Networks Project: Numerical analysis of the Barabási-Albert Model

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Abstract: We investigate the behaviour of the Barabási-Albert Model.

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1 Theory

The master equation that describes the evolution of the BA model is given by

$$n(k, t+1) = n(k, t) + m\Pi(k-1, t)n(k-1, t) - m\Pi(k, t)n(k, t) + \delta_{k,m} \tag{1}$$

where k is the total degree of a vertex, $n(k, t)$ is the number of nodes at time t with total degree k, and the probability Π for choosing the existing vertex depends on the model. In the pure preferential attachment model, we choose an existing edge with $\Pi_{pa} \propto k$, which after normalizing gives $\Pi_{pa} = k/2E(t)$ where $E(t)$ is the number of edges and $2E(t)$ is the normalization constant corresponding to the total degree of the network. Assuming $E(0) = mN(0)$, the number of edges at a given time t is given by $E(t) = mN(t)$, so we get $\Pi = k/2mnN(t)$. Since we are concerned with the degree distribution of the model at large t, we consider the long-time ansatz $n(k, t) \rightarrow N(t)p_\infty(k)$.

Substituting these terms into the master equation, we obtain

$$p_\infty(k) = \frac{1}{2}[(k-1)p_\infty(k-1) - kp_{infty}(k)] + \delta_{k,m} \tag{2}$$

It is clear that $p_\infty(k < m) = 0$, since m edges are added at every stage. So there are 2 cases to consider when solving for the above equation: $k = m$ and $k > m$.

We first consider the case when $k > m$. In this case, $\delta_{k,m} = 0$ and we can rearrange Equation 2 to get

$$\frac{p_\infty(k)}{p_\infty(k+1)} = \frac{k-1}{k+2} \tag{3}$$

To solve this equation, we can substitute in a trial solution of the form

$$f(z) = A\frac{\Gamma(z+1+a)}{\Gamma(z+1+b)} \tag{4}$$

where $\Gamma(z)$ is the Gamma function, which is an extension of the factorial function, with its argument shifted by one, to all real and complex nnumbers except the non-positive integers. Its central property is that

$$\Gamma(z+1) = z\Gamma(z), \ \Gamma(1) = 1. \tag{5}$$

Substituting the trial solution in Equation 4 gives

$$\frac{A\Gamma(z+1+a)}{\Gamma(z+1+b)} \times \frac{\Gamma(z+b)}{A\Gamma(z+a)} \tag{6}$$

which indeed simplifies to give $(z+a)/(z+b)$, using the the property in Equation 5 that $\Gamma(z+a+1)/\Gamma(z+a) = z+a$.

Substituting $a = -1$ and $b = 2$, we get the solution for Equation 3 in terms of A and the Gamma function:

$$p_\infty(k) = A\frac{\Gamma(k)}{\Gamma(k+3)} \tag{7}$$

which simplifies to

$$p_\infty(k) = \frac{A}{k(k+1)(k+2)}. \tag{8}$$

For the second case of $k = m$, Equation 2 becomes

$$p_\infty(m) = \frac{1}{2}[(m+1)p_\infty(m-1) - mp_\infty(m)] + 1. \tag{9}$$

However, we already know that $p_\infty(k < m) = 0$, that is, $p_\infty(m-1) = 0$. Using this, and rearranging Equation 9, we get

$$p_\infty(m) = \frac{2}{m+2}. \tag{10}$$

Substituting $k = m$ and Equation 10 into Equation 8, we get

$$\frac{A}{m(m+1)(m+2)} = \frac{2}{m+2}, \tag{11}$$

giving us the constant A as

$$A = 2m(m+1). \tag{12}$$

For this constant to be physically reasonable, we need to check that the probability satisfies normalization, that is, we need to prove

$$\sum_{k=m}^{\infty} p_\infty(k) = 2m(m+1)\sum_{k=m}^{\infty}\frac{1}{k(k+1)(k+2)} = 1. \tag{13}$$

The term in the summation of Equation 13 can be expanded as a partial fraction:

$$\sum_{k=m}^{\infty}\frac{1}{k(k+1)(k+2)} = \sum_{k=m}^{\infty}\frac{1}{2k} - \sum_{k=m}^{\infty}\frac{1}{k+1} + \sum_{k=m}^{\infty}\frac{1}{2(k+2)} \tag{14}$$

By writing out the first few terms of each summation, we can see that most terms cancel:

$$\begin{array}{cccccccccc}
\frac{1}{2m} & - & \frac{1}{m+1} & + & \cancel{\frac{1}{2(m+2)}} & & & & & \\
& + & \frac{1}{2(m+1)} & - & \cancel{\frac{1}{m+2}} & + & \cancel{\frac{1}{m+3}} & & & \\
& & & + & \cancel{\frac{1}{2(m+2)}} & - & \cancel{\frac{1}{m+3}} & + & \frac{1}{2(m+4)} & \\
& & & & & + & \cancel{\frac{1}{2(m+3)}} & - & \frac{1}{m+4} & + \ \frac{1}{2(m+5)} \\
& & & & & & & + & \dots &
\end{array} \tag{15}$$

and from the remaining terms we get the relation in Equation 13

$$\sum_{k=m}^{\infty} p_\infty(k) = 2m(m+1)\left(\frac{1}{2m} - \frac{1}{m} + \frac{1}{2(m+1)}\right) = 2m(m+1)\frac{1}{2m(m+1)} = 1 \tag{16}$$

Hence, we can confirm that the complete exact solution for the probability distribution in the long time limit is

$$p_\infty(k) = \frac{2m(m+1)}{k(k+1)(k+2)} \tag{17}$$