MATH412/COMPSCI434/MATH713
Fall 2025
Topological Data Analysis

Topic 1: Topology Basics

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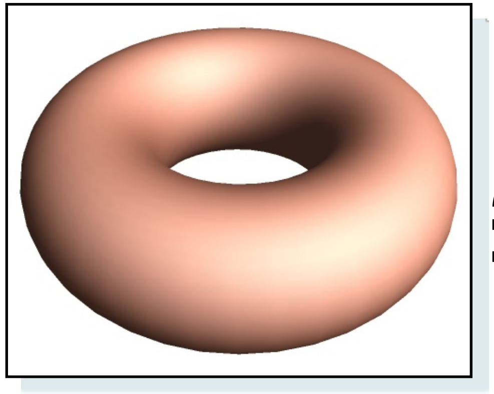
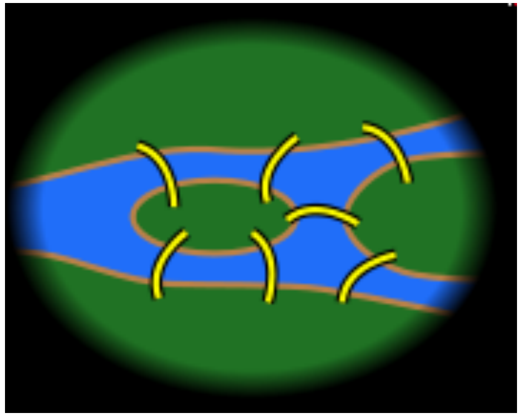
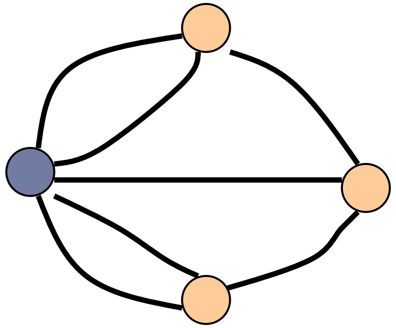
Check-in: Where are we?

- Fundamental concepts
  - Topological space → How we mathematically talk about space of interest
  - Continuous maps → Now we need ways to connect different spaces!
  - Homeomorphisms and homotopies → Describe relations of spaces
  - Manifolds

 $\cong$ 

 $\simeq$ 

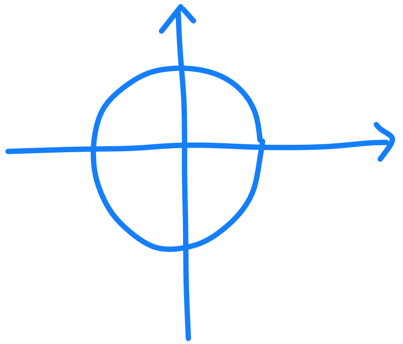
Homeomorphism = homoios + morphē = Similar shapes

> **Definition 5 (Homeomorphism)** *Given two topological spaces* X *and* Y*, a* homeomorphism *between them is a map* $h : \mathsf{X} \to \mathsf{Y}$ *such that* h *is bijection and the inverse of* h *is also continous.*
>
> *Two topological spaces are* X *and* Y *are* homeomorphic*, denoted by* $\mathsf{X} \cong \mathsf{Y}$*, if there is a homeomorphism between them.*

- Homeomorphic spaces are called *topologically equivalent*
  - Note that equivalent relations are transitive.
  - $X \cong Y$ and $Y \cong Z$ implies $X \cong Z$
- Layman's terms: two spaces are homeomorphic if one can continuously deform (stretch, compress) into the other without ever breaking or stitching them
  - Caveat: not always true
- Homeomorphism preserves all topological quantities: dimension, number of connected components, number of holes, voids, etc.

More Examples and Non-Examples

- The Euclidean space $\mathbb{R}^d$ is homeomorphic to any open ball $\mathbb{B}_o(c, r)$
  - Exercise: try to construct the homeomorphism by yourself



$c =$ origin , $r = 1$

Define $f: \mathbb{R}^d \longrightarrow B_o(\vec{0}, 1)$

$$\vec{x} \longmapsto \frac{\vec{x}}{1 + \|\vec{x}\|_2}$$

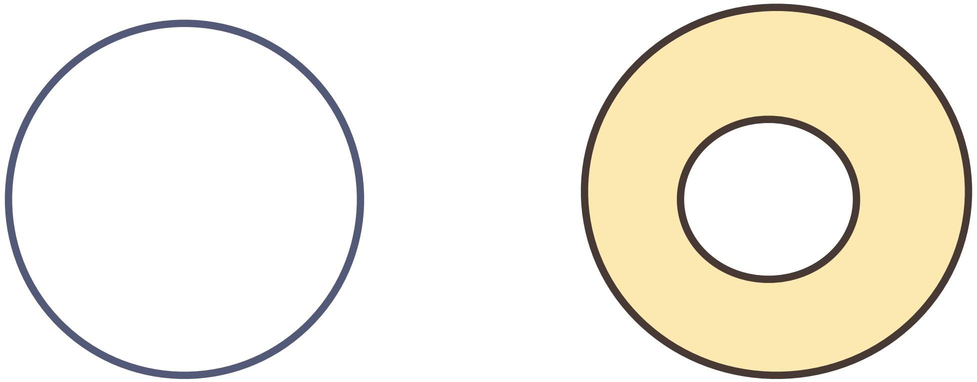
$$f^{-1}(\vec{y}) = \frac{\vec{y}}{1 - \|\vec{y}\|_2}$$

- Homeomorphism also preserves compactness, open sets, closed sets, boundary points, etc.
  - $\underbrace{[0,1]}_{\text{compact}}$, $\underbrace{(0,1]}_{\text{one boundary point}}$, $\underbrace{(0,1)}_{\text{open}}$

$$\mathbb{R}^m \ncong \mathbb{H}^m := \{(x_1, \cdots, x_m) \mid x_i \geq 0\}$$

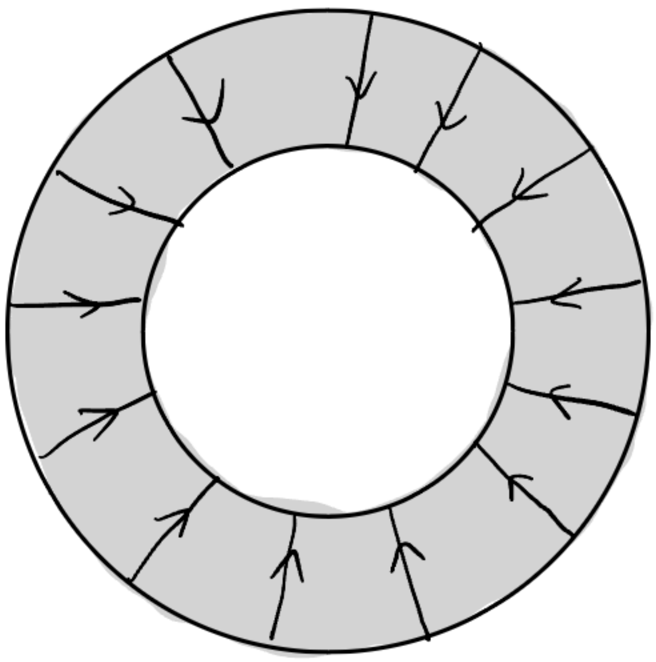
$\mathbb{H}^m$ has boundary points, but $\mathbb{R}^m$ does not.

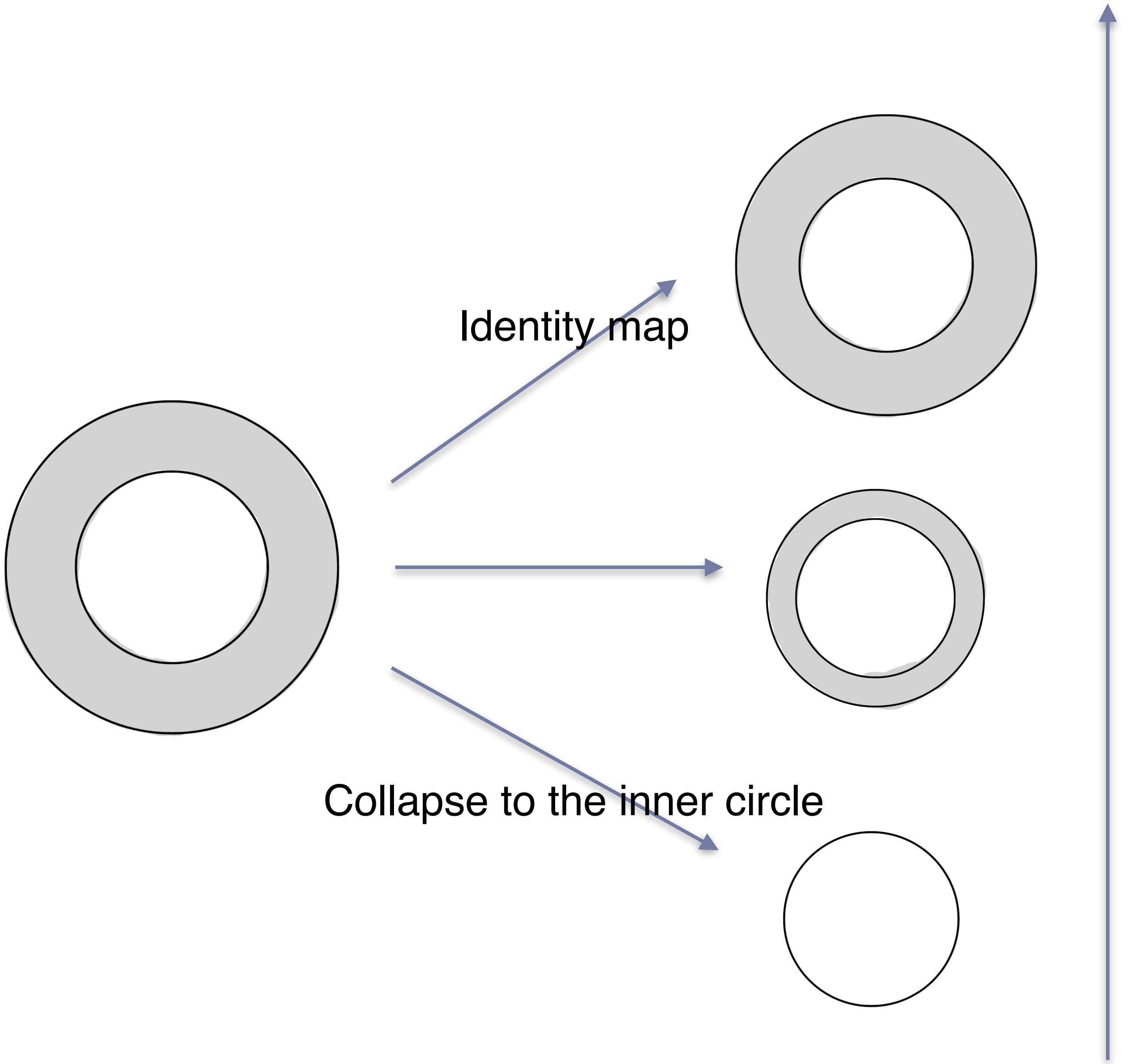
Another level of similarity



- They are not homeomorphic (why?)
- But they look very similar

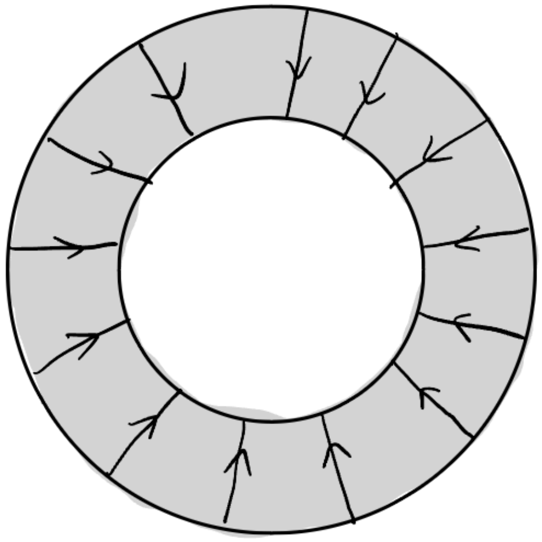
A closer look

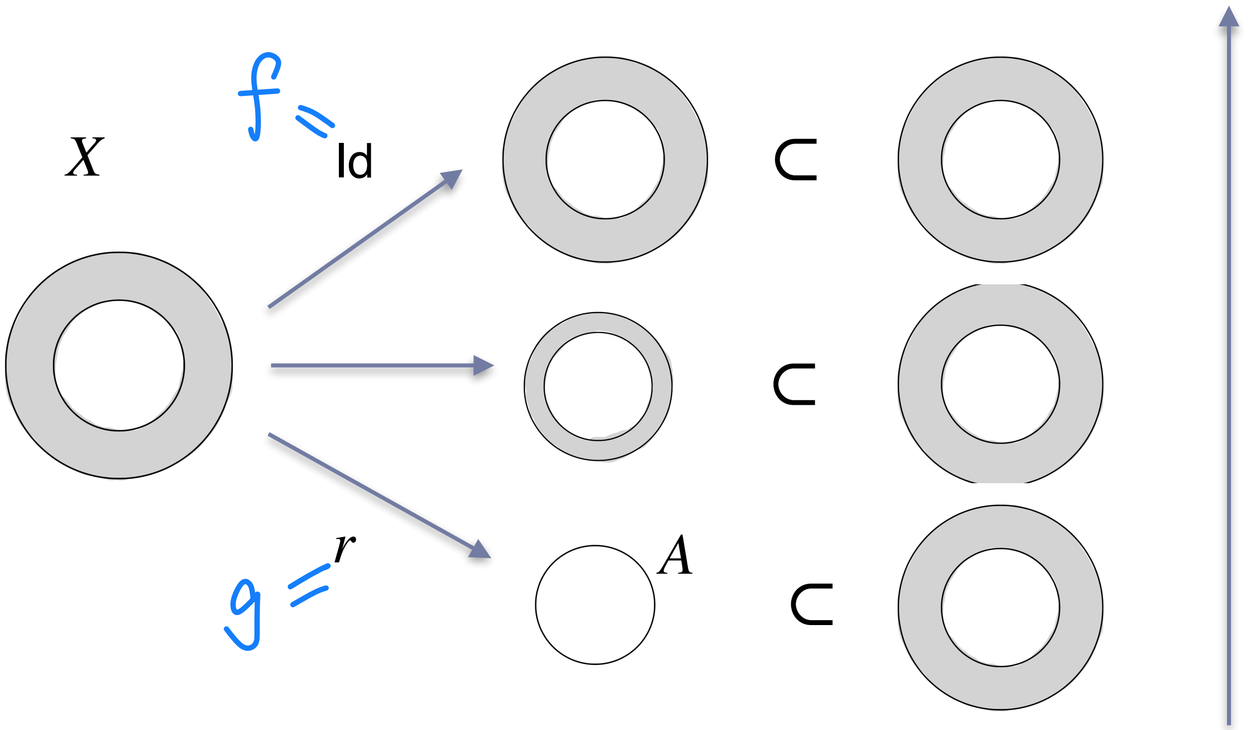




Homotopy

Definition 6 (Homotopy) *First, two* $f, g : X \to Y$ *are* homotopic *if there is a continuous map* $H : X \times [0,1] \to Y$ *such that* $H_0 = f$ *and* $H_1 = g$. *This map* H *is called a* homotopy *connecting* f *and* g. If H exists, we say f and g are homotopic, denoted by $f \simeq g$

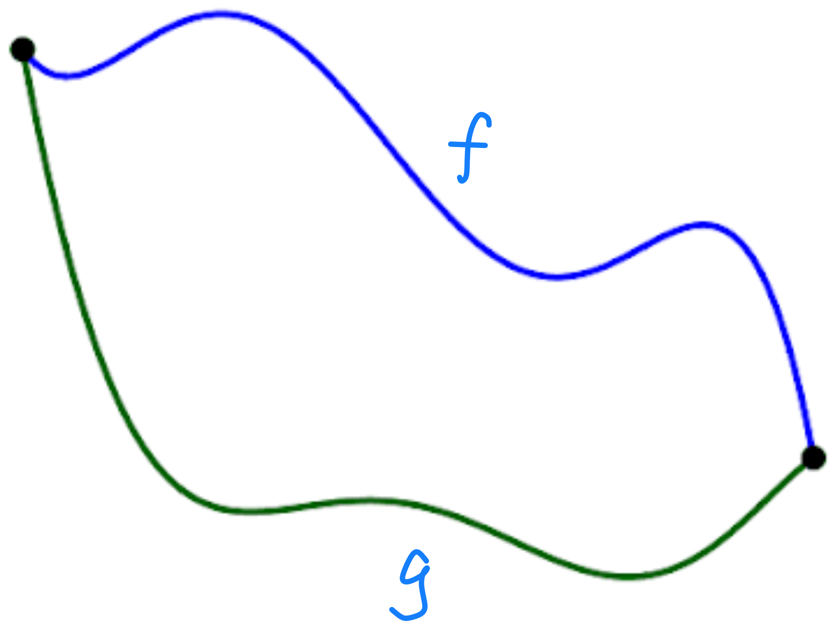




Homotopy

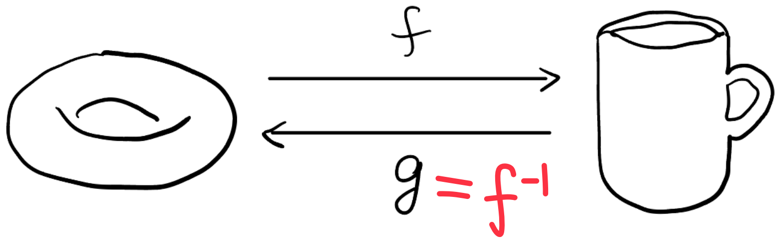
Definition 6 (Homotopy) *First, two* $f, g : \mathsf{X} \to \mathsf{Y}$ *are* homotopic *if there is a continuous map* $H : \mathsf{X} \times [0,1] \to \mathsf{Y}$ *such that* $H_0 = f$ *and* $H_1 = g$. *This map* H *is called a* homotopy *connecting* f *and* g. If H exists, we say f and g are homotopic, denoted by $f \simeq g$

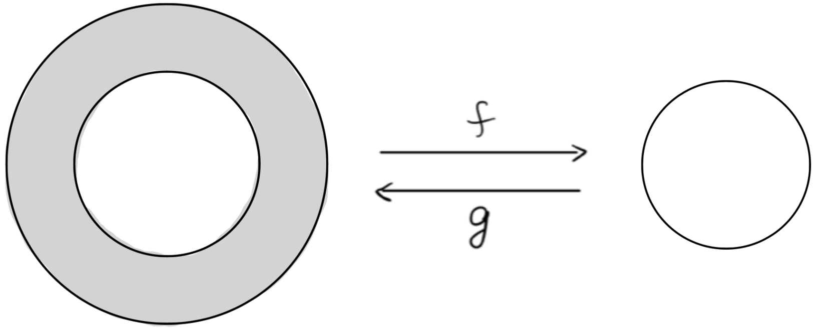
$$H : X \times [0,1] \longrightarrow Y$$
$$(x, 0) \longmapsto f(x)$$
$$(x, 1) \longmapsto g(x)$$



A weaker notion of similarity: Homotopy equivalent

- When X, Y are homeomorphic, there exists continuous functions $f : X \to Y, g : Y \to X$ such that $f \circ g = Id_Y$ and $g \circ f = Id_X$





homeomorphism.

$f \circ g = id_X$

$g \circ f = id_Y$

homotopy equivalent

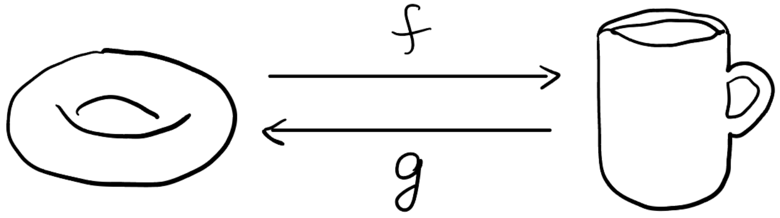
$f \circ g \simeq id_X$

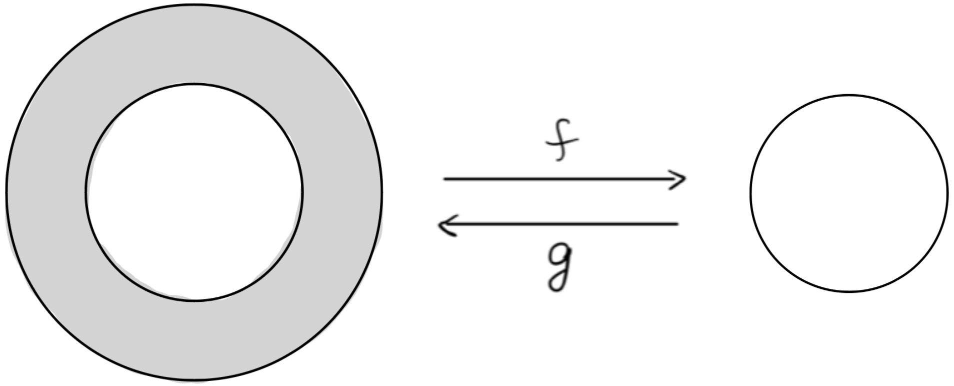
$g \circ h \simeq id_Y$

- We say X, Y are **homotopy equivalent**, if there exists continuous functions $f : X \to Y, g : Y \to X$ such that $f \circ g \simeq Id_Y$ and $g \circ f \simeq Id_X$

A weaker notion of similarity: Homotopy equivalent

- Homeomorphism allows stretching and shrinking





- Homotopy allows stretching, shrinking and **crushing/collapsing**

A weaker notion of similarity: Homotopy equivalent

- Theorem:
  - Two homeomorphic spaces X and Y are also homotopy equivalent. But the inverse may not hold.

- Homotopy equivalent relation is transitive.

$$X \cong Y \;\&\; Y \cong Z \implies X \cong Z$$

$$X \simeq Y \;\&\; Y \simeq Z \implies X \simeq Z$$

- In general, hard to establish homotopy equivalence as well

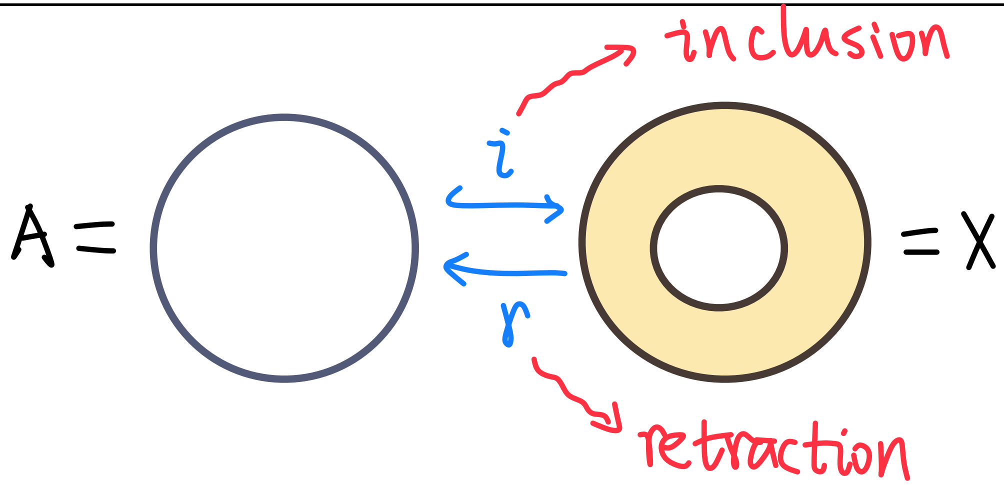
A special type of homotopy equivalence

Definition 7 (Deformation retraction) *Let $A \subseteq X$ be a subspace of topological space X. A* retraction (map) *r is a continuous map $r: X \to A$ such that $r(x) = x$ for any $x \in A$.*

We say that $A \subseteq X$ is a deformation retract of X *if there is a retraction r that is homotopic to the identity map in X. This retraction map is called a* deformation retraction.

Equivalently, a continuous map $R : X \times [0, 1] \to X$ is a deformation retraction of X onto A *if for every $x \in X$ and $a \in A$, $R(x, 0) = x$; $R(x, 1) \in A$ and $R(a, 1) = a$.*

- **Theorem:**
  - If Y is a deformation retract of X, then X and Y are homotopy equivalent.

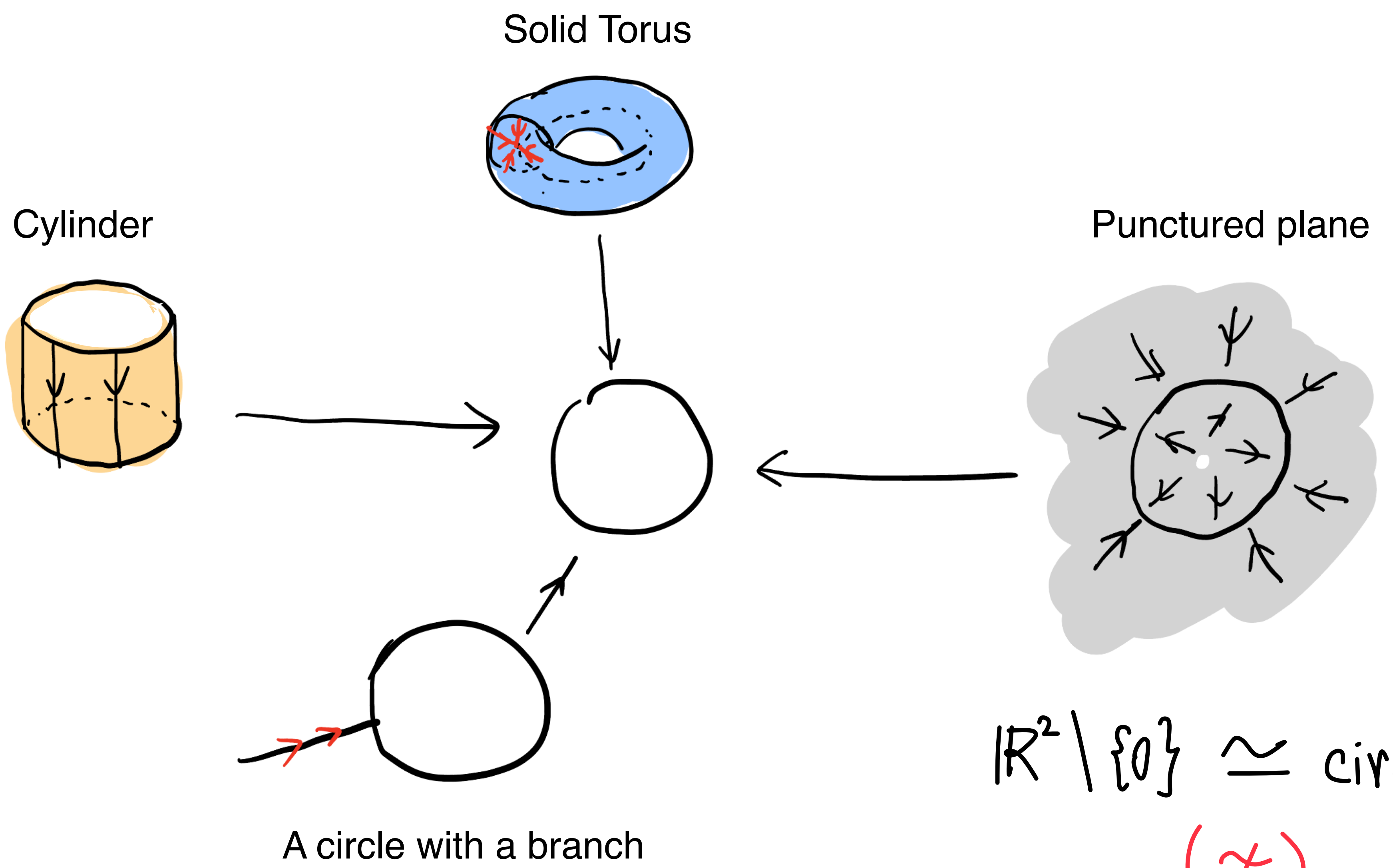


$r \circ i = id_A$ (always holds)

$i \circ r \simeq id_X$ (requirement)

(R is a homotopy between $i \circ r$ & id_X,
for $r = R(\cdot, 1)$)

Examples of Deformation Retraction

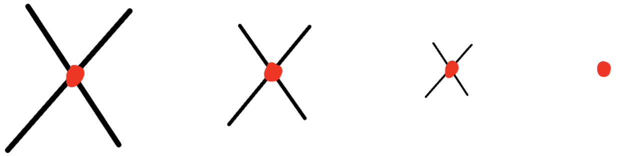


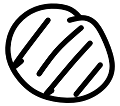
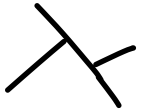
$\mathbb{R}^2 \setminus \{0\} \simeq$ circle

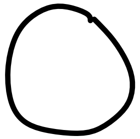
$(\not\cong)$

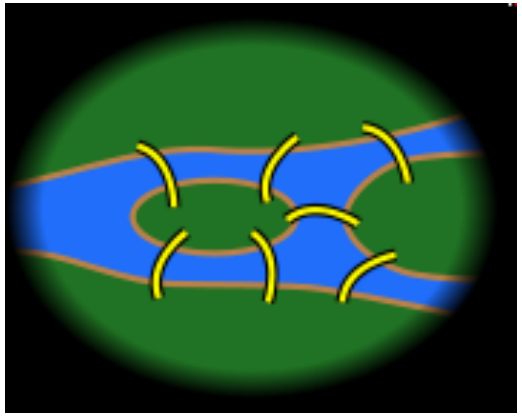
Examples of homotopy equivalence

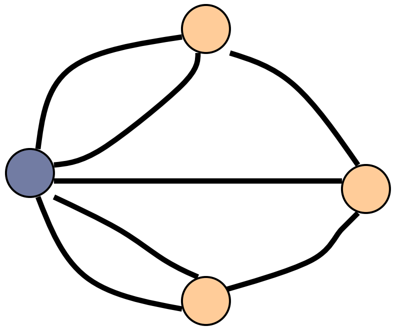
- X and Y are homotopy equivalent but not homeomorphic

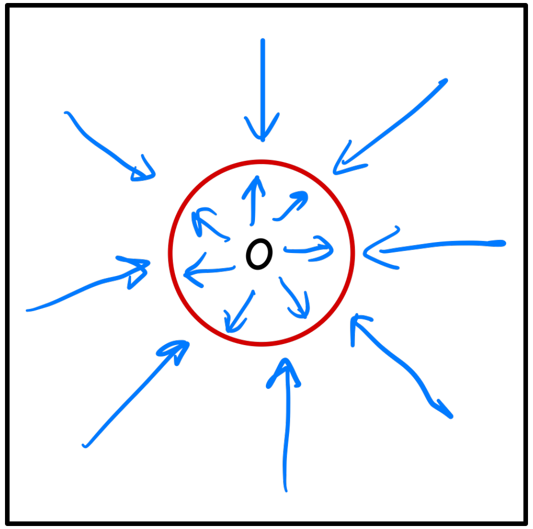


- A disk and a point  $\simeq \bullet$
- A tree and a point  $\simeq \bullet$
- A punctured plane and a circle

$\mathbb{R}^2 \setminus \{0\} \simeq$ 







$$R : [0,1] \times \mathbb{R}^2 \backslash 0 \to \mathbb{R}^2 \backslash 0$$

$$R(t, x) = (1 - t)x + t\frac{x}{\|x\|}$$

Examples

- A punctured hollow torus $\approx$?

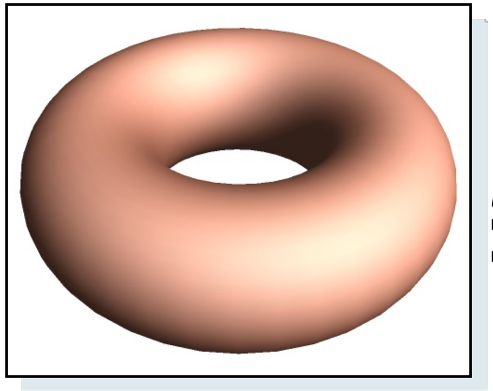
More Examples

Use homology groups

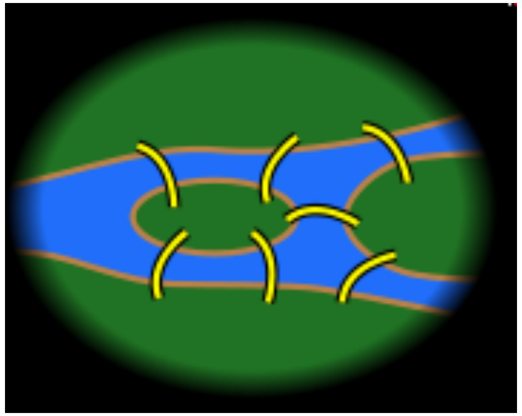
- The Euclidean space $\mathbb{R}^n$ is homotopy equivalent to a point
- But a sphere $\mathbb{S}^n$ is not homotopy equivalent to a point
  - Surprisingly we need advanced machinery to prove this
  - Take this for granted at this moment
- Are $\mathbb{S}^n$ and $\mathbb{S}^m$ homotopy equivalent?
- Can we use computer to determine whether two topological spaces are homotopy equivalent or not?

Both homeomorphism and homotopy equivalence can be hard to detect, and not computationally friendly in general. Later we will focus on homology, which is much easier to compute.

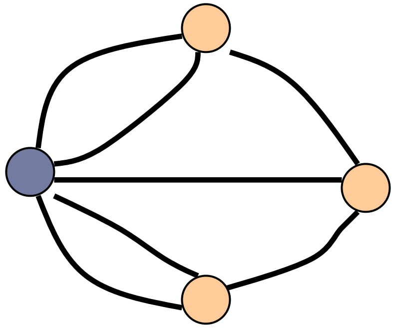
Summary



$\cong$



$\simeq$



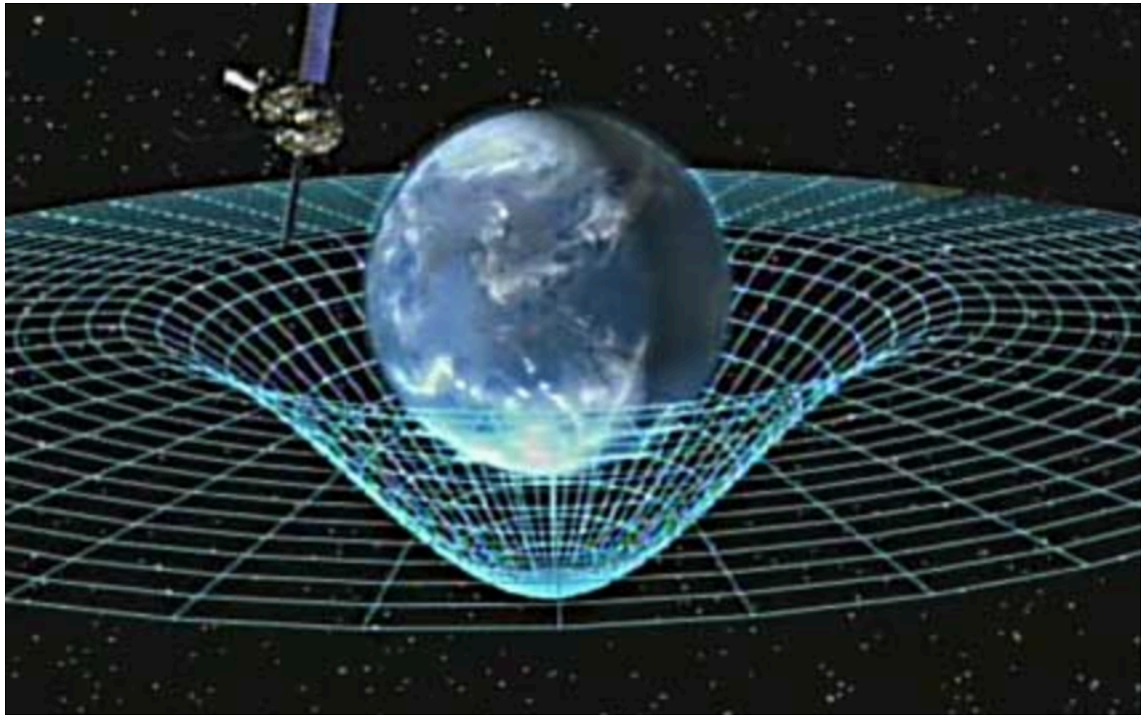
- Fundamental Questions
  - What is a topological space?
  - What is a "continuous" way of turning one space to another?
  - When can we say two spaces are the "same"?

Check-in: Where are we?

- Fundamental concepts
  - Topological space → How we mathematically talk about space of interest
  - Continuous maps → Now we need ways to connect different spaces!
  - Homeomorphisms and homotopies → Describe relations of spaces
  - Manifolds → A generalization of Euclidean space; a special type of topological space

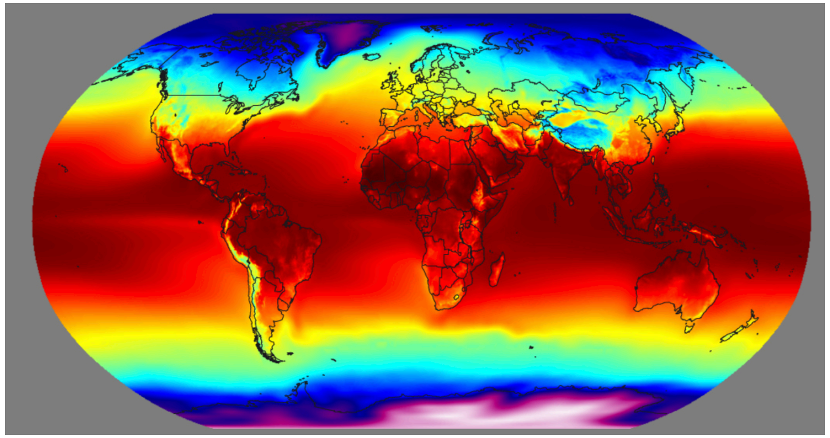
Why manifolds?

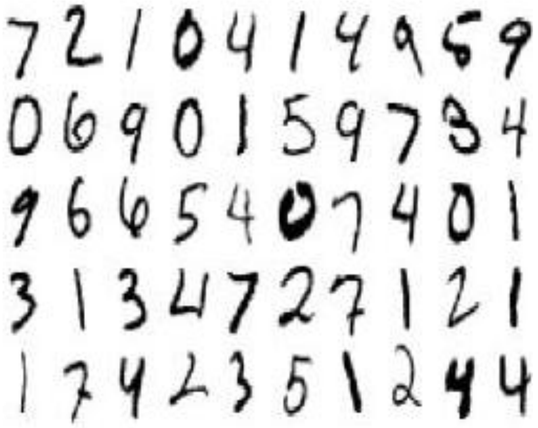
- We like Euclidean spaces. Why manifolds?
  - We live in non-Euclidean space

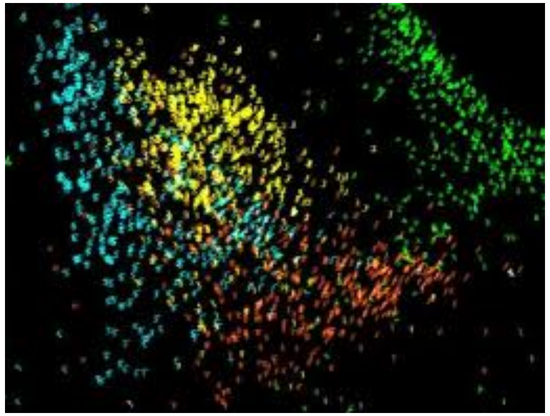


Why manifolds?

- Data and the space where data are sampled from often live in a subspace within the ambient space they are embedded in
  - Ambient space:
    - the space where data or the object/space of interest are embedded in.
  - Intrinsic space:
    - the space where data are sampled from **(manifold hypothesis)**

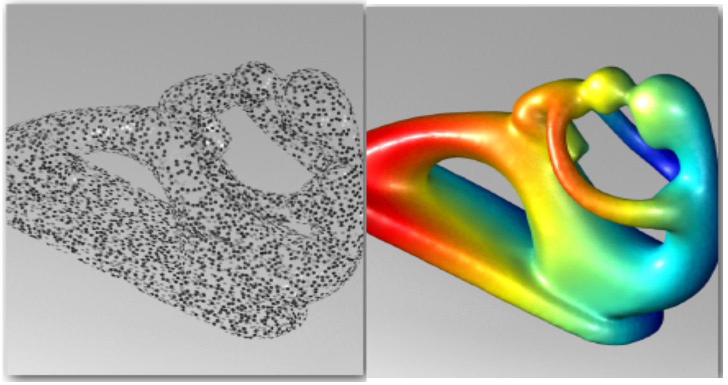






Why manifolds?

- Data and the space where data are sampled from often live in a subspace within the ambient space they are embedded in
  - Ambient space:
    - the space where data or the object/space of interest are embedded in.
  - Intrinsic space:
    - the space where data are sampled from
- The intrinsic space may not be a linear subspace of the ambient space
  - e.g, the surface of a bunny in $\mathbb{R}^3$



What are manifolds

open ball $= \mathbb{B}_o^m \cong \mathbb{R}^m \not\cong \mathbb{H}^m$

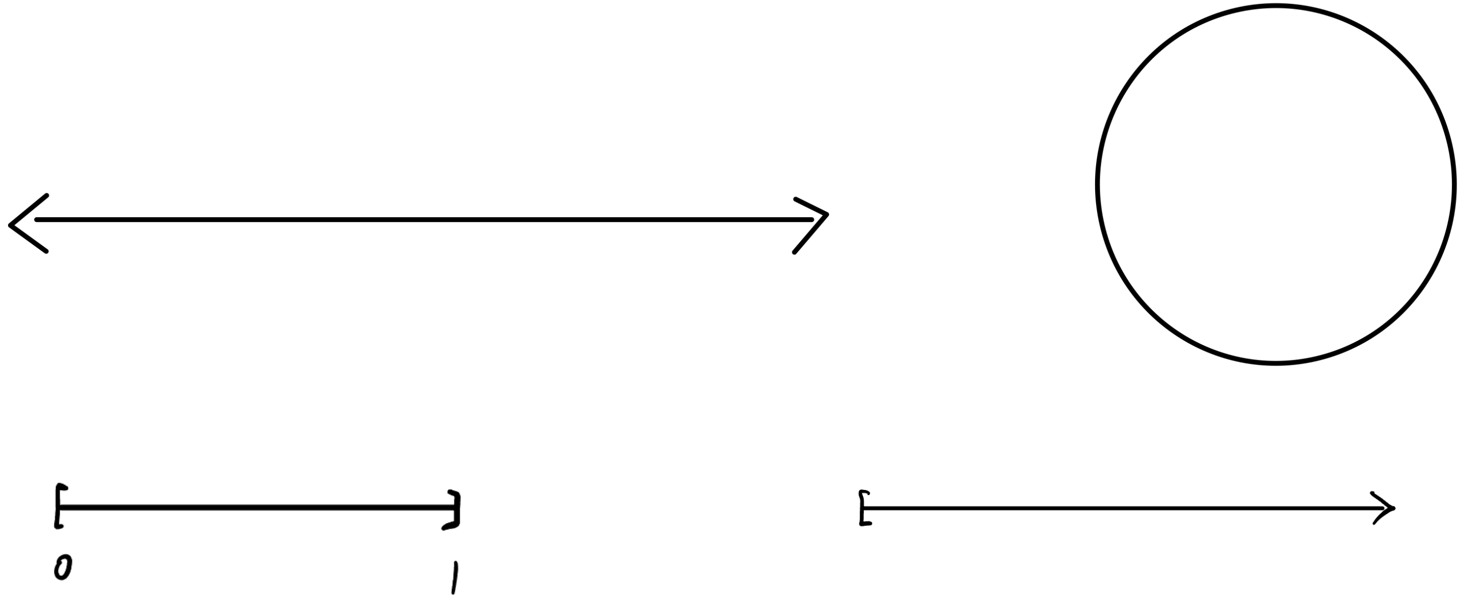
$\mathbb{H}^m = \{(x_1, \cdots, x_m) \mid x_m \geq 0\}$

- Non-linear, yet still well-behaved spaces!
- In particular, locally it behaves like a linear space.

Definition 8 (Manifold). A topological space M is a *m-manifold*, or simply *manifold*, if every point $x \in M$ has a neighborhood homeomorphic to $\mathbb{B}_o^m$ or $\mathbb{H}^m$. The *dimension* of M is m.

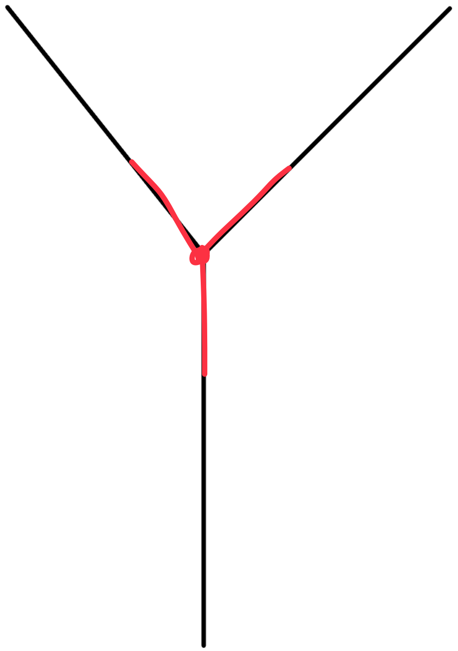
- Interior of M: points whose neighborhood is homeomorphic to $\mathbb{B}_o^d$
- Boundary of M: points whose neighborhood is homeomorphic to the half space.
- When referring to a manifold, usually it comes with no boundary, i.e., all neighborhood homeomorphic to the open ball instead of the half space
- Will emphasize "manifold with boundary" when necessary

1-dim manifold



- These are everything!
- Any connected 1-dim manifold (with or without boundary) is homeomorphic to one of the above

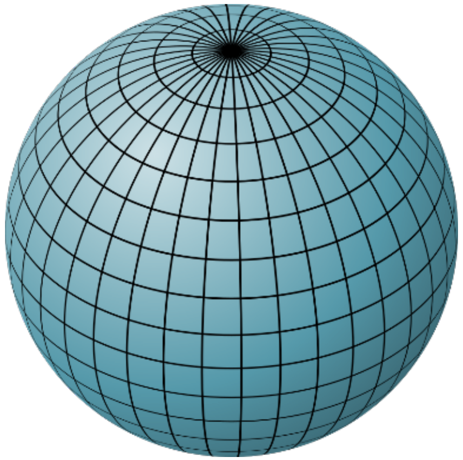
A non-example



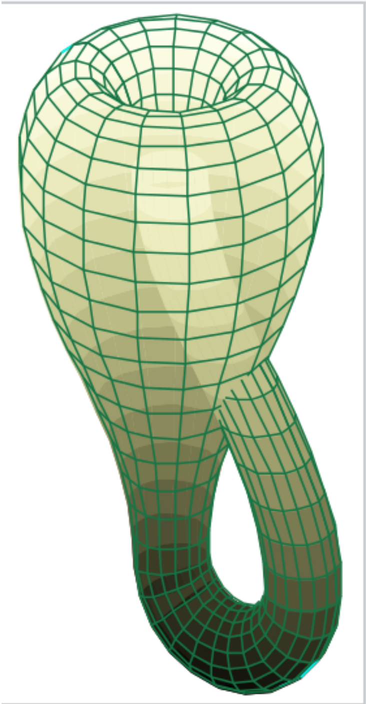
$Y \ncong \mathbb{R}^1$

2-dim manifold

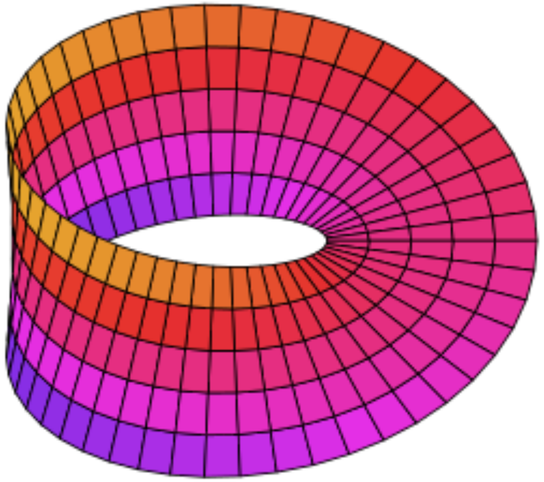
Sphere



Klein bottle

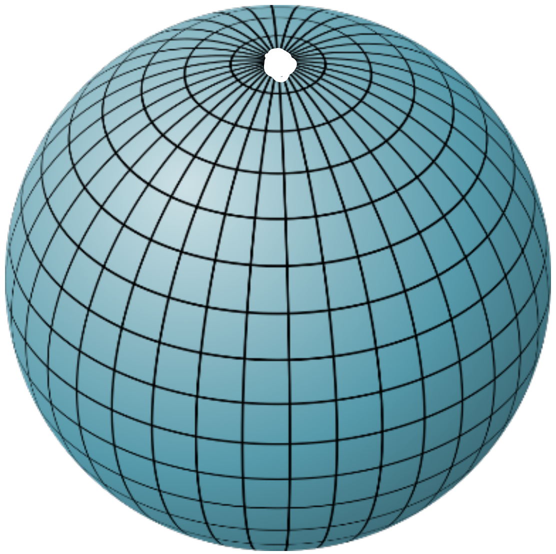


Möbius strip



Homeomorphism to $\mathbb{R}^d$

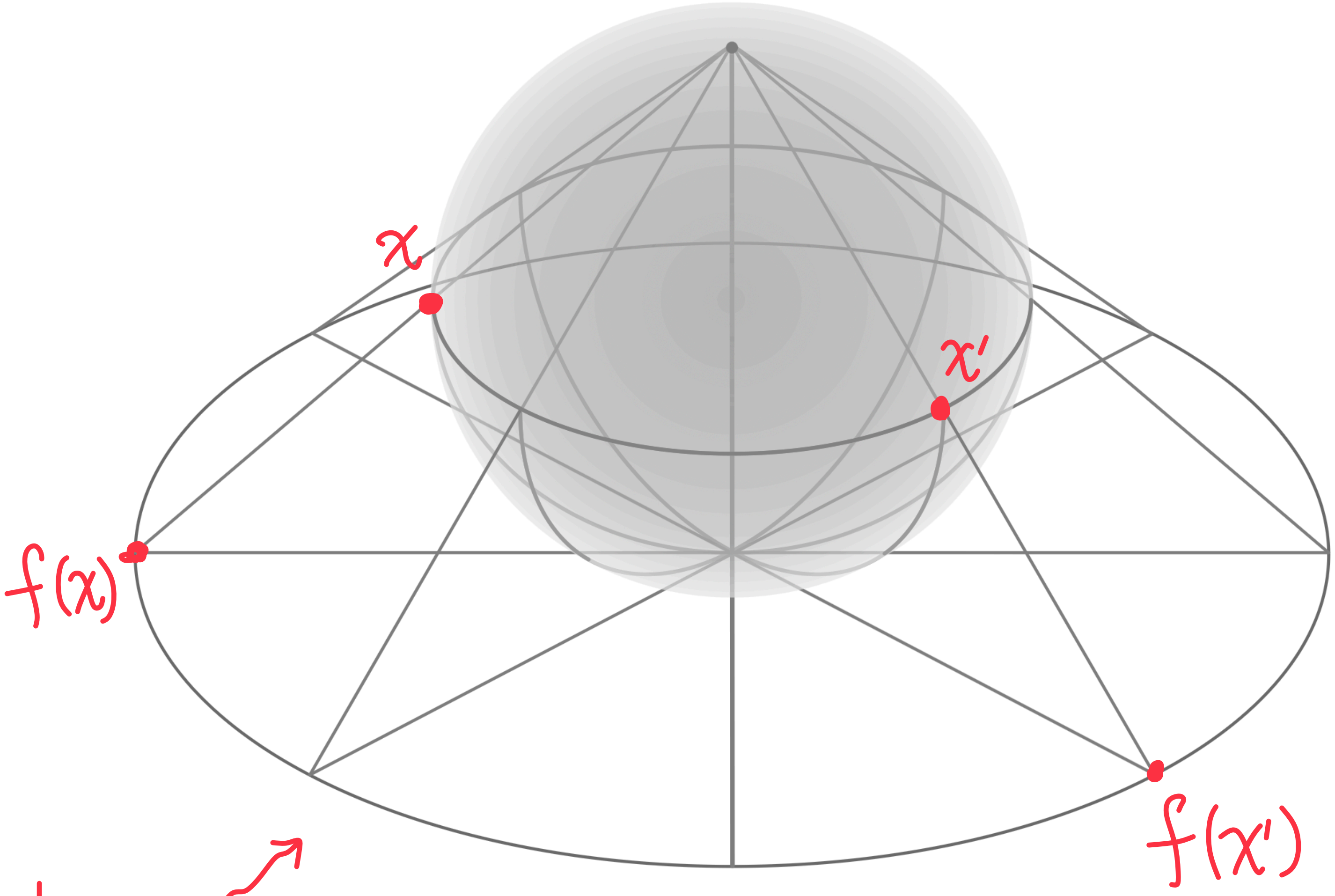
$X =$ sphere $\setminus$ {north pole}



$\exists$ a homeomorphism

$f: X \longrightarrow \mathbb{R}^2$ given by

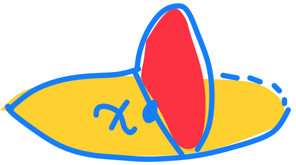
Stereographic projection (pic from wiki)

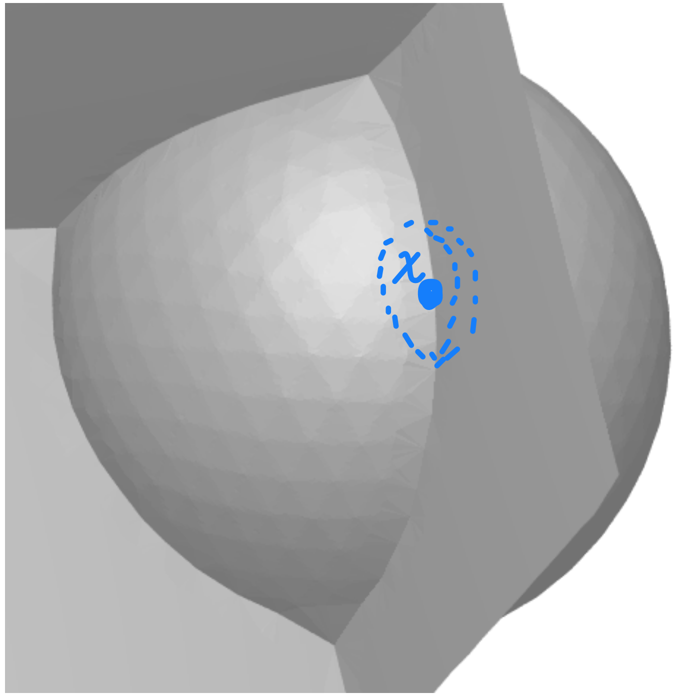


Is this a 2-d manifold?

open neighborhood

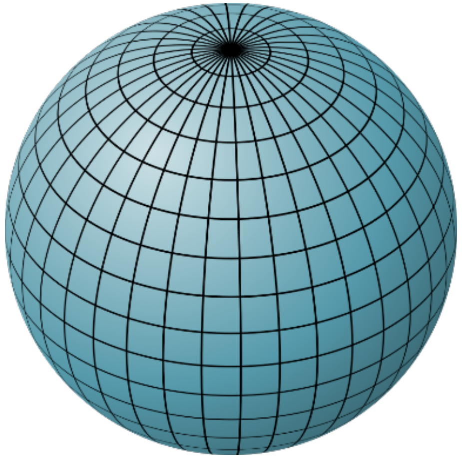
at x is

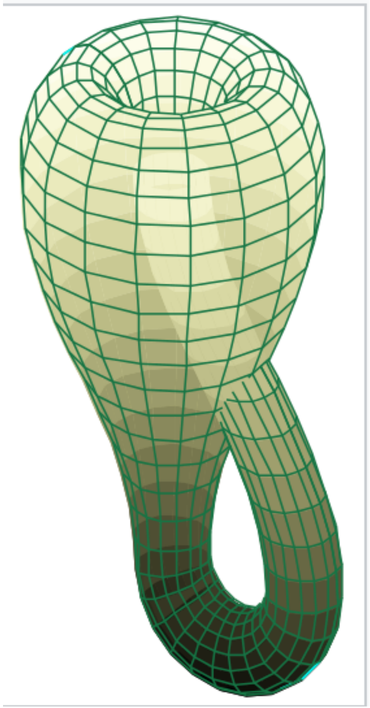




Classification of 2d manifolds?

- A 2d manifold without boundary is also called a **surface**
- All connected compact surfaces can be classified into a specific class of surfaces

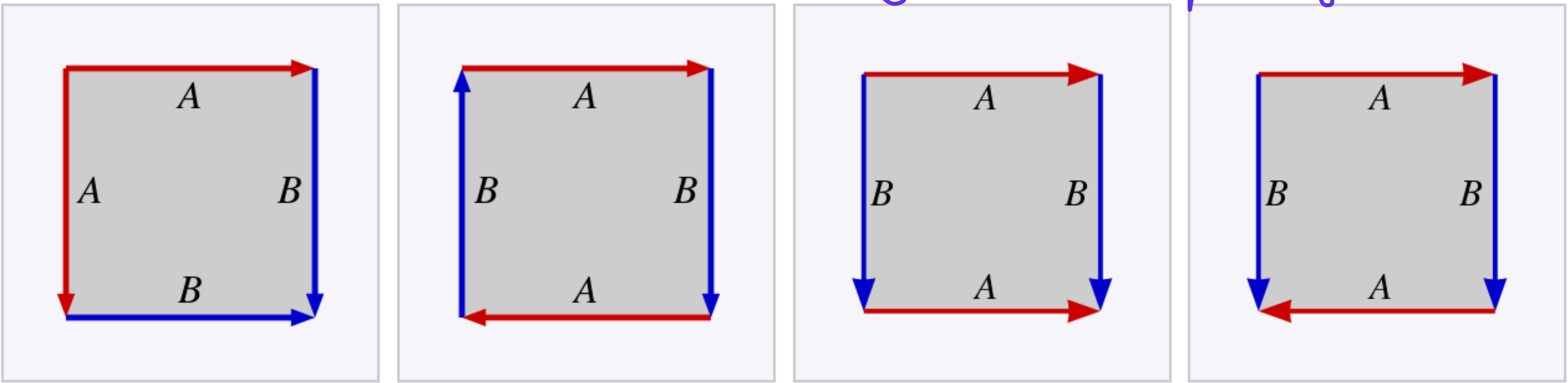




Classification of 2d manifolds?

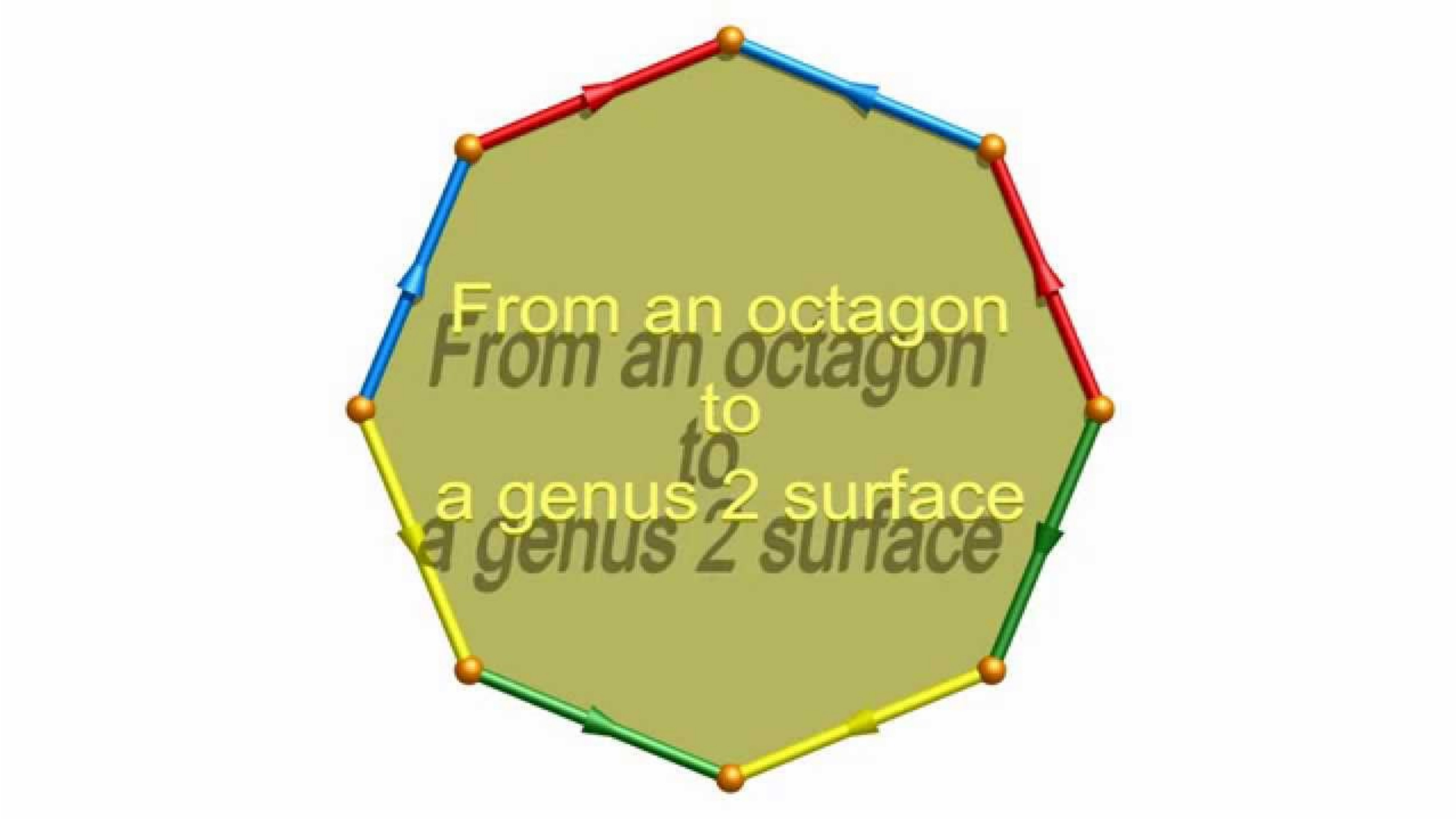
- A 2d manifold without boundary is also called a **surface**
- All connected compact surfaces can be classified into a specific class of surfaces constructed from polygons

(Glue sides with the same color along the corresponding direction)

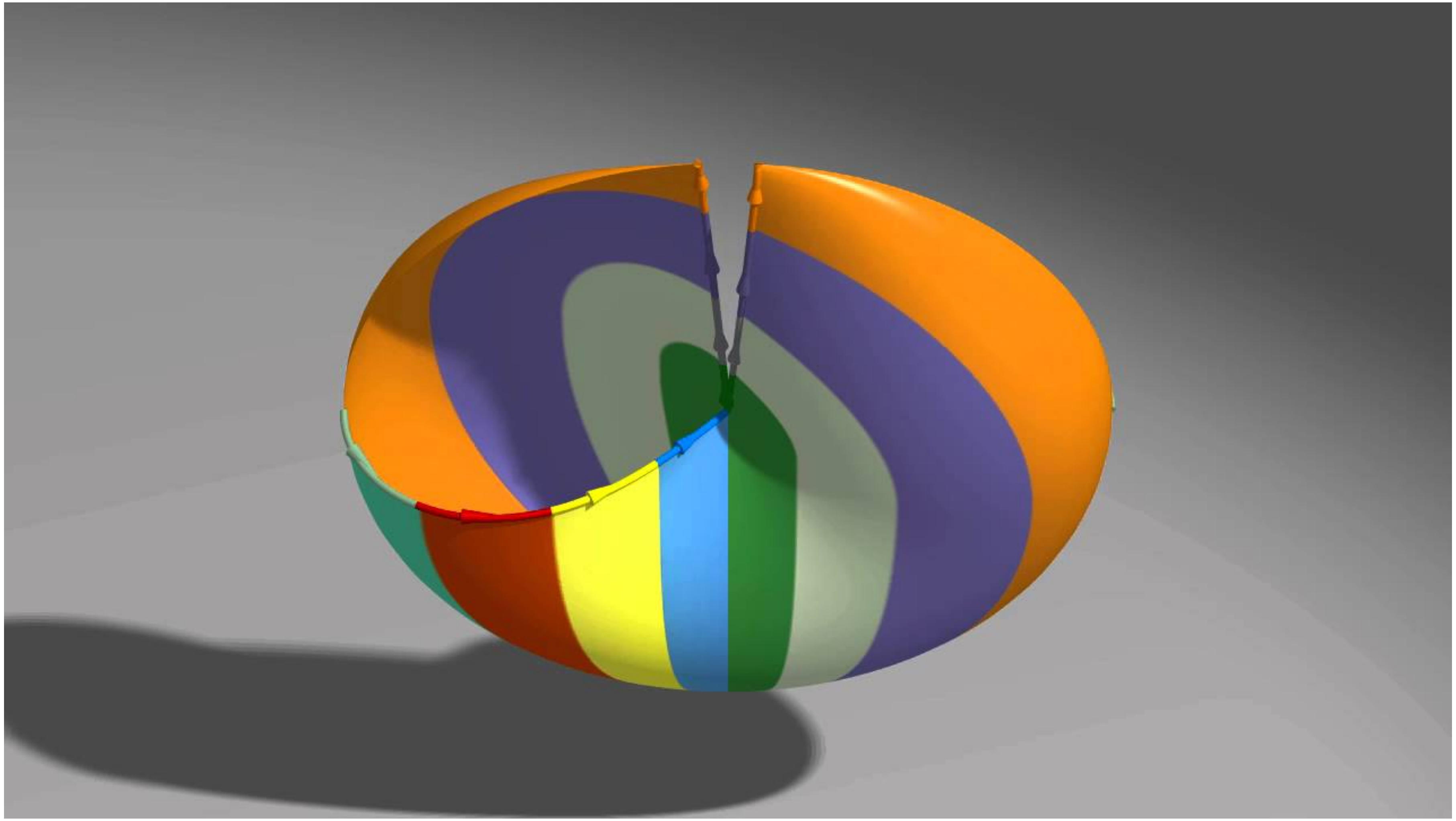


sphere real projective plane torus Klein bottle

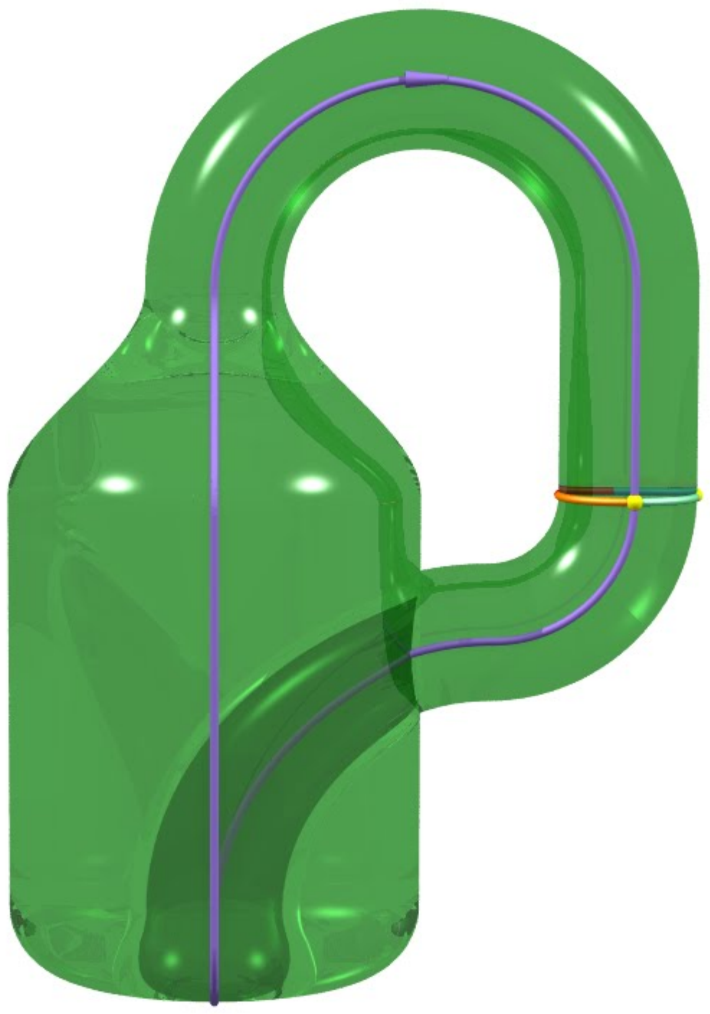
Visualization of double Torus (video)



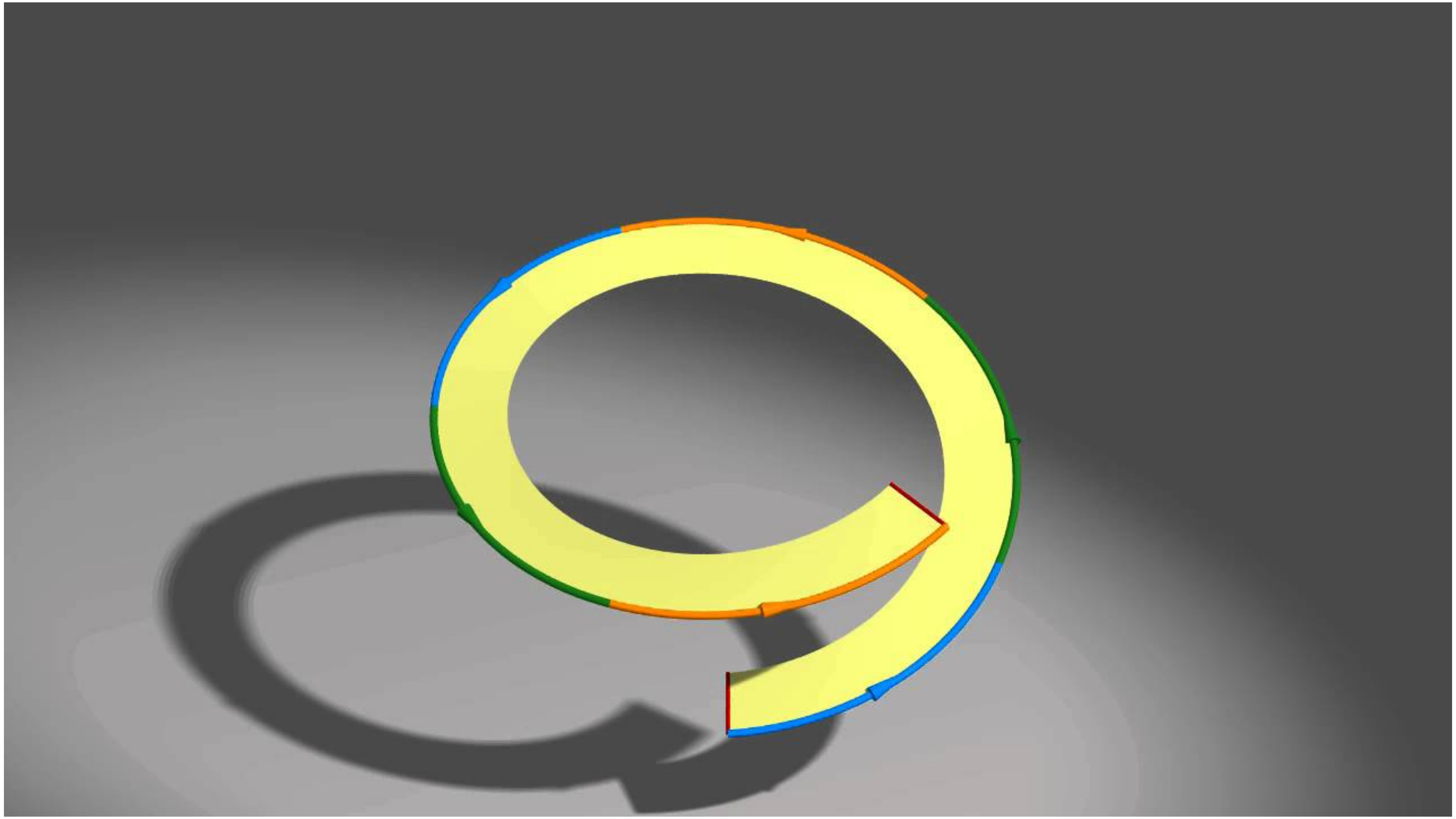
Visualization of projective plane (video)



Visualization of Klein bottle (video)

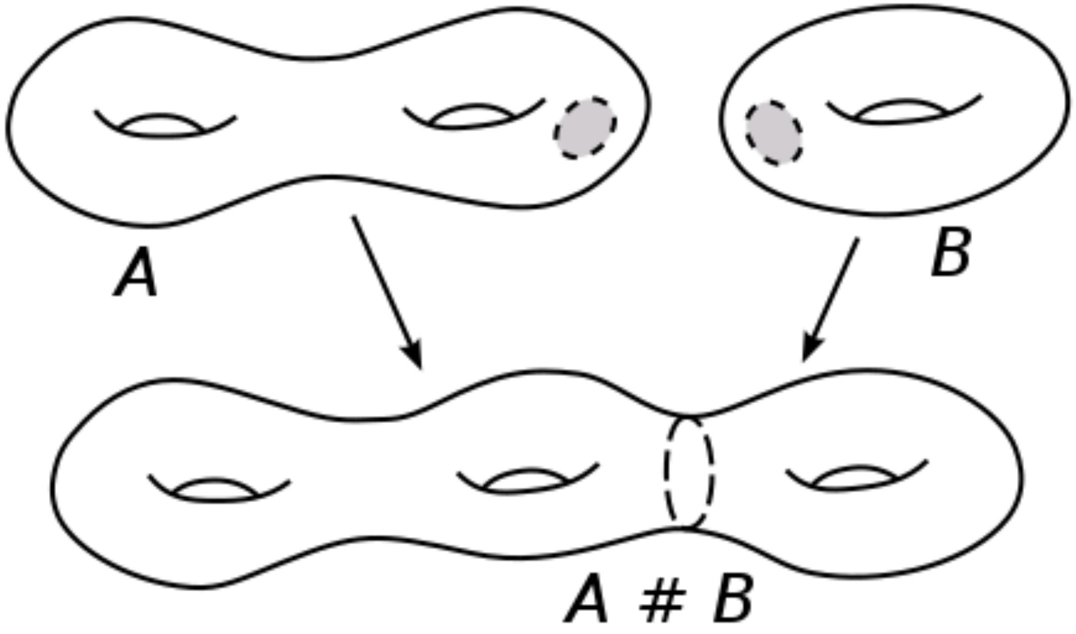


Projective plane = Mobius strip + a disk (video)



Connected sum operation

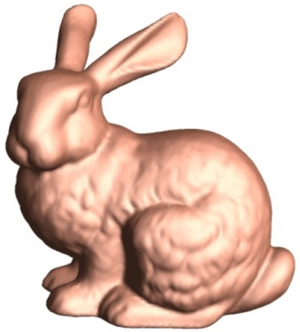
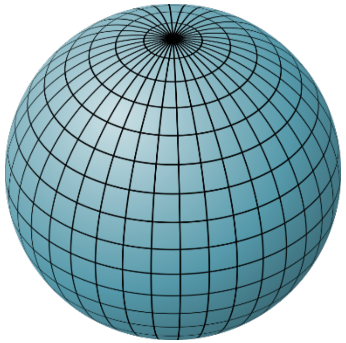
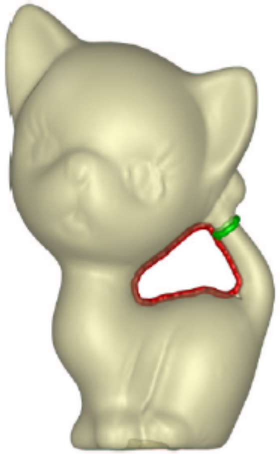
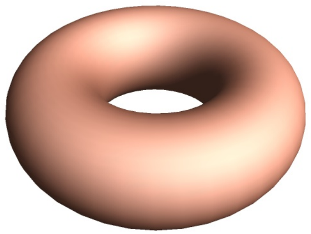
- Given two compact surface M and S, the connected sum $M\#S$ intuitively "merge" the two by cutting off a small disk (cap) from each surface, and the glue the remaining of the two surfaces along the boundary after the cutting.



Wikipedia

Classification of compact surfaces

Theorem 2 (Classification Theorem) *The two infinite families* $\mathbb{S}$, $\mathbb{T}$, $\mathbb{T}\#\mathbb{T}$, $\ldots$, *and* $\mathbb{P}$, $\mathbb{P}\#\mathbb{P}$, $\ldots$, *exhaust the family of compact 2-manifold without boundary (upto homeomorphism). The first family of surfaces are all orientable; while the second family are all non-orientable. Furthermore, no two surfaces in these sequences are homeomorphic.*

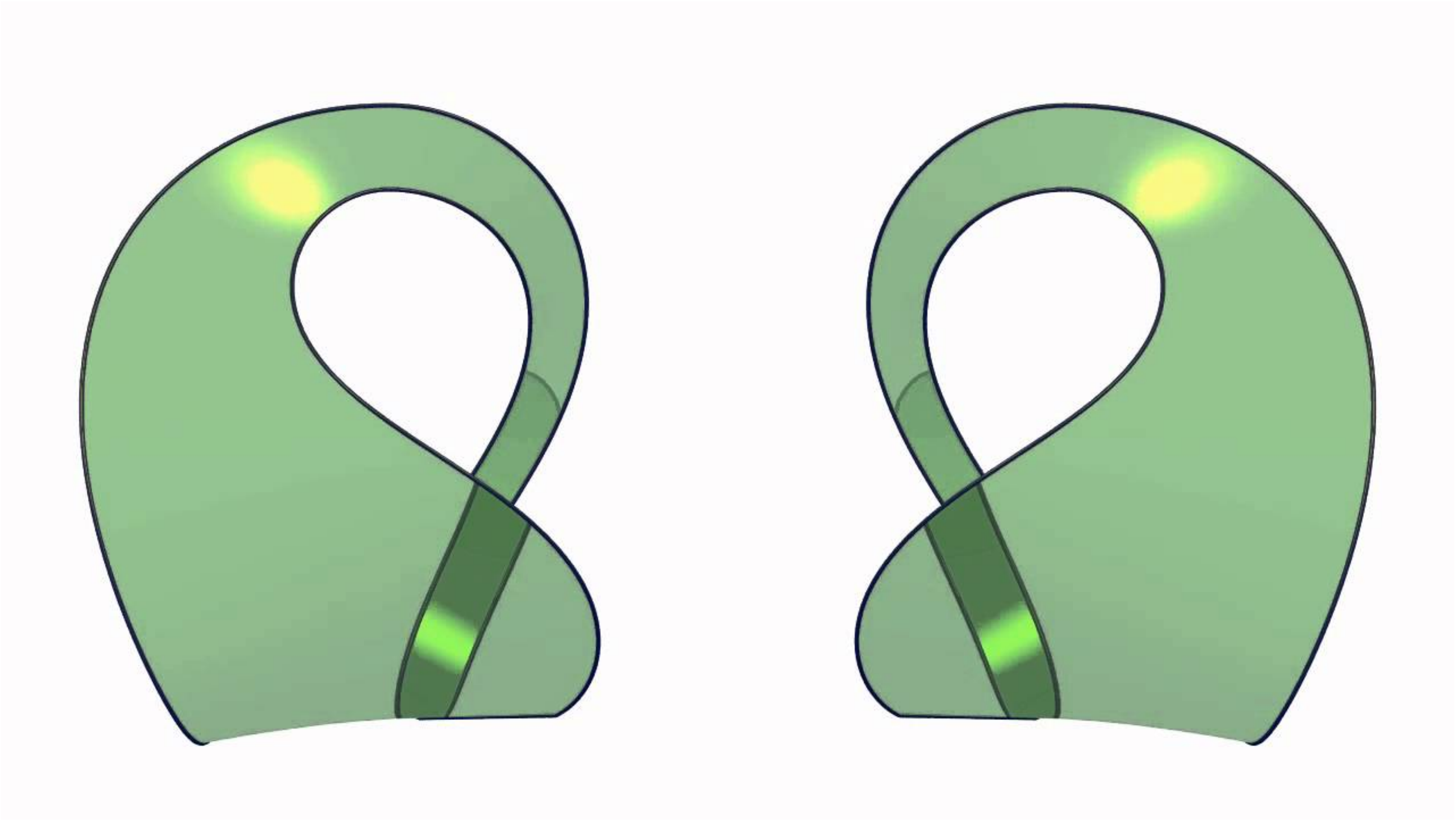
- A 2-manifold M is non-orientable if
  - Starting from some point $p \in M$, one can walk on one side of M and end up on the opposite side of M upon returning to p
- Otherwise, it is orientable.

Classification of compact surfaces

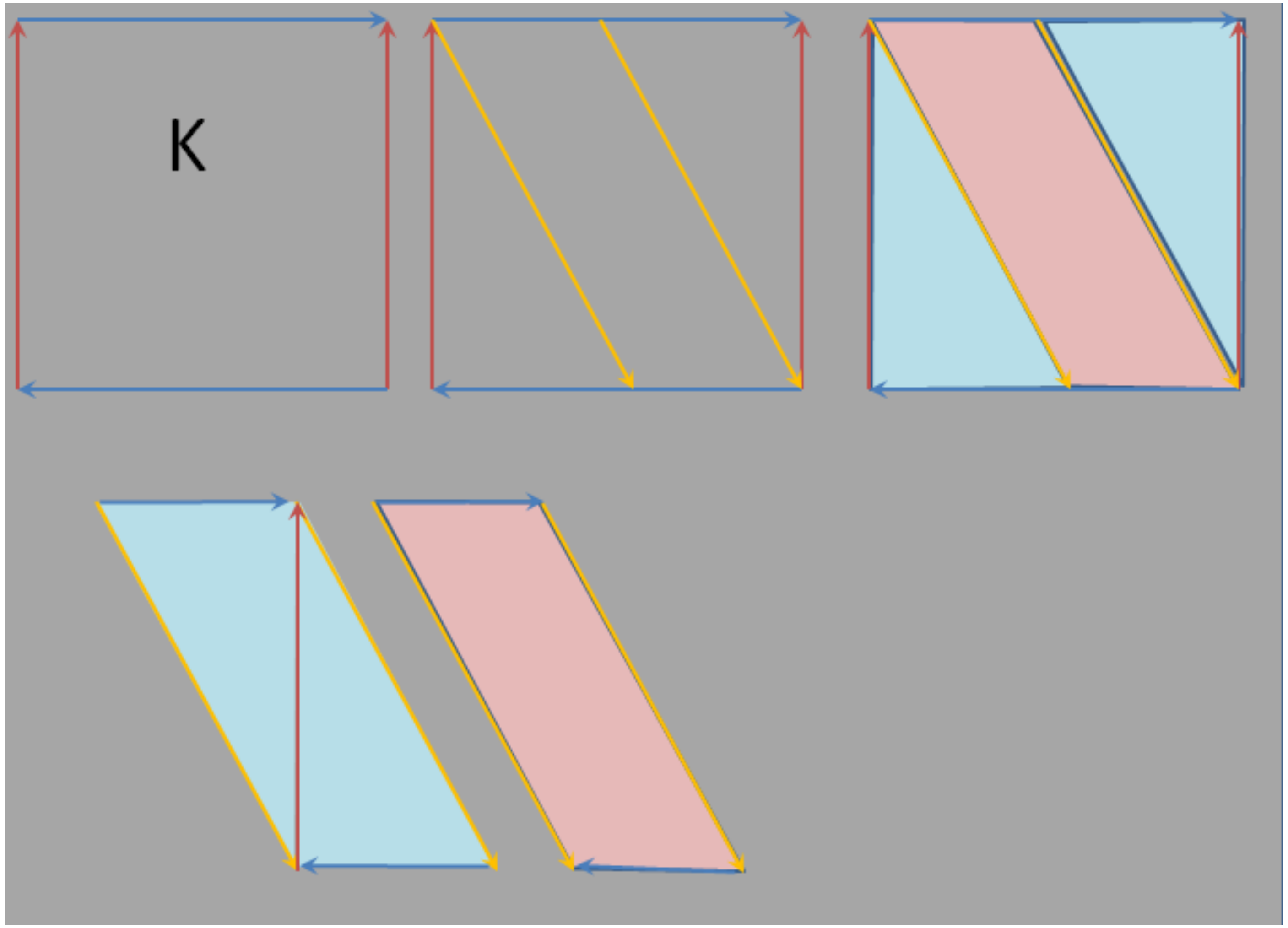
Theorem 2 (Classification Theorem) *The two infinite families* $\mathbb{S}$, $\mathbb{T}$, $\mathbb{T}\#\mathbb{T}$, $\ldots$, *and* $\mathbb{P}$, $\mathbb{P}\#\mathbb{P}$, $\ldots$, *exhaust the family of compact 2-manifold without boundary (upto homeomorphism). The first family of surfaces are all orientable; while the second family are all non-orientable. Furthermore, no two surfaces in these sequences are homeomorphic.*

- The number of $\mathbb{T}$ or $\mathbb{P}$ needed is called the genus g of the surface M
  - Sphere has genus 0, torus has genus 1, double-torus has genus 2.
- Hence the genus of a surface completely decides its topology up to homeomorphism
  - Any two compact surfaces with the same genus are homeomorphic

How does two Mobius strips make the Klein bottle? (video)



How does $\mathbb{P}\#\mathbb{P}$ become the Klein bottle?



3d and beyond?

- Super complicated
- Way way way beyond the scope of this class
- Fields Medal level of mathematics