MATH412/COMPSCI434/MATH713
Fall 2025

Topological Data Analysis

Topic 2: Simplicial Complexes

Instructor: Ling Zhou

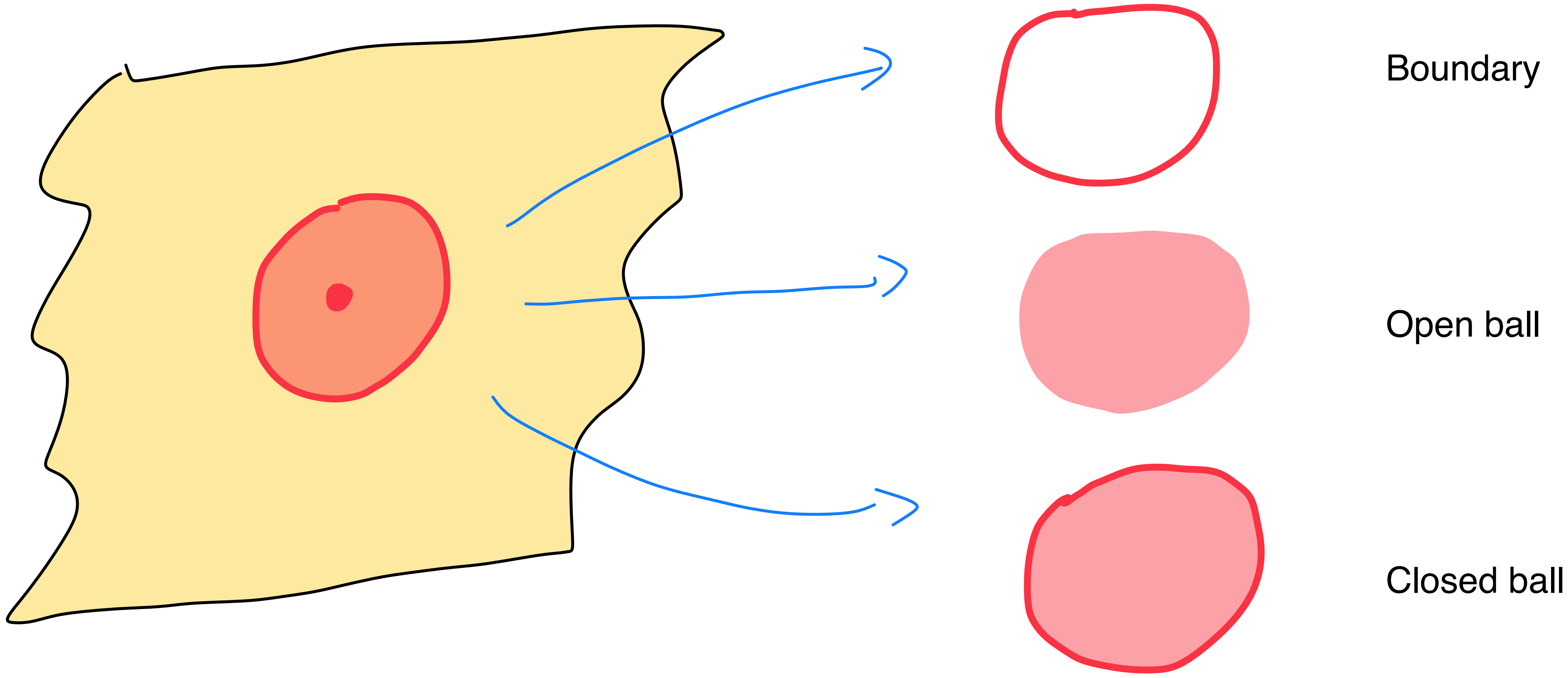
Overview

- Notions for simplicial complexes
  - Simplicial maps
  - Euler characteristic
  - Triangulation

- Commonly used simplicial complexes from point cloud data (PCD)
  - Nerve complex
  - Čech complex
  - Alpha complex
  - Delaunay Complex
  - Vietoris-Rips complex

Some notions related to simplicial complexes

Star and links



Star and links

$\tau \subset \sigma$

(face of σ) (coface of τ)

- Given a simplex $\tau \in K$

(open ball)
- Star: $St(\tau) = \{\sigma \in K \mid \tau \subset \sigma\}$
  - A star may not be a simplicial complex

$\|$

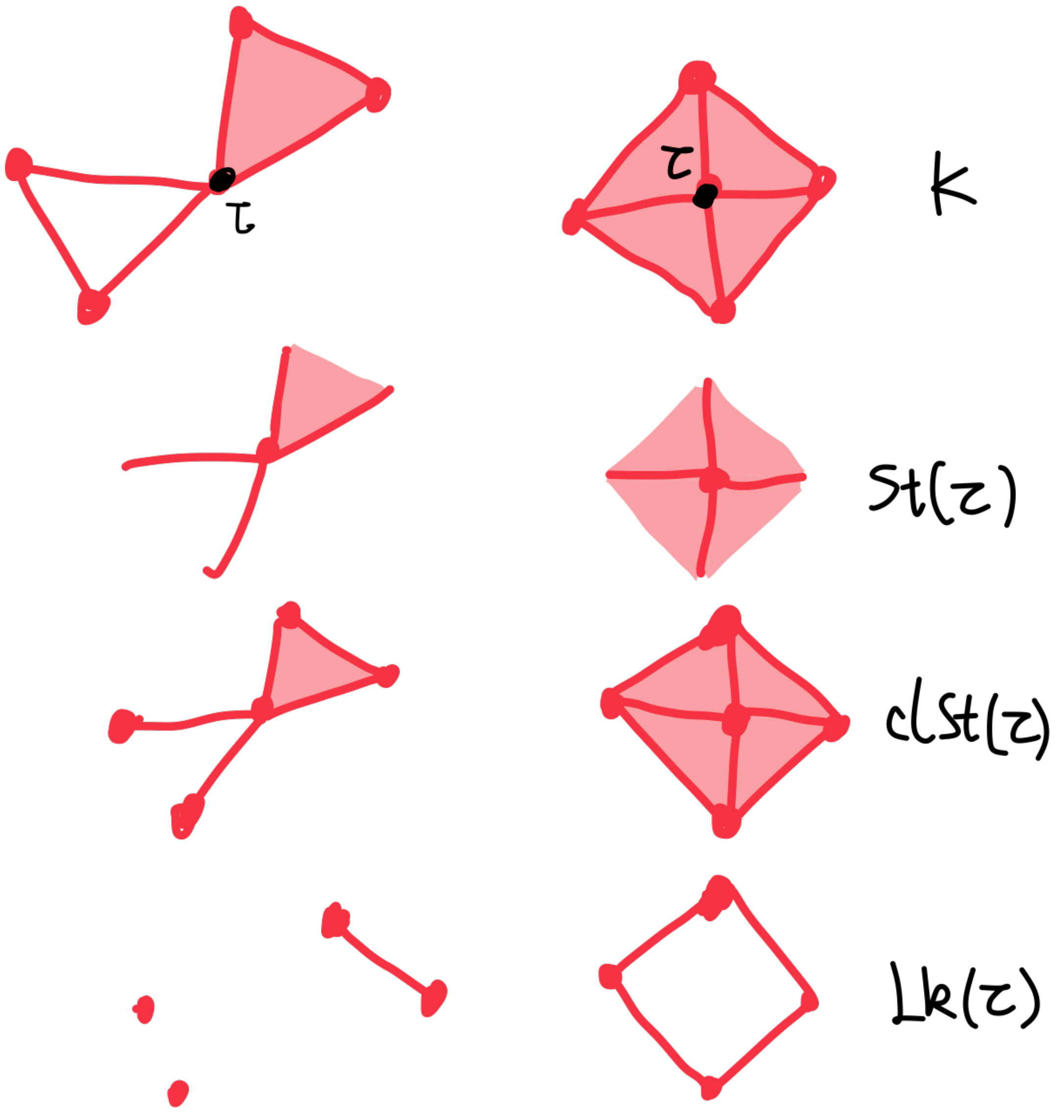
$\{$cofaces of $\tau\}$

(closed ball)
- Closed star: $clSt(\tau) = \bigcup_{\sigma \in St(\tau)} \{\sigma' \mid \sigma' \subset \sigma\}$

$\|$

$\{$faces of cofaces of $\tau\}$

(bdry of ball)
- Link: $Lk(\tau) = \{\sigma \in clSt(\tau) \mid \sigma \cap \tau = \varnothing\}$



Star and links

K

$St(\tau) = \{\text{all simplices containing } \tau\}$

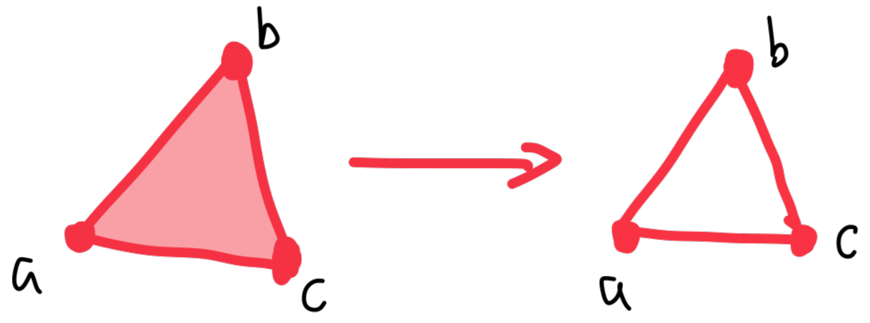
$clSt(\tau) = \{\text{faces of simplices in } St(\tau)\}$

$Lk(\tau) = \{\text{simplices in } clSt(\tau) \text{ not intersecting } \tau\}$

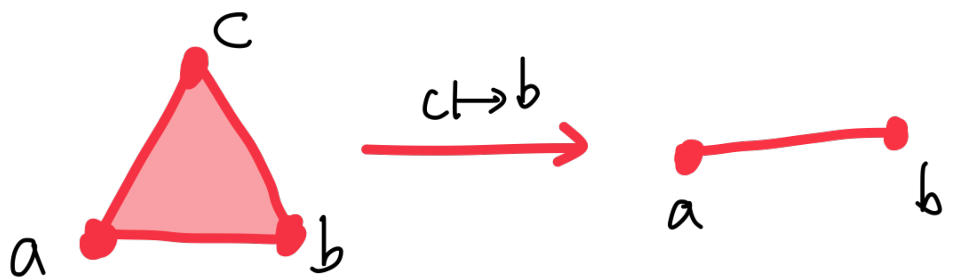
Simplicial map

- Intuitively, analogous to continuous maps between topological spaces
- Given simplicial complexes K and L
  - a function $f : V(K) \to V(L)$ is called a simplicial map if
    - for any $\sigma = \{p_0, \ldots, p_d\} \in \Sigma(K)$, $f(\sigma) = \left\{ f(p_0),\ \ldots,\ f(p_d) \right\}$ spans a simplex in L, i.e., $f(\sigma) \in \Sigma(L)$.
  - A simplicial map is also denoted $f : K \to L$

Need not to be independent



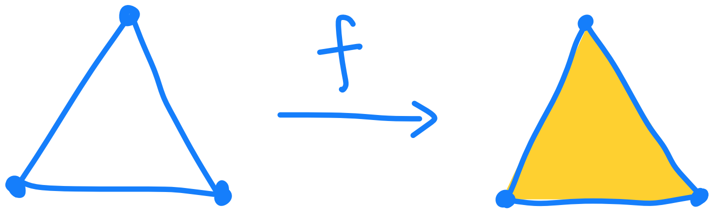
NON-example



example

Simplicial map

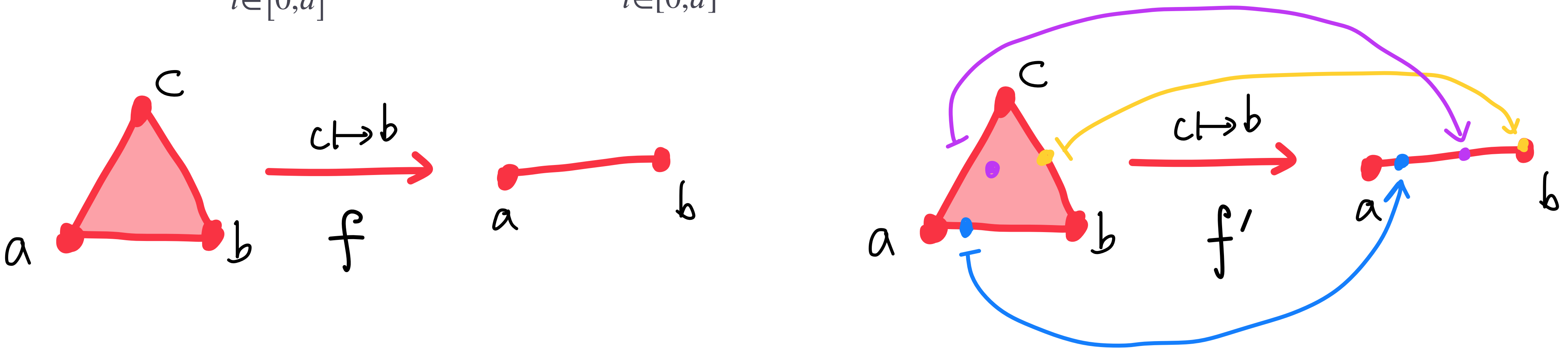
- Intuitively, analogous to continuous maps between topological spaces
- Given simplicial complexes K and L
  - a function $f : V(K) \to V(L)$ is called a simplicial map if
    - for any $\sigma = \{p_0, \ldots, p_d\} \in \Sigma(K)$, $f(\sigma) = \left\{ f(p_0), \ \ldots, \ f(p_d) \right\}$ spans a simplex in L, i.e., $f(\sigma) \in \Sigma(L)$.
  - A simplicial map is also denoted $f : K \to L$
- A simplicial map $f : K \to L$ is an **isomorphism**
  - if f is bijective between vertex sets and f^{-1} is a simplicial map



- f is bij on vertices
- f^{-1} is NOT simplicial

Simplicial map

- A simplicial map $f\colon K \to L$ induces a natural continuous function $f'\colon |K| \to |L|$
  - s.t $f'(x) = \sum_{i\in[0,d]} a_i f(p_i)$ for $x = \sum_{i\in[0,d]} a_i p_i \in \sigma = \{p_0, \ldots, p_d\}$



- **Theorem:**
  - An isomorphism $f\colon K \to L$ induces a **homeomorphism** $f'\colon |K| \to |L|$

A topological invariant – Euler Characteristic

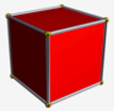
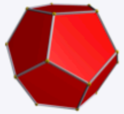
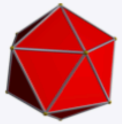
- For the surface of a polyhedron, the Euler Characteristic is defined as
  $\chi = V - E + F.$
- Euler's polyhedron formula:
  - $\chi = 2$ for surface of convex polyhedron

$n_0 = V$: # of vertices (0-simplices)

$n_1 = E$: # of edges (1-simplices)

$n_2 = F$: # of triangles (2-simplices)

$\chi = n_0 - n_1 + n_2$

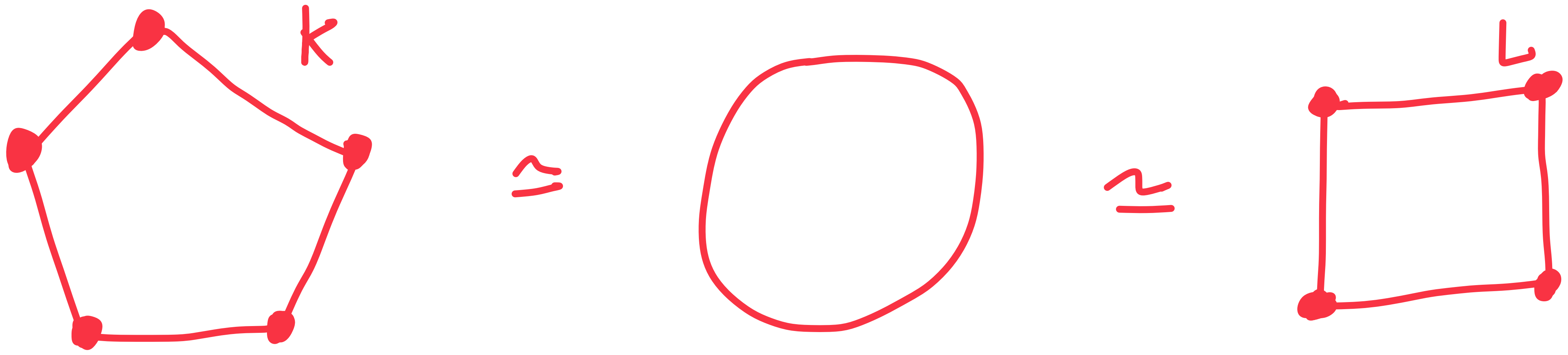
Name	Image	Vertices V	Edges E	Faces F	Euler characteristic: $V - E + F$
Tetrahedron		4	6	4	**2**
Hexahedron or cube		8	12	6	**2**
Octahedron		6	12	8	**2**
Dodecahedron		20	30	12	**2**
Icosahedron		12	30	20	**2**

A topological invariant – Euler Characteristic

- Given a d-dim simplicial complex K with n_i number of i-simplices
- the *Euler characteristic* of K is defined as:
  - $\chi(K) := \sum_{i=0} (-1)^i n_i = n_0 - n_1 + n_2 \cdots + (-1)^d n_d$

- Euler characteristic is both a topological invariant and a homotopy invariant, meaning that it does not change under homeomorphism or homotopy equivalence.

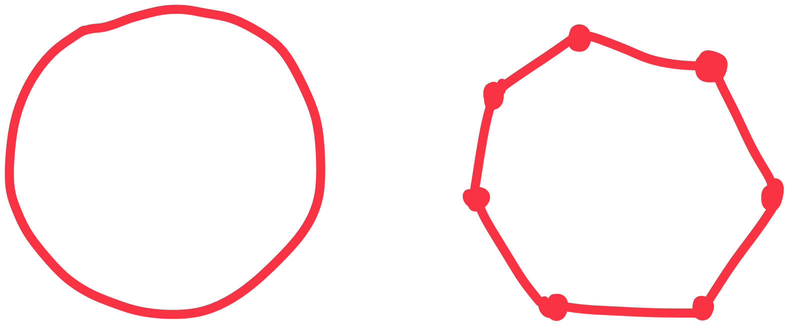
A topological invariant – Euler Characteristics

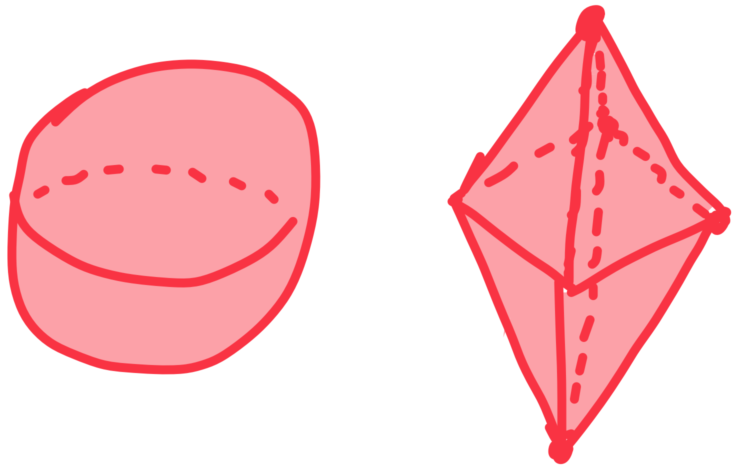


$\chi(K) = 5 - 5 = 0$ $\chi(\mathbb{S}^1) = 0?$ $\chi(L) = 4 - 4 = 0$

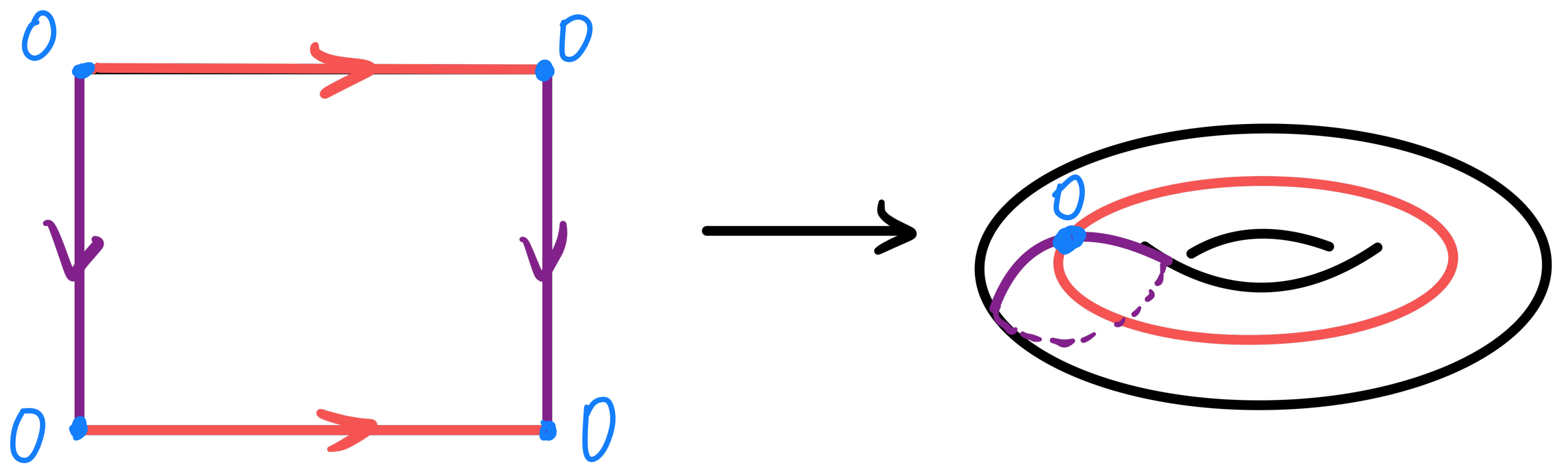
Triangulation of a manifold

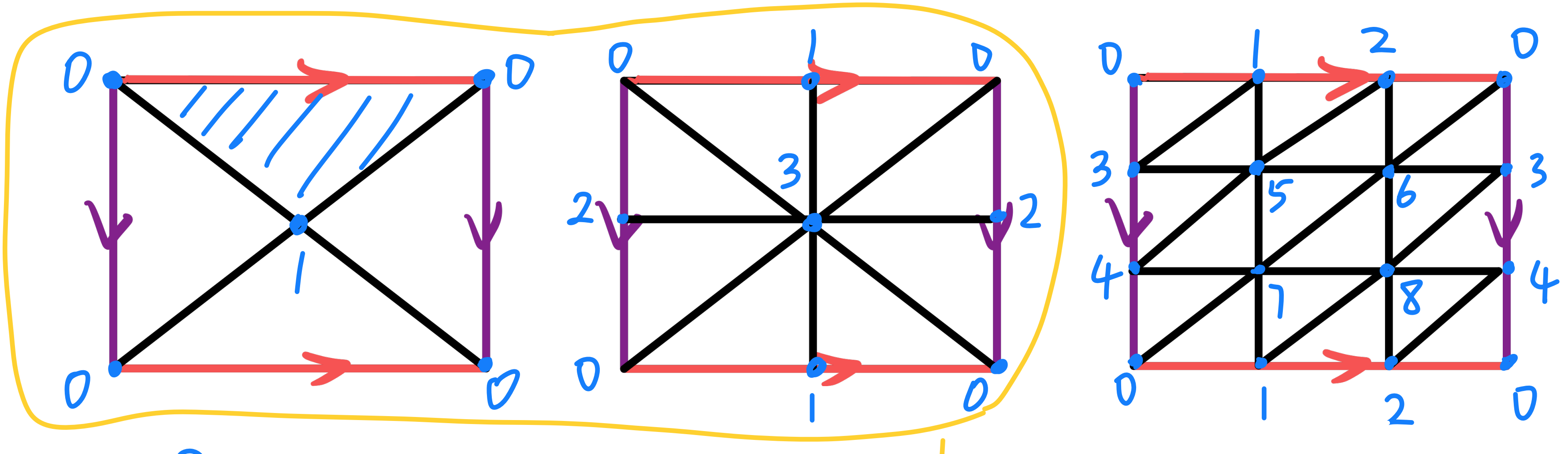
- Given a manifold (with or without boundary) M, a simplicial complex K is a triangulation of M
  - if the underlying space $|K|$ of K is homeomorphic to M

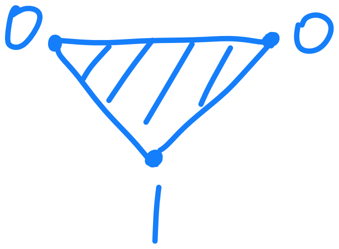




simplicial complex $\underset{\text{triangulation}}{\overset{\text{geometric realization}}{\rightleftarrows}}$ topological space



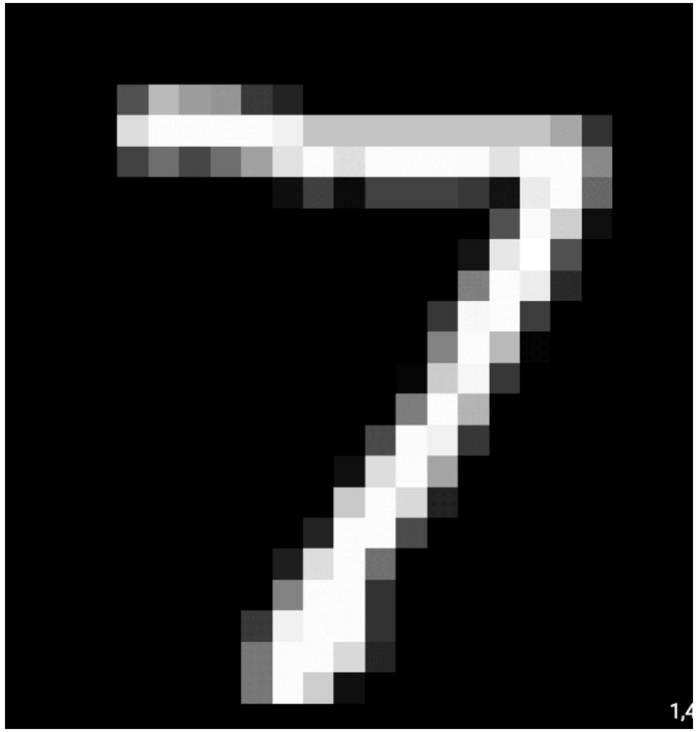




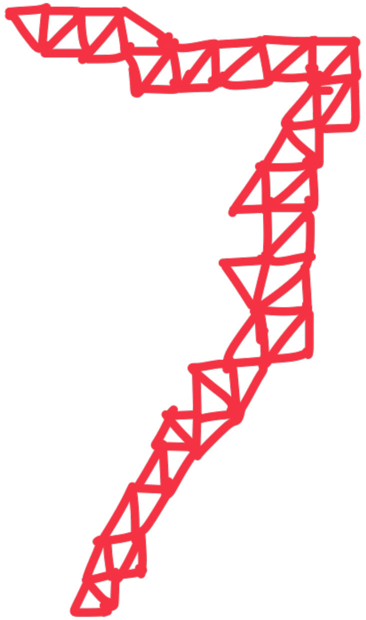
is NOT a simplex

NOT simplicial complex

Image Data



triangulation?



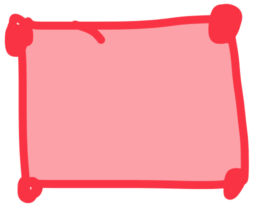
Cubical Complex

- Building blocks are "cubical" shapes

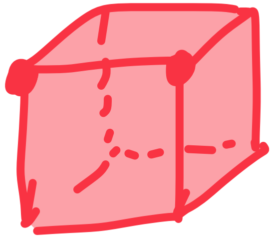
0-cube

1-cube

2-cube

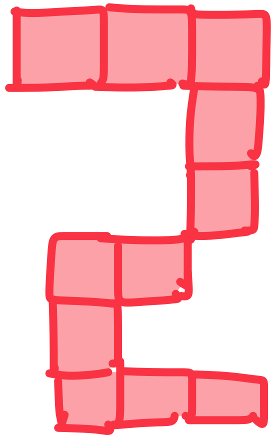


3-cube



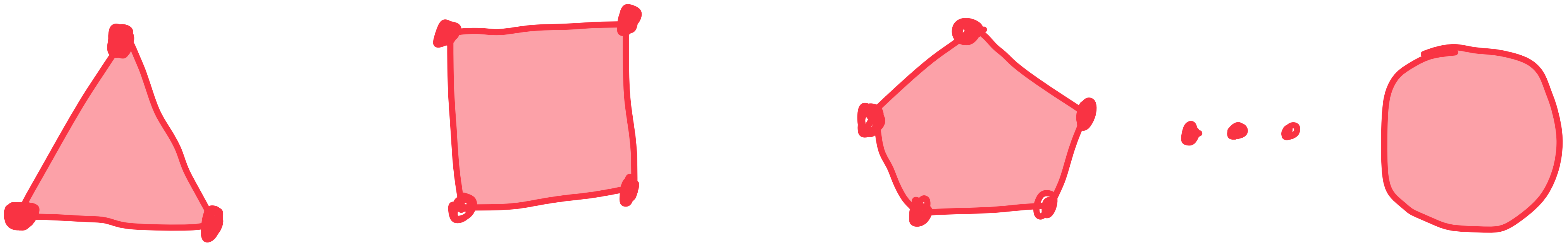
…

2-dim cubical complex



CW Complex

- Can we build spaces using “balls” instead of polygons?



Triangle Rectangle Pentagon Disk

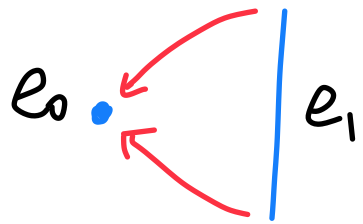
0-cell 1-cell 2-cell 3-cell K-cell

CW Complex

- A CW complex X is the union of a sequence of topological spaces
  - $\varnothing = X_{-1} \subset X_0 \subset X_1 \subset \cdots$
  - Such that X_k is obtained from X_{k-1} by "gluing" k-cells $\{e_\alpha^k\}_\alpha$, each homeomorphic to $\mathbb{D}^k$, by continuous maps $\partial e_\alpha^k \to X_{k-1}$
  - Each X_k is called the k-skeleton of X

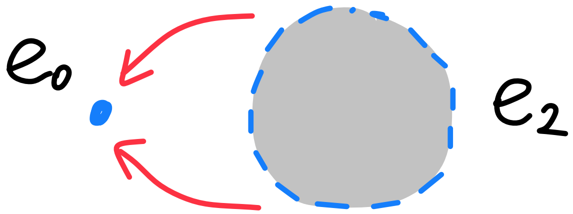
CW Complex

(glue boundary of 1-cell to 0-cell)



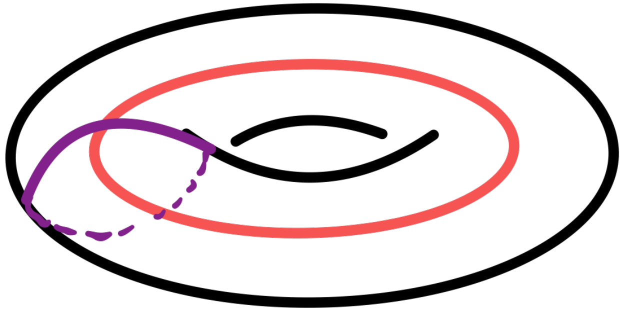
simplicial complex

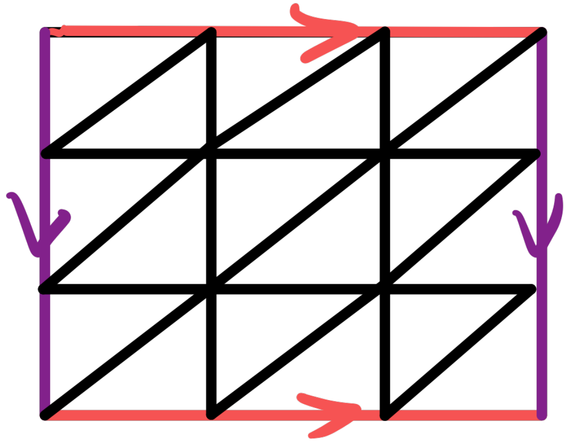
CW complex



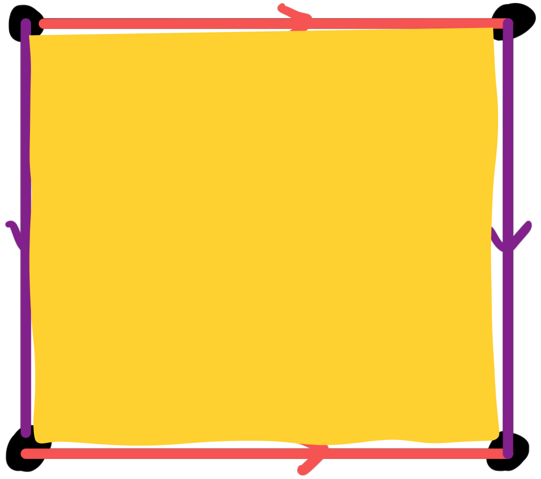
(glue boundary of 2-cell to 0-cell)

CW Complex





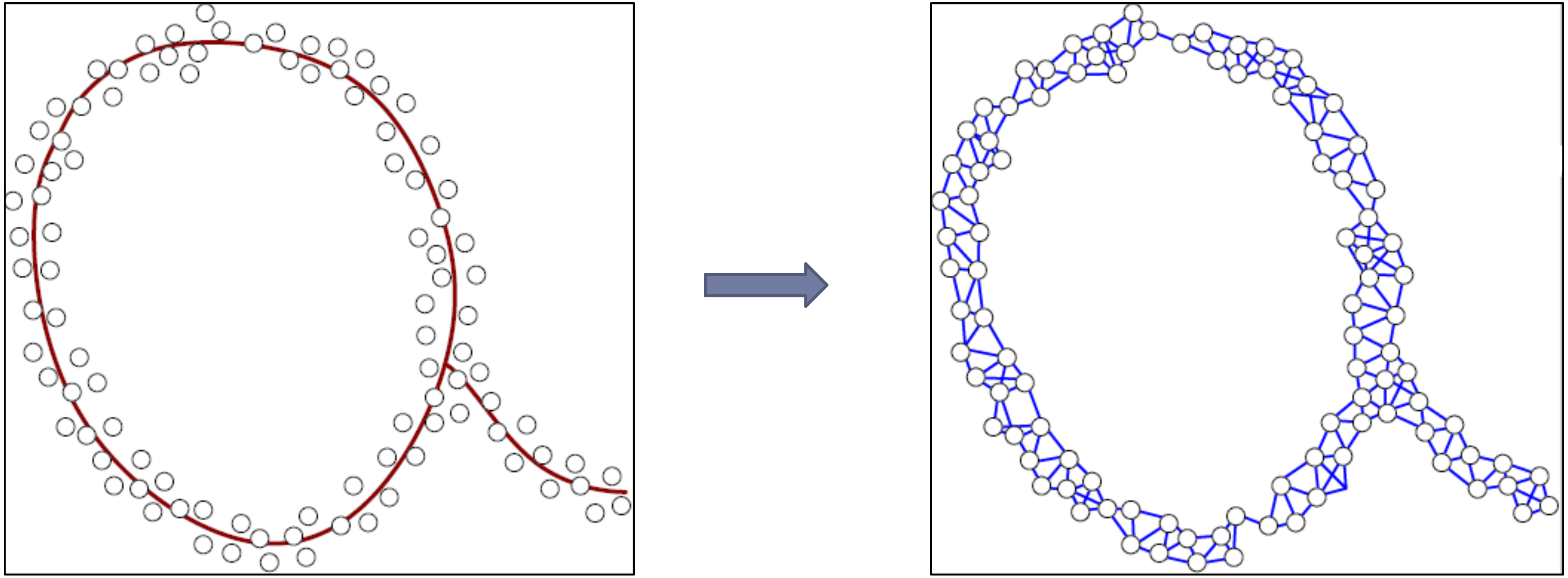
Triangulation of a torus



CW structure of a torus

Common Complexes

Goal: create simplicial complexes from dataset in order to use topological tools



Nerve complex

- Given a finite collection of sets $\mathscr{U} = \{U_\alpha\}_{\alpha \in A}$, its nerve complex $Nrv(\mathscr{U})$ is a simplicial complex
  - The vertex set $V = A$
  - $\{\alpha_0, \ldots, \alpha_k\} \in \Sigma$ iff $\cap_{i=0}^{k} U_{\alpha_i} \neq \emptyset$

$$\mathscr{U} = \{U_1, U_2, U_3\}$$

$$\Downarrow$$

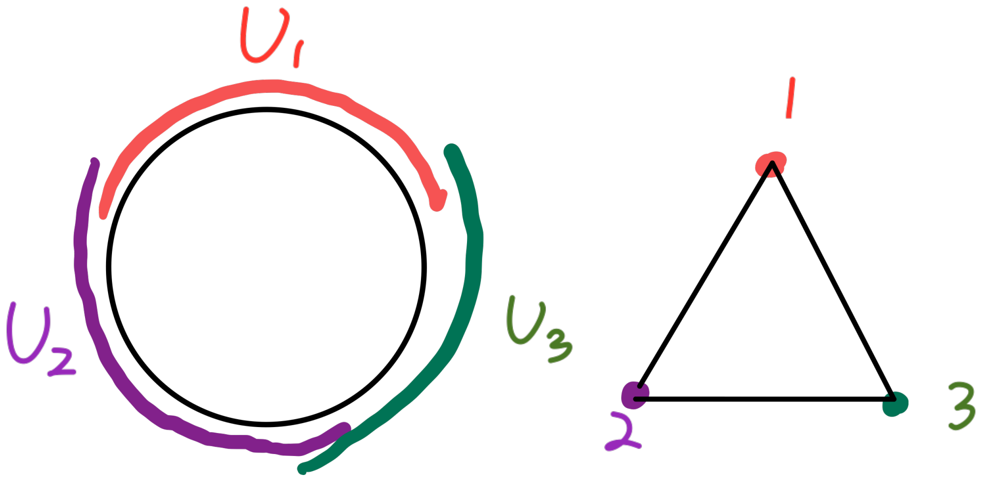
$$V = \{1, 2, 3\}$$

$$\Sigma = \{1, 2, 3, 12, 13, 23\}$$

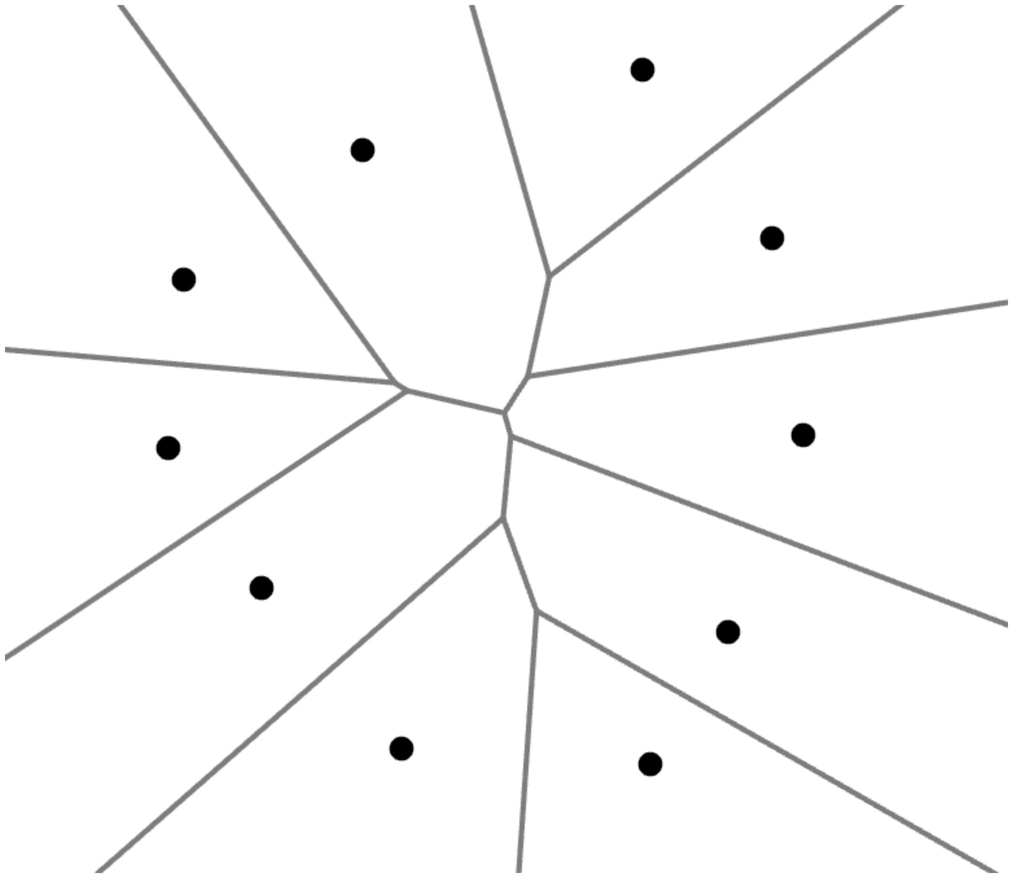
$U_1 \cap U_2 \neq \emptyset$

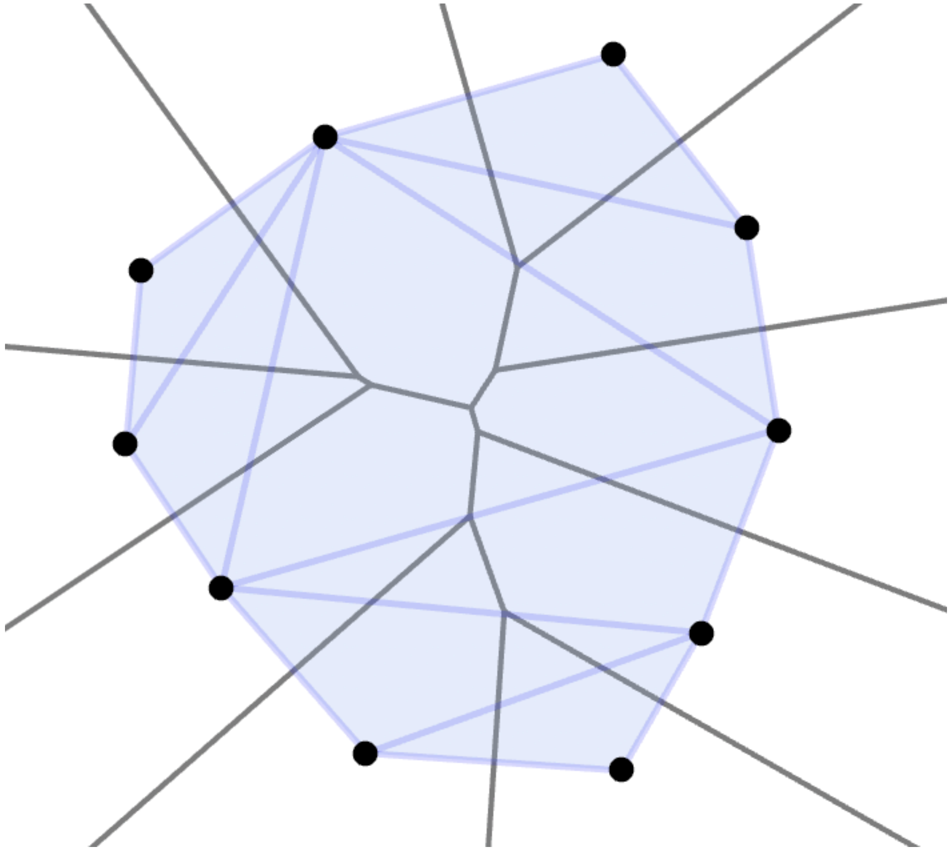
$U_1 \cap U_3 \neq \emptyset$

$U_2 \cap U_3 \neq \emptyset$



Example





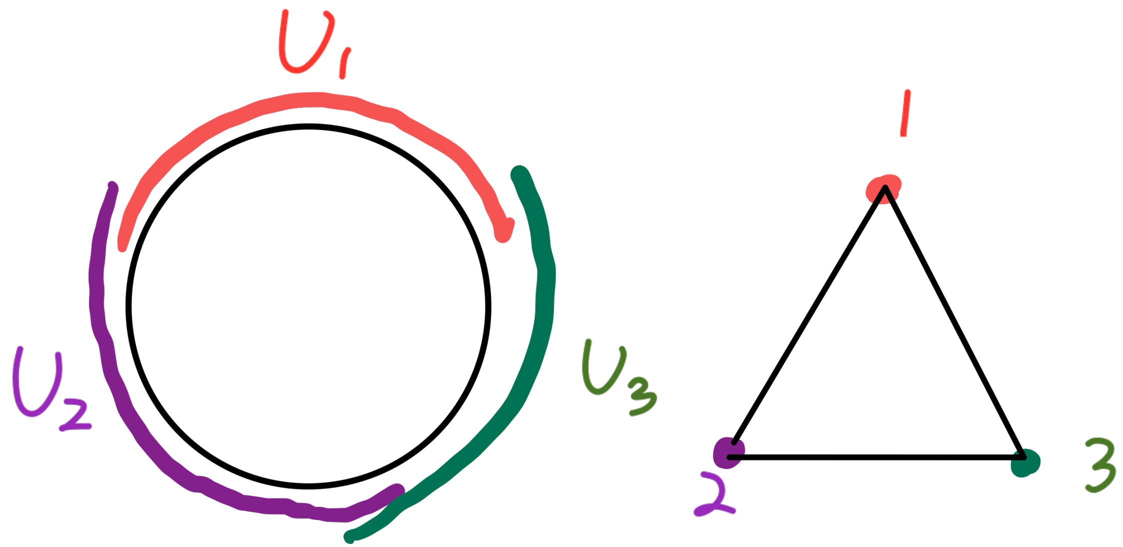
Nerve complex recovers homotopy type of the cover

- A topological space is said to be contractible if it is homotopy equivalent to a point.

- Nerve Lemma (intrinsic):
  - Let $\mathscr{U}$ be an **open** cover of a metric space X such that $\cap_{i=1}^{k} U_{\alpha_i}$ is contractible for any finite elements in $\mathscr{U}$.

    "good cover"
  - Then $|Nrv(\mathscr{U})| \simeq X$.

- Nerve Lemma (Euclidean version):
  - Let $\mathscr{U}$ be a finite collection of **closed**, **convex** subsets in R^d. Then $|Nrv(\mathscr{U})| \simeq \cup_{\alpha \in A} U_\alpha \subset \mathbb{R}^d$.



$X =$ circle

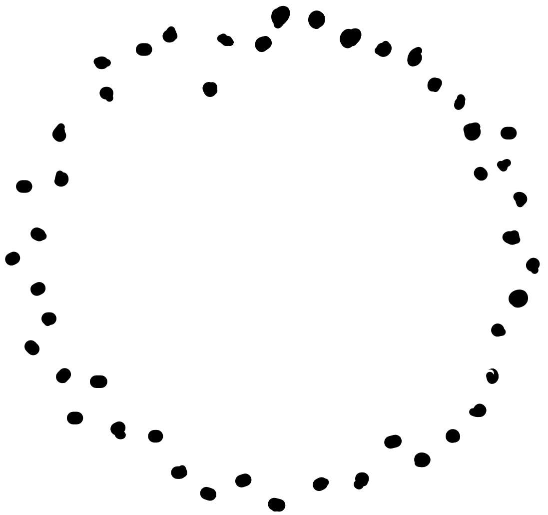
$\mathcal{U} = \{U_1, U_2, U_3\}$ covers X

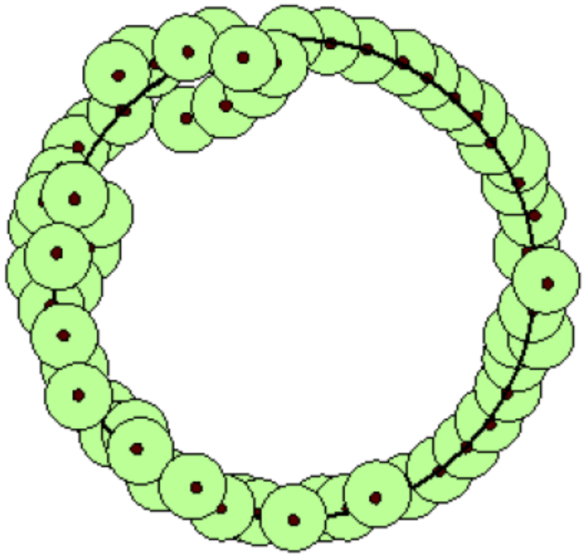
$\mathrm{Nrv}(\mathcal{U}) = \triangle$

Nerve lem. $\Longrightarrow$ $X \simeq |\mathrm{Nrv}(\mathcal{U})|$ $\quad (\bigcirc \simeq |\triangle|)$

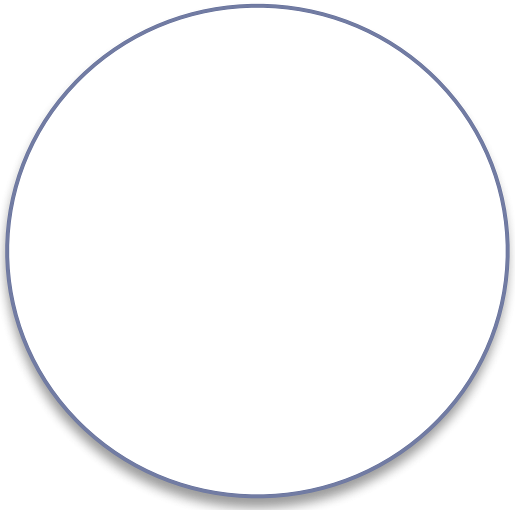
Čech complex: Thickening “recovers” the shape

- The topology of the data set is trivial since we only have finitely many points
- Thickening can be used to recover the shape of the underlying ground truth
  - Create a collection of balls with given radius $\longrightarrow$ gives “good cover”
  - Build the nerve of the given collection





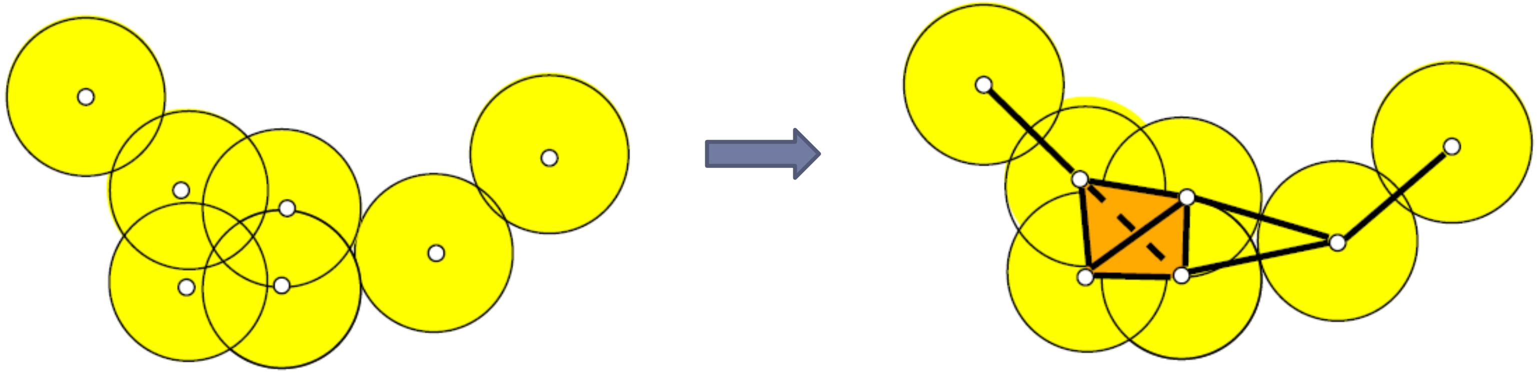
$\simeq$



Čech Complex

$C^r(P) := Nrv\left(\{B(p_i, r) : p_i \in P\}\right)$

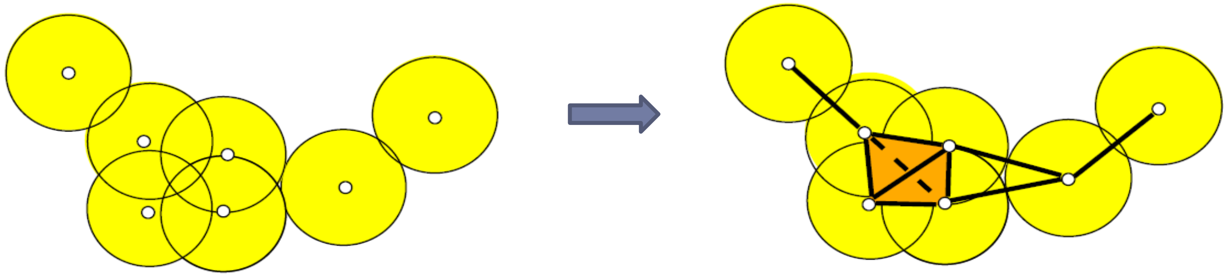
- Given a set of points $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^d$
- Given a real value $r > 0$, the *Čech complex* $C^r(P)$ is the nerve of the set of closed balls $\{B(p_i, r)\}_{i=1,\ldots,n}$, where $B(p, r) = \{x \in \mathbb{R}^d \mid d(p, x) \leq r\}$
  - i.e, $\sigma = \left\{p_{i_0}, \ \ldots, p_{i_s}\right\} \in C^r(P)$ iff $\bigcap_{j=0,\ldots,s} B(p_{i_j}, r) \neq \emptyset$
- The definition can be extended to a finite sample P of a metric space (X, d).



Čech Complex

- Nerve Lemma (Euclidean version):
  - Let $\mathscr{U}$ be a finite collection of closed, convex subsets in R^d. Then $|Nrv(\mathscr{U})| \simeq \cup_{\alpha \in A} U_\alpha \subset \mathbb{R}^d$.

- Corollary:
  - $|C^r(P)| \simeq \cup_{p \in P} B(p, r)$, i.e, $|C^r(P)|$ is homotopy equivalent to the union of r-balls around points in P

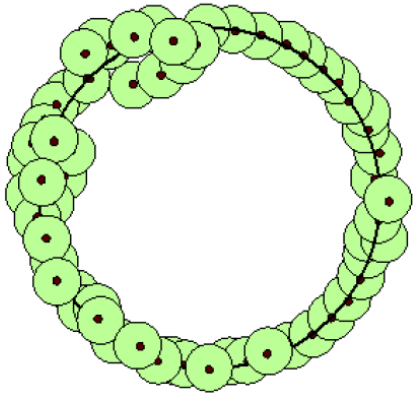


Čech Complex

- Nerve Lemma (Euclidean version):
  - Let $\mathscr{U}$ be a finite collection of closed, convex subsets in R^d. Then $|Nrv(\mathscr{U})| \simeq \cup_{\alpha \in A} U_\alpha \subset \mathbb{R}^d$.

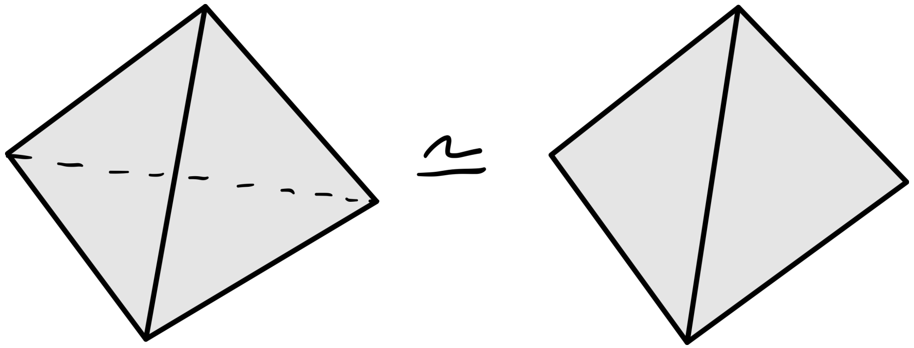
- Corollary:
  - $|C^r(P)| \simeq \cup_{p \in P} B(p, r)$, i.e, $|C^r(P)|$ is homotopy equivalent to the union of r-balls around points in P

- Given a set of points P
  - approximating a hidden domain M
  - $U^r(P) = \bigcup_{p \in P} B(p,\ r)$ approximates M
  - $C^r(P)$ approximates $U^r(P)$

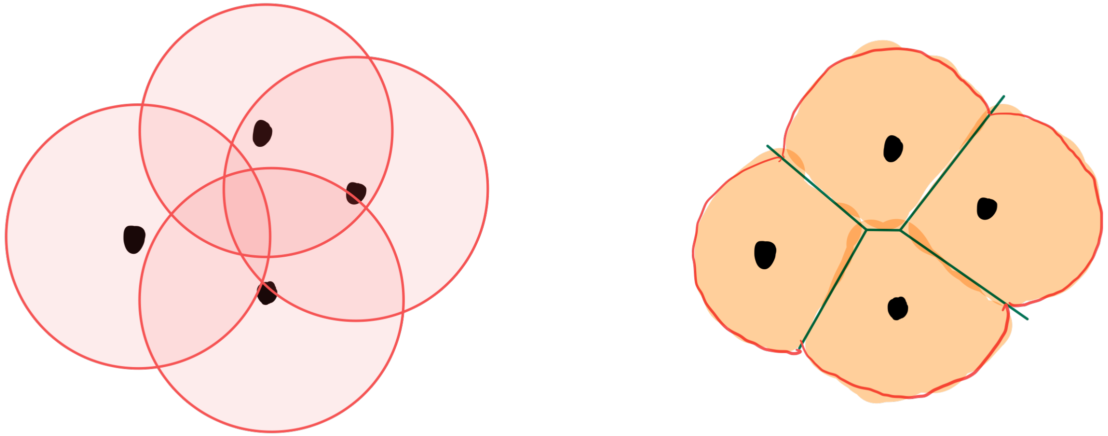


More on Čech

- Some high dimension simplicies are redundant

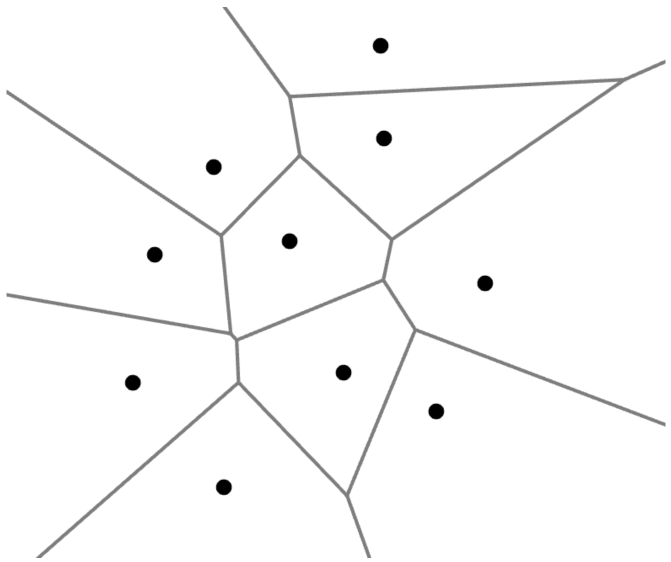


- Change the cover to create a simplified simplicial complex



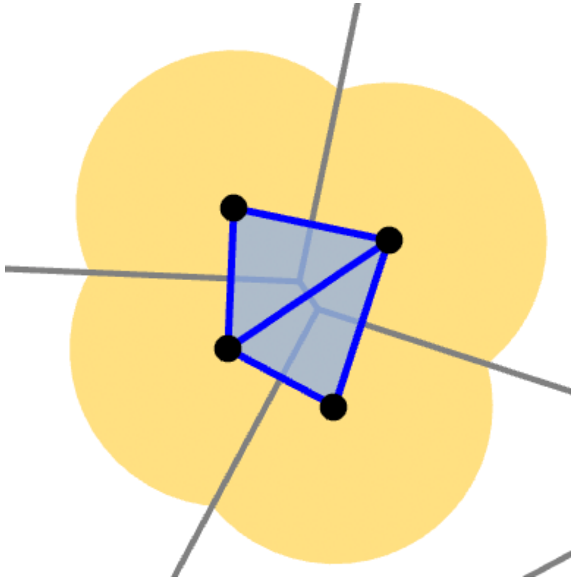
Voronoi Diagram

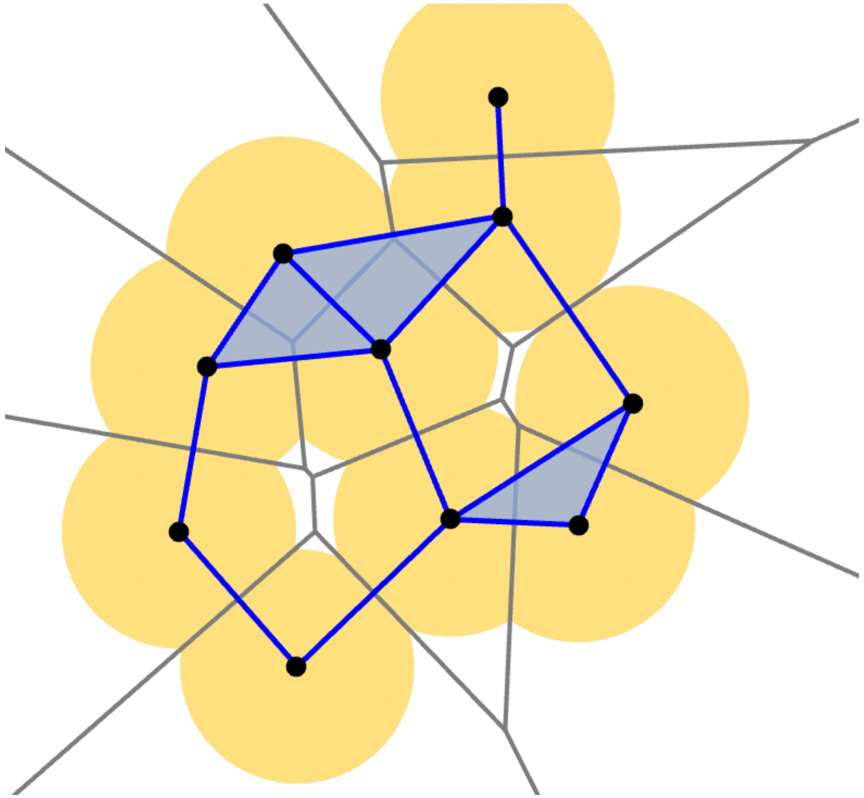
- Given a finite set $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^d$, the **Voronoi cell** of p_i is
  - $Vor(p_i) = \{x \in \mathbb{R}^d \mid \|x - p_i\| \leq \|x - p_j\|, \forall j \neq i\}$
- The **Voronoi Diagram** of P is the collection of all Voronoi cells.



Alpha complex

- Given a set of points $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^d$
- Given a real value $r > 0$, the *Alpha complex* $Del^r(P)$ is the nerve of the set $\{B(p_i, r) \cap Vor(p_i)\}_{i=1}^{n}$



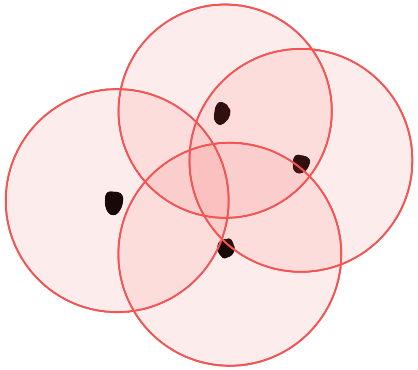


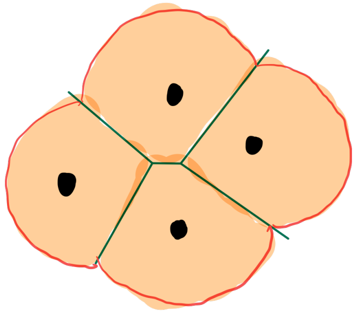
Alpha complex vs Čech complex

- $Del^r(P) \subset C^r(P)$
- $|Del^\infty(P)| = O(n^{\frac{d}{2}})$ whereas $|C^\infty(P)| = O(2^n)$
- $\dim Del^r(P) \leq d$ for generic P

$Del^r(P) := Nrv\left(\{B(p_i, r) \cap Vor(p_i) \mid p_i\}\right)$

$C^r(P) := Nrv\left(\{B(p_i, r) : p_i \in P\}\right)$





- **Proposition:**
  - $Del^r(P) \simeq C^r(P) \simeq \cup_p B(p, r)$, i.e, $C^r(P)$ and $Del^r(P)$ are homotopy equivalent.