Random matrix approach to estimation of high-dimensional factor models

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Abstract

In dealing with high-dimensional data sets, factor models are often useful for dimension reduction. The estimation of factor models has been actively studied in various fields. In the first part of this paper, we present a new approach to estimate high-dimensional factor models, using the empirical spectral density of residuals. The spectrum of covariance matrices from financial data typically exhibits two characteristic aspects: a few spikes and bulk. The former represent factors that mainly drive the features and the latter arises from idiosyncratic noise. Motivated by these two aspects, we consider a minimum distance between two spectrums; one from a covariance structure model and the other from real residuals of financial data that are obtained by subtracting principal components. Our method simultaneously provides estimators of the number of factors and information about correlation structures in residuals. Using free random variable techniques, the proposed algorithm can be implemented and controlled effectively. Monte Carlo simulations confirm that our method is robust to noise or the presence of weak factors. Furthermore, the application to financial time-series shows that our estimators capture essential aspects of market dynamics.

Keywords: random matrix theory, factor models, principal components, free random variables, Kullback-Leibler distance

1 Introduction

The increasing accessibility of 'big data' occurs also in economics and finance. In dealing with such high-dimensional data sets, factor models are often used, since they can

reduce the dimension and effectively extract relevant information. The estimation of high-dimensional factor models has been actively studied extensively in statistics and econometrics (Chamberlain and Rothschild [1983]; Connor and Korajczyk [1986]; Connor and Korajczyk [1993]; Stock and Watson [1988]; Bai and Ng [2002]; Kapetanios [2004]; Onatski [2010]; Ahn and Horenstein [2013]; Harding [2013]; Bai [2013]; Pelger [2015]). This paper provides a new approach to estimating high-dimensional factor models, using the eigenvalue distribution of residuals. From a minimum distance approach, we estimate the number of factors and the correlation structure of residuals. The proposed method is validated with Monte carlo simulations and, in most of the cases we consider, it outperforms other known methods. Furthermore, the results with financial data show that our estimators effectively capture structural market changes.

Consider a factor model that is as follows. For $i = 1, \cdots, N$ and $t = 1, \cdots, T$,

$$R_{it} = \sum_{j=1}^{p} L_{ij} F_{jt} + U_{it} \tag{1}$$

where R_{it} is the data of i-th unit (e.g.,asset return) at time t, F_{tj} is the j-th factor at time t, L_{ij} is the loading of j-th factor on i-th cross-sectional unit. U_{it} is the idiosyncratic component or residual of R_{it}. Usually, only R is observable, so L, F, and U must be estimated.

One way to deal with factors is to use the principal components to estimate them and determine the number by looking at singular values of covariance matrix of R and take some of them based on a given threshold for variance explanation. Then one usually assumes U as pure noises.

However, in this paper, we mainly focus on the residuals U, and their dynamics and dependence, to estimate the covariance structure of U and the number of factors p simultaneously.[1]. Our approach is based on the investigation of the empirical spectral distribution of covariance matrix of residuals.

The first contribution of this paper is that we connect the factor model estimation problems to the limiting empirical eigenvalue distribution of covariance matrices of residuals. Thus, the main focus of the proposed method is on residuals, U. Instead of requiring that the idiosyncratic components U_{it}'s are uncorrelated to each other, we assume there are cross- and auto-correlated structures, such that U is represented as $U = A_N^{1/2} \epsilon B_T^{1/2}$, where ϵ is an $N \times T$ ($T = T(N)$) matrix with i.i.d. Gaussian entries, and A_N and B_T are an $N \times N$ and $T \times T$ symmetric non-negative definite matrices, representing cross- and auto- covariances, respectively[2]. Then the empirical covariance matrix of U can be written as $C_N = \frac{1}{T} U U^T = \frac{1}{T} A_N^{1/2} \epsilon B_T \epsilon^T A_N^{1/2}$. In this paper, we restrict the

[1] In high dimensional settings: N and T go to infinity

[2] This is not the most general model, since cross- and auto-covariance contributions are decoupled: $cov(U_{it}, U_{js}) = A_{N\,ij} B_{T\,ts}$.

matrix structures of A_N and B_T, so that they are completely defined by simple parameter sets, θ_{A_N} and θ_{B_T} that are to be estimated along with the number of factors. For example, a simple case is that each residual has the same cross-correlation[3], β, to other residuals, and each residual has an exponentially decaying temporal autocorrelations with a parameter τ. Then two parameters $\theta_{A_N} = \beta$ and $\theta_{B_T} = \tau$, completely determine A_N and B_T, since $A_N = \left\{(A_N)_{ii} = 1, \quad (A_N)_{ij,i\neq j} = \beta, \quad i,j = 1,\cdots,N\right\}$ and $B_T = \left\{(B_T)_{st} = \exp(-|s-t|/\tau), \quad s,t = 1,\cdots,T\right\}$.[4]

Now the objective of our estimation method is to match the eigenvalue distribution of C_N to that of the empirical covariance matrix of residuals constructed from market data. The latter can be controlled by the number of principal components to be removed. The former depends on the modeling of A_N and B_T, but we assume a parsimonious matrix structure, determined by only a small parameter set, $(\theta_{A_N}, \theta_{B_T})$.

We search for the number of factors (p) and the parameter sets $(\theta_{A_N}, \theta_{B_T})$, such that the spectral distance between a model and real data is minimized. This spectrum-based approach is motivated by the two typical characteristic aspects in the spectrum of real data: a few spikes and bulk. The former represent factors that mainly drive the market features and the latter arises from idiosyncratic noise. It is also theoretically motivated by the results of Zhang [2006], which analyzes, under certain assumptions, the convergence of the empirical eigenvalue distribution of C_N to a suitable limiting distribution.

The factor model estimation problem is stated as follows.

$$\{\hat{p}, \hat{\theta}\} = \arg\min_{p,\theta} \mathcal{D}\Big(\rho_{\texttt{real}}(p), \rho_{\texttt{model}}(\theta)\Big) \tag{2}$$

where $\rho_{\texttt{real}}(p)$ is an empirical eigenvalue density of covariance matrix of residuals that are constructed by removing p principal components from original data, $\rho_{\texttt{model}}(\theta)$ is a limiting eigenvalue density of the general covariance matrix characterized by a parameter set $\theta = (\theta_{A_N}, \theta_{B_T})$, and $\mathcal{D}$ is a spectral distance measure or loss function we choose. The solution of this minimization problem gives the number of factors and parameters for the correlation structure of the residuals. As for estimating the number of factors, there are several methods proposed in previous literature (Bai and Ng [2002]; Kapetanios [2004]; Onatski [2010]; Ahn and Horenstein [2013]; Harding [2013]). The main difference from other estimators is that our method finds the best fit of the whole spectral distribution, which enables us to take into account both spikes and bulk of the distribution.

A difficulty in the implementation is the calculation of $\rho_{\texttt{model}}(\theta)$[5], since using the limiting distribution from the Stieltjes transform in Zhang [2006] for general A_N and B_T is very complicated. However, a recent work by Burda et al. [2010] provides an analytic derivation of limiting spectral density using free random variable techniques. In this paper, we

[3] We assume each time-series, U_{it} $(t = 1,\cdots,T)$, is normalized and has an unit variance.

[4] As we will discuss later, we mainly focus on the temporal correlation structure of residuals, so in our simplified model, we present the result of the case where $A_N = I_{N\times N}$, supposing the cross-correlations are effectively removed by the factors.

[5] $\rho_{\texttt{real}}(p)$ can be obtained easily with data. See Section 5.1 for details

use these results to calculate $\rho_{\texttt{model}}(\cdot)$. Furthermore, we propose a simplified estimation problem that considers parsimonious matrix structures for A_N and B_T. In particular, supposing that the cross-correlations are effectively removed by the factors, we assume that the cross-correlations among the normalized residuals are negligible: $A_N \approx I_{N\times N}$ (or $\beta = 0$ in the previous example). But we still assume they are serially-correlated, with exponential decays with respect to time lags: $(B_T)_{ij} = b^{|i-j|}$. Then the $\rho_{\texttt{model}}(\theta_{A_N}, \theta_{B_T})$ is replaced by $\rho_{\texttt{model}}(b)$, and the minimization problem has only two scalar variables, p and b. This parsimonious model has significance in two senses. First, it is good for calculability, as we adopt the free-random variable techniques. Second, the parameter b indicates global rate of mean-reversion of residuals. The mean-reversion property of residuals getting increasing attentions in the current financial markets, especially for statistical arbitrage strategy.

The second main contribution of our work is that the proposed methods are validated from tests with synthetic data, generated using known models. Monte Carlo simulations with synthetic data show that the finite-sample performances of the estimators are good. The number of factors and the autoregressive parameter are accurately estimated for various choices for N and T. We compare the estimated number of factors from our method with those from other methods in the literature, and show that our method is robust to noise and performs well in identifying weak factors.

The third contribution is that we find, with real market time-series data, that our estimators of the simplified problem successfully capture market dynamics. The estimation problem we propose is static, so in order to observe time-varying behaviors of parameters, we repeat the estimation procedures with moving windows. For market data, we use daily returns of S&P500 stocks in the period of 2000-2015. We compute time changes of the estimators. It turns out that the estimators reflect the regime-change information of the market. In particular, we find that during stress periods, the number of factors is decreasing, while the variance explained by the corresponding factors increases, which shows market condensation. Furthermore, the *global* mean-reversion time of residuals, represented by the estimated autoregressive coefficient b, tracks the volatility index very closely. We also find that during the crisis, the residuals are more trending, showing slower mean-reversions.

The rest of the paper consists of the following content. In Section 2, we review related literature. In Section 3, we consider a motivating example. Section 4 provides theoretical preliminaries. In Section 5, we develop an estimation method of factor models and describe the procedures used. Section 6 contains Monte Carlo analysis and comparisons with other methods. Section 7 shows applications with real data. We conclude in Section 8.

2 Related literature

Our method in high-dimensional settings is fundamentally based on random matrix theory. Random matrix theory, developed originally to study the interactions in complex quantum

systems (Wigner [1951]), can be used to identify non-random properties which are deviations from the universal predictions. Laloux et al. [1999] was the first that applied the random matrix theory to financial correlations, and myriads of papers have followed in the physics community. (Plerou et al. [1999]; Stanley et al. [2001]; Plerou et al. [2001]; Plerou et al. [2002]; Dimov et al. [2012]) They have analyzed eigenvalue distribution of empirical cross-correlation matrix from stock returns. They claimed that deviated eigenvalues from a theoretical expectation, Marchenko-Pastur law (Marchenko and Pastur [1967]), provides genuine market information, such as market mode or industrial sectors. Then the number of factors is determined by counting those deviating eigenvalues.

However, "no information" or "pure noise" assumption in the bulk region[6] is too strict and it turns out to be invalid in practice. As seen from the example in Section 3, the fit of the empirical spectral density of covariance matrix from real residual returns to the Marchenko-Pastur distribution is problematic. This implies that the residuals from real data are not necessarily pure noise, and more general correlation structure needs to be considered to assess the empirical densities.

The phenomenal work by Zhang [2006] provides a central theoretical foundation for our estimation method. The author considers a general covariance matrix, C_N, of the form $C_N = \frac{1}{T} A_N^{1/2} \epsilon B_T \epsilon^T A_N^{1/2}$, where A_N and B_T are non-negative definite matrices of size $N \times N$ and $T \times T$, respectively, and ϵ is an $N \times T$ Gaussian random matrix with i.i.d. entries. Let $c = N/T$. Zhang [2006] shows that, under certain assumptions, the empirical eigenvalue distribution of C_N converges weakly to a non-random distribution $\mathcal{F}^{c,A,B}$, and that the Stieltjes transform of $\mathcal{F}^{c,A,B}$, $m(z)$, together with other analytical function $p(z)$ and $q(z)$, constitutes a solution to the system

$$\begin{cases} m(z) = -z^{-1}(1-c) - z^{-1} c \int \dfrac{1}{1+q(z)x} d\mathcal{F}^A(x) \\ m(z) = -z^{-1} \int \dfrac{1}{1+p(z)y} d\mathcal{F}^B(y) \\ m(z) = -z^{-1} - p(z)q(z) \end{cases}$$

which is unique in the set $\{(m(z), p(z), q(z)) : Im(m(z)) > 0, Im(p(z)) > 0, Im(q(z)) > 0\}$.[7] Note that we actually do not solve this system of equations, but only use the fact that the empirical spectrum of C_N converges to a non-random spectrum. Assuming the residuals has the form $U = A_N^{1/2} \epsilon B_T^{1/2}$ in the factor model, this theory can be employed to show the consistency of our estimators from minimum spectral distance of residuals.

The paradigm of factor model approach in finance was initiated by Ross [1976], which proposed Arbitrage Pricing Theory that supports strict factor models. With relaxed assumptions allowing weak correlation in idiosyncratic components, the approximate factor models were introduced by Chamberlain and Rothschild [1983]. The dynamic factor models

[6] The eigenvalue distribution considered in this paper consists of many bounded small eigenvalues (bulk) and several large ones (spikes)

[7] Here $\mathcal{F}^A$ and $\mathcal{F}^B$ are limiting distributions of $\mathcal{F}^{A_N}$ and $\mathcal{F}^{B_T}$, respectively.

developed by Stock and Watson [2005] also received attentions.

The determination of the number of factors in high-dimensional factor models is one of the crucial issues in both theoretical and practical perspectives. The original work of Bai and Ng [2002] uses an information criterion to determine the number factors. Kapetanios [2004] is the first to use the idea of structure of idiosyncratic terms. He points out that the correlated assumption on idiosyncratic components implies a closed-form expression for a sharp asymptotic upper bound on the idiosyncratic eigenvalues of the sample covariance matrix. Thus, he claims that counting the eigenvalues above the bound gives an estimate of the number of factors. Onatski [2010] provides a criterion using the difference of two adjacent eigenvalues. The method based on the eigenvalues ratio is also developed in Ahn and Horenstein [2013], and Pelger [2015] for high-frequency data. Recently Harding [2013] also proposed a method for estimating the number of factors using spectrums. [8] The main difference from his work and ours is that Harding [2013] takes only the first few moments, while our method uses the whole probability density, and takes into account the characteristic aspects of both spikes and bulk of the covariance matrix. Thus, our method does not need to decide how many moments to take, and is free from the instability in using high-order moments.

Moreover, the consistency of the estimators based on largest eigenvalues is supported by the theoretical work of Paul and Silverstein [2009]. Motivated by Zhang [2006], Paul and Silverstein [2009] show, with additional assumptions, that there will be no eigenvalues in any closed interval outside the support of the limiting distribution.

The simplest non-null model mimicking the empirical eigenvalue distributions is that the population covariance is a finite rank perturbation from the identity matrix. This so-called spiked population model was proposed by Johnstone [2001] and extensively studied by Baik and Silverstein [2006]. The estimation of population convariance matrix eigenvalues has been studied in the literature before (Lawley [1956]; Mestre [2008]; El Karoui [2008]). A recent work by Ledoit and Wolf [2015] develops an estimator of the population eigenvalues that is consistent under large-dimensional asymptotics. In particular, the authors claim that the sample covariance matrix overestimate the variance of larger principal components, which result in selecting fewer principal components for the total-variation-explanation rule. The covariance matrix estimation using factor model has been studied in Fan et al. [2008] and Fan et al. [2013].

3 Example: problematic fit of MP-law to real data

In this example, we apply the Marchenko-Pastur (MP) law (Marchenko and Pastur [1967]) for residuals from real market data and from synthetic data. The real data is extracted from daily returns of 400 stocks in S&P500 during 2012-2015. The synthetic data is generated

[8] Our work was started from the mismatch between real data and Marchenko-Pastur law, and independently studied, we discovered Harding [2013] also employs the notions of spectrums.

by the following model[9]

$$R_{it} = \sum_{j=1}^{p} L_{ij} F_{jt} + U_{it} \tag{3}$$

where $F_{jt} \sim N(0, 0.1^2)$, $L_{ij}, U_{it} \sim N(0, 1)$ are independent, and the true number of factors p is set to be 3. With synthetic data, the spectral density of the residuals converges to the MP-law, when the true number of factors (3) are subtracted, as shown in Figure 1. On the contrary, as seen from Figure 2, the density with real data residual does not fit to the MP-law, no matter how many factors are subtracted.

We confirmed that using correlation matrix and its eigenvalues has the same problem.

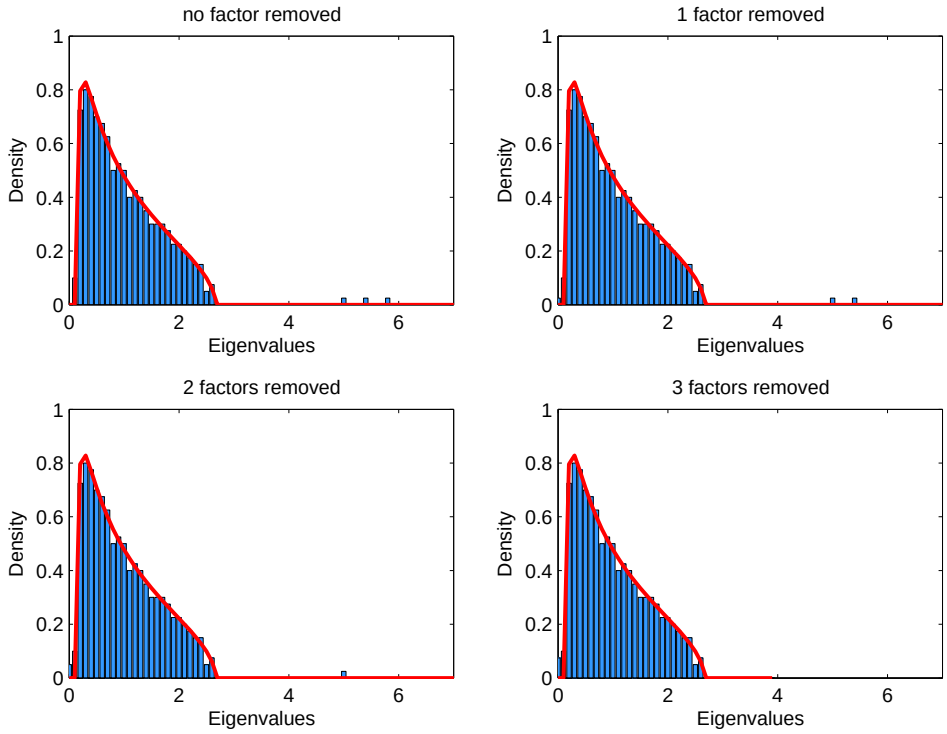

Figure 1
Eigenvalue distribution of covariance matrix of residuals from synthetic data, when few principal components are removed. The data is extracted with N=400 and T=1000. The Marchenko-Pastur (MP) law is plotted with and the bulk is fit by MP-law. The true number of factors, p, is set to be 3.

[9]This model is commonly used

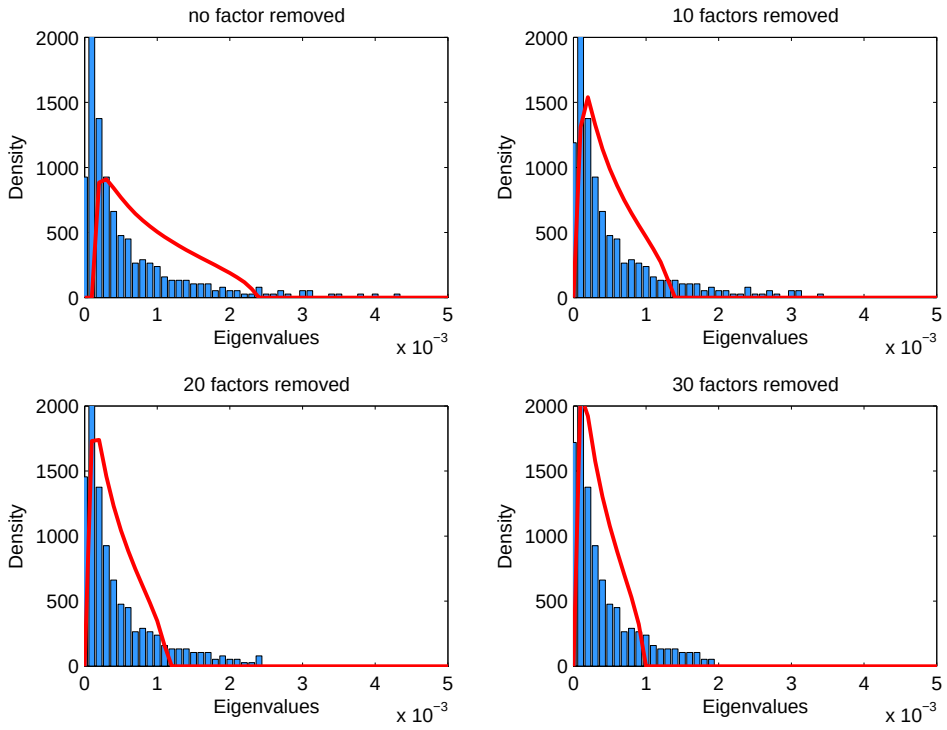

Figure 2
Eigenvalue distribution of covariance matrix of residuals from real data. No matter how many factors are removed, the residual parts cannot be fit by MP-law.

4 Preliminaries

This section provides preliminaries that are required for the supporting theory to our estimation method.

Definition 1 (Empirical spectral distribution)**.** *Let A_n be an $n \times n$ matrix having real eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$. Then the empirical spectral distribution of A_n is defined as*

$$\mathcal{F}^{A_n}(x) \quad = \quad \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{\{\lambda_i(A_n)\leq x\}} \tag{4}$$

where $\mathbf{1}_{\{.\}}$ denotes the indicator function of the set $\{\cdot\}$.

Definition 2 (The Stieltjes transform)**.** *Let $\mathcal{F}(x)$ be any function of bounded variation. Then the Stieltjes transform of $\mathcal{F}(x)$ is defined as*

$$m_F(z) = \int \frac{1}{x-z} d\mathcal{F}(x), \quad (Im(z) > 0) \tag{5}$$

Assumption 1. *The general covariance matrix C_N has the form*

$$C_N = \frac{1}{T} A_N^{1/2} \epsilon B_T \epsilon^T A_N^{1/2} \tag{6}$$

where ϵ is an $N \times T$ random matrix with i.i.d. entries, and A_N and B_T are deterministic symmetric semi-definite matrices of size $N \times N$ and $T \times T$, respectively.

Assumption 2. *$T = T(N)$ and there exists a positive constant c such that*

$$\frac{N}{T(N)} \to c, \quad as \quad N \to \infty, \tag{7}$$

Assumption 3. *ϵ_{it} are i.i.d and $\mathbf{E}\epsilon_{it} = 0$, $\mathbf{E}|\epsilon_{it}|^2 = 1$, $\mathbf{E}|\epsilon_{it}|^4 < \infty$.*

Assumption 4. *$\mathcal{F}^{A_N}$ and $\mathcal{F}^{B_T}$ weakly converge to non-random probability density functions $\mathcal{F}^A$ and $\mathcal{F}^B$, as $N \to \infty$.*

Assumption 5. *$\parallel A_N \parallel$ and $\parallel B_T \parallel$, the respective spectral norms of A_N and B_T, are bounded in N.*

Note that the class of matrices of the form C_N in Assumption 1 appears in various applications, such as multiple-input multiple-output (MIMO) system in wireless communications or in financial time series where A_N and B_T represent the cross- and serial-correlation structure of data. This class of model is also known as the separable covariance model, since there is no space-time interaction. As will be discussed in the following section, the motivation of this assumption is natural, as the approximate factor model allows

cross-sectional and serial correlations in residuals. The Assumption 2 requires that N and T are comparable asymptotically. Assumption 3 indicates moment conditions, so that the maximum eigenvalues of $\frac{1}{T}\epsilon\epsilon^T$ do not diverge. Assumption 4 restates the convergence of the empirical spectral distribution to non-random limiting distributions. Assumption 5 restricts unusual large variations of idiosyncratic components.

Now we are ready to state the main result of Zhang [2006].

Lemma 1 (Zhang [2006]). *If Assumptions 1 to 4 hold, the eigenvalue distribution of* $C_N = \frac{1}{T}A_N^{1/2}\epsilon B_T \epsilon^T A_N^{1/2}$ *converges weakly to a non-random distribution* $\mathcal{F}^{c,A,B}$. *The Stieltjes transform of* $\mathcal{F}^{c,A,B}$, $m(z)$, *together with other analytical function* $p(z)$ *and* $q(z)$, *constitutes a solution to the system*

$$\begin{cases} m(z) = -z^{-1}(1-c) - z^{-1}c\int \dfrac{1}{1+q(z)x} d\mathcal{F}^A(x) \\ m(z) = -z^{-1}\int \dfrac{1}{1+p(z)y} d\mathcal{F}^B(y) \\ m(z) = -z^{-1} - p(z)q(z) \end{cases}$$

which is unique in the set $\{(m(z), p(z), q(z)) : Im(m(z)) > 0, Im(p(z)) > 0, Im(q(z)) > 0\}$.

This lemma elegantly presents the convergence of the empirical spectral distributions of the general sample covariance matrices and identified the limiting spectral distribution through a system of equations determining its Stieltjes transform. Our method is based on this. Once we assume residuals have doubly-correlated structure, such that $U = A_N^{1/2}\epsilon B_T^{1/2}$, this lemma asserts the convergence of empirical spectrum of residuals. However, we will not solve this system of equations, since the analytic expression with general form of A_N and B_T is implicit as well as very complicated to deal with. Rather, we introduce an approximate model with simple parameterizations, and directly derive the probability distribution of eigenvalues by using the techniques introduced in Burda et al. [2010]. Then we relate the spectrum of the model to real data.

The boundedness of eigenvalues in the support of $\mathcal{F}^{c,A_N,B_T}$ is known as shown in the following lemma.

Lemma 2 (Paul and Silverstein [2009]). *Suppose Assumption 1 to 5 hold. Let* ϵ *have Gaussian entries, or either* A_N *or* B_T *be a diagonal matrix . Then*

$$\mathbf{P}(\text{no eigenvalue of } C_N \text{ appears in } [a,b] \text{ for all large } N) = 1$$

where the interval $[a,b]$ *with* $a > 0$ *lies in an open interval outside the support of* $\mathcal{F}^{c,A_N,B_T}$.

Note that the literatures using eigenvalue cutoff for estimating the number of factors rely on this lemma.

Notation 1 (Factor models)**.** *A factor model for N assets and T observations is written as*

$$R = LF + U \tag{8}$$

where R is an $N \times T$ matrix of data, p is the number of factors, L is an $N \times p$ matrix of factor loadings, F is a $p \times T$ matrix of factors, and U is an $N \times T$ matrix of the idiosyncratic components of residuals.

The rationale of this factor model is to linearly decompose the original signal into systemic components (factors) and idiosyncratic components (residuals). We consider an approximate factor model, which is widely used in financial econometrics and was originally introduced by Chamberlain and Rothschild [1983] and Connor and Korajczyk [1986]. This framework differs from classical static factor models in the sense that the idiosyncratic errors are allowed to be mildly correlated, when both N and T goes to infinity. There are certainly static and dynamic extension of approximate factor models, which are beyond the scope of this paper.

Definition 3 (Principal components)**.** *For any p, the methods of principal components minimizes*

$$\min_{L,F} (NT)^{-1} \|R - LF\|_{Frob} \tag{9}$$

$$s.t. \quad \frac{1}{T} F F^T = I_p \quad or \quad \frac{1}{N} L^L = I_p \tag{10}$$

One solution for the above problem is given as

$$\begin{aligned} \widehat{L} &= \sqrt{N} \times (\texttt{eigenvectors corresponding to} \\ &\quad \texttt{the } p \texttt{ largest eigenvalues of } R^T R) \\ \widehat{F} &= \frac{1}{N} \widehat{L}^T R \end{aligned}$$

Note that as $N, T \to \infty$, common components LF can be consistently estimated by $\widehat{L}\widehat{F}$ (Bai [2003]; Fan et al. [2013]).

5 Factor model estimations

Although Lemma 1 itself has nothing to do with factor models, it can be applied to the spectrum of covariance matrix of residuals in factor models. In this paper, we mainly discuss this connection, with high dimensional settings.

Our estimation method aims to find appropriate matches between two spectra. The method basically needs two quantities. One is the empirical eigenvalue distribution of

residuals that are obtained by removing factors from real data. The other is the empirical eigenvalue distribution of residuals of which the covariance structure is modeled by a parameter set. Once these two distributions are obtained, we minimize the distance between the two, so that we can estimate desired parameters. Figure 3 depicts our estimation procedure.

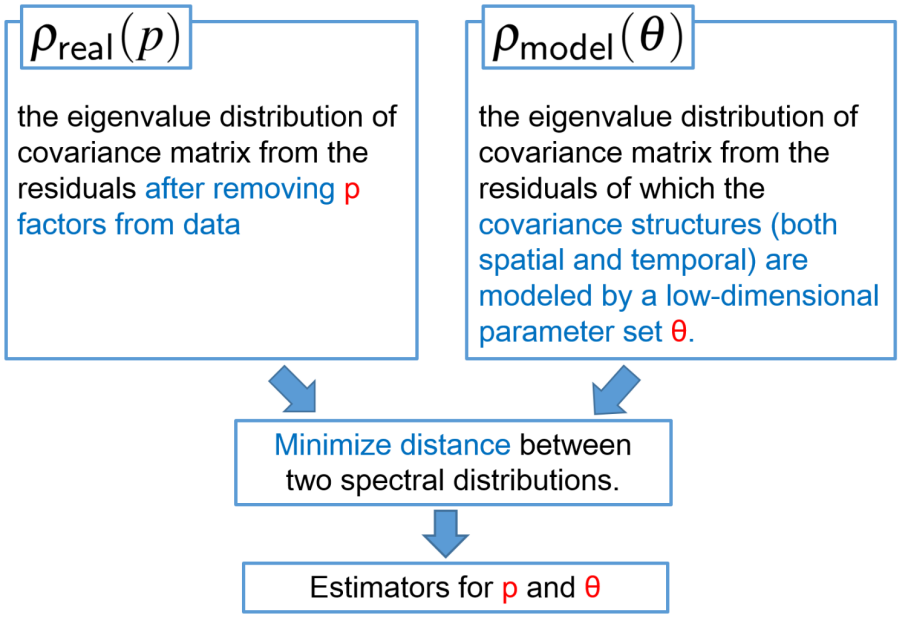

Figure 3
Factor model estimation procedure

5.1 $\rho_{\texttt{real}}(p)$: using principal components

The first step is to generate empirical residuals, by extracting p largest principal components from real data. Here we use principal components as factors. In large dimensional data, principal components determine portfolios that approximately mimic all true factors (Stock and Watson [2002]) up to rotations. Since residuals are defined by how many factors are subtracted, we introduce the following definition.

Definition 4 (p-level residual)**.** *Consider the factor model as in Eq.8. The p-level residual of R, denoted by $\hat{U}^{(p)}$ is defined by*

$$\hat{U}^{(p)} = R - \hat{L}^{(p)}\hat{F}^{(p)} \tag{11}$$

where $\hat{F}^{(p)}$ is an $p \times T$ matrix of p factors, each row of which is a $j(j = 1, \cdots, p)$-th principal component from correlation matrix of R, $\hat{L}^{(p)}$ is an $N \times p$ matrix of factor loadings, estimated by multivariate least squares regression of R on $\hat{F}^{(p)}$.

If more than one factor actually exists, $\hat{U}^{(p)}$ can be always calculated for $p \geq 1$. Then the covariance matrix from p-level residuals is obtained as

$$C^{(p)}_{\texttt{real}} = \frac{1}{T}\hat{U}^{(p)}\hat{U}^{(p)T}. \tag{12}$$

The subscript `real` indicates that it is constructed from real market data. We aim to find the number of factors from spectral distribution of $C^{(p)}_{\texttt{real}}$, by controlling p in our algorithm. The idea behind this is simple. We keep subtracting factors until the bulk spectrum from the residuals using real data becomes close to that from modeled residuals.

5.2 $\rho_{\texttt{model}}(\theta)$: modeling covariance of residuals

The next step is to model the covariance structure of residual processes. Let the residuals have a certain covariance structure, characterized by parameters θ_{A_N} and θ_{B_T}, for cross-covariance matrix A_N and auto-covariance matrix B_T, respectively. Then we can assume the residual term in Eq.8 has a structure of the form[10]

$$U = {A_N}^{1/2}\epsilon{B_T}^{1/2} \tag{13}$$

where ϵ is an $N \times T$ uncorrelated random matrix with i.i.d. entries, and A_N and B_T satisfy Assumption 1, representing the cross- and auto- covariance structures, respectively. For now, we keep using the notation for the arbitrary form of A_N and B_T.[11] Then the empirical covariance matrix of U is given as

$$C_N = \frac{1}{N}UU^T = \frac{1}{T}A_N^{1/2}\epsilon B_T\epsilon^T A_N^{1/2} \tag{14}$$

Note that if empirical spectral distribution of A_N and B_T converge, it is shown that the spectral distribution of C_N converges to a suitable limit, when N and T are large. (see Lemma 1.

5.3 Factor model estimation

The estimation problem is stated here. We solve a minimization problem which searches for an effective parameter set for covariance matrix of residual processes and the number of factors such that the distance between the spectrum from a model and that from real data is minimized.

$$\{\hat{p}, \hat{\theta}\} = \arg\min_{p,\theta} \mathcal{D}\Big(\rho_{\texttt{real}}(p), \rho_{\texttt{model}}(\theta)\Big) \tag{15}$$

[10] This model is known as Kronecker model, and widely used in communications (Kermoal et al. [2002]) and recently introduced in econometrics (Onatski [2010]).

[11] Later, we restrict the model by simplification. See Section 5.3.2 for simplified parametrization and approximation.

where $\rho_{\texttt{real}}(p)$ is the eigenvalue distribution of $C^{(p)}_{\texttt{real}}$, $\rho_{\texttt{model}}(\theta)$ is a limiting eigenvalue density of the general covariance matrix characterized by a parameter set $\theta = (\theta_{A_N}, \theta_{B_T})$, and $\mathcal{D}$ is a spectral distance measure or loss function we choose. This simultaneously estimates for the number of factors and parameters of residual correlations. We state the consistency of the estimators here.

5.3.1 Distance measure

Since the empirical spectrum contains spikes, not all distance measures are useful in this problem. Our method needs a measure that must be sensitive to the presence of spikes as well as account for correctly reflect the distribution from grouped eigenvalues. We tested several distance metrics[12], for the covariance matrices we consider. We use Jensen-Shannon divergence, which is a symmetrized version of Kullback-Leibler divergence.

$$\mathcal{D}_{JS}(P\|Q) = \frac{1}{2}\mathcal{D}_{KL}(P\|M) + \frac{1}{2}\mathcal{D}_{KL}(Q\|M) \tag{16}$$

where P and Q are probability densities, $M = \frac{1}{2}(P+Q)$ and $\mathcal{D}_{KL}(P\|Q)$ is the Kullback-Leibler divergence defined by $\mathcal{D}_{KL}(P\|Q) = \sum_i P_i \log \frac{P_i}{Q_i}$. Note that the Kullback-Leibler distance becomes larger if one density has a spike at a point while the other is almost zero at that point. Using this measure, in addition, the information disparity in the bulk region is also taken into account. Further discussion on its numerical calculation of Kullback-Leibler divergence with discretized grids is in Appendix B.

5.3.2 Simplified model

As discussed earlier, the calculation of $\rho_{\texttt{real}}(p)$ is straightforward when using principal components estimators as factors. A difficulty lies in the calculation of the limiting distributions, $\rho_{\texttt{model}}(\theta)$, for general $\theta = (\theta_{A_N}, \theta_{B_T})$. Although Lemma 1 guarantees the convergence of empirical spectral distribution to a suitable limit, and the Stieltjes transforms obtained by Lemma 1 provide useful information on the limiting distribution, the actual calculation of it is quite complex, which makes the implementation hard. However, a recent study of Burda et al. [2010] provides the direct derivation of spectral density using free random variable techniques. They particularly present analytic forms when the time-series follows vector autoregressive processes. In this paper, we employ this technique to calculate the spectrum $\rho_{\texttt{model}}(\cdot)$. For this, we propose a simplified modeling for A_N and B_T, from mean-field model for the residual processes.

[12] such as Kolmogorov-Smirnov metric, Levy metirc, Kullback-Leibler distance, p-norm, Chi-square metric, Bhattacharyya distance distance, and others.

Mean-field model on spectrum

A mean-field model is used to study the behavior of large and complex stochastic models by investigating a simpler model.[13] In factor models, each idiosyncratic return has its own driving force, namely a field. Analogous to traditional mean-field theory, rather than considering every individual residual separately, we consider single correlation structure that enables us to approximately replicate the spectral density of the original heterogenous correlation structures.

Assumption 6 (Mean-field model on spectrum)**.** *Suppose we have two $N \times T$ matrices, Y and Z, such that*

$$Y_{it} = b_i Y_{i,t-1} + \xi_{it} \tag{17}$$

$$Z_{it} = \bar{b} Z_{i,t-1} + \eta_{it} \tag{18}$$

where $|b_i| < 1$, $\bar{b} = \frac{1}{N}\sum b_i$, $\xi_{it} \sim N(0, \sigma_i^2)$ and $\eta_{it} \sim N(0, \overline{\sigma}^2)$. Let $\sigma_i^2 = 1 - b_i^2$ and $\overline{\sigma}^2 = 1 - \bar{b}^2$, so that $\mathbf{var}(Y) = \mathbf{var}(Z) = 1$. Consider two empirical spectral distributions, ρ_{C_Y} and ρ_{C_Z}, where $C_Y = \frac{1}{T} Y Y^T$ and $C_Z = \frac{1}{T} Z Z^T$. Then the distance between ρ_{C_Y} and ρ_{C_Z} becomes sufficiently small, as N, T are large.

$$\mathcal{D}\Big(\rho_{C_Y}, \rho_{C_Z}\Big) \approx 0 \tag{19}$$

For this assumption, we provide a numerical illustration as in Figure. 4.[14] We draw random samples for b_i, from uniform distribution between 0 and 1, and take $\bar{b} = 0.35, 0.5, 0.65$. The synthetic data sets for Y and Z are generated from the above autoregressive processes, and eigenvalue distributions of C_Y and C_Z are presented. Note that the spectrum of ρ_{C_Y} (red line) is the closest to ρ_{C_Z} when $\bar{b} = 0.5$ (black line), and the spectral distance (Kullback-Leibler distance in this case) is minimized at the same point.

Factor model estimation with simplified model

Now we propose a simplified model, which has much more simplified parameter sets, for A_N and B_T.

Assumption 7 (Reduced parameterizations)**.** *(1) The cross-correlations are effectively removed from p principal components, where p is the true number of factors, and the residual $U^{(p)}$ has sufficiently negligible cross-correlation: $A_N \approx I_{N\times N}$.*

[13] For example, in magnetism in quantum spin systems, mean-field theory says that spin moves in the average field produced by all other spins. Usually in high dimensional systems, mean field theory gives a good picture of phase transitions. Our regime-change argument will be also based on estimated value of this single parameter.

[14] The theoretical proof is not straightforward, since the correlation structure for Y is not separable, which is beyond the scope of this section.

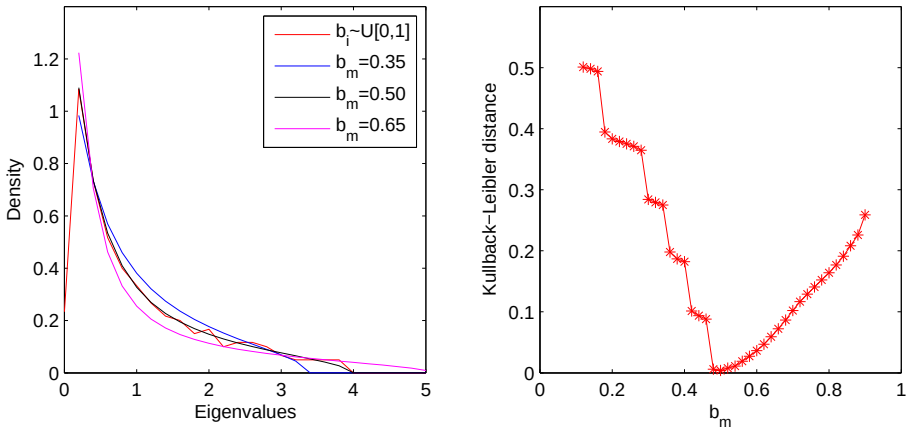

Figure 4
Numerical demonstration of Assumption 7. Left: eigenvalue distribution from heterogenous process Y with $b \sim U[0,1]$ (red), and from homogeneous autoregressive processes Z with $\bar{b}$=0.35, 0.50, and 0.65, for $N = 300, T = 600$. Right: plot of $\mathcal{D}\Big(\rho_{C_Y}, \rho_{C_Z}\Big)$. Note that the distance is minimized and almost zero near $\bar{b} = 0.50$.

(2) The autocorrelations of U are exponentially decreasing (by an identical rate) with respect to time-lags: $\{B_T\}_{ij} = b^{|i-j|}$, with $|b| < 1$. (This is equivalent to modeling residual returns as an AR(1) process: $U_{it} = bU_{i,t-1} + \xi_{it}$, where $\xi_{it} \sim N(0, 1-b^2)$ so that the variance of U_t is one.)

From Assumption 7 and Assumption 6, we approximate the original estimation by using only two control variables, the number of factors, p, and the global mean-reversion rate b. The estimation with simplified parameterizations is stated as

$$\{\hat{p}, \hat{b}\} = \arg\min_{p,b} \mathcal{D}\Big(\rho_{\texttt{real}}(p), \rho_{\texttt{model}}(b)\Big). \tag{20}$$

For numerical experiments, we work with this simplified model. Although it seems to be too simple at the first glance, we will show that it sufficiently improves the robustness to noise levels and the ability of detecting weak factors.

Remarks

Although this model came from our assumptions on covariance matrices, it actually has several benefits.

First, it makes the calculation of the density almost analytically. The numerical process to obtain the spectral density $\rho_{\texttt{model}}(b)$ is straightforward, if we use the free-random variable techniques.

Second, the two parameters reflect the essential features of typical spectra of covariance matrices we considers. As shown before, the spectrum is roughly decomposed into two parts: spikes and a bulk. The parameter p controls the number of spikes in the residuals. As we subtract p factors from data, then p spikes that correspond to the p largest eigenspaces are removed from the spectrum of the original data. At the same time, the parameter b controls the region of smaller eigenvalues. Although it does not represent all possible shapes of bulks, it can effectively emulate the variability of the bulk spectrum of residuals. Based on the numerical results, it turns out that the edge of the bulk is sufficiently controllable within the desired numerical precisions. In addition, we also found from the Monte Carlo simulations that the number of factors is still accurately estimated by the method that uses only b.

Third, the parameter b is an aggregate quantity that represents the rate of mean-reversion of residual returns. The dynamics of residual spaces has received a significant attention in recent years. Although it cannot directly be applied to any practical use such as trading, the characterization of residual subspace of real markets using this single parameter provides an insight into market dynamics.[15]

Calculation of $\rho_{\texttt{model}}(b)$

The simplified problem enables us to calculate the modeled spectral density, $\rho_{\texttt{model}}(b)$, more easily. It can be done by using the free random variable techniques proposed in Burda et al. [2010]. We briefly describe only the implementation steps here.

1. Get the mean spectral density from the Green's function $G(z)$ by using the Sokhotsky's formula:

$$\rho_{\texttt{model}}(\lambda; b) = -\frac{1}{\pi} \lim_{\epsilon \to 0^+} \mathfrak{Im} G_{\mathbf{c}}(\lambda + i\epsilon). \tag{21}$$

2. The green's function $G(z)$ can be obtained from the moments' generating function $M(z)$.

$$M(z) = zG(z) - 1 \tag{22}$$

3. Solve the polynomial equation for $M = M(z)$ ($a = \sqrt{1-b^2}$ and $c = N/T$):

$$\begin{aligned} a^4c^2M^4 + 2a^2c\big(-(1+b^2)z + a^2c\big)M^3 &+ \\ \big((1-b^2)^2z^2 - 2a^2c(1+b^2)z + (c^2-1)a^4\big)M^2 - 2a^4M - a^4 &= 0 \end{aligned} \tag{23}$$

See Appendix A for details.

[15] A possible outcome of the simplified model is that, if the true residual processes are only cross-correlated but not auto-correlated, then the solution of the minimization problem gives a number for b, which may be far different from the true value. From a spectrum point of view, this number is equivalent to the true number.

6 Monte Carlo analysis

6.1 Experiments setup

We first evaluate the performance of our estimation method by Monte Carlo studies. To obtain data sets, we generate synthetic data, using the following model:

$$
\begin{aligned}
X_{it} &= \sum_{j=1}^{p} L_{ij}F_{jt} + \sqrt{\theta}U_{it}; \\
&with \\
U_{it} &= \sqrt{\frac{1-\rho^2}{1+2J\beta^2}}e_{it} \\
&where \\
e_{it} &= \rho e_{i,t-1} + v_{it} + \sum_{h=\max(i-J,1)}^{i-1} \beta v_{ht} + \sum_{h=i+1}^{\min(i+J,N)} \beta v_{ht} \\
v_{ht}, L_{it}, F_{jt} &\sim N(0,1)
\end{aligned}
$$

This model is also used in Bai and Ng [2002], Onatski [2010], and Ahn and Horenstein [2013]. The rationale of this model is as follows.

1. The coefficient $\sqrt{\frac{1-\rho^2}{1+2J\beta^2}}$ makes the variance of U_{it} be always 1. This allows the model to control the variance (or noise) level of residuals only by θ.

2. θ controls the *signal-to-noise ratio* (SNR), where $SNR = \frac{\mathbf{var}\,(Factors)}{\mathbf{var}\,(Residuals)} = \frac{p}{\theta}$. We will use 1/SNR instead of θ to denote the noise level. For example, if 1/SNR = 0.25, this implies $\theta = 0.25 \times p$.

3. ρ controls the decaying rate of auto-correlations of residuals. ($|\rho| < 1$)

4. Cross-correlations of residuals are controlled by β for magnitudes $|\beta| \leq 1$ and by J for affecting ranges. Since this local cross-correlations can be broader for larger system in practice, we set J is proportional to N, i.e., $J = N/10$.

The model parameters used in our Monte Carlo analysis are summarized in Table 1.[16]

We first investigate the performance of our method. Next, the estimated number of factors from our method is compared with those came from other three methods of Bai and Ng [2002], Onatski [2010], and Ahn and Horenstein [2013]. Our numerical experiments have several perspectives. We first examine the convergence rate of error when the sample

[16] We also have tried other possible values, but there is no other significant implications found from those.

Sample sizes	N, T	$\{50, 100, 200, 300, 500\}$
Number of factors	p	$\{3, 4, 5\}$
1/SNR	θ	$\{0.1, 0.25, 0.5, 0.75, 1, 1.5, 2, 3\} \times p$
Correlations in residuals	(ρ, β, J)	$\{(0, 0, 0), (0.5, 0, 0), (0, 0.5, N/10), (0.5, 0.5, N/10)\}$

Table 1
Parameter configurations used in the Monte carlo experiments.

size becomes small or large. Next, the effect of the different residual correlation structures on the estimation error is investigated. The performance with various noise levels is tested. Lastly, we tested the detection ability in the presence of weak factors. As an error measure, the root mean squared error (RMSE) is obtained over 1000 replications. Before computing eigenvalues and eigenvectors, each series is demeaned and standardized to have unit variance.

6.2 Performance of our method

We first check the performance of our method on estimating factor models. Table 2 summarizes the averages of $\hat{p}$ and $\hat{b}$. We can first observe that the averages of our estimators, $\hat{p}$ and $\hat{b}$, are very close to the true number of factors and true auto-correlation coefficient for a broad range of N and noise 1/SNR. One exception is when the sample size is small and the noise amount is large, where our estimator starts to underestimate the true number of factors.

The true correlation structures are also varied in the test. ρ represents the identical auto-regressive coefficient for residuals and β represents the cross-correlation within the range of J in the matrix. For the first case where there is no correlation in residuals, as $(\rho, \beta) = (0, 0)$, the estimator $\hat{b}$ gives numbers between 0.03 and 0.05 which is close to the true value 0. When auto-correlations are imposed, as $(\rho, \beta) = (0.5, 0)$, $\hat{b}$ is also very close to the true value 0.5. Adding cross-correlation structure here, as $(\rho, \beta) = (0.5, 0.5)$, shifts the average value and decreases the accuracy, but not significantly. This is due to the fact that in our experiment setup the contribution of local cross-correlations on the spectrum is insignificant to that of auto-correlations. However, when only cross-correlations are forced in true residual processes, as $(\rho, \beta) = (0, 0.5)$, the average of $\hat{b}$ is going far from the true value 0, giving values between 0.1 and 0.25. We interpret that from a spectral point of view, this deviated $\hat{b}$ has an effect on the spectrum approximately [17] equivalent the contribution from cross-correlations. However, we emphasize that this cross-correlation-only structure does not decrease the accuracy $\hat{p}$, as seen from the table.

[17] within numerical tolerance on the spectral distance

		$\rho, \beta = (0,0)$		$\rho, \beta = (0.5,0)$		$\rho, \beta = (0.5,0.5)$		$\rho, \beta = (0,0.5)$	
N, T	1/SNR	$\hat{p}$	$\hat{b}$	$\hat{p}$	$\hat{b}$	$\hat{p}$	$\hat{b}$	$\hat{p}$	$\hat{b}$
50	0.10	4.000	0.048	4.006	0.489	4.006	0.495	4.000	0.217
50	0.25	4.000	0.047	4.024	0.483	4.031	0.489	4.000	0.200
50	0.50	4.011	0.047	4.041	0.478	4.052	0.484	4.002	0.194
50	0.75	4.034	0.047	3.999	0.478	4.002	0.484	4.002	0.192
50	1.00	4.069	0.047	3.817	0.489	3.826	0.493	3.987	0.192
50	1.50	4.086	0.047	3.657	0.495	3.564	0.502	3.822	0.211
50	2.00	4.030	0.046	3.604	0.485	3.616	0.490	3.541	0.242
50	3.00	3.665	0.045	3.653	0.456	3.560	0.465	3.434	0.251
100	0.10	4.000	0.050	4.000	0.504	4.000	0.505	4.000	0.172
100	0.25	4.000	0.049	4.000	0.503	4.000	0.503	4.000	0.162
100	0.50	4.000	0.049	4.000	0.502	4.000	0.502	4.000	0.159
100	0.75	4.000	0.049	4.000	0.501	4.001	0.501	4.000	0.158
100	1.00	4.000	0.049	4.000	0.501	4.001	0.501	4.000	0.159
100	1.50	4.002	0.049	4.001	0.500	4.001	0.500	4.000	0.158
100	2.00	4.002	0.049	3.973	0.501	3.991	0.500	4.000	0.157
100	3.00	4.008	0.049	3.640	0.517	3.813	0.511	3.992	0.158
150	0.10	4.000	0.039	4.000	0.505	4.000	0.505	4.000	0.132
150	0.25	4.000	0.038	4.004	0.505	4.004	0.504	4.000	0.119
150	0.50	4.000	0.038	4.015	0.505	4.009	0.504	4.000	0.113
150	0.75	4.000	0.038	4.017	0.505	4.015	0.504	4.000	0.112
150	1.00	4.017	0.038	4.019	0.505	4.020	0.504	4.001	0.111
150	1.50	4.061	0.039	4.038	0.504	4.037	0.503	4.001	0.110
150	2.00	4.065	0.039	4.060	0.503	4.061	0.502	4.005	0.109
150	3.00	4.060	0.039	4.114	0.501	4.099	0.501	4.017	0.106
200	0.10	4.000	0.050	4.000	0.506	4.000	0.507	4.000	0.118
200	0.25	4.000	0.050	4.000	0.506	4.000	0.506	4.000	0.114
200	0.50	4.000	0.050	4.000	0.505	4.000	0.506	4.000	0.112
200	0.75	4.000	0.050	4.000	0.505	4.000	0.506	4.000	0.112
200	1.00	4.000	0.050	4.002	0.505	4.001	0.506	4.000	0.112
200	1.50	4.000	0.050	4.002	0.505	4.002	0.506	4.000	0.112
200	2.00	4.005	0.050	4.001	0.505	4.002	0.505	4.000	0.112
200	3.00	4.061	0.050	4.008	0.505	4.004	0.505	4.000	0.111

Table 2
Average values of the estimated p and b over 1000 simulations. There are four different residual correlation structures: $(\rho, \beta) = (0,0), (0.5,0), (0.5,0.5), (0,0.5)$, and $J = N/10$. True number of factors is $p = 4$. Note that if $\beta = 0$, $\hat{b}$ must be an estimator of ρ, since in this case, the generating model for synthetic data and our assumed model for reduced problem are exactly the same. Otherwise, $\hat{b}$ does not necessarily converge to ρ, as seen from the last column, for example. The tables for RMSE for each estimate is provided in a supplemental report.

6.3 Comparison with other methods

In this section, we will compare estimators from our spectral distance (SD, hereafter) method with other methods, including the BIC3 estimator of Bai and Ng [2002] that uses information criteria, the ED estimator of Onatski [2010] that uses eigenvalue differences, and the ER estimator of Ahn and Horenstein [2013] that uses eigenvalue ratios.

6.3.1 Sample sizes and noise amounts

Figure 5 reports the convergence speed of estimators with respect to sample sizes. The true number of factors is 4, and we set $T = N$. Residuals have correlations, as $(\rho, \beta) = (0, 0.5)$ or $(0.5, 0.5)$, and $J = N/10$. As seen from the figure, it is clear that the estimators are generally converging to the true number of factors as N and T become large. When the amount of noise is small, BIC3 and ER converges the fastest. However, as the noise level increases, our estimator outperforms others especially with small sample sizes.

This result is also reflected in Figure 6, where the graphs of RMSE are drawn with respect to the noise level. Clearly, higher noise levels inhibit the estimation precisions. In addition, it is easy to observe that SD is less sensitive to noise amount than other methods, especially for smaller sample size ($N = 100$). For larger sample size ($N = 200$), ER shows the best performance, followed by SD which is still stable from noise disturbance. We also discovered that the considered cross-correlation structure is less affected than auto-correlation structure from increasing noise amounts. In the meantime, BIC3 is the most vulnerable to noise levels.

6.3.2 Presence of weak factors

Detecting weak factors is generally harder than strong factors. In this section, similar to the experiments in Ahn and Horenstein [2013], we study the influence of weak factors on the estimated number of factors.

To construct weak factors, we reduce the variance of f_{jt} in Eq.24 to be less than one: $f_{jt}^{weak} \sim N(0, \sigma_{weak}^2)$, with $\sigma_{weak} < 1$. We set four true factors, we consider the case where all four factors are weak and the case where only three factors are weak. The performance with weak factors is compared in Figure 7. Clearly, if factors get weaker (smaller σ_{weak}), it becomes harder to detect the those weak factors, which results in increasing estimation errors as presented in the figure. In addition, if there is one stronger factor and several weak factors, it is generally more difficult to distinguish weak ones. This explains the fact that the overall RMSE values on the left column is larger than that on the right column.

More importantly, this figure provides evidence that our method (SD) has more powerful ability to identify weak factors, compared to other methods, from all of the considered cases. There are several possible explanations for this result. Note that the spectral distance measure we consider has larger weights for the spikes in the spectrum. Therefore, if the eigenvalues corresponding to weak factors are not diverging much and staying outside

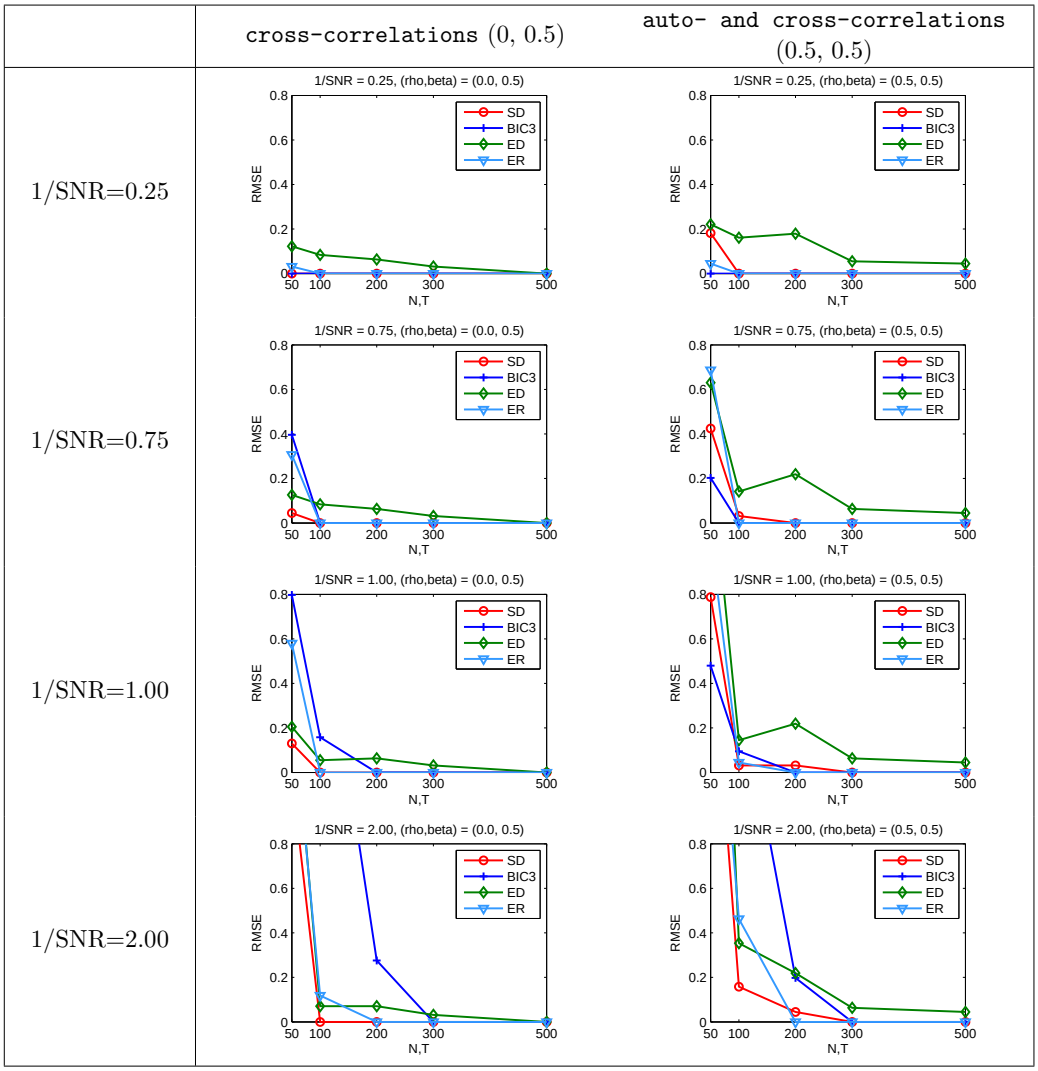

Figure 5
Root mean square errors (RMSE) for the estimated number of factors with respect to N. Each plot is generated with different noise level: 1/SNR=0.1, 0,25, 0.5, 0.75, 1, and 2. Here we set $p = 4$, $T = N$. The residuals have correlation structure: cross-correlations $(\rho, \beta) = (0, 0.5)$ (left) and auto- and cross-correlations $(\rho, \beta) = (0.5, 0.5)$ (right). We set $J = N/10$. Our estimator converges sufficiently well for $N \geq 100$, regardless of signal-to-noise ratios.

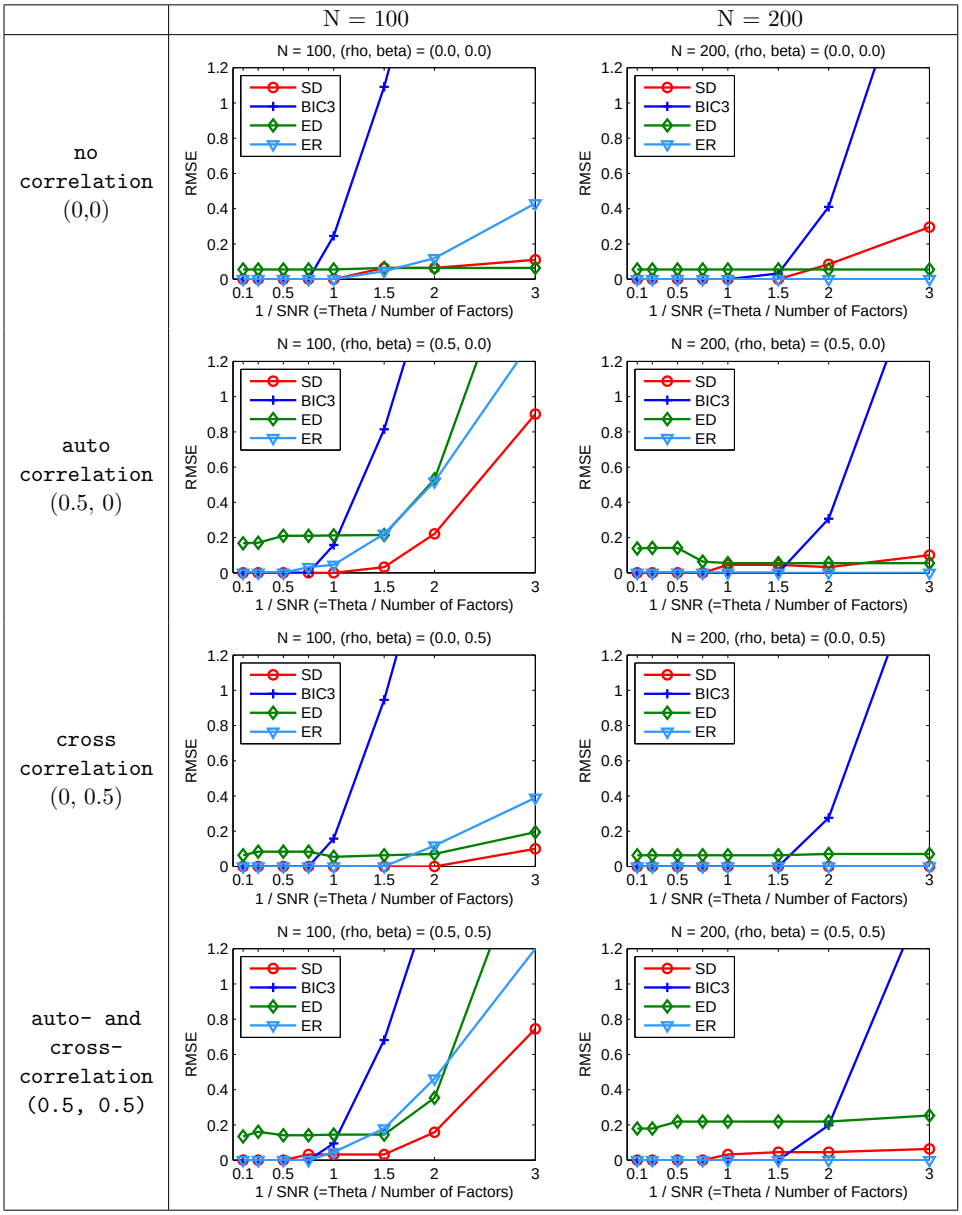

Figure 6

Root mean square errors (RMSE) with respect to noise level (1/SNR). Each column shows results from different $N = T = 100$ (left) and 200 (right). Each row represents different correlation structures in residuals: From top to bottom, $(\rho, \beta) = (0, 0), (0.5, 0), (0, 0.5), (0.5, 0.5)$. The estimation error is increasing when the noise amount of residual becomes larger. Note that for comparatively small sample size ($N \leq 100$), our estimation method is more robust to residual noises than other methods.

the bulk, our algorithm is likely to detect them as factors. Besides, the control parameter b allows to amplify the resolution of detecting those weaker factors. On the other hand, other methods do not have this mechanism in their algorithms.

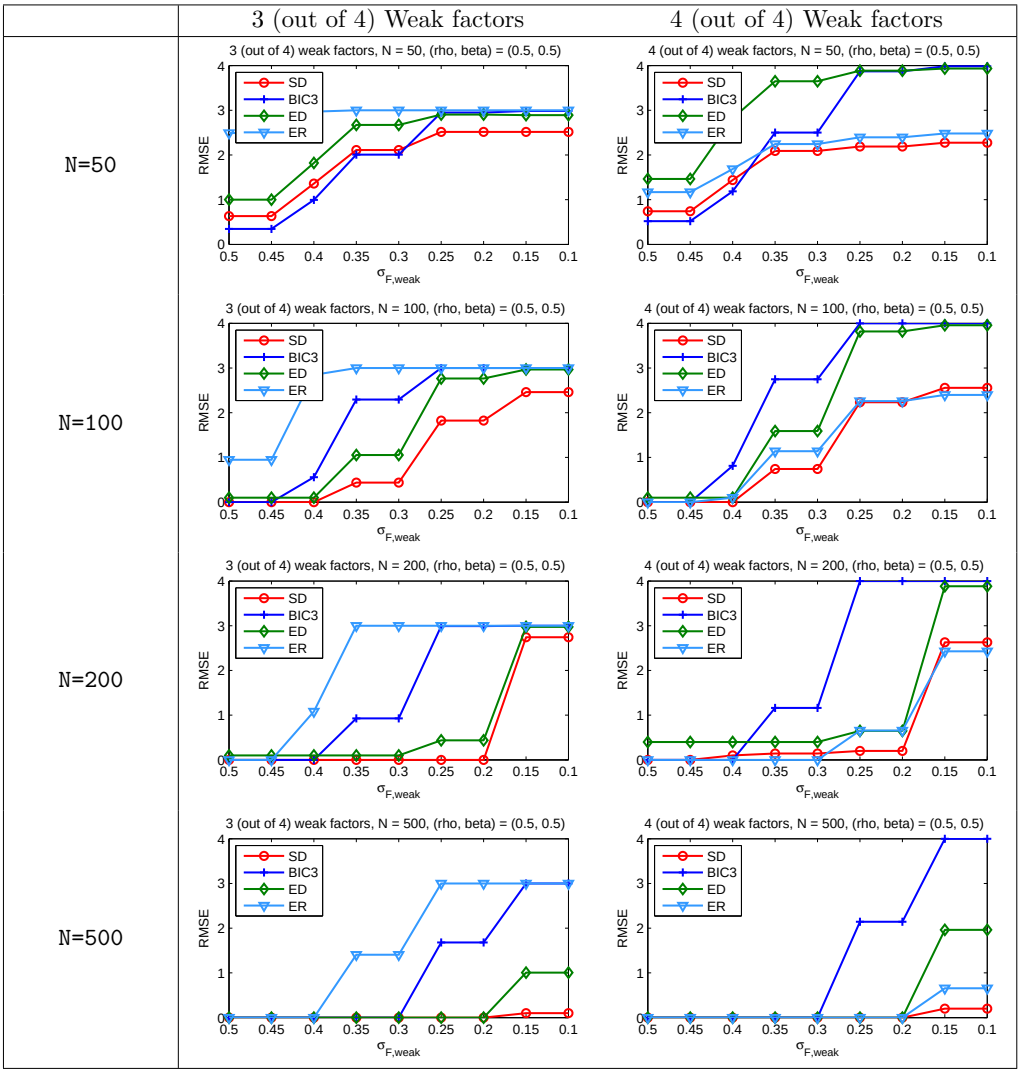

Figure 7
Root mean square errors (RMSE) with respect to variance of weak factors. The true number of factors is 4. The left column represents when 3 factors are weak, and the right column is for when all 4 factors are weak. Sample sizes are also varied: $N = 50, 100, 200, 500$, and $T = N$. Residuals have correlation structures: $(\rho, \beta) = (0.5, 0.5)$, and $J = N/10$. The ability of SD to detect the weak factors are significantly better than other methods.

7 Applications to real data

In this section, we apply the proposed methods to market data. Daily returns of 378 stocks[18] in S&P500 between 2000-2015 are used.

Instead of taking the entire time range at once, we use a certain length of estimation window, and move the window one day at a time. There is an overlap in the data contained in consecutive windows, which enables us to track the temporal evolution of number of factors and correlation structure of residuals. The estimation with moving windows produces $\hat{p}$ and $\hat{b}$ for each day, giving the time-series of estimators.

7.1 Fit of simplified model to real data

Before discussing dynamics of estimated parameters, we first check how well the simplified model can fit the real residuals. In Figure 8, we show several sample fitted results. Four random days are selected in the year of 2001, 2005, 2008, and 2011 and the factor model estimation using the simplified model is applied to each data. Note that the estimated p and b's are different for different data, but it explains well the eigenvalue distribution of correlation matrix of real residuals, compared to corresponding Marchenko-Pastur law.

7.2 Implications of $\hat{p}$

Repetitive estimation procedures with moving windows generates time-series of $\hat{p}$. In order to interpret $\hat{p}$, we compare the number of factors estimated from other methods. Figure 9 reports the estimators. It is clear that our estimator is between 4 and 12, which is mostly larger than others that display 1 to 7 factors. Note that the Fama-French three factor model (Fama and French [1992]) has been used widely in explaining the returns of equity securities.[19] We have not studied whether any of our estimators are identified as Fama-French factors (Mkt, SMD, and HML), but the most likely explanation is that our method identifies several weak factors in addition to strong factors, as we already have seen from the Monte carlo results in Section 6.3.2.

Before we close this section, we investigate what those factors actually consist of. To do this, we examine the components of eigenvectors corresponding to top eigenvalues of correlation matrix of returns. As seen from Table 3, the components in the first eigenvector are very uniformly contributed, which indicates the market mode. From the second, each eigenvector corresponds to business sector. For example, during 2004-2005, the major three factors are Energy, Financial REITs, and Information Technology. However, during 2008-2009, Energy sector takes the second and the third factors, and Financials are the third

[18] those who have survived persistently in the period

[19] But as reported Fan et al. [2016], testing Fama-French model is more often to be rejected when using the daily data, compared to the monthly data, due to a larger volatility of the unexplained factor components.

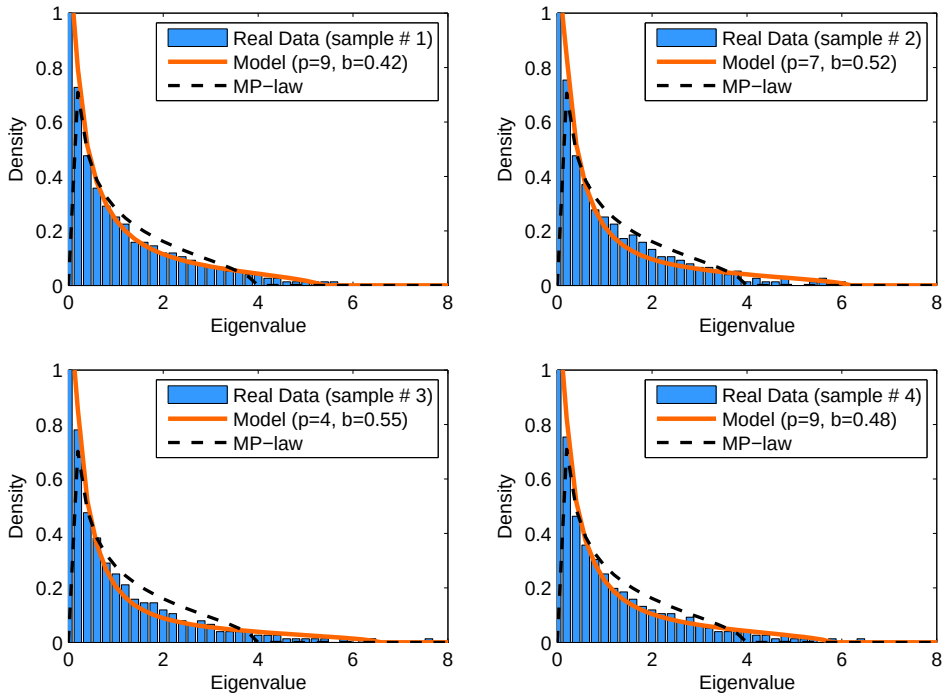

Figure 8
Fit of simplified model to real data. Real data is taken from four different day that is randomly chosen in the year of 2001, 2005, 2008, and 2011. The model density with estimated p and b generally fits the real residuals. For comparison, Marchenko-Pastur (MP) law for the correlation matrix is plotted.

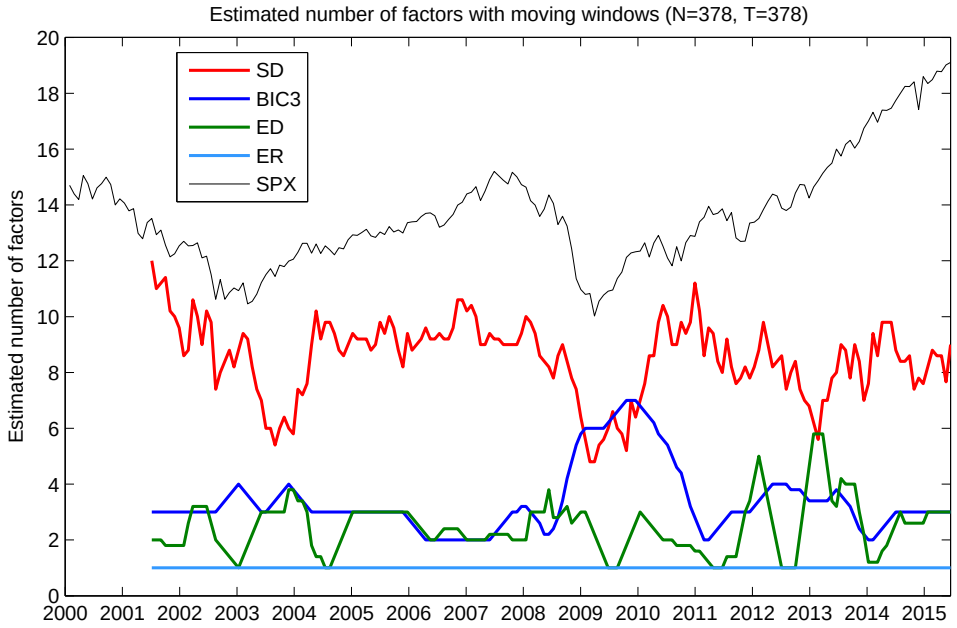

Figure 9
Estimated number of factors are compared with other methods. For real data, the results vary depending on methods. Overall, our estimator (SD) is larger than others. Eigenvalue Ratio (ER) method by Ahn and Horenstein [2013] gives always one factor throughout the whole investigated period. Information criteria based method (BIC3) by Bai and Ng [2002] and eigenvalue difference method (ED) by Onatski [2010] display almost opposite results during crisis (2008-2009) The estimated number of factors from BIC3 is increasing, while that of ED decreases at the same time. The two provide the same number of factors during 2005-2006. The investigated stocks are $N = 378$, time window is $T = 378$ business days.

factor. In any cases, although the principal component factors are constructed from purely statistical procedure, they are closely related to business sectors.

Meanwhile, it is interesting to note that eigenvalue ratio (ER) methods gives one factor all the time. Information criteria-based method (BIC3) shows the nearly opposite behaviors to other methods, especially estimating more factors in crisis. We have not found a clear reason for that.

7.3 Implications of $\hat{b}$

As seen from Figure 11, the estimator for "representative" autoregressive coefficient of residuals, $\hat{b}$, reflects the market movement. Most of time, $\hat{b}$ is mimicking the behaviors of SPX in the opposite direction. Thus, this estimator reflect essential information on market fluctuations.

To illustrate the meaning of $\hat{b}$, we consider $\hat{b}_i$ for each residual, where b_i is the estimated AR(1) coefficient for i-th residual, such that $U_{it} = b_i U_{i,t-1} + \epsilon_{it}$. Let us define an estimator $\hat{b_{ind}} := \frac{1}{N}\sum_{i=1}^{N} |\hat{b}_i|$. Here we take the absolute value for b_i, since the limiting spectrum for vector AR(1) processes depends only on the magnitude of coefficients (see Eq. 23). Figure 10 plots the evolutions of $\hat{b}$ and $\hat{b_{ind}}$. Except the scale difference, the patterns of the two quantities are very similar. They both indicate that during crisis, the residual returns tend to be trending (i.e., having longer mean-reversion times) than normal periods. To check whether these patterns are generic for residuals, we also estimated AR(1) coefficients from each *original* return. We found that although it also increases in crisis, its behavior is not close to $\hat{b}$ compared to that of residuals. Therefore, in the context of Section 5.3.2, $\hat{b}$ is the bulk coefficient that delivers compressed information from of the coefficients of all residuals. The scale of the discrepancy may be due to the fact that $\hat{b}$ coarsely integrates many outliers or complicated correlation cases.

7.4 Market dynamics from estimators

We also calculate the variance explained by $\hat{p}$ factors and the variance per factor. S&P500 index (SPX) and volatility index (VIX) are compared as well. Figure 11 displays time evolution of the above quantities with re-scaled values. Note that each quantity is closely related to each other. The number of factors sharply decreases in the crisis (2008-2009). At the same period, the variance explained per factor is sharply increasing, indicating the market condensation phenomenon. That is, during the major market events in 2008, correlations changed dramatically, even affecting previously uncorrelated sectors. Thus, the whole market moves together, which increases the largest eigenvalues of the correlation matrix.

It is also interesting to note that the estimated b mimics the behaviors of the VIX.

2004-2005						2008-2009					
#	Eigenvalue	Company	Sector	Components	Contributions	#	Eigenvalue	Company	Sector	Components	Contributions
1	0.2485	PPG Industries	Chemical	0.076	0.40%	1	0.5096	E.I. Du Pont De Nemours & Co.	Chemical	0.064	0.33%
		T. Rowe Price Group	Financials	0.074	0.39%			Walt Disney Company	Media	0.062	0.32%
		Northern Trust Corp.	Financials	0.073	0.39%			Illinois Tool Works Inc	Machinery	0.062	0.32%
		Emerson Electric Company	Industrials	0.073	0.39%			PPG Industries	Chemical	0.062	0.32%
		Praxair Inc.	Materials	0.073	0.38%			Franklin Resources	Financial	0.062	0.32%
		SunTrust Banks	Financials	0.072	0.38%			United Technologies	Industrials	0.062	0.32%
		PACCAR Inc.	Industrials	0.072	0.38%			Stanley Black & Decker	Consumer Discretionary	0.062	0.32%
		BB&T Corporation	Financials	0.072	0.38%			T. Rowe Price Group	Financials	0.061	0.32%
		M&T Bank Corp.	Financials	0.072	0.38%			Equifax Inc.	Financials	0.061	0.32%
		Realty Income Corporation	Real Estate	0.071	0.38%			Invesco Ltd.	Financials	0.061	0.31%
2	0.0571	Apache Corporation	Energy	0.167	1.26%	2	0.0499	Noble Energy Inc	Energy	0.112	0.72%
		Chesapeake Energy	Energy	0.165	1.25%			Apache Corp	Energy	0.112	0.72%
		Newfield Exploration Co	Energy	0.163	1.23%			Southwestern Energy Company	Energy	0.109	0.70%
		Diamond Offshore Drilling	Energy	0.161	1.22%			Anadarko Petroleum Corp	Energy	0.108	0.69%
		EOG Resources	Energy	0.161	1.22%			Entergy Corp	Energy	0.108	0.69%
		Noble Energy Inc	Energy	0.161	1.22%			National Oilwell Varco Inc.	Energy	0.106	0.68%
		Ensco plc	Energy	0.161	1.22%			Diamond Offshore Drilling	Energy	0.105	0.68%
		Hess Corporation	Energy	0.160	1.21%			Cabot Oil & Gas	Energy	0.105	0.67%
		Occidental Petroleum	Energy	0.158	1.20%			Chesapeake Energy	Energy	0.104	0.67%
		Transocean	Energy	0.158	1.19%			Kimco Realty	Financials	(0.104)	(0.67%)
3	0.0345	Boston Properties	Financials REITs	(0.161)	(1.12%)	3	0.0304	Freeport-Mcmoran Inc	Mining	0.139	0.90%
		Macerich	Financials REITs	(0.159)	(1.11%)			Consol Energy	Energy	0.133	0.86%
		Simon Property Group Inc	Financials REITs	(0.156)	(1.09%)			Southern Co.	Energy	(0.128)	(0.83%)
		Public Storage	Financials REITs	(0.151)	(1.05%)			Cameron International	Energy	0.126	0.82%
		Kimco Realty	Financials REITs	(0.148)	(1.03%)			Newfield Exploration Co	Energy	0.124	0.80%
		Vornado Realty Trust	Financials REITs	(0.147)	(1.02%)			Consolidated Edison	Utilities	(0.118)	(0.77%)
		SL Green Realty	Financials REITs	(0.147)	(1.02%)			Duke Energy	Energy	(0.117)	(0.76%)
		AvalonBay Communities, Inc.	Financials REITs	(0.146)	(1.02%)			Campbell Soup	Consumer Staples	(0.116)	(0.76%)
		Equity Residential	Financials REITs	(0.146)	(1.02%)			Range Resources Corp.	Energy	0.116	0.75%
		HCP Inc.	Financials REITs	(0.144)	(1.00%)			Abbott Laboratories	Health Care	(0.113)	(0.73%)
4	0.0192	Altera Corp	Information Technology	0.163	1.09%	4	0.0189	KeyCorp	Financials	(0.195)	(1.33%)
		Xilinx Inc	Information Technology	0.154	1.03%			Morgan Stanley	Financials	(0.187)	(1.27%)
		Lam Research	Information Technology	0.146	0.98%			XL Capital	Financials	(0.173)	(1.18%)
		Broadcom Corporation	Information Technology	0.145	0.97%			Hartford Financial Svc.Gp.	Financials	(0.173)	(1.18%)
		KLA-Tencor Corp.	Information Technology	0.144	0.96%			SunTrust Banks	Financials	(0.170)	(1.16%)
		Microchip Technology	Information Technology	0.143	0.96%			Citigroup Inc.	Financials	(0.159)	(1.08%)
		Skyworks Solutions	Information Technology	0.143	0.96%			Bank of America Corp	Financials	(0.156)	(1.06%)
		Applied Materials Inc	Information Technology	0.122	0.82%			Zions Bancorp	Financials	(0.150)	(1.02%)
		Linear Technology Corp.	Information Technology	0.120	0.80%			Goldman Sachs Group	Financials	(0.148)	(1.01%)
		Motorola Solutions Inc.	Information Technology	0.118	0.79%			Comerica Inc.	Financials	(0.138)	(0.94%)
5	0.0164	TJX Companies Inc.	Consumer Discretionary	(0.159)	(1.06%)	5	0.0137	Realty Income Corporation	Financials REITs	0.157	1.02%
		Urban Outfitters	Consumer Discretionary	(0.130)	(0.87%)			Equity Residential	Financials REITs	0.147	0.96%
		Nordstrom	Consumer Discretionary	(0.126)	(0.84%)			Public Storage	Financials REITs	0.147	0.96%
		Applied Materials Inc	Information Technology	0.123	0.81%			Coca-Cola Enterprises'	Consumer Staples	(0.144)	(0.94%)
		Prologis	Financials REITs	(0.116)	(0.77%)			Welltower Inc.	Financials REITs	0.142	0.93%
		Lowe's Cos.	Consumer Discretionary	(0.116)	(0.77%)			Priceline.com Inc	Consumer Discretionary	0.133	0.87%
		Broadcom	Information Technology	0.115	0.76%			Vornado Realty Trust	Financials	0.130	0.85%
		Target Corp.	Consumer Discretionary	(0.110)	(0.73%)			Universal Health Services	Health Care	(0.121)	(0.79%)
		Ross Stores	Consumer Discretionary	(0.110)	(0.73%)			Simon Property Group Inc	Financials	0.119	0.78%
		Amazon.com Inc	Consumer Discretionary	(0.109)	(0.72%)			AvalonBay Communities, Inc.	Financials	0.117	0.77%
6	0.013	Parker-Hannifin	Industrials	0.136	0.86%	6	0.0109	Broadcom Corporation	Information Technology	(0.213)	(1.39%)
		Dow Chemical	Materials	0.135	0.85%			Altera Corp	Information Technology	(0.164)	(1.07%)
		Cummins Inc.	Industrials	0.130	0.82%			Apple Inc.	Information Technology	(0.147)	(0.96%)
		Eastman Chemical	Materials	0.122	0.77%			Microchip Technology	Information Technology	(0.144)	(0.94%)
		FMC Corporation	Materials	0.121	0.76%			NetApp	Information Technology	(0.142)	(0.93%)
		American International Group, Inc.	Financials	(0.120)	(0.76%)			Xilinx Inc	Information Technology	(0.140)	(0.92%)
		Stryker Corp.	Health Care	(0.119)	(0.75%)			F5 Networks Inc.	Information Technology	(0.139)	(0.91%)
		Ingersoll-Rand PLC	Industrials	0.113	0.71%			Intel Corp.	Information Technology	(0.139)	(0.91%)
		Deere & Co.	Industrials	0.112	0.70%			Citrix Systems	Information Technology	(0.136)	(0.89%)
		PPG Industries	Materials	0.110	0.69%			Nvidia Corporation	Information Technology	(0.125)	(0.82%)
7	0.012	Nordstrom	Consumer Discretionary	(0.152)	(0.98%)	7	0.0097	DaVita Inc.	Health Care	0.146	0.95%
		Kohl's Corp.	Consumer Discretionary	(0.135)	(0.88%)			niversal Health Services, Inc.	Health Care	0.142	0.93%
		Macy's Inc.	Consumer Discretionary	(0.134)	(0.87%)			SL Green Realty	Financials	0.132	0.86%
		Target Corp.	Consumer Discretionary	(0.133)	(0.86%)			Cardinal Health Inc.	Health Care	0.116	0.76%
		L Brands Inc.	Consumer Discretionary	(0.128)	(0.83%)			Dentsply Sirona	Health Care	0.113	0.74%
		Baxter International Inc.	Health Care	0.126	0.82%			Prologis	Financials	0.110	0.72%
		TJX Companies Inc.	Consumer Discretionary	(0.116)	(0.75%)			Welltower Inc.	Financials	0.110	0.72%
		Urban Outfitters	Consumer Discretionary	(0.111)	(0.72%)			Patterson Companies	Health Care	0.109	0.71%
		Vertex Pharmaceuticals Inc	Health Care	0.110	0.72%			TECO Energy	Utilities	(0.109)	(0.71%)
		Costco Co.	Consumer Staples	(0.107)	(0.70%)			Humana Inc.	Health Care	0.109	0.71%
8	0.0116	Aon plc	Financials Insurance	(0.177)	(1.17%)	8	0.0088	Tegna	Consumer Discretionary	0.155	0.99%
		Hartford Financial Svc.Gp.	Financials Insurance	(0.160)	(1.06%)			Dollar Tree	Consumer Discretionary	(0.148)	(0.95%)
		Chubb Corp	Financials Insurance	(0.153)	(1.01%)			Wal-Mart Stores	Consumer Discretionary	(0.143)	(0.91%)
		Marsh & McLennan	Financials Insurance	(0.147)	(0.97%)			Ross Stores	Consumer Discretionary	(0.129)	(0.83%)
		Abbott Laboratories	Health Care	0.139	0.92%			Celgene Corp.	Health Care	(0.121)	(0.77%)
		ACE Limited	Financials Insurance	(0.132)	(0.87%)			Costco Co.	Consumer Staples	(0.114)	(0.73%)
		American International Group, Inc.	Financials Insurance	(0.130)	(0.86%)			AutoZone Inc	Consumer Discretionary	(0.113)	(0.72%)
		Smucker (J.M.)	Consumer Staples	0.130	0.86%			Interpublic Group	Consumer Discretionary	0.111	0.71%
		Merck & Co.	Health Care	0.125	0.83%			Gilead Sciences	Health Care	(0.111)	(0.71%)
		CVS Health	Consumer Staples	0.125	0.82%			Kroger Co.	Consumer Staples	(0.110)	(0.71%)
9	0.0104	Unum Group	Financials Insurance	(0.181)	(1.26%)	9	0.0077	Hormel Foods Corp.	Consumer Discretionary	0.166	1.08%
		United Health Group Inc.	Health Care	(0.170)	(1.18%)			Vertex Pharmaceuticals Inc	Health Care	(0.145)	(0.94%)
		Marsh & McLennan	Financials Insurance	(0.167)	(1.16%)			PPL Corp.	Utilities	(0.128)	(0.83%)
		ACE Limited	Financials Insurance	(0.164)	(1.14%)			Kellogg Co.	Consumer Staples	0.123	0.80%
		XL Capital	Financials Insurance	(0.158)	(1.09%)			Alexion Pharmaceuticals	Pharmaceutical	(0.121)	(0.79%)
		Hartford Financial Svc.Gp.	Financials Insurance	(0.158)	(1.09%)			Bed Bath & Beyond	Consumer Discretionary	(0.119)	(0.77%)
		Aon plc	Financials Insurance	(0.156)	(1.08%)			Pulte Homes Inc.	Consumer Discretionary	(0.118)	(0.77%)
		CIGNA Corp.	Health Care	(0.149)	(1.03%)			American Electric Power	Utilities	(0.117)	(0.76%)
		Humana Inc.	Health Care	(0.132)	(0.91%)			Celgene Corp.	Health Care	(0.115)	(0.75%)
		BB&T Corporation	Financials Insurance	0.131	0.91%			Gap (The)	Consumer Discretionary	0.114	0.74%
10	0.009	Sysco Corp.	Consumer Staples	0.177	1.17%	10	0.0074	Alexion Pharmaceuticals	Pharmaceutical	0.164	1.04%
		St Jude Medical	Health Care	(0.152)	(1.00%)			Gap (The)	Consumer Discretionary	(0.145)	(0.92%)
		The Hershey Company	Consumer Staples	0.152	1.00%			Interpublic Group	Consumer Discretionary	(0.135)	(0.86%)
		PepsiCo Inc.	Consumer Staples	0.151	1.00%			Union Pacific	Industrials	0.132	0.84%
		Northrop Grumman Corp.	Industrials	0.142	0.94%			Ross Stores	Consumer Discretionary	(0.131)	(0.83%)
		Celgene Corp.	Health Care	(0.138)	(0.91%)			Harman Int'l Industries	Consumer Discretionary	(0.128)	(0.81%)
		Expeditors Int'l	Industrials	(0.132)	(0.87%)			AutoZone Inc	Consumer Discretionary	(0.116)	(0.74%)
		C. H. Robinson Worldwide	Industrials	(0.128)	(0.85%)			Autodesk Inc	Information Technology	0.116	0.74%
		Reynolds American Inc.	Consumer Staples	0.127	0.84%			Precision Castparts Corp.	Aerospace and defense	0.116	0.74%
		Johnson Controls	Consumer Discretionary	0.126	0.84%			Smucker (J.M.)	Consumer Staples	(0.115)	(0.73%)

Table 3
Eigenvectors corresponding to top 10 largest eigenvalues are displayed. For each eigenvector, largest (in absolute value) components are listed. The contribution of each firm in the first eigenvector is uniform, which implies the market mode. Other eigenvectors represent business sectors.

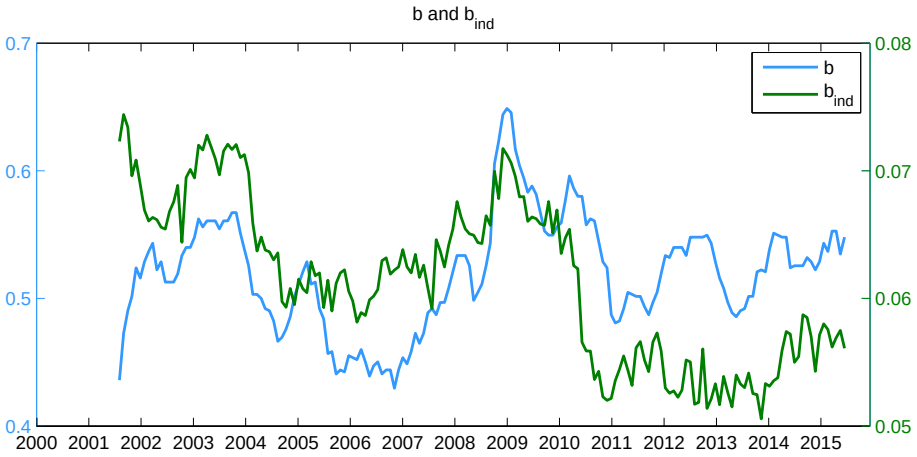

Figure 10
Our estimator for b and the average of estimated individual AR(1) coefficients (denoted as b_{ind} in the plot). The two quantities behaves very similarly, although there is a scale difference, showing slower mean-reversions of residual returns in crisis.

8 Conclusions

Random matrix theory is gaining increasing attentions for analyzing complex high-dimensional data. This section relates the factor model estimation problem to fitting empirical eigenvalue distribution of the covariance matrix. The spectrum from real data is complex and cannot be trivially dissected by traditional usage of the Marchenko-Pastur law or mere counting of the largest eigenvalues. Instead, we present a new approach to estimate factor models, by allowing control for both the number of factors and the correlation structure of residuals. Under reasonable assumptions for approximate factor models, we show that our estimators are consistent. In addition, by using the free random variable techniques and modified estimation problem, the implementation of our method is done efficiently. Monte Carlo analysis shows that the proposed method boosts up the power of identification of weak factors and that the performance is less affected by signal-to-noise ratios. Furthermore, from the application to real data with moving windows, we monitor how our estimators effectively characterize the market dynamics.

Several future studies are planned. Clearly, further research will be needed to employ the more general residual modeling, for which we can calculate the distribution readily. For example, as described in Burda et al. [2010], if we consider vector ARMA(1,1) processes, we have up to 6th-order polynomial equations. The covariance matrix estimations have also interesting connections. The covariance matrix estimations via factor models have been investigated by Fan et al. [2008], Fan et al. [2011], and Fan et al. [2013]. In particular, in those papers, with sparsity assumptions, the error covariance matrix is estimated by

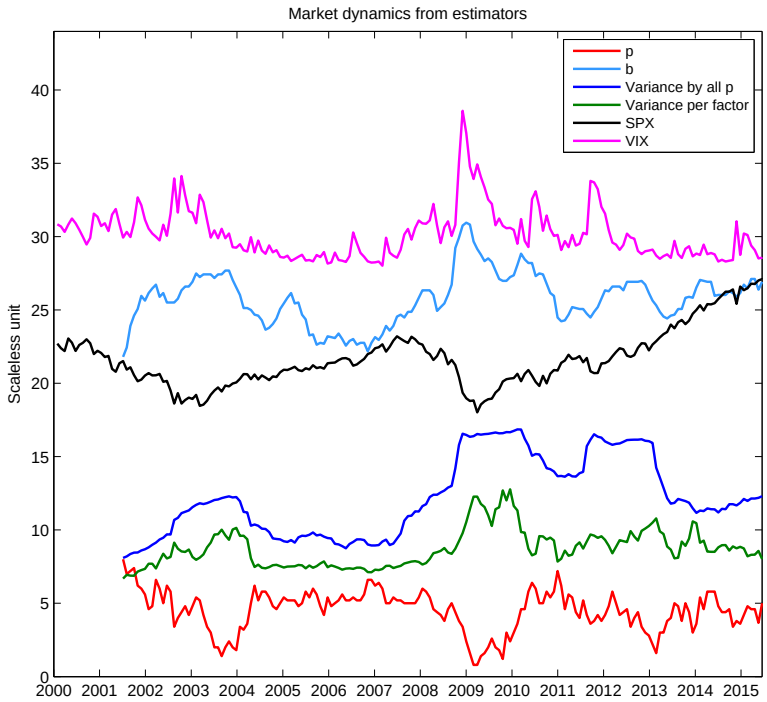

Figure 11
Market dynamics captured by estimators and variance explanations: $\hat{p}$, $\hat{b}$, variance explained by $\hat{p}$, variance explained per factor, S&P500 index (SPX), and volatility index (VIX). Each quantity is obtained with moving windows, and rescaled for comparisons. It is clear that the variance of the whole market is the most condensed during the crisis (2008-2009). In addition, the overall trends after the crisis is different, showing sporadic market condensations. The number of investigated stocks is $N = 378$ and the time window is also set to be the same length, $T = 378$ business days (~ 1.5 years).

thresholdings. A possible extension is to develop a more delicate method to dynamically estimate the residual covariance matrix, so that the residual processes can be exploited for more practical purposes, such as mean-reversion dependence structures in large dimensions. To apply our method to different frequency data would be also interesting.

A A brief overview of free random variables techniques

Key concepts

In this section, we summarize main concepts and key results of the technique that we employed. We follow the notations and derivations from Burda et al. [2005] and Burda et al. [2010]. Throughout this section, we assume a simple decomposition of covariance structures

$$Cov_{ia,jb} = A_{ij}B_{ab}$$

$i, j = 1, \ldots, N$, $a, b = 1, \ldots, T$, A is a $N \times N$ cross-covariance matrix and B is a $T \times T$ auto-covariance matrix. Suppose ϵ is $N \times T$ uncorrelated Gaussian random matrix. Then a correlated Gaussian random matrix U (e.g., $N \times T$ time series) can be written as

$$U = A^{1/2}\epsilon B^{1/2}$$

Define the sample (empirical) covariance matrix C as

$$C = \frac{1}{T}UU^T,$$

We will show the relation between C and A, B, using free random variable techniques. It generalizes the results for the eigenvalue density of large-dimensional empirical covariance matrices with doubly-correlated structure.

Consider a real symmetric $N \times N$ random matrix H.

Definition 5 (Mean spectral density)**.**

$$\begin{aligned} \rho_H(\lambda) &= \frac{1}{N}\sum_{i=1}^{N}\langle\delta(\lambda - \lambda_i)\rangle \\ &= \frac{1}{N}\langle \mathcal{T}\mathcal{r}(\lambda \mathbf{1}_N - H)\rangle \end{aligned}$$

where the expectation $\langle\cdots\rangle$ is taken w.r.t. the rotationally invariant probability measure, $\delta(\cdot)$ is a Dirac delta function, and $\mathbf{1}_N$ is a $N \times N$ unit matrix.

Definition 6 (The Green's Function (or Stieltjes Transform))**.**

$$\begin{aligned} G_H(z) &= \frac{1}{N}\sum_{i=1}^{N}\Big\langle \frac{1}{z-\lambda_i}\Big\rangle \\ &= \frac{1}{N}\Big\langle \frac{1}{z\mathbf{1}_N - H}\Big\rangle \\ &= \int \frac{\rho_H(\lambda)}{z-\lambda}d\lambda \end{aligned}$$

The relationship between $\rho_H(\lambda)$ and $G_H(z)$ is the following:

$$\rho_H(\lambda) = -\frac{1}{\pi}\lim_{\epsilon\to 0^+}\mathfrak{Im}G_H(\lambda+i\epsilon).$$

This Green's function generates moments of a probability distribution, where the n-th moment is defined by

Definition 7 (Moment)**.**

$$\begin{aligned} m_n &= \frac{1}{N}\langle \mathcal{T}\mathcal{r}H^n\rangle \\ &= \int \rho_H(\lambda)\lambda^n d\lambda \end{aligned}$$

Definition 8 (Moment generating function)**.**

$$\begin{aligned} G_H(z) &= \sum_{n\geq 0}\frac{m_n}{z^{n+1}} \\ M_H(z) &= \sum_{n\geq 1}\frac{m_n}{z^{n+1}} \end{aligned}$$

This suggest the the relation between $G_H(z)$ and $M_{H,n}$ as

$$M_H(z) = zG_H(z) - 1.$$

There is the inverse transform of the Green's function and moment generating function.

Definition 9 (Blue's function and N-transform)**.**

$$\begin{aligned} G_H(B_H(z)) = B_H(G_H(z)) = z \\ M_H(N_H(z)) = N_H(M_H(z)) = z \end{aligned}$$

Now we return to our original objective, empirical covariance matrix, C. Recall that it can be expressed as

$$\begin{aligned} C &= \frac{1}{T}UU^T \\ &= \frac{1}{T}A^{1/2}\epsilon B\epsilon^T A^{1/2} \end{aligned}$$

For arbitrary A and B, the N-transform of C can be derived as

$$\begin{aligned} N_C(z) &= N_{\frac{1}{T}A^{1/2}\epsilon B\epsilon^T A^{1/2}}(z) \\ &= N_{\frac{1}{T}\epsilon B\epsilon^T B}(z) \quad \because \textit{cyclic property of trace} \\ &= \frac{z}{1+z}N_{\frac{1}{T}\epsilon B\epsilon^T}(z)N_A(z) \quad \because \textit{FRV multiplication law} \\ &= \frac{z}{1+z}N_{\frac{1}{T}\epsilon^T\epsilon B}(rz)N_B(z) \quad \because \textit{cyclic property of trace + rescaling} \\ &= \frac{z}{1+z}\frac{rz}{1+rz}N_{\frac{1}{T}\epsilon^T\epsilon}(rz)N_B(rz)N_A(z) \quad \because \textit{FRV multiplication law} \\ &= rzN_B(rz)N_A(z) \quad \because N_{\frac{1}{T}\epsilon^T\epsilon}(z) = \frac{(1+z)(r+z)}{z} \end{aligned}$$

Using the moments' generating function $M \equiv M_C(z)$ and its inverse relation to N-transform, this can be written as,

$$\begin{aligned} N_C(z) &= rzN_B(rz)N_A(z) \\ \Longleftrightarrow \quad z &= rMN_B(rM)N_A(M) \end{aligned}$$

We will use this equation to compute the spectral density for given matrix A and B.

The case of $U_{it} = bU_{i,t-1} + \xi_{it}$

Suppose U_{it} $(n = 1, \cdots, N,\ t = 1, \cdots, T)$ be a time-series, following the autoregressive model:

$$U_{it} = bU_{i,t-1} + \xi_{it}$$

where $|b| < 1$ and $\xi_{it} \sim N(0, 1-b^2)$. Let $c = \frac{N}{T}$. The free random variables technique provide analytic derivation for the eigenvalue distribution of correlation matrix $C = \frac{1}{T}UU^T$.

Our goal is to find $\rho_C(\lambda)$. The strategy is the following:

1. Find $M_C(z)$, from the equation for N-transform.
2. Find $G_C(z)$, by $M_C(z) = zG_C(z) - 1$.
3. Find $\rho_C(\lambda)$, by $\rho_C(\lambda) = -\frac{1}{\pi}\lim_{\epsilon\to 0^+}\mathfrak{Im}G_C(\lambda + i\epsilon)$.

Other than the first part is straightforward, so let us examine the equation for N-transform.

Recall that we have the equation for N-transform for arbitrary matrix A and B as

$$z = rMN_B(rM)N_A(M)$$

For the vector AR(1) process considered above, the cross-correlation matrix $A = I_N$. Then $N_A(z) = N_I(z) = 1 + 1/z$. Thus, the above equation can be rewritten as

$$\begin{aligned} z &= rMN_B(rM)(1 + 1/M) = r(1 + M)N_B(rM) \\ \iff \frac{z}{r(1+M)} &= N_B(rM) \\ \iff rM &= M_B\left(\frac{z}{r(1+M)}\right). \end{aligned}$$

Now we will need to find M_B. The two-point covariance function for VAR(1) is the following. Note that the auto-covariance matrix of vector AR(1) process we consider has a simple form:

$$B_{st} = \frac{\mathbf{var}\,(\xi)}{1-b^2} b^{|s-t|} = b^{|s-t|} \quad \because \mathbf{var}\,(\xi) = 1 - b^2.$$

Using Fourier-transform of the matrix B, it can be shown that the moment generating function of B is give by

$$M_B(z) = -\frac{1}{\sqrt{1-z}\sqrt{1 - \frac{(1+b^2)^2}{1-b^2} z}}.$$

Now we solve Eq. 24 for M_B, which leads to the following polynomial equation (with $a^2 = 1 - b^2$):

$$\begin{aligned} &a^4c^2M^4 + 2a^2c(-(1+b^2)z + a^2c)M^3 + ((1-b^2)^2z^2 - 2a^2c(1+b^2)z \\ &+(c^2-1)a^4)M^2 - 2a^4M - a^4 = 0 \end{aligned}$$

Thus, we obtain the first step. The other steps are followed straightforwardly.

B Numerical calculation of Kullback-Leibler divergence

The spectral distance measure we use requires the calculation of Kullback-Leibler (KL) divergence.

$$\mathcal{D}_{KL}(P\|Q) = \sum_i P_i \log \frac{P_i}{Q_i}$$

where P and Q are probability densities, and $P_i = P(ih)$ with grid size h. To deal with zero elements of P_i that possibly appear due to the spectral characteristics of empirical covariance matrix, we use $\widetilde{P}_i$ from the following manipulation. For a small $\varepsilon > 0$,

$$\begin{aligned} \widetilde{P}_i &= \begin{cases} \alpha P_i, & \textit{if } P_i > 0 \\ \varepsilon, & \textit{if } P_i = 0 \end{cases} \\ \textit{where} & \\ \alpha &= 1 - \#(\textit{zeros of } P_i)\varepsilon \end{aligned}$$

where we use the fact that $\sum_i \widetilde{P}_i = 1$ and ε is assumed to be small enough such that $\varepsilon \ll 1/\#(\textit{zeros of } P_i)$.

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