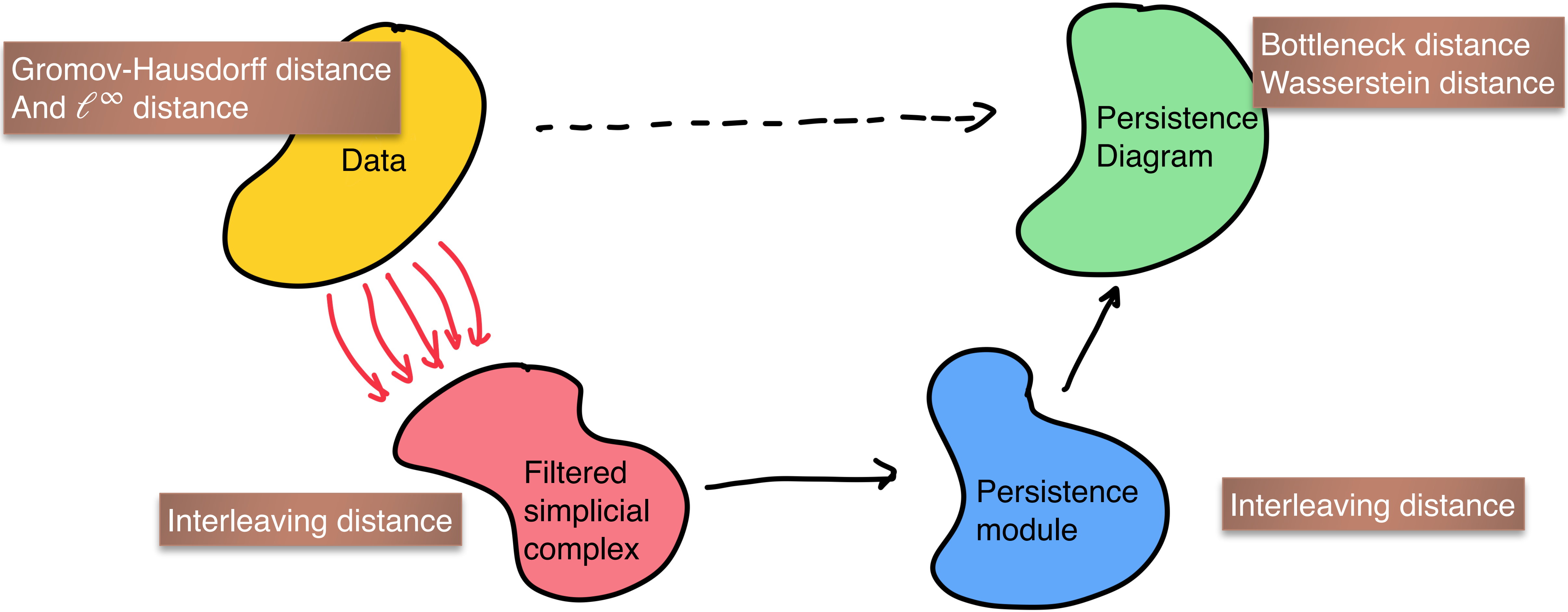
MATH412/COMPSCI434/MATH713
Fall 2025
Topological Data Analysis

Topic 5: Stability

Instructor: Ling Zhou

Using metrics to measure perturbations



Recall

Stability Theorem 1 [Chazal, Cohen-Steiner, Gliss, Guibas, Oudot 2009]

Given two finitely represented persistence modules U and V, let D_U and D_V be their corresponding persistence diagrams. We then have:

$$d_B(D_U, D_V) \leq d_I(U, V)$$

Isometry Theorem [Lesnick 2015], [Chazal, de Silva, Gliss and Oudot, 2016]

Given two finitely represented persistence modules U and V, let D_U and D_V be their corresponding persistence diagrams. We then have:

$$d_B(D_U, D_V) = d_I(U, V)$$

Stability Theorem 2

Given two simplicial filtrations $\mathcal{X}$ and $\mathcal{Y}$, let $PH_p(\mathcal{X})$ and $PH_p(\mathcal{Y})$ be the corresponding p-th persistence homology induced by them. We then have:

$$d_I(PH_p(\mathcal{X}), PH_p(\mathcal{X})) \leq d_I(\mathcal{X}, \mathcal{Y})$$

Stability Theorem 3

Given a topological space X and two functions $f, g : X \to \mathbb{R}$,

$$d_I(X_f, X_g) \leq \|f - g\|_\infty$$

Recall: Stability - Function induced persistence

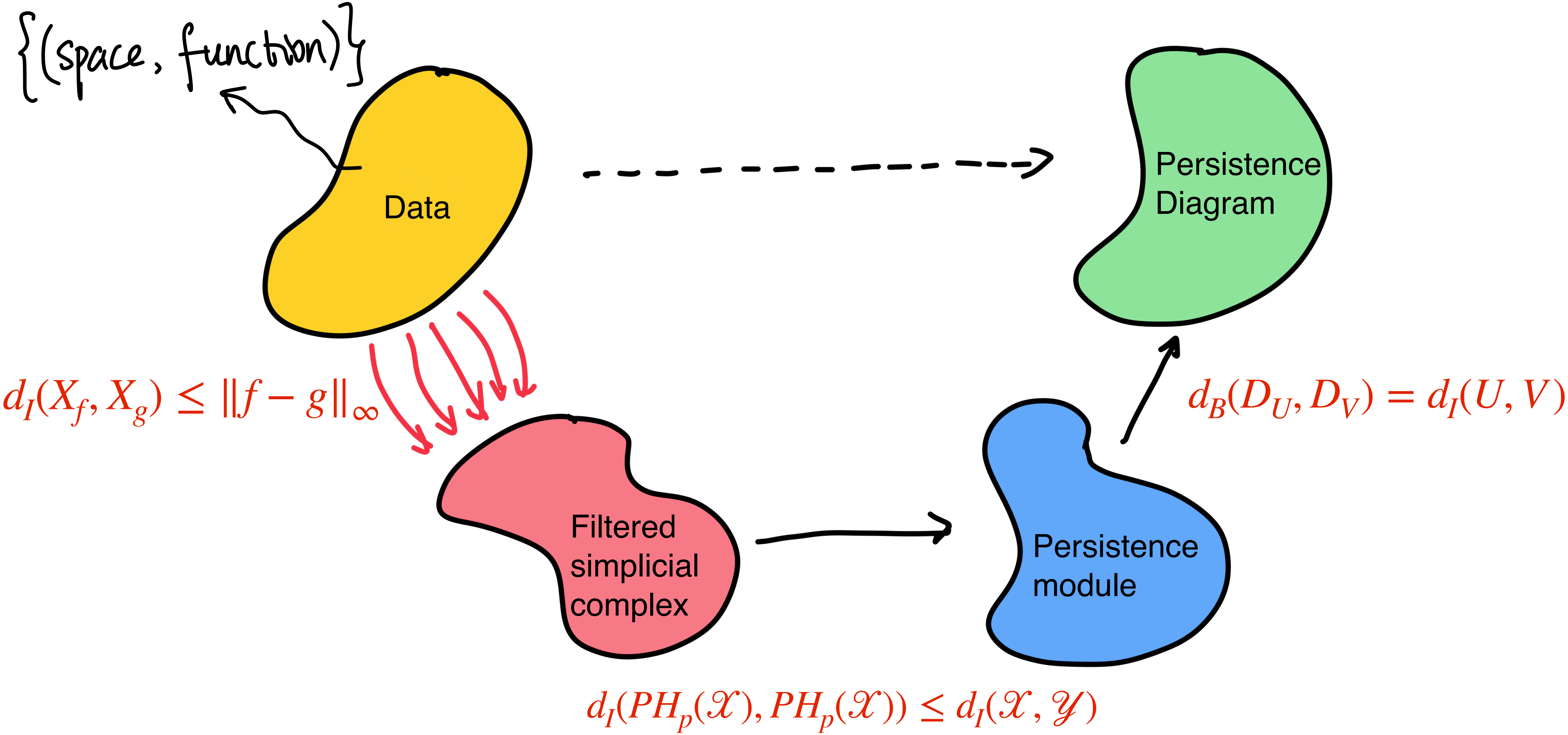
Stability Theorem for Function-Induced Persistence [Cohen-Steiner et al 2007]

Given two "nice" functions $f,\ g : X \to R$, let D^*_f and D^*_g be the persistence diagrams for the persistence modules induced by the sub-level set (resp. super-level set) filtrations w.r.t f and g, respectively. We then have:

$$d_B(D^*_f, D^*_g) = d_I(PH_*(X_f), PH_*(X_g)) \leq \|f - g\|_\infty$$

Isometry Theorem

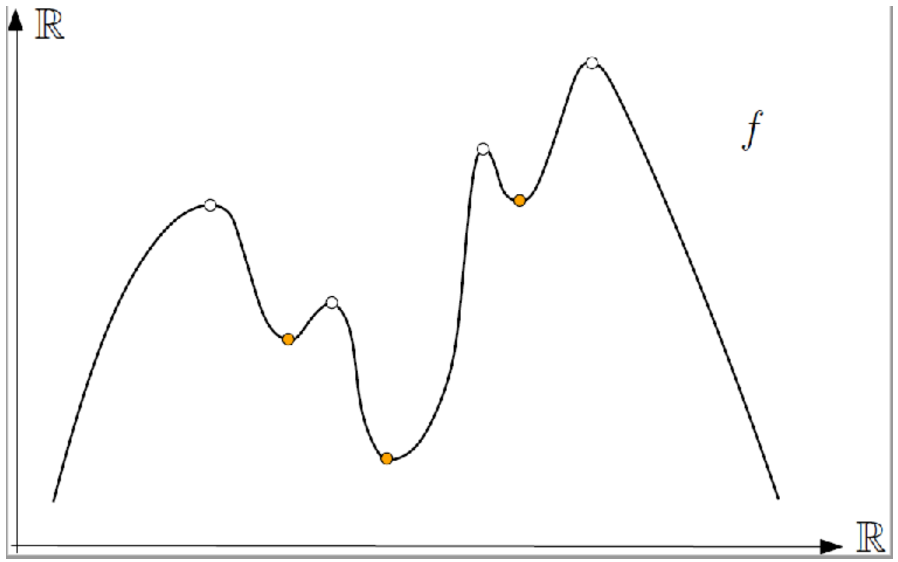
Stability Theorem 2 and Stability Theorem 3


{(space, function)}
Data
Persistence Diagram
$d_I(X_f, X_g) \leq \|f - g\|_\infty$
Filtered simplicial complex
Persistence module
$d_B(D_U, D_V) = d_I(U, V)$
$d_I(PH_p(\mathscr{X}), PH_p(\mathscr{X})) \leq d_I(\mathscr{X}, \mathscr{Y})$

Remarks on function-induced persistence

Gradients and critical points

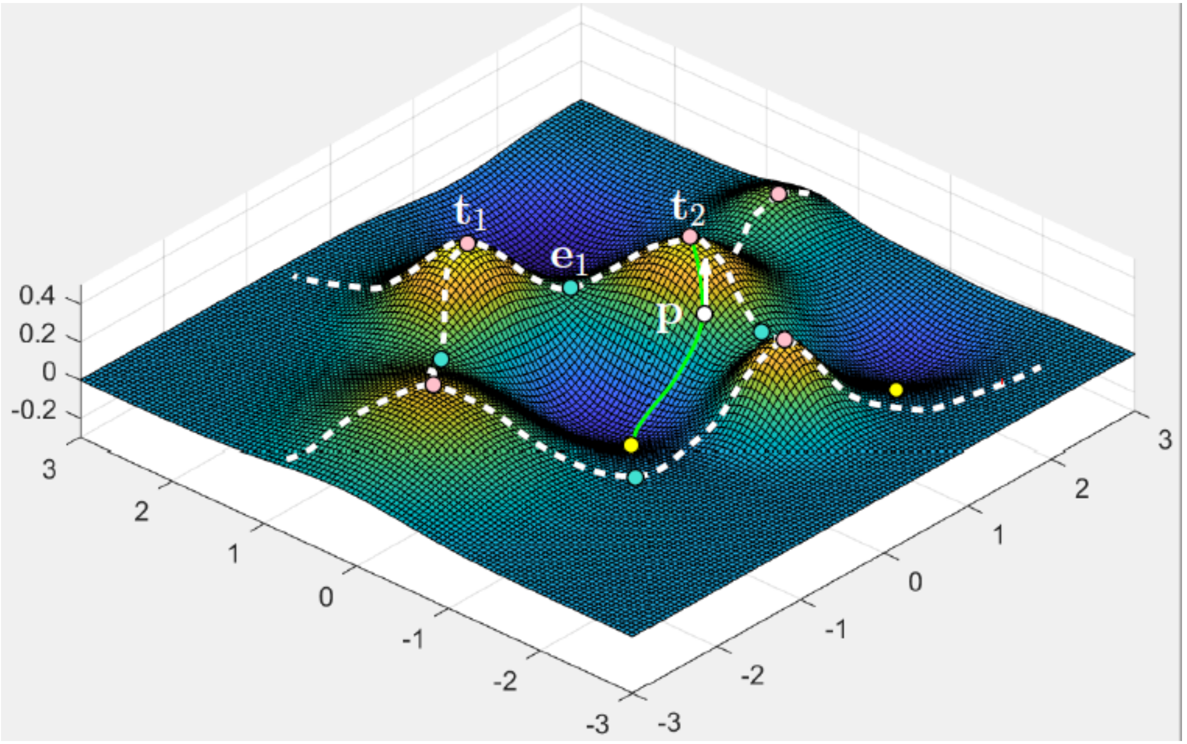
- 1D case: $f: \mathbb{R} \to \mathbb{R}$



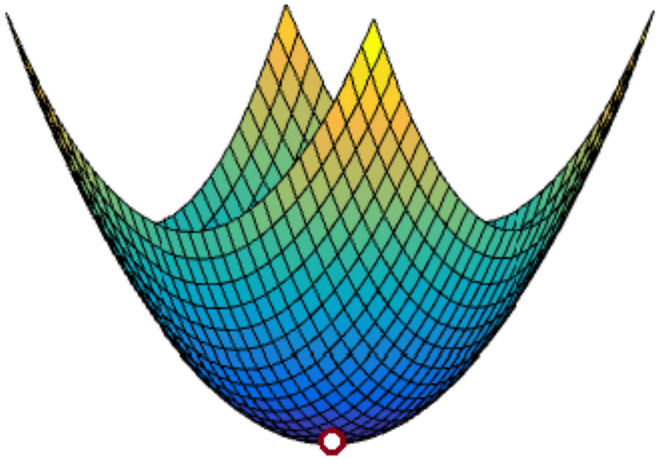
- Derivative
  - $\nabla f(x) = \lim_{t \to 0} \dfrac{f(x+t) - f(x)}{t}$
  - measures rate of change
- Critical points:
  - A point $x \in \mathbb{R}$ is a *critical point* w.r.t. f if $\nabla f(x) = 0$
- A non-critical point is called a *regular point*.

Gradients and critical points

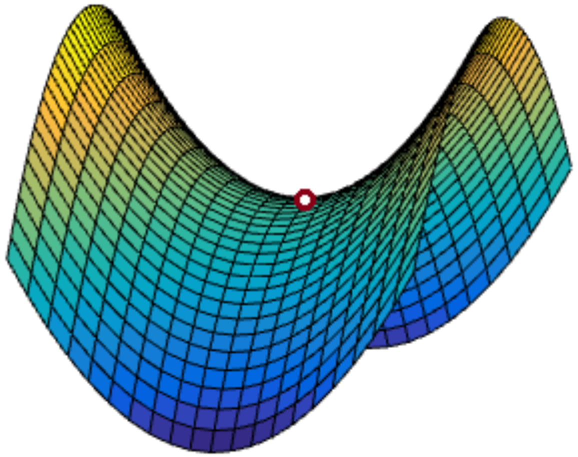
- dD case: $f\colon \mathbb{R}^d \to \mathbb{R}$
- Directional derivative:
  - $D_v f(p) = \lim_{t\to 0} \frac{f(p+tv) - f(p)}{t}$
  - measures rate of change in direction v
- *Gradient vector* at p
  - $\nabla f(p) = \left[\frac{\partial f}{x_1}, \ldots, \frac{\partial f}{x_d}\right]^T$, where $x_1, .. x_d$ form an orthonormal coordinate system
    - It is in the direction with largest directional derivative (with steepest rate of increase)
    - The magnitude is that largest rate of increase.
- Critical points:
  - A point p is *critical* if $\nabla f(p) = [0, \ldots, 0]^T$; that is, where gradient vanishes.



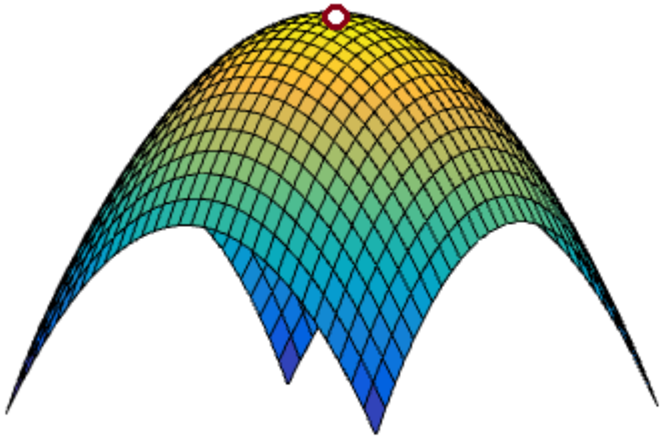
Examples for a 2D function



minimum



saddle



maximum

Gradients and critical points for manifolds

- d-manifold case: $f\colon M \to R$
- Same intuition, simply *within a small neighborhood at each point*

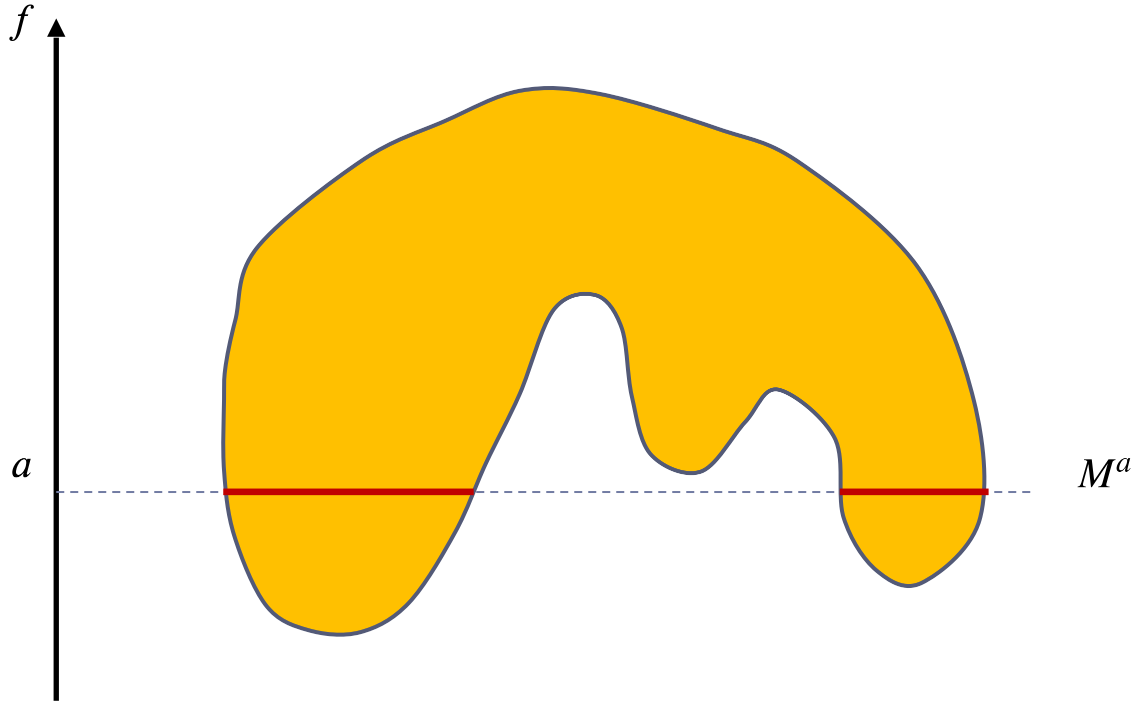
Definition 8 (Gradient vector field; Critical points)**.** Given a smooth function $f : M \to \mathbb{R}$ defined on a smooth m-dimensional Riemannian manifold M, the *gradient vector field* $\nabla f : M \to$ TM is defined as follows: for any $x \in M$, let $(x_1, x_2, \ldots, x_m)$ be a local coordinate system in a neighborhood of x with orthonormal unit vectors x_i, the gradient at x is

$$\nabla f(x) = [\frac{\partial f}{\partial x_1}(x),\ \frac{\partial f}{\partial x_2}(x),\ \cdots\ \frac{\partial f}{\partial x_m}(x)]^T.$$

A point $x \in M$ is *critical* if $\nabla f(x)$ vanishes, in which case $f(x)$ is called a *critical value* for f. Otherwise, x is *regular*.

Notations

- Function: $f: M \to R$
- Level set: $M^a = \left\{x \in M \mid f(x) = a\right\}$,
- Sub-level set: $M^{\leq a} = \{x \in M \mid f(x) \leq a\}$
  - $M^{\leq a} \subseteq M^{\leq b}$ for any $a \leq b$



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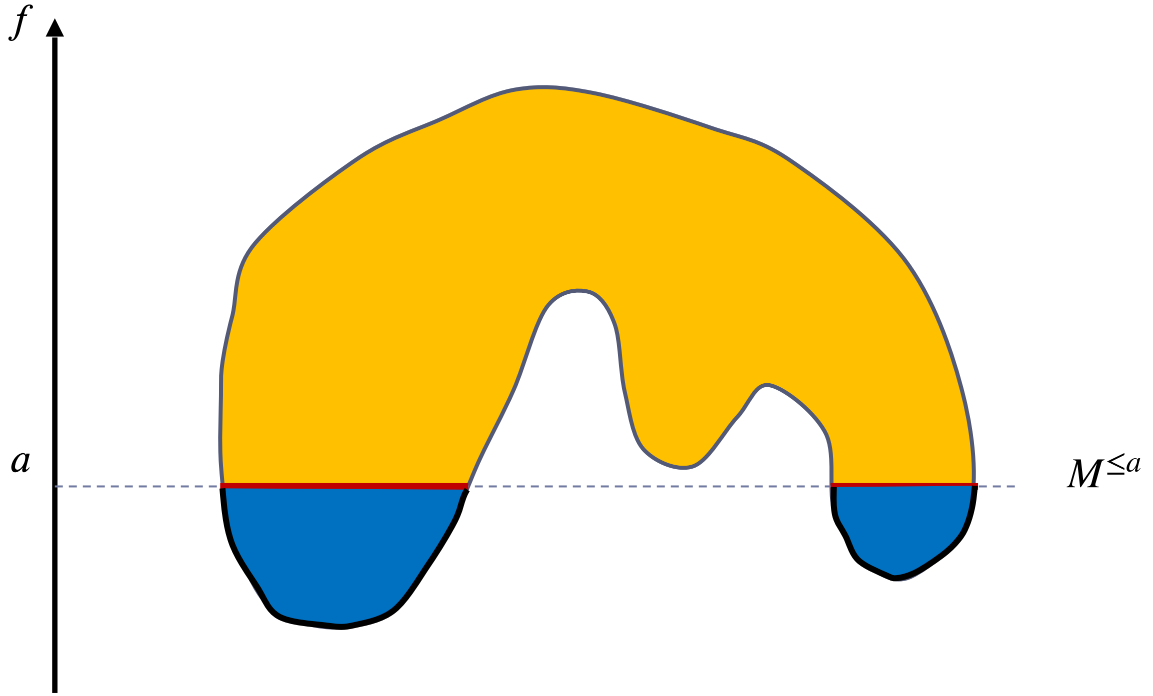
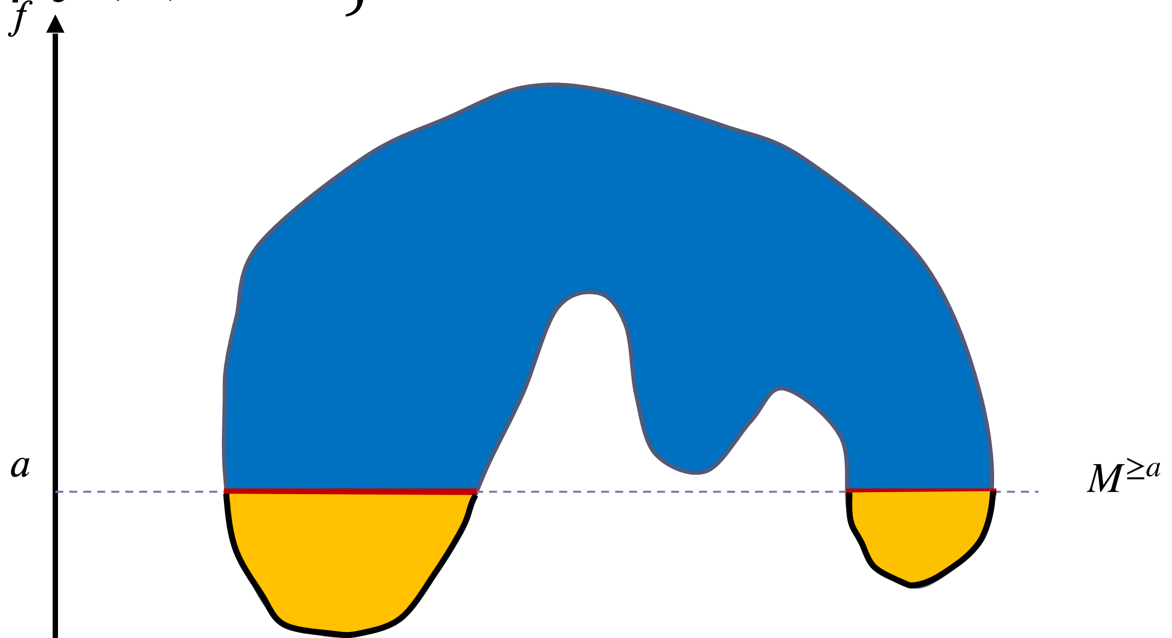


Notations

- Function: $f: M \to R$
- Level set: $M^a = \{x \in M \mid f(x) = a\}$,
- Sub-level set: $M^{\leq a} = \{x \in M \mid f(x) \leq a\}$
- Super-level set: $M^{\geq a} = \{x \in M \mid f(x) \geq a\}$
  - $M^{\geq a} \supseteq M^{\geq b}$ for any $a \leq b$

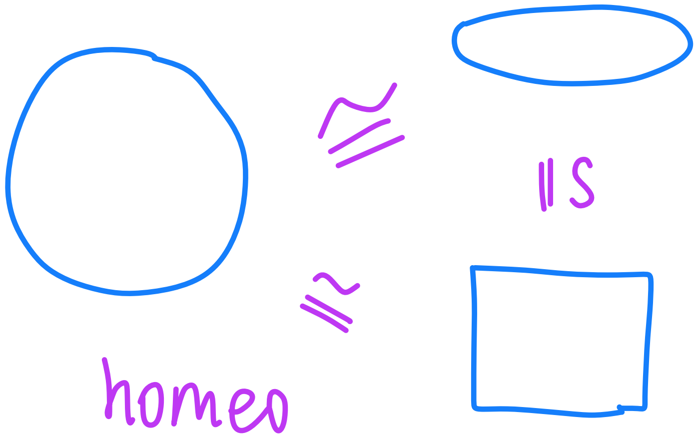


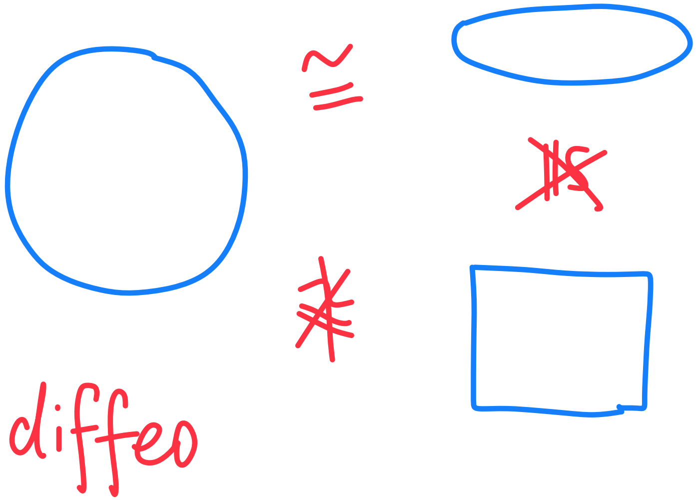
Critical points and topology

Theorem 3 (Homotopy type of sub-level sets). *Let $f : M \to \mathbb{R}$ be a smooth function defined on a manifold M. Given $a < b$, suppose the interval-level set $M_{[a,b]} = f^{-1}([a,b])$ is compact and contains no critical points of f. Then $M_{\leq a}$ is diffeomorphic to $M_{\leq b}$.*

Furthermore, $M_{\leq a}$ is a deformation retract of $M_{\leq b}$, and the inclusion map $i : M_{\leq a} \hookrightarrow M_{\leq b}$ is a homotopy equivalence.

- diffemorphism => homeomorphism => homotopy equivalence
- diffemorphism = homeomorphism + smooth





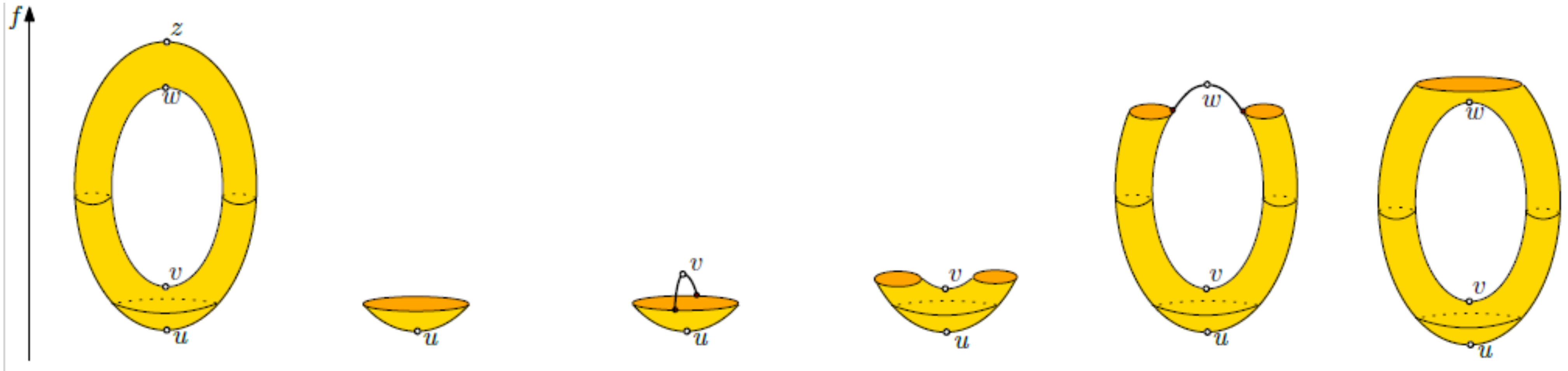
Critical points and topology

diffeomorphism > homeomorphism

> homotopy equivalence

Theorem 3 (Homotopy type of sub-level sets). *Let $f : M \to \mathbb{R}$ be a smooth function defined on a manifold M. Given $a < b$, suppose the interval-level set $M_{[a,b]} = f^{-1}([a,b])$ is compact and contains no critical points of f. Then $M_{\le a}$ is diffeomorphic to $M_{\le b}$.* $\Rightarrow M_{\le a} \simeq M_{\le b}$

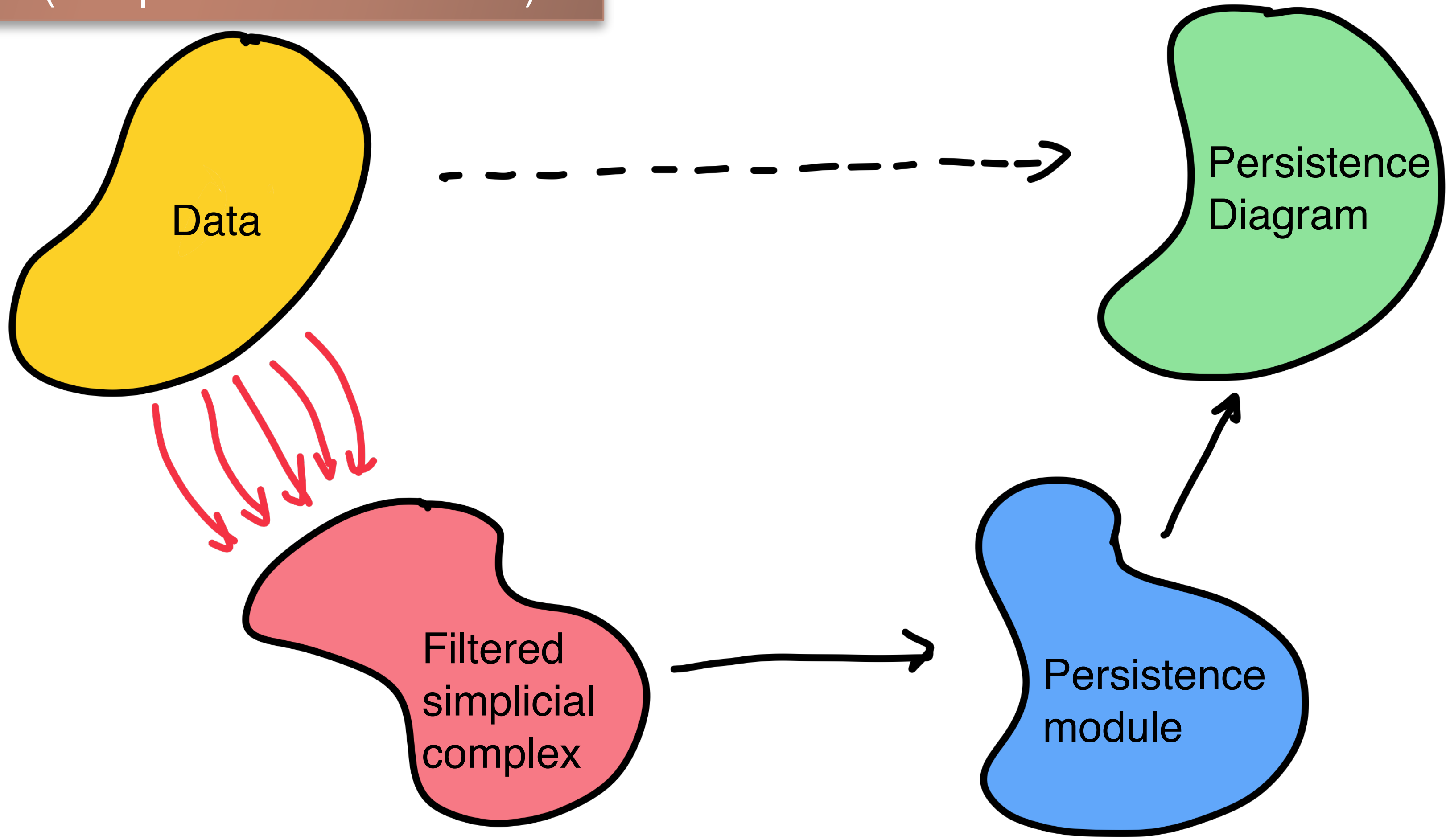
Furthermore, $M_{\le a}$ is a deformation retract of $M_{\le b}$, and the inclusion map $i : M_{\le a} \hookrightarrow M_{\le b}$ is a homotopy equivalence. $\Rightarrow M_{\le a} \simeq M_{\le b}$ via the inclusion map



(a) (b) (c) (d) (e) (f)

Stability for point cloud and general metric spaces

Gromov-Hausdorff distance (for metric spaces)
And ℓ^∞ distance (for spaces with functions)

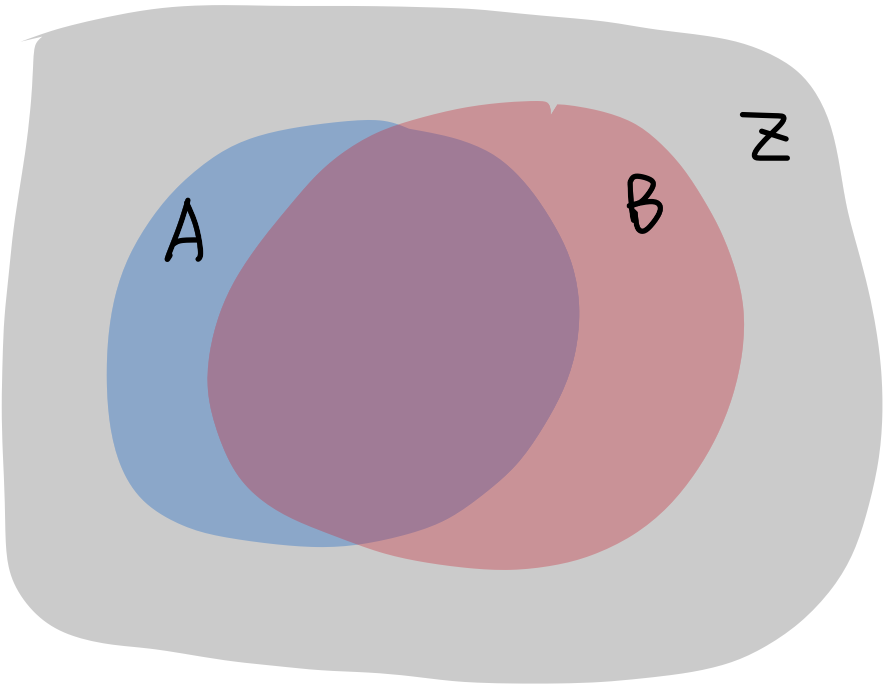


Hausdorff distance between subsets

- **Hausdorff distance** between two sets $A,\ B \subset \left(Z,\ d_Z\right)$
  - $d_H^Z(A,B) := \inf\{r : A \subseteq B^r, B \subseteq A^r\}$, where A^r is the r-neighborhood of A.
  - Equivalently, $d_H(A,\ B) = \max\{\max_{a\in A}\min_{b\in B} d_Z(a,b), \max_{b\in B}\min_{a\in A} d_Z(a,b)\}$

distance between a & B

$\max_{a\in A} d_Z(a,B)$



- For simplicity, we often omit Z and just write $d_H(A,B)$

Hausdorff distance between subsets

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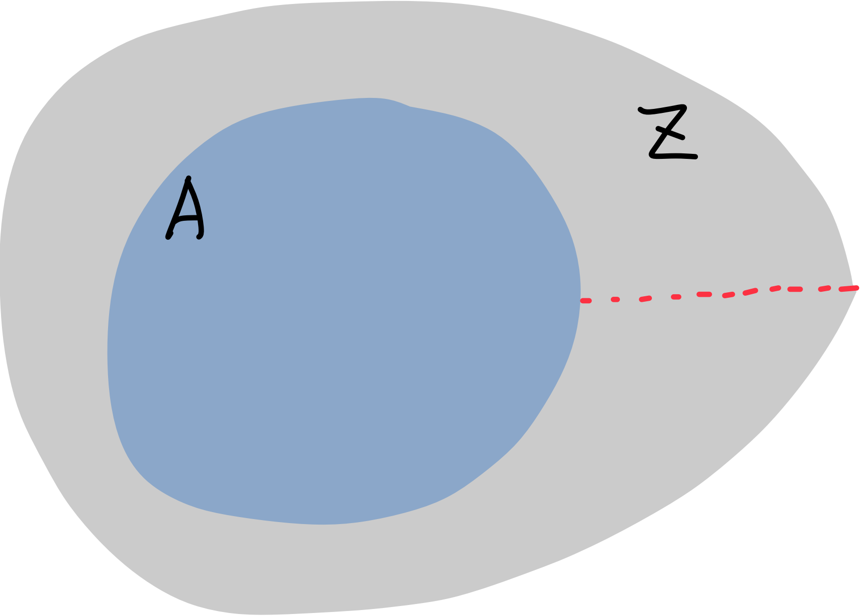
- If B=Z, then $d_H(A,Z) := \inf\{r : Z \subseteq A^r\}$

How much should we "grow" A to cover Z.



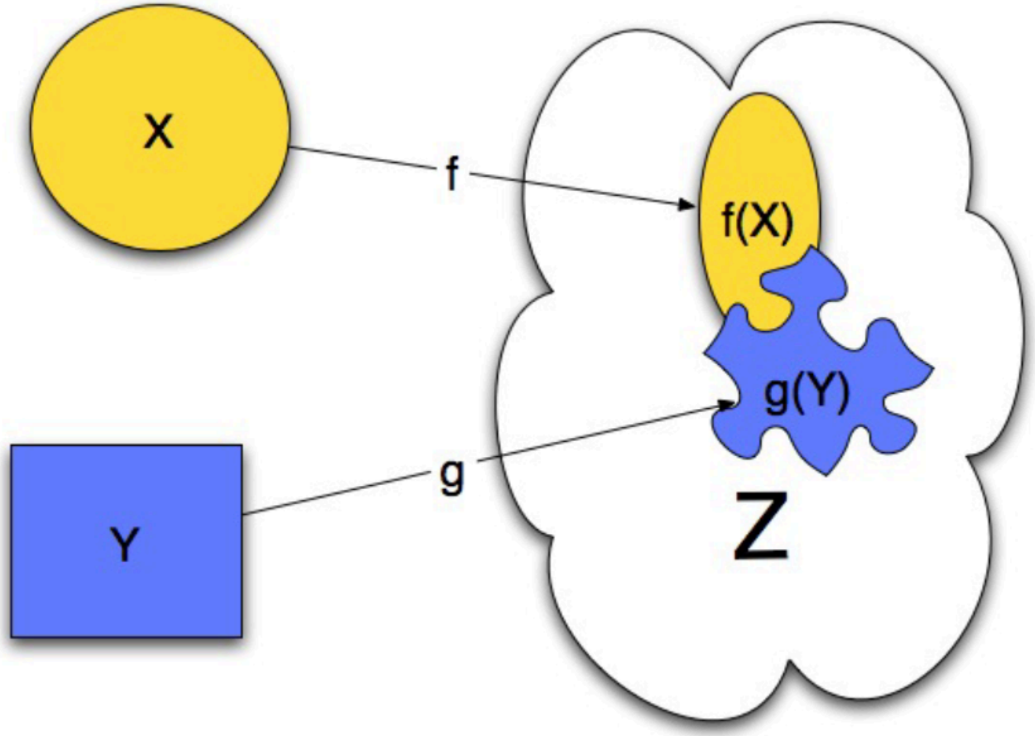
Gromov-Hausdorff distance between metric spaces

- Definition: Given two metric spaces X and Y, the **Gromov-Hausdorff distance** between them is defined as

$$d_{GH}(X,Y) := \inf_{X \hookrightarrow Z,\, Y \hookrightarrow Z} d_H^Z(X,Y)$$

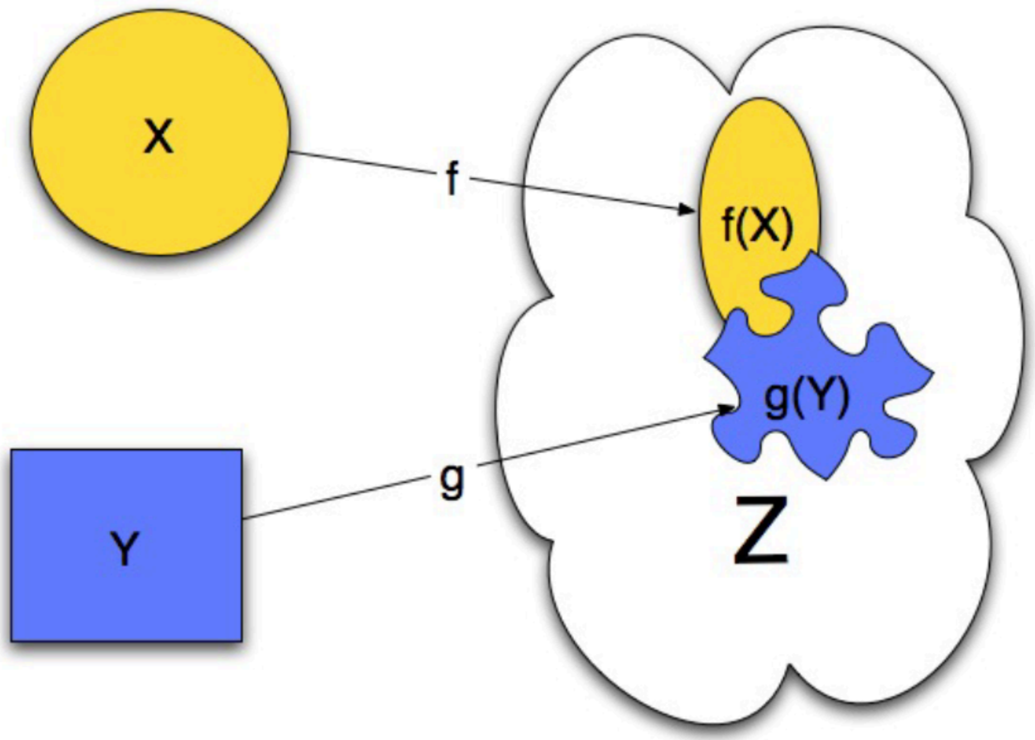
↓

X and Y are both sub-metric spaces of Z.



Gromov-Hausdorff distance between metric spaces

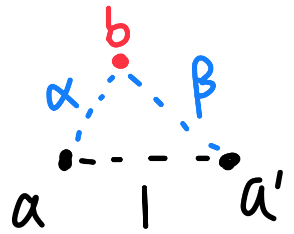
- Fact: $d_{GH}(X, Y) = 0$ implies that X is isometric to Y
- Fact: The Gromov-Hausdorff distance is a metric on the class of compact metric spaces



Examples

$X = \underset{a}{\bullet} \overset{1}{\cdots\cdots} \underset{a'}{\bullet}$ $\quad Y = \underset{b}{\bullet}$ $\quad d_{GH}(X,Y) = \inf\limits_{X \hookrightarrow Z \hookleftarrow Y} d_H^Z(X,Y) = \,?$

It suffices to consider Z with ≤ 3 points.



s.t. $\begin{cases} 1 \leq \alpha + \beta \\ \alpha \leq 1 + \beta \\ \beta \leq 1 + \alpha \end{cases}$

WLOG, suppose $\alpha \leq \beta$

$\xRightarrow{1 \leq \alpha + \beta} \beta \geq \frac{1}{2}$

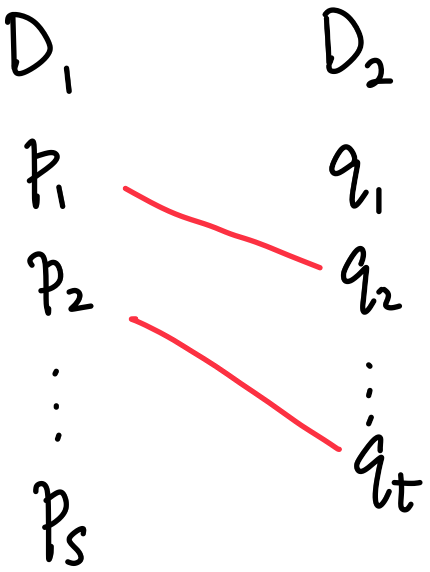
$d_H^Z(X,Y) = \beta \quad (\leftsquigarrow \text{smallest } r \text{ s.t. } X \subset Y^r \ \& \ Y \subset X^r)$

$\geq \frac{1}{2}$

equality can be acheived at $\alpha = \beta = \frac{1}{2}$. $\Rightarrow d_{GH} = \frac{1}{2}$

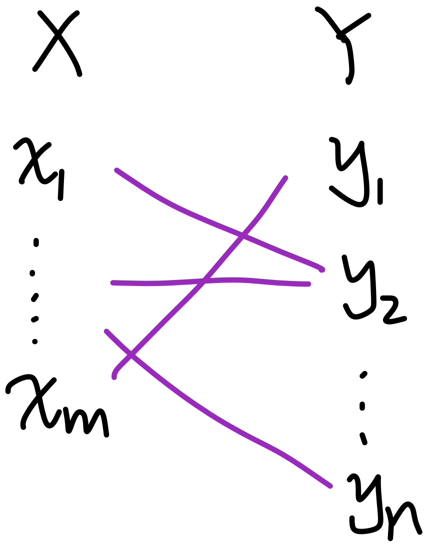
Gromov-Hausdorff distance: alternative definition

- A **correspondence** $R \subset X \times Y$ between two sets
  - For **every** $x \in X$, there exists $y \in Y$ such that $(x, y) \in R$
  - For **every** $y \in Y$, there exists $x \in X$ such that $(x, y) \in R$
  - (this is different from partial-matching)



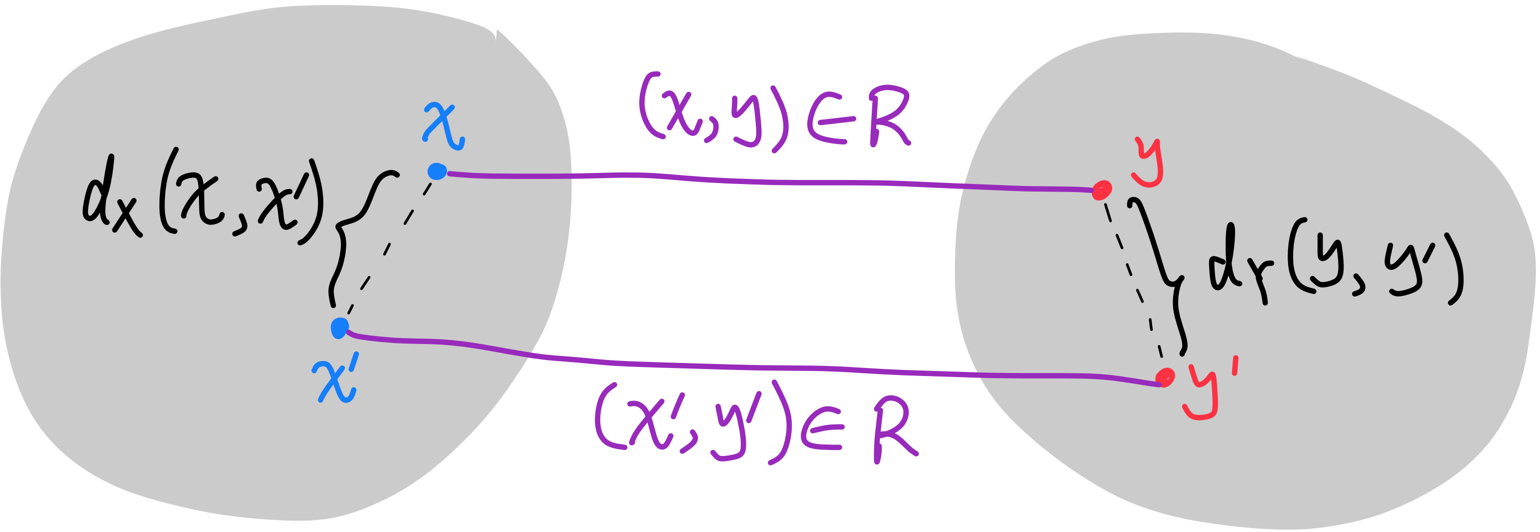
Each node has ≤ 1 edges

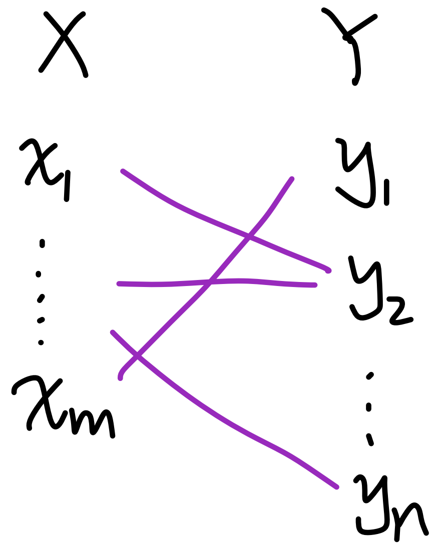
Each node has ≥ 1 edges



Gromov-Hausdorff distance: alternative definition

- **Cost (or distortion)** of a correspondence R
  - $dis(R) = \max_{(x,y),(x',y')\in R} |d_X(x,x') - d_Y(y,y')|$
- Fact: $d_{GH}(X,Y) = \frac{1}{2}\inf\{dis(R) \mid R \text{ a correspondence}\}$.





$$d_{GH}(X,Y) = \frac{1}{2}\inf\{dis(R) \mid R \text{ a correspondence}\}.$$

Example

single-point space $Y = \{y\}$

- $d_{GH}(X, *) = ?$

$\exists!$ correspondence $R = \{(x,y) \mid x \in X\}$

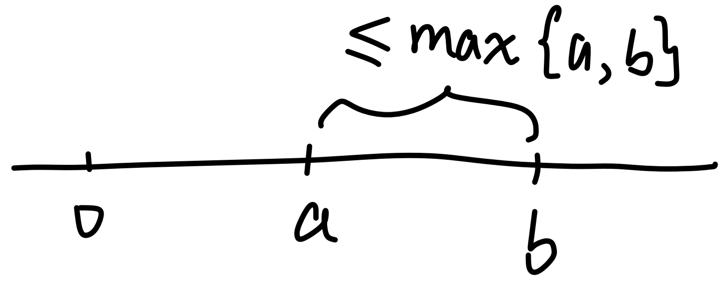
Compute $dis(R) = \max_{\substack{(x,y)\in R \\ (x',y')\in R}} |d_X(x,x') - d_Y(y,y')|$

$= \max_{x,x'\in X} |d_X(x,x') - 0| = diam(X)$

$\Rightarrow d_{GH} = \frac{1}{2} diam(X)$

$$d_{GH}(X,Y) = \frac{1}{2}\inf\{dis(R) \mid R \text{ a correspondence}\}.$$

Basic bounds



- $\operatorname{diam}(X) := \sup_{x,x' \in X} d_X(x,x')$
- $d_{GH}(X,Y) \leq \frac{1}{2}\max(\operatorname{diam}(X), \operatorname{diam}(Y))$

proof | $\forall$ correspondence $R \subset X \times Y$

$$dis(R) = \max_{\substack{(x,y)\in R \\ (x',y')\in R}} |d_X(x,x') - d_Y(y,y')|$$

$$\leq \max\{d_X(x,x'), d_Y(y,y')\}$$

$$\leq \max\{\operatorname{diam}(X), \operatorname{diam}(Y)\}$$

$$\Rightarrow d_{GH} \leq \frac{1}{2} dis(R) \leq \frac{1}{2}\max\{\operatorname{diam}(X), \operatorname{diam}(Y)\}$$

$$d_{GH}(X, Y) = \frac{1}{2}\inf\{dis(R) \mid R \text{ a correspondence}\}.$$

Basic bounds

- $\mathrm{diam}(X) := \sup_{x,x' \in X} d_X(x, x')$
- $d_{GH}(X, Y) \geq \frac{1}{2}|\mathrm{diam}(X) - \mathrm{diam}(Y)|$

proof] $\forall$ correspondence $R \subset X \times Y$. Claim: $dis(R) \geq |\mathrm{diam}(X) - \mathrm{diam}(Y)|$

Let $(x, y), (x', y') \in R$ be s.t. $d_X(x, x') = \mathrm{diam}(X)$

$\Rightarrow \mathrm{diam}(X) = d_X(x, x') \leq |d_X(x, x') - d_Y(y, y')| + d_Y(y, y')$

$\leq dis(R) + \mathrm{diam}(Y)$

$\Rightarrow \mathrm{diam}(X) - \mathrm{diam}(Y) \leq dis(R)$

Similarly, $\mathrm{diam}(Y) - \mathrm{diam}(X) \leq dis(R)$

Gromov-Hausdorff distance vs Quadratic Assignment problem

- $d_{GH}(X, Y) = \frac{1}{2} \inf_R dis(R) = \frac{1}{2} \min_{i,k,j,l} \max \Gamma_{ikjl} \delta^R_{ij} \delta^R_{kl}$, where
  - $\Gamma_{ikjl} = |d_X(x_i, x_k) - d_Y(y_j, y_l)|$
  - $\sum_i \delta^R_{ij} \geq 1$ and $\sum_j \delta^R_{ij} \geq 1$, $\delta^R_{ij} \in \{0,1\}$
- Computational complexity of d_{GH} :
  - Computing d_{GH} between finite metric spaces is NP-hard [Agarwal et al. 2018; Schmiedl 2017]
  - FPT algorithm exists for computing d_{GH} between ultrametric spaces [Mémoli et al. 2021] [link to github]

Stability of persistence diagrams for PCD (metric spaces)

Stability Theorem for Point Cloud Data (PCD)

Given two metric spaces X and Y, let $VR(\,\cdot\,)$ and $C(\,\cdot\,)$ denote the Vietoris-Rips and the Čech filtrations, respectively.

- $d_I(VR(X), VR(Y)) \leq d_{GH}(X, Y)$
- $d_I(C(X), C(Y)) \leq 2d_{GH}(X, Y)$.

Thus,

- $d_B(Dgm_*(VR(X)), Dgm_*(VR(Y))) = d_I(VR(X), VR(Y)) \leq d_{GH}(X, Y)$
- $d_B(Dgm_*(C(X)), Dgm_*(C(Y))) = d_I(C(X), C(Y)) \leq 2d_{GH}(X, Y)$

Sketch of proof using Correspondences

- Let R be a correspondence with distortion 2ϵ, i.e.,

$$\sup_{(x,y),(x',y')\in R} |d_X(x,x') - d_Y(y,y')| = 2\epsilon$$

- For each t, create $f_t : VR^t(X) \to VR^{t+\epsilon}(X)$ and $g_t : VR^t(Y) \to VR^{t+\epsilon}(Y)$ based on R
- Prove that these give rise to ϵ-interleaving

[77] Frédéric Chazal, David Cohen-Steiner, Marc Glisse, Leonidas J. Guibas, and Steve Oudot. Proximity of persistence modules and their diagrams. In *Proc. 25th Annu. Sympos. Comput. Geom. (SoCG)*, pages 237–246, 2009.

[78] Frédéric Chazal, David Cohen-Steiner, Leonidas J. Guibas, Facundo Mémoli, and Steve Y. Oudot. Gromov-Hausdorff stable signatures for shapes using persistence. *Comput. Graphics Forum*, 28(5):1393–1403, 2009.

[81] Frédéric Chazal, Vin de Silva, and Steve Oudot. Persistence stability for geometric complexes. *Geometriae Dedicata*, 173(1):193–214, Dec 2014.

part

full

Alternative proof using Hausdorff Distance

- Proposition 10.2.9 of Žiga Virk's book: https://zigavirk.gitlab.io/PhBook.pdf

Examples from Section 10.4 of Žiga Virk's book

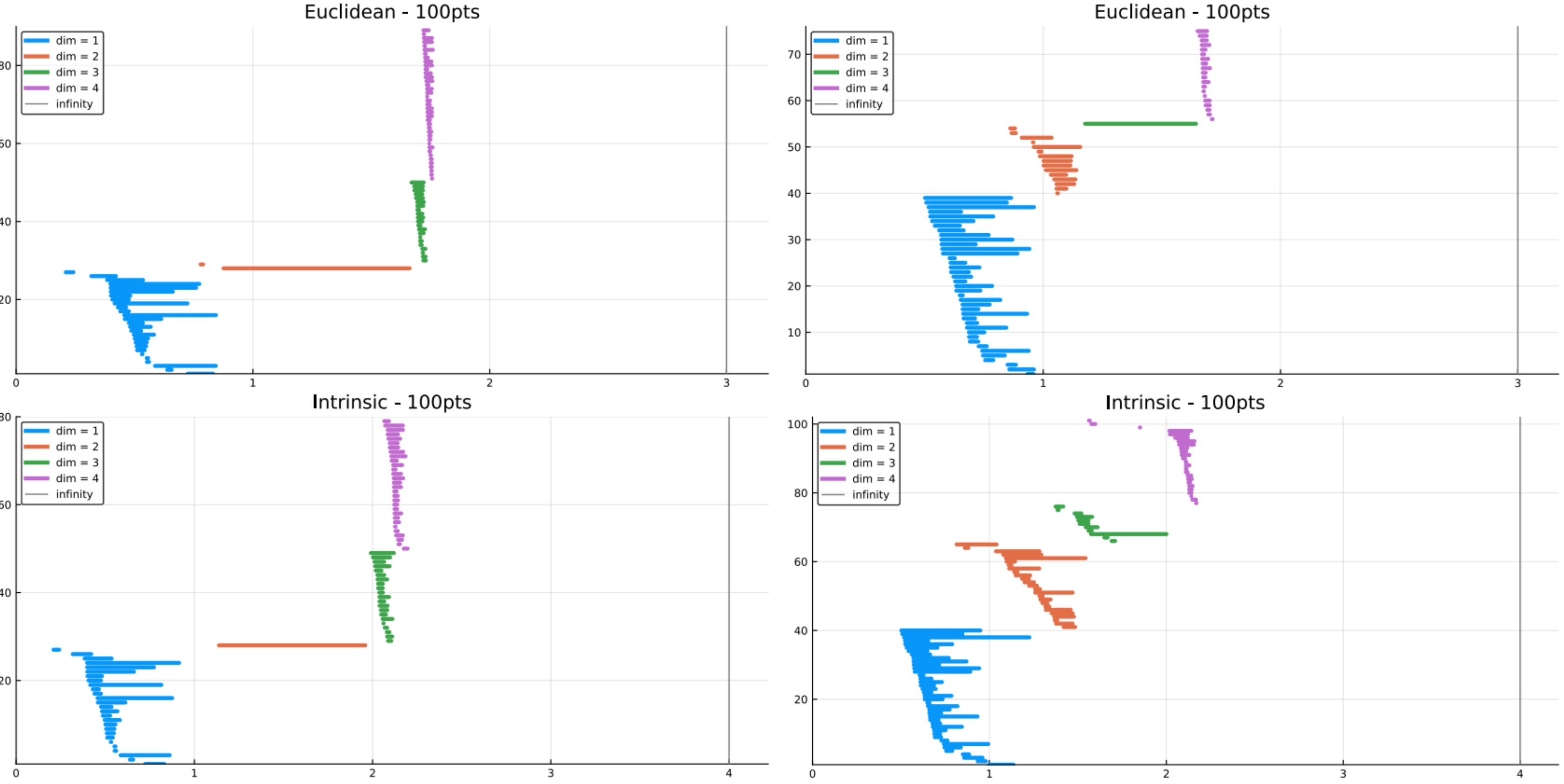
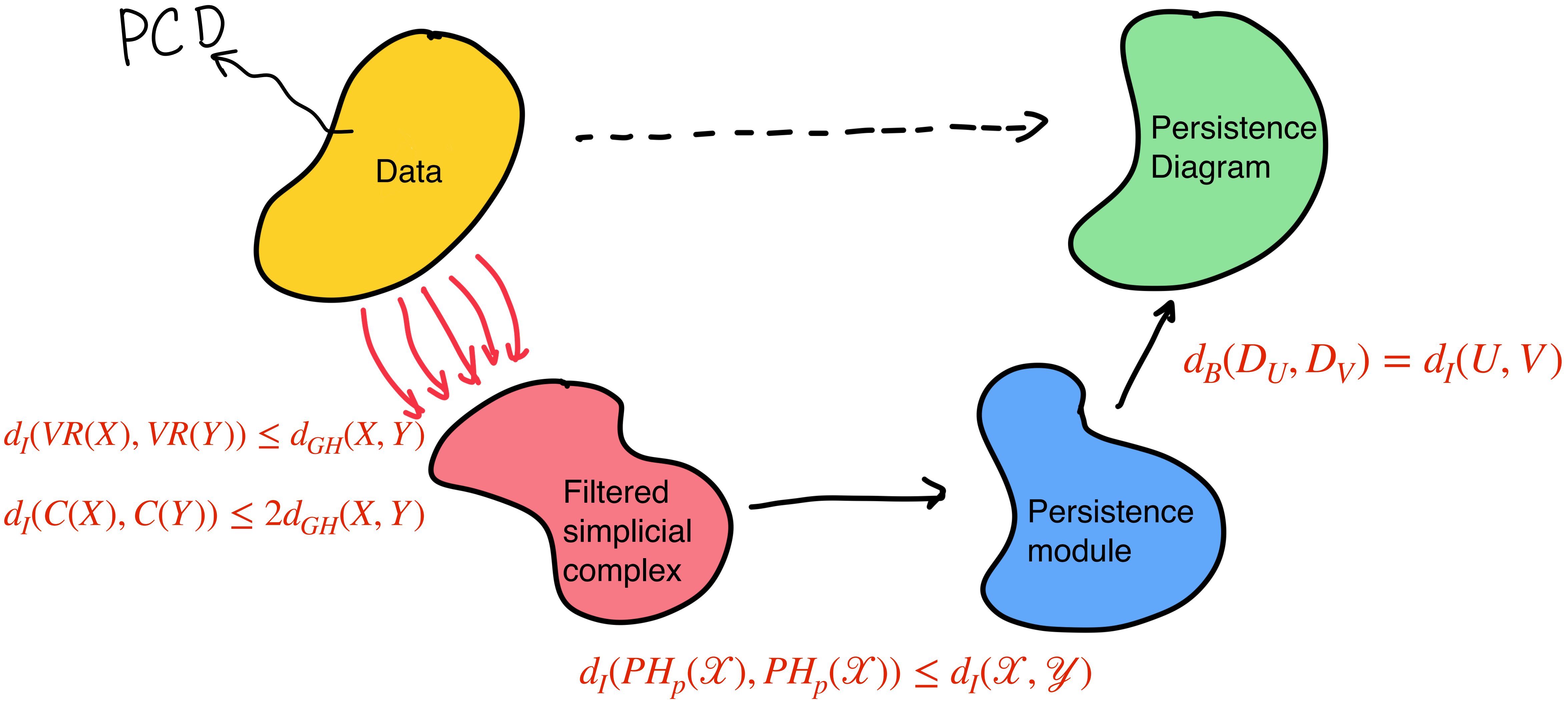
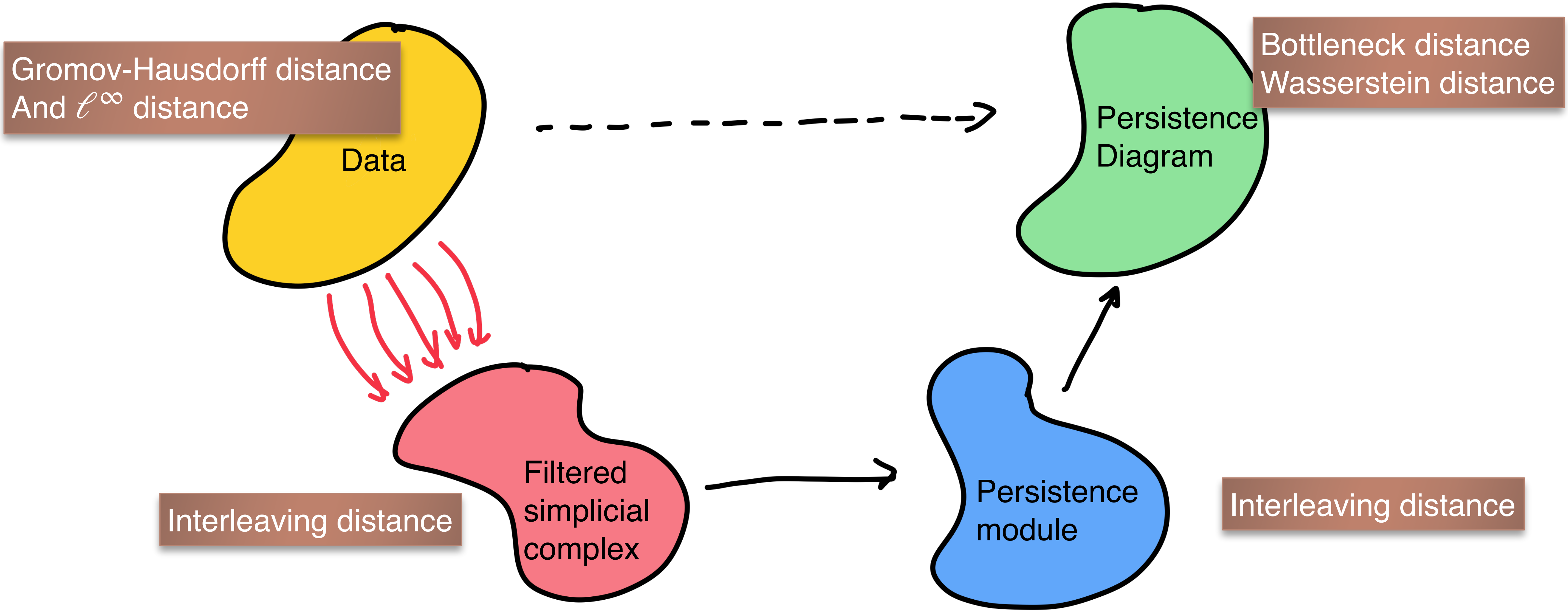


Figure 10.15: Persistence diagrams via Rips complexes of samples of one hundred points sampled from unit S^2 (on left) and S^3 (on right) using Euclidean or intrinsic (geodesic) distance.


PCD
Data
Persistence Diagram
Filtered simplicial complex
Persistence module
$d_B(D_U, D_V) = d_I(U, V)$
$d_I(VR(X), VR(Y)) \leq d_{GH}(X, Y)$
$d_I(C(X), C(Y)) \leq 2d_{GH}(X, Y)$
$d_I(PH_p(\mathscr{X}), PH_p(\mathscr{X})) \leq d_I(\mathscr{X}, \mathscr{Y})$

Stability


Gromov-Hausdorff distance
And ℓ^∞ distance
Data
Bottleneck distance
Wasserstein distance
Persistence
Diagram
Filtered
simplicial
complex
Interleaving distance
Persistence
module
Interleaving distance