A Comparative Study of Algorithms and Proofs in the Nine Chapters on the Mathematical Art and Euclid's Elements

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Abstract

This paper presents a comparative study between the foundations of ancient Eastern and Western mathematics. The goal is a better contextual understanding of the independent development of two major systems of mathematical knowledge. The primary texts to be studied will be *The Nine Chapters on the Mathematical Art* and *Euclid's Elements*. This paper offers an analysis of both the algorithms and proofs presented. The methodology will be to specifically analyze algorithms and proofs put forward by both texts that address identical problems and investigate their respective methodologies, benefits and limitations. After analysis, it can be concluded that early Western mathematics was defined by a proof-first epistemic approach, whereas early Eastern mathematics was defined by an algorithm-first pragmatic approach.

1. Introduction

The Nine Chapters on the Mathematical Art (hereby referred to in this papers as *Nine Chapters*) is well regarded as the canonical text of Ancient Chinese Mathematics. References to it throughout history indicate that it was the dominant authoritative text on mathematics not only in China but throughout Asia until around the 14th Century CE. It served not only as the leading textbook for education in mathematics, but also as the cornerstone work upon which other mathematical work was built. In this sense, it served a role similar to that of Euclid's *Elements* (Boyer 424).

Euclid's Elements (referred to in this paper as *Elements*) is often regarded as the foundation of modern mathematics. For more than 2000 years after its creation, it was regarded as the cornerstone text on the fundamental principles of mathematics, including an extensive discourse on elementary arithmetic, geometry and algebra. (Boyer 94) The Greek philosopher Proclus likened the relationship between *Elements* and the rest of Mathematics to that between the letters of the alphabet and language itself. One of its greatest innovations and the source of much of its praise was Euclid's use of the 'axiomatic method,' which has now established itself at the core of modern mathematics (Mueller 290).

The suitability of these two texts for a comparative study if a product of the fact that they were the latest texts thought to have been developed in isolation, before trade pathways opened up and knowledge was exchanged. Thus we can examine the independent development of two of the major ancient systems of mathematical knowledge through our comparative study of Element's and the *Nine Chapters*.

2. Historical Background

2.1 *Nine Chapters*

The *Nine Chapters* is difficult to date as it was originally produced on bamboo strips, none of which have survived. Most historians estimate that the problems and methods in the text were completed around 100B.C., with further commentary added later (Chelma 424).

In total, the book includes 246 problems divided into chapters based on subject matter. Most of the text focuses on practical problems of the day, including surveying, goods exchange, engineering, and right-angled triangles (O'Connor "Nine Chapters on the Mathematical Art).

Each problem is presented as an imagined scenario; it describes the situation with real world terms (doors, fields, grain) and then asks the reader to 'Tell,' requesting some specific answer to a problem posed.

In the tradition of Chinese classical texts, the *Nine Chapters* has no specific author, but rather follows the ancient collaborative tradition in which many writers build upon the work of their predecessors (Boyer 424). However, Liu Hui, a Chinese mathematician born around 220 AD, is of special notability as his commentary in the *Nine Chapters* provided many of the proofs that we are to analyze in this paper.

2.2 *Elements*

Euclid of Alexandria, a Greek mathematician, originally authored *Elements* around 300 BC. Though Euclid is often the sole author to whom the *Elements* are attributed, he himself made no claim to originality. It is believed that Euclid drew heavily upon the work of his predecessors, though the arrangement of *Elements* and many of its proofs were supplied by Euclid himself.

Elements is partitioned into thirteen chapters, which cover plane geometry, number theory, solid geometry and incommensurables. The axiomatic style is clear, as *Elements* has no known preface and rather begins with twenty-three definitions followed by five axioms and five postulates. It is from this basis that Euclid begins the rigorous mathematical discourse that covers over 450 propositions, each one progressing in small steps, building upon knowledge established in previous episodes (Joyce *Euclid's Elements*).

3. Analysis of Algorithms and Proofs

The three problems to be comparatively analyzed in both *Elements* and *Nine Chapters* are (a) Pythagoras' theorem, (b) square roots and (c) quadratic equations. These three problems were chosen because of the fundamental role they play in modern and ancient mathematics. Furthermore, they are all well-defined problems, which allow us to confidently identify equivalent sections of both texts, allowing for a more specific comparative study. Finally, the problems are addressed in different sections throughout both texts, and together they represent a good cross sectional representation of the style and form of both *Elements* and *Nine Chapters*.

The analysis will consist of defining a problem, making observations about the approach both texts take, and then analyzing the benefits and limitations of each approach in the context of the other. At this point it is appropriate to introduce some extra terminology the Chinese used to establish the required context. The word 'gou' was used to refer to the shorter orthogonal side of a right-angled triangle and the word 'gu' used to refer to the longer orthogonal side. The word 'xian' was used to refer to the hypotenuse, but for this discussion we will continue to call it the hypotenuse. Pythagorean triples were known in Chinese as 'gougu numbers' (Shen 439). Furthermore, the Chinese units of length presented in the questions analyzed must be explained. These are shown in a table in Figure 1. The smallest unit presented is cun, which is equal to 33.3mm. Next, chi is defined as 10 cun, and similarly, zhang is defined as 10 chi. 6 chi make up one bu.

Name	Symbol	Relative unit	Metric unit
cun	寸	1/10	32 mm
chi	尺	1	0.32 m
bu	步	6	1.6 m
zhang	丈	10	3.2 m

Figure 1

3.1 Pythagoras' Theorem

A starting point for our investigation is an analysis of Pythagoras' theorem; that the square of the length of the hypotenuse of the triangle is equal to the sum of the square of the other two sides. Though Pythagoras of Samos, born circa 570 BC, is credited as the namesake for this theorem in modern times, there is evidence that Indian, Babylonian and Chinese cultures had discovered the theorem and Pythagorean triples much earlier. In fact, a Babylonian tablet named Plimpton 322, dated between 1600 to 1900 BC, shows an early understanding of Pythagorean triples (Boyer 32). Indeed, Pythagoras' theorem has played a prominent role in most ancient mathematical systems. The class of problems we are examining take the form:

$$a^2 + b^2 = c^2 \qquad (1)$$

3.1.1 *Nine Chapters*: Chapter Nine, Problems 1-3

The Pythagoras rule was also well known in China under the name of the 'gougu rule.' The gougu rule was:

> *Add the squares of the gou and the gu, take the square root of the sum giving the hypotenuse'. Further, if the gu is subtracted from the square on the hypotenuse. The square root of the remainder is the gou. Further, if the gou is subtracted from the square of the hypotenuse. The square root of the remainder is the gu.*

Liu's commentary adds that:

> *The gou is shorter than the gu. The gou is shorter than the hypotenuse. They apply in various problems in terms of rates of proportion*

Here already we can see a strong understanding of algebra, in that one rule is sufficient to draw three corollaries from. The explanation of the gougu rule gives three alternative methods for its use, and then proves each method to be valid. The proof is as follows:

> *Let the square on the gou be red in colour, the square on the gu be blue. Let the deficit and excess parts be mutually substituted into corresponding positions, the other parts remain unchanged.* (Shen 459)

Liu's original diagram has since been lost, and there is ambiguity as its original structure. However, three academics have attempted valid reconstructions. Li Huang's diagram is shown in Figure 2, (Shen 46) whereas Figure 3 is conceived by Donald B. Wanger, and Figure 4 is suggested by Professor Jöran Friberg (Wagner "A proof of the Pythagorean Theorem").

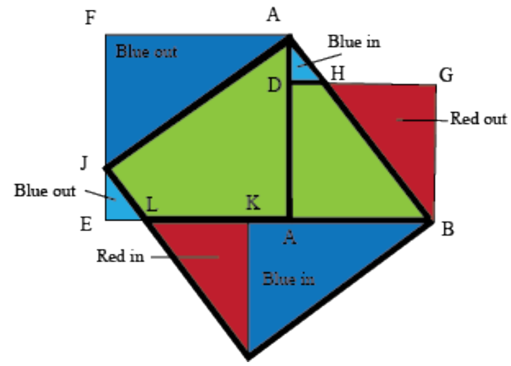

Figure 2

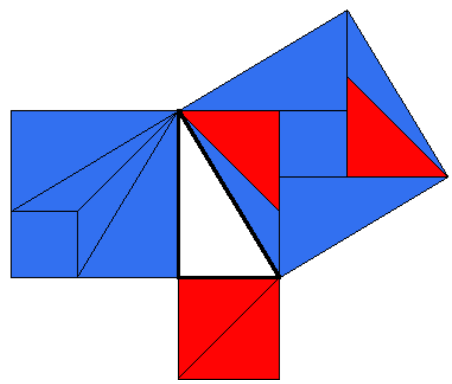

Figure 3

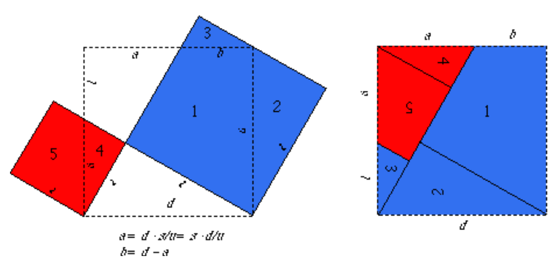

Figure 4

It is of particular interest that Liu's statements are vague and concise to the extent that we are unable to ascertain the exact proof to which he was referring.

3.1.2 *Elements*: Book I, Proposition 47

Under Euclid, the Pythagoras rule is introduced in proposition 47 of Book I. That is, Euclid proposes that that:

In right-angled triangles, the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.

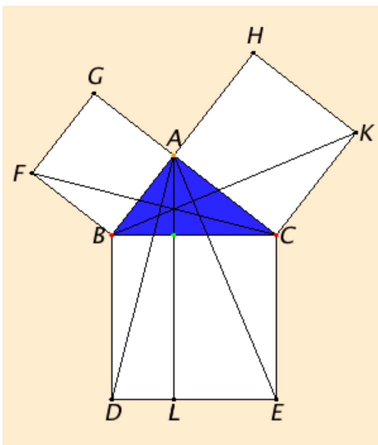

Figure 5

The proof is shown in Figure 5. The proof rests upon four earlier propositions (I.4, I.14, I.31 and I.46). A redacted explanation of Euclid's proof is as follows:

1. The area of triangle ABD is equal to half of parallelogram BL as they both share the base BD and the perpendicular height of BL.
2.The area of triangle ABD, is equal to half the area of the square BG for ABD has the perpendicular height BA. Thus, the square BG has equal area to the parallelogram BL.
3. By the same proof, the square HC has area equal to the parallelogram CL. QED.
(Joyce *Euclid's Elements*)

Here, Euclid's proof is in his regular rigorous, axiomatic in style, in that he takes small but concrete steps, using his previous work as stepping-stones. The rigor of Euclid is made even evident in that he found it necessary to prove the converse statement, that 'If in a triangle the square on one of the sides equals the sum of the squares on the remaining two sides of the triangle, then the angle contained by the remaining two sides of the triangle is right,' in proposition 48, rather than take it for granted as obvious.

3.1.3 Comparative Analysis

The proof Liu gives is a geometric one that uses the in-out principle, meaning it juxtaposes geometric figures. Because the proof is visual, it is easier to understand and is more intuitive, especially for those not well versed in Mathematics. In each of the versions proposed, the proof relies on 'seeing' the relation between shapes - and not upon any earlier established principles. Thus this style gives the *Nine Chapters* a degree of modularity that Element's lacks. Thus, for a textbook of study, the *Nine Chapters* would provide the same idea and convincing proof for the Pythagorean rule but at a much lower level of difficulty and esotericism than that offered by *Elements*.

In support of *Elements*, the proof given has far less ambiguity than that presented in the *Nine Chapters*. The proof can be understood entirely in abstract terms, and any drawing of the diagram would essentially yield the same geometric figure after applying simple transformations. In contrast, at least three figures have been proposed for Liu's proof, evidence that his proof is not as epistemically certain as that of Euclid as it leaves freedom for interpretation.

It is of interest to note that in proposition 47 no specific method is provided for calculating the gou, gu or hypotenuse given any of the other two. These algorithms must instead be logically derived. This would be non-obvious to a reader without foundations in algebra. For these reasons, it could be said that the *Nine Chapters* has a higher accessibility and would be more effective as an instructive mathematical textbook, whilst the proofs offered by *Elements* have less ambiguity and are more concrete.

3.2 Square Roots

The method for the extraction of a square root is of particular importance as it is a prominent sub-routine for numerous other algorithms. Thus, the method given to solve for a square root has widespread implications across the entire system of mathematical knowledge. Here we are looking for methods that solve for the class of problem expressed as:

$$x^2 = c \qquad (2)$$

3.2.1 *Nine Chapters* – Chapter 4, Problems 12- 16

The challenge of extracting a square root was first addressed in problems 12 to 16 of Chapter 4 of the *Nine Chapters*. In this chapter, Liu goes so far as to introduce the notion of limits and infinitesimals, showing that the epistemic capabilities of the Chinese mathematicians at this time were not inferior to those of their European counterparts; rather, their approach to mathematics was fundamentally different. (O'Connor "Nine Chapters on the Mathematical Art). In fact, the algorithm given is numerical in nature and is introduced in a series of three problems, each one in the same form.

Problem 12: Now given an area 55225 square bu, Tell: what is the side of the square?
Answer: 235 bu.
Problem 13: Now given an area 25281 square bu, Tell: what is the side of the square?
Answer: 159 bu.
Problem 14: Now given an area 55225 square bu, Tell: what is the side of the square?
Answer: 268 bu.
Problem 15: Now given an area 564725 (1/4) square bu, Tell: what is the side of the square?
Answer: 751(1/2) bu.
Problem 16: Now given an area 55225 square bu, Tell: what is the side of the square?
Answer: 63025 bu.

What stands out in these problems is the degree of redundancy provided by the *Nine Chapters*. Also apparent in the gougu rule, it is a style consistent throughout the text in order to guide the reader through multiple problems with the same concept in order to aid in their comprehension. This is a sharp departure from Euclid's reductionist style, each proposition containing the minimum proved, and the goal of each proposition being only to prove a statement true. The method provided to extract the square root is as follows:

Lay down the given area as shi. Borrow a counting rod to determine the digital place. SEt it under the unit place of the shi. Advance to the left every two digital places as one step. Estimate the first digit of the root. The estimated number multiplied by the borrowed rod is regarded as fa. Then carry out the subtraction. After that, double the fa as determined fa. Prepare the second subtraction. Move the determined fa one digit to the right and set the borrowed rod as before. Estimate the second digit of the root. Multiply it by the borrowed rod and subjoin the product to determine fa. Then carry out the second subtraction. Subjoin to the determined fa a second time. Proceed with the operation in the same manner.

The method given to find the square root gives a numeric solution to an arbitrary degree of accuracy, though interestingly the method involves guessing and estimation. The method is derived from the geometric notion of what a square root means, that is in order to find the side length of a square, we only need to cut away increasingly smaller gonoms and squares. This can be done infinitely to find an arbitrarily accurate square root. (Yong "The Development of Polynomial Equations in Traditional China")

Liu discusses this geometric understanding in depth, explaining in eight logical steps why the methodology presented gives the square root. Throughout his explanation, he constantly refers to a diagram reconstructed in Figure 6. He uses colors to describe sections of the square and how they are 'cut away' at each step of the algorithm given. Again, his discussion of correctness is highly visual, and, when followed step-by-step, is very intuitive. It requires no prior knowledge of other sections of the *Nine Chapters*.

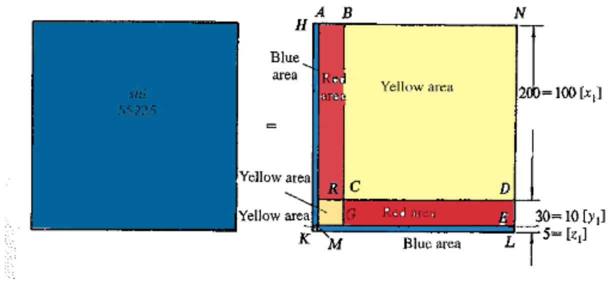

Figure 6

Note also how the method given for square roots as described has ample use of variable names, loops and small but simple steps. It is not dissimilar to modern day programming, where computers are given concrete algorithms with no ambiguity to their execution. In this manner, even without an understanding of the notion of square roots, anyone can find an arbitrarily accurate square root by following the instructions closely.

3.2.2 *Elements*: Book VI, Proposition 13

Euclid's method for finding a square root can be deduced from Proposition 13, in Book VI. Here Euclid proposes a method for finding the mean proportional between two given straight lines. Based on Figure 7 his method is as follows:

1. *Place AB, and BC on a straight line.*
2. *Describe the semicircle ADC*
3. *Draw BD, at right angles to the line AC, join AD, and DC.*
4. *ADC is a right angle for it is an angle in a semicircle.*
5. *Therefore BD is the mean proportional between base segments AB and BC.*

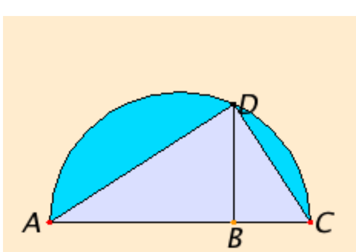

Figure 7

Algebraically stated, taking length AB to be a, BC to be b, and BD to be x, this proposition can be expressed algebraically as:

$$\sqrt{\mathrm{ab}} = c \qquad (3)$$

Immediately, we see an elegant method that is able to produce a perfect geometric root of a number. This again, in Euclid's traditional axiomatic style, draws upon previous work in propositions: I. 11, III.31 and VI.8. Furthermore, the method to produce a square root of one number is non-obvious. It requires that we set either AB or BC to a unit length and construct the other of length equal to the square root we are trying to find. It is also worth nothing that though geometrically we produce a perfect root, there is evidence to suggest that the Greeks had trouble in actual calculation. This method requires measuring the length BD against the unit length, inherently reaching an upper bound to potential accuracy.
(Joyce *Euclid's Elements*)

3.2.3 Comparative Analysis

The contrast is evident here. Euclid's methods gave a perfect geometric square root, a powerful tool for later proofs; though for computational purposes it proved ineffective. To numerically calculate a root, one would need to construct Figure 7 and then take a unit rule to measure the length of line BD. Hence; we are inherently introducing an upper bound to the accuracy this method could produce. Evidence suggests that the Greek's were often met with difficulty when extracting roots, especially for irrational results such as the $\sqrt{2}$.

The Chinese method, though it only gives approximations of square roots, is able to do so with an unbounded degree of accuracy. The method is inherently numerical, and thus serves as a powerful computational method. Though the discussion of correctness is not as concrete as Euclid's, it is still convincing whilst having a higher degree of intuition and accessibility in its visual nature.

This method of extracting roots did not appear in the western world until several hundred years later. It was so powerful and easy to practice that it was commonly taught in American elementary schools up till the advent of the digital calculator, most commonly learnt through rote memorization.

3.3 Quadratic Equations

Finally, let us analyze problems involving solutions to quadratic equations, an advanced issue addressed in both texts. Both *Elements* and *Nine Chapters* approach this problem expressed in a geometric form. In this analysis, we are looking for solutions to problems that can be algebraically expressed in the form:

$$x^2 + bx - c = 0 \qquad (4)$$

3.3.1 *Nine Chapters*: Chapter Nine, Problem 20

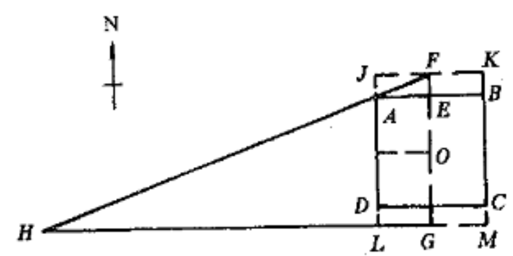

Figure 8

The *Nine Chapters* introduces a problem involving general quadratic equations at problem 20. It is depicted in Figure 8 (Shen 508). It reads:

> *Now given a square city of unknown side with gates opening in the middle. 20 bu from te north gate is a tree, which is visible when one goes 14 bu from the south ate and then 1775 bu westward. Tell: what are the lengths of each side*
> *Answer: 250 bu*

The solution given is as follows:

> *Method: Take the distance from the north gate to multiply the westward distance. Double the product as the shi. Take the sum of the distance from the north and south gates as the linear coefficient (congfa). Extract the root to obtain the side of the city.*

Algebraically we are solving the equation:

$$\mathrm{x}(\mathrm{x} + 34) = 2 * 20 * 1775 \qquad (5)$$

$$x^2 + 34x - 71000 = 0 \qquad (6)$$

Liu's solution uses the property of similar triangles. He asks us to observe that ΔHFG is congruent to ΔAFE and ΔHAL. We know the gou and gu of triangle ΔHFG, as well as the gu of ΔAFE, therefore we can deduce the gou of ΔAFE. Using the notion of ratios, we can understand that AE therefore is equal to $AE = HG \times \frac{FE}{FG}$. However, algebra was not developed to this point yet, and the *Nine Chapters* rather views $AE \times FG$ as the rectangle depicted in Figure 9 (Shen 508). Here, the rectangle is doubled to make the city, with an area equal to $2 \times AE \times FG = 2 \times 20 \times 1775$. This rectangle contains the square of the city, and thus the method given relies on extracting the middle square and applying a square root operation on it.

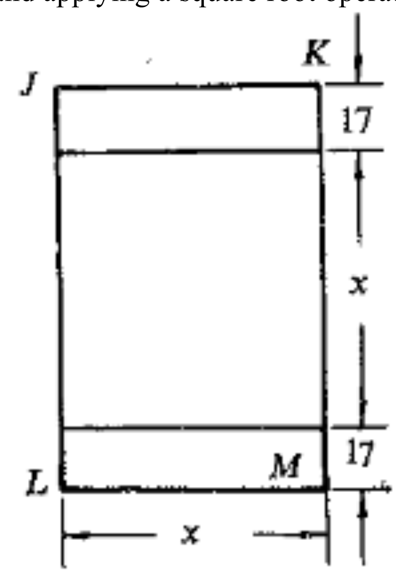

Figure 9

This problem shows the *Nine Chapters* is based upon an advanced understanding of geometric properties such as similar triangle and ratios, as well as drawing relations between shapes. The method provided is in the regular algorithmic style, and can be generalized to solve any quadratic equation of the form specified earlier.

3.3.2 *Elements*: Book VI, Proposition 29

A method for solving the general quadratic equations can be deduced from proposition 29, given in book VI. Euclid offers us the following challenge:

> *To apply a parallelogram equal to a given rectilinear figure to a given straight line but exceeding it by a parallelogram similar to a given one.*

Though Euclid's method is generalized for any rectilinear figure and parallelogram, in order to maintain comparability with the *Nine Chapters*, as well as avoid excessive complexity, let us consider the rectilinear figure to be a square and the parallelogram to be a rectangle. The method and reasoning behind the solution of a general quadratic solution remains the same. Thus the problem is depicted in Figure 10.

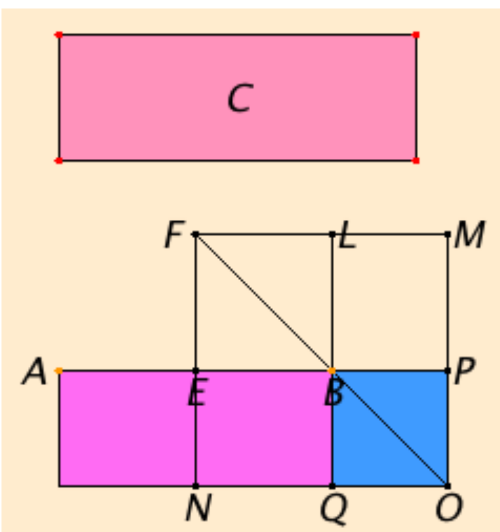

Figure 10

The challenge is to construct rectangle AO such that the area AQ is equal to the area C, and BO is a square. Given that AB represents a, and BP represents x, the proposition can be expressed algebraically as:

$$(a + x)x = C \qquad (7)$$

$$x^2 + ax - C = 0 \qquad (8)$$

The solution is as follows:

1. Bisect AB at the point E,
2.Cconstruct the square BEFL
3. Construct FNOM such that the gnomon LMONEB is equal to the area of C
4. BO is the square that exceeds the parallelogram, finding BP from this gives x.

The solution draws upon 6 previous propositions (VI.25, VI.21, VI.26, I.36, I.34, and VI.24). Note that in order to set up this problem to find numerical solutions, the value of c must be given as the area of a shape, and the length a be equal to the length of the line. We presented here the simplified problem, but the visual proof for the generic class of problem originally stated is shown in Figure 11. (Joyce *Euclid's Elements*)

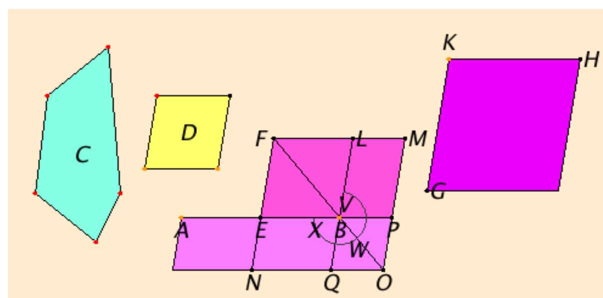

Figure 11

3.3.3 Comparative Analysis

Nine Chapters offers a simple numerical method for finding the solution to a quadratic equation using geometric principles. The class of problem presented is an easily understood positioning problem with a triangle and square. Euclid's construction, on the other hand, is much more powerful in that it represents a much larger class of problems, with C being any rectilinear figure and BO being any parallelogram. In the usual axiomatic style, its proof is convincing in that it rests upon many previously established principles, though especially in its generalized form it is much more difficult to understand and the method and proof are less intuitive.

Another limitation of the *Nine Chapters* method is that it only gives a solution to the problem in the specific form $x^2 + bx - c = 0$ where b and c are positive numbers. In contrast, Euclid, in other propositions, introduces concepts that can produce methods that solve quadratic equations in all other combinations of positive and negative coefficients for b and c, including the forms: $x^2 - bx + c = 0$, $x^2 + bx + c = 0$, and $x^2 - bx - c = 0$. These are given in Book II, Proposition 5, and Book VI propositions 28 and 29. Here we see Euclid's rigor lead to a more complete set of solutions. Of course, given that Euclid's method of square root is fundamentally not numeric, we again produce perfect geometric results at the cost of computational difficulty (Laubenbacher).

4. Discussion

Through our analysis, we have seen a fundamental difference in mathematical philosophy between ancient Western and Eastern civilizations, as revealed by our investigation of *Elements* and *Nine Chapters*. Critical to our analysis is the fact that we are not establishing the superiority of one system of knowledge over the other, but rather that their way of thinking of Mathematics was fundamentally different, each one having its respective advantages and disadvantages. Overall, it can be seen that *Elements* takes an epistemic, proof-first approach, whereas the *Nine Chapters* takes a pragmatic, algorithm-first approach.

First there is commonality seen between the texts. Both discovered numerous fundamental principles of mathematics. These include notions such as right-angled triangles, ratios, geometric relations, algebra, square roots, and quadratic equations. Thus, it cannot be said that either text is epistemically inferior to the other; rather it is their fundamental mathematical philosophy that differed.

Both civilizations understood well the notion of a subroutine and building upon previously established knowledge. This allowed both systems of knowledge the sufficient degree of abstraction to reach high levels of complexity. Euclid does this as a fundamental part of every proposition in that every statement made in the proof of a proposition had been earlier proved true. This is central to his axiomatic style. The Eastern use of subroutines is more oriented to provide tools required to complete subsections of algorithm rather than to prove the correctness of a statement. For example, problem 11 of Chapter 9 of the *Nine Chapters*, relies on the square root subroutine introduced in Chapter 4, problem 12, however problem 11 of Chapter 9 features an independent geometric proof of correctness. Here, we see Euclid's approach of subroutines is for a more epistemic purpose, whereas the *Nine Chapters* use of subroutines is for a more pragmatic purpose.

Present in both texts, though in different forms, is the notion that algorithms should be generalized to solve a class of problems. Euclid does this strongly, in that none of the propositions use any numerical data, but rather all lengths and figures are described in purely abstract form`s. This often means that the method presented in *Elements* solves a wider class of problems, such as we saw in our analysis VI.29. Fundamental to the *Nine Chapters* also is the notion of generality, thought it makes a significantly greater effort to guide the reader to understand the methods given. Though *Nine Chapters* provides only a single method for each class of problems addressed, it produces several redundant numerical examples of questions and answers are produced for each method. This gives it strength in that, though a generic method is given, the reader can work through specific examples to ensure understanding of how the method is to be applied. In this sense, *Elements* can be seen to have taken a reductionist approach to avoid making unnecessary assumptions, and to preserve the tightness of proofs. In contrast, *Elements* features much redundancy in order to guide the reader and give the algorithms presented greater accessibility.

We can immediately see differences in mathematical purpose from the layout of the two texts. Euclid's proofs are systematically arranged in such a way that they form a comprehensive logical discourse, with each proposition resting on the last. In contrast, the Chinese, similar to the Babylonians, grouped problems by subject matter addressed (Boyer 176). For instance, in the *Nine Chapters*, chapter one focuses on land surveying, chapter five is on civil engineering and chapter nine deals with right-angled triangles. The writers and commentators of the *Nine Chapters* themselves were not simply producing ad-hoc solutions to prevalent problems. It is clear from their work that they had an organized mathematical framework. They frequently used specific operations in their proofs, for which they had developed a unique lexicon. It can thus be been seen that these writers had established a system of conventions, not unlike the approach taken by Euclid in *Elements* (Boyer 425).

Finally, we see a marked difference in the philosophy of algorithms and proofs between the ancient Western and Eastern civilizations. Even in introducing a problem, *Elements* beings by making a statement and then using previous propositions to prove it to be true. The advantage of this, is that when proofs are taken in atomic steps, they can be seen as very convincing with little ambiguity. The disadvantage is that understanding of a single proposition may involve in-depth study of previous sections of *Elements* as well as a strong understanding in mathematics.

In contrast, *Nine Chapters* offers first a set of problems that are to be solved, then the method for solving them, and finally followed by a discussion of the correctness of the method provided. In offering a method directly, *Nine Chapters* has great applicability to even a layperson, who may be interested only in practical applications of mathematics. In contrast, many of the algorithms in *Elements* must be derived from one or several propositions, and these are often non-obvious as we saw in all three of the problems analyzed.

Also, with the focus on algorithms, *Nine Chapters* often produces computationally superior tools as seen in the square root method. Whereas the Chinese method was able to produce arbitrarily accurate results through a method involving estimation, Euclid's method produced perfect square roots, though an upper bound on accuracy is introduced when trying to produce a numerical solution. The proofs of *Nine Chapters* are often less rigorous than those provided in *Elements*, the goal being to only convince the reader that the method is correct, rather than establish an infallible mathematical axiom on which to build further knowledge.

To the *Nine Chapters* benefit, the proofs offered are often highly visual and thus more easily understood. Take for example in Chapter 4, problem 12 - Liu asks the reader to imagine a triangle and then begins to describe sections in terms of colors. Though less comprehensive, this form of proofs can be more intuitively understood by the reader and provides *Nine Chapters* with a degree of modularity and accessibility not shared by the *Elements*.

5. Conclusion

In our analysis, we have identified major differences in the fundamental mathematical philosophy between ancient Eastern and Western civilizations, as expelled in *Elements* and the *Nine Chapters*. Overall, it can be said that the Western philosophy of Mathematics lied a proof-first approach, which had a primary goal of producing an epistemic library of knowledge. Its strength lies in the rigor of proofs, its degree of abstraction and its reductionist elegance. The limitations lie in the difficulty in understanding the proofs, and the non-obvious and sometimes absent nature of the methodology to solve numerical problems. In contrast, the Eastern philosophy of Mathematics can be seen as an algorithm-first approach, which had a primary goal of producing a library of correct algorithms. This approach excels in pragmatism and providing computationally superior algorithms that are easily executed. It also produces proofs that are more intuitive and accessible, though that are limited by their visual and sometimes ambiguous nature. It is understood that neither approach was superior to the other, but rather that each civilization had a fundamentally different approach to mathematical knowledge.

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