}{2}e^{X_t}\sigma^2(t)]dt + e^{X_t}\sigma(t)dW_t & (25.72)\\ &= e^{X_t}(\alpha(t)dt + \sigma(t)dW_t) & (25.73)\\ \Rightarrow dS_t &= S_0 e^{X_t}(\alpha(t)dt + \sigma(t)dW_t) & (25.74)\\ &= S_t(\alpha(t)dt + \sigma(t)dW_t) & (25.75)\\ \Rightarrow \frac{dS_t}{S_t} &= \alpha(t)dt + \sigma(t)dW_t & (25.76) \end{aligned}$$

where $\alpha(t)$ denotes expected returns and $\sigma(t)$ denotes volatility.

LEMMA 25.3. *Induce the moments generating function of normal distribution. Let $I(t) = \int_0^t \Delta(s)dW_s, \phi(s) = E[e^{uI(t)}]$, then $\phi(s) = exp\{\frac{1}{2}u^2 \int_0^t \Delta^2(s)ds\}$.*

Proof. Basically, if $X \sim N(0, \sigma^2)$, then $E[e^{uX}] = e^{\frac{1}{2}\sigma^2 u^2}$. Let

$$M(t) = \exp\{uI(t) - \frac{1}{2}u^2 \int_0^t \Delta^2(s)ds\} \tag{25.77}$$

We just need to show $M(t)$ is a martingale, then $M(t) = M(0) = 1$ as we want. Let $X_t = u\int_0^t \Delta(s)dW_s - \frac{1}{2}u^2 \int_0^t \Delta^2(s)ds$, then $dX_t = u\Delta(t)dW_t - \frac{1}{2}u^2\Delta^2(t)dt$, then

$$\begin{aligned} dM_t &= de^{X_t} & (25.78)\\ &= e^{X_t}dX_t + \frac{1}{2}e^{X_t}d^2X_t & (25.79)\\ &= e^{X_t}(u\Delta(t)dW_t - \frac{1}{2}u^2\Delta^2(t)) + \frac{1}{2}e^{X_t}u^2\Delta^2(t)dt & (25.80)\\ &= e^{X_t}u\Delta(t)dW_t & (25.81)\\ &= u\Delta(t)M(t)dW_t & (25.82)\\ \Rightarrow M(t) - M(0) &= \int_0^t u\Delta(t)M(t)dW_t & (25.83) \end{aligned}$$

where $u\Delta(t)M(t)$ is a stochastic function, then $M(t) - M(0)$ is an Ito's integral, so is a martingale. Then $E[M(t) - M(0)] = 0$, i.e., $E[M(t)] = 1$. So,

$$E[e^{uI(t)}] = \exp\{\frac{1}{2}u^2 \int_0^t \Delta^2(s)ds\} \tag{25.84}$$

□

Example 25.3. *(Vasicek Short Rate Model) The Vasicek short rate model:*

$$dR_t = (\alpha - \beta R_t)dt + \sigma dW_t \tag{25.85}$$

where $\aleph, \beta, \sigma$ are all positive constants. Given R_0, we want to find R_t, show

$$R_t = e^{-\beta t}R_0 + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + \sigma e^{-\beta t}\int_0^t e^{\beta s}dW_s \tag{25.86}$$

Proof. If we eliminate dW_t, then $dR_t = \alpha dt - \beta R_t dt$. Let $f(t,r) = e^{\beta t} r$, then

$$\begin{aligned}
de^{\beta t} dR_t &= (\beta e^{\beta t} dt)[(\alpha - \beta R_t)dt + \sigma dW_t] = 0 && (25.87)\\
df(t, R_t) &= R_t de^{\beta t} + e^{\beta t} dR_t + de^{\beta t} dR_t && (25.88)\\
&= \beta e^{\beta t} R_t d_t + e^{\beta t}[(\alpha - \beta t)dt + \sigma W_t] && (25.89)\\
&= \alpha e^{\beta t} dt + e^{\beta t} \sigma dW_t && (25.90)\\
\Rightarrow e^{\beta t} R_t - R_0 &= \in_0^t \alpha e^{\beta t} dt + e^{\beta t} \sigma dW_t && (25.91)\\
&= \frac{\alpha}{\beta}(e^{\beta t} - 1) + \int_0^t \sigma e^{\beta s} dW_s && (25.92)\\
\Rightarrow R_t &= e^{-\beta t} R_0 + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + \sigma e^{-\beta t} \int_0^t e^{\beta s} dW_s && (25.93)
\end{aligned}$$

Then we have

$$\begin{aligned}
E[R_t] &= e^{-\beta t} R_0 + \frac{\alpha}{\beta}(1 - e^{-\beta t}) && (25.94)\\
Var[R_t] &= Var[\sigma e^{-\beta t} \int_0^t e^{\beta s} dW_s] = \int_0^t \sigma^2 e^{2\beta(\sigma - t)} dt && (25.95)\\
\Rightarrow R_t &\sim N(\mu_t, v_t), \forall t && (25.96)
\end{aligned}$$

The above model specifies a mean-reverting short rate process and the sort rate can become negative in the model (long-run mean $\lim_{t\to\infty} \frac{\alpha}{\beta}$). □

We can simulate the Vasicek short rate model in two approaches.

- intergel:

$$\begin{aligned}
f(t_{i+1}, R_{i+1}) - f(t_i, R_i) &= \int_{t_i}^{t_{i+1}} \alpha e^{\beta t} + \int_{t_i}^{t_{i+1}} e^{\beta t} \sigma dW_t && (25.97)\\
\Rightarrow e^{\beta t_{i+1}} R_{i+1} - e^{\beta t_i} R_i &= \frac{\alpha}{\beta}(e^{\beta t_{i+1}} - e^{\beta t_i}) + \int_{t_i}^{t_{i+1}} e^{\beta t} \sigma dW_t && (25.98)
\end{aligned}$$

  we need to simulate $\int_{t_i}^{t_{i+1}} e^{\beta t} \sigma dW_t$, which is normally distributed with zero mean and $\int_{t_i}^{t_{i+1}} e^{2\beta t} \sigma^2 dt$ variance.

- difference:

$$\begin{aligned}
dR_t &= (\alpha - \beta R_t)dt + \sigma dW_t && (25.99)\\
\Rightarrow R_{i+1} - R_i &= (\alpha - \beta R_i)\Delta t_i + \sigma(W_{i+1} - W_i) && (25.100)
\end{aligned}$$

  we need to simulate $\sigma(W_{i+1} - W_i)$ which follows $N(0, (t_{i+1} - t_i)\sigma)$.

25.2 Black-Scholes-Merton Formula

Let S_t, the price of stocks, follows $dS_t = \alpha S_t dt + \sigma S_t dW_t, S_0 = X$. The risk free rate is r, i.e., $dB_t = rB_t dt$ where B_t is a bond. Assume no trade friction and no leverage constraint.

We construct a self-financing portfolio (no capital addition or withdraw):

$$X_t = \Delta_t S_t + X_t - \Delta_t S_t \tag{25.101}$$

where $\Delta_t S_t$ is the risk investment and $X_t - \Delta_t S_t$ is the risk-free investment. Then,

$$\begin{aligned}
dX_t &= \Delta_t dS_t + r(X_t - \Delta_t S_t)dt && (25.102)\\
&= rX_t dt + \Delta_t(dS_t - rS_t dt) && (25.103)
\end{aligned}$$

Now we calculate the discounted prices, $de^{-rt}dX_t = 0$, then

$$de^{-rt}X_t = X_t de^{-rt} + e^{-rt}dX_t \tag{25.104}$$
$$= -re^{-rt}X_t dt + e^{-rt}dX_t \tag{25.105}$$
$$= -re^{-rt}X_t dt + e^{-rt}[rX_t dt + \Delta_t(dS_t - rS_t dt)] \tag{25.106}$$
$$= e^{-rt}\Delta_t(dS_t - rS_t dt) \tag{25.107}$$
$$= \Delta_t de^{-rt}S_t \tag{25.108}$$

where $de^{-rt}S_t = e^{-rt}(dS_t - rS_t dt) = e^{-rt}[(\alpha - r)S_t dt + \sigma S_t dW_t]$.

Consider a European call option with payoff $(S_T - K)^+$ at time T where K is the strick price. Its price $c(t, S_t)$ depends on the calender time t and the spot price S_t.

Assume $c(t, x)$ is continuously differentiable w.r.t t and second-order continuously differentiable w.r.t x (smooth enough to apply Ito's formula). Then

$$dc(t, S_t) = c_t(t, S_t)dt + c_x(\alpha S_t dt + \sigma S_t dW_t) + \frac{1}{2}c_{xx}\sigma^2 S_t^2 dt \tag{25.109}$$
$$= (c_t(t, S_t) + \alpha S_t c_x + \frac{1}{2}\sigma^2 S_t^2 c_{xx})dt + \sigma S_t c_x dW_t \tag{25.110}$$
$$de^{-rt}c(t, S_t) = -re^{-rt}c(t, S_t)dt + e^{-rt}dc(t, S_t) \tag{25.111}$$
$$= -re^{-rt}c(t, S_t)dt + e^{-rt}(c_t(t, S_t) + \alpha S_t c_x + \frac{1}{2}\sigma^2 S_t^2 c_{xx})dt + \sigma S_t c_x dW_t \tag{25.112}$$
$$= e^{-rt}[-rc + c_t + \alpha S_t c_x + \frac{1}{2}\sigma^2 S_t^2 c_{xx}]dt + e^{-rt}\sigma S_t c_x dW_t \tag{25.113}$$

Financial derivatives are usually priced by replicating, which is equivalent to the no-arbitrage principle. We want to find in initial capital X_0 s.t. we want construct a portfolio process X_t whose value coincides with the option's value at any time t, i.e., $X_t = c(t, S_t), \forall t$, or we say

$$X_0 = c(0, S_0) \tag{25.114}$$
$$d[e^{-rt}X_t] = d[e^{-rt}c(t, S_t)] \tag{25.115}$$

Thus,

$$\Delta_t e^{-rt}[(\alpha - r)S_t dt + \sigma S_t dW_t] = e^{-rt}[-rc + c_t + \alpha S_t c_x + \frac{1}{2}\sigma^2 S_t^2 c_{xx}]dt + e^{-rt}\sigma S_t c_x dW_t \tag{25.116}$$
$$\Rightarrow \begin{cases} \Delta_t \sigma S_t = \sigma S_t C_x(t, S_t) \\ -rc + c_t + \alpha S_t c_x + \frac{1}{2}\sigma^2 S_t^2 c_{xx} = \Delta(\alpha - r)S_t \end{cases} \tag{25.117}$$
$$\Rightarrow \begin{cases} \Delta_t = c_x(t, S_t) \\ -rc + c_t + rS_t c_x + \frac{1}{2}\sigma^2 S_t^2 c_{xx} = 0 \end{cases} \tag{25.118}$$

that hold for all t and $-rc + c_t + rS_t c_x + \frac{1}{2}\sigma^2 S_t^2 c_{xx} = 0$ is a partial differential equation (PDE), called Black-Scholes option pricing equation. Now the boundary condition is natural

$$c(T, S_T) = (S_T - K)^+ \tag{25.119}$$

Solve

$$\begin{cases} -rc + c_t + rS_t c_x + \frac{1}{2}\sigma^2 S_t^2 c_{xx} = 0 \\ c(T, S_T) = (S_T - K)^+ \end{cases} \tag{25.120}$$

we obtain

$$c(t, x) = xN(d_+) - Ke^{-r(T-t)}N(d_-) \tag{25.121}$$
$$N(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}du, d_+ = \frac{\log(x/K) + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}, d_- = d_+ - \sigma\sqrt{T - t} \tag{25.122}$$

The Black-Scholes option pricing equation is applicable for all European options, i.e., the pricing equation for a European option with payoff function $g(x)$ is

$$rf(t, x) = f_t(t, x) + rxf_x(t, x) + \frac{1}{2}\sigma^2 x^2 f_{xx}(t, x) \tag{25.123}$$
$$f(T, x) = g(x) \tag{25.124}$$

where $f(t,x)$ is the price of option at t with stock price x.

Take $g(x) = (K-x)^+$, we obtain the pricing function for a put option is

$$
\begin{aligned}
p(t,x) &= K^{-r(T-t)}N(-d_-) - xN(-d_+) && (25.125)\\
\Rightarrow p(t,S_t) - c(t,S_t) &= K^{-r(T-t)}N(-d_-) - xN(-d_+) - xN(d_+) + Ke^{-r(T-t)}N(d_-) && (25.126)\\
&= Ke^{-r(T-t)} - x && (25.127)
\end{aligned}
$$

Then the put-call parity,

$$
c(t,S_t) - p(t,S_t) = S_T - Ke^{-r(T-t)} \tag{25.128}
$$

There are two alternative approaches to obtain the Black-Scholes PDE.

Alternative Approach 1, construct a portfolio $\Pi(t,S_t) = c(t,S_t) - \Delta_t S_t$, then the instantaneous change in the portfolio value is

$$
\begin{aligned}
d\Pi(t,S_t) &= dc(t,S_t) - \Delta_t dS_t && (25.129)\\
&= c_t dt + c_t(\alpha S_t dt + \sigma S_t dW_t) + \frac{1}{2}c_{xx}\sigma^2 S_t dt - \Delta_t S_t(\alpha dt + \sigma dW_t) && (25.130)
\end{aligned}
$$

Eliminate the riskiness of the portfolio by the properly choosing Δ_t (Delta Hedging)

$$
\begin{aligned}
c_x\sigma S_t - \Delta_t\sigma S_t &= 0 && (25.131)\\
\Rightarrow \Delta_t &= c_x(t,S_t) && (25.132)
\end{aligned}
$$

Then the risk-free portfolio should earn a risk-free rate of return

$$
\begin{aligned}
d\Pi_t &= r\Pi_t dt && (25.133)\\
\Rightarrow c_t dt + c_x\alpha S_t st + \frac{1}{2}c_{xx}\sigma^2 S_t^2 dt - \alpha S_t c_x dt &= r(c - S_t c_x)dt && (25.134)\\
\Rightarrow c_t + \frac{1}{2}\sigma^2 S_t^2 + rS_t c_x - rc &= 0 && (25.135)
\end{aligned}
$$

Alternative Approach 2 (Black's original idea), calculate the expected return and volatility of the option

$$
\begin{aligned}
dc(t,S_t) &= (c_t(t,S_t) + \alpha S_t c_x + \frac{1}{2}\sigma^2 S_t^2 c_{xx})dt + \sigma S_t c_x dW_t && (25.136)\\
\frac{dc(t,S_t)}{c(t,S_t)} &= \frac{c_t(t,S_t) + \alpha S_t c_x + \frac{1}{2}\sigma^2 S_t^2 c_{xx}}{c}dt + \frac{\sigma S_t c_x}{c}dW_t && (25.137)
\end{aligned}
$$

In equilibrium, the risk-return profile of the option should not differ from that of the underling asset, because they are exposed to the same risk factor W_t. So

$$
\begin{aligned}
\frac{\frac{c_t(t,S_t)+\alpha S_t c_x + \frac{1}{2}\sigma^2 S_t^2 c_{xx}}{c}}{\frac{\sigma S_t c_x}{c}} &= \frac{\alpha - r}{\sigma} && (25.138)\\
\Rightarrow c_t\alpha S_t c_x + \frac{1}{2}\sigma^2 S_t^2 c_{xx} 0 rc &= (\alpha - r)c_x S_t && (25.139)\\
\Rightarrow -rc + c_t + rS_t c_x + \frac{1}{2}\sigma^2 S_t^2 c_{xx} &= 0 && (25.140)
\end{aligned}
$$

Under the Black-Scholes model, there is a dynamic portfolio that can replicate the payoff of the call option under any state of the world. The option's price should be equal to the value of the replicating portfolio. According to the BS formula, the replicating portfolio consists of $\Delta_t = c_x(t,S_t) = N(d_+)$ shares of the underlying asset. As $\frac{\partial \Delta^t}{\partial S_t} > 0$, then the hedger needs to buy more shares of the underlying asset when its prices rises and to sell share of the underlying asset then its prices drops (buy-high-sell-low).

LEMMA 25.4. *(Positive Vega of Vanilla Options) The Black-Scholes option pricing formula is an increasing function of σ. For either a call or a put, its vega is*

$$
\frac{\partial c}{\partial \sigma} = \frac{\partial p}{\partial \sigma} = xN'(d_+)\sqrt{T-t} > 0 \tag{25.141}
$$

DEFINITION 25.3. *(Implied Volatility, IV) Given a valid market price of a call option $C_0(t)$, its Black-Scholes implied volatility is the quantity σ^* which solves,*

$$BSM(T-t, S_t; K, r, \sigma^*) = C_0(t) \tag{25.142}$$

where $BSM(T-t, S_t; K, r, \sigma)$ is the Black-Scholes formula for a call option.

LEMMA 25.5. *Consider a call and a put option on the same underlying asset, with the same time-to-maturity and strike price. Then their market prices satisfy put-call parity if and only if their implied volatilities are the same.*

Proof.

$$c(t, S_t; \sigma_c^*) - p(t, S_t; \sigma_p^*) = S_t - Ke^{-r(T-t)} \tag{25.143}$$

$$c(t, S_t; \sigma_c^*) - p(t, S_t; \sigma_c^*) = S_t - Ke^{-r(T-t)} \tag{25.144}$$

$$\Rightarrow p(t, S_t; \sigma_p^*) = p(t, S_t; \sigma_c^*) \tag{25.145}$$

$$\frac{\partial p}{\sigma^*} > 0 \Rightarrow \sigma_p^* = \sigma_c^* \tag{25.146}$$

□

The spread between the implied volatility of calls and of puts (implied volatility spread) can be used to empirically measure the relative price of calls and puts. If A call has a higher IV than a put with same characteristics does, then it is interpreted as the call is relatively more expensive than the put.

One empirical observation: IVS seems to be informative. If the implied volatility of calls is higher (lower) than that of puts (on the same underlying asset), then the underlying is likely to perform well (poorly) in the near future.

Before Oct 1987, the implied volatility curves observed in the market are indeed almost flat. However, after Oct 1987, the implied volatility curves suddenly became curved. These curves are called implied volatility smile On its face, implied volatility smile suggests that options with different moneyness are priced by assuming the underlying has different volatility.

The presence of implied volatility smile (or skew) simply means that the Black-Scholes model is no longer used as a pricing model in the options market. Instead, it has become a quoting model.

> A professional trader will say "I am selling the option at 25%" instead of "I am selling the option at 4.52 dollars".

The most problematic assumption of Black-Scholes model is the normal log-return assumption, which cannot capture the non-normality of asset returns in time series. Advanced option price models usually focus on describing the dynamics of underlying asset more accurately (for example, stochastic volatilities, jumps in prices, and so on)

25.3 High-dimensional Brownian Motion

DEFINITION 25.4. *(Multi-domensional Brownian Motion) Let $W_i(t)$ be a standard 1-dimensional Brownian Motion for $i = 1, 2, \cdots, d$, and assume they are independent of each other. The vector*

$$W(t) = (W_1(t), W_2(t), \cdots, W_d(t))' \tag{25.147}$$

is called a d-dimensional standard Brownian Motion.

LEMMA 25.6. *(Quadratic Variation of high-dimensional BM) Let $W(t)$ be a d-dimensional standard Brownian Motion. When $i \neq j$, then cross variation of $W_i(t)$ and $W_j(t)$, over any interval $[0, T]$, is zero.*

Proof. Let $\Pi : 0 = t_0 < t_1 < \cdots < t_n t = T$ be a partition of $[0, T]$, then let $S_\Pi = \sum_{k=1}^{n-1} (W_i(k+1) -$

$W_i(k))(W_j(k+1) - W_j(k))$. So

$$E[S_\Pi] = E[\sum_{k=0}^{n-1}(W_i(k+1) - W_i(k))(W_j(k+1) - W_j(k))] \tag{25.148}$$
$$= E[\sum_{k=0}^{n-1}(W_i(k+1) - W_i(k))]E[\sum_{k=0}^{n-1}(W_j(k+1) - W_j(k))] \tag{25.149}$$
$$= 0 \tag{25.150}$$
$$Var[S_\Pi] = Var[\sum_{k=0}^{n-1}(W_i(k+1) - W_i(k))(W_j(k+1) - W_j(k))] \tag{25.151}$$
$$= \sum_{k=0}^{n-1} Var[(W_i(k+1) - W_i(k))(W_j(k+1) - W_j(k))] \tag{25.152}$$
$$= \sum_{k=0}^{n-1} E[(W_i(k+1) - W_i(k))^2(W_j(k+1) - W_j(k))^2] \tag{25.153}$$
$$= \sum_{k=0}^{n-1}(t_{k+1} - t_k)^2 \tag{25.154}$$
$$\leq |\Pi|T \tag{25.155}$$
$$\Rightarrow \lim_{|\Pi|\to 0} Var[S_\Pi] = \lim_{|\Pi|\to 0} |\Pi|T = 0 \tag{25.156}$$
$$\Rightarrow \lim_{|\Pi|\to 0} S_\Pi = 0, a.s. \tag{25.157}$$

We rewrite the statement as

$$dW_i(t)dW_j(t) = 0, \forall i \neq j \tag{25.158}$$

□

Now the Box algebra in high dimensions is,

$$dW_i(t)dW_i(t) = dt, \forall i \tag{25.159}$$
$$dW_i(t)dW_j(t) = 0, \forall i \neq j \tag{25.160}$$
$$dW_i(t)dt = 0, \forall i \tag{25.161}$$
$$dtdt = 0 \tag{25.162}$$

Assume we have an n-dimensional Ito's process $\mathbf{X}(t) = [X_1(t), X_2(t), \cdots, X_n(t)]'$, driven by a d-dimensional Brownian Model $\mathbf{W}(t) = [W_1(t), W_2(t), \cdots, W_d(t)]'$,

$$X(t) = X(0) + \int_0^t \Theta(s)ds + \int_0^t \Sigma_{n,d}(t)dW(s) \tag{25.163}$$

where $\Theta(s)$ is an $n \times 1$ vector and $\Sigma_{n,d}$ is an $n \times d$ matrix.

Assume we have a smooth function f, then how do we derive the Ito's formula for the stochastic process $f(t, \mathbf{X}(t))$?

step (1). write down the differential form of the Ito's process as

$$dX_1(t) = \Theta_1(t)dt + \sigma_{1,1}(t)dW_1(t) + \cdots + \sigma_{1,d}(t)dW_d(t) \tag{25.164}$$
$$dX_2(t) = \Theta_2(t)dt + \sigma_{2,1}(t)dW_1(t) + \cdots + \sigma_{2,d}(t)dW_d(t) \tag{25.165}$$
$$\cdots \tag{25.166}$$
$$dX_n(t) = \Theta_n(t)dt + \sigma_{n,1}(t)dW_1(t) + \cdots + \sigma_{n,d}(t)dW_d(t) \tag{25.167}$$
$$\tag{25.168}$$

step (2). use the taylor expansion to expand the function f,

$$df(t,\mathbf{X}(t)) = f_t(t,\mathbf{X}(t)) + \sum_{i=1}^{n} f_{X_i}(t,\mathbf{X}(t))dX_i(t) \tag{25.169}$$

$$+ \frac{1}{2}\sum_{i,j=1}^{n} f_{X_i,X_j}(t,\mathbf{X}(t))dX_i(t)dX_j(t) \tag{25.170}$$

step (3). substitute the expansion $dX_i(t)$ into the above formula and use the Box-algebra;

step (4). integrate the differential form to obtain the integral form.

Let $f(t,x,y) = xy$, we obtain the product rule by $df(t,X(t),Y(t)) = d[X(t)Y(t)]$,

$$d[X(t)Y(t)] = X(t)dY(t) + Y(t)dX(t) + dX(t)dY(t) \tag{25.171}$$

THEOREM 25.3. *(Levy's Theorem) If a stochastic process has continuous sample paths, is a martingale and accumulates unit quadratic variation per unit of time, then it must be a standard Brownian Motion.*

Levy's theorem in high dimension: if we have several stochastic processes, each of which has continuous paths, is a martingale, and accumulates unit quadratic variation per unit of time. If, moreover, the cross variation of each pair of the processes is zero, then these stochastic processes form a high dimensional standard BM.

In reality, the prices of different assets can be correlated. Consider the following two-stock model,

$$\frac{dS_1(t)}{dS_1(t)} = \alpha_1 dt + \sigma_1 dW_1(t) \tag{25.172}$$

$$\frac{dS_2(t)}{dS_2(t)} = \alpha_2 dt + \sigma_2(\rho dW_1(t) + \sqrt{1-\rho^2}dW_t(t)) \tag{25.173}$$

Let $W(t) = \rho dW_1(t) + \sqrt{1-\rho^2}dW_t(t)$, then

(1) $W(t)$ has continuous sample space;

(2) $W(t)$ is a martingale;

(3) quadratic variation, $dW(t) = \rho dW(t) + \sqrt{1-\rho^2}dW_t(t), dW_(t)dW(t) = \rho^2 dt + (1-\rho^2)dt = dt.$

Thus, $W(t)$ is standard BM (Levy's Theorem). $\rho \in [-1,1]$ is interpreted as the correlation between the return of two stocks.

Now assume we have a derivative security with a payoff of $g(S_1(T),S_2(T))$, then we want to find its price by replicating. Let $f(t,x,y)$ be its pricing function and we construct a replicating portfolio,

$$\Pi(t) = f(t,S_1(t),S_2(t)) - \Delta_1(t)S_1(t) - \Delta_2(t)S_2(t) \tag{25.174}$$

So,

$$d\Pi(t) = df(t,S_1(t),S_2(t)) - \Delta_1(t)dS_1(t) - \Delta_2(t)dS_2(t) \tag{25.175}$$

$$= f_t dt + f_x dS_1(t) + f_y dS_2(t) \tag{25.176}$$

$$+ \frac{1}{2}f_{xx}\sigma_1^2 S_1(t)^2 dt + \frac{1}{2}f_{yy}\sigma_2^2 S_2(t)^2 dt + f_{xy}\rho\sigma_1\sigma_2 S_1(t)S_2(t)dt \tag{25.177}$$

$$- \Delta_1(t)dS_1(t) - \Delta_2(t)dS_2(t) \tag{25.178}$$

Let $\Delta_1(t) = f_x, \Delta_2(t) = f_y$, then $d\Pi(t) = r\Pi dt$,

$$f_t dt + \frac{1}{2}f_{xx}\sigma_1^2 S_1(t)^2 dt + \frac{1}{2}f_{yy}\sigma_2^2 S_2(t)^2 dt + f_{xy}\rho\sigma_1\sigma_2 S_1(t)S_2(t)dt = r(f - f_x S_1(t) - f_y S_2(t)) \tag{25.179}$$

We obtain,

$$f_t + rS_1(t)f_x + rS_2(t)f_y - rf + \frac{1}{2}f_{xx}\sigma_1^2 S_1(t)^2 + \frac{1}{2}f_{yy}\sigma_2^2 S_2(t)^2 + f_{xy}\rho\sigma_1\sigma_2 S_1(t)S_2(t) = 0 \tag{25.180}$$

In general form, plus the boundary condition,

$$\begin{cases} -rf + f + rxf_x + ryf_y + \frac{1}{2}\sigma_1^2 x^2 f_{xx} + \frac{1}{2}\sigma_2^2 y^2 f_{yy} + \rho\sigma_1\sigma_2 xyf_{xy} = 0 \\ f(T,x,y) = g(x,y) \end{cases} \tag{25.181}$$

26 Risk-Neutral Pricing

26.1 Pricing on a Binomial Tree

In a dividend-discount model, a stock's fair price is the discounted value of its future dividends,

$$p = \sum_{i=1}^{\infty} \beta^i E[D_i] \tag{26.1}$$

where the discounting factor β measures the investor's risk appetite. However, a derivative security con not be priced by calculating the expected value of its future cash-flows (otherwise arbitrage opportunities emerge).

Risk-neutral principle: a derivative's price can be written as a expectation of its discounted cash-flows by assuming that we were living in a hypothetical world.

We first price ingredients of a simplest binomial tree model. Let $t = 0, 1$ and there are two tradable assets, a risk-free bond which earns constant interest rate r, and a risky asset S with price S_0 on day 0.

The risk of S is characterized by a binomial distribution of its price on day 1,

$$S_1 = \begin{cases} S_0 u, \text{ probability } p \\ S_0 d, \text{ probability } 1-p \end{cases} \tag{26.2}$$

where $u > 1 + r, d < 1 + r$ (no arbitrage condition). The market is completely frictionless.

A derivative security on the stock is represented by a payoff function $g(S_1)$, which can be replicated by a self-financing portfolio Π. Let $\Pi_0 = \alpha S_0 + \beta$, then

$$\Pi_1 = \begin{cases} \alpha S_0 u + \beta(1+r), p \\ \\ alpha S_0 u + \beta(1+r), 1-p \end{cases} \tag{26.3}$$

Let Π track the payoff of the derivative,

$$\begin{cases} \alpha S_0 u + \beta(1+r) = g(S_0 u) \\ \alpha S_0 d + \beta(1+r) = g(S_0 d) \end{cases} \tag{26.4}$$

Solve the equations, we have

$$\begin{cases} \alpha = \frac{g(S_0 u) - g(S_0 d)}{S_0(u-d)} \\ \beta = \frac{1}{1+r}[\frac{-d}{u-d} g(S_0 u) + \frac{u}{u-d} g(S_0 d)] \end{cases} \tag{26.5}$$

Then, the value of the replicating portfolio is

$$\Pi_0 = \frac{1}{1+r}[\frac{1+r-d}{u-d} g(S_0 u) + \frac{u-1-r}{u-d} g(S_0 d)] \tag{26.6}$$

Thus, the fair price of the derivative must be Π_0, i.e.,

$$p_0 = \frac{1}{1+r}[\frac{1+r-d}{u-d} g(S_0 u) + \frac{u-1-r}{u-d} g(S_0 d)] = \hat{p} g(S_0 u) + \hat{q} g(S_0 d) \tag{26.7}$$

where $\begin{cases} \hat{p} = \frac{1+r-d}{u-d} \\ \hat{q} = \frac{u-1-r}{u-d} \end{cases}$ is the risk-neutral probabilities ($\hat{p} + \hat{q} = 1, \hat{p}, \hat{q} \in (0, 1)$). Under this probability measurable, the fair price of the derivative is the discounted expected value of its payoff.

$$\begin{aligned} \hat{E}[S_1] &= \hat{p} S_1 u + \hat{q} S_1 d && (26.8) \\ &= (\frac{1+r-d}{u-d} u + \frac{u-1-r}{u-d} d) S_0 && (26.9) \\ &= \frac{u + ru - du + du - d - dr}{u-d} S_0 && (26.10) \\ &= \frac{(1+r)(u-d)}{u-d} S_0 && (26.11) \\ &= (1+r) S_0 && (26.12) \end{aligned}$$

So ”investor” requires no risk premium (risk-neutral world)!

However, on a trinomial tree, replication is impossible and there are many risk neutral probabilities.

Let S_1 follow

$$S_1 = \begin{cases} S_0 u, p \\ S_0 m, q \\ S_0 d, 1-p-q \end{cases} \tag{26.13}$$

and we construct a self-financing portfolio Π, s.t. $\Pi_0 = \alpha S_0 + \beta$, then

$$\Pi_1 = \begin{cases} \alpha S_0 u + \beta(1+r), p \\ \alpha S_0 m + \beta(1+r), q \\ \alpha S_0 d + \beta(1+r), 1-p-q \end{cases} \tag{26.14}$$

Then we need to solve,

$$\begin{cases} \alpha S_0 u + \beta(1+r) = g(S_0 u) \\ \alpha S_0 m + \beta(1+r) = g(S_0 m) \\ \alpha S_0 d + \beta(1+r) = g(S_0 d) \end{cases} \tag{26.15}$$

Two variables but three equations, there is no solution in most time. In other words, we can not replicate the derivative when the market is incomplete (three states but two assets).

However, the risk-neutral probabilities exist and not unique. Let

$$S_1 = \begin{cases} S_0 u, \hat{p} \\ S_0 m, \hat{q} \\ S_0 d, 1-\hat{p}-\hat{q} \end{cases} \tag{26.16}$$

then,

$$\hat{E}(S_1) = (1+r)S_0 \tag{26.17}$$

$$\hat{p}S_0 u + \hat{q}S_0 m + (1-\hat{p}-\hat{q})S_0 d = (1+r)S_0 \tag{26.18}$$

$$\Rightarrow \hat{p}u + \hat{q}u + (1-\hat{p}-\hat{q})d = 1+r \tag{26.19}$$

Two variables with one equation, we have infinite solutions!

26.2 Change of Measure and Girsanov Theorem

Given an probability measurable P and any non-negative r.v. Z with $E[Z] = 1$, we can define a new probability measure $\tilde{P}$ through

$$\tilde{P}(A) = \int_A Z(\omega) dP(\omega) \tag{26.20}$$

The expectation of any r.v. Y under th new measure, can be calculated by

$$\tilde{E}[Y] = E[YZ] \tag{26.21}$$

The information flow allows us to construct a Randon-Nikodym derivative process as follows

$$Z(t) = E[Z|\mathcal{F}_t] \tag{26.22}$$

Then $Z(t)$ is a martingale as

$$E[Z(s)|\mathcal{F}_t] = E[E[Z|\mathcal{F}_s]|\mathcal{F}_t] \tag{26.23}$$

$$= E[Z|\mathcal{F}_t] \tag{26.24}$$

For a $\mathcal{F}_t$-measurable r.v. Y, we have

$$\tilde{E}[Y] = E[YZ(t)] \tag{26.25}$$

as

$$E[YZ(t)] = E[YE[Z(t)|\mathcal{F}_t]] = E[E[YZ|\mathcal{F}_t]] = E[YZ] = \tilde{E}[Y] \tag{26.26}$$

Let $0 \leq s \leq t \leq T$, for a $\mathcal{F}_t$-measurable r.v. Y, we have

$$\tilde{E}[Y|\mathcal{F}_s] = \frac{1}{Z(s)} E[YZ(t)|\mathcal{F}_s] \tag{26.27}$$

We now explain this. Let $X = E[Y\frac{Z(t)}{Z(s)}|\mathcal{F}_s]$, then we want to show X is the expectation of Y with $\mathcal{F}_s$ under $\tilde{P}$, i.e., show

(1) X is $\mathcal{F}_s$-measurable;

(2) $\forall A \in \mathcal{F}_s$, we have $\int_A X d\tilde{P} = \int_A Y d\tilde{P}$.

(1) is trivial. We focus on (2). $\forall A \in \mathcal{F}_s$ and $s \leq t$ implies $A \in \mathcal{F}_t$.

$$\begin{aligned}
\int_A X d\tilde{P} &= \int_\Omega I_A X d\tilde{P} && (26.28)\\
&= \tilde{E}[I_A X] && (26.29)\\
&= E[I_A X Z(s)] && (26.30)\\
&= E[I_A E[Y\frac{Z(t)}{Z(s)}|\mathcal{F}_s]Z(s)] && (26.31)\\
&= E[I_A E[YZ(t)|\mathcal{F}_s]] && (26.32)\\
&= E[E[I_A YZ(t)|\mathcal{F}_s]] && (26.33)\\
&= E[I_A YZ(t)] && (26.34)\\
&= \int_\Omega I_A YZ(t) dP && (26.35)\\
&= \int_A Y d\tilde{P} && (26.36)\\
& && (26.37)
\end{aligned}$$

This outcome matters, we summarize as, if Y is $\mathcal{F}_t$-measurable r.v. and $0 \leq s \leq t$, the

$$\tilde{E}[Y] = E[YZ(t)] \tag{26.38}$$

$$\tilde{E}[Y|\mathcal{F}_s] = E[Y\frac{Z(t)}{Z(s)}|\mathcal{F}_s] \tag{26.39}$$

THEOREM 26.1. *(Girsanov Theorem) Let W_t be a standard Brownian Motion with a filtration $\mathcal{F}(t)$ and $\Theta(t)$ be a stochastic process adapted to $\mathcal{F}_t$. Then we make a process*

$$\tilde{W}(t) \equiv W(t) + \int_0^t \Theta(u)du \tag{26.40}$$

Define

$$Z(t) = e^{-\int_0^t \Theta(u)dW(t) - \frac{1}{2}\int_0^t \Theta(u)^2 du} \tag{26.41}$$

and set $Z = Z(T)$. Then

(1) $E[Z] = 1$;

(2) under the probability $\tilde{P}$ induced by Z, the process $\tilde{W}(t)$ is a standard Brownian Motion.

Proof. (1) We first show $Z_t = Z(t)$ is a martingale under P. Let $X_t = -\int_0^t \Theta(u)dW(t) - \frac{1}{2}\int_0^t \Theta(u)^2 du$, then $dX_t = -\Theta_t dt - \frac{1}{2}\Theta_t^2 dt$ and $Z_t = X_t$.

$$\begin{aligned}
dZ_t &= e^{X_t}dt + \frac{1}{2}e^{X_t}dX_t dX_t && (26.42)\\
&= Z_t(-\Theta_t dW_t - \frac{1}{2}\Theta_t^2 dt + \frac{1}{2}\Theta_t^2 dt) && (26.43)\\
&= -\Theta_t Z_t dW_t && (26.44)\\
Z_0 &= e^{X_0} = 1 && (26.45)\\
\Rightarrow Z_t &= Z_0 + \int_0^t dZ_t = 1 - \int_0^t \Theta(s)Z_s ds && (26.46)
\end{aligned}$$

which implies Z_t is an Ito's integral, so a martingale under P. Then $E[Z_T] = E[Z_0] = 1$.

(2) We use Levy's Theorem to show $\tilde{W}$ is a BM under $\tilde{P}$.

a. $d\tilde{W}_t = dW_t + \Theta_t dt$, then

$$d\tilde{W}_t d\tilde{W}_t = (dW_t)^2 + (\Theta_t dt)^2 + 2dW_t\Theta_t dt = dt \tag{26.47}$$

which indicates the quadratic variation of $\tilde{W}_s$ is one unit per unit of time.

b. $\tilde{W}_t$ is a MG under $\tilde{P}$.

a) Z_t is a MG under P;

b) $\tilde{W}_t Z_t$ is a MG under P.

$$\begin{aligned} d\tilde{W}_t Z_t &= Z_t d\tilde{W}_t + \tilde{W}_t dt + d\tilde{W}_t dZ_t && (26.48)\\ &= Z_t(dW_t + \Theta_t dt) + \tilde{W}_t(-\Theta_t Z_t dW_t) - \Theta_t Z_t dt && (26.49)\\ &= Z_t(1 - \Theta_t \tilde{W}_t)dW_t && (26.50)\\ \tilde{W}_t Z_t &= \tilde{W}_0 Z_0 + \int_0^t d\tilde{W}_t Z_t && (26.51)\\ &= \tilde{W}_0 Z_0 + \int_0^t Z_t(1 - \Theta_t \tilde{W}_t)dW_t && (26.52)\\ & && (26.53) \end{aligned}$$

so $\tilde{W}_t Z_t$ is an Ito's integral, i.e., a MG under P.

c) $\forall 0 \le s \le t, \tilde{E}[\tilde{W}_t|\mathcal{F}_s] = \tilde{W}_s$.

$$\begin{aligned} \tilde{E}[\tilde{W}_t|\mathcal{F}_s] &= E[\tilde{W}_t \frac{Z_t}{Z_s}|\mathcal{F}_s] && (26.54)\\ &= \frac{1}{Z_s}E[\tilde{W}_t Z_t|\mathcal{F}_s] && (26.55)\\ &= \frac{1}{Z_s}\tilde{W}_s Z_s && (26.56)\\ &= \tilde{W}_s && (26.57) \end{aligned}$$

Thus, $\tilde{W}_s$ is a MG under $\tilde{P}$.

c. $\tilde{W}_s$ has continuous path (no matter with probability measures).

□

26.3 Risk-Neutral Measure in Continuous-Time

Underlying asset price dynamics under the true measure P is

$$dS_t = \alpha_t S_t dt + \sigma_t S_t dW_t \tag{26.58}$$

where $\sigma_t > 0, \forall t$ (Generalized GBM). Interest rate is assumed to be an adapted stochastic process R_t, a money market account,

$$M_t = e^{\int_0^t R_s ds} \tag{26.59}$$

and the discounted process

$$D_t = \frac{1}{M_t} = e^{-\int_0^t R_s ds} \tag{26.60}$$

Then, changes in the discount process,

$$dD_t = -R_t D_t dt \tag{26.61}$$

D_t is smooth, and has a derivative.

The dynamics of the discounted stock price,

$$
\begin{aligned}
d[D_tS_t] &= D_tdS_t + S_tdD_t + dS_tdD_t &(26.62)\\
&= D_t(\alpha_tS_tdt + \sigma_tS_tdW_t) - S_tR_tD_tdt + 0 &(26.63)\\
&= D_tS_t[(\alpha_t - R_t)dt + \sigma_tdW_t] &(26.64)
\end{aligned}
$$

Let $\tilde{W}_t = W_t + \int_0^t \Theta_u du, \Theta_u = \frac{\alpha_u - R_u}{\sigma_u}$ (the market price of risk), then

$$
d[D_tS_t] = \sigma_tD_tS_td\tilde{W}_t \tag{26.65}
$$

We want to find a new measure $\tilde{P}$ s.t. $\tilde{W}_t$ is a SBM, then D_tS_t is a MG (D_tM_t always equal to 1, then a MG).

With Girsanov Theorem, we have, let

$$
Z = Z(t) = e^{-\int_0^t \Theta(u)dW(u) - \frac{1}{2}\int_0^t \Theta(u)^2du} \tag{26.66}
$$

and define

$$
\frac{d\tilde{P}}{dP} = Z \tag{26.67}
$$

This new measure $\tilde{P}$ is called the risk-neutral measure. Under $\tilde{P}$, D_tS_t, D_tM_t are MGs, then any self-financing portfolio is a MG, e.g., let

$$
X_t = \Delta_tS_t + X_t - \Delta_tX_t \tag{26.68}
$$

then,

$$
\begin{aligned}
dX_t &= \Delta_tdS_t + R_t(X_t - \Delta_tS_t)dt &(26.69)\\
&= \Delta_t(\alpha_tS_td_t + \sigma_tS_tdW_t) + R_t(X_t - \Delta_tS_t)dt &(26.70)\\
&= R_tX_tdt + \Delta_tS_t((\alpha_t - R_t)dt + \sigma_tdW_t) &(26.71)
\end{aligned}
$$

Then,

$$
\begin{aligned}
dD_tX_t &= X_tdD_t + D_tdX_t + dD_tdX_t &(26.72)\\
&= -R_tX_tD_tdt + D_t(R_tX_tdt + \Delta_tS_t((\alpha_t - R_t)dt + \sigma_tdW_t)) &(26.73)\\
&= D_t\Delta_tS_t((\alpha_t - R_t)dt + \sigma_tdW_t) &(26.74)\\
&= D_t\Delta_tS_t\sigma_t(\Theta_tdt + dW_t) &(26.75)\\
&= D_t\Delta_tS_t\sigma_td\tilde{W}_t &(26.76)
\end{aligned}
$$

$$
\Rightarrow D_tX_t = D_0X_0 + \int_0^t D_t\Delta_tS_t\sigma_td\tilde{W}_t \tag{26.77}
$$

Thus, D_tX_t is a MG under $\tilde{P}$.

Now consider a derivative, mature at T and payoff is $V(T)$. Assume there is a self-financing portfolio X_t s.t. $X(T) = V(T)$ a.s.. As D_tX_t is a MG under $\tilde{P}$, then

$$
\begin{aligned}
D_tX_t &= \tilde{E}[D_TX_T|\mathcal{F}_t] &(26.78)\\
&= \tilde{E}[D_TV_T|\mathcal{F}_t] &(26.79)\\
\Rightarrow X_t &= \tilde{E}[\frac{D_T}{D_t}V_T|\mathcal{F}_t] &(26.80)\\
&= \tilde{E}[e^{-\int_t^T R_sds}V_T|\mathcal{F}_t] &(26.81)
\end{aligned}
$$

Then, the risk-neutral pricing formula is

$$
V_t = \tilde{E}[e^{-\int_t^T R_sds}V_T|\mathcal{F}_t] \tag{26.82}
$$

Given an initial capital $X_0 = V_0$, how can we replicate the derivative ? X_t is uniquely determined by Δ_t, then we just need to find Δ_t, s.t.

$$
d[D_tX_t] = d[D_tV_t] \tag{26.83}
$$

We know $d[D_tX_t] = D_t\Delta_tS_t\sigma_t d\tilde{W}_t$. But

$$D_tV_t = e^{-\int_0^t R_sds}\tilde{E}[e^{-\int_t^T R_sds}V_T|\mathcal{F}_t] \tag{26.84}$$
$$= \tilde{E}[e^{-\int_0^T R_sds}V_T|\mathcal{F}_t] \tag{26.85}$$
$$= \tilde{E}[D_TV_T|\mathcal{F}_t] \tag{26.86}$$

Thus, D_tV_t is a martingale.

THEOREM 26.2. *(Martingale Representation Theorem) Let W_t be a BM and let $\mathcal{F}_t$ be the filtration generated by W_t. Let M_t be an MG w.r.t. the filtration. Then there is an adapted process Γ_u s.t.*

$$M_t = M_0 + \int_0^t \Gamma_u dW_u \tag{26.87}$$

Under $\mathcal{F}_t$, any martingale is an Ito's integral.

The above MRT can be extended to the case with a changed measure. Then we say there is an adapted process γ_t s.t.

$$D_tV_t = D_0V_0 + \int_0^t \sigma_s\Gamma_s d\tilde{W}_s \tag{26.88}$$

i.e., $d[D_tV_t] = \sigma_t\Gamma_t d\tilde{W}_t$. Then we need

$$D_t\Delta_tS_t\sigma_t = \sigma_t\Gamma_t \tag{26.89}$$

so

$$\Delta_t = \frac{\Gamma_t}{D_tS_t}, \forall t \tag{26.90}$$

When pricing financial derivatives, it is more often that we directly build the model under the isk-neutral measure $\tilde{P}$, rather than starting from the physical measure P and seeking for a change of measure.

$$dW_t = d\tilde{W}_t - \Theta_t dt \tag{26.91}$$
$$dS_t = \alpha_tS_tdt + \sigma_tS_tdW_t \tag{26.92}$$
$$= \alpha_tS_tdt + \sigma_tS_t(d\tilde{W}_t - \theta_tdt) \tag{26.93}$$
$$= \alpha_tS_tdt + \sigma_tS_td\tilde{W}_t - (\alpha_t - R_t)S_tdt \tag{26.94}$$
$$= R_tS_tdt + \sigma_tS_td\tilde{W}_t \tag{26.95}$$

where R_t is the risk free rate and $\tilde{W}_t$ is a BM in the risk-neutral world.

The value of derivatives can be obtained by Monte-carlo simulation,

$$V_t = \tilde{E}[e^{-\int_0^t R_sds}V_T|\mathcal{F}_t] \tag{26.96}$$

Example 26.1. *Induce BSM by the risk-neutral approach. Let $dS_t = \alpha S_tdt + \sigma S_tdW_t$ and risk-free rate r (constant) and the derivative's payoff $V_T = (S_T - K)^+$.*

Proof. Let $\tilde{P}$ denote the risk-neutral probability measure. Then,

$$V_t = \tilde{E}[e^{-r(T-t)}(S_T - K)^+|\mathcal{F}_t] \tag{26.97}$$
$$dS_t = rS_tdt + \sigma S_td\tilde{W}_t \tag{26.98}$$
$$\Rightarrow S_T = S_te^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(\tilde{W}_T-\tilde{W}_t)} \tag{26.99}$$

Let $\tilde{W}_T - \tilde{W}_t = \sqrt{T-t}y, y \sim N(0,1), \tau = T-t$, then

$$S_T = S_t e^{(r-\frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}y} \tag{26.100}$$

$$V_t = e^{-r\tau}\tilde{E}[(S_T-K)^+|\mathcal{F}_t] \tag{26.101}$$

$$= e^{-r\tau}\tilde{E}[(S_T-K)I_{S_T>K}|\mathcal{F}_t] \tag{26.102}$$

$$= e^{-r\tau}\tilde{E}[S_T I_{S_T>K} - K I_{S_T>K}|\mathcal{F}_t] \tag{26.103}$$

$$= e^{-r\tau}I_1 - Ke^{-r\tau}I_2 \tag{26.104}$$

$$I_2 = \tilde{E}[I_{S_T>K}|\mathcal{F}_t] \tag{26.105}$$

$$= \tilde{E}[I_{y>\frac{\ln\frac{K}{S_t}-(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}}] \tag{26.106}$$

$$= \tilde{P}[y > \frac{\ln\frac{K}{S_t}-(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}] \tag{26.107}$$

$$= \tilde{P}[y < \frac{\ln\frac{S_t}{K}+(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}] \tag{26.108}$$

$$= N(d_-) \tag{26.109}$$

$$I_1 = \tilde{E}[S_T I_{S_T>K}|\mathcal{F}_t] \tag{26.110}$$

$$= \tilde{E}[S_t e^{(r-\frac{1}{2}\sigma^2)\tau+\sigma\sqrt{\tau}y} I_{y>\frac{\ln\frac{K}{S_t}-(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}}|\mathcal{F}_t] \tag{26.111}$$

$$= S_t\tilde{E}[e^{(r-\frac{1}{2}\sigma^2)\tau+\sigma\sqrt{\tau}y} I_{y>\frac{\ln\frac{K}{S_t}-(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}}] \tag{26.112}$$

$$= S_t\int_{-\infty}^{\infty} e^{(r-\frac{1}{2}\sigma^2)\tau+\sigma\sqrt{\tau}y} I_{y>\frac{\ln\frac{K}{S_t}-(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}} \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy \tag{26.113}$$

$$= S_t\int_{\frac{\ln\frac{K}{S_t}-(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}}^{\infty} e^{(r-\frac{1}{2}\sigma^2)\tau+\sigma\sqrt{\tau}y} \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy \tag{26.114}$$

$$= S_t e^{rt}\int_{\frac{\ln\frac{K}{S_t}-(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}}^{\infty} e^{-\frac{1}{2}(y-\sigma\sqrt{\tau})^2}dy \tag{26.115}$$

$$u = y-\sigma\sqrt{\tau} \Rightarrow I_1 = S_t e^{rt}\int_{\frac{\ln\frac{K}{S_t}-(r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}du \tag{26.116}$$

$$= S_t e^{rt}\int_{-\infty}^{\frac{\ln\frac{S_t}{K}+(r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}du \tag{26.117}$$

$$= e^{rt}S_t N(d_+) \tag{26.118}$$

$$\Rightarrow V_t = S_t N(d_+) - Ke^{-rt}N(d_-) \tag{26.119}$$

□

Now we extend the Girsanov Theorem and Martingale Representation Theorem to high dimensions.

Let $\mathbf{W}_t = [W_1(t), W_2(t), \cdots, W_d(t)]'$ be a d-dimensional BM, with filtration $\mathcal{F}_t$. Consider a drifted BM,

$$\tilde{\mathbf{W}}_t = \mathbf{W}_t + \int_0^t \boldsymbol{\Theta}_u du \tag{26.120}$$

where $\boldsymbol{\Theta}_{\mathbf{u}} = [\Theta_1(t), \Theta_2(t), \cdots, \Theta_d(t)]'$. Then, let

$$\frac{d\tilde{P}}{dP} = Z = e^{-\int_0^t \boldsymbol{\Theta}_u \cdot d\mathbf{W}_u - \frac{1}{2}\int_0^t ||\boldsymbol{\Theta}_{\mathbf{u}}||^2 du} \tag{26.121}$$

where $\boldsymbol{\Theta}_u \cdot d\mathbf{W}_u = \sum_{i=1}^d \Theta_{i,u}dW_{i,u}$, $||\boldsymbol{\Theta}_{\mathbf{u}}||^2 = \sum_{i=1}^d \theta_{i,u}^2$. Then, under $\tilde{P}$, $\tilde{\mathbf{W}}$ is a standard Brownian Motion.

Let $\mathbf{W}_t$ be a d-dimensional BM and $\mathcal{F}_t$ be its associated filtration. Let $\mathbf{M_t}$ be a MG associated with filtration $\mathcal{F}_t$. Then, there exists a d-dimensional adapted process $\boldsymbol{\Gamma}_t$ s.t.

$$\mathbf{M}_t = \mathbf{M}_0 + \int_0^t \boldsymbol{\Gamma}_u \cdot d\mathbf{W}_u = \mathbf{M}_0 + \sum_{i=1}^d \int_0^t \Gamma_{i,u}dW_{i,u} \tag{26.122}$$

26.4 Fundamental Theorems of Asset Pricing

A model with multiple assets and multiple sources of risk. Let $\mathbf{W}_t = [W_{1,t}, W_{2,t}, \cdots, W_{d,t}]'$ be a BM, sources of risk, associated filtration $\mathcal{F}_t$. Tradable assets include risk-free bond with rate R_t, and m risk assets $S_{i,t}, i = 1, 2, \cdots, m$. Dynamics of risk assets are

$$dS_{i,t} = \alpha_1(t)S_{i,t} + S_{i,t}\sum_{i=1}^{d}\sigma_{ij,t}dW_{j,t}, i = 1, 2, \cdots, m \tag{26.123}$$

The total volatility of stock i,

$$\sigma_{i,t} = \sqrt{\sum_{j=1}^{d}\sigma_{ij,t}^2} \tag{26.124}$$

$$\frac{dS_{i,t}}{S_{i,t}} = \alpha_{i,t}dt + \sum_{i=1}^{d}\sigma_{ij,t}dW_{j,t}, i = 1, 2, \cdots, m \tag{26.125}$$

The stock returns are correlated. Now discount these price processes.

$$\begin{aligned}
M_t &= e^{\int_0^t R_s ds} && (26.126)\\
D_t M_t &= 1 && (26.127)\\
D_t &= \frac{1}{M_t} = e^{-\int_0^t R_s ds} && (26.128)\\
\Rightarrow dD_t &= -R_t D_t dt && (26.129)\\
d[D_t S_{i,t}] &= D_t dS_{i,t} + S_{i,t}dD_t + dS_{i,t}dD_t && (26.130)\\
&= D_t(\alpha_{1,t}S_{i,t} + S_{i,t}\sum_{i=1}^{d}\sigma_{ij,t}dW_{j,t}) + S_{i,t}(-R_t D_t dt) && (26.131)\\
&= D_t S_{i,t}[(\alpha_{i,t} - R_t)dt + \sum_{j=1}^{d}\sigma_{ij,t}dW_{j,t}] && (26.132)
\end{aligned}$$

DEFINITION 26.1. *(Risk-Neutral Measure) A probability measure $\tilde{P}$ is a risk-neutral measure if two conditions are met,*

(1) $\tilde{P}$ and P are equivalent;

(2) under $\tilde{P}$, we have $D_t S_{i,t}$ is an MG, for $i = 1, 2, \cdots, m$.

That is, we need to find a $\tilde{P}$ s.t. $(\alpha_{i,t} - R_t)dt + \sum_{j=1}^{d}\sigma_{ij,t}dW_{j,t}$ be a Brownian Motion for $\forall i$.

$$(\alpha_{i,t} - R_t)dt + \sum_{j=1}^{d}\sigma_{ij,t}dW_{j,t} = \sum_{i=1}^{d}\sigma_{ij,t}(dW_{j,t} + \Theta_{j,t}dt) \tag{26.133}$$

$$\alpha_{i,t} - R_t = \sum_{j=1}^{d}\sigma_{ij,t}\Theta_{j,t}, i = 1, 2, \cdots, m \tag{26.134}$$

$\Theta_{i,t}$ denotes the risk premium of $W_{i,t}, i = 1, 2, \cdots, d$. So, we get the equations for market price of risk and need to find $\Theta^*_{j,t}, j = 1, 2, \cdots, d$ s.t.,

$$\begin{cases}
\alpha_{1,t} - R_t & = \sum_{j=1}^{d}\sigma_{1j,t}\Theta_{j,t}\\
\alpha_{2,t} - R_t & = \sum_{j=1}^{d}\sigma_{2j,t}\Theta_{j,t}\\
\cdots & \\
\alpha_{m,t} - R_t & = \sum_{j=1}^{d}\sigma_{mj,t}\Theta_{j,t}
\end{cases} \tag{26.135}$$

Rewrite,

$$
\begin{aligned}
B &= \begin{bmatrix} \sigma_{11,t} & sigma_{12,t} & \cdots & \sigma_{1d,t} \\ \sigma_{21,t} & sigma_{22,t} & \cdots & \sigma_{2d,t} \\ & & \cdots & \\ \sigma_{m1,t} & sigma_{m2,t} & \cdots & \sigma_{md,t} \end{bmatrix} && (26.136)\\
\Theta &= [\Theta_{1,t}, \Theta_{2,t}, \cdots, \Theta_{d,t}]' && (26.137)\\
C &= [\alpha_{1,t} - R_t, \alpha_{2,t} - R_t, \cdots, \alpha_{m,t} - R_t]' && (26.138)\\
\Rightarrow B\Theta &= C && (26.139)
\end{aligned}
$$

d variables and m equations. If equations 26.135 has solution, then, let

$$\frac{d\tilde{P}}{dP} = Z = e^{-\int_0^t \boxtimes_u \cdot d\mathbf{W}_u - \frac{1}{2}\int_0^t ||\boxtimes_\mathbf{u}||^2 du} \tag{26.140}$$

(Note that Θ decides a uniquely Z, but a Z may be induced by different Θ!) Thus, under $\tilde{P}$, $D_t S_{i,t}$ is a MG for $\forall i = 1, 2, \cdots, m$, i.e.,

$$d[D_t S_{i,t}] = D_t S_{i,t} \sum_{i=1}^{d} \sigma_{ij,t} d\tilde{W}_{j,t} \tag{26.141}$$

Further, under $\tilde{P}$, any self-financing portfolio X_t is a MG. Let

$$X_t = \sum_{i=1}^{m} \Delta_{i,t} S_{i,t} + X_t - \sum_{i=1}^{m} \Delta_{i,t} S_{i,t} \tag{26.142}$$

Then

$$
\begin{aligned}
dX_t &= \sum_{i=1}^{m} \Delta_{i,t} dS_{i,t} + R_t(X_t - \sum_{i=1}^{m} \Delta_{i,t} S_{i,t}) && (26.143)\\
&= \sum_{i=1}^{m} \Delta_{i,t}(\alpha_1(t) S_{i,t} + S_{i,t} \sum_{i=1}^{d} \sigma_{ij,t} dW_{j,t}) + R_t(X_t - \sum_{i=1}^{m} \Delta_{i,t} S_{i,t}) && (26.144)\\
&= R_t X_t dt + \sum_{i=1}^{m} \Delta_{i,t} S_{i,t}[(\alpha_{i,t} - R_t)dt + \sum_{j=1}^{d} \sigma_{ij,t} dW_{j,t}] && (26.145)\\
d[D_t X_t] &= X_t dD_t + D_t dX_t && (26.146)\\
&= X_t(-R_t D_t dt) + D_t[R_t X_t dt + \sum_{i=1}^{m} \Delta_{i,t} S_{i,t}[(\alpha_{i,t} - R_t)dt + \sum_{j=1}^{d} \sigma_{ij,t} dW_{j,t}]] && (26.147)\\
&= D_t \sum_{i=1}^{m} \Delta_{i,t} S_{i,t}[(\alpha_{i,t} - R_t)dt + \sum_{j=1}^{d} \sigma_{ij,t} dW_{j,t}] && (26.148)\\
&= \sum_{i=1}^{m} \Delta_{i,t} D_t S_{i,t}[(\alpha_{i,t} - R_t)dt + \sum_{j=1}^{d} \sigma_{ij,t} dW_{j,t}] && (26.149)\\
&= \sum_{i=1}^{m} \Delta_{i,t} D_t S_{i,t}[\sum_{j=1}^{d} \sigma_{ij,t}(dW_{j,t} + \Theta_{j,t} dt)] && (26.150)\\
&= \sum_{i=1}^{m} \Delta_{i,t} D_t S_{i,t}[\sum_{j=1}^{d} \sigma_{ij,t} d\tilde{W}_{j,t}] && (26.151)
\end{aligned}
$$

So,

$$D_t X_t = D_0 X_0 + \int_0^t \sum_{i=1}^{m} \Delta_{i,t} D_t S_{i,t}[\sum_{j=1}^{d} \sigma_{ij,t} d\tilde{W}_{j,t}] \tag{26.152}$$

i.e., $D_t X_t$ is a MG under $\tilde{P}$.

DEFINITION 26.2. *(Arbitrage) An arbitrage is a self-financing portfolio processes X_t with $X_0 = 0$, and for some $T > 0$, we have*

- $P(X_T \geq 0) = 1$ *(no loss almost surely);*
- $P(X_T > 0) > 0$ *(have a chance to profit).*

Note that these conditions are imposed under the true probability measure P.

An arbitrage exists if there is a way to surely beat the money market account.

THEOREM 26.3. *(Fundamental Theorems of Asset Pricing 1) If a market model has a risk-neutral probability measure, then it does not permit arbitrage opportunities.*

Proof. Suppose not, let $\tilde{P}$ be a risk-neutral probability measure and X_t s.t. $P(X_T \geq 0) = 1, X_0 = 0$. As P and $\tilde{P}$ are equivalent, then $\tilde{P}(X_T \geq 0) = 1$, i.e., $\tilde{P}(D_T X_T \geq 0) = 1$ and $\tilde{E}[D_T X_T] = D_0 X_0 = 0$, so

$$\tilde{P}(D_T X_T > 0) = 0 \tag{26.153}$$
$$\Rightarrow \tilde{P}(X_T > 0) = 0 \tag{26.154}$$
$$\Rightarrow P(X_T > 0) = 0 \tag{26.155}$$

contradicting to $P(X_T > 0) > 0$. □

Example 26.2. *Let W_t be a Brownian Motion under P, r be the risk free rate and $S_{1,t}, S_{2,t}$ s.t.*

$$dS_{i,t} = \alpha_i S_{i,t} dt + \sigma_i S_{i,t} dW_{i,t}, i = 1, 2 \tag{26.156}$$

Then, the model permits no arbitrage if and only if $\frac{\alpha_1 - r}{\sigma_1} = \frac{\alpha_2 - r}{\sigma_2}$.

Proof. To obtain the risk-neutral probability measure, we need to solve the equations for market prices of risk,

$$\begin{cases} \alpha_1 - r = \sigma_1 \theta \\ \alpha_2 - r = \sigma_2 \theta \end{cases} \tag{26.157}$$

Then when $\frac{\alpha_1 - r}{\sigma_1} = \frac{\alpha_2 - r}{\sigma_2}$, we have $\theta^* = \frac{\alpha_1 - r}{\sigma_1} = \frac{\alpha_2 - r}{\sigma_2}$, so there is a risk-neutral probability measure $\tilde{P}$. Otherwise, θ^* does exist, so does $\tilde{P}$.

Now we explain the Fundamental Theorem of Asset Pricing 1. Let $\frac{\alpha_1 - r}{\sigma_1} > \frac{\alpha_2 - r}{\sigma_2}$, and we show the arbitrage. Let $\Pi_t = S_{1,t} - \Delta_t S_{2,t}$, then,

$$\begin{aligned} d\Pi_t &= dS_{1,t} - \Delta_t S_{2,t} &(26.158)\\ &= \alpha_1 S_{1,t} st + \sigma_1 S_{1,t} dW_t - \Delta(\alpha_2 S_{2,t} dt + \sigma_2 S_{2,t} dW_t) &(26.159)\\ &= (\alpha_1 S_{1,t} - \Delta_t \alpha_2 S_{2,t}) dt + (\sigma_1 S_{1,t} - \Delta_t \sigma_2 S_{2,t}) dW_t &(26.160) \end{aligned}$$

Let Π be risk-free, i.e.,

$$\begin{aligned} \sigma_1 S_{1,t} - \Delta_t \sigma_2 S_{2,t} = 0 &\qquad (26.161)\\ \Delta_t = \frac{\sigma_1 S_{1,t}}{\sigma_2 S_{2,t}} &\qquad (26.162)\\ \Rightarrow d\Pi_t = (\alpha_1 S_{1,t} - \Delta_t \alpha_2 S_{2,t}) dt &\qquad (26.163)\\ = \alpha_1 S_{1,t} d_t - \frac{\sigma_1 S_{1,t}}{\sigma_2 S_{2,t}} \alpha_2 S_{2,t} dt &\qquad (26.164)\\ = (\alpha_1 - \frac{\sigma_1}{\sigma_2} \alpha_2) S_{1,t} dt &\qquad (26.165) \end{aligned}$$

$\frac{\alpha_1 - r}{\sigma_1} > \frac{\alpha_2 - r}{\sigma_2}$ implies $\alpha_1 - \frac{\sigma_1}{\sigma_2}\alpha_2 > (1 - \frac{\sigma_1}{\sigma_2}) r$. And $\Pi_t = S_{1,t} - \Delta_t S_{2,t} = S_{1,t} - \frac{\sigma_1 S_{1,t}}{\sigma_2 S_{2,t}} S_{2,t} = (1 - \frac{\sigma_1}{\sigma_2}) S_{1,t}$, thus,

$$\begin{aligned} d\Pi_t &= (\alpha_1 - \frac{\sigma_1}{\sigma_2} \alpha_2) S_{1,t} dt &(26.166)\\ &\geq (1 - \frac{\sigma_1}{\sigma_2}) r S_{1,t} dt &(26.167)\\ &= r\Pi dt &(26.168) \end{aligned}$$

That implies we construct a risk-free portfolio which earns return higher than the risk-free account! Let $Y_t = X_t - X_t$ ($-X_t$ denotes risk-free account), then $Y_0 = 0$ and $dY_t = dX_t - rX_t dt > 0$, i.e., $P(Y_T \geq 0) = 1, P(Y_T > 0) > 0$, an arbitrage.

This example sheds light on the Black's original idea to induce BSM. □

DEFINITION 26.3. *(Completeness) A market model is complete if every $\mathcal{F}_T$-measurable payoff V_T can be replicated (hedged). That is, $\exists X_t$ (self-financing portfolio) s.t. $X_T = V_T, \forall t, a.s.$.*

THEOREM 26.4. *Assume a market model ha at least one risk-neutral probability measure, then the model is complete if and only if the risk-neutral probability measure if unique.*

Proof. Let $\tilde{P}$ denote the risk-neutral probability measure. Given a derivative definedbt a $\mathcal{F}_T$-measurable payoff V_T, then we want to find a self-financing portfolio X_t s.t. $X_T = V_T$. Let $S_{i,t}, i = 1, 2, \cdots, m$ be m risky assets and $D_t = \frac{1}{M_t}$ be the discounting process.

(1) Let $X_t = \sum_{i=1}^m \Delta_{i,t} S_{i,t} + X_t - \sum_{i=1}^m \Delta_{i,t} S_{i,t}$, then,

$$dX_t D_t = \sum_{j=1}^d \sum_{i=1}^m \Delta_{i,t} D_t S_{i,t} \sigma_{ij,t} d\tilde{W}_{j,t} \tag{26.169}$$

The fair price of the derivative is,

$$V_t = \tilde{E}[\frac{D_T}{D_t} V_T | \mathcal{F}_t] \tag{26.170}$$

then $D_t V_t = \tilde{E}[D_T V_T | \mathcal{F}_t]$ is a MG under $\tilde{P}$. The MRT tells, $\exists \Gamma_{j,t}, j = 1, 2, \cdots, d$ s.t.

$$D_t V_t = D_0 V_0 + \int_0^t \sum_{j=1}^d \Gamma_{j,t} d\tilde{W}_{j,t} \tag{26.171}$$

To ensure $X_T = V_T$ ($D_T X_T = D_T V_T$), we need $X_0 = V_0$ and $dD_t X_t = dD_t V_t$, thus,

$$\sum_{j=1}^d \sum_{i=1}^m \Delta_{i,t} D_t S_{i,t} \sigma_{ij,t} d\tilde{W}_{j,t} = \sum_{j=1}^d \Gamma_{j,t} d\tilde{W}_{j,t} \tag{26.172}$$

that is,

$$\sum_{i=1}^m \Delta_{i,t} D_t S_{i,t} \sigma_{ij,t} d\tilde{W}_{j,t} = \Gamma_{j,t} d\tilde{W}_{j,t}, j = 1, 2, \cdots, d \tag{26.173}$$

We want to solve $\Delta_{i,t}, i = 1, 2, \cdots, m$. Let $y_{i,t} = \Delta_{i,t} S_{i,t}$, then we need to solve,

$$\begin{cases} y_{1,t}\sigma_{11,t} + y_{2,t}\sigma_{21,t} + \cdots + y_{m,t}\sigma_{m1,t} = \frac{\Gamma_{1,t}}{\Delta_t} \\ y_{1,t}\sigma_{12,t} + y_{2,t}\sigma_{22,t} + \cdots + y_{m,t}\sigma_{m2,t} = \frac{\Gamma_{2,t}}{\Delta_t} \\ \cdots \\ y_{1,t}\sigma_{1d,t} + y_{2,t}\sigma_{2d,t} + \cdots + y_{m,t}\sigma_{md,t} = \frac{\Gamma_{d,t}}{\Delta_t} \end{cases} \tag{26.174}$$

Let

$$A_{d\times m} = \begin{bmatrix} \sigma_{11,t} & \sigma_{21,t} & \cdots & \sigma_{m1,t} \\ \sigma_{12,t} & \sigma_{22,t} & \cdots & \sigma_{m2,t} \\ & & \cdots & \\ \sigma_{1d,t} & \sigma_{2d,t} & \cdots & \sigma_{md,t} \end{bmatrix} \tag{26.175}$$

$$y = [y_{1,t}, y_{2,t}, \cdots, y_{m,t}] \tag{26.176}$$

$$D = [\frac{\Gamma_{1,t}}{\Delta_t}, \frac{\Gamma_{2,t}}{\Delta_t}, \cdots, \frac{\Gamma_{d,t}}{\Delta_t}]' \tag{26.177}$$

$$\Rightarrow Ay = D \tag{26.178}$$

Note that V_T is arbitrary, then $\Gamma_{j,t}, j = 1, 2, \cdots, d$ is arbitrary and so is D.

Refer to $A'\Theta = C$ (the Equations 26.135,Θ deciding $\tilde{P}$ uniquely). Under the condition of $A'\Theta = C$ has a solution, we have if Θ^* is unique, then for any $D \in \mathbb{R}^d, Ay = D$ has solution (proof provided latter) (We can not make it conversely as a $\tilde{P}$, or Z directly, can be induced by different Θ, i.e., the uniqueness of $\tilde{P}$ does ensure an unique Θ^*). That is if the risk-neutral probability measure is unique, then market is complete.

(2) Now we show if the market is complete, then the risk-neutral probability measure is unique is unique.

Let $\tilde{P}_1, \tilde{P}_2$ be two risk-neutral probability measure, then we show $\forall A \in \mathcal{F}_T, \tilde{P}_1(A) = \tilde{P}_2(A)$. Let $V_T = \frac{1}{D_T} I_A$, then it can be replicated by X_t, i.e., $X_T = V_T$. Under both $\tilde{P}_1, \tilde{P}_2$, $D_t X_t$ is a martingale, so

$$D_0 X_0 = \tilde{E}_1[D_T X_T] = \tilde{E}_1[D_T V_T] = \tilde{E}_1[I_A] = \tilde{P}_1(A) \tag{26.179}$$

Then $X_0 = \tilde{P}_1(A)$ and analogously, $X_0 = \tilde{P}_1(A)$. Thus, $\forall A \in \mathcal{F}_T, \tilde{P}_1(A) = \tilde{P}_2(A)$, so $\tilde{P}_1 = \tilde{P}_2$. ☐

LEMMA 26.1. *Let $B \in \mathbb{R}^{m \times d}, C \in \mathbb{R}^m$, and $Bx = C$ has a solution. We have $Bx = C$ has q unique solution then $\forall D \in \mathbb{R}^d$, $B'y = D$ has solutions.*

Proof. $Bx = C$ has a solution, denoted x^*, then $R(B) = R([B, C]) \leq d$.

x^* is unique $\Rightarrow y^*$ exists. The unique x^* implies $R(B) = d$, so $d \leq m$ as $R(B) \leq m$. Then, $R([B', D]) \leq \min\{d, m+1\} = d$ but $R(B') = R(B) = d$, so $R([B', D]) = R(B') = d \leq m$, i.e., y^* exists. ☐

Example 26.3. *Heston Stochastic Volatility model. Let $W_{1,t}, W_{2,t}$ be two independent Brownian Motions and r be the risk free rate.*

$$dS_t = \alpha S_t + \sqrt{V_t} S_t dW_{1,t} \tag{26.180}$$

$$dV_t = \kappa(\theta - V_t)dt + \nu\sqrt{V_t}(\rho dW_{1,t} + \sqrt{1-\rho^2} dW_{2,t}) \tag{26.181}$$

(1) Assume S_t is tradable but V_t is not tradable. Let $\tilde{P}$ be a risk-neutral probability measure. Then $D_t S_t, D_t M_t$ are MGs, i.e.,

$$d[D_t S_t] = D_t S_t[(\alpha - r)dt + \sqrt{V_t} dW_{1,t}] \tag{26.182}$$

Let

$$\tilde{W}_{1,t} = W_{1,t} + \int_0^t \frac{\alpha - r}{\sqrt{V_t}} dW_{1,t} \tag{26.183}$$

So,

$$d[D_t S_t] = D_t S_t \sqrt{V t} d\tilde{W}_{1,t} \tag{26.184}$$

We just need $\tilde{W}_{1,t}$ to be a MG. Then, let

$$\tilde{W}_{1,t} = W_{1,t} + \int_0^t \frac{\alpha - r}{\sqrt{V_t}} dW_{1,t} \tag{26.185}$$

$$\tilde{W}_{2,t} = W_{2,t} + \int_0^t \Theta_{2,t} dW_{2,t}, \forall \Theta_{2,t} \tag{26.186}$$

From the Girsanov Theorem, we say

$$\frac{d\tilde{P}}{dP} = Z = e^{\int_0^t \frac{\alpha - r}{\sqrt{V_s}} dW_{1,s} + \Theta_{2,s} dW_{2,s} - \frac{1}{2}\int_0^t [(\frac{\alpha - r}{\sqrt{V_s}})^2 + \Theta_{2,s}^2] ds} \tag{26.187}$$

Then, under $\tilde{P}$, $D_t S_t, D_t M_t$ are MGs. Given $\frac{\alpha - r}{\sqrt{V_t}}$ $\tilde{P}$ is decided by $\Theta_{2,s}$ uniquely, so there are infinitely risk neutral probability measures. In this situation, V_t can not be replicated.

(2) If V_t is tradable, we want to show $\tilde{P}$ is unique. Let $D_t M_t, D_t S_t, D_t V_t$ be MGs under $\tilde{P}$. Then,

$$d[D_t S_t] = D_t S_t \sqrt{V_t} d\tilde{W}_{1,t} \tag{26.188}$$

$$d[D_t V_t] = V_t dD_t + D_t dV_t \tag{26.189}$$

$$= -rD_t V_t dt + D_t[d\kappa(\theta - V_t)dt + \nu\sqrt{V_t}(\rho dW_{1,t} + \sqrt{1-\rho^2} dW_{2,t})] \tag{26.190}$$

$$= [\kappa(\theta - V_t) - rD_t V_t]dt + \nu\rho D_t \sqrt{V_t} dW_{1,t} + \nu\sqrt{1-\rho^2} D_t \sqrt{V_t} dW_{2,t} \tag{26.191}$$

$$= [\kappa(\theta - V_t) - rD_t V_t]dt + \nu\rho D_t \sqrt{V_t}(d\tilde{W}_{1,t} - \frac{\alpha - r}{\sqrt{V_t}}) + \nu\sqrt{1-\rho^2} D_t \sqrt{V_t} dW_{2,t} \tag{26.192}$$

$$= \nu\rho D_t \sqrt{V_t} d\tilde{W}_{1,t} + \nu\sqrt{1-\rho^2} D_t \sqrt{V_t}(dW_{2,t} + \Theta_{2,t} dt) \tag{26.193}$$

where

$$\Theta_{2,t} = \frac{\kappa D_t(\theta - V_t) - rD_tV_t}{\nu\sqrt{1-\rho^2}D_t\sqrt{V_t}} - \frac{\nu\rho D_t\sqrt{V_t}\frac{\alpha-r}{\sqrt{V_t}}}{\nu\sqrt{1-\rho^2}D_t\sqrt{V_t}} \tag{26.194}$$

$$= \frac{\kappa(\theta - V_t) - rV_t}{\nu\sqrt{1-\rho^2}\sqrt{V_t}} - \frac{\rho(\alpha - r)}{\sqrt{1-\rho^2}\sqrt{V_t}} \tag{26.195}$$

Let

$$\begin{cases} \tilde{W}_{1,t} = W_{1,t} + \int_0^t \frac{\alpha-r}{\sqrt{V_s}}ds \\ \tilde{W}_{2,t} = W_{2,t} + \int_0^t \Theta_{2,s}ds \end{cases} \tag{26.196}$$

Therefore,

$$\frac{d\tilde{P}}{dP} = Z = e^{\int_0^t \frac{\alpha-r}{\sqrt{V_s}}dW_{1,s} + \Theta_{2,s}dW_{2,s} - \frac{1}{2}\int_0^t[(\frac{\alpha-r}{\sqrt{V_s}})^2 + \Theta_{2,s}^2]ds} \tag{26.197}$$

is unique, i.e., $\tilde{P}$ is unique. To replicate V_T we just need to hold $X_t = V_t$.

26.5 Handle Dividends

There are two approaches to handle dividends, continuous dividends and discrete dividends.

Dividend payout reduces stock price. Consider the following model of a stock with a continuous dividend stream,

$$dS_t = \alpha_t S_t dt + \sigma_t S_t dW_t - A_t S_t dt \tag{26.198}$$

where A_t is the dividend ratio. For a self-financing portfolio, $X_t = \Delta_t S_t + X_t - \Delta_t S_t$, its evolution is,

$$dX_t = \Delta_t dS_t + \Delta_t A_t S_t dt + R_t(X_t - \Delta_t S_t) \tag{26.199}$$

$$= \Delta_t(\alpha_t S_t dt + \sigma_t S_t dW_t - A_t S_t dt) + \Delta_t A_t S_t dt + (X_t - \Delta_t S_t)R_t dt \tag{26.200}$$

$$= \Delta_t(\alpha_t S_t dt + \sigma_t S_t dW_t) + R_t(X_t - \Delta_t S_t)dt \tag{26.201}$$

$$= \Delta_t[(\alpha_t - R_t)S_t dt + \sigma_t S_t dW_t] + R_t X_t dt \tag{26.202}$$

$$\Rightarrow d[D_t X_t] = X_t dD_t + D_t dX_t + dD_t dX_t \tag{26.203}$$

$$= -R_t D_t X_t dt + D_t[\Delta_t[(\alpha_t - R_t)S_t dt + \sigma_t S_t dW_t] + R_t X_t dt] + 0 \tag{26.204}$$

$$= \Delta_t D_t S_t \sigma_t(\frac{\alpha_t - R_t}{\sigma_t}dt + dW_t) \tag{26.205}$$

Let $\tilde{W}_t = W_t + \int_0^t \frac{\alpha_t - R_t}{\sigma_t}dt$. Then to ensure $\tilde{W}_t$ is a BM, let

$$\frac{d\tilde{P}}{dP} = Z = e^{-\int_0^t \frac{\alpha_s - R_s}{\sigma_s}dW_s - \frac{1}{2}(\frac{\alpha_t - R_t}{\sigma_t})^2 ds} \tag{26.206}$$

Under $\tilde{P}$, the dynamic of S_t is

$$dS_t = \alpha_t S_t dt + \sigma_t S_t dW_t - A_t S_t dt \tag{26.207}$$

$$= \alpha_t S_t dt + \sigma_t S_t(d\tilde{W}_t - \frac{\alpha_t - r}{\sigma_t}dt) - A_t S_t dt \tag{26.208}$$

$$= (R_t - A_t)S_t dt + \sigma_t S_t d\tilde{W}_t \tag{26.209}$$

Now we modify the BM model with dividends. Let $A_t = q$, then

$$dS_t = (R - q)S_t dt + \sigma_t S_t d\tilde{W}_t \tag{26.210}$$

The call option's fair price,

$$c(t, S_t) = \tilde{E}[e^{-r(T-t)}(S_T - K)^+ | \mathcal{F}_t] \tag{26.211}$$

where $S_T = S_t e^{(r-q-\frac{1}{2}\sigma^2)(T-t)+\sigma(W_T - W_t)}$.

Continuous dividends can describe the dividend distribution pattern of multiple stocks portfolios.

Discrete dividends. Let $0 < t_1 < t_2 < \cdots < t_n < T$ is a sequence of dividend-paying dates and on date t_j, a dividend of $a_j(S_{t_j-})$ is paid, where t_j- denotes the left limiting time at t_j.

Then,

$$S_{t_j} = (1 - a_j)S_{t_j-} \tag{26.212}$$

and durning $[t_j, t_{j+1})$

$$dS_t = \alpha_t S_t dt + \sigma_t S_t dt dW_t \tag{26.213}$$

Let $X_t = \Delta S_t + X_t - \Delta_t S_t$, then durning $[t_i, t_{i+1})$,

$$\begin{aligned} dX_t &= \Delta dS_t + R_t(X_t - \Delta_t S_t)dt && (26.214)\\ &= R_t X_t dt + \Delta_t S_t((\alpha_t - R_t)dt + \sigma_t dW_t) && (26.215)\end{aligned}$$

and when $t = t_i$, receive a dividend,

$$\begin{aligned} X_{t_i} &= \Delta_{t_i}(1 - a_i)S_{t_i} + \Delta_{t_i-} a_i S_{t_i-} + X_{t_i-} - \Delta_{t_i} S_{t_i} && (26.216)\\ &= X_{t_i-} && (26.217)\end{aligned}$$

Portfolio value will not jump and the change of measure is the same as before.

26.6 Forwards and Futures

The model: a financial market with a finite horizon of $\bar{T}$, a risk-neutral probability measure $\tilde{P}$ (no arbitrage) and a discounting process D_t.

The price of a zero coupon bond matured at T with face value 1, is given by

$$B(t,T) = \tilde{E}[e^{-\int_t^T R_s ds} \cdot 1|\mathcal{F}_t] = \tilde{E}[\frac{D_T}{D_t} \cdot 1|\mathcal{F}_t] = \frac{1}{D_t}\tilde{E}[D_T \cdot 1|\mathcal{F}_t] \tag{26.218}$$

(risk-neutral pricing formula)

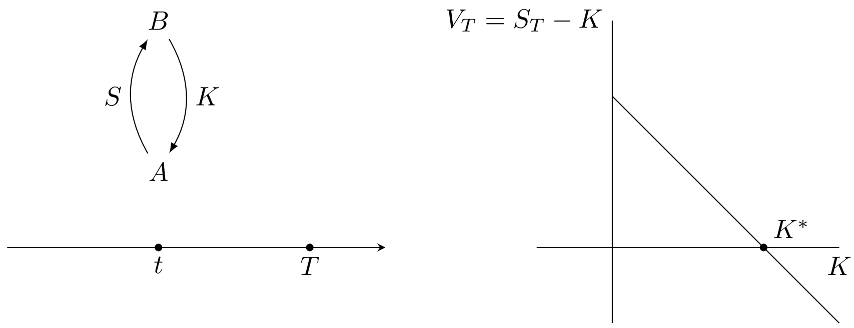


Figure 36: Foward

DEFINITION 26.4. *(Forward Contract) A forward contract is an agreement between two counterparties. One counterparty of (the buyer of the contract) pays a pre-specified amount of K at T in exchange for one share of the underlying asset. See Figure 36.*

DEFINITION 26.5. *(Forward Price) The forward price of an asset is the contractual price K which makes the forward contract has zero initial value at inception.*

From B's perspective, the value of the trade is $V_T = S_T - K$, then

$$\begin{aligned} V_t &= \tilde{E}[\frac{D_T}{D_t} V_T|\mathcal{F}_t] && (26.219)\\ &= \tilde{E}[\frac{D_T}{D_t}(S_T - K)|\mathcal{F}_t] && (26.220)\\ &= \tilde{E}[\frac{D_T}{D_t} S_T|\mathcal{F}_t] - \tilde{E}[\frac{D_T}{D_t} K|\mathcal{F}_t] && (26.221)\\ &= \frac{1}{D_t}\tilde{E}[D_T S_T|\mathcal{F}_t] - K\tilde{E}[\frac{D_T}{D_t}|\mathcal{F}_t] && (26.222)\\ &= \frac{1}{D_t} S_t D_t - KB(t,T) && (26.223)\\ &= S_t - KB(t,T) && (26.224)\end{aligned}$$

note that $D_T S_T$ is a MG under $\tilde{P}$. Let $V_t = 0$, we have the forward price of an asset at t is given by

$$For_S(t,T) = \frac{S_t}{B(t,T)} \tag{26.225}$$

Although the forward trade is fair at t, but it becomes unfair as time goes. Consider B, owning a long position of the forward position, assume settled at t_k. At $t_j \geq t_k$, the value of the position is,

$$\begin{aligned}
V_{t_j} &= \tilde{E}[\frac{D_T}{D_t}(S_T - K)|\mathcal{F}_{t_j}] &(26.226)\\
&= \tilde{E}[\frac{D_T}{D_t}(S_T - \frac{S_{t_k}}{B(t_k,T)})|\mathcal{F}_{t_j}] &(26.227)\\
&= \tilde{E}[\frac{D_T}{D_t}S_t|\mathcal{F}_{t_j}] - \frac{S_{t_k}}{B(t_k,T)}\tilde{E}[\frac{D_T}{D_t}|\mathcal{F}_{t_j}] &(26.228)\\
&= S_{t_j} - S_{t_k}\frac{B(t_j,T)}{B(t_k,T)} &(26.229)
\end{aligned}$$

which varies in $\mathbb{R}$, not zero necessarily, thus default of the counterparty becomes a concern. With higher V_{t_j}, the default risk of A faced by B is bigger.

An intuitive way to reduce exposure to the counterparty risk is to roll-over the forward contract, that is, each day the investor terminates an old forward contract and initiates a new one, in order to spilt potential large gain / loss into small ones. But that requires highly liquid market.

A better way is to create a futures prices $Fut_S(t,T)$, and directly mark the gains / losses of holding the futures to the changes in the futures price.

If the investor holds one share of the futures at t_k, then her / his gains / losses over the period $[t_k, t_{k+1}]$ is

$$Fut_S(t_{k+1},T) - Fut_S(t_k,T) \tag{26.230}$$

such gains / losses are immediately realized through the margin account to minimize the counterparty default risk.

At the exercise day of the futures contract, the futures price must coincide with the spot price, i.e., $Fut_S(T,T) = S_T$.

The futures price is set to ensure the cost of entering a futures trading on each day is zero. In practice, futures prices are affected by the pressure of purchasing and selling. Assume the risk free rate R is constant, then

$$\begin{aligned}
\tilde{E}[e^{\int_{t_k}^{t_{k+1}} R_s ds}Fut_S(t_{k+1},T) - Fut_S(t_k,T)|\mathcal{F}_{t_k}] = 0 &&(26.231)\\
\tilde{E}[e^{-R(t_{k+1}-t_k)}Fut_S(t_{k+1},T) - Fut_S(t_k,T)|\mathcal{F}_{t_k}] = 0 &&(26.232)\\
e^{-R(t_{k+1}-t_k)}\tilde{E}[Fut_S(t_{k+1},T) - Fut_S(t_k,T)|\mathcal{F}_{t_k}] = 0 &&(26.233)\\
\tilde{E}[Fut_S(t_{k+1},T) - Fut_S(t_k,T)|\mathcal{F}_{t_k}] = 0 &&(26.234)\\
\Rightarrow Fut_S(t_k,T) = \tilde{E}[Fut_S(t_{k+1},T)|\mathcal{F}_{t_k}] &&(26.235)
\end{aligned}$$

With the boundary condition $Fut_S(T,T) = S_T$, we can solve $Fut_S(t,T)$,

$$\begin{aligned}
Fut_S(t_k,T) &= \tilde{E}[Fut_S(t_{k+1},T)|\mathcal{F}_{t_k}] &(26.236)\\
&= \tilde{E}[\tilde{E}[Fut_S(t_{k+2},T)|\mathcal{F}_{t_{k+1}}]|\mathcal{F}_{t_k}] &(26.237)\\
&= \tilde{E}[Fut_S(t_{k+2},T)|\mathcal{F}_{t_k}] &(26.238)\\
&\cdots &(26.239)\\
&= \tilde{E}[Fut_S(T,T)|\mathcal{F}_{t_k}] &(26.240)\\
&= \tilde{E}[S_T|\mathcal{F}_{t_k}] &(26.241)
\end{aligned}$$

Thus, the futures prices reflect the expected price for the asset of the market.

Now we define the futures price process as

$$Fut_S(t,T) = \tilde{E}[S_T|\mathcal{F}_t] \tag{26.242}$$

we want to find Fut_S under the BS model. Recall the forward price,

$$For_S(t,T) = \frac{S_t}{B(t,T)} \tag{26.243}$$

The spread between forward price and futures price,

$$For_S(0,T) - Fut_S(0,T) = \frac{S_t}{B(t,T)} - \tilde{E}[S_T|\mathcal{F}_t] \tag{26.244}$$
$$= \frac{1}{D_t}\tilde{E}[D_T S_T|\mathcal{F}_t]\frac{1}{\frac{\tilde{E}[D_T|\mathcal{F}_t]}{D_t}} - \tilde{E}[S_T|\mathcal{F}_t] \tag{26.245}$$
$$= \frac{\tilde{E}[D_T S_T|\mathcal{F}_t]}{\tilde{E}[D_T]|\mathcal{F}_t} - \tilde{E}[S_T|\mathcal{F}_t] \tag{26.246}$$
$$= \frac{1}{\tilde{E}[D_T|\mathcal{F}_t]}[\tilde{E}[D_T S_T|\mathcal{F}_t] - \tilde{E}[S_T|\mathcal{F}_t]\tilde{E}[D_T|\mathcal{F}_t]] \tag{26.247}$$
$$= \frac{\tilde{Cov}(D_T, S_T|\mathcal{F}_t)}{\tilde{E}[D_T|\mathcal{F}_t]} \tag{26.248}$$

where $\tilde{E}[D_T|\mathcal{F}_t] > 0$. If D_T, S_T are positively correlated, then $For_S(0,T) > Fut_S(0,T)$, i.e., forward is more favorable. In BS model, the D_T is constant, so $\tilde{Cov}(D_T, S_T|\mathcal{F}_t) = 0$, i.e., $For_S(0,T) = Fut_S(0,T)$.

27 Feynman-Kac Theorem

DEFINITION 27.1. *(Stochastic Ddifferential Equation, SDE) An equation like*

$$dX_u = \beta(u, X_u)du + \gamma(u, X_u)dW_u, X_t = x \tag{27.1}$$

where W_u is a one-dimensional standard Brownian Motion,β, γ are deterministic functions ($\beta(u, X_u), \gamma(u, X_u)$ both stochastic processes), is a stochastic differential equation.

Interpret such a SDE by discrete types. Let $\Pi : 0 = t_0 < t_1 < \cdots < t_n < T$ be a partition of $[0, T]$, let

$$\tilde{X}_{t_0} = x \tag{27.2}$$
$$\tilde{X}_{t_1} = \tilde{X}_{t_0} + \beta(t_0, \tilde{X}_{t_0})(t_1 - t_0) + \gamma(t_0, \tilde{X}_{t_0})(W_{t_1} - W_{t_0}) \tag{27.3}$$
$$\tilde{X}_{t_2} = \tilde{X}_{t_1} + \beta(t_1, \tilde{X}_{t_1})(t_2 - t_1) + \gamma(t_1, \tilde{X}_{t_1})(W_{t_2} - W_{t_1}) \tag{27.4}$$
$$\cdots \tag{27.5}$$
$$\tilde{X}_{t_n} = \tilde{X}_{t_{n-1}} + \beta(t_{n-1}, \tilde{X}_{t_{n-1}})(t_n - t_{n-1}) + \gamma(t_{n-1}, \tilde{X}_{t_{n-1}})(W_{t_n} - W_{t_{n-1}}) \tag{27.6}$$
$$\tag{27.7}$$

Let $|\Pi| \to 0$, then $\tilde{X}_{t_k} \to \{X\}_t$, which is the solution of Equation 27.1.

Now we integrate at both sides of Equation 27.1, i.e.,

$$X_T - X_t = \int_t^T \beta(u, X_u)du + \int_t^T \gamma(u, X_u)dW_u \tag{27.8}$$
$$\Rightarrow X_T = X_t + \int_t^T \beta(u, X_u)du + \int_t^T \gamma(u, X_u)dW_u \tag{27.9}$$

Given $X_0 = x$, Equation 27.1 defines a random variable. Let $\mathcal{F}_t$ be the associated filtration of W_t, and $h(\cdot)$ be a $\mathcal{F}_t$-measurable, then we define a new r.v., $h(X_T)$ and

$$g(t, x) = E_{t,x}[h(X_T)] = E[h(X_T)|X_t = x] \tag{27.10}$$

(precisely, we assume X_t is a Markov process here, to ensure g exists)

THEOREM 27.1.

$$g(t, X_t) = E[h(X_T)|\mathcal{F}_t] \tag{27.11}$$

where $\mathcal{F}_t$ implies we know X_t but not fixed x necessarily.

When $t = T$, we have

$$g(T, X_T) = E[h(X_T)|\mathcal{F}_T] \tag{27.12}$$
$$= h(X_T) \tag{27.13}$$
$$x = X_T \Rightarrow g(T, x) = h(x) \tag{27.14}$$

When $t < T$, define

$$Y_t = g(t, X_t) = E[h(X_T)|\mathcal{F}_t] \tag{27.15}$$

then, $\forall 0 \leq s < t \leq T, E[h(X_t)|\mathcal{F}_s] = E[E[h(X_T)|\mathcal{F}_t]|\mathcal{F}_s] = E[h(X_T)|\mathcal{F}_s] = Y_s$, so Y_t is a martingale.

Now

$$dY_t = dg(t, X_t) \tag{27.16}$$

$$= g_t dt + g_x dX_t + \frac{1}{2} g_{xx}(dX_t)^2 \tag{27.17}$$

$$= g_t dt + g_x(\beta(u, X_u)du + \gamma(u, X_u)dW_u) + \frac{1}{2} g_{xx}\gamma^2 dt \tag{27.18}$$

$$= (g_t + \beta g_x + \frac{1}{2}\gamma^2 g_{xx})dt + \gamma g_x dW_t \tag{27.19}$$

as Y_t is a MG, so,

$$g_t + \beta g_x + \frac{1}{2}\gamma^2 g_{xx} = 0 i.e., g_t(t, X_t) + \beta(t, X_t)g_x(t, X_t) + \frac{1}{2}\gamma(t, X_t)^2 g_{xx}(t, X_t) \qquad = 0, \forall t \tag{27.20}$$

Thus, let $x = X_t$

$$g_t(t, x) + \beta(t, x)g_x(t, x) + \frac{1}{2}\gamma(t, x)^2 g_{xx}(t, x) = 0 \tag{27.21}$$

which is a partial differential equation (PDE), named the Feynman-Kac equation associated with the SDE.

Plus the boundary condition, we have

$$\begin{cases} g_t(t, x) + \beta(t, x)g_x(t, x) + \frac{1}{2}\gamma(t, x)^2 g_{xx}(t, x) = 0, \forall t < T \\ g(T, x) = h(x), t = T \end{cases} \tag{27.22}$$

Three steps of finding the Feynman-Kac equation,

step (1) find the MG;

step (2) calculate the differential using the Ito's formula;

step (3) set the drift term to 0.

The price of a derivative security is the expected value of its discounted payoff under the risk-neutral measure. Then we are interested in,

$$g(t, x) = E_{t,x}[e^{-\delta(T-t)}h(X_T)] = E[^{-\delta(T-t)}h(X_T)|X_t = x] \tag{27.23}$$

LEMMA 27.1. *Let*

$$g(t, X_t) = E[e^{-\delta(T-t)}h(X_T)|\mathcal{F}_t] \tag{27.24}$$

then $e^{-\delta t}g(t, X_t)$ *is a martingale.*

Proof. When $t = T, g(T, X_T) = E[e^{-\delta(T-T)}h(X_T)|\mathcal{F}_T] = h(X_T)$, so $g(T, x) = h(x)$, the boundary condition.

When $t < T$, try $Z_t = g(t, X_t) = E[e^{-\delta(T-t)}h(X_T)|\mathcal{F}_t]$, we need to drop $e^{\delta t}$. Let $Z_t = e^{-\delta t}g(t, X_t) = E[e^{-\delta T}h(X_T)|\mathcal{F}_t]$, we show Z_t is a MG.

$$dZ_t = d[e^{-\delta}g(t, X_t)] \tag{27.25}$$

$$= g(t, X_t)de^{-\delta t} + e^{-\delta t}dg(t, X_t) + de^{-\delta t}dX_t \tag{27.26}$$

$$= -\delta e^{-\delta t}g(t, X_t)dt + e^{-\delta t}[g_t + \beta g_x + \frac{1}{2}\sigma^2 g_{xx}] \tag{27.27}$$

$$= e^{-\delta t}[(-\delta g + g_t + \beta g_x + \frac{1}{2}\gamma^2 g_{xx})dt + r g_x dW_t] \tag{27.28}$$

Then we let the coefficient of dt be zero,

$$-\delta g + g_t + \beta g_x + \frac{1}{2}\gamma^2 g_{xx} = 0 \tag{27.29}$$

$$g_t(t, X_t) + \beta(t, X_t)g_x(t, X_t) + \frac{1}{2}\gamma(t, X_t)^2 g_{xx}(t, X_t) - \delta g(t, X_t) = 0, \forall t \tag{27.30}$$

Then the associated Feynman-Kac equation (plus the boundary condition) becomes

$$\begin{cases} g_t(t,X_t)+\beta(t,X_t)g_x(t,X_t)+\frac{1}{2}\gamma(t,X_t)^2g_{xx}(t,X_t)-\delta g(t,X_t) & =0 \\ g(T,x)=h(x) \end{cases} \tag{27.31}$$

□

Consider the classic Black-Scholes model. Under the risk-neutral probability measure $\tilde{P}$, the underlying asset has a dynamics of

$$dS_t = rS_tdt + \sigma S_t d\tilde{W}_t \tag{27.32}$$

so $\beta(t,x)=rx, \gamma(t,x)=\sigma x$. The European call option's price is

$$v(t,S_t)=\tilde{E}[e^{-r(T-t)}(S_T-K)^+|\mathcal{F}_t] \tag{27.33}$$

so $h(x)=(x-K)^+, \delta=r$ (the discounting factor). Now apply the discounted Feynman-Kac Theorem, we say

$$\begin{cases} v_t(t,x)+rxv_x(t,x)+\frac{1}{2}\sigma^2x^2v_{xx}(t,x)=rv(t,x) \\ v(T,x)=(x-K)^+ \end{cases} \tag{27.34}$$

which is exactly the Black-Scholes equation. Feynman-Kac theorem provides a direct way to link the expectation of a random variable (generated by an SDE) to the solution of a SDE.

Give as SDE

$$dX_u=\beta(u,X_u)du+\gamma(u,X_u)dW_u, X_t=x \tag{27.35}$$

we have two ways to compute the expectation,

$$g(t,x)=E_{t,x}[h(X_T)] \tag{27.36}$$

- Monte-Carlo simulation;
- Numerically, solve the Feynman-Kac PDE.

We provide two example to show how to solve the Feynman-Kac PDE.

Example 27.1. *(Zero-coupon Bond) Let R_t be the interest rate process (assume Markov process), s.t.*

$$dR_t=\beta(t,R_t)dt+\gamma(t,R_t)d\tilde{W}_t \tag{27.37}$$

where $\tilde{W}_t$ is a Brownian Motion under $\tilde{P}$. The price of a zero coupon bond (only yield 1 at T), is

$$B(t,T)=\tilde{E}[e^{-\int_t^T R_s ds}|\mathcal{F}_t] \tag{27.38}$$

Where R_t is a Markov process, then there is a f s.t. $B(t,T)=f(t,R_t)$. We want to construct a MG. Try $Z_t=\tilde{E}[e^{-\int_t^T R_s ds}|\mathcal{F}_t]=e^{\int_0^t}\tilde{E}[e^{-\int_0^T R_s ds}|\mathcal{F}_t]$, not a MG. To drop $e^{-\int_0^t}$, let $Z_t=e^{-\int_0^t}f(t,R_t)=\tilde{E}[e^{-\int_0^T R_s ds}|\mathcal{F}_t]$, then Z_t is a MG under $\tilde{P}$. Thus, let $D_t=e^{-\int_0^t R_s ds}$,

$$\begin{aligned} dZ_t &= dD_tf(t,R_t) && (27.39)\\ &= f(t,R_t)dD_t+f(t,R_t)dD_t+dD_tdf(t,R_t) && (27.40)\\ &= f(t,R_t)(-R_tD_tdt)+f(t,R_t)[f_tdt+f_r\beta dt+\gamma d\tilde{W}_t+\frac{1}{2}f_{rr}\gamma^2dt]D_t && (27.41)\\ &= D_t[(f_t+f_r\beta+\frac{1}{2}f_{rr}\gamma^2-R_tf)dt+f_r\gamma d\tilde{W}_t] && (27.42) \end{aligned}$$

So,

$$f_t(t,R_t)+f_r(t,R_t)\beta(t,R_t)+\frac{1}{2}\gamma(t,R_t)^2f_{rr}(t,R_t)-R_tf(t,R_t)=0, \forall t \tag{27.43}$$

Then, the associated Feynman-Kac SDE equation becomes

$$\begin{cases} f_t(t,r)+f_r(t,r)\beta(t,r)+\frac{1}{2}\gamma(t,r)^2f_{rr}(t,r)-rf(t,r)=0, \forall t f(T,r)=B(T,T)=1 \end{cases} \tag{27.44}$$

Example 27.2. *(Vasicek Short Rate Model, affine yield models) Let* $dR_t = \kappa(\theta - R_t)dt + \nu d\tilde{W}_t$*, the we want to find* $f(t,r)$ *to solve*

$$\begin{cases} f_t + \kappa(\theta - r)f_r + \frac{1}{2}\nu^2 f_{rr} - rf = 0 \end{cases} \tag{27.45}$$

Conjecture $f(t,r) = e^{-A(t)r - B(t)}$*, then*

$$f(T,r) = 1 \tag{27.46}$$
$$\Rightarrow e^{-A(T)r - B(T)} = 1 \tag{27.47}$$
$$\Rightarrow A(T) = B(T) = 0 \tag{27.48}$$
$$f_t(t,r) = e^{-A(t)r - B(t)}(-A'(t)r - B'(t)) \tag{27.49}$$
$$f_r(t,r) = e^{-A(t)r - B(t)}(-A(t)) \tag{27.50}$$
$$f_{rr}(t,r) = e^{-A(t)r - B(t)}A^2(t) \tag{27.51}$$
$$\tag{27.52}$$

From $f_t + \kappa(\theta - r)f_r + \frac{1}{2}\nu^2 f_{rr} - rf = 0, \forall r$*, we have the coefficient of r.v. and constant are zero, so*

$$\begin{cases} -A'(t) + \kappa A(t) - 1 = 0 \\ -B'(t) - \kappa\theta A(t) + \frac{1}{2}\nu^2 A^2(t) = 0 \\ A(T) = B(T) = 0 \end{cases} \tag{27.53}$$

We first solve $A(t)$ *and then* $B(t)$ *is obvious.*

$$-A'(t) + \kappa A(t) - 1 = 0 \tag{27.54}$$
$$e^{-\kappa t}A'(t) - \kappa e^{-\kappa t}A(t) = -e^{-\kappa t} \tag{27.55}$$
$$\frac{d}{dt}(e^{-\kappa t}A(t)) = -e^{-\kappa t} \tag{27.56}$$
$$e^{-\kappa T}A(T) - e^{-\kappa t}A(t) = -\int_t^T e^{-\kappa s}ds \tag{27.57}$$
$$e^{-\kappa T}A(T) - e^{-\kappa t}A(t) = \frac{1}{\kappa}\int_t^T (-\kappa)e^{-\kappa s}ds \tag{27.58}$$
$$-e^{-\kappa t}A(t) = \frac{1}{\kappa}(e^{-\kappa T} - e^{-\kappa t}) \tag{27.59}$$
$$\Rightarrow A(t) = \frac{1}{\kappa}(1 - e^{-\kappa(T-t)}) \tag{27.60}$$

Then from $B'(t) + \kappa\theta A(t) - \frac{1}{2}\nu^2 A^2(t) = 0$*, we can solve* $B(t)$*.*

So $B(t,T) = f(t,R_t) = e^{-A(t)R_t - B(t)}$ *(price of bond). In this model, the maturity yield return* Y *s.t.*

$$B(t,T)e^{Y(T-t)} = 1 \tag{27.61}$$
$$Y = \frac{1}{T-t}ln(\frac{1}{B(t,T)}) \tag{27.62}$$
$$= \frac{1}{T-t}ln(e^{A(t)R_t + B(t)}) \tag{27.63}$$
$$= \frac{1}{T-t}[A(t)R_t + B(t)] \tag{27.64}$$
$$= \frac{A(t)}{T-t}R_t + \frac{B_t}{T-t} \tag{27.65}$$

So Y *is a linear function of* R_t *(affine).*

Another famous short rate model satisfies the same property, named CIR (its adavantage is ensure $R_t > 0$ *by adding* $\sqrt{R_t}$*), s.t.* $dR_t = \kappa(\theta - R_t)dt + \nu\sqrt{R_t}dt$*, the associated Feynman-Kac equation is*

$$\begin{cases} f_t + \kappa(\theta - r)f_r + \frac{1}{2}\nu^2 r f_{rr} - rf = 0 \\ f(T,r) = 0 \end{cases} \tag{27.66}$$

The solution is also the $f(t,r) = e^{-A(t)r - B(t)}$ *type.*

Some SDE may not permit solution, like

$$dR_t = \sigma R_t d\tilde{W}_t \tag{27.67}$$

the associated Feynman-Kac equation

$$f_t + \frac{1}{2}\sigma^2 r^2 f_{rr} - rf = 0 f(T, r) = 1 \tag{27.68}$$

has no numerical solution.

A bond option is an option whose underlying asset is a bond (compound derivative). Assume the underlying bond's maturity is T_2, the bond option's exercise date is T_1 ($< T2$), a call option is struck at a fixed price K. Now pricing the bond option by Feynman-Kac equations. The payoff of the bond option is $(B(T_1, T_2) - K)^+$ and let f denote the price of the bond maturity at T_2, then $f(T_1, R_{T_1}) = (B(T_1, T_2) - K)^+$ (known, decided by R_t).

So the payoff becomes $(f(T_1, R_{T_1}) - K)^+$. Let $g(t, R_t)$ denote its price, When $t < T_1$, the fair price of the bond option is given by

$$g(t, R_t) = \tilde{E}[e^{-\int_t^{T_1} R_s ds}(f(T_1, R_{T_1}) - K)^+ | \mathcal{F}_t] \tag{27.69}$$

Let $t = T_1, g(T_1, R_{T_1}) = (f(T_1, r) - K)^+$. Also, it is trivial to verify

$$e^{-\int_0^t R_s ds} g(t, R_t) = \tilde{E}[e^{-\int_0^{T_1} R_s ds}(f(T_1, R_{T_1}) - K)^+ | \mathcal{F}_t] \tag{27.70}$$

is a MG. Then calculate $e^{-\int_0^t R_s ds} g(t, R_t)$, we need to drop dt terms, so the Feynman-Kac equation is

$$\begin{cases} g_t + \beta g_r + \frac{1}{2}\gamma^2 g_{rr} - rg = 0 \\ g(T_1, r) = (B(T_1, r) - K)^+ \end{cases} \tag{27.71}$$

References

[1] Claude Berge. *Topological Spaces: including a treatment of multi-valued functions, vector spaces, and convexity.* Courier Corporation, 1997.

[2] Russell Davidson, James G MacKinnon, et al. *Econometric theory and methods*, volume 5. Oxford University Press New York, 2004.

[3] Gerard Debreu. *Theory of value: An axiomatic analysis of economic equilibrium*, volume 17. Yale University Press, 1959.

[4] Yongmiao Hong. *Foundations of Modern Econometrics: A Unified Approach.* World Scientific, 2020.

[5] David M Kreps. *Microeconomic foundations I: choice and competitive markets*, volume 1. Princeton university press, 2013.

[6] Per Krusell. Real macroeconomic theory. *Working manuscript*, 2014.

[7] Steven E Shreve et al. *Stochastic calculus for finance II: Continuous-time models*, volume 11. Springer, 2004.

[8] Nancy L Stokey. *Recursive methods in economic dynamics.* Harvard University Press, 1989.

[9] Carl E Walsh. *Monetary theory and policy.* MIT press, 2017.