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Light bending in general relativity with a cosmological constant: a literature review

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Abstract: In this review, I survey some of the work analyzing the effect of the cosmological constant on gravitational deflection of light, a debate started mainly in a paper by Rindler and Ishak (2007) which challenged the conventional opinion. Since then, numerous papers have been written both in support and rebuttal of Rindler and Ishak. I go through the main arguments for and against a Λ -dependence in the lensing equation, identify limitations of some approaches, and look at the direction in which further work can take to settle this debate.

1 Introduction

It has been established that the Universe is accelerating in its expansion, based on various complementary observations (Spergel et al., 2003; Riess et al., 2004). In the Λ CDM cosmological model, this acceleration is powered by a cosmological constant Λ , that dominates the Universe's current energy budget.

An active dispute that has been the subject of several papers in the last decade is whether the cosmological constant alters the bending of light. Given the undisputed success of general relativity, for which the deflection of light is one of the three classical tests (Will, 1993), and the popularity of the Λ CDM model, one would think that the effect of a cosmological constant on gravitational lensing is well known. However, this is not the case.

While there is general agreement that the influence of the cosmological constant on the bending of light, if any, is small (Rindler and Ishak (2007) in Eq. 17 estimates the influence of Λ at roughly 10^{-28} of the neighbouring mass term), the possibility of doing precision cosmology with weak lensing as a tool renders the proposed difference significant enough. Importantly, if the cosmological constant is found to influence the deflection of light, gravitational lensing observations could allow for a new and independent constraint on Λ .

2 Conventional argument using the null geodesic

We are concerned about whether Λ contributes to the bending of light around a concentrated mass. Conventional view, first put forth by Islam (1983), is that it does not, and classical lensing is correct as it is. This view, supported by multiple authors (Lake, 2002; Finelli et al., 2007; Park, 2008; Simpson et al., 2008), argues that the equations describing the path followed by a photon, the null geodesic equations, takes the same form with or without Λ in the Kottler metric.

The Kottler metric (Kottler, 1918) represents the Schwarzschild vacuum metric extended to include a cosmological constant. In terms of curvature coordinates (r, θ, ϕ, t) , the line element in Kottler space is given by

$$ds^2 = f(r)dt^2 - \frac{dr^2}{f(r)} - r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (1)$$

where

$$f(r) = 1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}, \quad (2)$$

and m is the mass of the central object and relativistic units ($C = G = 1$) are used.

There is general agreement (Lake, 2002; Rindler and Ishak, 2007) that when we consider the null geodesic of such a metric, the cosmological constant drops out of the r, ϕ second-order differential equation in polar coordinates without approximation, and therefore does not contribute to light bending. Islam (1983) found that the differential equation governing light rays moving in a Kottler metric simplifies to

$$\frac{d^2u}{d\phi^2} + u = 3mu^2 \quad (3)$$

where $u = 1/r$. This formula is independent of Λ , therefore the resulting solution to this differential equation is the same as in a pure Schwarzschild background. For years this has been the accepted view, supported by the intuition that the cosmological constant is uniform but gravitational lensing occurs as a result of inhomogeneities in the spacetime.

3 Λ contribution to light bending in Kottler space-time

This official opinion was however challenged by Rindler and Ishak (2007), who argue that while the Λ term drops out of the null geodesic, Λ still affects light bending through the metric itself, since the photon is moving in Λ -dependent geometry. As space is not asymptotically flat, the process of measurement causes the cosmological constant to creep into the light bending angle.

As noted by Gibbons et al. (2008), the confusion with the conventional literature lies mainly in the question of what precisely (3) represents. (3) merely governs the projection of null rays onto the spatial sections of the metric, but not the more complex geometrical features. Rindler and Ishak's paper attempt to address this.

They started with the Kottler metric and noted that due to the spherical symmetry of the spatial part of the metric, any orbit must lie entirely in one of the central spatial equatorial coordinate "plane" $\theta = \pi/2$, and in Kottler spacetime this is given by the 2-metric

$$dl^2 = \left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2 d\phi^2. \quad (4)$$

Without the Λ term, this reduces to the Schwarzschild 2-metric. Near the central mass, where the $2m/r$ term dominates over $\Lambda r^2/3$, we get the geometry of the upper half of Flamm Paraboloid (Flamm, 1916), which has the equation

$$z^2 = 8m(r - 2m) \quad (5)$$

The dependence was also confirmed by Sereno (2008), who studied perturbations of the equations of motion in the weak deflection limit using the Kottler metric. Using observer coordinates $\{r_o, \phi_o = 0\}$ and source coordinates $\{r_s, \phi_s\}$, he wrote down the orbital equation for a light ray from source to observer, and integrated the expression term by term to obtain

$$\begin{aligned} \phi_s = & -\pi - \frac{4m}{b} + b \left(\frac{1}{r_s} + \frac{1}{r_o} \right) - \frac{15m^2\pi}{4b^2} - \frac{128m^3}{3b^3} + \frac{b^3}{6} \left(\frac{1}{r_s^3} + \frac{1}{r_o^3} \right) - \frac{3465m^4\pi}{64b^4} \\ & - \frac{3584m^5}{5b^5} - \frac{2mb}{r_\Lambda^2} - \frac{mb^3}{4} \left(\frac{1}{r_s^4} + \frac{1}{r_o^4} \right) + \frac{3b^5}{40} \left(\frac{1}{r_s^5} + \frac{1}{r_o^5} \right) - \frac{b^3}{2r_\Lambda^2} \left(\frac{1}{r_s} + \frac{1}{r_o} \right) + \mathcal{O}(\epsilon^6). \end{aligned} \quad (8)$$

where the cosmological constant appears through terms of order $\mathcal{O}(\epsilon^5)$ and also in $2bm/r_\Lambda^2$. Sereno considered the latter term local since it does not contain positional coordinates of the observer. He then derived the lens equation and when evaluated at the same conditions as in the Rindler and Ishak (2007) paper, he arrived at a cosmological constant term in the deflection, in agreement with (7).

The limitation of the analysis with Kottler spacetime is that this model assumes a static universe and a point mass as the lens. This is an unrealistic model for practical gravitational lensing since our universe is not static, and neither are gravitational lenses point masses. This may lead to misleading results when drawing conclusions.

4 Λ lensing and perturbation in a FRW model

The main criticism against the findings of Rindler and Ishak is that the relative movements of the source, observer, and lens were not incorporated into their analysis. In order to accomodate such a feature, most opponents of the Rindler-Ishak method use the Friedmann-Robertson-Walker (FRW) metric to look at the effect of Λ on lensing. Studies by Park (2008), Khriplovich and Pomeransky (2008), and Simpson et al. (2008) using this model led to the conclusion that classical bending is practically unaffected by Λ .

The FRW is the most general metric possible that describes a homogeneous, isotropic, and expanding universe. Simpson et al. (2008) compared the perturbed FRW, in particular the Newtonian gauge, to the Kottler metric and found an expression for the Kottler metric components, and concluded the Λ -dependent term obtained in Rindler and Ishak (2007) is merely a gauge artefact. Assuming a universe without anisotropic stress, the Newtonian gauge is given by

$$ds^2 = (1 + 2\Phi)dt^2 - a^2(t)(1 - 2\Phi)(d\chi^2 + \chi^2 d\psi^2) \quad (9)$$

where $a(t)$ is the scale factor, χ is the comoving radius, $d\psi$ is an element of angle on the sky, and Φ is the scalar potential that describes the scalar metric fluctuation. The cosmological constant Λ appears implicitly through the scale factor $a(t)$.

The authors consider a Kottler vacuole embedded in FRW spacetime. After some approximations and a coordinate transformation, the authors arrived at the following equation for f in the Kottler metric (described by (1)), to first order in Φ :

$$f = 1 - 2\chi\Phi' - \dot{a}^2\chi^2 \left[1 - 2\Phi \left(\frac{\partial \ln |\Phi|}{\partial \ln a} + 2 \right) \right]. \quad (10)$$

After dropping the term proportional to Φ in the square brackets and using the Friedmann equation to express (10) in terms of Λ , they arrive at

$$f = 1 - 2r\Phi'/a - 2mr^2R^3 - \Lambda r^2/3. \quad (11)$$

Comparing this equation with (2), they found the explicit expression for the perturbing potential responsible for light bending in the vacuole to be

$$\Phi = -\frac{m}{r} - \frac{mr^2}{2R^3} + \frac{3m}{2R} \quad (12)$$

where Λ has cancelled out with the corresponding Λ -term in the Kottler expression for $f(r)$ in 2, and the potential is independent of Λ .

However, this analysis is perturbative and the expression for Φ is only correct to lowest order. It is possible that Λ may appear in higher-order corrections to Φ , but even so, by this analysis the higher-order corrections are expected to be of smaller magnitude than the correction predicted by Rindler and Ishak.

Ishak et al. (2008b) criticized these studies and reported new calculations in which they estimated the bending by considering a model where a Kottler vacuole is embedded in a FRW background. According to them, the argument presented by Simpsons et al. (Simpson et al., 2008) eliminates the contribution of Λ a priori due to the too stringent assumption that the Kottler vacuole around the lens is negligibly small in comparison with the Hubble length. They proceeded differently from (10) and, without neglecting the size of the vacuole, they showed that their proposed Λ contribution is restored.

Park (2008) begins his analysis by using the McVittie (McVittie, 1933) solutions for embedding a mass (the lens) in a FRW spacetime. This metric interpolates between the Schwarzschild and FRW geometries, and contains the Kottler solution as a special case:

$$ds^2 = - \left(\frac{1-\mu}{1+\mu} \right)^2 dt^2 + (1+\mu)^4 a(t)^2 d\vec{X}^2 \quad (13)$$

where \vec{X} is the comoving coordinate,

$$\mu = \frac{m}{4a(t)|\vec{X} - \vec{X}_0|} \quad (14)$$

where \vec{X}_0 is the location of a mass of Schwarzschild radius m and $a(t)$ is the scale factor. He noted that the angle ψ in Rindler and Ishak (2007) is one measured by a static observer, but for actual observations all three objects (lens, source, object) are moving relative to each other due to expansion. Therefore, to solve for the null geodesics connecting a source to an observer in this background, he defined "physical" spatial coordinates by $\vec{x} = e^{Ht}\vec{X}$. He worked up to $\mathcal{O}(m)$, arriving at the following metric

$$ds^2 = - \left(1 - \frac{m}{\sqrt{(x + e^{Ht}q)^2 + y^2 + z^2}} \right) dt^2 + \left(1 + \frac{m}{\sqrt{(x + e^{Ht}q)^2 + y^2 + z^2}} \right) (d\vec{x} - H\vec{x}dt)^2 \quad (15)$$

where the coordinates are aligned to put the lens on the x -axis. He derived the null geodesic equations for this metric, and with some simplifying assumptions and approximations, he arrived at the following lens equation

$$\theta = \beta + \frac{2md_{SL}}{\beta d_S d_L} (1 + \mathcal{O}(H^3) + \mathcal{O}(\beta^2)) + \mathcal{O}(m^2) \quad (16)$$

where

$$\mathcal{O}(\beta^2) = -\beta^2 \frac{x_S^2 - 4x_S d_L + 2d_L^2}{4(x_S - d_L) + \dots} \quad (17)$$

This contradicts the conclusion of Rindler and Ishak (2007) since there is no $\mathcal{O}(\Lambda) \sim \mathcal{O}(H^2)$ correction as predicted by Rindler and Ishak. However, this was questioned by Ishak et al. (2008b), where they noted that second-order terms including $H^2 = \Lambda/3$ terms were dropped in the calculations due to the smallness assumption, which may explain the absence of a Λ contribution. Sereno (2008) also argued that the Λ does not appear explicitly in Park (2008) and Khriplovich and Pomeransky (2008) as it is already included in the angular diameter distances.

More recently, Faraoni and Lapierre-Léonard (2017) revisited this problem, using also the perturbed FRW metric but taking a different approach to the notion of “mass” as compared to the simple Newtonian central mass m used in previous investigations. The authors note that defining mass in General Relativity is problematic, since the Equivalence Principle allows us to locally eliminate gravity. This is especially so in non-asymptotically flat spacetimes (which is implied by the addition of the cosmological constant), and there are several definitions of mass that may be applicable in their own circumstances. In this analysis, they adopt the Hawking-Hayward quasilocal energy construct (Hawking, 1968; Hayward, 1994), specifically the Misner-Sharp-Hernandez mass (Misner and Sharp, 1964), M_{MSH} . Under conformal transformations, this mass is

$$\tilde{M}_{MSH} \simeq ma(t) + \frac{H^2 \tilde{R}^3}{2}, \quad (18)$$

where \tilde{R} is the conformal areal radius of the spherically symmetric spacetime in which the Misner-Sharp-Hernandez mass is defined.

The authors use the conformal Newtonian gauge, in a similar form to (9), only they used conformal time η instead of comoving time t . These are related by $dt = a d\eta$ where a is the scale factor.

After some conformal transformations and approximations, the authors obtained the deflection to be

$$\Delta\tilde{\phi} = \frac{4\tilde{M}_{MSH}}{\tilde{R}} - 2H^2 \tilde{R}^3 \quad (19)$$

to first order in Φ (in (9)) and $H^2 \tilde{R}^3$, where the quantities with tilde hat represent conformal quantities, and $\Delta\tilde{\phi} = \Delta\phi$ since angle are left invariant by conformal transformations. After substituting in M_{MSH} from (18), the contributions to $\Delta\tilde{\phi}$ due to the cosmological background vanishes. The authors then argue that the question of whether the cosmological constant directly affects the light bending angle depends on whether one identifies the lens mass with the Newtonian mass m or with the a quasi-local mass construct.

5 Other approaches to Λ contribution in light deflection

Arakida (2016) approached the problem by looking at the time-transfer function (Le Poncin-Lafitte et al., 2004) instead of solving the null-geodesic equation, as is done conventionally. He concludes that the deflection angle in a Kottler metric, using a time-transfer function, is the same as in the Schwarzschild case, where there is no cosmological constant.

In a more numerical approach, Beynon (2012) examine the LTB dust model (Lemaître and MacCallum, 1997; Tolman, 1934; Bondi, 1947) and numerically integrated the null geodesic equations with a Adams-Bashforth-Moulton solver, a multistep ODE solver. She followed the trajectory of several light rays by propagating them backwards with the null geodesic equations and used these to calculate lensing quantities such as the bend angle. In the LTB dust model, she models the lens as an overdensity in the FRW metric. However, results were only presented for the LTB model without pressure, whereas to fully investigate the effect of the cosmological constant, numerical simulations need to be run for the generalized LTB model with pressure.

There have also been some arguments that the cosmological constant enters into the null geodesic equation directly at the first-order differential equation level, directly challenging the conventional view outlined in Section 2. The main argument is that the cosmological constant is hidden implicitly in the null geodesic equation since the coordinate distance of closest approach depends on Λ .

This was first proposed by Bhadra et al. (2010), who reason that the first order differential equation for the null geodesic in the Kottler metric (described by (3)) contains a Λ -term that drops out at second order. The second-order solution must also satisfy the parent first-order differential equation from which (3) was derived:

$$\frac{1}{r^4} \left(\frac{dr}{d\phi} \right)^2 + \frac{f(r)}{r^2} - \frac{1}{b^2} = 0 \quad (20)$$

where $f(r)$ is given by (2). Therefore the authors assert that Λ should reappear in the null geodesic solution as an integration constant, and together with (20) it implies Λ -dependence should appear in the distance of closest approach.

Hammad (2013) agrees with this, and reiterated this claim in his paper where he claims the effects of the cosmological constant lies in the impact parameter $b = L/E$ where E is the total energy and L is the total angular momentum. Solving for the null geodesic in the Kottler metric and introducing b , he arrived at

$$\left(\frac{dr}{d\phi} \right)^2 = \left(\frac{1}{R_0^2} - \frac{2M}{R_0^3} \right) r^4 - r^2 + 2Mr \quad (21)$$

where he claims that Λ also appears implicitly in the nearest distance of approach to the mass R_0 , in the form of

$$\frac{1}{b^2} = \frac{\alpha_0}{R_0^2} = \frac{1}{\sin^2 \psi} \left(\frac{1}{R_1^2} - \frac{2M}{R_1^3} - \frac{\Lambda}{3} \right) \quad (22)$$

Substituting (22) into (21), we can see explicitly the contribution of Λ to the geodesic. The author then extracted the bending angle $\sin \psi$ from the null geodesic to get

$$\sin \psi = \sqrt{\frac{2M}{R_1}} + \frac{15\pi M}{32R_1} - \left(\sqrt{2} + \frac{675\pi^2}{2048\sqrt{2}} \right) \left(\frac{M}{R_1} \right)^{3/2} - \frac{1}{3\sqrt{2}} \Lambda \sqrt{MR_1^3} + \mathcal{O}(M^2, \Lambda M), \quad (23)$$

where he recovered the first- and second-order mass-terms as well as the first order Λ -term that appeared in Rindler and Ishak (2007).

6 Conclusion

While most agree that the cosmological constant drops out of the null geodesic equations of the light ray (with some exceptions, for example Bhadra et al. (2010) and Hammad (2013)), the debate was sparked again by Rindler and Ishak (2007) who argue that the cosmological constant shows itself in other ways in the metric. The discussion ensued, with many references agreeing with their conclusion while several others questioned it. Some references also suggest that the disagreement may be due to different interpretations of the Λ terms in the lensing equation.

The majority of work done in investigating the effect of a cosmological constant on the bending of light has been analytical rather than numerical. There have been attempts at settling the debate through numerical simulations (Beynon, 2012), but they were neither complete nor conclusive enough. Indeed, current studies have already suggested directions in which numerical simulations can proceed.

While of no immediate practical significance, answering the question of whether the cosmological constant affects gravitational lensing will be important for the next-generation precision cosmological. Given the importance of the cosmological constant in our current model of the universe, and as cosmological measurements become more precise, whether gravitational lensing observations can constrain the cosmological constant is a debate that deserves to be settled.

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