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Contents

1 Introduction

There has been a debate over the past decade about whether the cosmological constant enters directly into the gravitational lensing equation.

1.1 Previous work

1.2 A note on units and notation

I use a comma to denote partial derivative and an overdot to denote derivative with respect to the affine parameter λ. For example, $x_{,t}$ refers to $\frac{\partial x}{\partial t}$ and $\dot{x} = \frac{dx}{d\lambda}$. Throughout this work I use natural units such that $c = G = 1$.

1.3 Background

In General Relativity, spacetime is described by a metric tensor g.

The Einstein field equations (EFE)

This is the analogue of Poisson's equation in Newtonian gravity.

The motion of a partice is described by a trajectory $x^\mu(\lambda)$

Objects move on a geodesic, which is a generalisation of the notion of "straight lines" to a curved spacetime. The equations can be derived

The geodesic equation is

$$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0 \tag{1.1}$$

where an overdot represents a derivative with respect to the affine parameter λ, and Γ are the Christoffel symbols given by

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\rho}(g_{\rho\alpha,\beta} + g_{\rho\beta,\alpha} - g_{\alpha\beta,\rho}). \tag{1.2}$$

$$g_{\mu\nu}dx^\mu dx^\nu \tag{1.3}$$

2 Gravitational lensing formalism

It is useful to first revise gravitational lensing in a universe without Λ, in a Schwarzschild metric, which is well understood. The Schwarzschild metric, one of the first known solutions to Einstein's field equations, describes the vacuum that lies outside a spherically symmetric distribution of matter. Its line element is given by

$$ds^2 = -\left(1 - \frac{2M}{R}\right)dt^2 + \left(1 - \frac{2M}{R}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.1}$$

where M is the central mass.

Due to spherical symmetry, we can restrict ourselves to the equatorial plane $\theta = \pi/2$ without loss of generality. This metric is asymptotically flat as $r \to \infty$. We can then find the total deflection angle α experienced by a particle that comes in from $r = -\infty$, gets deflected, and travels on towards $r = +\infty$ as

$$\alpha = 2\int_{r_0}^{\infty} \left|\frac{d\phi}{dr}\right| dr - \pi \tag{2.2}$$

where r_0 is the distance of closest approach.

The static nature and spherical symmetry of the Schwarzschild metric implies that there are two constants of motion for any particle traveling in this geometry. These can be obtained from the E-L equation [??]

$$E = \left(1 - \frac{2M}{R}\right)\dot{t}, \;\; L = r^2\dot{\phi}. \tag{2.3}$$

By applying the null condition (Equation 1.3) on the metric, we obtain an expression for $\frac{d\phi}{dr}$

$$\frac{d\phi}{dr} = \pm\frac{1}{r^2}\sqrt{\frac{1}{\frac{1}{b^2} - \left(1 - \frac{2M}{r}\right)\frac{1}{r^2}}} \tag{2.4}$$

where $b = L/E$ is the impact parameter (since $\frac{d\phi}{dr} = \dot{\phi}/\dot{r}$). Integrating this (for a detailed derivation see Keeton and Petters, 2005), we obtain an expression for the bending angle α as a series expansion in M/r_b,

We can define another constant of the motion R

The purpose of expressing the bending angle in terms of R instead of b is that The purpose of doing is this is

3 Description of the Swiss Cheese model

3.1 Spacetime patches

Swiss-cheese models model were first introduced by Einstein and Straus (1945) to investigate the gravitational field of a mass well described by the Schwarzschild metric but embedded in a non-Minkowski background spacetime. Such a model is constructed by removing non-overlapping comoving spheres from the background cheese and replacing them with a metric representing an appropriate condensed spherical mass distribution. This stitching together of two metrics is of course not arbitrary, and matching conditions will impose restriction on the parameters of the two metrics. This will be detailed in the next section.

There are several reasons why this model was chosen. First, this is an exact solution of Einstein's equations, and hence it will allow us to properly investigate the higher order corrections that have been debated in literature. Previously some research (Simpson et al., 2010) has been done using a perturbative approach, but others (Ishak et al., 2010) have contested whether the approximations were valid. Secondly, by putting observers in the homogeneously expanding "cheese", it accounts for observers moving with the Hubble flow, which is a common objection to Rindler and Ishak's use of a static metric (Simpson et al., 2010; Butcher, 2016; Park, 2008; Khriplovich and Pomeransky, 2008). Lastly, this model also takes the finite range of the mass into account by confining the influence of the central mass to the size of the hole.

Previously, this model has been used to investiate the effect of local inhomogeneities on luminosity-redshift relations (Kantowski, 1969; Fleury et al., 2013).

3.1.1 Friedmann-Robertson-Walker geometry

Outside the hole, geometry is described by the Friedmann-Robertson-Walker (FRW) metric, given by

$$ds^2 = -dt^2 + a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{3.1}$$

where a is the scale factor and k represents the curvature. This is the line element for a homogeneous and expanding universe, with general spatial curvature. The scale factor a parametrizes the relative expansion of the universe, such that the relationship between physical distance and comoving distance between two points at a certain cosmic time t is given as

$$d_{\text{physical}} = a(t)d_{\text{comoving}}. \tag{3.2}$$

The scale factor also satisfies the Friedmann equation

$$H^2 \equiv \left(\frac{a_{,t}}{a}\right) = \frac{8\pi G\rho}{3} + \frac{\Lambda}{3} - \frac{k}{a^2} \tag{3.3}$$

where ρ is the energy density of a pressureless fluid and H is the Hubble parameter.

It is common to introduce the cosmological parameters, where a subscript 0 refers to quantities evaluated today:

$$\Omega_m = \frac{8\pi G\rho_0}{3H_0^2},\ \Omega_\Lambda = \frac{\Lambda}{3H_0^2},\ \Omega_k = -\frac{k}{a_0^2 H_0^2} \tag{3.4}$$

and rewrite the Friedmann equation as

$$H^2 = H_0^2\left[\Omega_m\left(\frac{a_0}{a}\right)^3 + \Omega_k\left(\frac{a_0}{a}\right)^2 + \Omega_\Lambda\right]. \tag{3.5}$$

Subsequently in this dissertation instead of working directly with Λ I will work with Ω_Λ instead.

3.1.2 Kottler geometry

In this project we use a Kottler condensation in the Swiss-cheese for the central lensing mass. This is described by a Kottler metric (Kottler, 1918), which is the extension of the famous Schwarzschild metric to include a cosmological constant, given by

$$ds^2 = -f(R)dT^2 + \frac{dR^2}{f(R)} + R^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.6}$$

with

$$f(R) = 1 - \frac{2M}{R} - \frac{\Lambda R^2}{3}, \tag{3.7}$$

where M is the mass of the central object. Unlike the FRW, this metric describes a static spacetime.

3.2 Matching conditions

Two geometries can be matched across the boundary to form a well defined spacetime only if and only if they satisfy the Israel junction conditions [??]. These conditions dictate that the first and second fundamental forms of the two metrics must match on the matching hypersurface Σ, that is, both metrics must induce (i) the same metric, and (ii) the same extrinsic curvature.

and we arrive at the somewhat intuitive result that the central mass M in Kottler space must be equal to the original mass inside the homogeneous comoving sphere of radius r_h

$$M = \frac{4\pi}{3}a^3 r_h^3. \tag{3.8}$$

3.3 Light propagation

Light propagation is governed by the geodesic equation.

Due to spherical symmetry, we can restrict ourselves to the $\theta = \pi/2$ plane without loss of generality.

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