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Contents

1 Introduction

There has been a debate over the past decade about whether the cosmological constant enters directly into the gravitational lensing equation.

Till now, there is still no consensus as to whether Λ contributes to lensing.

1.1 Previous work

We are concerned about whether Λ directly contributes to the bending of light around a concentrated mass. Conventional view, firt put forth by Islam (1983), is that it does not, and classical lensing is correct as it is.

Note that

1.2 Structure of this report

In the remaining portions of this chapter I give an introduction of General Relativity and the basics of light propagation, which lays the mathematical foundation for this work. In chapter 2, I provide an overview of the current established literature on gravitational lensing in a Schwarzschild spacetime where $\Lambda = 0$ and derivation of the key equations, before moving on to the case of a non zero Λ.

The bulk of the work is in chapter 3, where I give the mathematical derivation of the equations which form the basis of this project, some of which do not appear explicitly in literature. In particular, in this chapter I describe the construction and mathematical properties of the Swiss-Cheese model with a Kottler condensation, and based on that, obtain the equations for light propagation in such a universe.

Finally, in chapter 4, I present my numerical results for light propagation in such a universe and discuss their significance in the context of some of the analytical analyses that have been previously done.

1.3 A note on units and notation

I use a comma to denote partial derivative and an overdot to denote derivative with respect to the affine parameter λ. For example, $x_{,t}$ refers to $\frac{\partial x}{\partial t}$ and $\dot{x} = \frac{dx}{d\lambda}$. Throughout this work I use natural units such that $c = G = 1$.

1.4 Background

The essence of General Relativity is very well summarized in the quote by Wheeler (Wheeler et al., 2000, pg.235): "Space-time tells matter how to move, matter-energy tells space-time how to curve.".

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8 \tag{1.1}$$

In Genera Relativity (GR), Einstein's Field Equations (EFEs) describe the relation between matter and the geometry of spacetime.

Einstein field Equa

In General Relativity, spacetime is described by a metric tensor g.

This is the analogue of Poisson's equation in Newtonian gravity.

The motion of a partice is described by a trajectory $x^\mu(\lambda)$

Objects move on a geodesic, which is a generalisation of the notion of "straight lines" to a curved spacetime. The equations can be derived

The geodesic equation is

$$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0 \tag{1.2}$$

where an overdot represents a derivative with respect to the affine parameter λ, and Γ are the Christoffel symbols given by

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\rho}(g_{\rho\alpha,\beta} + g_{\rho\beta,\alpha} - g_{\alpha\beta,\rho}). \tag{1.3}$$

$$g_{\mu\nu}dx^\mu dx^\nu = 0 \tag{1.4}$$

2 Gravitational lensing formalism

It is useful to first revise gravitational lensing in a universe without Λ, in a Schwarzschild metric, which is well understood. The Schwarzschild metric, one of the first known solutions to Einstein's field equations, describes the vacuum that lies outside a spherically symmetric distribution of matter. Its line element is given by

$$ds^2 = -\left(1 - \frac{2M}{R}\right)dt^2 + \left(1 - \frac{2M}{R}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.1}$$

where M is the central mass.

Due to spherical symmetry, we can restrict ourselves to the equatorial plane $\theta = \pi/2$ without loss of generality. This metric is asymptotically flat as $r \to \infty$. We can then find the total deflection angle α experienced by a particle that comes in from $r = -\infty$, gets deflected, and travels on towards $r = +\infty$ as

$$\alpha = 2\int_{r_0}^{\infty} \left|\frac{d\phi}{dr}\right| dr - \pi \tag{2.2}$$

where r_0 is the distance of closest approach.

The static nature and spherical symmetry of the Schwarzschild metric implies that there are two constants of motion for any particle traveling in this geometry. These can be obtained from the E-L equation [??]

$$E = \left(1 - \frac{2M}{R}\right)\dot{t}, \;\; L = r^2\dot{\phi}. \tag{2.3}$$

By applying the null condition (Equation 1.4) on the metric, we obtain an expression for $\frac{d\phi}{dr}$

$$\frac{d\phi}{dr} = \pm\frac{1}{r^2}\sqrt{\frac{1}{\frac{1}{b^2} - \left(1 - \frac{2M}{r}\right)\frac{1}{r^2}}} \tag{2.4}$$

where $b = L/E$ is the impact parameter (since $\frac{d\phi}{dr} = \dot{\phi}/\dot{r}$). Integrating this (for a detailed derivation see Keeton and Petters, 2005), we obtain an expression for the bending angle α as a series expansion in M/r_b,

We can define another constant of the motion R which corresponds to the unperturbed trajectory of light [see diagram ??] by (see Eq. 6 of Ishak, Rindler, Dossett, et al. (2008))

The purpose of expressing the bending angle in terms of R instead of b is that it has been pointed out in literature Hence we use The purpose of doing is this is

3 Description of the Swiss Cheese model

3.1 Spacetime patches

Swiss-Cheese (SC) models model were first introduced by Einstein and Straus (1945) to investigate the gravitational field of a mass well described by the Schwarzschild metric but embedded in a non-Minkowski background spacetime. Such a model is contructed by removing a coming sphere from the homogeneous and replacing it with an inhomogeneous mass distribution. In this case, we use a mass distribution that is vacuum everywhere except for a point mass at the centre. In principle, since the sphere is comoving, multiple spheres can be inserted in the cheese as long as they are initially non-overlapping. In our model, only one hole is needed to model the lens. This stitching of two metrics on the 'cheese'-'hole' boundary is of course not arbitrary, and matching conditions will impose restriction on the parameters of the two metrics. This will be discussed in detail in the following sections.

There are several reasons why this model was chosen. First, this is an exact solution of Einstein's equations which preserves the global dynamics, and hence it will allow us to properly investigate the higher order corrections that have been oft-debated in literature. Previously some research (Simpson et al., 2010)[??] has been done using a perturbative approach, but others (Ishak, Rindler, and Dossett, 2010) have contested whether the approximations were valid. Secondly, by putting observers in the homogeneously expanding "cheese", it accounts for observers moving with the Hubble flow, which is a common objection to Rindler and Ishak's use of a static metric (Simpson et al., 2010; Butcher, 2016; Park, 2008; Khriplovich and Pomeransky, 2008). Lastly, this model also takes the finite range of the mass into account by confining the influence of the central mass to the size of the hole.

Light propagation in SC models has been extensively studied (Szybka, 2011; Vanderveld et al., 2008; Fleury, 2014), but not particularly so in the subject of Lambda's dependence on gravitational lensing. Some of the areas that Swiss-Cheese models have been commonly used include investigating the effect of local inhomogeneities on luminosity-redshift relations (Kantowski, 1969; Fleury et al., 2013) and studying fluctuations in redshift and distance of the cosmic microwave background (Bolejko, 2009; Valkenburg, 2009; Bolejko, 2011).

3.1.1 Friedmann-Robertson-Walker geometry

Outside the hole, geometry is described by the Friedmann-Robertson-Walker (FRW) metric, the simplest homogeneous and isotropic model of the universe. Its line element is given by

$$ds^2 = -dt^2 + a(t)^2\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) \tag{3.1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the metric on a 2-sphere, a is the scale factor, and k represents the curvature. This is the line element for a homogeneous and expanding universe, with general spatial curvature. The scale factor a parametrizes the relative expansion of the universe, such that the relationship between physical distance and comoving distance between two points at a certain cosmic time t is given as

$$d_{\text{physical}} = a(t) d_{\text{comoving}}. \tag{3.2}$$

The scale factor also satisfies the Friedmann equation

$$H^2 \equiv \left(\frac{a_{,t}}{a}\right) = \frac{8\pi G\rho}{3} + \frac{\Lambda}{3} - \frac{k}{a^2} \tag{3.3}$$

where ρ is the energy density of a pressureless fluid and H is the Hubble parameter.

It is common to introduce the cosmological parameters, where a subscript 0 refers to quantities evaluated today:

$$\Omega_m = \frac{8\pi G\rho_0}{3H_0^2},\ \Omega_\Lambda = \frac{\Lambda}{3H_0^2},\ \Omega_k = -\frac{k}{a_0^2 H_0^2} \tag{3.4}$$

and rewrite the Friedmann equation as

$$H^2 = H_0^2\left[\Omega_m\left(\frac{a_0}{a}\right)^3 + \Omega_k\left(\frac{a_0}{a}\right)^2 + \Omega_\Lambda\right]. \tag{3.5}$$

Subsequently in this report instead of working directly with Λ I will work with Ω_Λ instead.

3.1.2 Kottler geometry

In this project we use a Kottler condensation in the Swiss-Cheese for the central lensing mass. This is described by a Kottler metric (Kottler, 1918), which is the extension of the famous Schwarzschild metric to include a cosmological constant, given by

$$ds^2 = -f(R)dT^2 + \frac{dR^2}{f(R)} + R^2 d\Omega^2 \tag{3.6}$$

with

$$f(R) = 1 - \frac{2M}{R} - \frac{\Lambda R^2}{3}, \tag{3.7}$$

where M is the mass of the central object. Unlike the FRW, this metric describes a static spacetime.

3.2 Matching conditions

Two geometries can be matched across the boundary to form a well defined spacetime only if and only if they satisfy the Darmois-Israel junction conditions (Darmois, 1927; Israel, 1966). These conditions dictate that the first and second fundamental forms of the two metrics must match on the matching hypersurface Σ, that is, both metrics must induce (i) the same metric, and (ii) the same extrinsic curvature.

3.2.1 Continuity of the induced metric

We match the FRW and Kottler metrics on a surface of a comoving 2-sphere, Σ, which is defined by $r = r_h =$ constant in FRW coordinates and $R = R_h(T)$ in Kottler coordinates.

The induced metric is the quantity

$$h_{ab} = g_{\alpha\beta} j_a^\alpha j_\beta^\alpha \tag{3.8}$$

where j_a^α is defined as

$$j_a^\alpha = \frac{\partial \bar{X}^\alpha}{\partial \sigma^a}. \tag{3.9}$$

Here we have introduced X^α to represents coordinates of the original metric. We define σ^a to be natural intrinsic coordinates for Σ, and $\bar{X}^\alpha(\sigma^a)$ is the parametric equation of the hypersurface.

More concretely, using the coordinates defined previously in Equation 3.1, these quantities are

$$X^\alpha = \{t, r, \theta, \phi\} \tag{3.10a}$$
$$\sigma^a = \{t, \theta, \phi\} \tag{3.10b}$$
$$\bar{X}^\alpha(\sigma^a) = \{t, r_h, \theta, \phi\}. \tag{3.10c}$$

Similarly, in the Kottler region, we have

$$X^\alpha = \{T, R, \theta, \phi\} \tag{3.11a}$$
$$\sigma^a = \{T, \theta, \phi\} \tag{3.11b}$$
$$\bar{X}^\alpha(\sigma^a) = \{T, R_h(T), \theta, \phi\}. \tag{3.11c}$$

Using these definitions, the 3-metric induced by the FRW geometry on Σ is

$$ds_\Sigma^2 = -dt^2 + a^2(t) r^2 d\Omega^2, \tag{3.12}$$

while the induced metric on the Kottler metric is

$$ds_\Sigma^2 = -\kappa^2(T) dT^2 + R_h^2(T) d\Omega^2, \tag{3.13}$$

where

$$\kappa \equiv \sqrt{\frac{f^2[R_h(T)] - R_{h,T}^2(T)}{f[R_h(T)]}}. \tag{3.14}$$

Equating the components of Equation 3.12 and Equation 3.13, we obtain the following:

$$R_h(T) = a(t)r, \tag{3.15}$$

$$\frac{dt}{dT} = \kappa(T). \tag{3.16}$$

These two relationships relate the radial and time coordinates of the two metrics respectively.

3.2.2 Continuity of the extrinsic curvature

The second condition equates extrinsic curvature of the two geometries. By definition, the extrinsic curvature K_{ab} of a hypersurface is given by

$$K_{ab} = n_{\alpha;\beta} j_a^\alpha j_a^\beta \tag{3.17}$$

where n_μ is the unit vector normal to Σ, j is as defined previously in Equation 3.9,and the semicolon notation ";" denotes a covariant derivative, for example, $n_{\alpha;\beta} = \nabla_\beta n_\alpha$. For any vector V^ν, the covariant derivative is defined as

$$\nabla_\mu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho. \tag{3.18}$$

For a hypersurface defined by a function $q = 0$, the unit vector normal to it is

$$n_\mu = \frac{q_{,\mu}}{\sqrt{g^{\alpha\beta} f_{,\alpha} f_{,\beta}}}. \tag{3.19}$$

In our case $q = r - r_h$ in FRW coordinates and $q = R - R_h(T)$ in Kottler coordinates. For example, the unit vector in the FRW region is trivial to calculate, and we get $n_\mu^{(\mathrm{FRW})} = \delta^r_\mu / a$. Applying this formula, the extrinsic curvature induced by the FRW geometry is

$$K_{ab} dx^a dx^b = \frac{a(t)r}{\sqrt{1 - kr^2}} d\Omega^2 \tag{3.20}$$

while the extrinsic curvature induced by the Kottler geometry is

$$K_{ab} dx^a dx^b = \frac{1}{\kappa}\left[R_{h,tt} + \frac{f'}{2f}(f^2 - 3R_{h,t}^2)\right] dT^2 + \frac{R_h f}{\kappa} d\Omega^2 \tag{3.21}$$

where $f' = \partial f / \partial R$, and all quantities are evaluated at $R = R_h(T)$.

Equating the components of Equation 3.20 and Equation 3.21, we obtain

$$\frac{R_h f}{\kappa} = \frac{a(t)r}{\sqrt{1 - kr^2}} \tag{3.22}$$

and

$$R_{h,tt} + \frac{f'}{2f}(f^2 - 3R_{h,t}^2) = 0. \tag{3.23}$$

The second equation is provided for completeness although it is not needed for subsequent derivations.

3.2.3 Consequences on the property of the hole

Combining Equation 3.22 with Equation 3.14, we can eliminate κ. We can also replace $R_{h,T}$ using the relation obtained in Equation 3.15, since

$$\frac{dR_h}{dT} = \frac{d(ar)}{dT} = \frac{da}{dt}\frac{dt}{dT}r. \tag{3.24}$$

where da/dt is given by the Friedmann equation 3.3.

Following through the algebra, we arrive at the somewhat intuitive result that both regions must have the same cosmological constant Λ and that the central mass M in the Kottler region must be equal to the original mass inside the homogeneous comoving sphere of radius r_h

$$M = \frac{4\pi}{3}a^3 r_h^3. \tag{3.25}$$

The last thing we need from the boundary conditions is the relate the tangent vectors between the two metrics. The continuity of the metric, imposed by the first junction condition, implies that the connection does not diverge across the boundary. Therefore, light is not deflected as it crosses the boundary and we just need to convert the components of the tangent vector between the two coordinate systems. To obtain $\dot{R}$ in terms of FRW tangent vectors $\dot{r}$ and $\dot{t}$, we differentiate Equation 3.15 and substitute a_t with the Friedmann equation Equation 3.3. Keeping in mind the boundary conditions, we get an expression for $\dot{R}$. The angular coordinates and angular tangent vectors are unchanged moving from Kottler to FRW coordinates, and vice versa. With $\dot{R}$ and $\dot{\phi}$, $\dot{T}$ then can be easily obtained from the null condition Equation 1.4. The result is

$$\dot{T} = \frac{1}{f}\sqrt{1-kr^2}\dot{t} + \frac{a}{f\sqrt{1-kr^2}}\sqrt{\frac{2M}{ar} - kr^2 + \frac{\Lambda}{3}a^2r^2}\dot{r} \tag{3.26a}$$

$$\dot{R} = \sqrt{\frac{2M}{ar} - kr^2 + \frac{\Lambda}{3}a^2r^2}\dot{t} + a\dot{r} \tag{3.26b}$$

$$\dot{\phi} = \dot{\phi} \tag{3.26c}$$

$$\dot{\theta} = \dot{\theta} \tag{3.26d}$$

where for completeness I have also given the trivial relations between the angular tangent vectors. The quantities above are all evaluated at the boundary of the hole. This result is given for flat space in Fleury et al. (2013) and Schücker (2009), but here it has been extended to allow for arbitrary spatial curvature. The reverse transformation is easily obtained by inverting the Jacobian from above.

In summary, given a FRW spacetime with pressureless matter and a cosmological constant Λ, a spherical hole, whose geometry is described by the Kottler metric, can be constructed which contains a constant mass $M = 4\pi\rho a^3 r_h^3/3$ at its centre. The geometry resulting from combining the two metrics at the boundary is an exact solution of the Einstein field equations. Applying the boundary conditions, we can obtain all the necessary transformations needed for continuation of light propagation at the boundary.

3.3 Light propagation

Light propagation is governed by the geodesic equation.

Due to spherical symmetry, we can restrict ourselves to the $\theta = \pi/2$ plane without loss of generality.

4 Results and discussion

A graph of results when we keep the lensing mass M constant and vary Ω_Λ can be seen in [fig??]. On the y-axis, we have plotted the the deviation of α as a fraction of the standard FRW lensing case (given by [eq??]), in order to put them on the same scale. [State te mass and zlens parameters]

By varying the integration step size, we are able to estimate numerical errors on the integration. For a certain step size, we group the results obtained from the vicinity of step sizes together and find the variance in bending angles in that range. Fig [??] shows how the α obtained varies with step size. As is expected, the precision increases as we reduce the step size.

Our results seem to follow the trend of Kantowski's predictions most closely, with a gap that reduces towards higher Λ. A possible explanation of this gap can be found by examining the neglected higher order term in Kantowski's predicted bending angle [eq??]. When $\Lambda = 0$, the ratio of this term to the leading order $(4M/r_0^2)\cos^3\tilde{\phi_1}$ term is of the same order of magnitude as the fractional deviation of our numerical results from Kantowski's predictions. As is expected, this ratio decreases as Λ when mass is kept constant, as can be seen from [fig??].

From the graph, we can see that even for the $\Lambda = 0$ case there is an offset between the numerical Swiss-Cheese result and the FRW prediction. Qualitatively, this is due to the fact that conventional lensing analyis assumes a mass superimposed on the homogeneous background, and this mass has infinite range. However, in the Swiss-Cheese model, the influence of the mass is limited, and bending stops once it leaves the Kottler hole. This is the main effect that Kantowski quantified in his paper Ronald Kantowski et al. (2010). This then begs the question of which model is a more accurate description of our physical universe, but this is not our primary concern. We are concerned about whether Λ has an influence on this effect.

There are a few different factors at play here. In discussing the results of this numerical integration, let us take a step back to look at the specific parts of ray-tracing that have a Λ-dependence. These are:

1. The size of the hole. This is governed by Equation 3.25. In flat space, increasing Ω_Λ implies decreasing Ω_Λ, which corresponds to the matter density of the universe. If we are to keep the mass constant, the hole size would have to increase as we increase Ω_Λ.

2. The rate of expansion of the hole in static Kottler coordinates, given by [ref??].

3. The Jacobian at the boundary, given by [??]

The first effect does not seem to be a truly direct Λ effect, merely a side effect that in a flat universe, changing Ω_Λ must imply a change in matter density, but ultimately, it is the size of the hole that is the true determining factor.

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