MATH412/COMPSCI434/MATH713
Fall 2025

Topological Data Analysis

Topic 12: Persistent Cohomology

Instructor: Ling Zhou

Cohomology

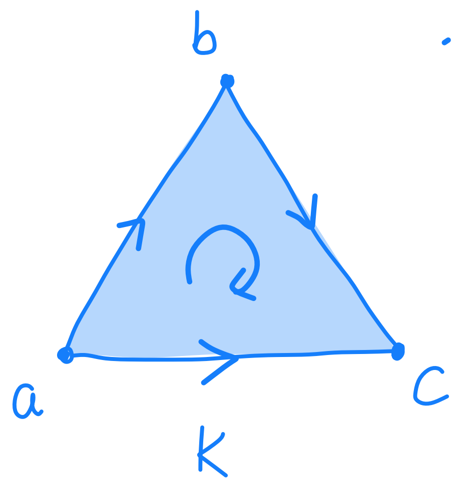
Review: Chain complex

Theorem (Fundamental Boundary Property):
$$\partial_p \circ \partial_{p+1} = 0$$

- A **p-chain** is a formal linear sum of p-simplices $c = \sum c_i \sigma_i$.
- The **p-th chain space** of K is the linear space of p-chains in K, denoted $C_p(K)$.
- The p-th **boundary operator** (*a linear map*) $\partial_p : C_p \to C_{p-1}$
  - For a simplex $\sigma = \{v_0, \ldots, v_p\}$, define
    - $\partial_p(\sigma) := \sum_{i=0}^{p} (-1)^i \{v_0, \ldots, \hat{v}_i, \ldots, v_p\}$
  - For a general chain $c = \sum_j c_j \sigma_j$, define $\partial_p(c) := \sum_j c_j \partial_p(\sigma_j)$
- **Chain complex:** $\cdots \xrightarrow{\partial_{p+2}} C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \xrightarrow{\partial_{p-1}} \cdots$.

Review: Chain complex

$$\cdots \xrightarrow{\partial_{p+2}} C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \xrightarrow{\partial_{p-1}} \cdots .$$



$$\begin{array}{ccccccccc}
\cdots \longrightarrow C_3 & \xrightarrow{\partial_3} & C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{\partial_0} & 0 \\
\| & & \| & & \| & & \| & & \\
0 & & \langle abc \rangle & & \langle ab, bc, ac \rangle & & \langle a, b, c \rangle & &
\end{array}$$

$$abc \longmapsto ab - ac + bc$$

$$ab \longmapsto b - a$$

$$bc \longmapsto c - b$$

$$ac \longmapsto c - a$$

- We define $C_{-1}(K) := 0$

Cochain complex

- A **p-cochain** is a linear map from $C_p(K)$ to the base field (e.g. R).
- The **p-th cochain space** is the linear space of p-cochains in K, denoted $C^p(K)$.
- The p-th **coboundary operator** (*a linear map*) $\delta^p : C^p(K) \to C^{p+1}(K)$ such that
  - $\delta^p(f)(\alpha_{p+1}) := f(\partial_{p+1}(\alpha_{p+1}))$ for any $f \in C^p(K), \alpha_{p+1} \in C_{p+1}(K)$.
- **Cochain complex:**

$$0 \longrightarrow C^0(X) \xrightarrow{\delta^0} C^1(X) \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{p-1}} C^p(K) \xrightarrow{\delta^p} C^{p+1}(K) \to \cdots$$

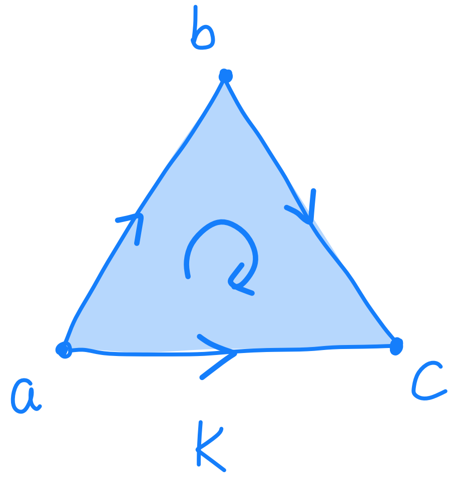
$$C^0(X) = \{C_0(X) \to \mathbb{F}\} \qquad C^1(X) = \{C_1(X) \to \mathbb{F}\} \qquad C^p(K) = \{C_p(X) \to \mathbb{F}\} \qquad C^{p+1}(K) = \{C_{p+1}(X) \to \mathbb{F}\}$$

Cosimplex: basis of cochain space

- A **p-cosimplex** is the dual of a simplex σ, i.e., the linear map $\sigma^* : C_p(K) \to \mathbb{F}$ such that

$$\sigma^*(\tau_p) := \begin{cases} 1, & \text{if } p\text{-simplex } \tau_p = \sigma \\ 0, & \text{if } p\text{-simplex } \tau_p \neq \sigma \end{cases}$$

Example



ab is a 1-simplex

$$(ab)^* : C_1(K) \longrightarrow \mathbb{F}$$

$$ab \longmapsto 1$$

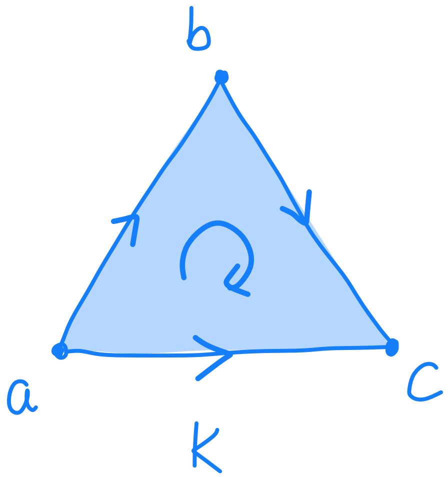
$$bc \longmapsto 0$$

$$ac \longmapsto 0$$

- The set of p-cosimplices form a basis of the p-cochain space

Coboundary map

- The p-th **coboundary operator** (*a linear map*) $\delta^p : C^p(K) \to C^{p+1}(K)$ such that
  - $\delta^p(f)(\alpha_{p+1}) := f(\partial_{p+1}(\alpha_{p+1}))$ for any $f \in C^p(K)$, $\alpha_{p+1} \in C_{p+1}(K)$.



$$\delta^1 : C^1(K) \longrightarrow C^2(K)$$

$$C^1(K) = \langle ab^*, bc^*, ac^* \rangle \qquad C^2(K) = \langle abc^* \rangle$$

$$C_2(K) = \langle abc \rangle$$

$$\delta^1(ab^*) : C_2(K) \longrightarrow \mathbb{F}$$

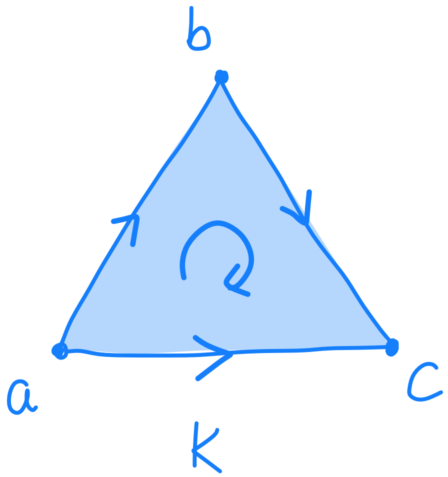
$$abc \longmapsto \delta^1(ab^*)(abc)$$

$$\delta^1(ab^*)(abc) = ab^*(\partial_2(abc)) = ab^*(bc - ac + ab) = 0 - 0 + 1 = 1$$

$$\Longrightarrow \delta^1(ab^*) = abc^*$$

Coboundary map

- The p-th **coboundary operator** (*a linear map*) $\delta^p : C^p(K) \to C^{p+1}(K)$ such that
  - $\delta^p(f)(\alpha_{p+1}) := f(\partial_{p+1}(\alpha_{p+1}))$ for any $f \in C^p(K)$, $\alpha_{p+1} \in C_{p+1}(K)$.



$$\delta^1 : C^1(K) \longrightarrow C^2(K)$$

$$C^1(K) = \langle ab^*, bc^*, ac^* \rangle \qquad C^2(K) = \langle abc^* \rangle$$

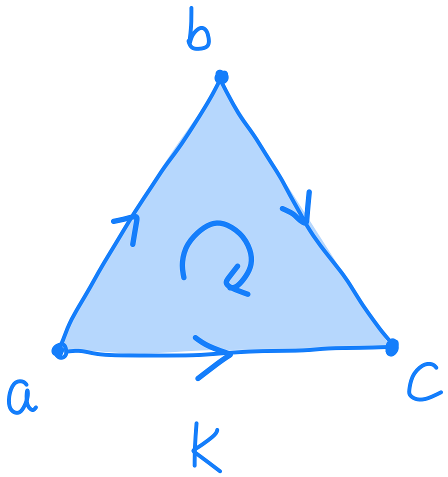
$$ab^* \longmapsto \delta^1(ab^*) = abc^*$$

$$bc^* \longmapsto -abc^*$$

$$ac^* \longmapsto abc^*$$

Coboundary map

- The p-th **coboundary operator** (*a linear map*) $\delta^p : C^p(K) \to C^{p+1}(K)$ such that
  - $\delta^p(f)(\alpha_{p+1}) := f(\partial_{p+1}(\alpha_{p+1}))$ for any $f \in C^p(K)$, $\alpha_{p+1} \in C_{p+1}(K)$.



$$\begin{array}{ccc} \delta^0 : C^0(K) & \longrightarrow & C^1(K) \\ \| & & \| \\ \langle a^*, b^*, c^* \rangle & & \langle ab^*, bc^*, ac^* \rangle \end{array}$$

Coboundary map

- The p-th **coboundary operator** (*a linear map*) $\delta^p : C^p(K) \to C^{p+1}(K)$ such that
  - $\delta^p(f)(\alpha_{p+1}) := f(\partial_{p+1}(\alpha_{p+1}))$ for any $f \in C^p(K), \alpha_{p+1} \in C_{p+1}(K)$.
- For a p-cosimplex σ^* (with σ a p-simplex), we have

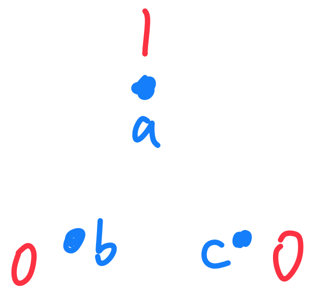
$$\Big(\underbrace{\delta^p(\sigma_p^*)}_{\text{is a map on } C_{p+1}(K)}\Big)(\tau_{p+1}) = \begin{cases} (-1)^j, & \text{if } \underline{\sigma_p \text{ is the j-th face of } \tau_{p+1}} \\ 0, & \text{otherwise} \end{cases}$$

[σ_p is obtained from τ_{p+1} by removing the j-th vertex.]

$$\Rightarrow \delta^p(\sigma_p^*) = \sum_{\sigma_p \text{ is j-th face of } \tau_{p+1}} (-1)^j \tau_{p+1}^*$$

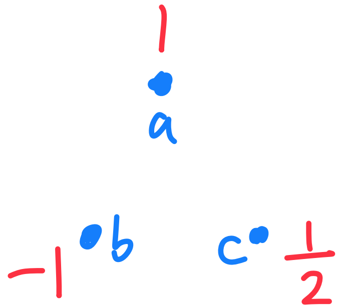
Coboundary map

Visual representation of cosimplex



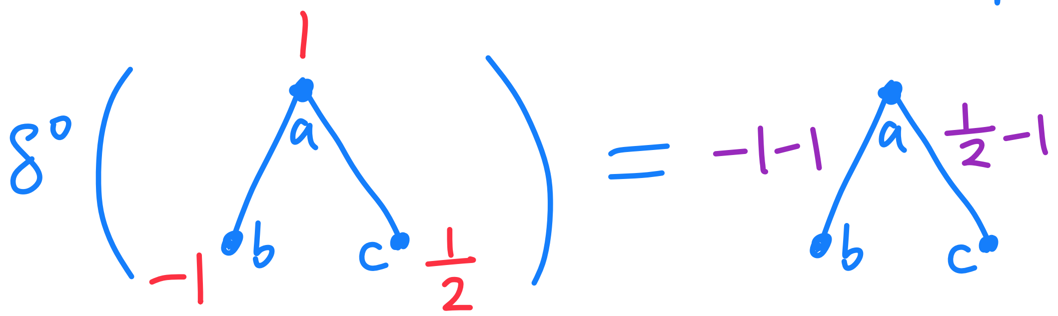
a^*

Visual representation of cochain



$a^* - b^* + \frac{1}{2} c^*$

Understand coboundary map



$f := a^* - b^* + \frac{1}{2} c^*$

(Assign value to each edge ij as $f(j) - f(i)$

$$\delta^p(f)(\alpha_{p+1}) := f(\partial_{p+1}(\alpha_{p+1}))$$
)

Cocycle, coboundary and cohomology

$$0 \longrightarrow C^0(X) \xrightarrow{\delta^0} C^1(X) \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{p-1}} C^p(K) \xrightarrow{\delta^p} C^{p+1}(K) \rightarrow \cdots$$

Theorem (Fundamental Coboundary Property):

$$\delta^p \circ \delta^{p-1} = 0$$

- The p-th **cocycle group** $Z^p(K)$ is defined as $\ker(\delta^p) \subset C^p(K)$.
- The p-th **coboundary group** $B^p(K)$ is defined as $\mathrm{im}(\delta^{p-1}) \subset C^p(K)$.
- The p-th **cohomology group** $H^p(K)$ is defined as $\ker(\delta^p)/\mathrm{im}(\delta^{p-1})$.

Cohomology and homology

(A_{p+1}: matrix representation of ∂_{p+1})

$$\overbrace{\boxed{\quad A_{p+1} \quad}}^{(p+1)\text{-simplices}} \Big\}\ p\text{-simplices}$$

$$\begin{tikzcd}
0 & C_0(X) \arrow[l] & C_1(X) \arrow[l, "\partial_1"'] & \cdots \arrow[l, "\partial_2"'] & & C_p(K) \arrow[ll, "\partial_p"'] \arrow[d] & C_{p+1}(K) \arrow[l, "\partial_{p+1}"'] \arrow[d] & \cdots \arrow[l] \\
0 \arrow[r] & C^0(X) \arrow[r, "\delta^0"] & C^1(X) \arrow[r, "\delta^1"] & \cdots \arrow[rr, "\delta^{p-1}"] & & C^p(K) \arrow[r, "\delta^p"] & C^{p+1}(K) \arrow[r] & \cdots
\end{tikzcd}$$

$$\sigma \mapsto \sigma^*$$

Fact: matrix representation of δ^p is A_{p+1}^T

$$\boxed{\quad A_{p+1}^T \quad}$$

Cohomology and homology

Recall: $\mathrm{Nul}(A) = \mathrm{Col}(A^T)^\perp$

$$\boxed{A_{p+1}}$$

$$\begin{tikzcd} C_p(K) \arrow[d] & C_{p+1}(K) \arrow[l, "\partial_{p+1}"'] \arrow[d] \\ C^p(K) \arrow[r, "\delta^p"] & C^{p+1}(K) \end{tikzcd}$$

$$\sigma \mapsto \sigma^*$$

$$\boxed{A_{p+1}^T}$$

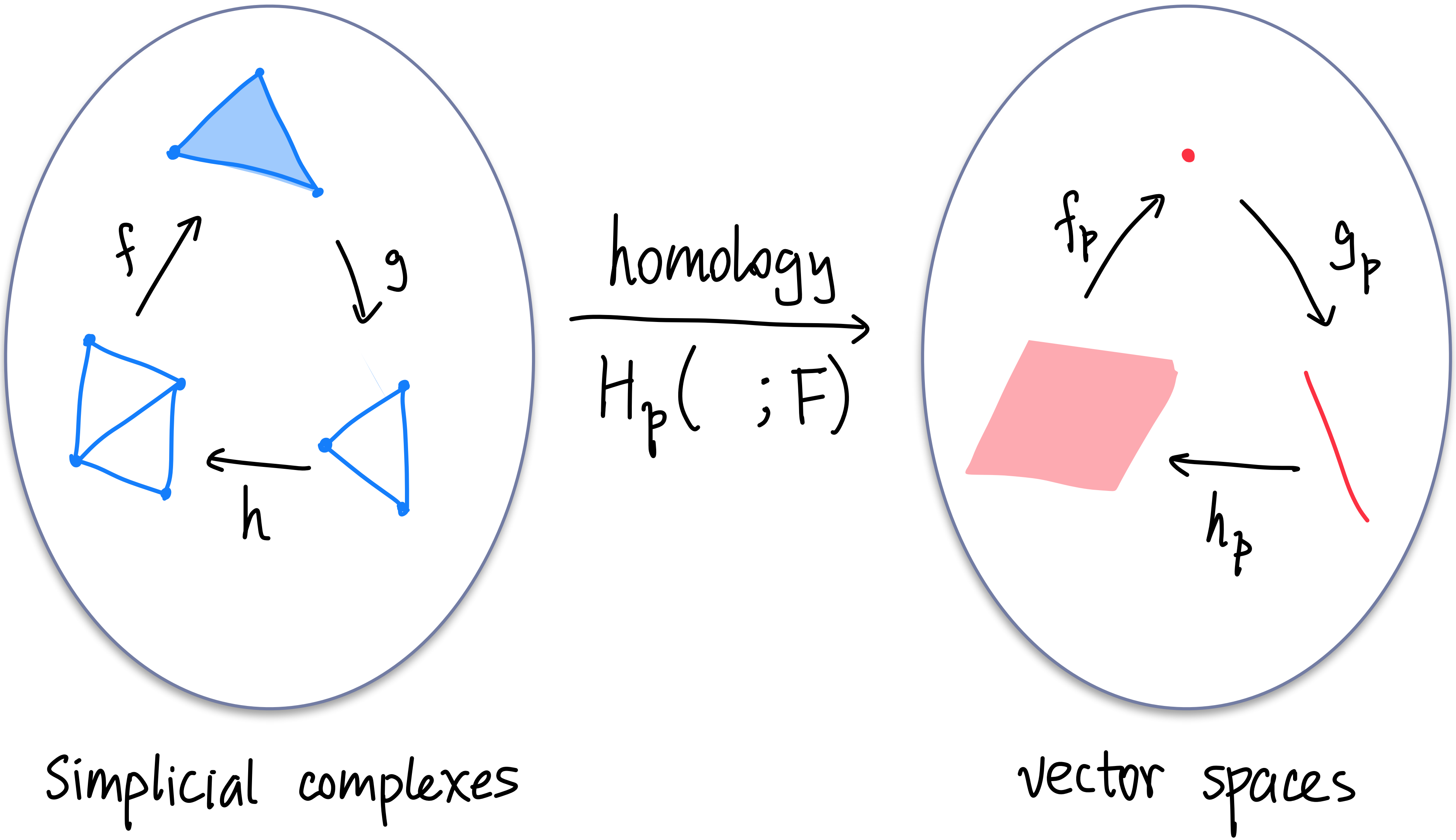
(1) $$H_p(K) = \ker(\partial_p) / \mathrm{im}(\partial_{p+1}) \cong \ker(A_p) / \mathrm{im}(A_{p+1})$$

(2) $$H^p(K) = \ker(\delta^p) / \mathrm{im}(\delta^{p-1}) \cong \ker(A_{p+1}^T) / \mathrm{im}(A_p^T)$$

(3) $$\begin{aligned} \dim H^p(K) &= \dim\left(\mathrm{Col}(A_{p+1})^\perp / \mathrm{Nul}(A_p)^\perp \right) \\ &= (n_p - \mathrm{rk}(A_{p+1})) - (n_p - \mathrm{nul}(A_p)) \\ &= \mathrm{nul}(A_p) - \mathrm{rk}(A_{p+1}) \\ &= \dim H_p(K) \end{aligned}$$

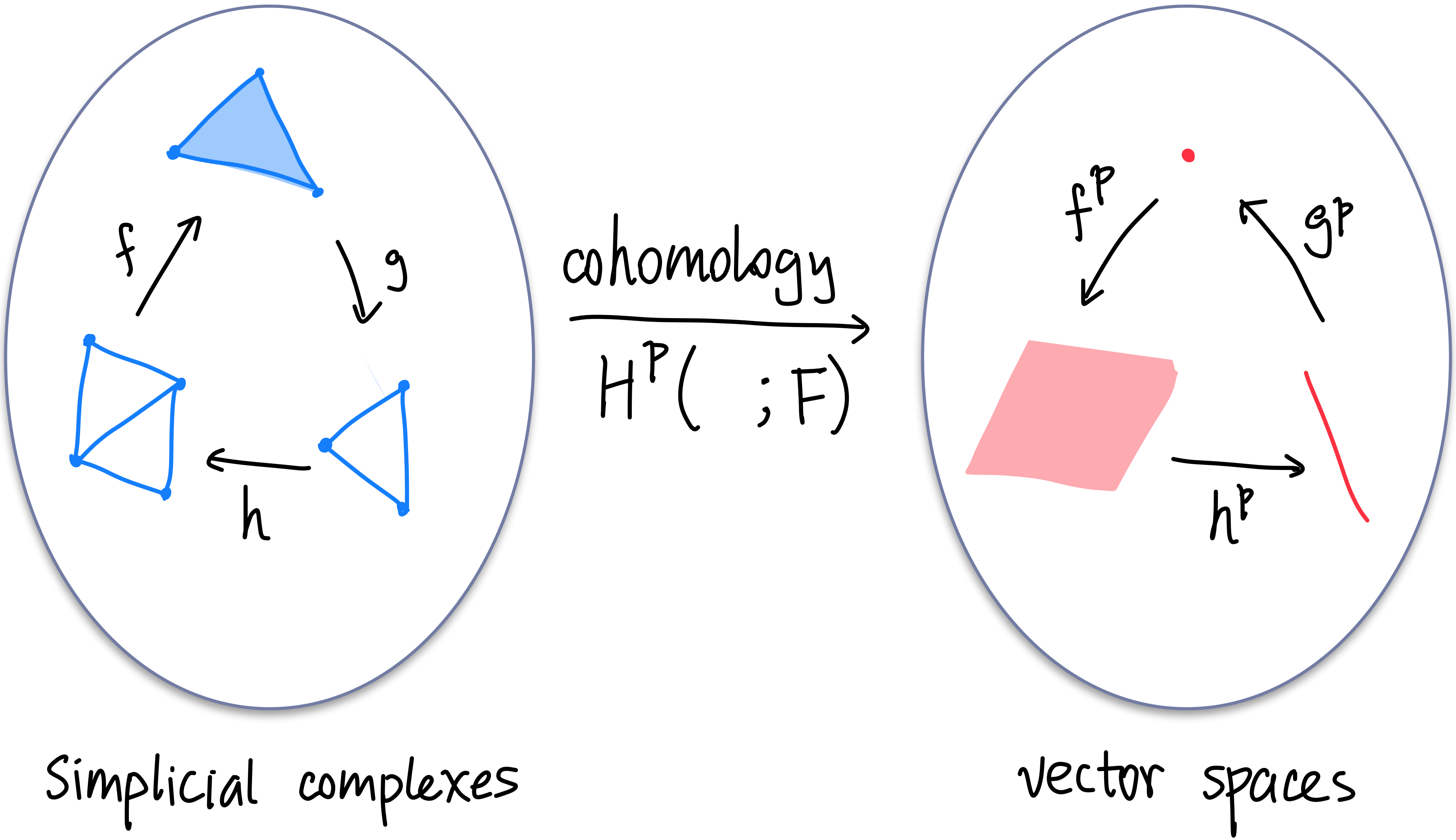
Persistent Cohomology

Recall: Functoriality of homology



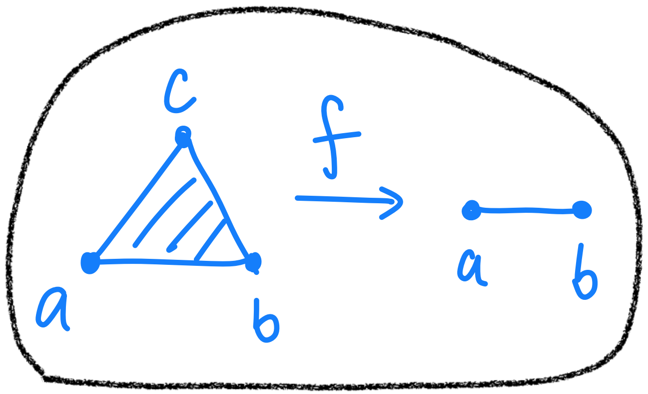
New: Functoriality of cohomology

Directions are reversed!



Construction of f^p

- First, f induces linear maps on chain spaces $\bar{f}^p : C^p(K') \to C^p(K)$
  - given by the restriction of linear maps
- Then, $\bar{f}^p : C^p(K') \to C^p(K)$ induces $f^p : H^p(K') \to H^p(K)$
  - $f^p([c]) := [\bar{f}^p(c)]$

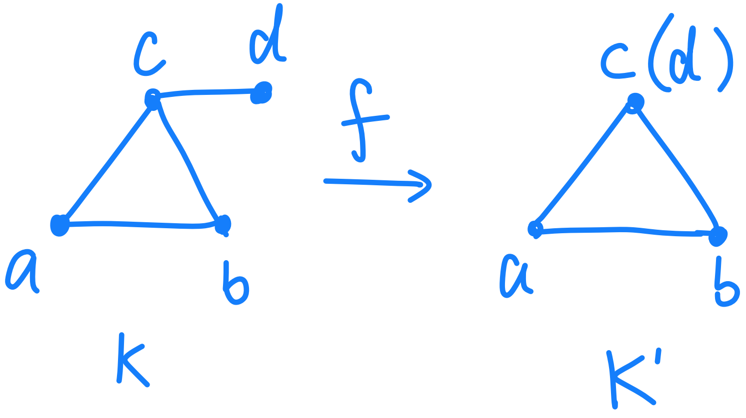


$$\begin{tikzcd}
\cdots \arrow[r] & C^{p-1}(K) \arrow[r] & C^p(K) \arrow[r] & \cdots \\
\cdots \arrow[r] & C^{p-1}(K') \arrow[r] \arrow[u] & C^p(K') \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

$f^p : H^p(K') \to H^p(K)$

for every p

Construction of f^p



$$\begin{tikzcd}
 & & \langle ab^*, ac^*, bc^*, cd^* \rangle \arrow[d, phantom, "\parallel"] & \\
\cdots \arrow[r] & C^0(K) \arrow[r] & C^1(K) \arrow[r] & \cdots \\
\cdots \arrow[r] & C^0(K') \arrow[r] \arrow[u] & C^1(K') \arrow[r] \arrow[u] \arrow[d, phantom, "\parallel"] & \cdots \\
 & & \langle ab^*, ac^*, bc^* \rangle &
\end{tikzcd}$$

$$\begin{aligned} f^1 : H^1(K') &\longrightarrow H^1(K) \\ [\alpha] &\longmapsto [\bar{f}^1(\alpha)] \end{aligned}$$

e.g: for $\alpha = ab^*$

$$\Rightarrow f^1([\alpha]) = [\bar{f}^1(\alpha)] = [ab^*]$$

$\bar{f}^0 : a^* \mapsto a^*,\ b^* \mapsto b^*,\ c^* \mapsto c^*,\ d^* \mapsto c^*$

$\bar{f}^1 : ab^* \mapsto ab^*,\ ac^* \mapsto ac^*,\ bc^* \mapsto bc^*$
$\quad cd^* \mapsto 0$

Persistent Cohomology

- $\imath : K \hookrightarrow K' \implies \imath^p : H^p(K') \to H^p(K)$
  - Simplicial maps (e.g. the above inclusion) induce homomorphisms in cohomology groups (under Z_2-coefficients, linear maps in vector spaces)
- Let $K_\bullet : K_0 \hookrightarrow K_1 \hookrightarrow \ldots \hookrightarrow K_n$ be a **simplicial filtration**, i.e, a sequence of simplicial complexes connected by inclusions $\imath^{i,i+1} : K_i \hookrightarrow K_{i+1}$.
- **Persistent cohomology** of the filtration $K_\bullet$ is

$$H^p(K_0) \leftarrow H^p(K_1) \leftarrow H^p(K_2) \leftarrow \cdots \leftarrow H^p(K_n)$$

(Compare with persistent homology

$$H_p(K_0) \longrightarrow H_p(K_1) \longrightarrow H_p(K_2) \longrightarrow \cdots \longrightarrow H_p(K_n)$$)

Advantage of Persistent Cohomology 1

Dualities in persistent (co)homology
Vin de Silva, Dmitriy Morozov and Mikael Vejdemo-Johansson
Published 10 November 2011 • 2011 IOP Publishing Ltd
Inverse Problems, Volume 27, Number 12
Citation Vin de Silva *et al* 2011 *Inverse Problems* **27** 124003
DOI 10.1088/0266-5611/27/12/124003

- Persistent cohomology and persistent homology give the same barcodes
- But the former one is more efficient to compute.
  - Ripser using a 'clearing' technique to make the matrix reduction more efficient
  - 'Clearing' essentially allows one to skip certain columns in the reduction
  - Coboundary matrix reduction benefits from clearing more than the boundary matrix

Ripser: efficient computation of Vietoris–Rips persistence barcodes

Ulrich Bauer[1]

Received: 11 August 2019 / Revised: 25 February 2021 / Accepted: 14 April 2021 /
Published online: 17 June 2021

Advantage of Persistent Cohomology 1

- standard matrix reduction:

$$\sum_{d=1}^{k+1} \underbrace{\binom{n}{d+1}}_{\dim C_d(K)} = \sum_{d=1}^{k+1} \underbrace{\binom{n-1}{d}}_{\dim B_{d-1}(K)} + \sum_{d=1}^{k+1} \underbrace{\binom{n-1}{d+1}}_{\dim Z_d(K)}$$

$k = 2, n = 192$: $\quad 56\,050\,096 = 1\,161\,471 + 54\,888\,625$

- using clearing:

$$\sum_{d=1}^{k+1} \underbrace{\binom{n-1}{d}}_{\dim B_{d-1}(K)} + \underbrace{\binom{n-1}{k+2}}_{\dim H_{k+1}(K)} = \sum_{d=1}^{k+2} \binom{n-1}{d}$$

$k = 2, n = 192$: $\quad 54\,888\,816 = 1\,161\,471 + 53\,727\,345$

- standard matrix reduction:

$$\sum_{d=0}^{k} \underbrace{\binom{n}{d+1}}_{\dim C^d(K)} = \sum_{d=0}^{k} \underbrace{\binom{n-1}{d+1}}_{\dim B^{d+1}(K)} + \sum_{d=0}^{k} \underbrace{\binom{n-1}{d}}_{\dim Z^d(K)}$$

$k = 2, n = 192$: $\quad 1\,179\,808 = 18\,337 + 1\,161\,471$

- using clearing:

$$\sum_{d=0}^{k} \underbrace{\binom{n-1}{d+1}}_{\dim B^{d+1}(K)} + \underbrace{\binom{n-1}{0}}_{\dim H^0(K)} + = \sum_{d=0}^{k+1} \binom{n-1}{d}$$

$k = 2, n = 192$: $\quad 1\,161\,472 = 1 + 1\,161\,471$

The above is taken from Ulrich Bauer's talk: https://github.com/ubauer/ubauer.github.io/blob/master/ripser-talk.pdf

Advantage of Persistent Cohomology 2

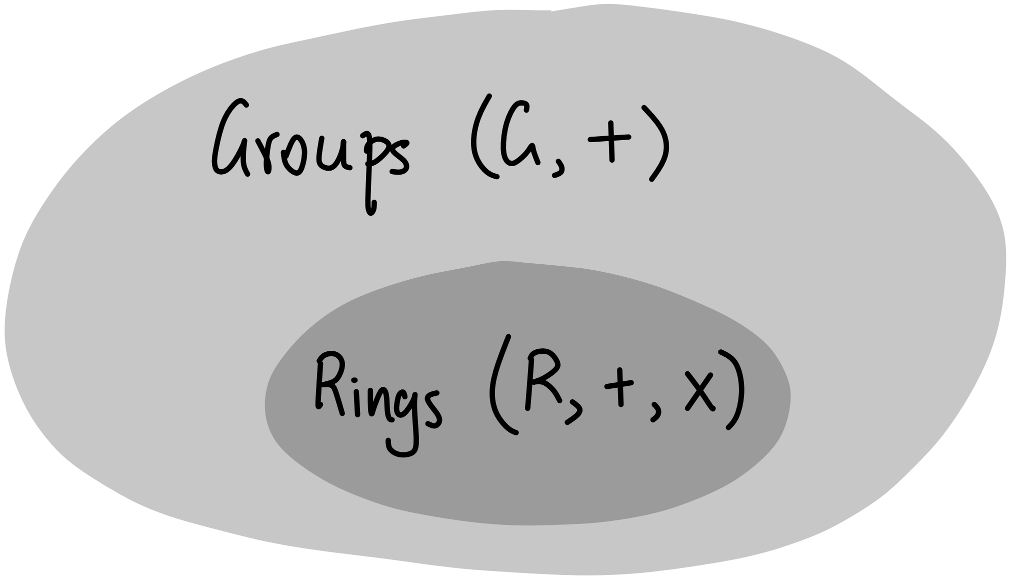
- Despite having the same barcodes, persistent cohomology can be enriched with a multiplicative structure, and thus gives rise to a richer invariant

Persistent Cohomology Ring

Review of algebraic tools - Group and Ring

- A **group** is a tuple $(G, +)$ where G is a set and $+ : G \times G \to G$ is a binary operation
  - (Associativity) $a + (b + c) = (a + b) + c$
  - (Identity) There exist $0 \in G$ such that $a + 0 = 0 + a = a$
  - (Inverse) For any $a \in G$, there exist $-a \in G$ such that $a + (-a) = 0$
- A **ring** is a tuple $(F, +, \times)$ where $(F, +)$ is an abelian group and $\times : F \times F \to F$ is another binary operation such that
  - (Associativity) $a \times (b \times c) = (a \times b) \times c$
  - (Multiplicative identity)There exist 1 in F such that $a \times 1 = a$
  - (Distributivity) $a \times (b + c) = (a \times b) + (a \times c)$

Review of algebraic tools - Group and Ring



- Examples:
  - $(\mathbb{Z}, +, \times)$ is a commutative ring
  - $(\mathbb{R}, +, \times)$ is a commutative ring

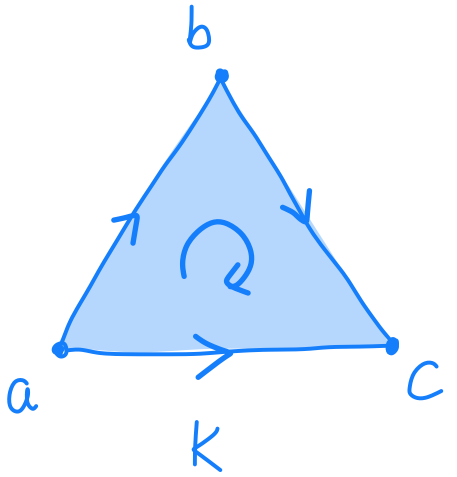
Cup product on cochains

Definition

Given a p_1-cochain σ_1 and a p_2-cochain σ_2, the cup product $\sigma_1 \smile \sigma_2$ is a linear map from $C_{p_1+p_2}(X)$ to $\mathbb{F}$ such that

$$(\sigma_1 \smile \sigma_2)\left(\left[\underbrace{v_{i_0}, \ldots, v_{i_{p_1+p_2}}}_{(p_1+p_2)\text{-simplex } \tau}\right]\right) := \sigma_1\left(\left[\underbrace{v_{i_0}, \ldots, v_{i_{p_1}}}_{\text{front } p_1\text{-face of } \tau}\right]\right) \cdot \sigma_2\left(\left[\underbrace{v_{i_{p_1}}, \ldots, v_{i_{p_1+p_2}}}_{\text{back } p_2\text{-face of } \tau}\right]\right).$$

Example



$$ab^* \smile bc^* : \underset{\langle abc \rangle}{\overset{\parallel}{C_2(X)}} \longrightarrow \mathbb{F}$$

Cup product on cohomology classes

- Leibniz rule: $\delta(\varphi \smile \psi) = (\delta\varphi) \smile \psi + (-1)^p \varphi \smile (\delta\psi)$.

φ, ψ are cocycles $\Rightarrow \delta(\varphi \smile \psi) = 0 \Rightarrow \varphi \smile \psi$ is a cocycle.

φ, ψ are coboundaries $\Rightarrow \varphi \smile \psi = (\delta f) \smile \psi$
(assume $\varphi = \delta f$)
$= \delta(f \smile \psi) - (-1)^p f \smile \delta\psi$
$= \delta(f \smile \psi)$

- Cup product induces an operation on cohomology classes:

$$[\varphi] \smile [\psi] := [\varphi \smile \psi].$$

- The **cohomology ring is $(H^*(X), +, \smile)$.**

Cohomology ring

- The cohomology ring $(H^*(X), +, \smile)$ is a **graded commutative ring**.

$$H^*(X) = \bigoplus_{p \geq 0} H^p(X)$$

with the cup product has these properties:

$$[\varphi] \smile [\psi] = [\varphi \smile \psi] \qquad \text{(definition)}$$

$$([\alpha] + [\beta]) \smile [\gamma] = [\alpha] \smile [\gamma] + [\beta] \smile [\gamma] \qquad \text{(bilinearity)}$$

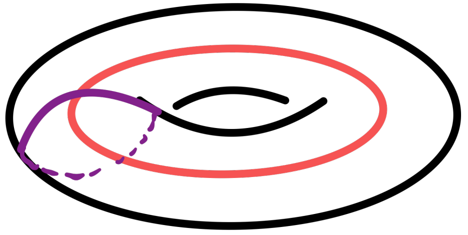
$$([\varphi] \smile [\psi]) \smile [\theta] = [\varphi] \smile ([\psi] \smile [\theta]) \qquad \text{(associativity)}$$

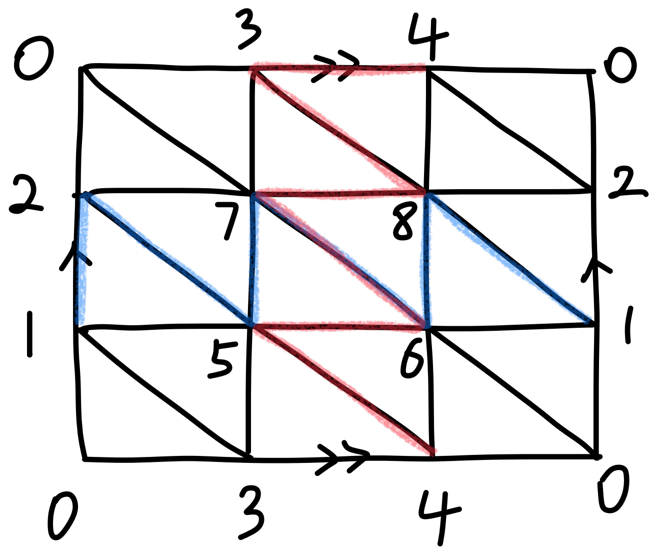
$$[\varphi] \smile [\psi] = (-1)^{pq} [\psi] \smile [\varphi] \qquad \text{(graded commutativity)},$$

where $p = \deg([\varphi])$ and $q = \deg([\psi])$.

Example

Note: $01+12+02$ is a 1-cycle, but
$01^*+12^*+02^*$ is NOT a 1-cocycle





$$\alpha = 34^* + 38^* + 78^* + 67^* + 56^* + 45^*$$

$$\beta = 12^* + 25^* + 57^* + 67^* + 68^* + 18^*$$

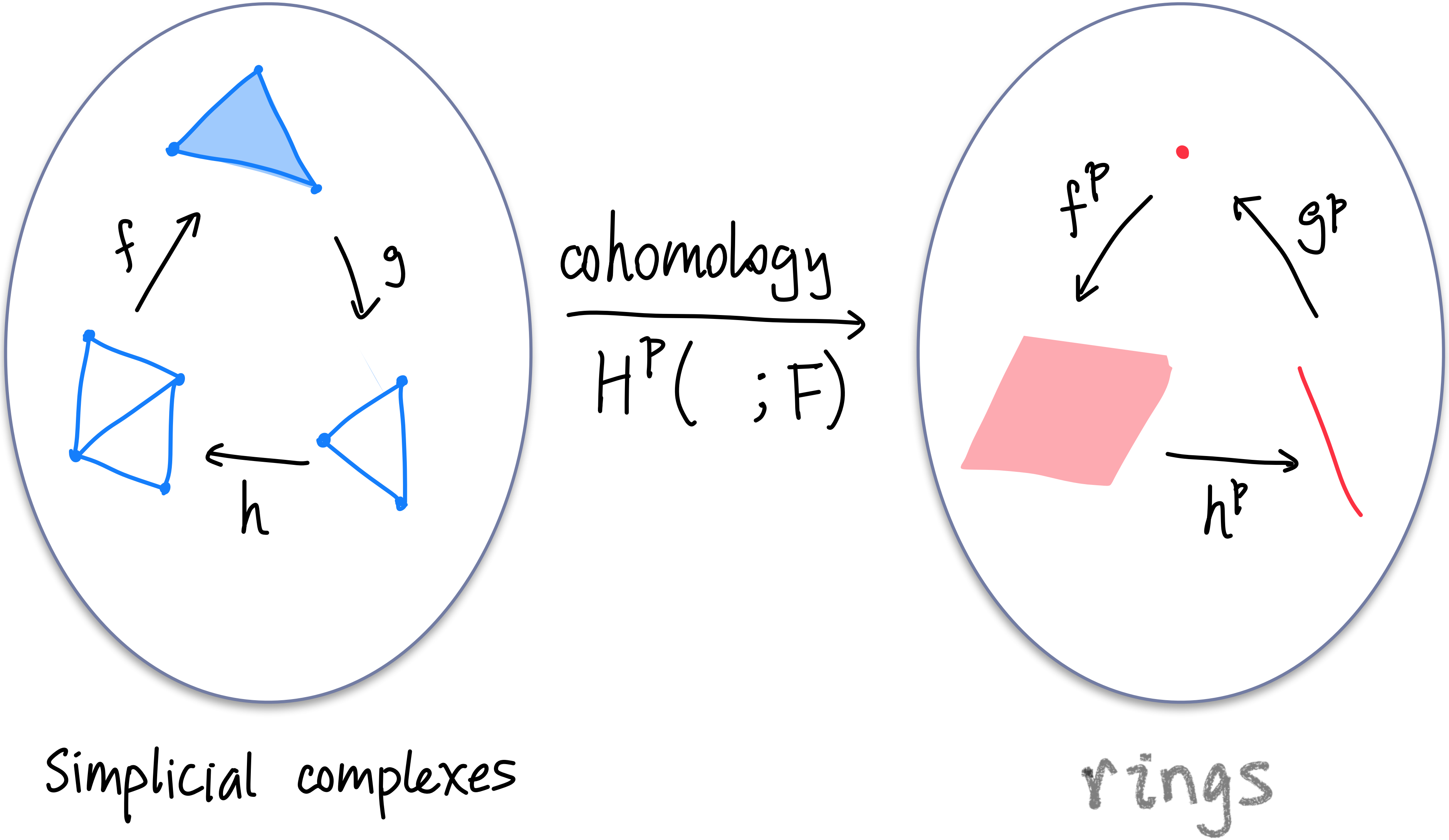
$$\alpha \smile \beta = 56^* \smile 67^* + 56^* \smile 68^* + 45^* \smile 57^*$$
$$= 567^* + 568^* + 457^*$$

$$H^*(T^2; \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha, \beta] \big/ \langle \alpha^2, \beta^2 \rangle$$

(elements in the ring are like
$\lambda\alpha + \mu\beta + \nu\alpha\beta$,
for $\lambda, \mu, \nu \in \mathbb{Z}_2$)

Functoriality of cohomology **ring**

Directions are reversed!

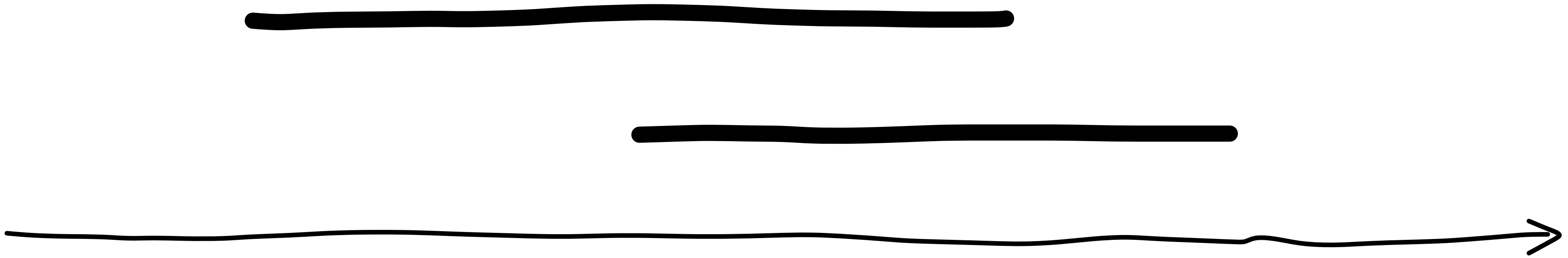


Persistent cohomology ring

- Let $K_\bullet : K_0 \hookrightarrow K_1 \hookrightarrow \dots \hookrightarrow K_n$ be a **simplicial filtration.**
- **Persistent cohomology ring** of the filtration $K_\bullet$ is

$$H^*(K_0) \leftarrow H^*(K_1) \leftarrow H^*(K_2) \leftarrow \cdots \leftarrow H^*(K_n)$$

$$\mathrm{Barc}\Big(\bigoplus_{p>0} H^p(K_\bullet)\Big)$$



Persistent cup product

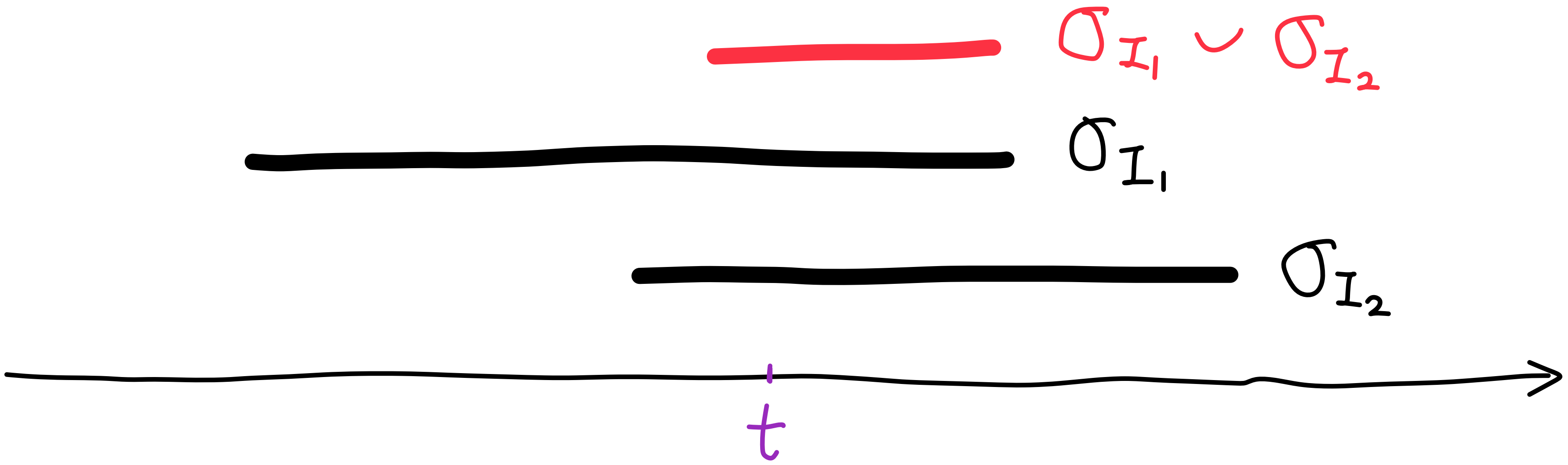
Persistent Cup-Length

Authors Marco Contessoto, Facundo Mémoli, Anastasios Stefanou, Ling Zhou

- Let $\mathscr{B}^+(K_\bullet)$ be the barcode of the filtration in positive degrees
- Let $\sigma := \{\sigma_I \mid I \in \mathscr{B}^+(K_\bullet)\}$ be a set of representative cocycles

Define the support of $\sigma_{I_1} \smile \cdots \smile \sigma_{I_\ell}$ to be

$$\mathrm{supp}(\sigma_{I_1} \smile \cdots \smile \sigma_{I_\ell}) := \{t \in \mathbb{R} \mid [\sigma_{I_1}|_{K_t} \smile \cdots \smile \sigma_{I_\ell}|_{K_t}] \neq 0\}$$



Persistent cup product

Persistent Cup-Length

Authors Marco Contessoto, Facundo Mémoli, Anastasios Stefanou, Ling Zhou

- Let $\mathscr{B}^+(K_\bullet)$ be the barcode of the filtration in positive degrees
- Let $\sigma := \{\sigma_I \mid I \in \mathscr{B}^+(K_\bullet)\}$ be a set of representative cocycles

Define the support of $\sigma_{I_1} \smile \cdots \smile \sigma_{I_\ell}$ to be

$$\operatorname{supp}(\sigma_{I_1} \smile \cdots \smile \sigma_{I_\ell}) := \{t \in \mathbb{R} \mid [\sigma_{I_1}|_{K_t} \smile \cdots \smile \sigma_{I_\ell}|_{K_t}] \neq 0\}$$

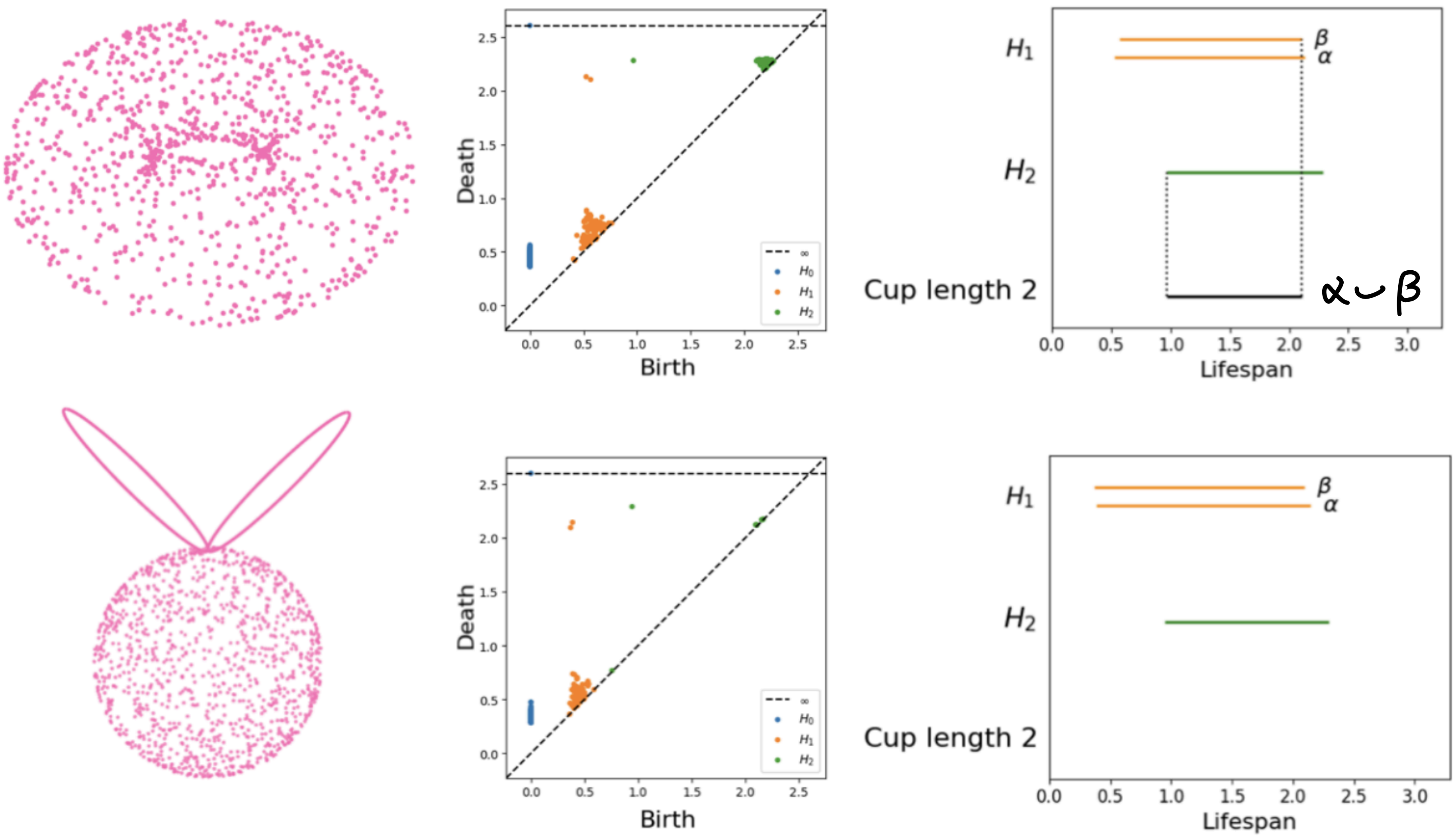
- Call $\{(\sigma_I, I) \mid I \in \mathscr{B}^+(K_\bullet)\}$ the annotated barcode and each element in it an annotated bar
- The **persistent cup product** of annotated bars is a new annotated bar:

$$(\sigma_{I_1} \smile \cdots \smile \sigma_{I_\ell},\ \operatorname{supp}(\sigma_{I_1} \smile \cdots \smile \sigma_{I_\ell}))$$

Persistent cup product

Doughnut or Mickey Mouse? Detecting Toroidal Structure in Data through Persistent Cup-Length

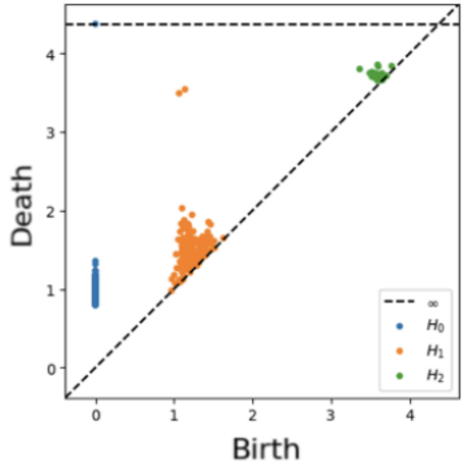
Ekaterina S. Ivshina, Galit Anikeeva, Ling Zhou

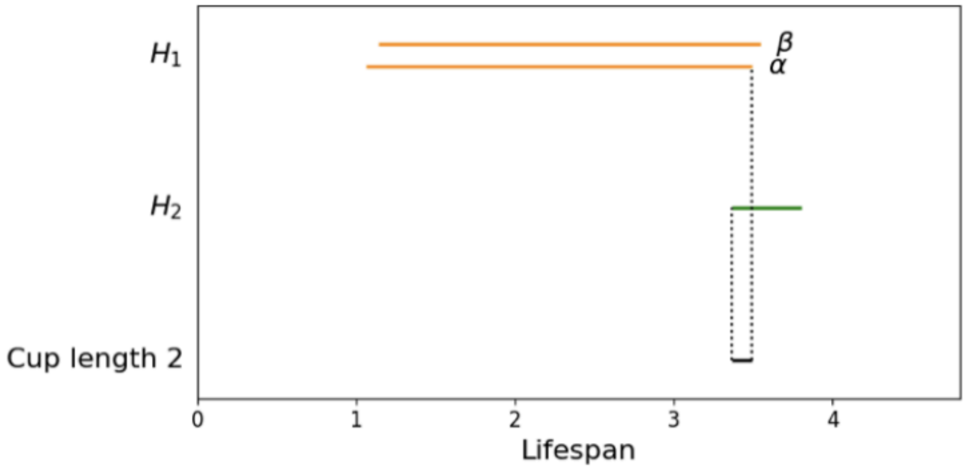


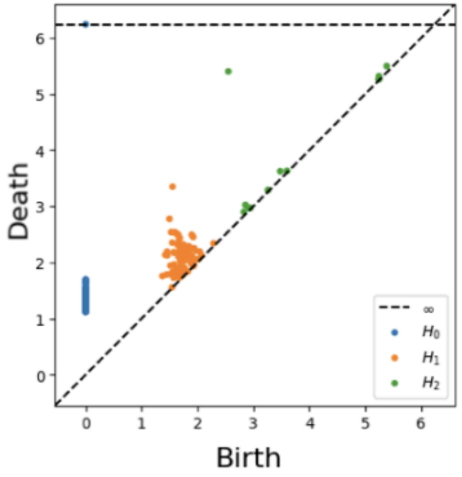
Persistent cup product

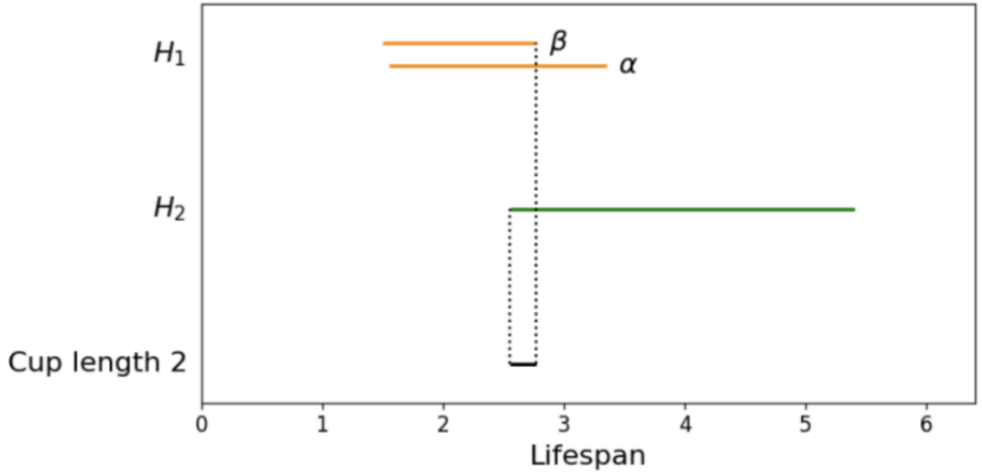
Doughnut or Mickey Mouse? Detecting Toroidal Structure in Data through Persistent Cup-Length

Ekaterina S. Ivshina, Galit Anikeeva, Ling Zhou









Applications of Persistent Cohomology Ring

Neuroscience background

Brain GPS system

Place cells

Grid cells

1971

2005

The 2014 Nobel Prize in Physiology or Medicine:

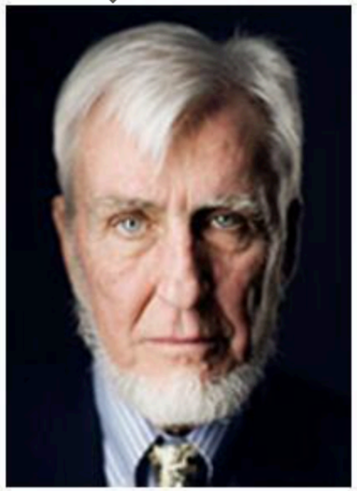


Photo: A. Mahmoud
John O'Keefe
Prize share: 1/2

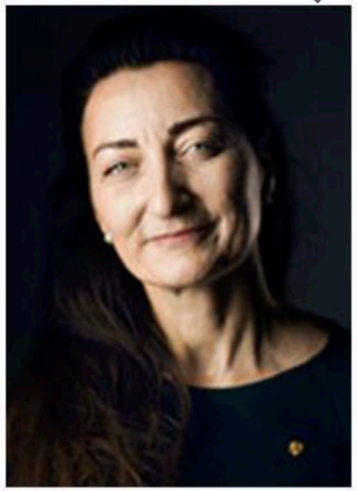


Photo: A. Mahmoud
May-Britt Moser
Prize share: 1/4

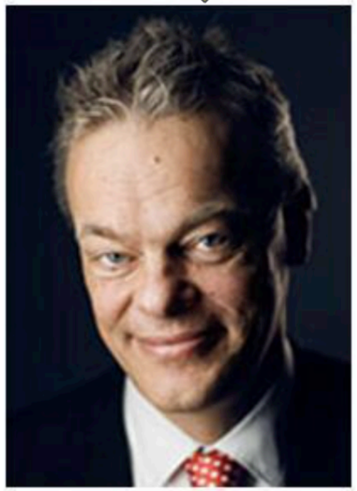


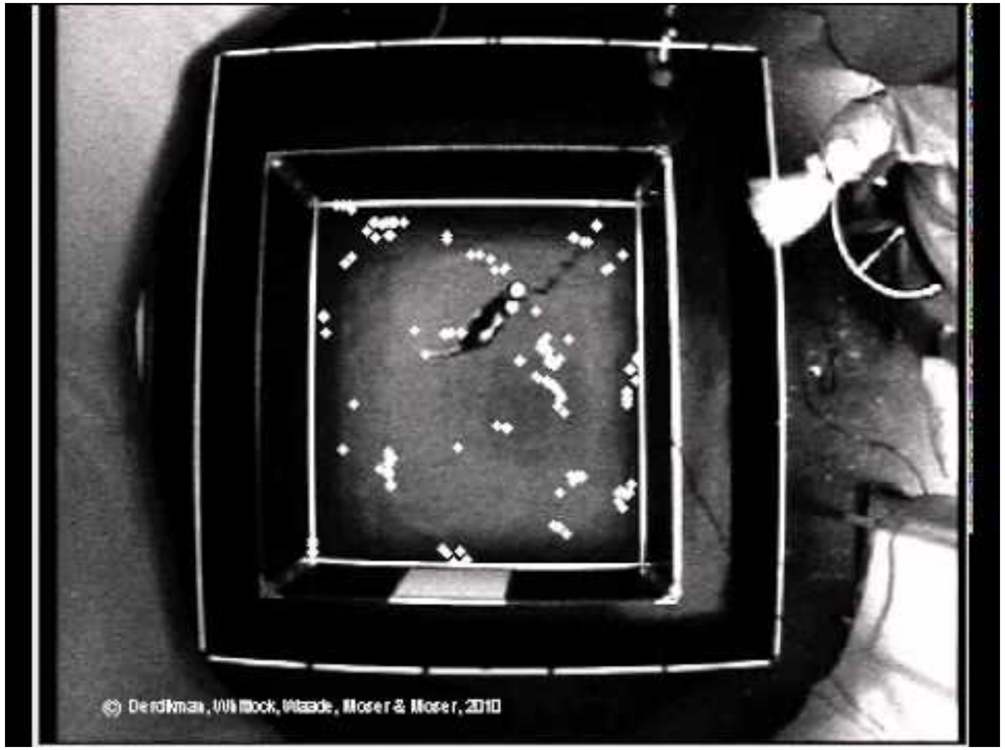
Photo: A. Mahmoud
Edvard I. Moser
Prize share: 1/4

Neuroscience background

Place cells → 'You are here'

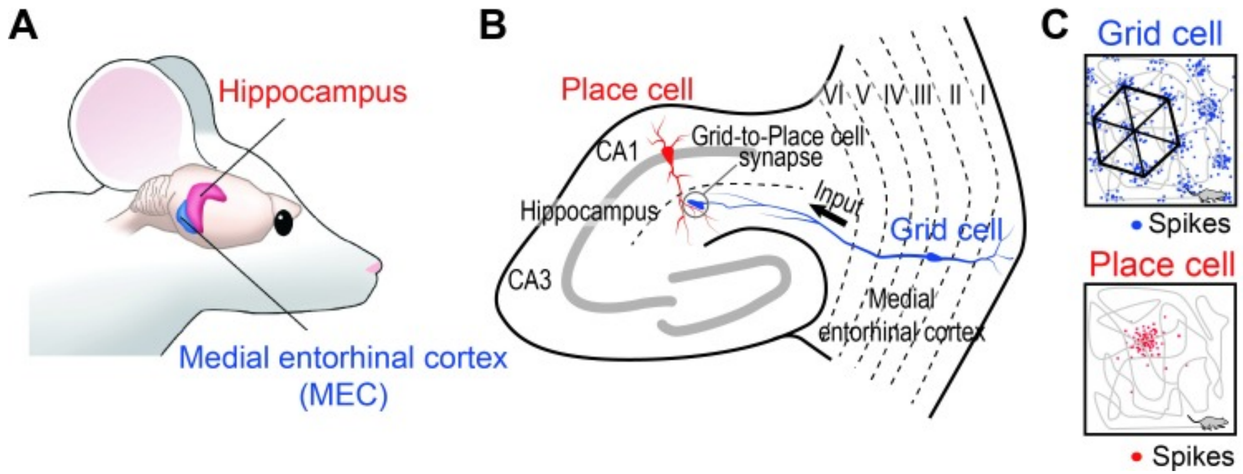
Grid cells → Map

Head-direction cells → Compass



An online lecture for head direction cells, grid cells, and others: https://www.youtube.com/watch?v=CQPswblUCkk&ab_channel=MITCBMM

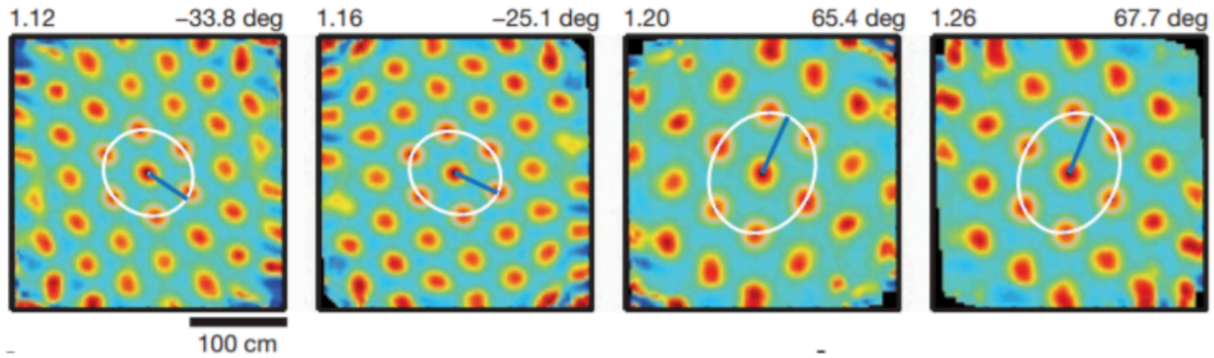
Neuroscience background

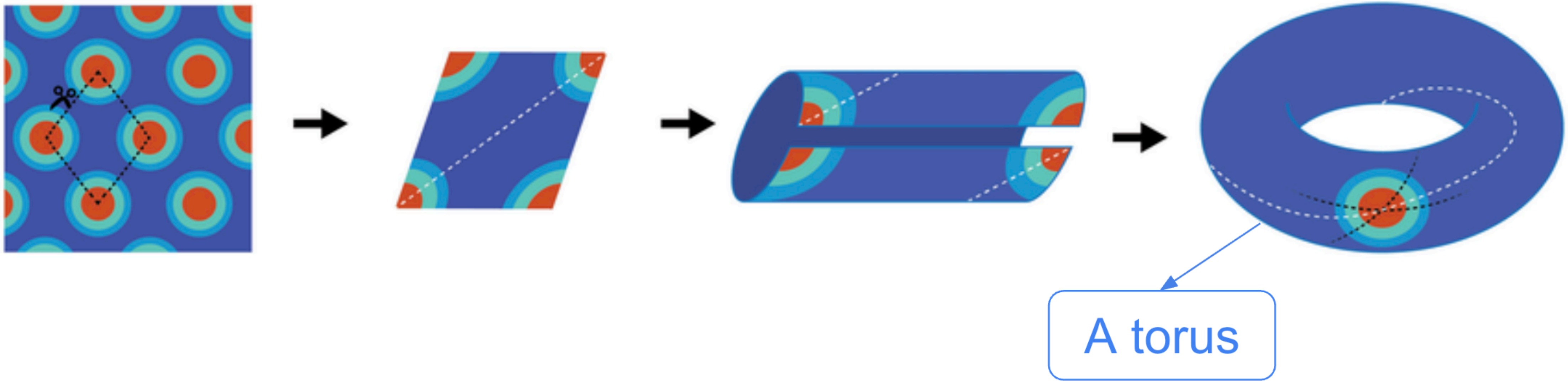


- Place cell fires when animal is at a particular location
- This is like 'You are here' mark on a map.
- The region that a place cell become active is called the **place field** of that place cell.
- Place cells of different place cells together form a good cover. By nerve lemma, the topology of the nerve complex of the place fields give the topology of the environment

Neuroscience background

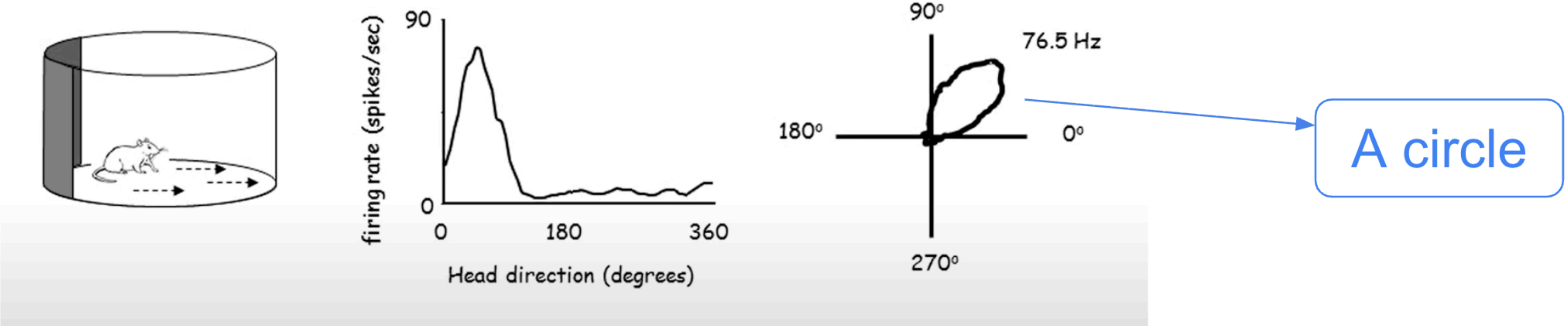
- Grid cell fires at multiple locations, typically in a grid-like triangular lattice
- Firing fields have different modules (depending on scales and orientations).





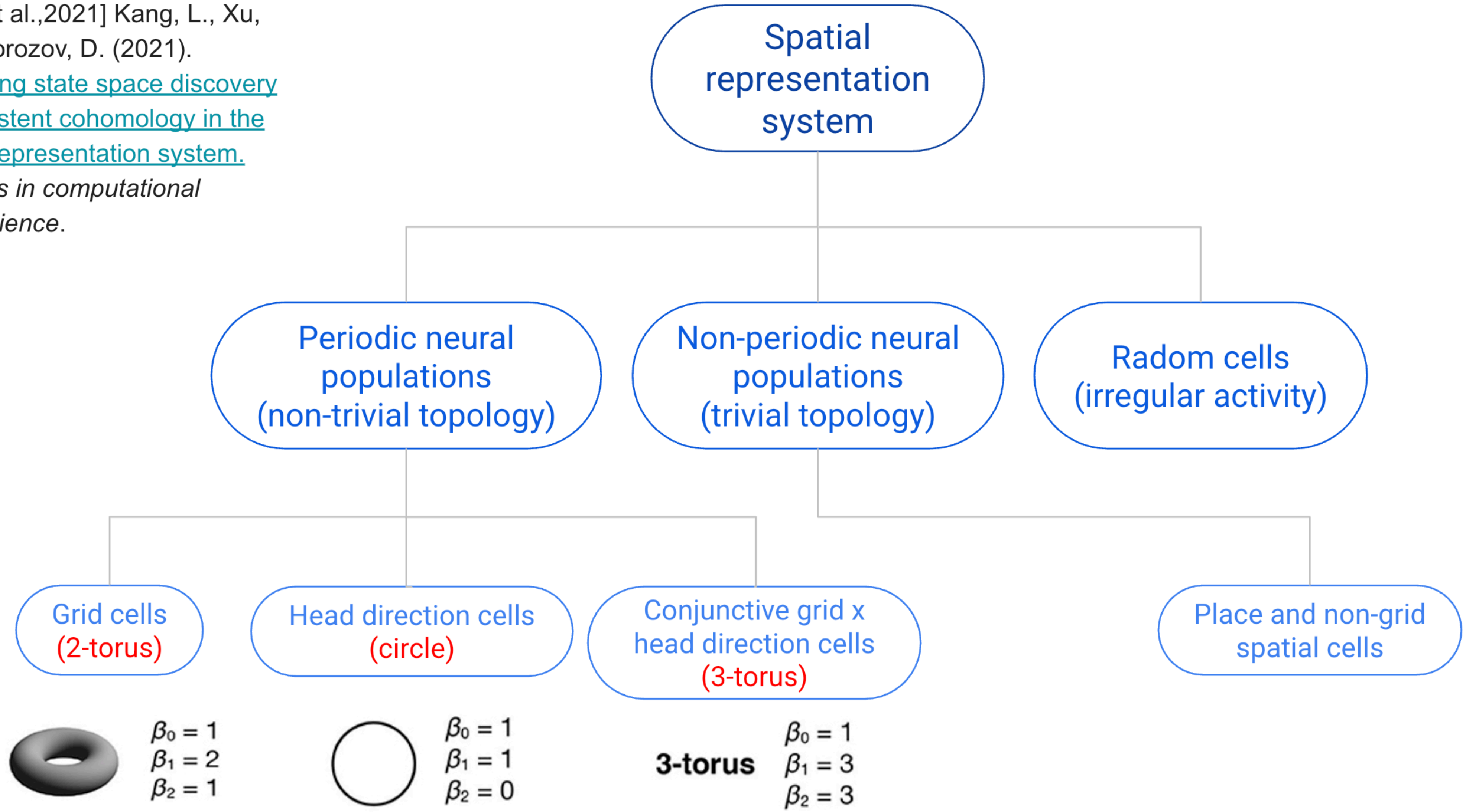
Neuroscience background

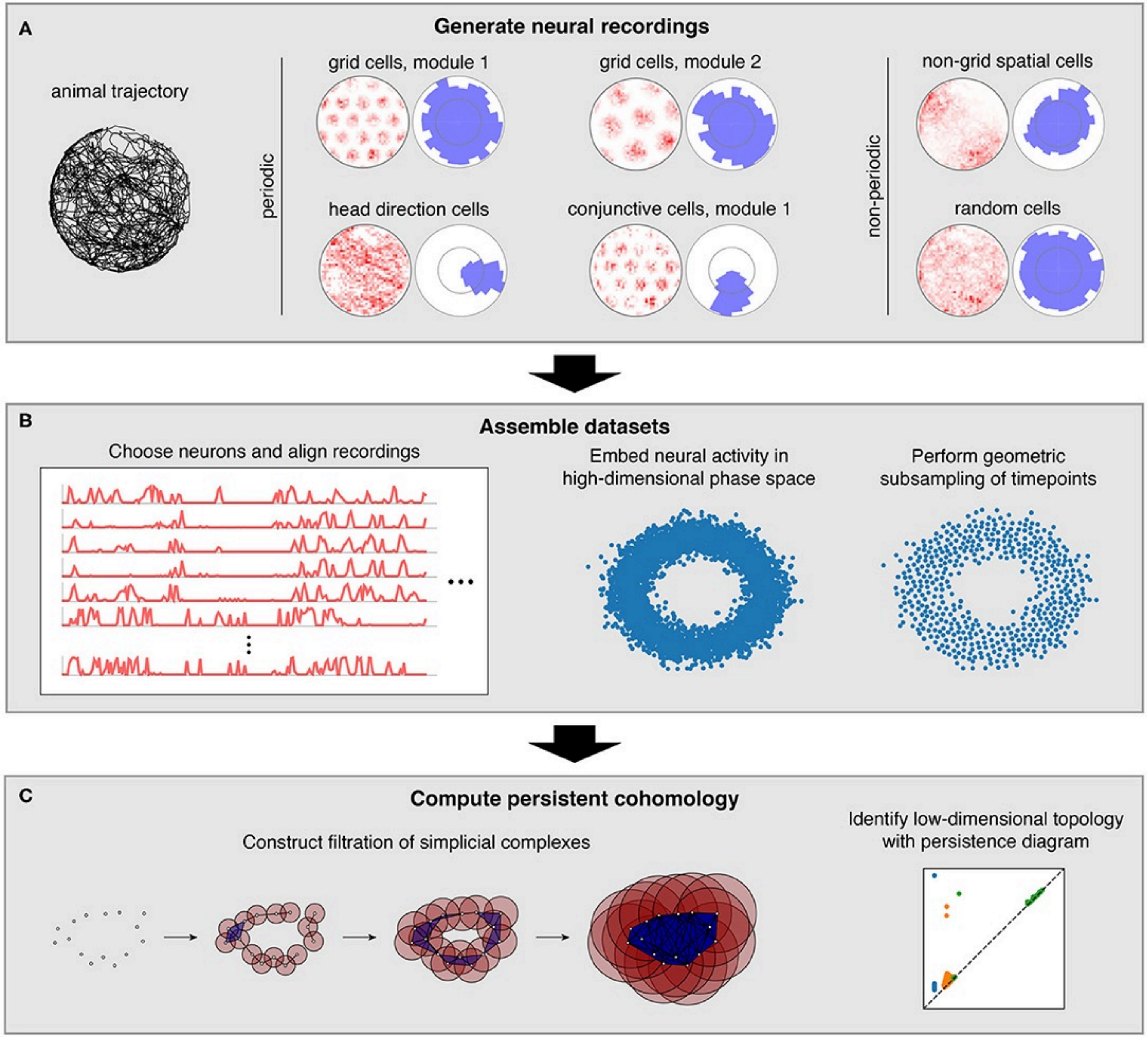
- Head direction cell fires when the head is pointing to a certain direction (independent of the location).
- Firing field for head-direction cells is a circle.



Use PH to analyze grid-cell and head direction cell activity

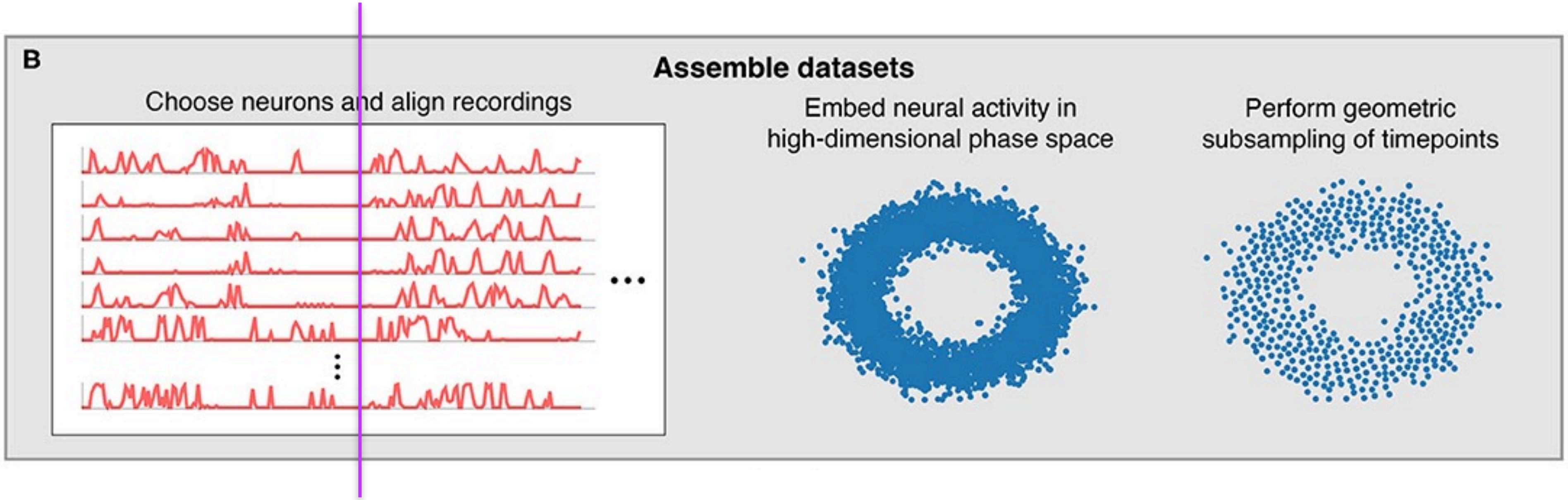
[Kang et al.,2021] Kang, L., Xu, B., & Morozov, D. (2021). [Evaluating state space discovery by persistent cohomology in the spatial representation system.]() *Frontiers in computational neuroscience*.





- Animal activity: a rat explores a circular enclosure of diam 1.8 m
- Recording time: 1,000 s
- Data
  - Position: Average of the animal's position within each 0.2 s time bin
  - Direction: Circular mean of the velocity vector angle within each 0.2 s bin
  - The distinction between body direction and head direction is ignored
- Each cell produces a vector of length 5,000

Assemble datasets & reduce size of data (5000 -> 1000)



1. Remove timepoints when all neurons are not active enough
2. Geometric subsampling, using furthest points algorithm
3. The above steps do not lose important information (due to stability)

Detecting grid cells

Goal: verify that the given cells are grid cells.

Method:

- Compute persistent cohomology of the Vietoris-Rips filtration of datasets and pick cocycles above a threshold.
- ‘Success’, if there are two 1-cocycles for grid cells.
- Experiment 100 times and compute ‘success rate’.

Conclusion:

longer memory
more neurons
$\Big\}$ higher success rates

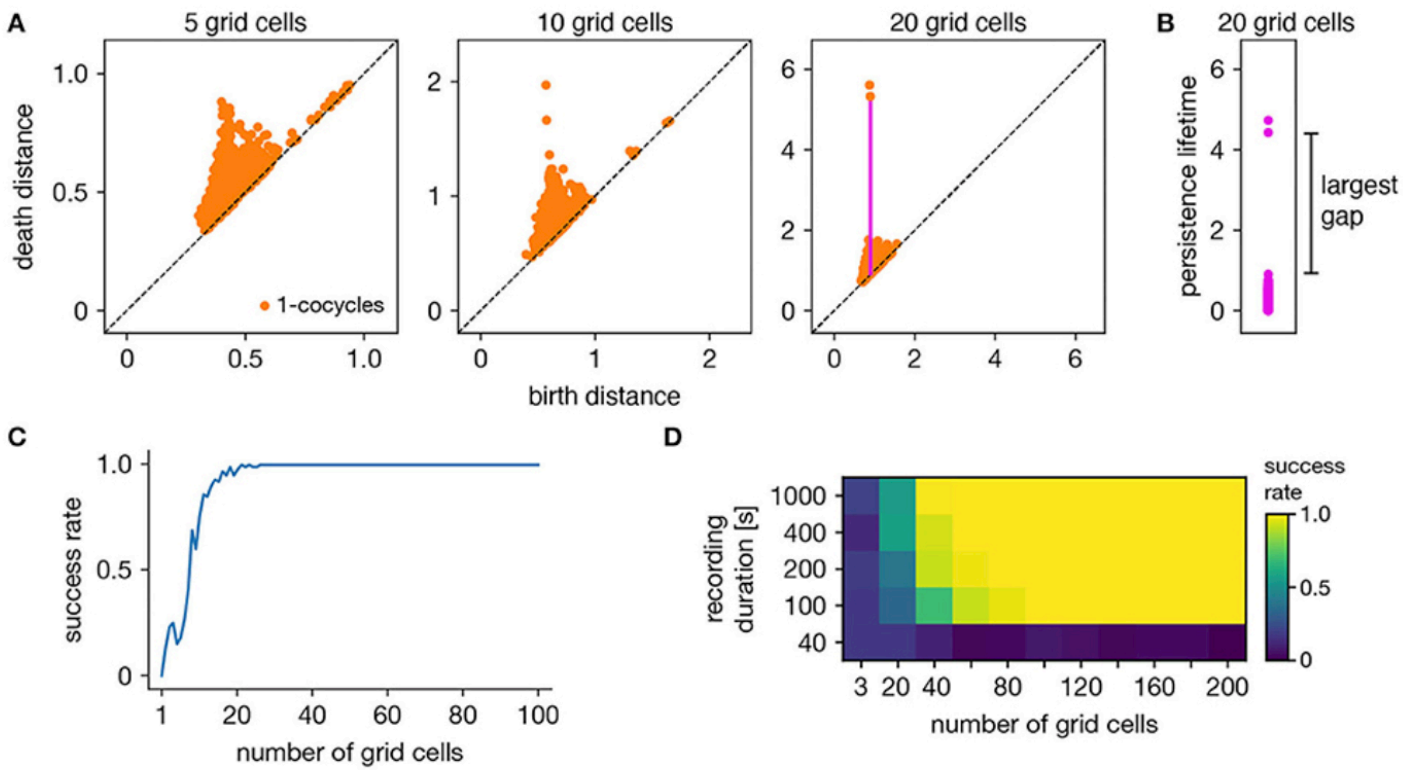


Figure 3

Detecting grid cells with mixed signals

Goal: verify that the given cells are grid cells.

Variant: assuming multi-neuron units, obtained by

- linearly combining spike trains of grid cells;
- coefficients are drawn from a uniform random distribution and then normalized

Method: compute the success rate of having two significant 1-cocycle

Conclusion:

- Figure 4.A: multi-neuron units have different behavior from sing neurons
- Figure 4.B: the toroidal topology can still be discovered and success rate is independent of the number of grid cells in each unit.

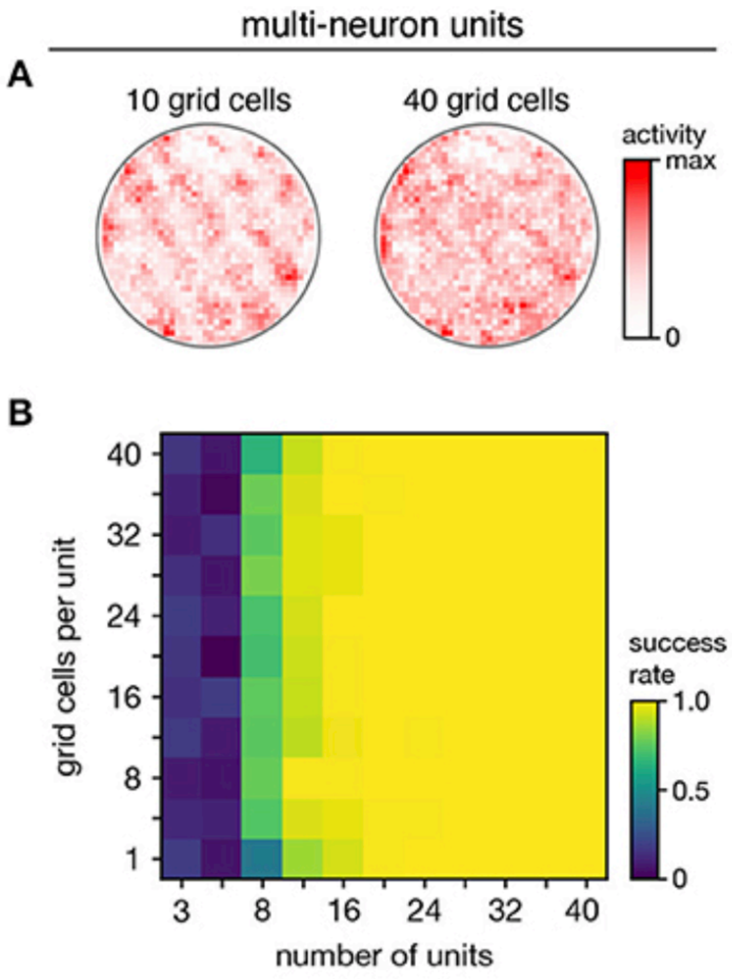


Figure 4

Cell detection with mixtures of neural populations

Goal: detect the types of grid cells or conjunctive cells, when mixed with other types of neurons.

Method: compute the success rate of having two significant 1-cocycles for grid cells, or having three significant 1-cocycles for conjunctive cells.

Conclusion:

$$\frac{\#\text{ other cells}}{\#\text{ grid cells}} < 2 \Rightarrow \text{reliable discovery of grid cells;}$$

Detection of conjunctive cells requires more neurons.

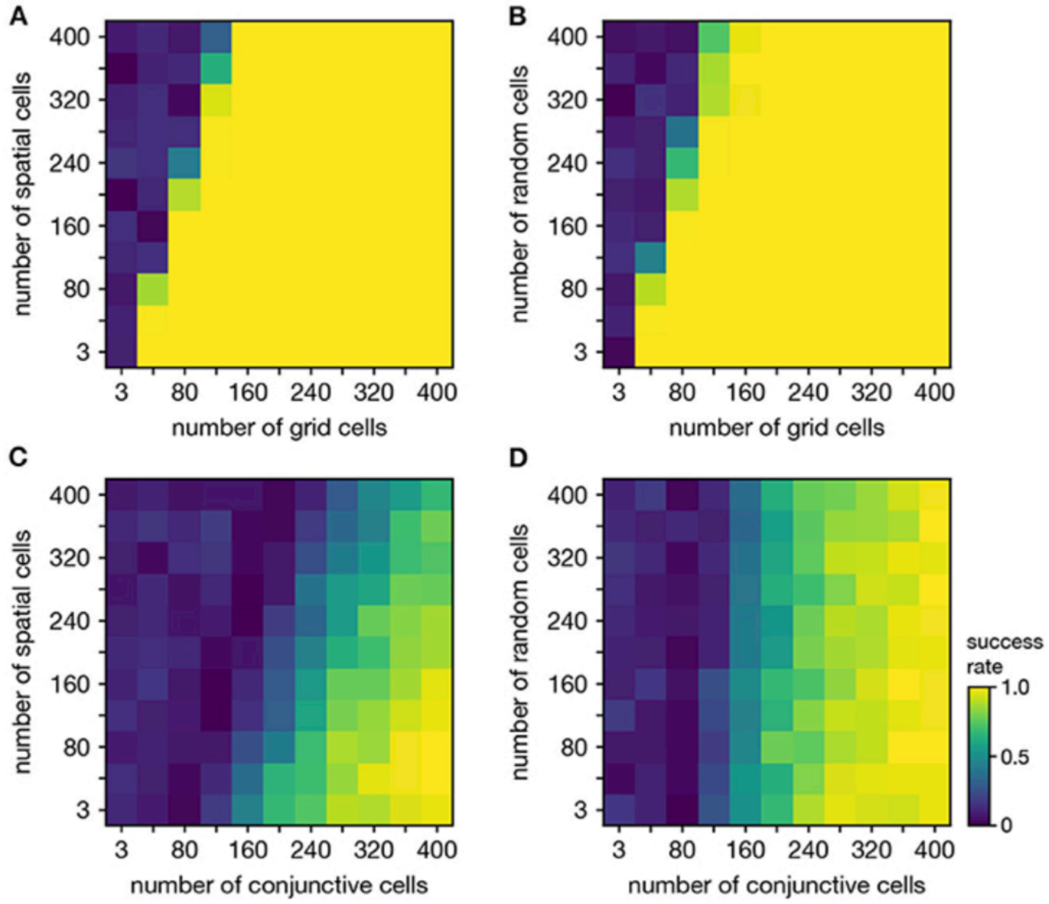
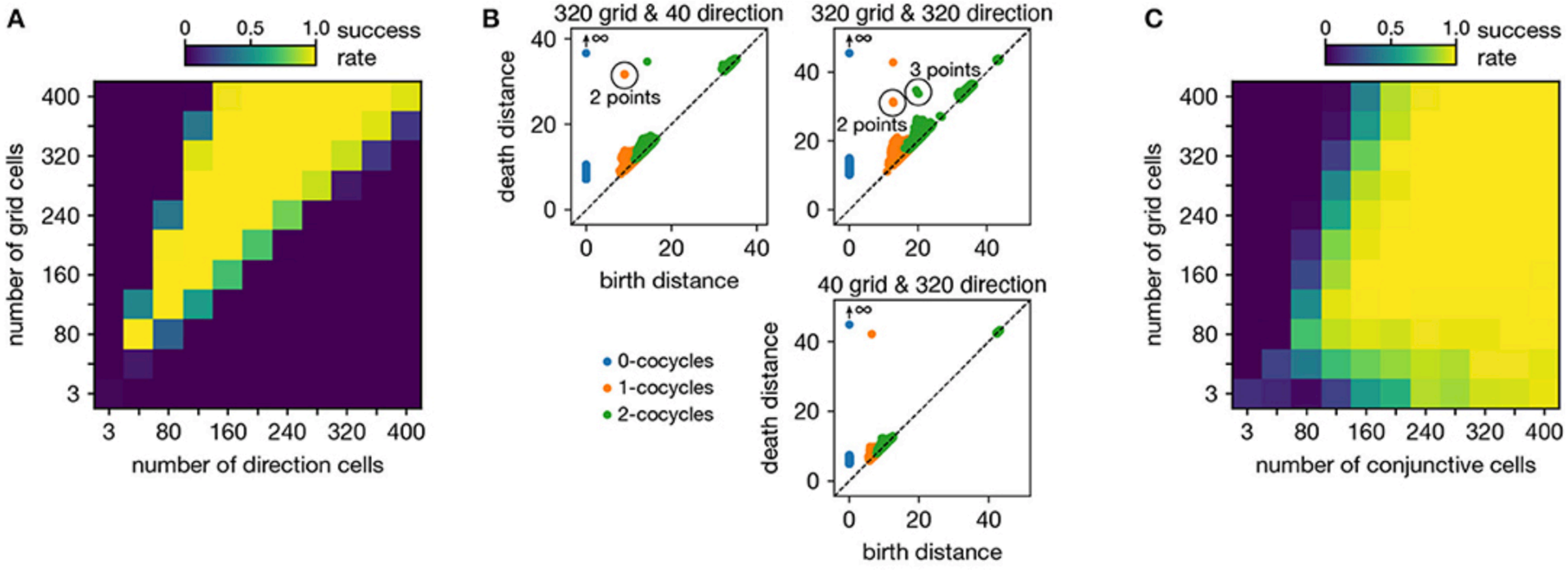


Figure 6

Cell detection with mixtures of neural populations

Goal: detect conjunctive cells under different ways of mixture.

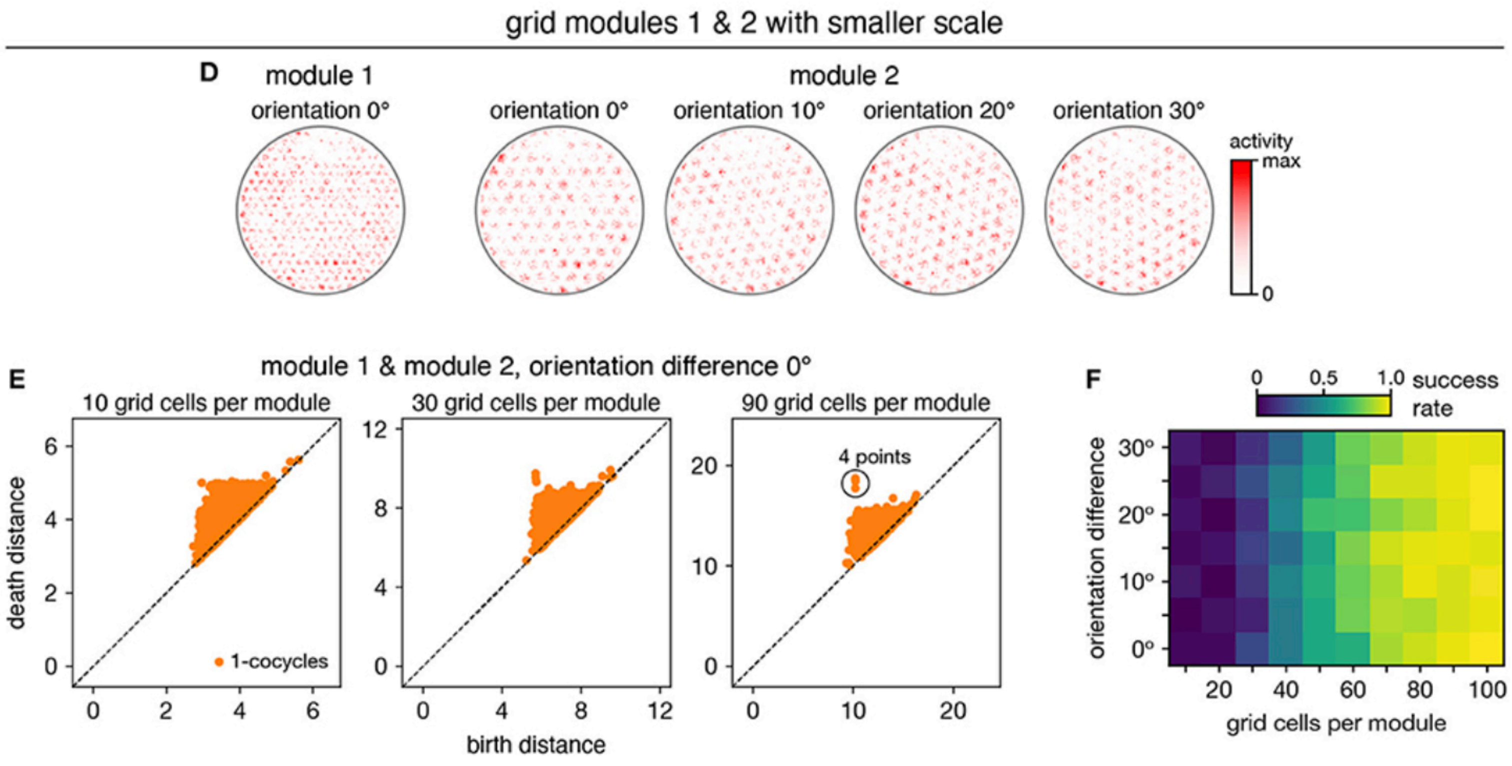
Conclusion: detection is the best when there is no dominating population.



Cell detection with mixtures of neural populations

Goal: detect mixed grid cells from multiple modules (e.g. mixing 2 modules results into a 4-torus).

Conclusion: larger environments are needed to fully sample the 4-torus structure.



Article | Open access | Published: 12 January 2022

Toroidal topology of population activity in grid cells

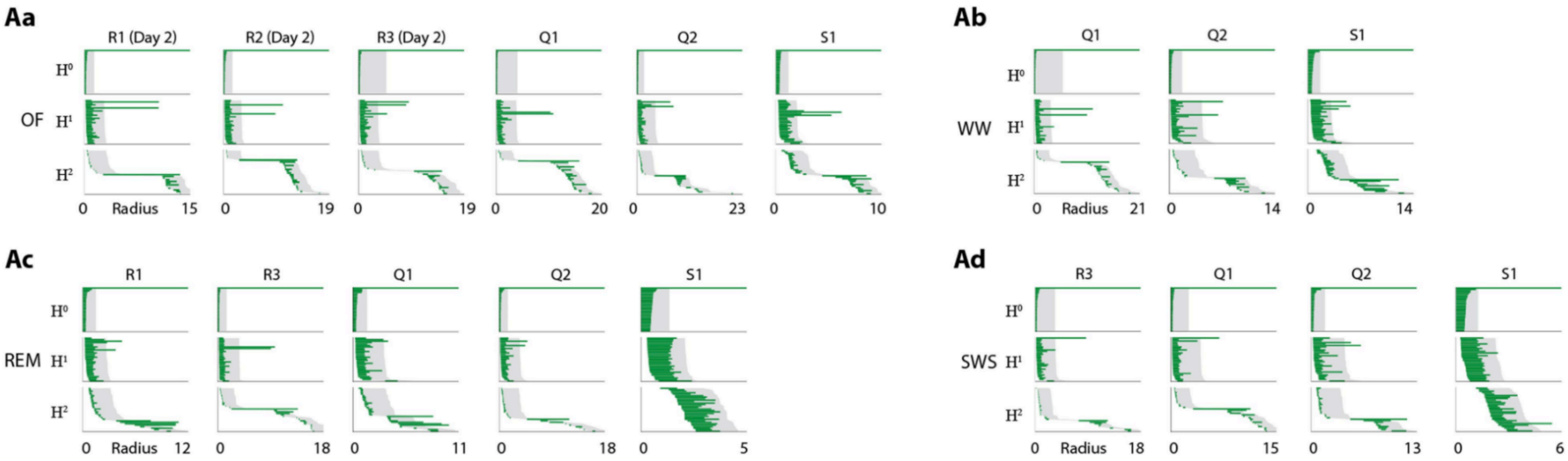
Richard J. Gardner, Erik Hermansen, Marius Pachitariu, Yoram Burak, Nils A. Baas, Benjamin A. Dunn, May-Britt Moser & Edvard I. Moser

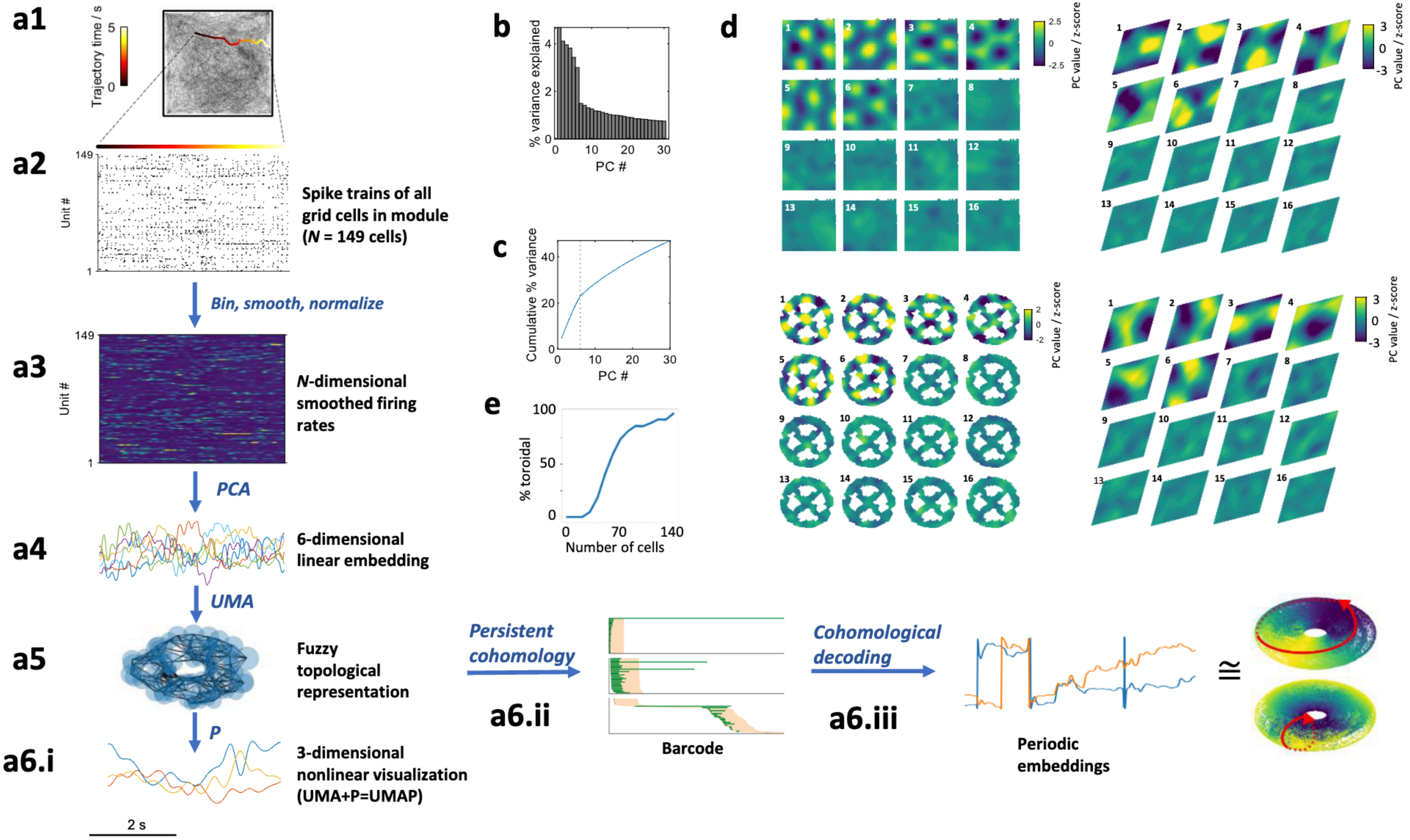
Nature 602, 123–128 (2022) | Cite this article

Naming convention: identifying each module by

- rat (R, Q, or S),
- module number (e.g., 1, 2, 3),
- condition (OF, WW, REM, or SWS)
    - OF: open field
    - WW: wagon-wheel track
    - REW: rapid-eye-movement sleep
    - SWS: slow-wave sleep

In total there were 27 combinations of module (Q1, Q2, R1, R2, R3, S1) and experimental condition (OF day 1, OF day 2, WW, REM, SWS).




a1
Trajectory time / s
a2
Spike trains of all grid cells in module (N = 149 cells)
Unit #
Bin, smooth, normalize
a3
N-dimensional smoothed firing rates
PCA
a4
6-dimensional linear embedding
UMA
a5
Fuzzy topological representation
P
a6.i
3-dimensional nonlinear visualization (UMA+P=UMAP)
2 s
b
% variance explained
PC #
c
Cumulative % variance
PC #
e
% toroidal
Number of cells
d
PC value / z-score
Persistent cohomology
a6.ii
Barcode
Cohomological decoding
a6.iii
Periodic embeddings

Doughnut or Mickey Mouse? Detecting Toroidal Structure in Data through Persistent Cup-Length

Ekaterina S. Ivshina, Galit Anikeeva, Ling Zhou

Data Processing Pipeline:

1. Spike trains $\rightarrow$ binned firing rates (20ms bins)
2. Gaussian smoothing ($\sigma = 50$ms) + z-scoring
3. PCA to 6 components (captures $> 90\%$ variance)
4. Geometric subsampling: 1,200 timepoints
5. Distance matrix: cosine distance

Persistent Cup-Length Computation:

- 500 landmarks via maxmin selection
- Vietoris-Rips filtration via Ripser
- Cup products of all H^1 representatives
- Identify cup-length 2 intervals

Detection Criteria

Module has toroidal topology iff $\exists$ interval I with:

- $\mathrm{cup}(\mathbb{X})(I) = 2$
- $|I| > \tau$ (persistence threshold)

Fewer than the original paper: 24.
The missing ones are:

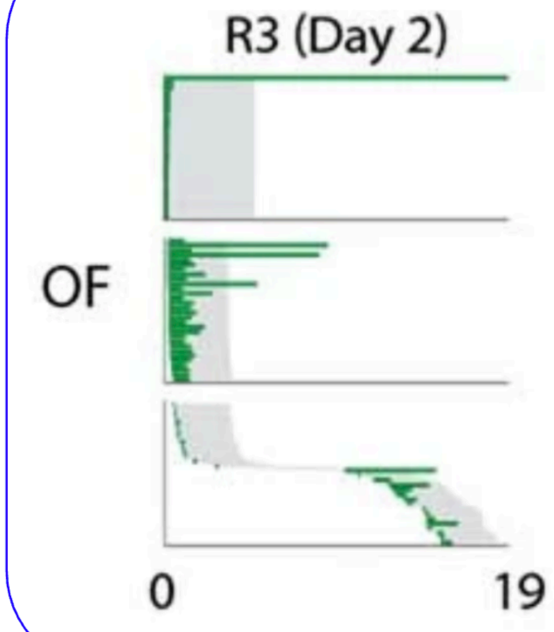
- REM in S1, SWS in R1, SWS in S1.

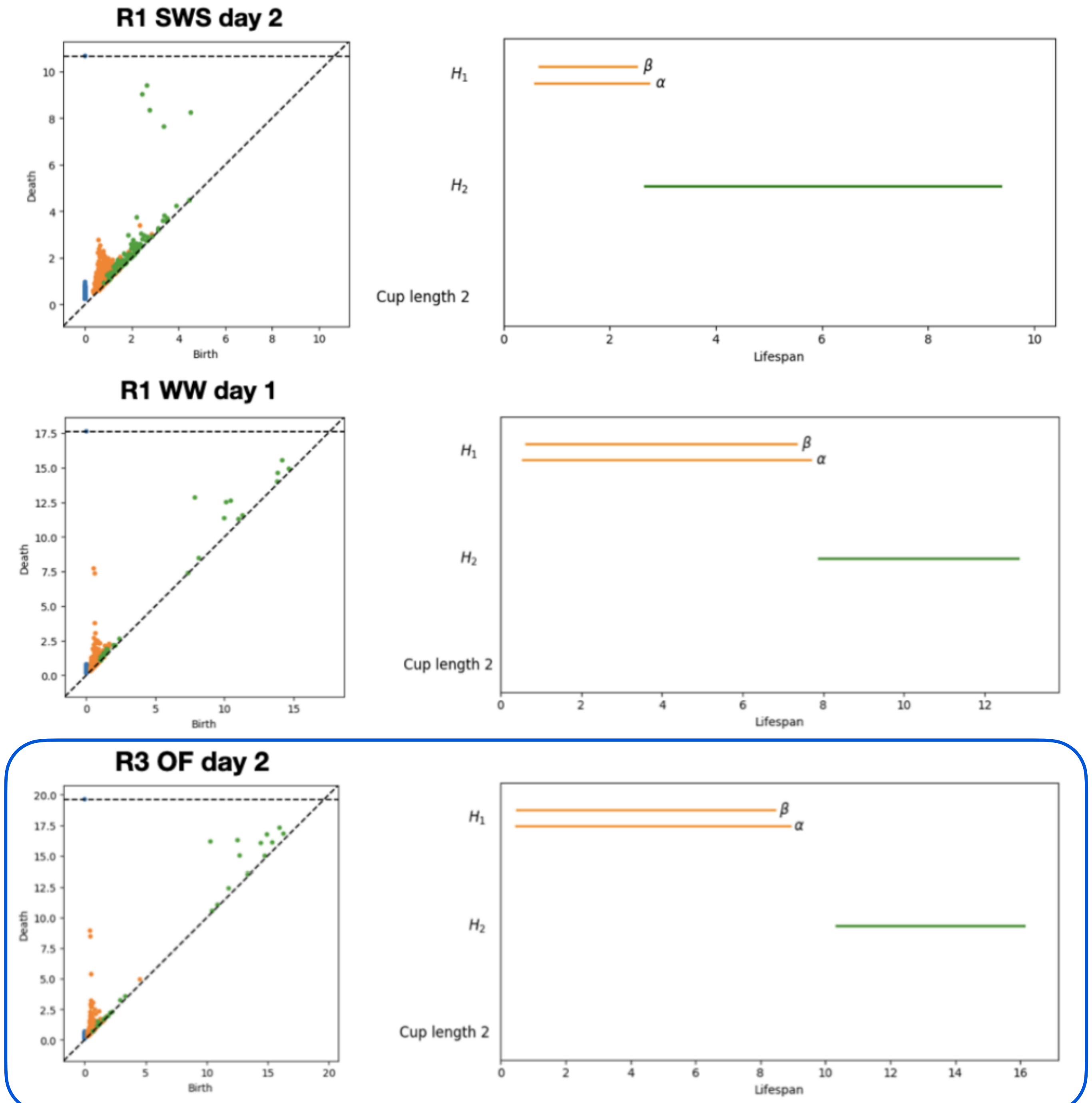
17/27 modules show persistent cup-length 2

Rat	OF	WW	REM	SWS
Q	2/2	1/1	0/2	2/2
R	6/7	1/2	2/4	2/3
S	1/1	0/1	0/1	0/1
Total	9/10	2/4	2/7	4/6

Failed detections correlate with:

- Low cell count (< 100 cells)
- Heterogeneous populations
- Short recording duration


R3 (Day 2)
OF
0
19


R1 SWS day 2
Death
Birth
H_1
β
α
H_2
Cup length 2
Lifespan
R1 WW day 1
Death
Birth
H_1
β
α
H_2
Cup length 2
Lifespan
R3 OF day 2
Death
Birth
H_1
β
α
H_2
Cup length 2
Lifespan

More about persistent cup product

Persistent cup product

Persistent Cup-Length

Authors Marco Contessoto, Facundo Mémoli, Anastasios Stefanou, Ling Zhou

- Let $\mathscr{B}^{*_\sigma \ell}(K_\bullet)$ be the collection of l-fold products of annotated bars

$(\sigma_{I_1} \smile \cdots \smile \sigma_{I_\ell},\ \mathrm{supp}(\sigma_{I_1} \smile \cdots \smile \sigma_{I_\ell}))$

Initialize $\ell = 1$ and $\mathcal{B}^{*_\sigma 0} = \emptyset$ for the following.

```
1 while B^{*σ ℓ} ≠ ∅ do                                        // O(k)
2     for (I_1, σ_1) ∈ B and (I_2, σ_2) ∈ B^{*σ ℓ} do          // O(q · q^{k−1})
3         if I_1 *_σ I_2 ≠ ∅ then                               // O(m^2 · max{c, q})
4             Append (I_1 *_σ I_2, σ_1 ⌣ σ_2) to B^{*σ (ℓ+1)}
```

Line 3 includes two main steps:

(1) compute the cup product of cocycles at cochain level, at cost $O(m^2 \cdot c)$;

(2) check whether a cocycle is a coboundary at a certain collection of birth times, at cost $O(m^2 \cdot q)$.

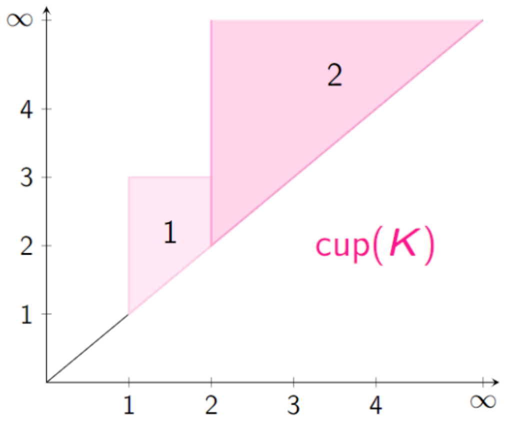
Persistent cup-length function

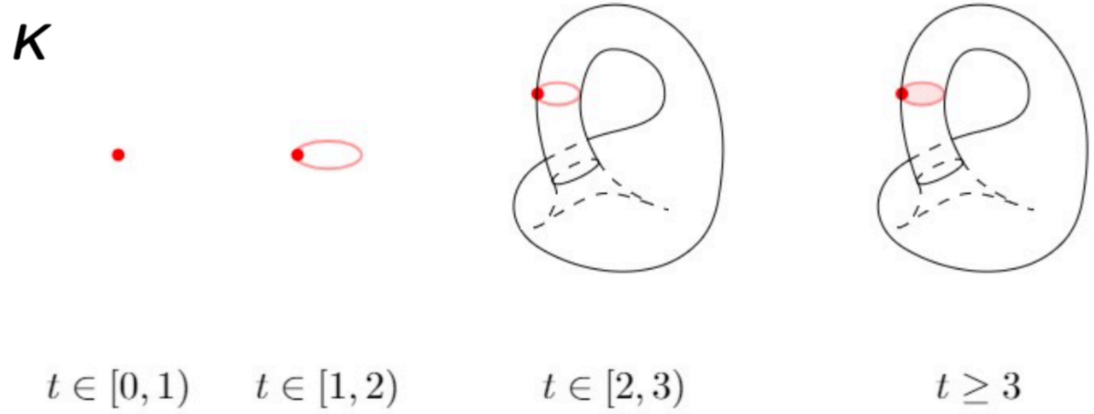
For a filtration $\boldsymbol{X} = \{X_t\}_{t\in\mathbb{R}}$, define the **(persistent) cup-length function of $\boldsymbol{X}$** to be the functor $\mathrm{cup}(\boldsymbol{X}) : \mathrm{Int} \to (\mathbb{N}, \geq)$, given by

$$[t, s] \mapsto \mathrm{cup}\left(\mathbf{Im}(\boldsymbol{H}^*(X_s) \to \boldsymbol{H}^*(X_t))\right).$$

analogous

Cup-length function
$I \mapsto \mathrm{cup}(K)(I)$





[P. betti numbers : $[t, s] \longmapsto \dim(\mathrm{Im}(H^p(X_s) \to H^p(X_t)))$.]

Persistent cup-length diagram

The **(persistent) cup-length diagram of $\boldsymbol{X}$ (associated to $\boldsymbol{\sigma}$)** is the function $\mathrm{dgm}^{\smile}_{\boldsymbol{\sigma}}(\boldsymbol{X}) : \mathrm{Int} \to \mathbb{N}$, given by

$$I \mapsto \max\{\ell \in \mathbb{N}^* \mid I \in \mathcal{B}^{*\boldsymbol{\sigma}^{\ell}}(\boldsymbol{X})\}.$$

Remark

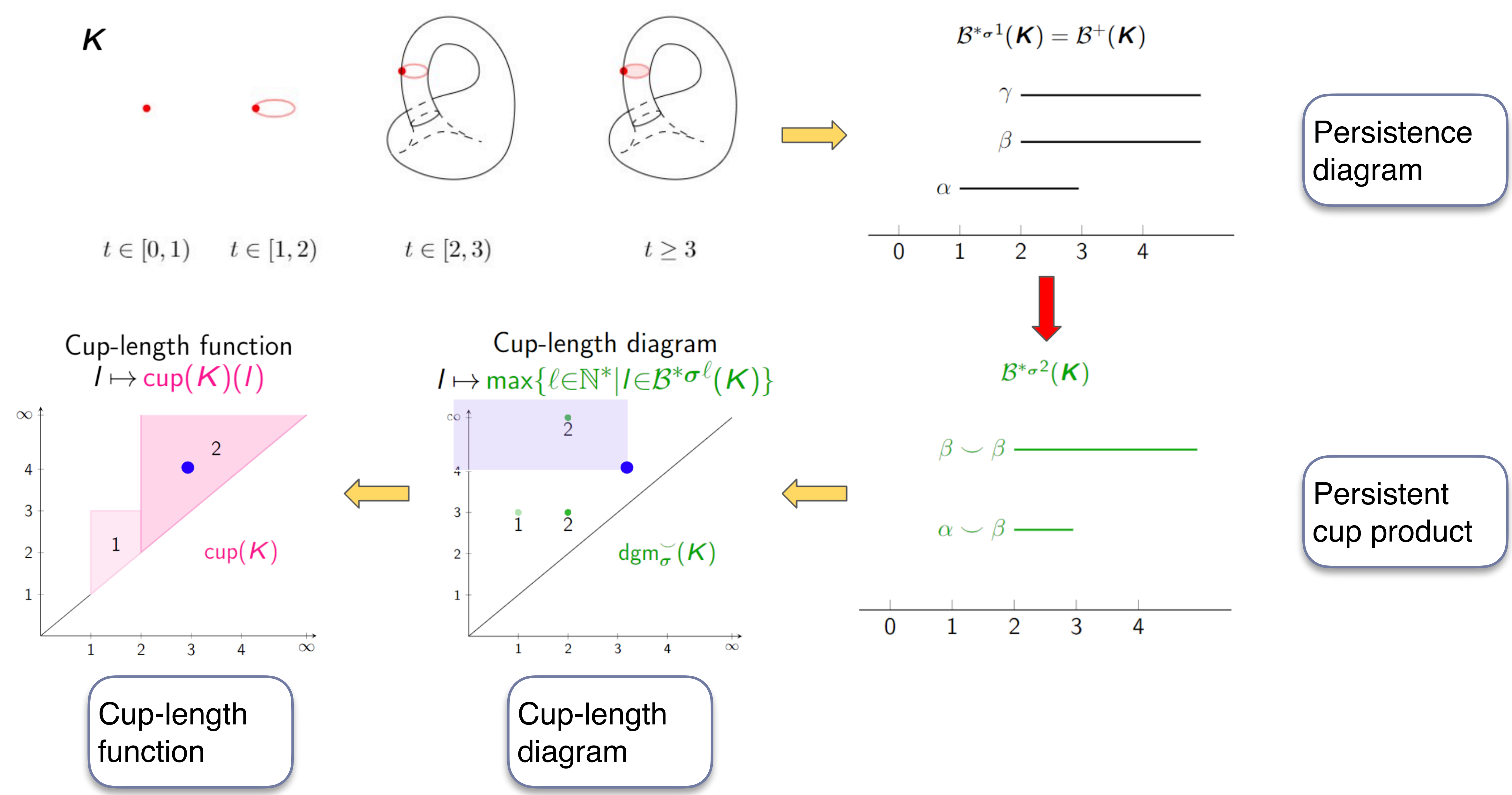
The cup-length diagram depends on the choice of representative cocycle. Example:

$$\bullet \hookrightarrow \mathbb{RP}^2 \hookrightarrow \mathbb{RP}^2 \vee \mathbb{RP}^2 \hookrightarrow \mathrm{Skeleton}_2(\mathbb{RP}^2 \times \mathbb{RP}^2)$$

Theorem

Let $\boldsymbol{X}$ be a filtration. For an interval I,

$$\mathrm{cup}(\boldsymbol{X})(I) = \max_{\bar{I} \supseteq I} \mathrm{dgm}^{\smile}_{\boldsymbol{\sigma}}(\boldsymbol{X})(\bar{I}).$$



K
$t \in [0,1)$
$t \in [1,2)$
$t \in [2,3)$
$t \geq 3$
$\mathcal{B}^{*_\sigma 1}(K) = \mathcal{B}^{+}(K)$
γ
β
α
0 1 2 3 4
Persistence diagram
$\mathcal{B}^{*_\sigma 2}(K)$
$\beta \smile \beta$
$\alpha \smile \beta$
0 1 2 3 4
Persistent cup product
Cup-length diagram
$I \mapsto \max\{\ell \in \mathbb{N}^* \mid I \in \mathcal{B}^{*_\sigma \ell}(K)\}$
$\mathrm{dgm}_\sigma^{\smile}(K)$
Cup-length diagram
Cup-length function
$I \mapsto \mathrm{cup}(K)(I)$
$\mathrm{cup}(K)$
Cup-length function