MATH412/COMPSCI434/MATH713 Fall 2025

Topological Data Analysis

Topic 2: Simplicial Complexes

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Overview

‣ **Simplicial complex**

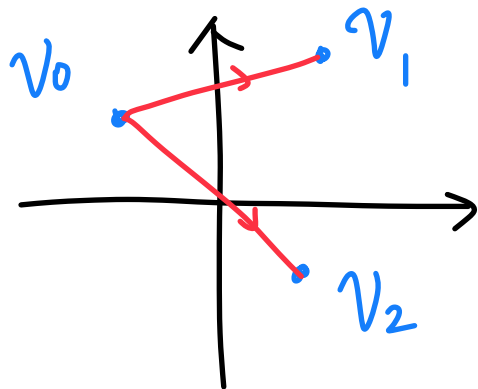   ‣ a specific type of topological space commonly used in practice to model data

‣ **Notions**

    ‣ **Geometric realization**

    ‣ **Stars and links**

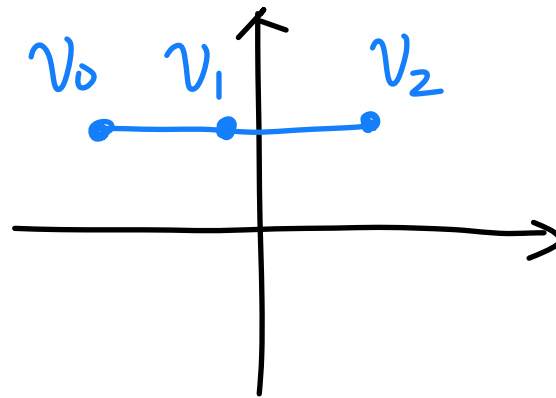# Introduction to Simplicial Complex

Geometric Simplices

▸ Points $\{v_0, \ldots, v_p\} \subset \mathbb{R}^N$ are (affinely) independent

    ▸ if vectors $v_i - v_0, i = 1, \ldots, p$ are linearly independent

$\{v_1 - v_0, v_2 - v_1\}$ linearly ind.

$\Rightarrow \{v_0, v_1, v_2\}$ ind.

$\{v_0, v_1, v_2\}$ NOT ind.

Geometric Simplices

▸ Points $\{v_0, \ldots, v_p\} \subset \mathbb{R}^N$ are (affinely) independent

  ▸ if vectors $v_i - v_0, i = 1, \ldots, p$ are linearly independent

▸ Geometric *p*-simplex is the convex hull of $p + 1$ *(affinely) independent* points in R^N.
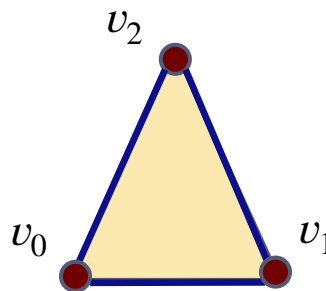
▸ In other words, a p-simplex is

▸

$$\sigma = \{ \sum_{i=0}^{p} a_i v_i \mid a_i \geq 0, \sum a_i = 1 \}$$
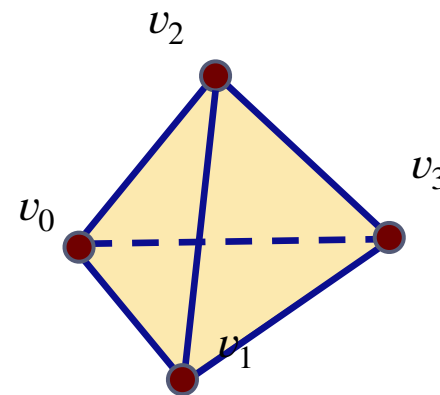

0-simplex   1-simplex   2-simplex   3-simplex

Geometric Simplices

▸ Points $\{v_0, \ldots, v_p\} \subset \mathbb{R}^N$ are (affinely) independent

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▸ Geometric *p*-simplex is the convex hull of $p + 1$ *(affinely) independent* points in R^N.

▸ In other words, a p-simplex is

▸
$$\sigma = \left\{ \sum_{i=0}^{p} a_i v_i \mid a_i \geq 0, \; \sum a_i = 1 \right\}$$

▸ We write $\sigma = \{ v_0, v_1, \ldots, v_p \}$ and define $(\dim(\sigma) = p)$

Geometric Simplices

▸ Simplex τ is a face of σ, if it is formed by a subset of $\{\ v_0, v_1, \ldots, v_p\}$. We write $\tau \subseteq \sigma$.

  ▸ τ is a proper face of σ if $\dim(\tau) < \dim(\sigma)$

  ▸ τ is a facet of σ if $\dim(\tau) = \dim(\sigma) - 1$

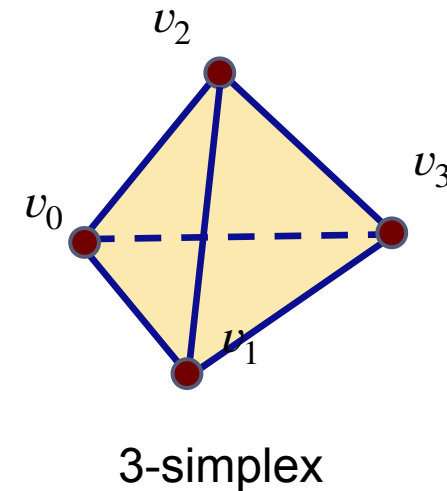 ▸ $\partial\sigma =$ collection of **all** proper faces of σ

▸ For a d-simplex σ

  ▸ $\sigma \cong \mathbb{B}^d, \ \partial\sigma \cong \mathbb{S}^{d-1}, \ \sigma^o \cong \mathbb{B}_0^d \cong \mathbb{R}^d$
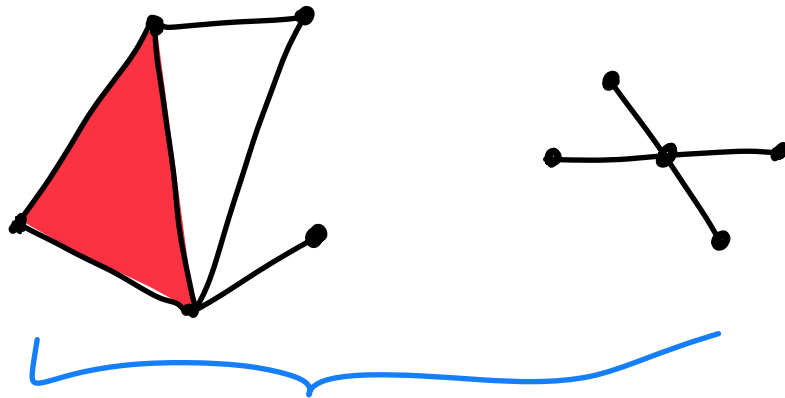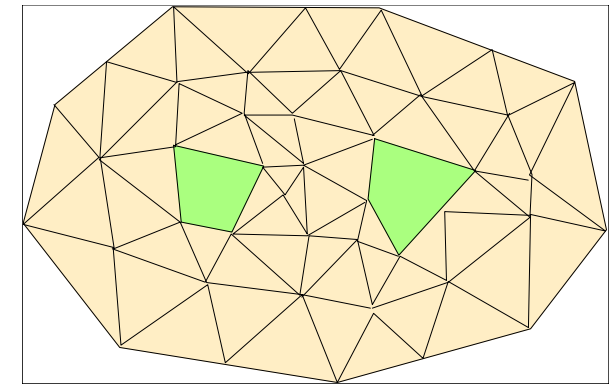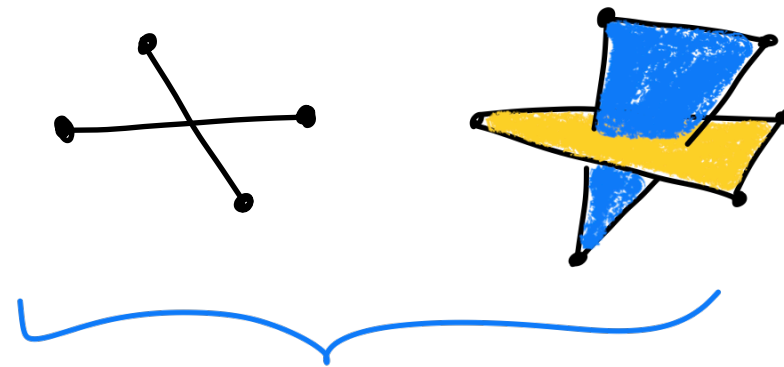
    ↓ closed ball ↓ sphere ↓ open ball

v_2

v_3

v_0

v_1

3-simplex

Geometric simplicial complex

‣ **A (geometric) simplicial complex K**

  ‣ A collection of simplices such that

    ‣ If $\sigma \in K$, then any fact $\tau \subseteq \sigma$ is also in K (all faces of a simplex are in)

    ‣ If $\sigma \cap \sigma' \neq \varnothing$, then $\sigma \cap \sigma'$ is a face of both simplices. (intersection of simplices) is a simplex)

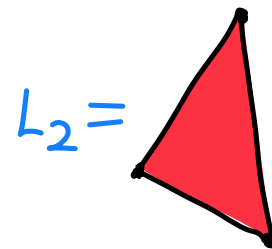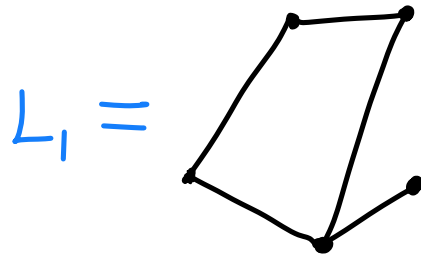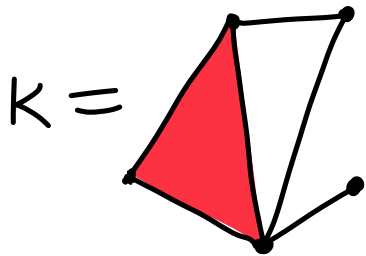 ‣ $\dim(K)$ = highest dim of any simplex in K

example

NON - example

Geometric simplicial complex

- Subcomplex: $L \subseteq K$ and L is a complex
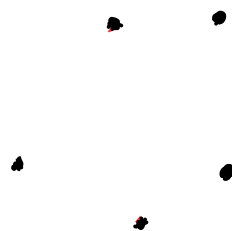
$$K = $$

$$L_1 = $$

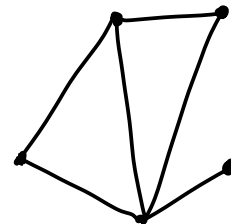$$L_2 = $$

NOT subcpx of K

- The p-**skeleton** of K consists of all simplices in K of dimension at most p
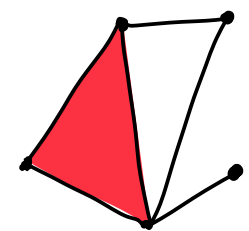
0-skeleton:

1-skeleton:

2-skeleton:

Geometric simplicial complex

▸ **Underlying space** $|K|$ of K

   ▸ is the union of all points in all simplices of K,

   ▸ i.e, $|K| = \bigcup_{\sigma \in K} \{x \mid x \in \sigma\}$

$K = \{\text{simplices}\}$

$|K| = \{\text{points of simplices}\}$

$K \xrightarrow{\text{"assemble"}} |K|$

$\left(\begin{array}{c} \text{geometric} \\ \text{simp. cpx.} \end{array} \right)$ $\left(\begin{array}{c} \text{topological} \\ \text{space} \end{array} \right)$

- Geometric simplicial completes are nice for intuition / having a mental picture. But we are interested in topology

- Distinct geometrically but the same topologically (i.e., they are homeomorphic)

- A graph can be abstractly defined as $G = (V, E)$

Abstract simplicial complex

‣ An (abstract) p-simplex $\sigma = \{ v_0, v_1, \ldots, v_p \}$

   ‣ a set of cardinality $p + 1$

   ‣ A subset $\tau \subseteq \sigma$ is a face of σ

‣ An (abstract) simplicial complex $K = (V, \Sigma)$

   ‣ A vertex set V

   ‣ A collection Σ of simplices such that

     ‣ If $\sigma \in \Sigma$, then any fact $\tau \subseteq \sigma$ is also in Σ
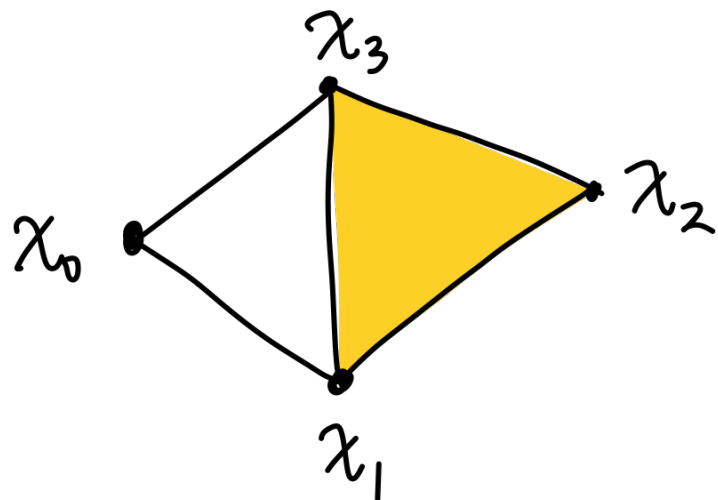
Different from geometric:

$$\left(\sigma = \left\{ \sum_{i=0}^{p} a_i v_i \;\middle|\; a_i \geq 0, \sum_{i=0}^{p} a_i = 1 \right\} \right)$$

Similar to geometric

$$\left(K = \{ simplices \} \right)$$

Abstract simplicial complex

$$V = \{x_0, x_1, x_2, x_3\}$$

$$\Sigma = \{ \{x_0\}, \{x_1\}, \{x_2\}, \{x_3\},$$
$$\{x_0,x_1\}, \{x_0,x_3\}, \{x_1,x_2\}, \{x_1,x_3\}, \{x_2,x_3\}$$
$$\{x_1,x_2,x_3\} \}$$

We may also use
- sequence of points, or
- indices

to represent simplices,

e.g: $x_1 x_2 x_3$ for $\{x_1, x_2, x_3\}$

12 for $\{x_1, x_2\}$

123 for $\{x_1, x_2, x_3\}$

Geometric realization

‣ **Geometric realization** of an abstract simplicial complex K

    ‣ Is a geometric simplicial complex S whose associated abstract simplicial complex $(V(S), \Sigma(S))$ is the "same" as $(V(K), \Sigma(K))$

> **Theorem**
> Any abstract simplicial complex K has a geometric realization. Any two geometric realizations of K have homeomorphic underlying spaces
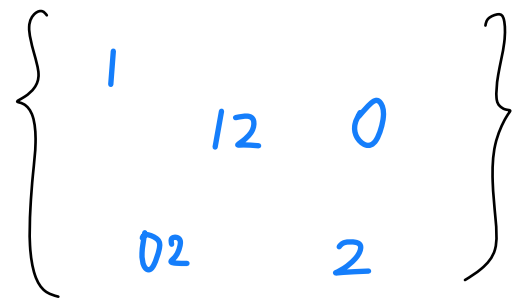
    ‣ We use $|K|$ to denote the underlying space of a geometric realization of K and call $|K|$ the underlying space of K.

Geometric realization

▸ **Geometric realization** of an abstract simplicial complex K
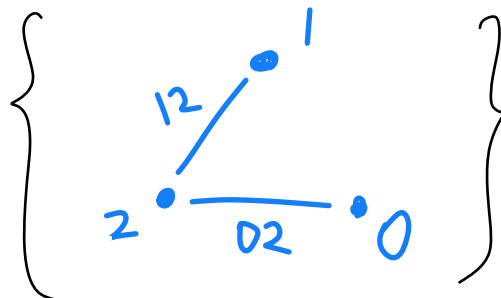
Simplices do not have geometric locations
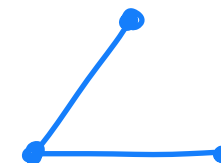
Simplices have geometric locations but not assembled

$$\left\{ \begin{array}{cc} 1 & \\ 12 & 0 \\ 02 & 2 \end{array} \right\}$$

$$K \xrightarrow[\text{realization}]{\text{geometric}} S \xrightarrow{\text{"assemble"}} |S| \overset{\text{def}}{=\!=\!=} |K|$$

$\begin{pmatrix} \text{abstract} \\ \text{simp. cpx.} \end{pmatrix}$

$\begin{pmatrix} \text{geometric} \\ \text{simp. cpx.} \end{pmatrix}$

$\begin{pmatrix} \text{topological} \\ \text{space} \end{pmatrix}$
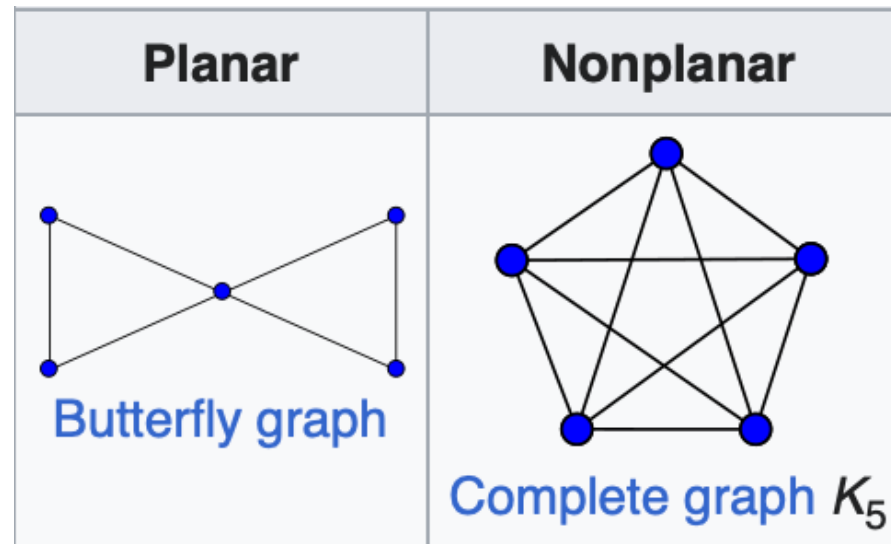
Geometric realization

> **Theorem**
>
> Any abstract simplicial complex K has a geometric realization. Any two geometric realizations of K are homeomorphic to each other.

‣ Attempt:

  ‣ For $V = \{v_0, \ldots, v_n\}$, embed V into $\mathbb{R}^{n+1}$ by $v_i \mapsto (0, \cdots, 1 \cdots, 0) = e_i$

  ‣ For each simplex $\sigma = \{v_{i_0}, \ldots, v_{i_k}\}$, add geometric simplex $cnx\{e_{i_0}, \ldots, e_{i_k}\}$ to the realization

‣ There may not work because simplices can overlap.

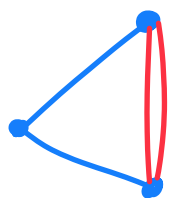# Geometric realization

‣ The recipe in the proof is not efficient in terms of ambient dimension

‣ Any finite d-dim abstract simplicial complex has a geometric realization in $\mathbb{R}^{2d+1}$ but may not have a geometric realization in $\mathbb{R}^{2d}$ or lower dimensions

   ‣ A graph (1-d simplicial complex) can be plotted in $\mathbb{R}^3$ but not necessarily in $\mathbb{R}^2$

Planar	Nonplanar
Butterfly graph	Complete graph K_5

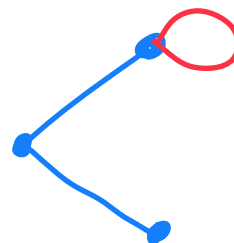# Graphs and Simplicial Complexes

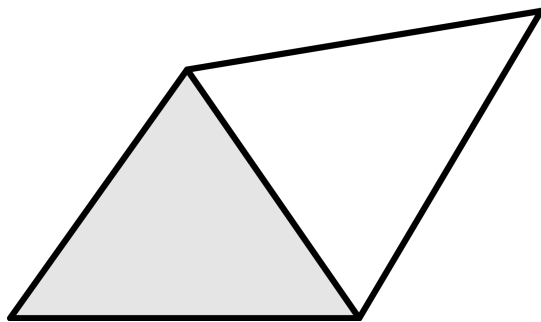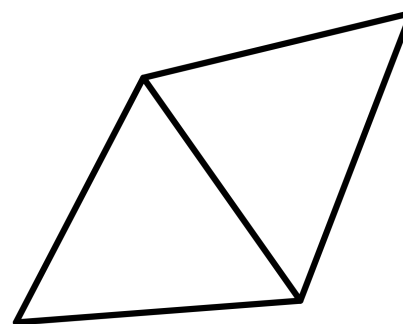▸ Any simple graph (without double edge and self-loop) is a simplicial complex

double edge

self loop

▸ The 1-skeleton of a simplicial complex is a graph (can be used as the definition of graph)
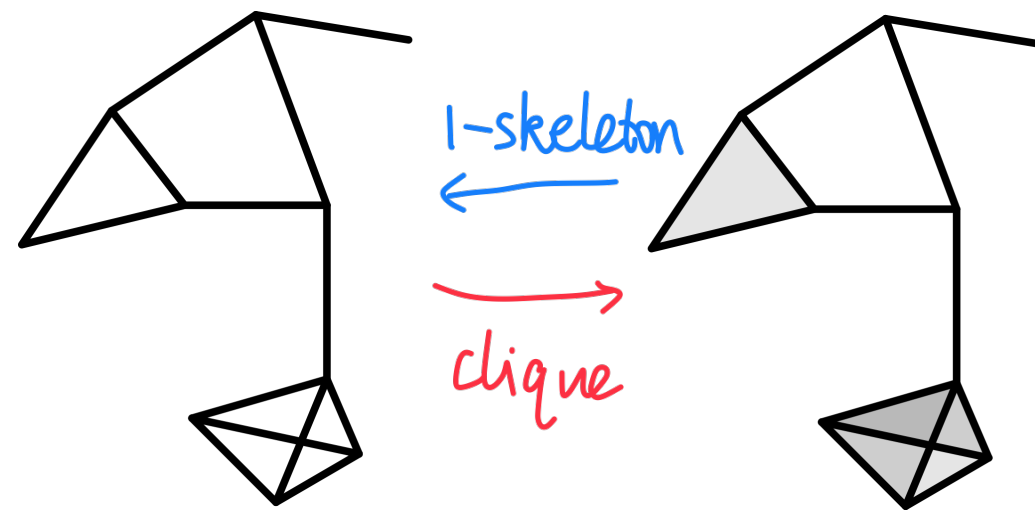
1-skeleton

Graphs and Simplicial Complexes

▸ **Clique complex** induced by a graph

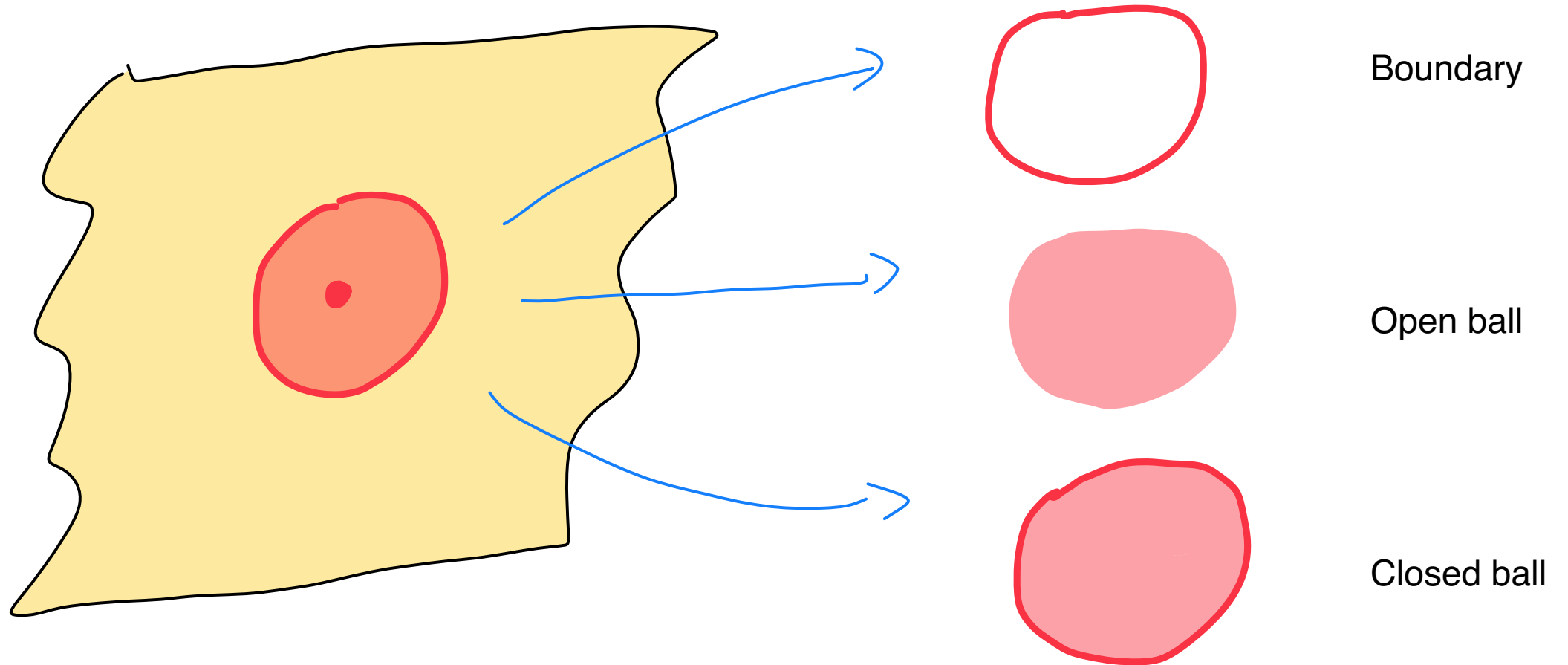Given a graph G with vertex set V & edge set E, its clique complex is a simplicial complex K

$$V(K) := V$$

$$\Sigma(K) := \left\{ \sigma = \{v_0, \cdots, v_p\} \mid v_i v_j \in E, \; \forall \; 0 \leq i < j \leq p \right\}$$

1-skeleton

clique

Some notions related to simplicial complexes

Star and links

Boundary

Open ball

Closed ball

Star and links

▸ Given a simplex $\tau \in K$

  ▸ Star: $St(\tau) = \{ \sigma \in K \mid \tau \subset \sigma \}$

    ▸ A star may not be a simplicial complex

  Closed star: $cl\,St(\tau) = \bigcup_{\sigma \in St(\tau)} \{ \sigma' \mid \sigma' \subset \sigma \}$

  ▸ Link: $Lk(\tau) = \{ \sigma \in cl\,St(\tau) \mid \sigma \cap \tau = \varnothing \}$

boundary of the ball

closed ball

open ball

K

$St(\tau)$

$cl\,St(\tau)$

$Lk(\tau)$