

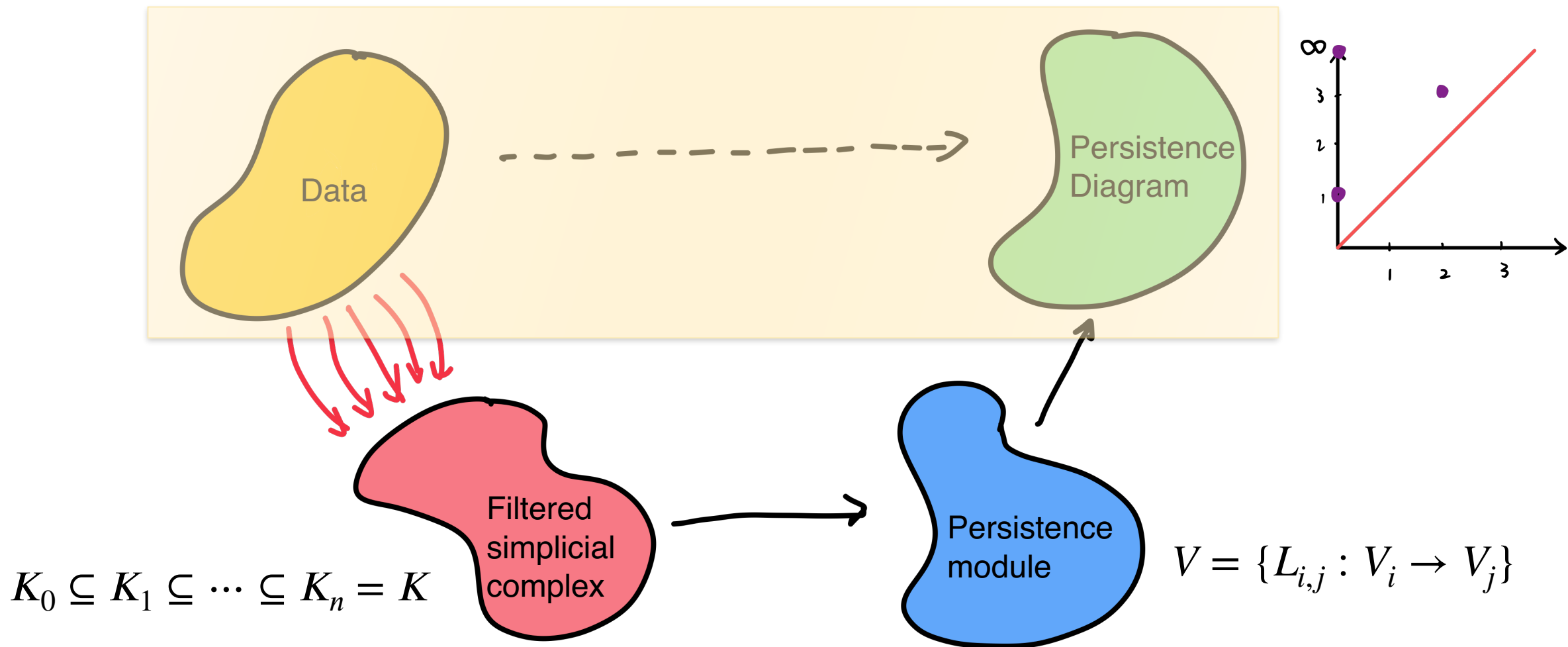
MATH412/COMPSCI434/MATH713
Fall 2025

Topological Data Analysis

Topic 4: Introduction to Persistent Homology - Part 2

Instructor: Ling Zhou

TDA in a nutshell



Summary

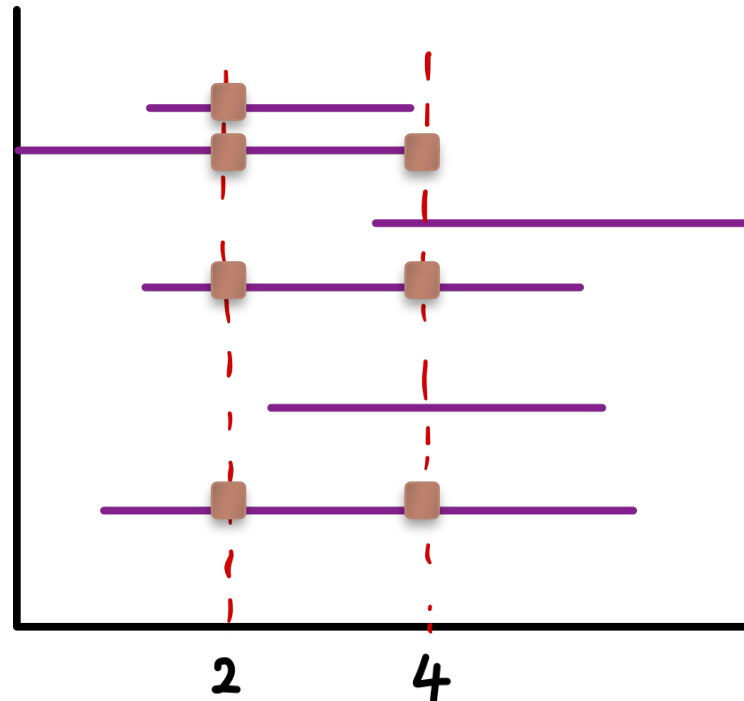
- ▶ Create a filtered simplicial complex $K_\bullet = \{K_i\}$ out of data
- ▶ Let $V_\bullet = \{V_i = H_p(K_i)\}_{i=0}^n$ be the p -dim persistence homology of K_\bullet
- ▶ Decompose $V_\bullet \cong I[b_1, d_1) \oplus I[b_2, d_2) \oplus \cdots \oplus I[b_M, d_M)$
- ▶ The multiset $Dgm_p(K_\bullet) = \{(b_j, d_j)\}_{j=1, \dots, M} \subseteq \mathbb{R}^2$ is called the **degree p persistence diagram** of K_\bullet .
- ▶ p -th **persistent homology group** from i to j : $H_p^{i,j} = \text{Im}(\iota_p^{i,j}) (\subset H_p(K_j))$
- ▶ p -th **persistent betti number** from i to j : $\beta_p^{i,j} = \dim H_p^{i,j}$
- ▶ $\beta_p^{i,j}$ denotes the number of homology classes co-existing at both K_i and K_j

Persistent Betti Number vs Barcode

- ▶ Let $V_\bullet = \{V_i = H_p(K_i)\}_{i=0}^n \cong I[b_1, d_1) \oplus I[b_2, d_2) \oplus \cdots \oplus I[b_M, d_M)$
- ▶ $\beta_p^{i,j} = \#$ of bars crossing both vertical lines at i and at j
- ▶ $\beta_p^{i,j} = \#$ of bars that are born $\leq i$ and die $> j$

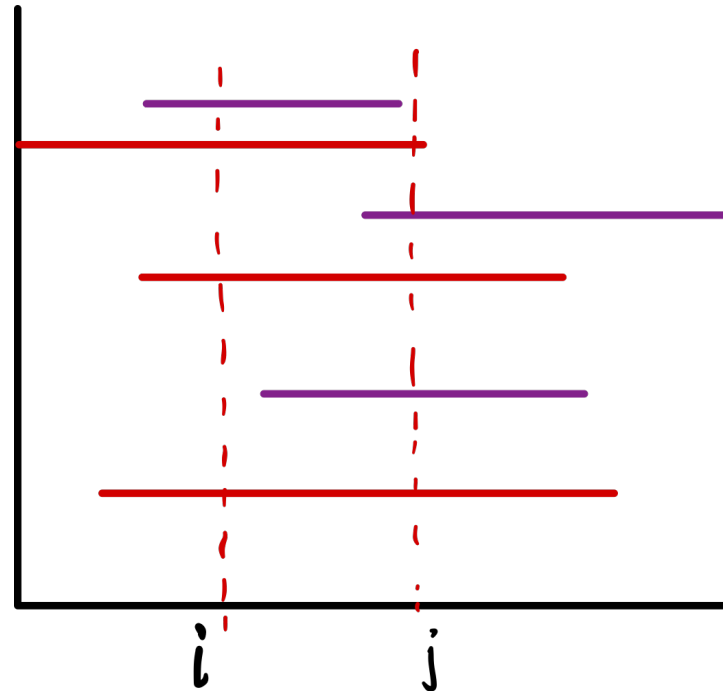
$$\beta_p^{i,j} = \dim H_p^{i,j}$$

$$\beta_p^{2,4} = 3$$



Persistent Betti Number vs Barcode

- ▶ $\mu^{b,d} :=$ multiplicity of the interval $[b, d)$ in the barcode,
 - ▶ $\mu^{b,d} : \text{Intervals} \rightarrow \mathbb{Z}$ is also called the **persistent pairing function**
- ▶ Theorem: $\beta_p^{i,j} = \sum_{k \leq i, j < l} \mu_p^{k,l} =$ number of intervals $[k, l)$ that contains $[i, j)$



Persistent Betti Number vs Barcode

► Theorem: Assume $\beta_p^{-1,j} = \beta^{i,n+1} = 0$. For

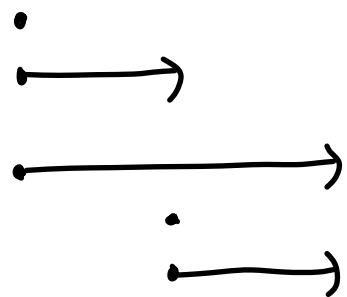
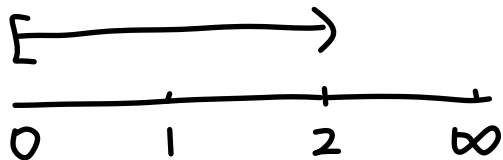
$0 \leq i < j \leq n + 1$, we have

$$\mu_p^{i,j} = (\beta_p^{i,j-1} - \beta_p^{i,j}) - (\beta_p^{i-1,j-1} - \beta_p^{i-1,j})$$

► $\mu^{b,d} :=$ multiplicity of the interval $[b, d)$ in the barcode,

► $\mu^{b,d} : \text{Intervals} \rightarrow \mathbb{Z}$ is also called the **persistent pairing function**

► Theorem: $\beta_p^{i,j} = \sum_{k \leq i, j < l} \mu_p^{k,l} =$ number of intervals $[k, l)$ that contains $[i, j]$



$$0 \ 1 = \underline{(0, 0)} - \underline{(0, 1)} - \cancel{(-1, 0)} + \cancel{(-1, 1)}$$

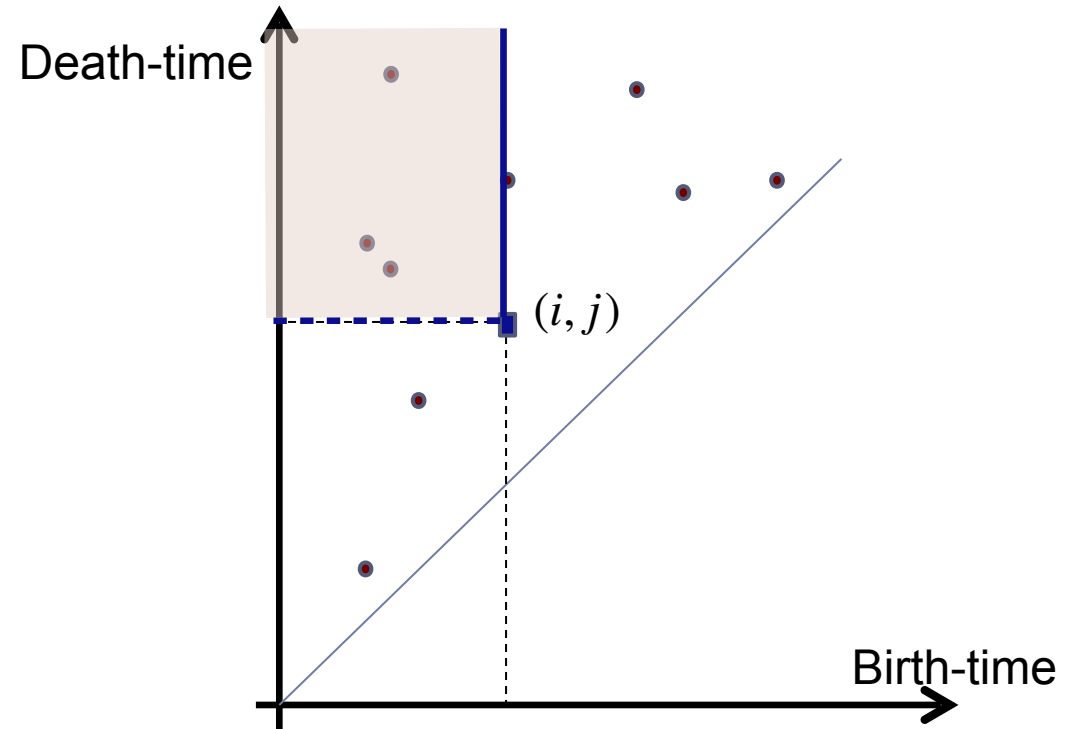
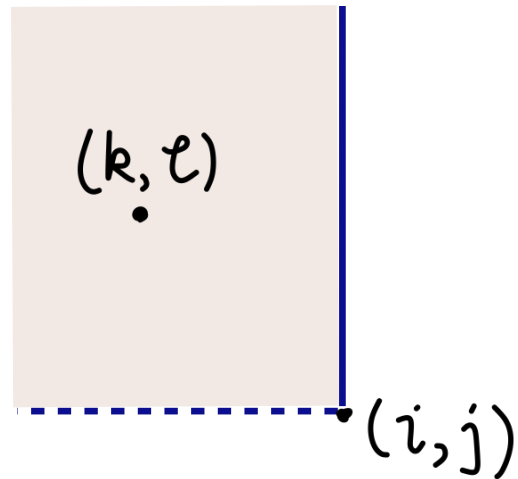
$$1 \ 2 = \underset{\checkmark}{(1, 1)} - \cancel{(1, 2)} - \underset{\checkmark}{(0, 1)} + \cancel{(0, 2)}$$

$$0 \ 2 = \underline{(0, 1)} - \cancel{(0, 2)} - \cancel{(-1, 1)} + \cancel{(-1, 2)}$$

Persistent Betti Number vs Persistence Diagram

- ▶ $\mu^{b,d} :=$ multiplicity of the point (b, d) in the persistence diagram
- ▶ Theorem: $\beta_p^{i,j} = \sum_{k \leq i, j < l} \mu_p^{k,l} =$ number of points (k, l) s.t. $k \leq i, j < l$

$$\begin{aligned} k &\leq i \\ j &< l \end{aligned}$$



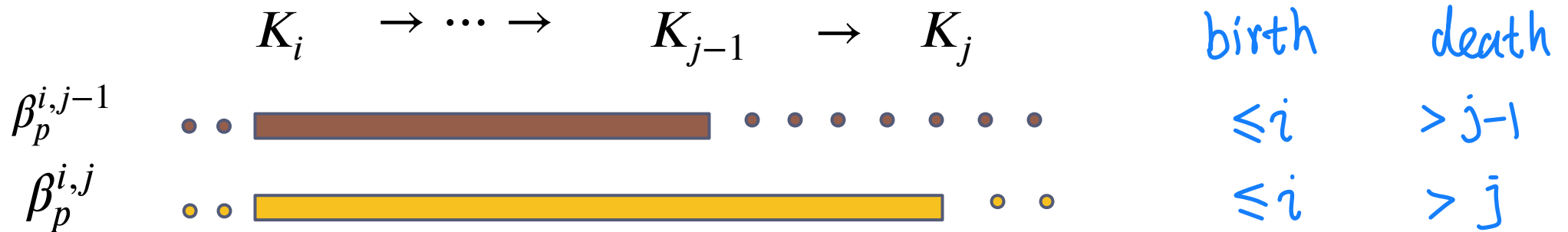
$$\beta^{i,j} = \# \text{ bars born } \leq i \text{ \& die } > j$$

Möbius Inversion: Compute μ from β

- Theorem: Assume $\beta_p^{-1,j} = \beta^{i,n+1} = 0$. For $0 \leq i < j \leq n+1$, we have

$$\mu_p^{i,j} = (\beta_p^{i,j-1} - \beta_p^{i,j}) - (\beta_p^{i-1,j-1} - \beta_p^{i-1,j})$$

Number of independent
homology classes from
 K_i but **died** entering K_j



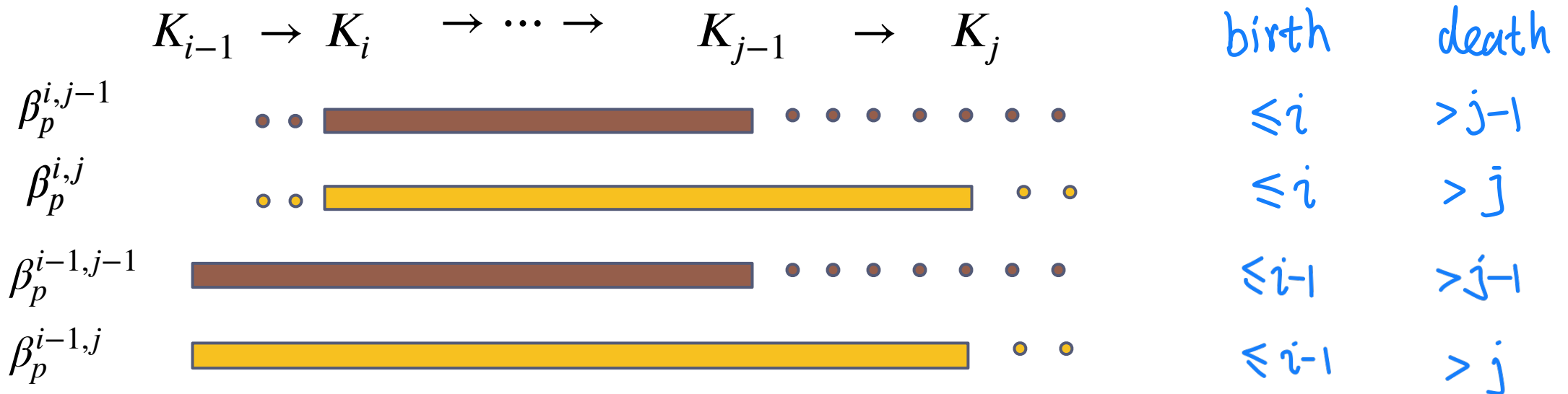
Möbius Inversion: Compute μ from β

- Theorem: Assume $\beta_p^{-1,j} = \beta^{i,n+1} = 0$. For $0 \leq i < j \leq n+1$, we have

$$\mu_p^{i,j} = (\beta_p^{i,j-1} - \beta_p^{i,j}) - (\beta_p^{i-1,j-1} - \beta_p^{i-1,j}) = (\text{born} \leq i, \text{dies at } j) - (\text{born} \leq i-1, \text{dies at } j) = (\text{born} = i, \text{dies at } j)$$

Number of independent homology classes from K_i but **died** entering K_j

Number of independent homology classes from K_{i-1} but **died** entering K_j



Example

$$\beta^{i,j} = \# \text{ bars born } \leq i \text{ \& die } > j$$

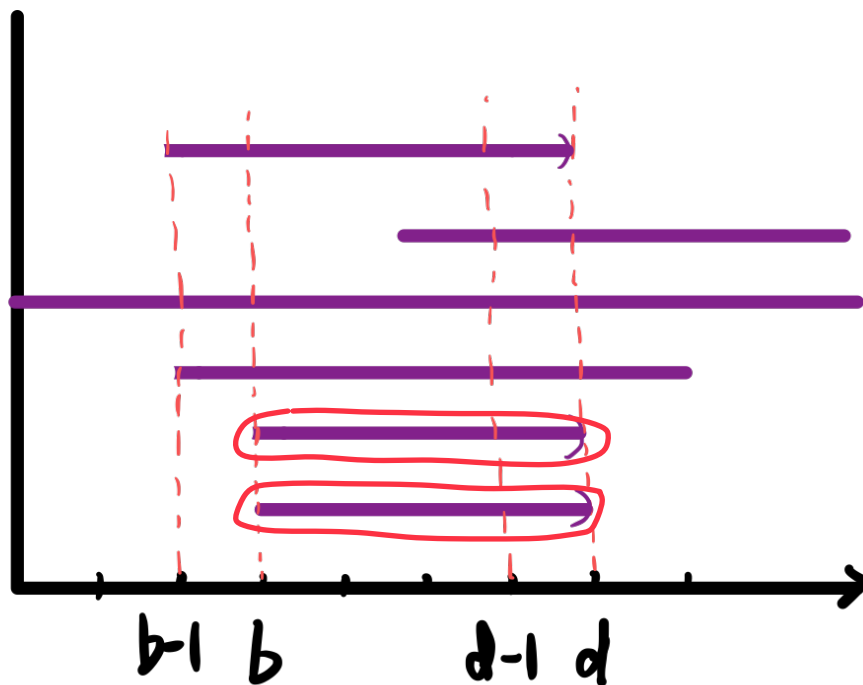
$$\mu^{b,d} = (\beta^{b,d-1} - \beta^{b,d}) - (\beta^{b-1,d-1} - \beta^{b-1,d})$$

\parallel

$\# [b, d)$

\parallel

2



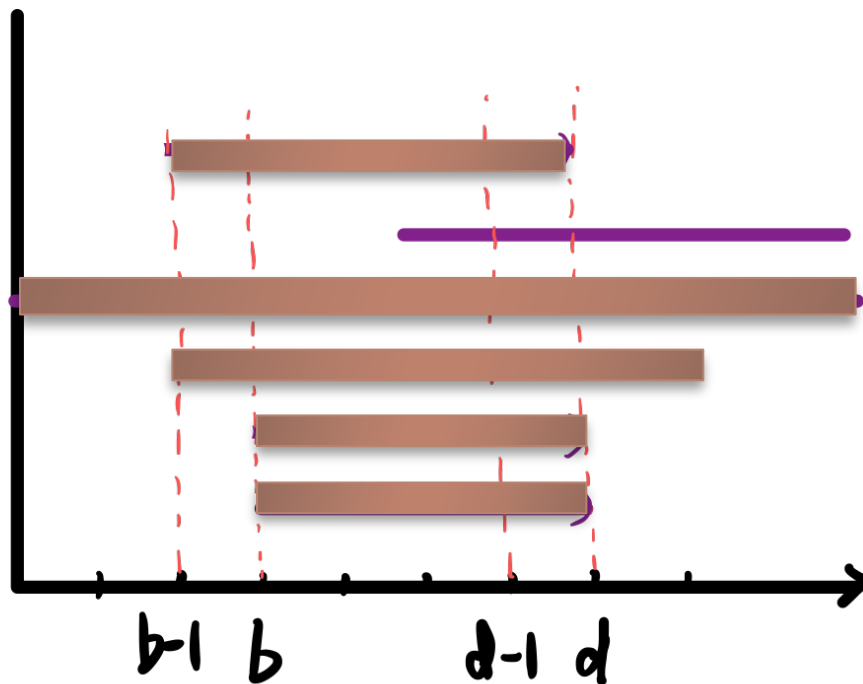
Example

$\beta^{i,j} = \# \text{ bars born } \leq i \text{ \& die } > j$

$$\mu^{b,d} = (\beta^{b,d-1} - \beta^{b,d}) - (\beta^{b-1,d-1} - \beta^{b-1,d})$$

$\mu^{b,d}$
 \parallel
 $\# [b, d)$
 \parallel
 2

\parallel
 5



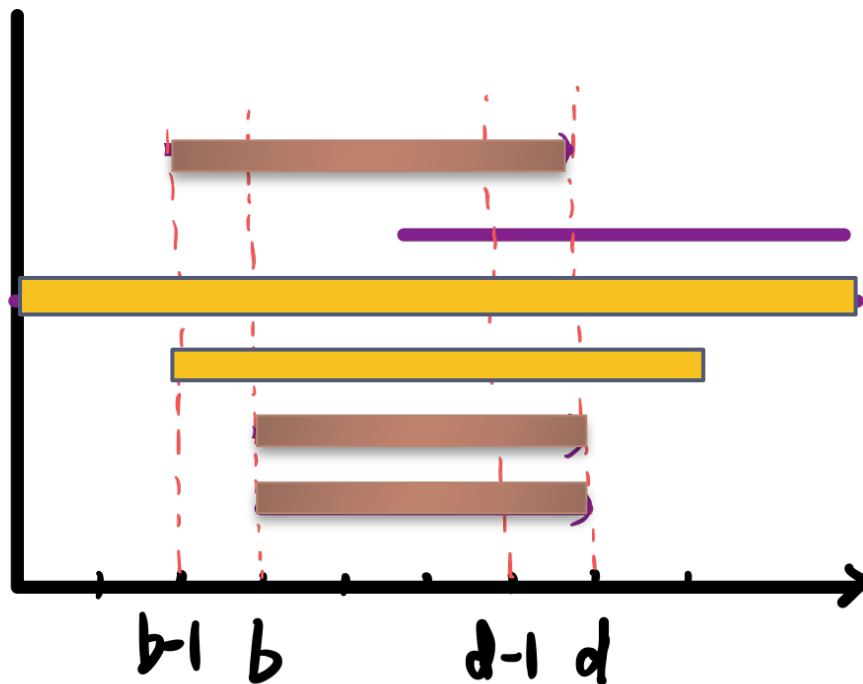
Example

$$\beta^{i,j} = \# \text{ bars born } \leq i \text{ \& die } > j$$

$$\mu^{b,d} = (\beta^{b,d-1} - \beta^{b,d}) - (\beta^{b-1,d-1} - \beta^{b-1,d})$$

$$\begin{array}{c} \text{||} \\ \hline \# [b, d) \\ \text{||} \\ 2 \end{array}$$

$$\begin{array}{c} \text{||} \\ \hline 5 \\ \text{||} \\ 2 \end{array}$$



Example

$$\beta^{i,j} = \# \text{ bars born } \leq i \text{ \& die } > j$$

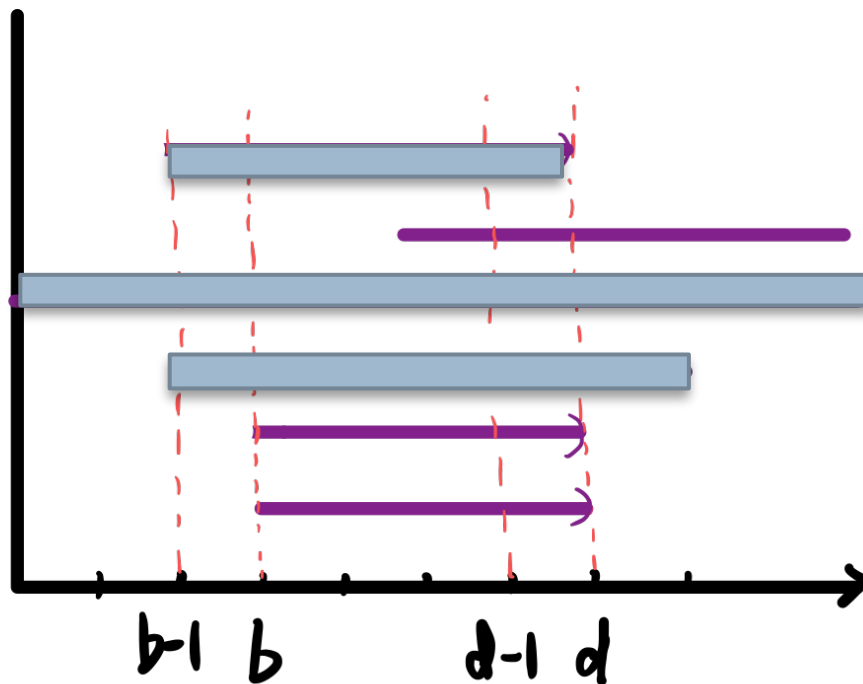
$$\mu^{b,d} = (\beta^{b,d-1} - \beta^{b,d}) - (\beta^{b-1,d-1} - \beta^{b-1,d})$$

$$\begin{array}{c} \text{||} \\ \hline \text{||} \\ \hline \# [b, d) \\ \text{||} \\ \hline 2 \end{array}$$

$$\begin{array}{c} \text{||} \\ \hline \text{||} \\ \hline 5 \end{array}$$

$$\begin{array}{c} \text{||} \\ \hline \text{||} \\ \hline 2 \end{array}$$

$$\begin{array}{c} \text{||} \\ \hline \text{||} \\ \hline 3 \end{array}$$



Example

$$\beta^{i,j} = \# \text{ bars born } \leq i \text{ \& die } > j$$

$$\mu^{b,d} = (\beta^{b,d-1} - \beta^{b,d}) - (\beta^{b-1,d-1} - \beta^{b-1,d})$$

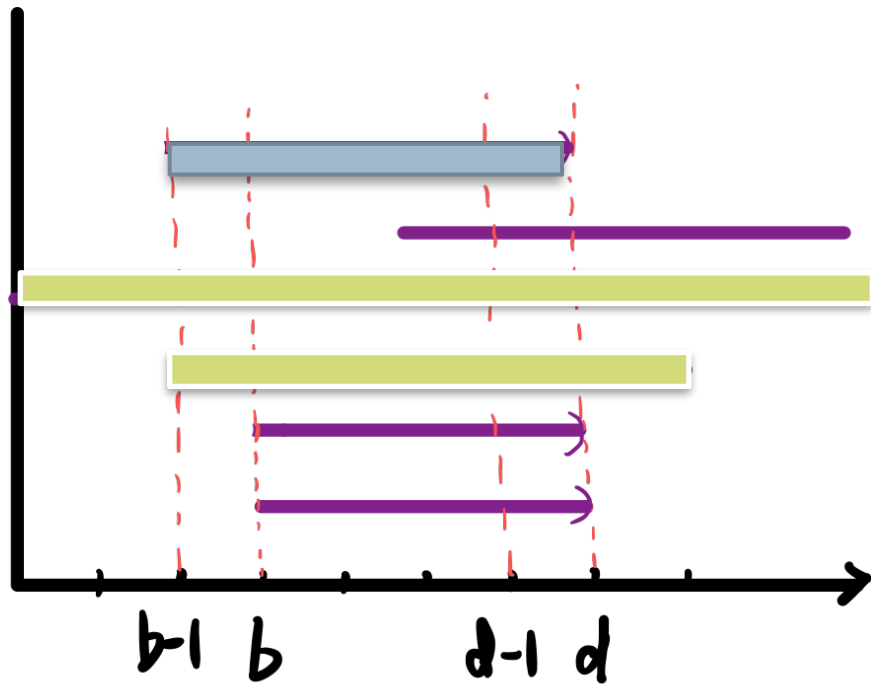
$$\begin{array}{c} \text{||} \\ \text{||} \\ \# [b, d) \\ \text{||} \\ 2 \end{array}$$

$$\begin{array}{c} \text{||} \\ \text{||} \\ 5 \end{array}$$

$$\begin{array}{c} \text{||} \\ \text{||} \\ 2 \end{array}$$

$$\begin{array}{c} \text{||} \\ \text{||} \\ 3 \end{array}$$

$$\begin{array}{c} \text{||} \\ \text{||} \\ 2 \end{array}$$

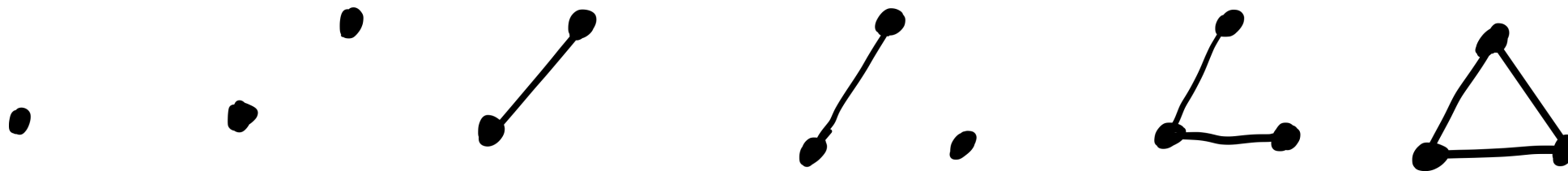


Topological View of Barcodes

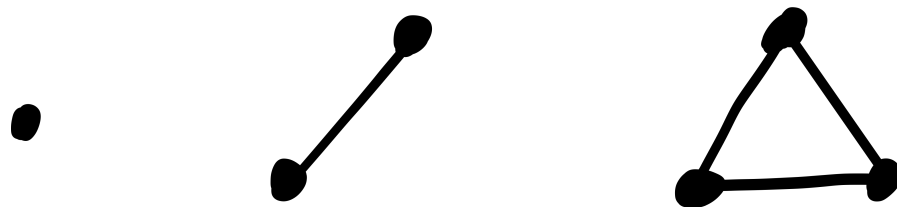
An alternative view

- ▶ **Simplex-wise filtration:** at each step, only one simplex is added.
- ▶ That is, $\emptyset = K_0 \subseteq K_1 \subseteq K_2 \subseteq \dots \subseteq K_n = K$, s.t. $\sigma_i = K_i \setminus K_{i-1}$

example



non-example



An alternative view

- ▶ At K_i , consider the next added p -simplex $\sigma = \sigma_{i+1}$. It can be
 - ▶ **A creator: adding σ creates a 'new' p -cycle**
 - ▶ Addition of σ generates a homology class which is not in $\text{Im}(H_p(K_i) \rightarrow H_p(K_{i+1}))$
 - ▶ hence $\beta_p \rightarrow +$

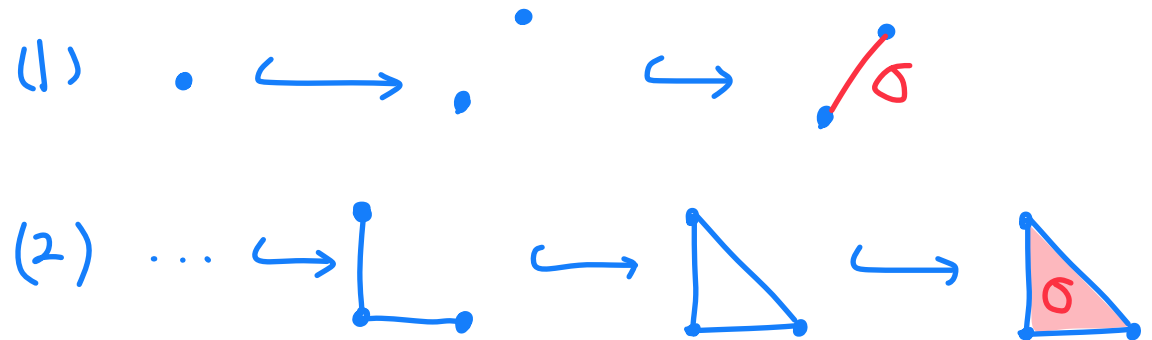
Examples



An alternative view

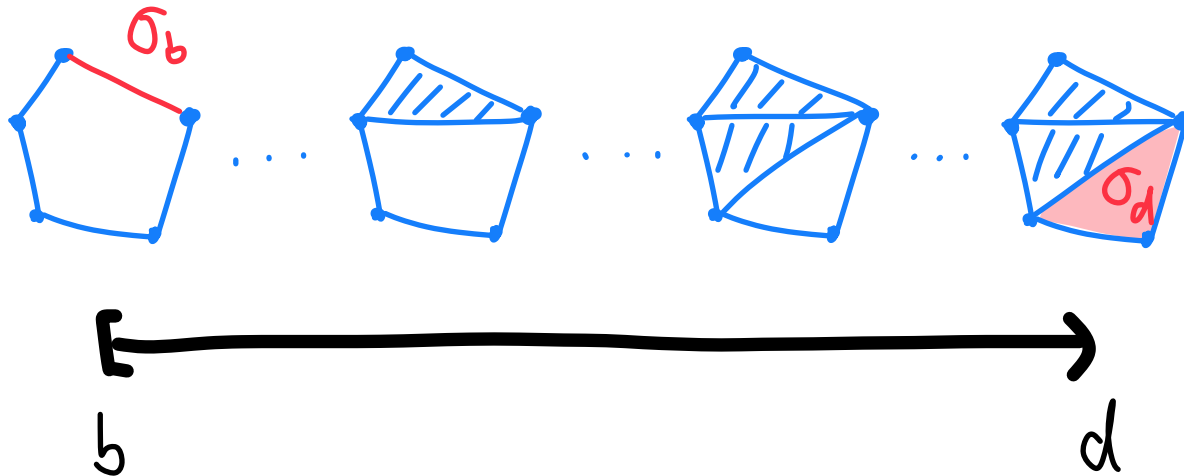
- ▶ At K_i , consider the next added p -simplex $\sigma = \sigma_{i+1}$. It can be
 - ▶ **A creator: adding σ creates a 'new' p -cycle**
 - ▶ Addition of σ generates a homology class which is not in $\text{Im}(H_p(K_i) \rightarrow H_p(K_{i+1}))$
 - ▶ hence $\beta_p \uparrow$
 - ▶ **A destroyer: adding σ kills an 'old' $(p-1)$ -cycle**
 - ▶ This $(p-1)$ -cycle is not trivial in $H_{p-1}(K_i)$, but becomes trivial in $H_{p-1}(K_{i+1})$
 - ▶ hence $\beta_{p-1} \downarrow$

Examples



Interpretation of Barcode

- ▶ Let $V_\bullet = \{V_i = H_p(K_i)\}_{i=0}^n \cong I[b_1, d_1) \oplus I[b_2, d_2) \oplus \cdots \oplus I[b_M, d_M)$, assuming a simplex-wise filtration
- ▶ Each $I[b, d)$ corresponds to
 - ▶ adding a p simplex σ_b at time b to create a p -cycle
 - ▶ adding a $p + 1$ simplex σ_d at time d to kill the above p -cycle



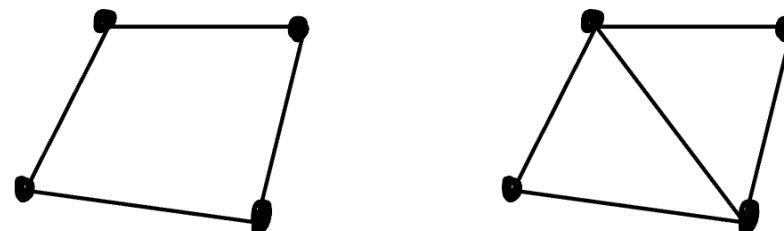
Subtlety of non-uniqueness

- ▶ Which cycle is killed?

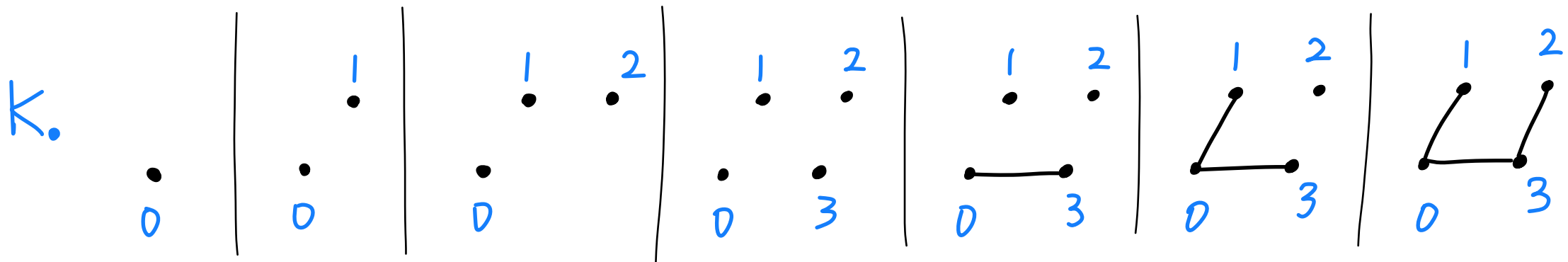


- ▶ The younger one will be killed

- ▶ Which cycle is created?



- ▶ Several cycle classes are born
- ▶ But the dimension only increases by 1



[0]

[1]

[2]

[3]

barcode of $H_0(K_\bullet)$

Persistence Algorithm

Recall: Matrix Reduction

- ▶ Turn A_p into the **column reduced form**
 - ▶ Each non-zero column has a unique lowID/pivot: index of lowest 1-entry
- ▶ **Column operations** in Gaussian elimination:
 - ▶ scaling (not needed over \mathbb{Z}_2)
 - ▶ swap (not necessary)
 - ▶ add one column to another
- ▶ Do **left-to-right column reduction** and get bases of
 - ▶ $B_{p-1} = \text{Im } \partial_p$: the reduced columns
 - ▶ $Z_p = \ker \partial_p$: the column operations

$$\begin{bmatrix} * & * & * & 0 \\ * & 1 & * & 0 \\ 1 & 0 & * & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Column reduced form

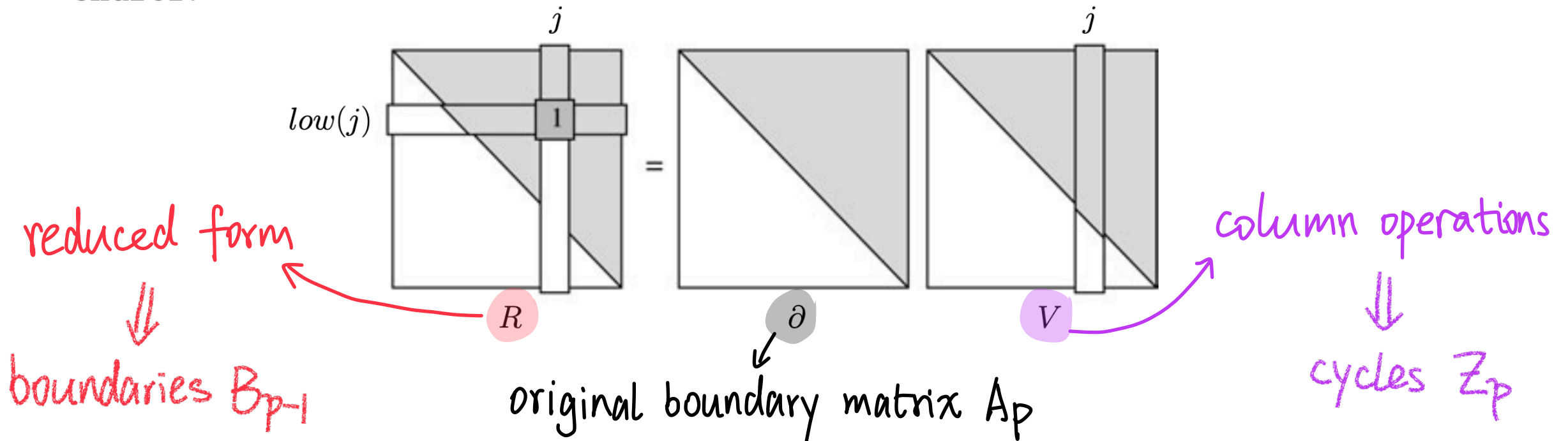
$$\text{low}[i] \neq \text{low}[j]$$

Recall: Left-to-right Column Reduction Algorithm

```

 $R = \partial;$ 
for  $j = 1$  to  $m$  do
  while there exists  $j_0 < j$  with  $low(j_0) = low(j)$  do
    add column  $j_0$  to column  $j$ 
  endwhile
endfor.

```

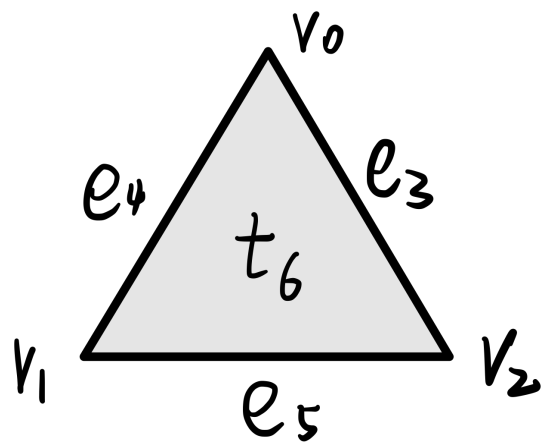
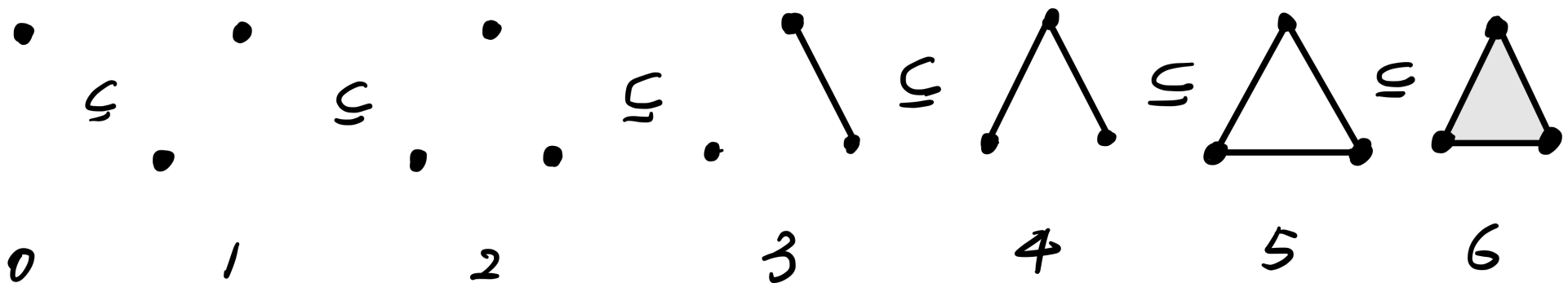


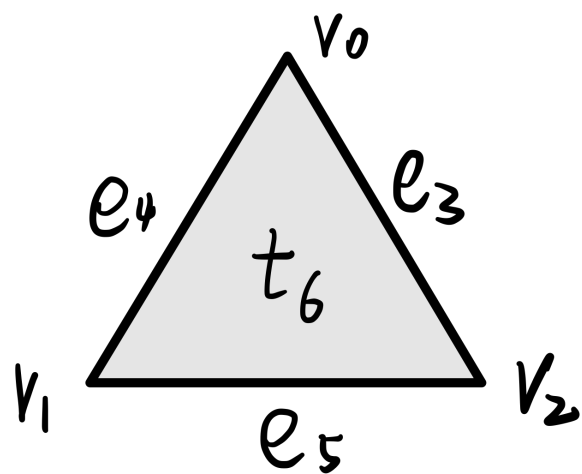
Persistent Algorithm

- ▶ Assume simplex-wise filtration $\emptyset = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n = K$
 - ▶ i.e., the filtration induced by **an ordered sequence of simplices $\sigma_1, \sigma_2, \dots, \sigma_n$**
s.t. $K_i = \{\sigma_1, \dots, \sigma_i\}$
- ▶ Let A be boundary matrix for K (the ambient one) with $Col_A[i] = \partial\sigma_i$
- ▶ Use the previous reduction algorithm to reduce A :

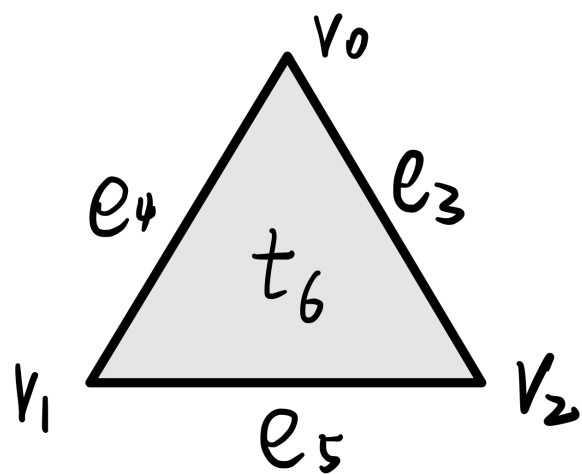
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 $R = \partial;$   
for  $j = 1$  to  $m$  do  
  while there exists  $j_0 < j$  with  $low(j_0) = low(j)$  do  
    add column  $j_0$  to column  $j$   
  endwhile  
endfor.
```

Example

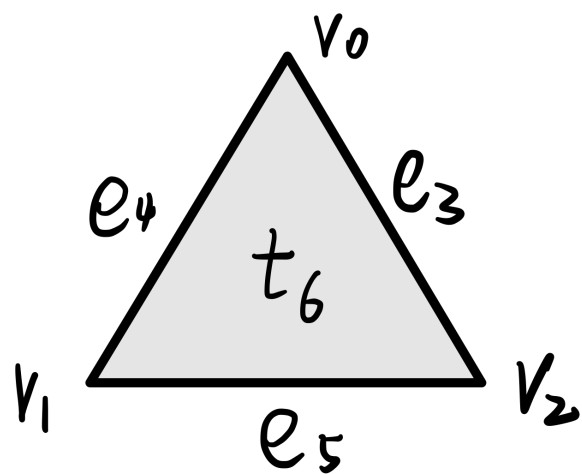




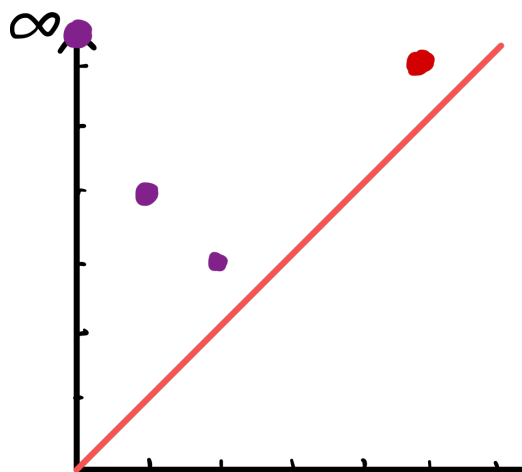
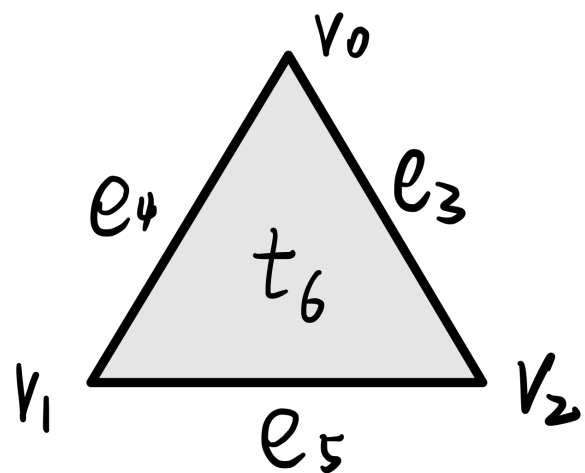
	v0	v1	v2	e3	e4	e5	t6
v0	0	0	0	1	1	0	0
v1	0	0	0	0	1	1	0
v2	0	0	0	1	0	1	0
e3	0	0	0	0	0	0	1
e4	0	0	0	0	0	0	1
e5	0	0	0	0	0	0	1
t6	0	0	0	0	0	0	0



	v0	v1	v2	e3	e4	e5+e3	t6
v0	0	0	0	1	1	1	0
v1	0	0	0	0	1	1	0
v2	0	0	0	1	0	0	0
e3	0	0	0	0	0	0	1
e4	0	0	0	0	0	0	1
e5	0	0	0	0	0	0	1
t6	0	0	0	0	0	0	0



	v0	v1	v2	e3	e4	e5+e3+ e4	t6
v0	0	0	0	1	1	0	0
v1	0	0	0	0	1	0	0
v2	0	0	0	1	0	0	0
e3	0	0	0	0	0	0	1
e4	0	0	0	0	0	0	1
e5	0	0	0	0	0	0	1
t6	0	0	0	0	0	0	0



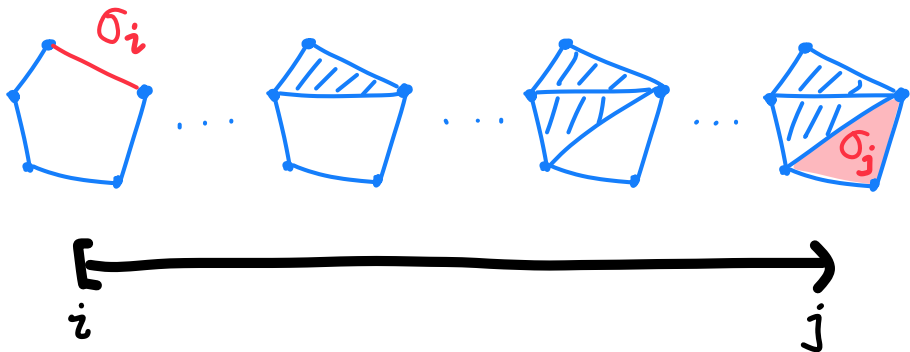
	v0	v1	v2	e3	e4	e5+e3+ e4	t6
v0	0	0	0	1	1	0	0
v1	0	0	0	0	1	0	0
v2	0	0	0	1	0	0	0
e3	0	0	0	0	0	0	1
e4	0	0	0	0	0	0	1
e5	0	0	0	0	0	0	1
t6	0	0	0	0	0	0	0



Persistent Pairings

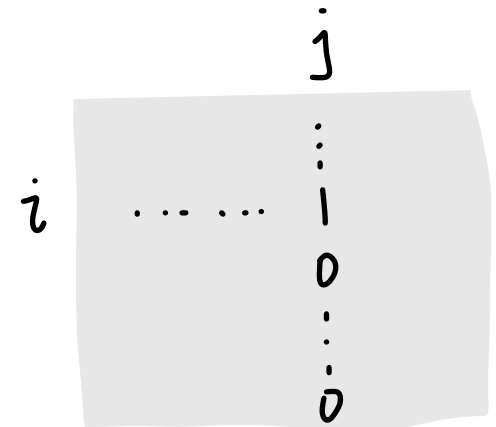
Definition 3.11 (Persistence pairs). Given a simplex-wise filtration $\mathcal{F} : K_0 \hookrightarrow K_1 \hookrightarrow \dots \hookrightarrow K_n$, for $0 < i < j \leq n$, we say a p -simplex $\sigma_i = K_i \setminus K_{i-1}$ and a $(p+1)$ -simplex $\sigma_j = K_j \setminus K_{j-1}$ form a persistence pair (σ_i, σ_j) if and only if $\mu_p^{i,j} > 0$.

- ▶ Theorem: Assume a simplex-wise filtration.
- ▶ Consider the output matrix R of the left-to-right column reduction algorithm. Then $\mu^{i,j} = 1$ **iff** $lowId_R(j) = i$



(σ_i, σ_j) is a p. pair

$$\mu^{i,j} > 0 \iff$$



Persistent Pairings

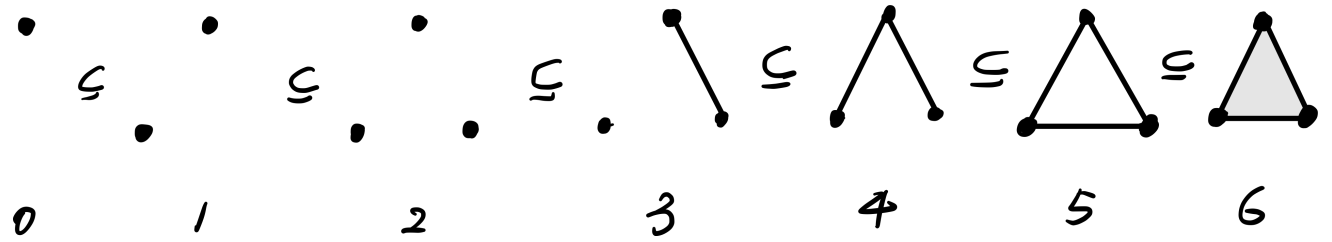
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▶ Theorem: Assume a simplex-wise filtration.

▶ Consider the output matrix R of the left-to-right column reduction algorithm. Then $\mu^{i,j} = 1$ **iff** $lowId_R(j) = i$

- ▶ Rules to compute persistence diagram (for simplex-wise filtration) after reduction:
 - ▶ Every i appears exactly once, either as birth or as death
 - ▶ Get (i, j) iff $lowId_R(j) = i$
 - ▶ Get (i, ∞) if i is not paired with anything after the previous step

Persistent Pairings



- ▶ Theorem: Assume a simplex-wise filtration.
 - ▶ Consider the output matrix R of the left-to-right column reduction algorithm. Then $\mu^{i,j} = 1$ **iff** $lowId_R(j) = i$

	v0	v1	v2	e3	e4	e5+e3+ e4	t6
v0	0	0	0	1	1	0	0
v1	0	0	0	0	1	0	0
v2	0	0	0	1	0	0	0
e3	0	0	0	0	0	0	1
e4	0	0	0	0	0	0	1
e5	0	0	0	0	0	0	1
t6	0	0	0	0	0	0	0

- ▶ Homology classes born at 0,1,2,5
- ▶ $(v_0, \infty), (v_1, e_4), (v_2, e_3), (e_5, t_6)$
- ▶ $Dgm_0 = \{(0, \infty), (1, 4), (2, 3)\}$
- ▶ $Dgm_1 = \{(5, 6)\}$

Algorithm 3 MATPERSISTENCE(D)

Input:

Boundary matrix D of a complex with columns and rows ordered by a given filtration

Output:

Reduced matrix with each column j either being empty or having a unique $\text{low}_D[j]$ entity

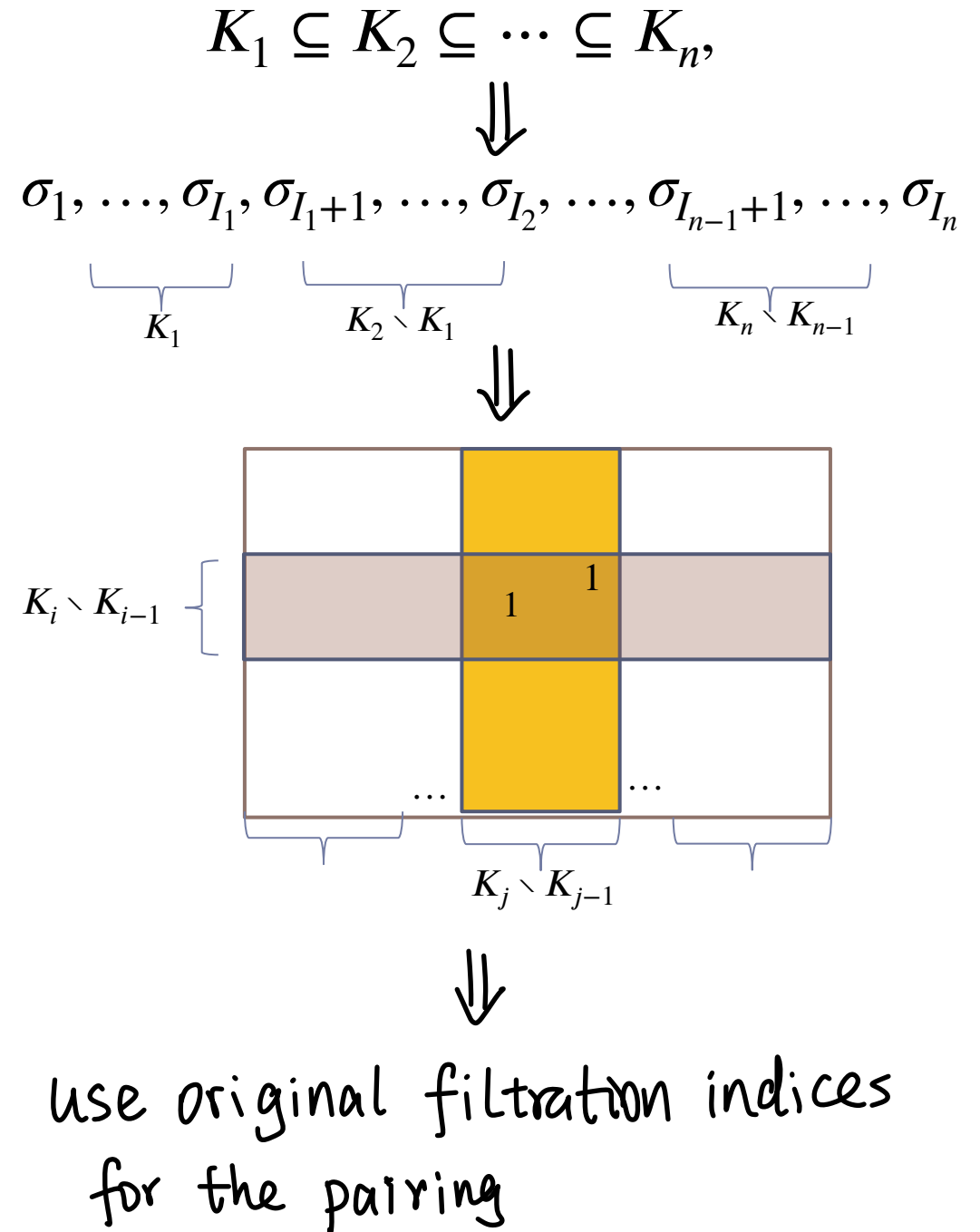
```
1: for  $j = 1 \rightarrow |\text{col}_D|$  do
2:   while  $\exists j' < j$  s.t.  $\text{low}_D[j'] == \text{low}_D[j]$  and  $\text{low}_D[j] \neq -1$  do
3:      $\text{col}_D[j] := \text{col}_D[j] + \text{col}_D[j']$ 
4:   end while
5:   if  $\text{low}_D[j] \neq -1$  then
6:      $i := \text{low}_D[j] \setminus *$  generate pair  $(\sigma_i, \sigma_j) * \setminus$ 
7:   end if
8: end for
```

reduction

extract persistent pairs

General Filtration

- ▶ Given a filtration $K_1 \subseteq K_2 \subseteq \dots \subseteq K_n$, we make it simplex-wise, by **introducing an ordering of the simplices** $\sigma_1, \sigma_2, \dots, \sigma_N$ consistent with the filtration.
- ▶ For practical reasons, when there is a tie, it is common to require
 - ▶ faces of a simplex appear before the simplex
 - ▶ lower dimensions enter first



Demo of Ripser

<https://colab.research.google.com/drive/1P6JBRiCMroQSZUwaXnayysEuvzdY7tyM?usp=sharing>