APPM 2360 project 1: Radioactive Decay

Recitation Section: 242 Wanqi Yu
Recitation Section: 281 Zongyi Huang
Recitation Section: 253 Lingyin Lu

I.   Introduction

Radioactive decay is a natural phenomenon that can be described by first order differential equations. In this report, we will study the radioactive decay chains with numerical solutions and analytical methods with graphs.

The radioactive decay will happen with the time because atomic combinations of protons and neutrons are unstable. And the half-life is a significant part in radioactive decay. The half-time is the time value when the half amount of atoms. In the radioactive decay process, we can define the "rate of decay" as the number of atoms that decay per time period is portional to the number of atoms present. The differential equation model of radioactive decay will be a rate of change with conservation laws. The rate of change of substances can be written by the following:

$$Rate \ of \ change = Rate \ in - Rate \ out \qquad (1)$$

And the element A is undergoing radioactive decay. We define the $N_A(t)$ as the number of atoms present at any time. So, the equation depends on time because particles will change during the decay process with the "rate constant" k_A . The rate of change equations can be written as:

$$\tfrac{d}{dt}N_A(t) =- k_A N_A(t) \qquad (2)$$

II. Analytical calculations for radioactive decay

1.   The differential equation is $\tfrac{d}{dt}N_A(t) =- k_A N_A(t)$. The initial condition is $N_A(0) = A_0$. So, when t is equal to 0, the N_A is equal to A_0 . It means that $N_A'(t) = - k_A N_A(t)$. Using the integrating factor method to solve this IVP equation as the following:

$$N_A'(t) =- k_A N_A(t)$$

$$\int \tfrac{N_A'(t)}{N_A(t)} dt = \int - k_A dt$$

$$\int \tfrac{1}{N_A} dN_A =- k_A t + c$$

$$ln \left| N_A \right| =- k_A t + c$$

Multiply by e with both 2 sides:

$$N_A = ce^{-k_A t}$$

And the initial condition is t=0, $N_A = A_0$:

So, we get   c= A_0

The answer is: $N_A(t) = A_0 e^{-k_A t}$

2. a). Solve for k_A

From the above question, we get $N_A(t) = A_0 e^{-k_A t}$ equation. And in this part, k_A is the "rate constant" in this equations. So, the result of k_A can be solved by following steps:

$$N_A(t) = A_0 e^{-k_A t}$$
$$\frac{N_A(t)}{A_0} = e^{-k_A t}$$

Take ln() for both sides:

$$ln(\frac{N_A(t)}{A_0}) = ln\ e^{-k_A t}$$
$$-k_A t = ln(\frac{N_A(t)}{A_0})$$
$$k_A = \frac{-ln(\frac{N_A(t)}{A_0})}{t}$$

b). Solve for k_A **with** $N_A(t_{A,1/2}) = \frac{1}{2}A_0$

From question 2a), we get the general expression of $k_A = \frac{-ln(\frac{N_A(t)}{A_0})}{t}$.

And the condition $N_A(t_{A,1/2}) = \frac{1}{2}A_0$, it represents that this time value should be the "half-life" time. And in this case, the amount of element A is half of the initial amount: A_0.

So, the k_A can be solved with following steps:

$$N_A(t_{A,1/2}) = \frac{1}{2}A_0$$
$$\frac{1}{2}A_0 = A_0 e^{-k_A t_{A,1/2}}$$

A_0 can be canceled,

$$\frac{1}{2} = e^{-k_A t_{A,1/2}}$$
$$ln(\frac{1}{2}) = -k_A t_{A,1/2}$$

So, the result of k_A is:

$$k_A = \frac{-ln(\frac{1}{2})}{t_{A,1/2}} = \frac{ln(2)}{t_{A,1/2}}$$

c) The units of the proportionality constant k_A should be the number of atoms that change during decay process or the number of atoms that decay per time period.

And now, $t_{A,1/2} = 5ln(2)$. So, the value of $k_A = \frac{-ln(\frac{1}{2})}{t_{A,1/2}} = \frac{ln(2)}{t_{A,1/2}}$ can be written as:

$$k_A = \frac{-ln(\frac{1}{2})}{5ln(2)} = \frac{ln(2)}{5ln(2)} = \frac{1}{5} = 0.2$$

3. a) In this part, the initial amount of element A is $A_0 = 15,000$ atoms. And then, we will use Euler's method to solve question 2 for the step sizes: $h_1 = 1, h_2 = 0.1, and\ h_3 = 0.01$.

In this case, the equation can be described as: $N_A(t) = 15000e^{-\frac{1}{5}t}$. The analytical graphs of Euler's method are shown as figure 1, figure 2, and figure 3.
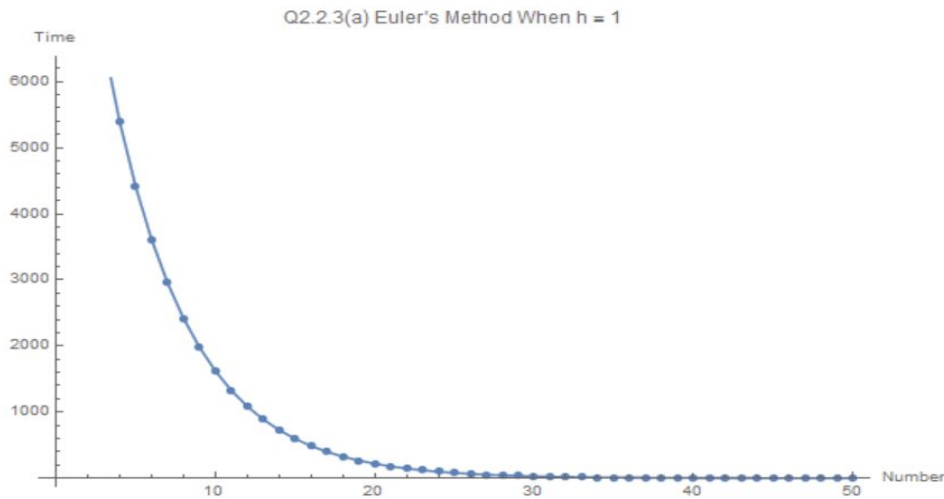

Figure 1: the numerical solution by Euler's Method with step size h=1

The figure 2 is the numerical solution by Euler's Method with step size h=0.1.
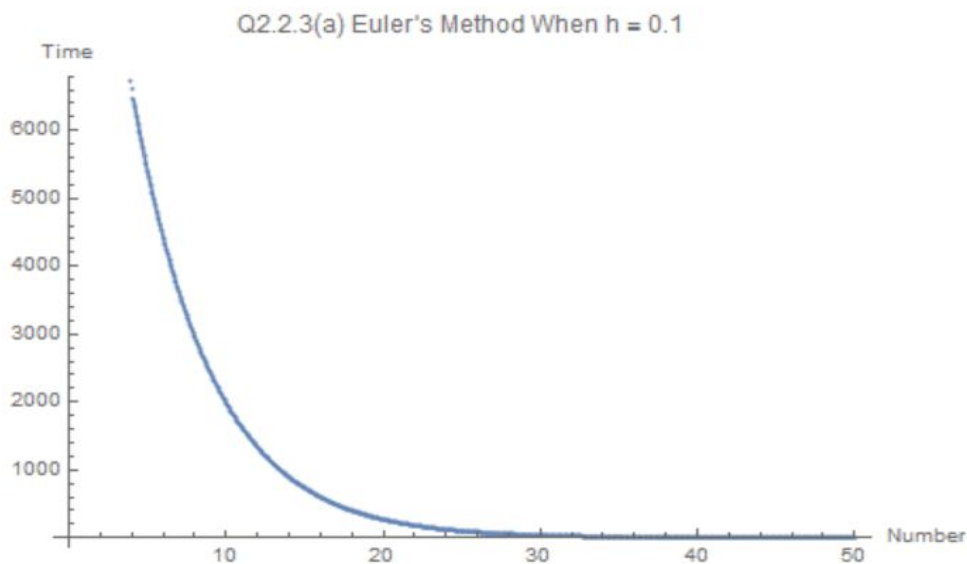

Figure 2: Euler's method when h=0.1

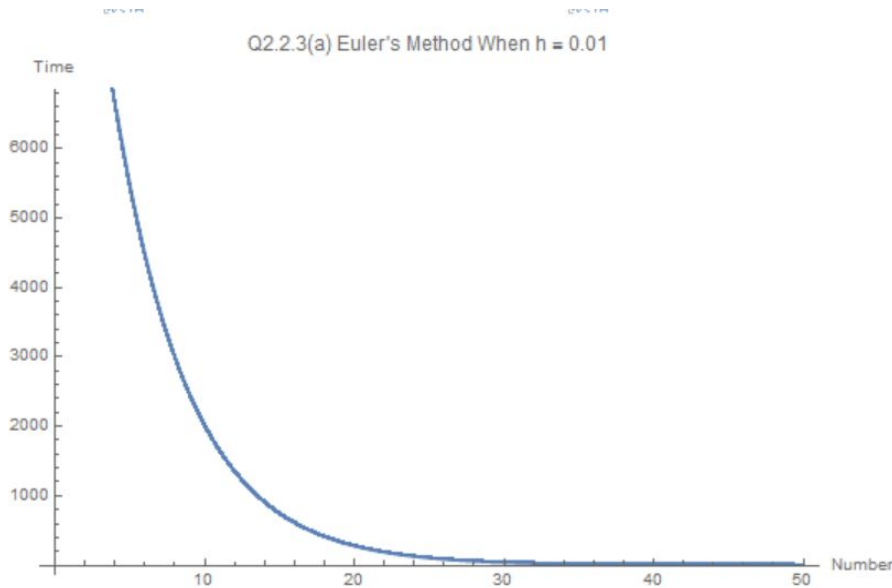The figure 3 shows the numerical solution by Euler's Method with step size h=0.01.

Figure 3: Euler's Method when h=0.01

b) <u>The absolute error</u> can be defined as:

$$Abs.Error = |Exact\ Solution - Approximate\ solution|$$

And the range of t is from 0 to 50. The errors will decrease with the increasing of t values. When step size h is smaller, the numerical solution will be closer with the exact solution. When h=1, the absolute error is the largest one. When h=0.01, the absolute error is the smallest one. It is almost overlapping with exact solution. So, the absolute errors of the numerical solutions are shown in figure 4.
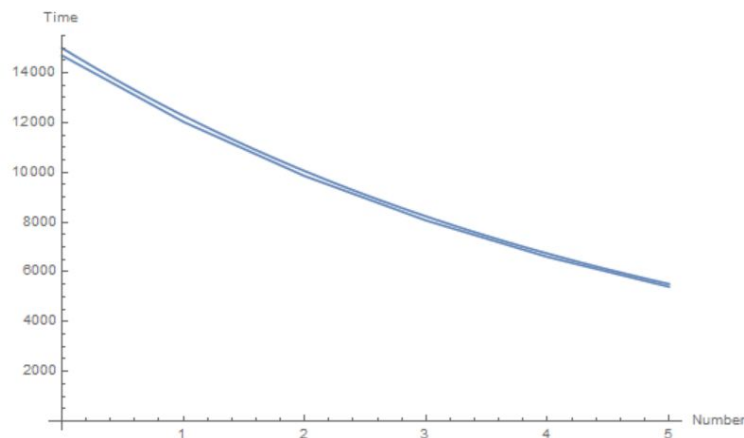

Figure 4: the absolute errors of numerical solutions for the t range

<u>c)</u> The numerical solutions will gradually approximate the exact solution. The smaller step size, the more accurate of the approximation. And the required time to find numerical solution will increase with the more accuracy with Euler's

method. When the step size is equal to 0.01, it will take almost 5000 times for errors. It needs large amount of calculations and time. When the step size is equal to 1, it is not very accurate because of the larger step size value, even it's the fastest one. So, h=0.1 is the best balance of accuracy and efficiency in these three step sizes.

4. A sample has completed decay away

According to the original function $N_A(t) = 15000e^{-\frac{1}{5}t}$, Assume atoms decay from 15000 to 1, let $N_A(t) = 1$ atom, we can get t = 48 seconds. In this case, we determine that sample of A has completely decayed away.

III. Analytical calculations for decay chain

1. Set-up differential equation for $N_B(t)$

For decay chain, the unstable element can take steps into a stable species during the decay process. And the setting up differential equations for species in the chain:

$$N'(t) = Rate\ in - Rate\ out \qquad (3)$$

So, the differential equation for $N_B(t)$ use the rate from A state to B state can be described by: $\frac{dN_{B(t)}}{dt} = k_A(N_A(t)) - k_B(N_B(t))$

2. Solve for $N_B(t)$ with integrating factor method

From the above question, the equation is :

$$N_B'(t) = k_A(N_A(t)) - k_B(N_B(t))$$

$$N_B'(t) + k_B(N_B(t)) = k_A(N_A(t))$$

By integrating factor method:

$$u(t) = e^{\int k_B dt} = e^{k_B t}$$

$$e^{k_B t}(N_B(t)) = \int e^{k_B t} k_A A_0 e^{-k_A t} dt$$

$$e^{k_B t}(N_B(t)) = k_A A_0 \int e^{k_B t - k_A t} dt$$

$$e^{k_B t}(N_B(t)) = \frac{k_A A_0}{k_B - k_A} e^{k_B t - k_A t} + c$$

Divide $e^{k_B t}$ with both sides:

$$N_B(t) = \frac{k_A A_0}{k_B - k_A} e^{-k_A t} + c e^{-k_B t}$$

And the initial condition for t=0, $N_B(t) = N_B(0) = 0$

$$N_B(0) = \frac{k_A A_0}{k_B - k_A} e^0 + c e^0$$

$$N_B(0) = \frac{k_A A_0}{k_B - k_A} + c$$

So, the $N_B(t) = \frac{k_A A_0}{k_B - k_A} (e^{-k_A t} - e^{-k_B t})$

3. Solve for $N_C(t)$

The differential equation for the $N_C(t) = Rate\ in - Rate\ out$. Since there is no rate out in C state. So, the differential equation of $N_C(t)$ can be written as:

$$\frac{dN_C(t)}{dt} = rate\ in = k_B N_B(t)$$

$$N_C'(t) = k_B \frac{k_A A_0}{k_B - k_A} (e^{-k_A t} - e^{-k_B t})$$

$$\int N_C'(t) dt = k_B \frac{k_A A_0}{k_B - k_A} \int (e^{-k_A t} - e^{-k_B t}) dt$$

$$N_C(t) = \frac{k_B k_A A_0}{k_B - k_A} (\frac{-1}{k_A} e^{-k_A t} - \frac{-1}{k_B} e^{-k_B t}) + c$$

When the initial condition is $N_C(0)$, it shows that t is equal to 0, and there is no C substance:

$$N_C(0) = \frac{k_B k_A A_0}{k_B - k_A} (\frac{-1}{k_A} - \frac{-1}{k_B}) + c$$

$$c = -\frac{k_B k_A A_0}{k_B - k_A} (\frac{-1}{k_A} - \frac{-1}{k_B}) = \frac{k_B A_0}{k_B - k_A} - \frac{k_A A_0}{k_B - k_A}$$

So, the equation is $N_C(t) = \frac{k_B A_0}{k_B - k_A} (1 - e^{-k_A t}) + \frac{k_A A_0}{k_B - k_A} (e^{-k_B t} - 1)$

4. The amount for B substance increases at first time period until a peak(local maximum), and then, the B species amount decreases as t approaches infinity. Because there are no B substance at beginning of the decay chain for B; later, the B substance begins to change to C substance, so the amount of B substance decreases with t increasing. The $N_B(t)$ will have a local maximum because the "rate in" is larger than "rate out" at first, and then, "rate in" is smaller than "rate out". Therefore, $N_B'(t)$ will change from positive to negative.

The condition is that $t_{B,\frac{1}{2}} = 15ln(2)$, and $N_B(15ln(2)) = \frac{1}{2}B_0$

And the k_B can be solved by:

$$N_B(t) = B_0 e^{-k_B t}$$

$$t_{B,\frac{1}{2}} = 15ln(2) \Rightarrow \quad N_B(15ln(2)) = \frac{1}{2}B_0$$

$$\frac{1}{2}B_0 = B_0 e^{-k_B t}$$

$$k_B = \frac{1}{15}$$

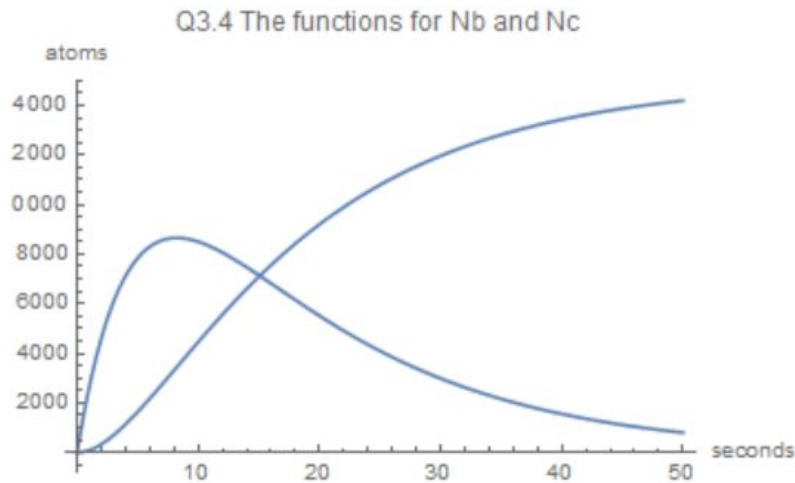And this relationship is shown as figure 5.

Q3.4 The functions for Nb and Nc

Figure 5: the graphs of $N_B(t)$ and $N_C(t)$

5.-- 6. Set-up differential equations $N_A(t), N_B(t)$, and $N_C(t)$ equations in modified scenario

As previous questions described as: $N'(t) = Rate\ in - Rate\ out$

(1). $N_A'(t) = P - k_A(N_A(t))$

$N_A'(t) + k_A(N_A(t)) = P$; P is constant and positive

$N_A(t) = \frac{P}{k_A} + \frac{C_A}{e^{kAt}}$

(2). As the equation which is written from previous question:

$N_B'(t) = k_A(N_A(t)) - k_B(N_B(t))$

By using the integrating factor method;

$N_B'(t) + k_B(N_B(t)) = k_A(N_A(t))$

$\int (N_B'(t) + k_B(N_B(t)))dt = \int k_A(N_A(t))dt$

$e^{k_B t}(N_B(t)) = \frac{P}{k_B}e^{k_B t} + \frac{C_A}{k_B - k_A}e^{-k_A t} + c_B$

So, $N_B(t) = \frac{P}{k_B} + \frac{C_A e^{-k_A t}}{k_B - k_A} + \frac{c_B}{e^{k_B t}}$

(3). And at the state for C, C is stable, it means that $Rate\ out = 0$

So, the equation which can be written from previous question:

$N_C'(t) = k_B N_B(t)$

$N_C'(t) = P + \frac{k_B c_A e^{-k_A t}}{k_B - k_A} + \frac{k_B c_B}{e^{k_B t}}$

$\int N_C'(t)dt = \int (P + \frac{k_B c_A e^{-k_A t}}{k_B - k_A} + \frac{k_B c_B}{e^{k_B t}})dt$

$N_C(t) = Pt - \frac{k_B c_A e^{-k_A t}}{k_A(k_B - k_A)} - c_B e^{k_B t} + c_c$

7. In this case, the production rate of A is $P = 5000\ atoms/s$; $k_A = \frac{ln(2)}{5ln(2)} = \frac{1}{5} = 0.2$; $k_B = \frac{1}{15}$.

   And then we will use Euler's method with a step size $h = 0.01$ to solve the differential equations for $N_A(t)$, $N_B(t)$, $and\ N_C(t)$ in the range $t \in [0, 100]$ seconds. From the graphs, it shows that the amount of substance A and B will increase in the early time range, and then, the increasing speed is smaller and smaller; it will gradually approach to 0. So, after some decay changing, the substance A and B will be changed from unstable to stable. The relationship is shown as following figure 6.
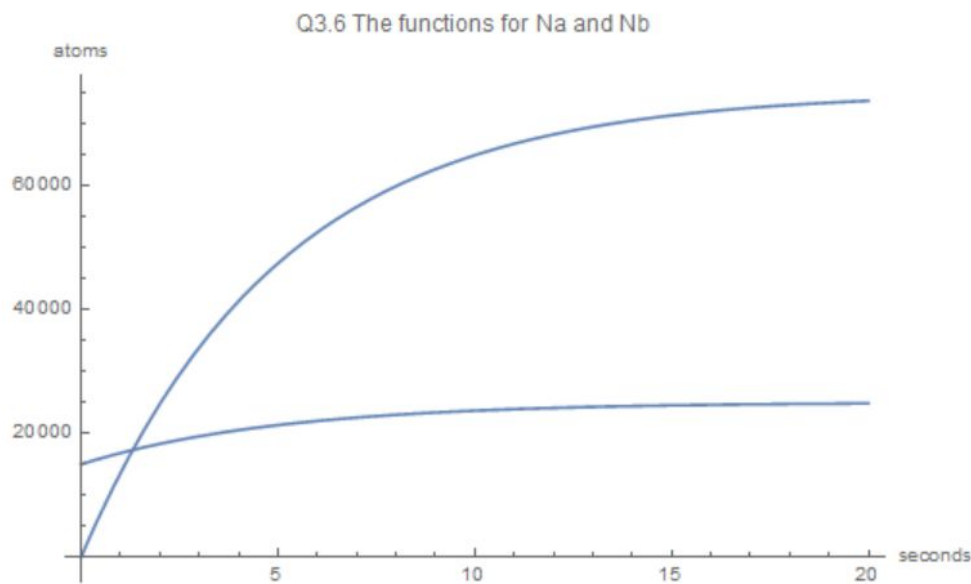

Figure 6: The graphs for substance A and substance B

The upper line is the graph of substance A. And the other one is the graph of substance B.

8. For C substance, the rate out is 0. So, the C substance is stable. The value of $N_C(t)$ will increase when t is approaching to infinity. And the graph of $N_C(t)$ is shown as figure 7 :
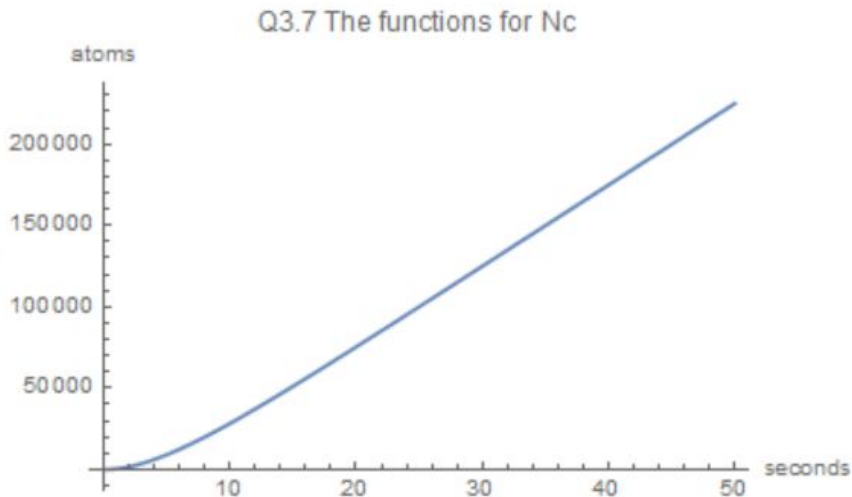
Q3.7 The functions for Nc

Figure 7: The behavior of substance C

9. The different types of different elements will have the different radioactive decay. And radioactive decay can be considered as the emission of energy of element radiation. And most of those unstable elements are always radioactive. So, the type of elements can affect the radioactive decay and decay chains. And previous models are only simple assumptions with radioactive decay. There are also some products by alpha, gamma and beta radiation. So, the total behaviors cannot easily be calculated by this model system because of the informative changes and influences by other sources that is difficult to calculate.

IV. Conclusion
In this project, we use analytical methods to calculate the results. And the numerical method results are similar with analytical method answers. We can use conversation laws and differential equations to describe the radioactive decay and decay chains. During the calculation, we use the Euler's method to approximate our solutions. And the smaller step size is, the more accurate result is. And for three chosen step size, h=0.1 is the best balance of accuracy and efficiency. And for the decay differential problems, the integrating factor method is a good way to get the original solutions.

Reference

1. Farlow, Jerry. *Differential Equations & Linear Algebra.* Prentice Hall, Harlow, England;Upper Saddle River, NJ;, 2007.