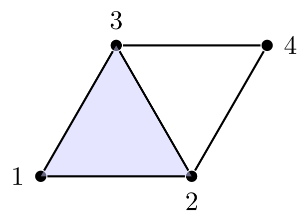
Please show your work to earn credit.

Homework should be submitted to Gradescope by 11:59 pm on **Friday, Sep 26**.

You can access the LaTeX source here:

1. (3 pt.) Use the fact that a p-simplex Δ^p is homotopy equivalent to a single point to compute its homology groups in all dimensions.

2. (4 pt.) We used the matrix reduction algorithm to find bases of B_0, Z_1, and H_1 for the simplicial complex K:



Assume $\mathbb{Z}_2$ coefficients. Find a suitable ordering of the edge set $\{12, 13, 34, 24, 23\}$ such that, after performing the left-to-right column reduction on the boundary matrix ∂_1, the basis obtained for Z_1 is

$$\{\, 12 + 13 + 23,\ 23 + 34 + 24 \,\}.$$

(This exercise illustrates that while the homology group $H_1(K)$ does not depend on ordering, the specific cycle representatives chosen by the reduction algorithm do.)

3. (a) (2 pt.) Let M and M' be the matrix representations of the boundary map ∂ based on two different orderings of the simplices. Show that there exist permutation matrices P and Q such that

$$M' = PMQ.$$

(b) (3 pt.) Using part (a), conclude that the Betti number

$$\beta_p = \dim H_p(K)$$

is independent of the ordering of simplices.

(c) **(Bonus, 1pt.)** A *chain map* between two chain complexes $\{C_p, \partial_p\}$ and $\{C'_p, \partial'_p\}$ is a collection of linear maps $\phi_p : C_p \to C'_p$ for all p such that

$$\partial'_p \circ \phi_p \;=\; \phi_{p-1} \circ \partial_p \quad \text{for all } p.$$

If each ϕ_p is an isomorphism, we call $\{\phi_p\}_p$ a *chain isomorphism*, and say the two chain complexes are *isomorphic*.

Define explicitly a chain isomorphism that permutes the basis elements of each C_p, and use it to justify that the homology groups $H_p(K)$ are independent of the ordering of simplices.

4. (10 pt.) Let K be the simplicial complex consisting of vertices $\{0, 1, 2, 3\}$ and edges $\{01, 12, 23, 03\}$.

(a) Write down the chain groups $C_0(K), C_1(K), C_2(K)$ over $\mathbb{Z}_2$.

(b) Find a matrix representation A_1 of the boundary map ∂_1 and compute a column reduced form of A_1. *(You may use a computer algebra system to do the matrix reduction.)*

(c) Find bases of $Z_1(K)$ and $B_0(K)$.

(d) Find a basis of $H_1(K)$.

5. (6 pt.) Let $f : K \to L$ be a simplicial map between simplicial complexes. Recall from lecture that f induces a linear map $\bar{f}_p : C_p(K) \to C_p(L)$ on the chain spaces.

(a) Show that $\bar{f}_{p-1} \circ \partial_p = \partial_p \circ \bar{f}_p$. *(i.e. $\{\bar{f}_p\}_p$ is a chain map.)*

(b) Deduce that if $c \in Z_p(K) = \ker(\partial_p)$ is a cycle, then $\bar{f}_p(c) \in Z_p(L)$.

(c) Deduce that if $b \in B_p(K) = \operatorname{im}(\partial_{p+1})$ is a boundary, then $\bar{f}_p(b) \in B_p(L)$.

(Notes: Parts (b) and (c) together explain why a chain map $\bar{f}_\bullet$ descends to a well-defined map on homology groups, i.e. on the quotient $H_p = Z_p/B_p$.)

6. (15 pt.) Read through the provided Jupyter notebook `homology_of_simplicial_complexes.ipynb`.

   (a) Run the examples already in the notebook (circle, sphere, torus, Klein bottle and real projective space) and record the Betti numbers of the examples computed.

   (b) Construct a simplicial complex that represents two circles joined at a single point. Compute its Betti numbers using the provided code. Submit a screenshot of your code along with your answer.

   (c) Modify the simplicial complex for the torus by removing the simplex [6, 7, 8]. Recompute the Betti numbers of this new complex and record your results. Submit a screenshot of your code along with your answer.

   (d) Compare the Betti numbers from part (b) and part (c). What do you observe? Briefly explain why this relation holds.

   (e) Use the provided code to compute the Betti numbers of the simplicial complex given in HW#1 Problem 8. Submit a screenshot of your code along with your answer.

   (f) **(Bonus, 2 pt.)** Modify the `reduce_matrix` Python function to be over real numbers instead of $\mathbb{Z}_2$. Use the modified code to recompute the Betti numbers for torus, Klein bottle and real projective space. Submit a screenshot of your code along with your answer.

7. (6 pt.) Let X_n denote the wedge of n circles, i.e. the space obtained by gluing n copies of S^1 together at a single common point. Compute the Betti numbers of X_n, and justify your answer by analyzing the chain groups and boundary maps arising from a suitable triangulation of X_n.