

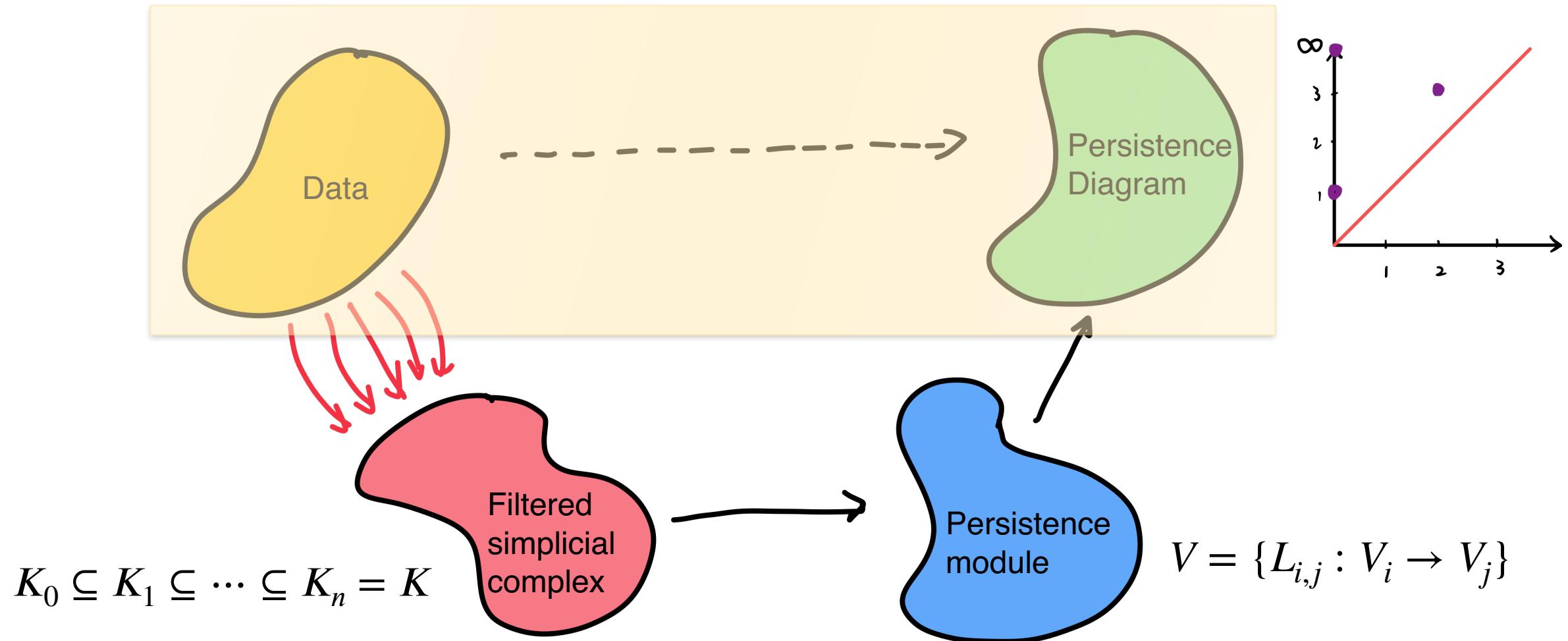
MATH412/COMPSCI434/MATH713
Fall 2025

Topological Data Analysis

Topic 4: Introduction to Persistent Homology - Part 2

Instructor: Ling Zhou

TDA in a nutshell



Summary

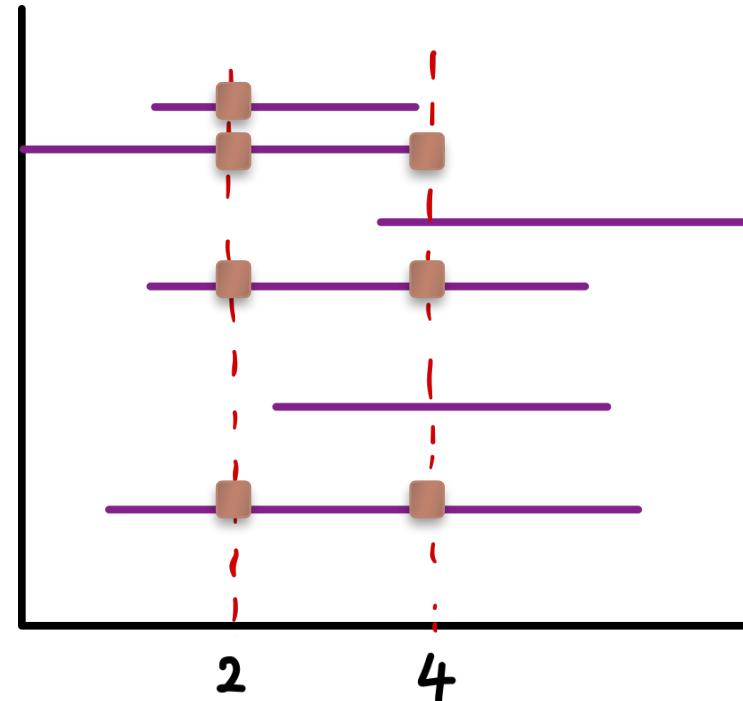
- ▶ Create a filtered simplicial complex $K_\bullet = \{K_i\}$ out of data
- ▶ Let $V_\bullet = \{V_i = H_p(K_i)\}_{i=0}^n$ be the p -dim persistence homology of K_\bullet
- ▶ Decompose $V_\bullet \cong I[b_1, d_1) \oplus I[b_2, d_2) \oplus \cdots \oplus I[b_M, d_M)$
- ▶ The multiset $Dgm_p(K_\bullet) = \{(b_j, d_j)\}_{j=1, \dots, M} \subseteq \mathbb{R}^2$ is called the **degree p persistence diagram** of K_\bullet .
- ▶ p -th **persistent homology group** from i to j : $H_p^{i,j} = \text{Im}(\iota_p^{i,j})$ ($\subset H_p(K_j)$)
- ▶ p -th **persistent betti number** from i to j : $\beta_p^{i,j} = \dim H_p^{i,j}$
- ▶ $\beta_p^{i,j}$ denotes the number of homology classes co-existing at both K_i and K_j

Persistent Betti Number vs Barcode

- Let $V_\bullet = \{V_i = H_p(K_i)\}_{i=0}^n \cong I[b_1, d_1) \oplus I[b_2, d_2) \oplus \dots \oplus I[b_M, d_M)$
- $\beta_p^{i,j} = \#$ of bars crossing both vertical lines at i and at j
- $\beta_p^{i,j} = \#$ of bars that are born $\leq i$ and die $> j$

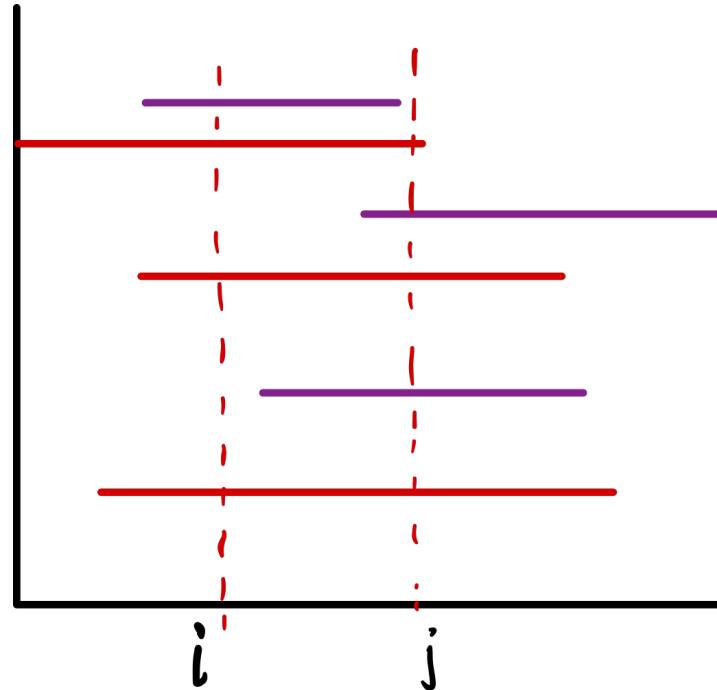
$$\beta_p^{i,j} = \dim H_p^{i,j}$$

$$\beta_p^{2,4} = 3$$



Persistent Betti Number vs Barcode

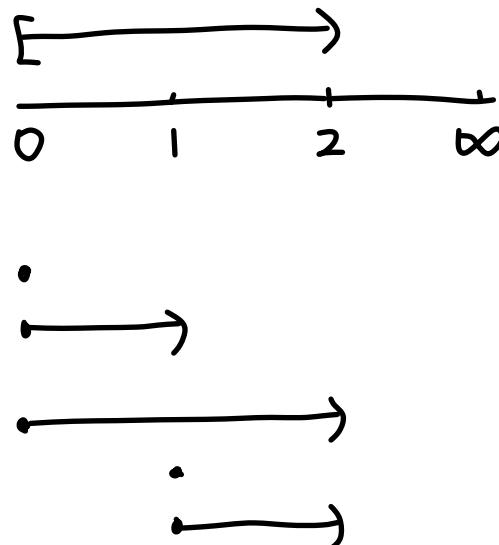
- ▶ $\mu^{b,d}$:= multiplicity of the interval $[b, d)$ in the barcode,
- ▶ $\mu^{b,d} : \text{Intervals} \rightarrow \mathbb{Z}$ is also called the **persistent pairing function**
- ▶ Theorem: $\beta_p^{i,j} = \sum_{k \leq i, j < l} \mu_p^{k,l}$ = number of intervals $[k, l)$ that contains $[i, j)$



Persistent Betti Number vs Barcode

► Theorem: Assume $\beta_p^{-1,j} = \beta^{i,n+1} = 0$. For $0 \leq i < j \leq n + 1$, we have
 $\mu_p^{i,j} = (\beta_p^{i,j-1} - \beta_p^{i,j}) - (\beta_p^{i-1,j-1} - \beta_p^{i-1,j})$

- $\mu^{b,d}$:= multiplicity of the interval $[b, d)$ in the barcode,
- $\mu^{b,d}$: Intervals $\rightarrow \mathbb{Z}$ is also called the **persistent pairing function**
- Theorem: $\beta_p^{i,j} = \sum_{k \leq i, j < l} \mu_p^{k,l}$ = number of intervals $[k, l)$ that contains $[i, j]$

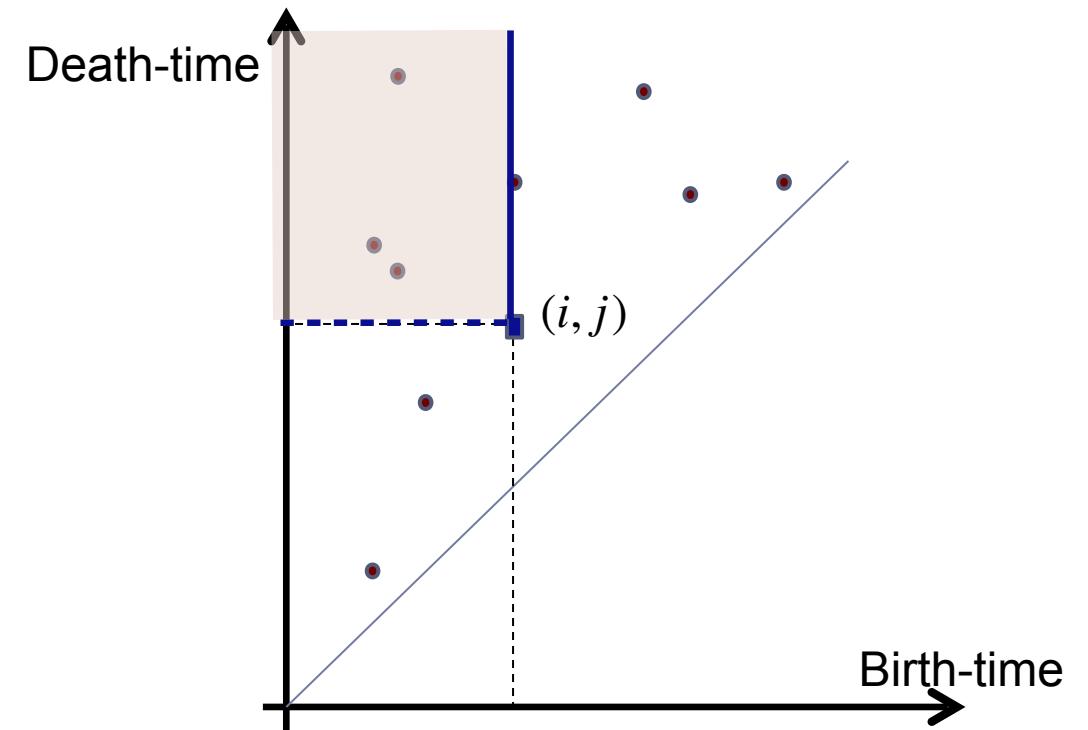


$$\begin{aligned}
 0|1 &= (0, 0) - \cancel{(0, 1)} - \cancel{(-1, 0)} + \cancel{(-1, 1)} \\
 1|2 &= (1, 1) - \cancel{(1, 2)} - \cancel{(0, 1)} + \cancel{(0, 2)} \\
 0|2 &= (0, 1) - \cancel{(0, 2)} - \cancel{(-1, 1)} + \cancel{(-1, 2)}
 \end{aligned}$$

Persistent Betti Number vs Persistence Diagram

- ▶ $\mu^{b,d}$:= multiplicity of the point (b, d) in the persistence diagram
- ▶ Theorem: $\beta_p^{i,j} = \sum_{k \leq i, j < l} \mu_p^{k,l}$ = number of points (k, l) s.t. $k \leq i, j < l$

$$k \leq i \iff \begin{array}{c} (k, \ell) \\ \bullet \\ \hline \end{array} \quad \begin{array}{c} (i, j) \\ \bullet \\ \hline \end{array}$$



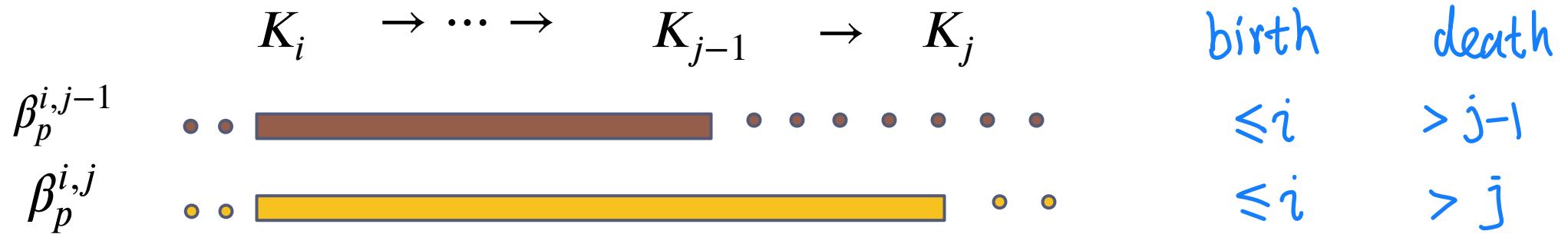
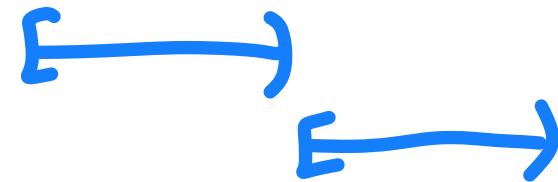
$$\beta^{i,j} = \# \text{ bars born } \leq i \text{ \& die } > j$$

Möbius Inversion: Compute μ from β

- Theorem: Assume $\beta_p^{-1,j} = \beta^{i,n+1} = 0$. For $0 \leq i < j \leq n + 1$, we have

$$\mu_p^{i,j} = (\beta_p^{i,j-1} - \beta_p^{i,j}) - (\beta_p^{i-1,j-1} - \beta_p^{i-1,j})$$

Number of independent homology classes from K_i but **died** entering K_j

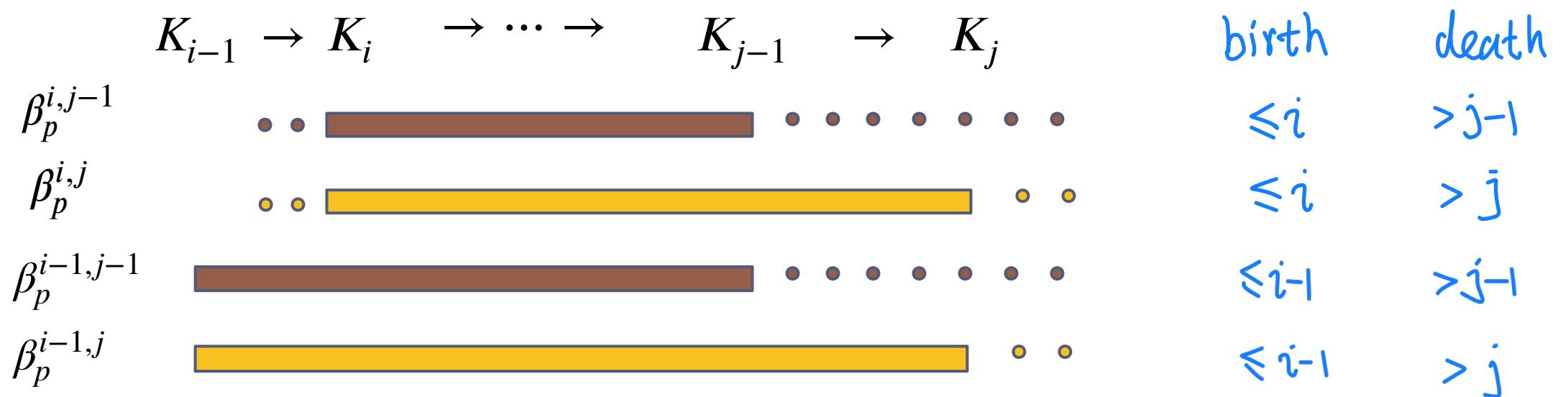


Möbius Inversion: Compute μ from β

- Theorem: Assume $\beta_p^{-1,j} = \beta_p^{i,n+1} = 0$. For $0 \leq i < j \leq n + 1$, we have

$$\mu_p^{i,j} = (\beta_p^{i,j-1} - \beta_p^{i,j}) - (\beta_p^{i-1,j-1} - \beta_p^{i-1,j}) = (\text{born } \leq i, \text{ dies at } j) - (\text{born } \leq i-1, \text{ dies at } j) = (\text{born } = i, \text{ dies at } j)$$

Number of independent homology classes from K_i but **died** entering K_j
Number of independent homology classes from K_{i-1} but **died** entering K_j



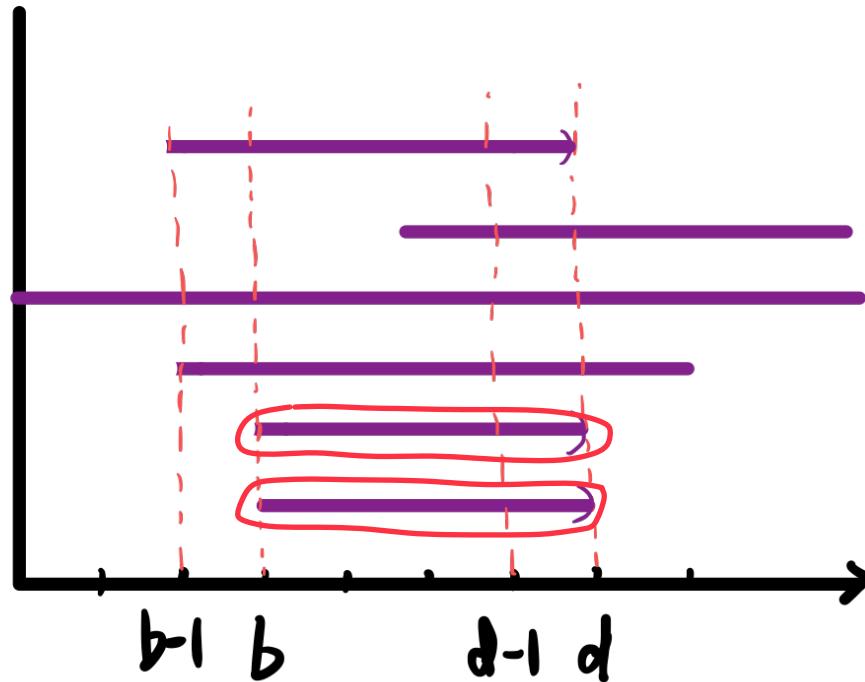
Example

$$\beta^{i,j} = \# \text{ bars born} \leq i \text{ \& die} > j$$

► $\frac{\mu^{b,d}}{||} = (\beta^{b,d-1} - \beta^{b,d}) - (\beta^{b-1,d-1} - \beta^{b-1,d})$

$\# [b, d)$

$\frac{||}{2}$



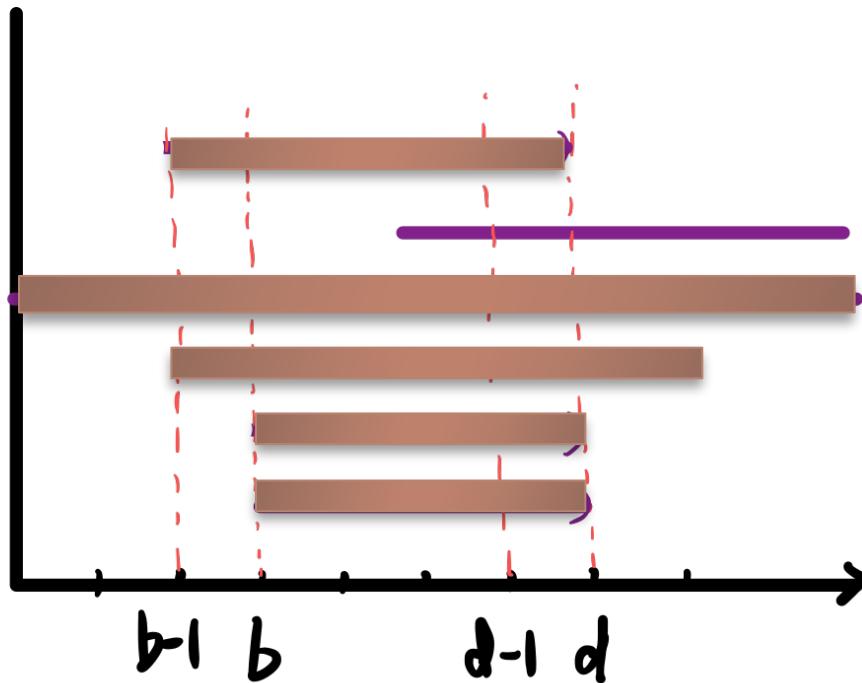
Example

$$\beta^{i,j} = \# \text{ bars born} \leq i \& \text{ die} > j$$

$$\triangleright \frac{\mu^{b,d}}{\# \{b, d\}} = (\underbrace{\beta^{b,d-1}}_{11} - \underbrace{\beta^{b,d}}_{5}) - (\underbrace{\beta^{b-1,d-1}}_{11} - \underbrace{\beta^{b-1,d}}_{5})$$

$\# \{b, d\}$

11
2



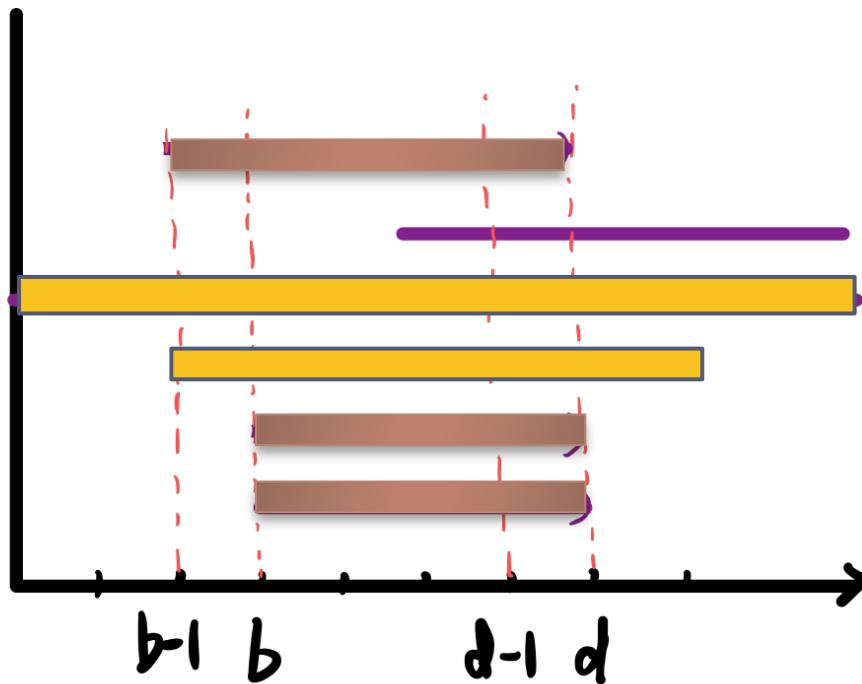
Example

$$\beta^{i,j} = \# \text{ bars born } \leq i \text{ & die } > j$$

$$\mu^{b,d} = \underbrace{\beta^{b,d-1}}_{11} - \underbrace{\beta^{b,d}}_{5} - \underbrace{(\beta^{b-1,d-1} - \beta^{b-1,d})}_{2}$$

$\# \Sigma_{b,d}$

11
2



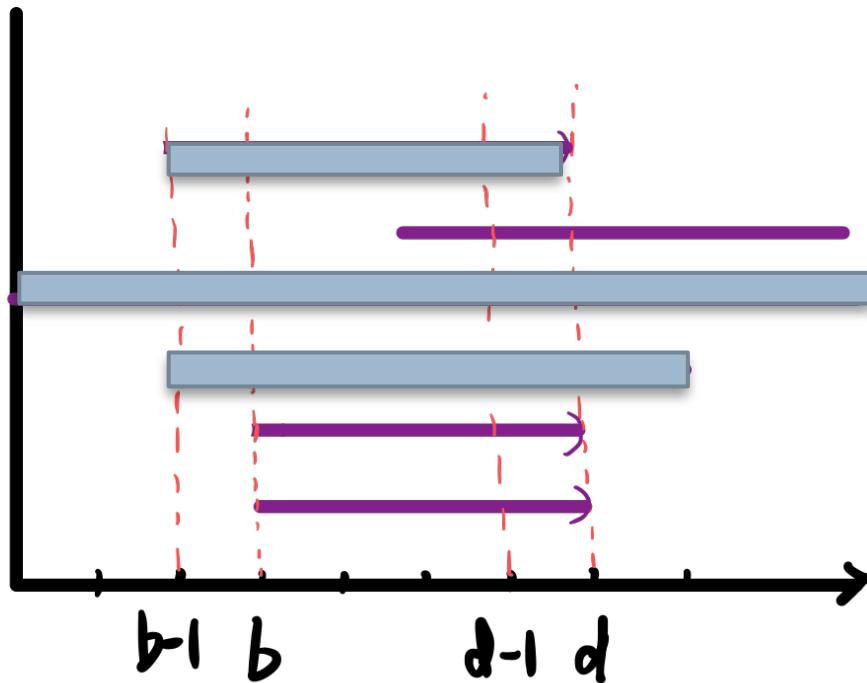
Example

$$\beta^{i,j} = \# \text{ bars born } \leq i \text{ & die } > j$$

$$\mu^{b,d} = \underbrace{\beta^{b,d-1}}_{\text{II}} - \underbrace{\beta^{b,d}}_{\text{5}} - \underbrace{(\beta^{b-1,d-1} - \beta^{b-1,d})}_{\text{II 3}}$$

$\# \Sigma_{b,d}$

II
2



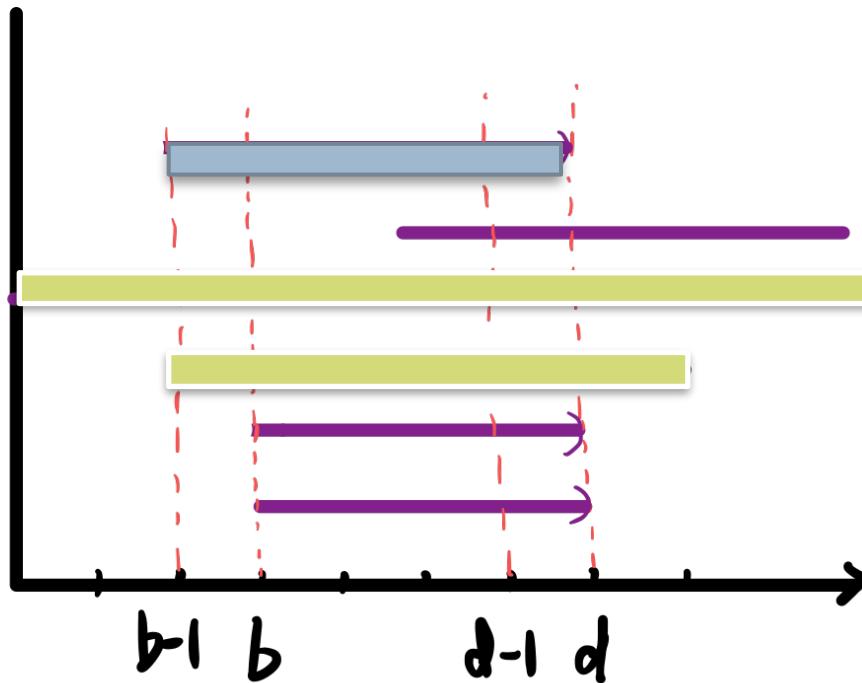
Example

$$\beta^{i,j} = \# \text{ bars born } \leq i \text{ & die } > j$$

$$\mu^{b,d} = \underbrace{\beta^{b,d-1}}_{\text{II}} - \underbrace{\beta^{b,d}}_{\text{II}} - (\underbrace{\beta^{b-1,d-1}}_{\text{II}} - \underbrace{\beta^{b-1,d}}_{\text{II}})$$

$\# \Sigma_{b,d}$

II
2

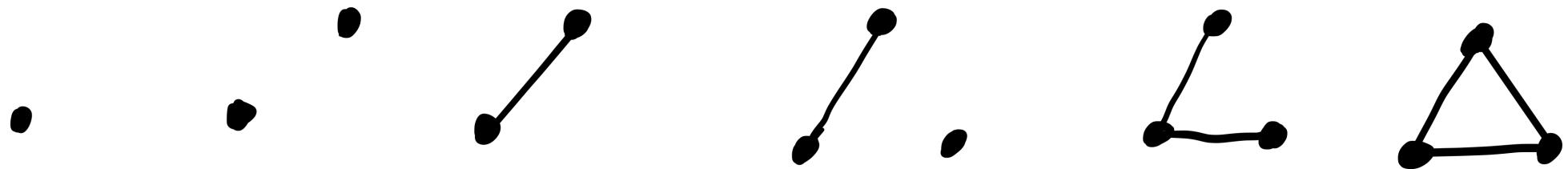


Topological View of Barcodes

An alternative view

- ▶ **Simplex-wise filtration:** at each step, only one simplex is added.
- ▶ That is, $\emptyset = K_0 \subseteq K_1 \subseteq K_2 \subseteq \dots \subseteq K_n = K$, s.t. $\sigma_i = K_i \setminus K_{i-1}$

example

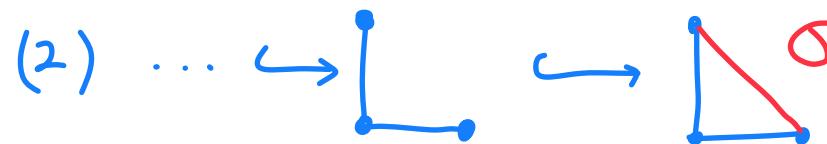


non-example

An alternative view

- ▶ At K_i , consider the next added p -simplex $\sigma = \sigma_{i+1}$. It can be
 - ▶ A creator: adding σ creates a ‘new’ p -cycle
 - ▶ Addition of σ generates a homology class which is not in $\text{Im}(H_p(K_i) \rightarrow H_p(K_{i+1}))$
 - ▶ hence β_p ++

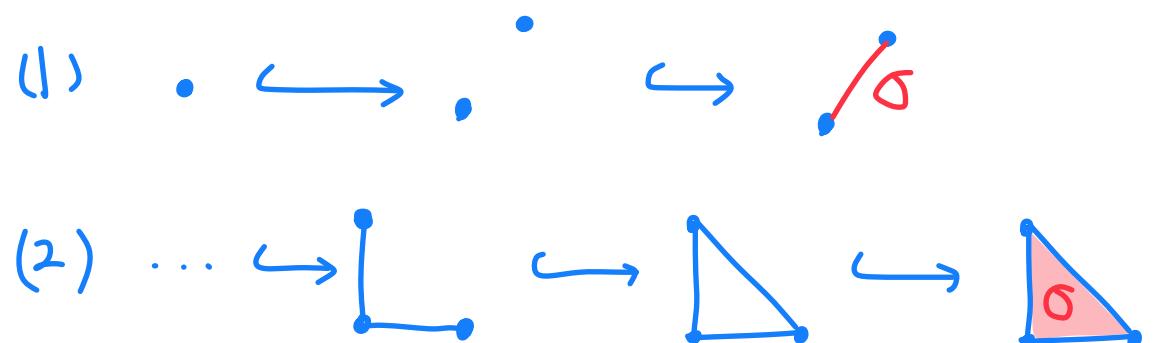
Examples



An alternative view

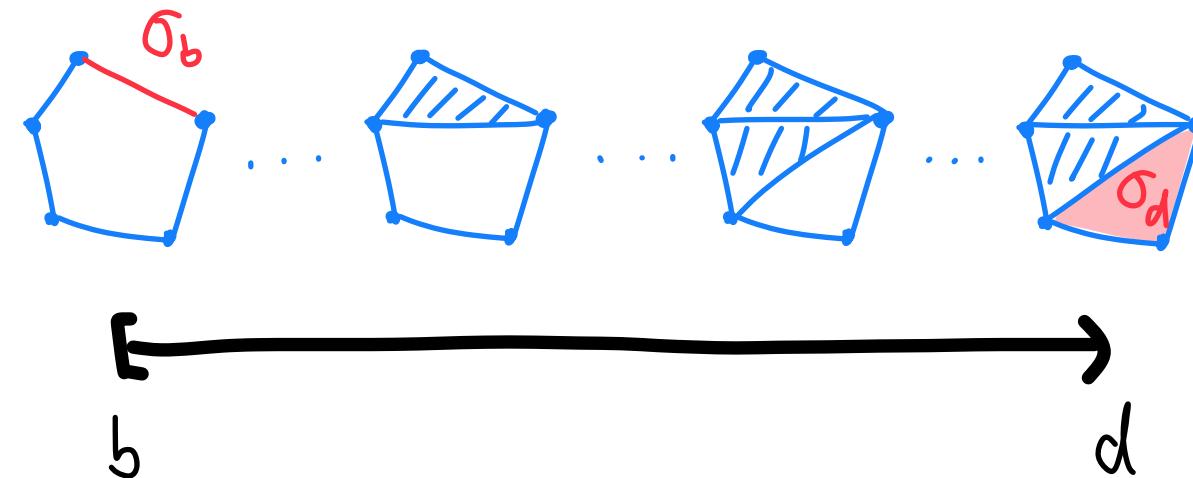
- ▶ At K_i , consider the next added p -simplex $\sigma = \sigma_{i+1}$. It can be
 - ▶ A creator: adding σ creates a ‘new’ p -cycle
 - ▶ Addition of σ generates a homology class which is not in $\text{Im}(H_p(K_i) \rightarrow H_p(K_{i+1}))$
 - ▶ hence $\beta_p ++$
 - ▶ A destroyer: adding σ kills an ‘old’ $(p-1)$ -cycle
 - ▶ This $(p-1)$ -cycle is not trivial in $H_{p-1}(K_i)$, but becomes trivial in $H_{p-1}(K_{i+1})$
 - ▶ hence $\beta_{p-1} --$

Examples



Interpretation of Barcode

- Let $V_{\bullet} = \{V_i = H_p(K_i)\}_{i=0}^n \cong I[b_1, d_1) \oplus I[b_2, d_2) \oplus \cdots \oplus I[b_M, d_M)$, assuming a simplex-wise filtration
- Each $I[b, d)$ corresponds to
 - adding a p simplex σ_b at time b to create a p -cycle
 - adding a $p + 1$ simplex σ_d at time d to kill the above p -cycle

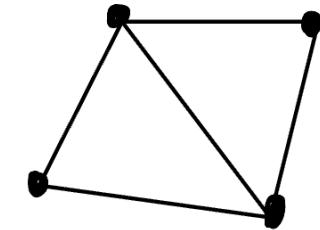
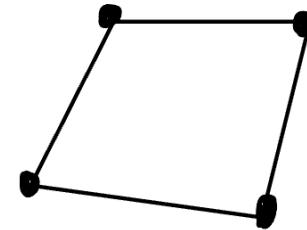


Subtlety of non-uniqueness

- ▶ Which cycle is killed?

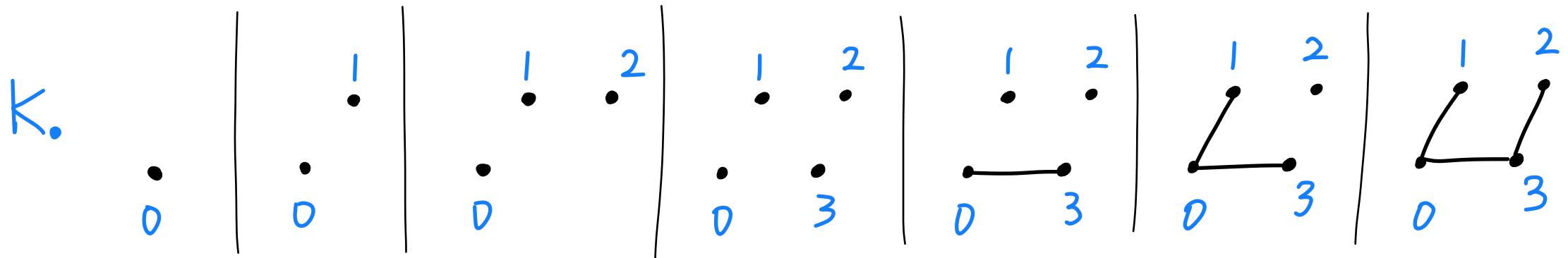


- ▶ Which cycle is created?



- ▶ The younger one will be killed

- ▶ Several cycle classes are born
- ▶ But the dimension only increases by 1



$[0]$

$[1]$

$[2]$

$[3]$

barcode of $H_0(K_*)$

Persistence Algorithm

Recall: Matrix Reduction

- ▶ Turn A_p into the **column reduced form**
 - ▶ Each non-zero column has a unique lowID/pivot: index of lowest 1-entry
- ▶ **Column operations** in Gaussian elimination:
 - ▶ scaling (not needed over \mathbb{Z}_2)
 - ▶ swap (not necessary)
 - ▶ add one column to another
- ▶ Do **left-to-right column reduction** and get bases of
 - ▶ $B_{p-1} = \text{Im } \partial_p$: the reduced columns
 - ▶ $Z_p = \ker \partial_p$: the column operations

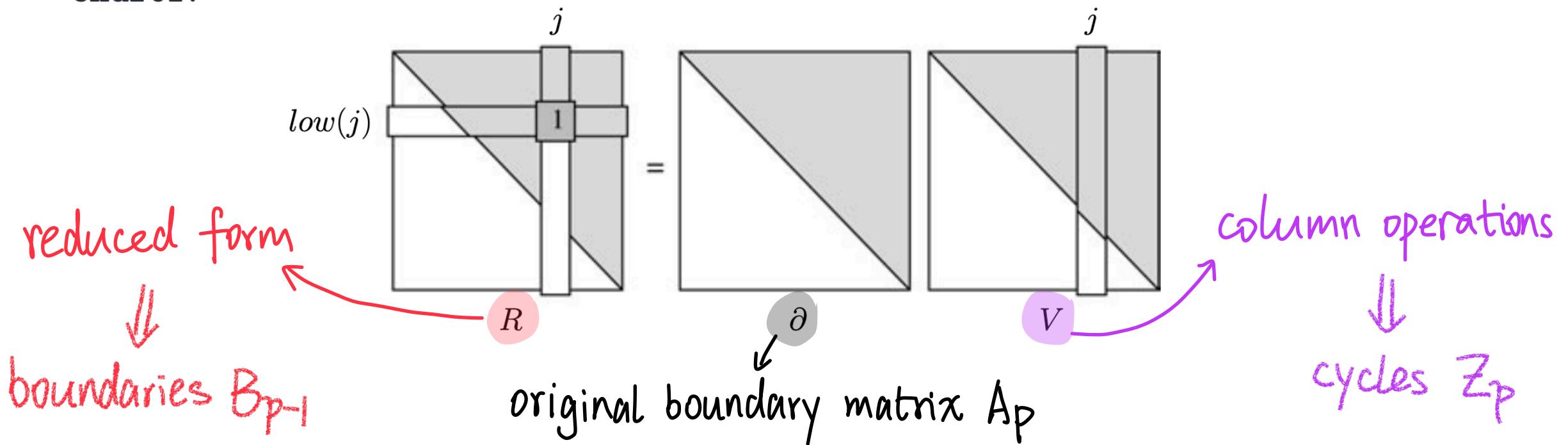
$$\begin{bmatrix} * & * & * & 0 \\ * & 1 & * & 0 \\ 1 & 0 & * & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Column reduced form

$\text{low}[i] \neq \text{low}[j]$

Recall: Left-to-right Column Reduction Algorithm

```
 $R = \delta;$   
for  $j = 1$  to  $m$  do  
  while there exists  $j_0 < j$  with  $low(j_0) = low(j)$  do  
    add column  $j_0$  to column  $j$   
  endwhile  
endfor.
```

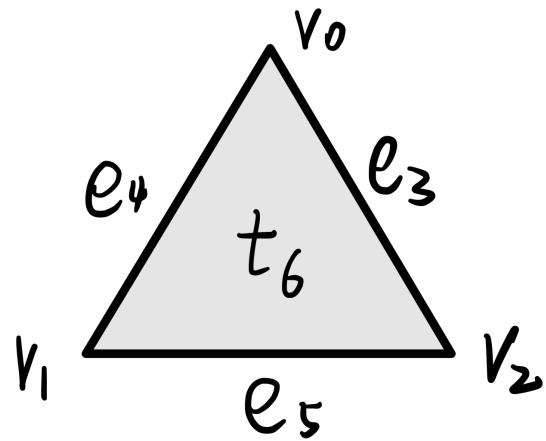
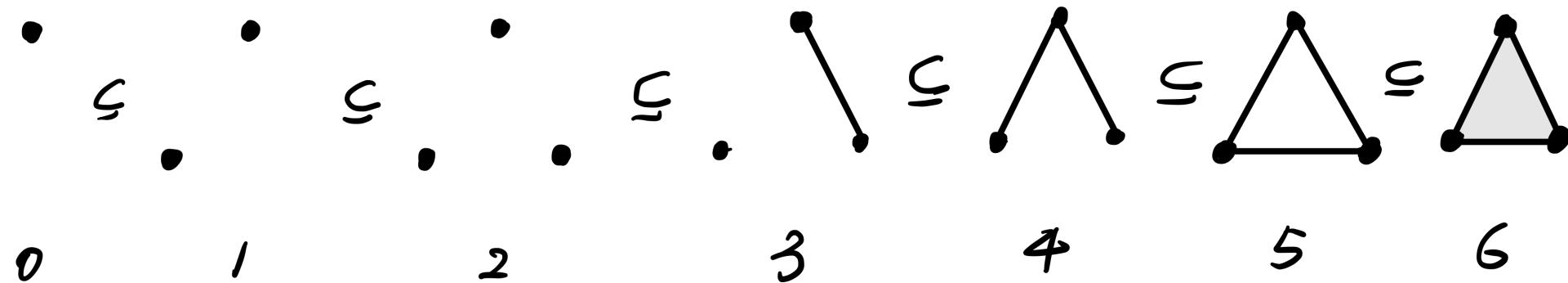


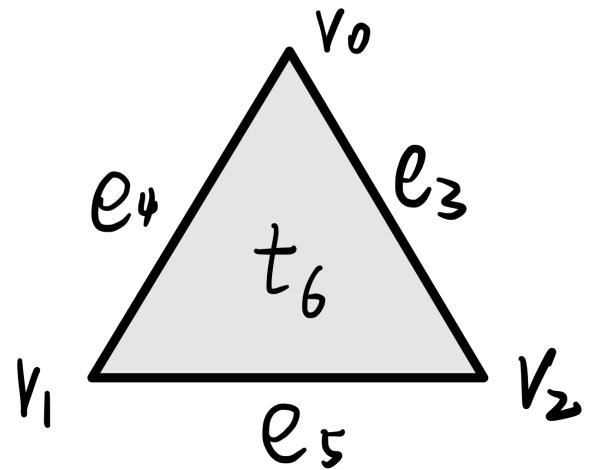
Persistent Algorithm

- ▶ Assume simplex-wise filtration $\emptyset = K_0 \subseteq K_1 \subseteq K_2 \subseteq \dots \subseteq K_n = K$
 - ▶ i.e., the filtration induced by **an ordered sequence of simplices $\sigma_1, \sigma_2, \dots, \sigma_n$**
s.t. $K_i = \{\sigma_1, \dots, \sigma_i\}$
- ▶ Let A be boundary matrix for K (the ambient one) with $Col_A[i] = \partial\sigma_i$
- ▶ Use the previous reduction algorithm to reduce A :

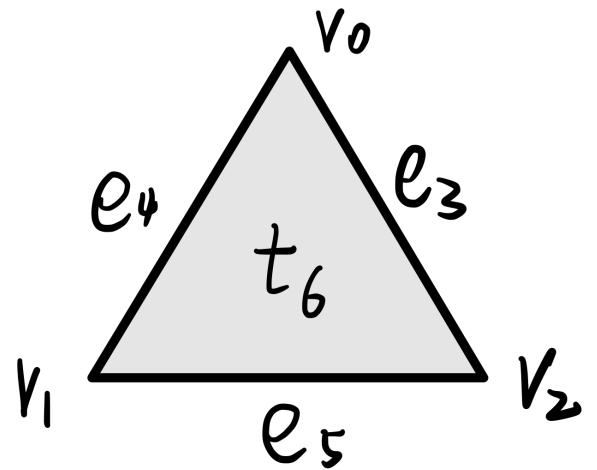
```
R = ∂;  
for j = 1 to m do  
  while there exists  $j_0 < j$  with  $low(j_0) = low(j)$  do  
    add column  $j_0$  to column  $j$   
  endwhile  
endfor.
```

Example

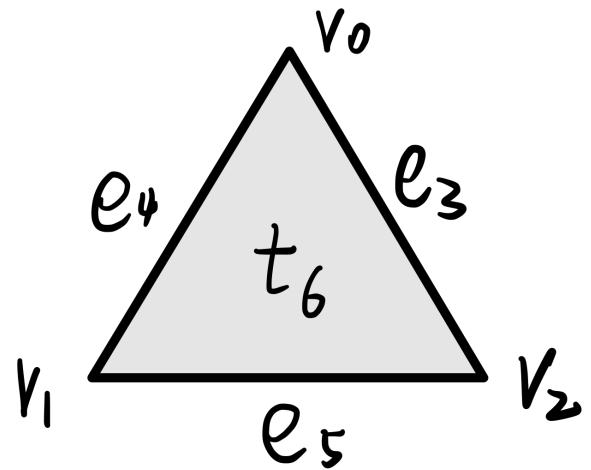




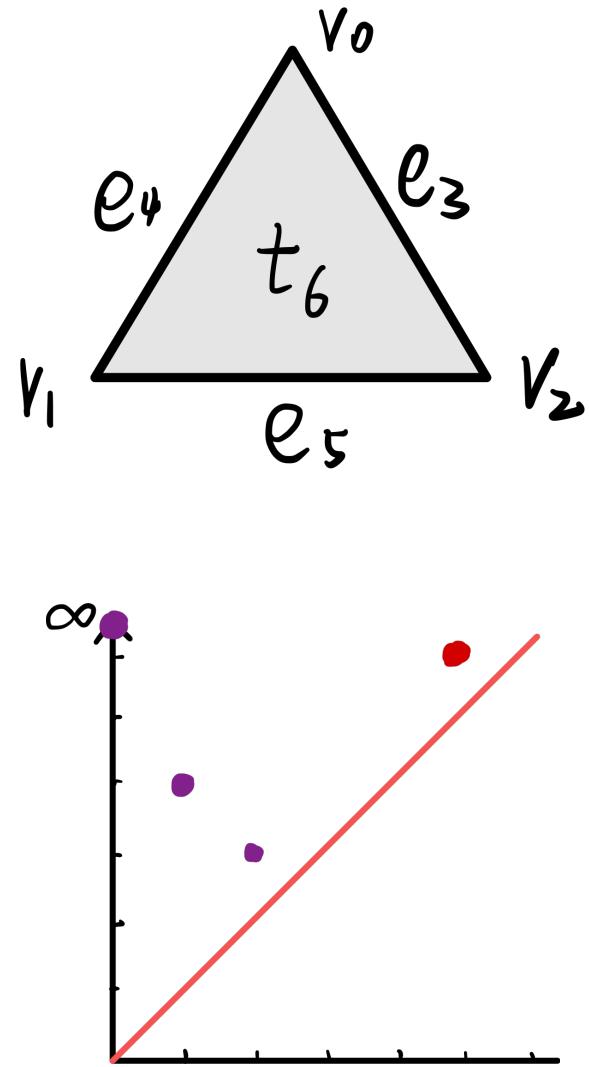
	v0	v1	v2	e3	e4	e5	t6
v0	0	0	0	1	1	0	0
v1	0	0	0	0	1	1	0
v2	0	0	0	1	0	1	0
e3	0	0	0	0	0	0	1
e4	0	0	0	0	0	0	1
e5	0	0	0	0	0	0	1
t6	0	0	0	0	0	0	0



	v0	v1	v2	e3	e4	e5+e3	t6
v0	0	0	0	1	1	1	0
v1	0	0	0	0	1	1	0
v2	0	0	0	1	0	0	0
e3	0	0	0	0	0	0	1
e4	0	0	0	0	0	0	1
e5	0	0	0	0	0	0	1
t6	0	0	0	0	0	0	0



	v0	v1	v2	e3	e4	e5+e3+ e4	t6
v0	0	0	0	1	1	0	0
v1	0	0	0	0	1	0	0
v2	0	0	0	1	0	0	0
e3	0	0	0	0	0	0	1
e4	0	0	0	0	0	0	1
e5	0	0	0	0	0	0	1
t6	0	0	0	0	0	0	0



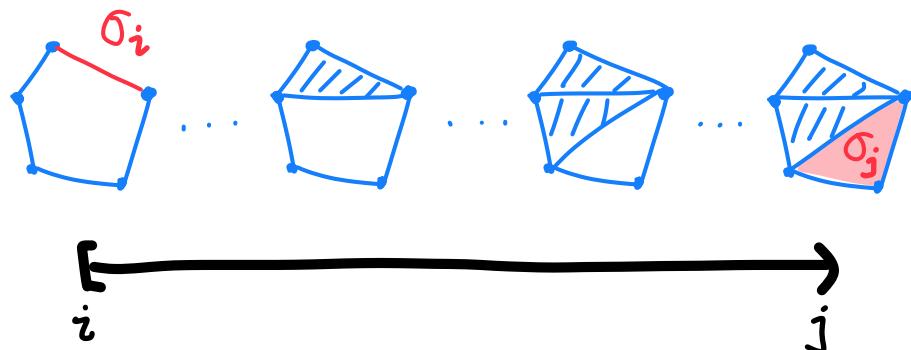
	v_0	v_1	v_2	e_3	e_4	$e_5 + e_3 + e_4$	t_6
v_0	0	0	0	1	1	0	0
v_1	0	0	0	0	1	0	0
v_2	0	0	0	1	0	0	0
e_3	0	0	0	0	0	0	1
e_4	0	0	0	0	0	0	1
e_5	0	0	0	0	0	0	1
t_6	0	0	0	0	0	0	0

?

Persistent Pairings

Definition 3.11 (Persistence pairs). Given a simplex-wise filtration $\mathcal{F} : K_0 \hookrightarrow K_1 \hookrightarrow \dots \hookrightarrow K_n$, for $0 < i < j \leq n$, we say a p -simplex $\sigma_i = K_i \setminus K_{i-1}$ and a $(p+1)$ -simplex $\sigma_j = K_j \setminus K_{j-1}$ form a persistence pair (σ_i, σ_j) if and only if $\mu_p^{i,j} > 0$.

- ▶ Theorem: Assume a simplex-wise filtration.
- ▶ Consider the output matrix R of the left-to-right column reduction algorithm. Then $\mu^{i,j} = 1$ iff $\text{lowId}_R(j) = i$



(σ_i, σ_j) is a p. pair
 \Updownarrow
 $\mu^{i,j} > 0 \iff$

		j

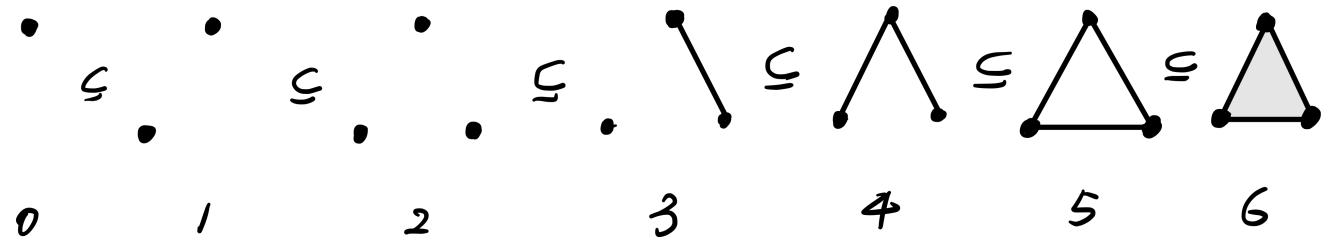
i		1
		0
		...
		0

Persistent Pairings

Definition 3.11 (Persistence pairs). Given a simplex-wise filtration $\mathcal{F} : K_0 \hookrightarrow K_1 \hookrightarrow \dots \hookrightarrow K_n$, for $0 < i < j \leq n$, we say a p -simplex $\sigma_i = K_i \setminus K_{i-1}$ and a $(p+1)$ -simplex $\sigma_j = K_j \setminus K_{j-1}$ form a persistence pair (σ_i, σ_j) if and only if $\mu_p^{i,j} > 0$.

- ▶ Theorem: Assume a simplex-wise filtration.
 - ▶ Consider the output matrix R of the left-to-right column reduction algorithm. Then $\mu^{i,j} = 1$ **iff** $lowId_R(j) = i$
- ▶ Rules to compute persistence diagram (for simplex-wise filtration) after reduction:
 - ▶ Every i appears exactly once, either as birth or as death
 - ▶ Get (i, j) iff $lowId_R(j) = i$
 - ▶ Get (i, ∞) if i is not paired with anything after the previous step

Persistent Pairings



- ▶ Theorem: Assume a simplex-wise filtration.
- ▶ Consider the output matrix R of the left-to-right column reduction algorithm. Then $\mu^{i,j} = 1$ iff $lowId_R(j) = i$

	v0	v1	v2	e3	e4	e5+e3+ e4	t6
v0	0	0	0	1	1	0	0
v1	0	0	0	0	1	0	0
v2	0	0	0	1	0	0	0
e3	0	0	0	0	0	0	1
e4	0	0	0	0	0	0	1
e5	0	0	0	0	0	0	1
t6	0	0	0	0	0	0	0

- ▶ Homology classes born at 0,1,2,5
- ▶ $(v_0, \infty), (v_1, e_4), (v_2, e_3), (e_5, t_6)$
- ▶ $Dgm_0 = \{(0, \infty), (1, 4), (2, 3)\}$
- ▶ $Dgm_1 = \{(5, 6)\}$

Algorithm 3 MATPERSISTENCE(D)**Input:**

Boundary matrix D of a complex with columns and rows ordered by a given filtration

Output:

Reduced matrix with each column j either being empty or having a unique $\text{low}_D[j]$ entity

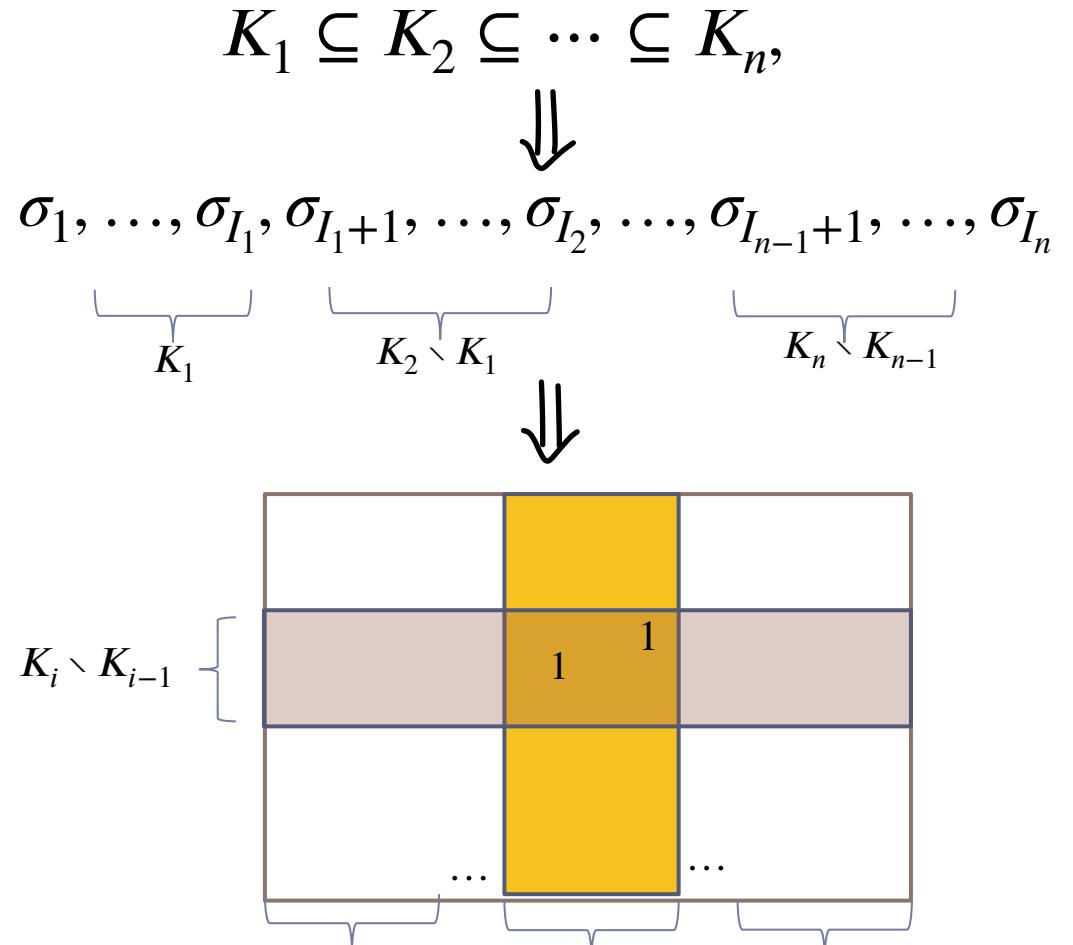
```
1: for  $j = 1 \rightarrow |\text{col}_D|$  do
2:   while  $\exists j' < j$  s.t.  $\text{low}_D[j'] == \text{low}_D[j]$  and  $\text{low}_D[j] \neq -1$  do
3:      $\text{col}_D[j] := \text{col}_D[j] + \text{col}_D[j']$ 
4:   end while
5:   if  $\text{low}_D[j] \neq -1$  then
6:      $i := \text{low}_D[j]$  /* generate pair  $(\sigma_i, \sigma_j)$  */
7:   end if
8: end for
```

reduction

extract persistent pairs

General Filtration

- Given a filtration $K_1 \subseteq K_2 \subseteq \dots \subseteq K_n$, we make it simplex-wise, by introducing an ordering of the simplices $\sigma_1, \sigma_2, \dots, \sigma_N$ consistent with the filtration.
- For practical reasons, when there is a tie, it is common to require
 - faces of a simplex appear before the simplex
 - lower dimensions enter first



use original filtration indices
for the pairing

Demo of Ripser

<https://colab.research.google.com/drive/1P6JBRiCMroQSZUwaXnayySEuvzdY7tyM?usp=sharing>