Please show your work to earn credit.
Homework should be submitted to Gradescope by 11:59 pm on **Thursday, Sep 11**.
You can access the LaTeX source here:

1. In every topological space X, the sets $\varnothing$ and X are both closed and open (often called *clopen* sets).
   (a) (2 pt.) Give an example of a space X that has more than two sets that are both clopen, and list all clopen sets in your example space.
   (b) (5 pt.) Prove that X is connected if and only if the only clopen sets in X are $\varnothing$ and X.
2. (5 pt.) Show that if X is a finite set equipped with any metric d, then the topology induced by d on X is the discrete topology (i.e., every subset of X is open).
3. A topological space X is called *path connected* if any two points $x, y \in X$ can be joined by a path, i.e. there exists a continuous map $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$. Here, $[0, 1]$ represents the closed interval between 0 and 1.
   (a) (4 pt.) Prove that every path connected space is connected.
   (b) (Bonus, 1 pt.) Prove that if X is finite, then connectedness and path connectedness are equivalent notions.
4. Consider $A = (0, 1] \cup [2, 3)$ as a subset of the real line $\mathbb{R}$ with the standard topology.
   (a) (3 pt.) Find $\mathrm{Cl}(A)$, $\mathrm{Int}(A)$, and $\mathrm{Bd}(A)$, which are the closure, interior and boundary of A, respectively.
   (b) (Bonus, 1 pt.) Repeat the same question for $A = \mathbb{Q} \cap [0, 1]$.
5. Let A, B be subsets of a topological space X.
   (a) (4 pt.) Prove that $\mathrm{Cl}(A \cup B) = \mathrm{Cl}(A) \cup \mathrm{Cl}(B)$.
   (b) (2 pt.) Show by example that $\mathrm{Int}(A \cup B)$ need not equal $\mathrm{Int}(A) \cup \mathrm{Int}(B)$.
   (c) (3 pt.) Prove that $\mathrm{Bd}(A \cup B) \subseteq \mathrm{Bd}(A) \cup \mathrm{Bd}(B)$.
6. (6 pt.) A relation $\sim$ on a set S is called an *equivalence relation* if it satisfies the following three properties:
   (a) Reflexivity: For all $a \in S$, we have $a \sim a$.
   (b) Symmetry: For all $a, b \in S$, if $a \sim b$ then $b \sim a$.
   (c) Transitivity: For all $a, b, c \in S$, if $a \sim b$ and $b \sim c$, then $a \sim c$.

   Prove that homeomorphisms and homotopy equivalence are both equivalence relations on the set of topological spaces.
7. (9 pt.) Construct the following required maps (no proofs are required):
   (a) A homeomorphism between the open interval $(0, 1)$ and the real line $\mathbb{R}$.
   (b) A deformation retraction of the closed unit disk with center removed $D^2 \setminus \{(0, 0)\} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \setminus \{(0, 0)\}$ onto its boundary circle $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.
   (c) Find explicit continuous maps $f : \mathbb{R}^2 \setminus \{0\} \to S^1$ and $g : S^1 \to \mathbb{R}^2 \setminus \{0\}$ such that $g \circ f \simeq \mathrm{id}_{\mathbb{R}^2 \setminus \{0\}}$ and $f \circ g \simeq \mathrm{id}_{S^1}$. In other words, f and g realize the homotopy equivalence between $\mathbb{R}^2 \setminus \{0\}$ and S^1.
8. (10 pt.) Answer the following questions about abstract simplicial complexes.
   (a) Let $V = \{v_0, v_1, v_2\}$. List all simplicial complexes that can be formed on V.

(b) Consider the simplicial complex

$$K = \Big\{\{v_0\}, \{v_1\}, \{v_2\}, \{v_3\},$$
$$\{v_0, v_1\}, \{v_1, v_2\}, \{v_2, v_0\}, \{v_0, v_3\}, \{v_1, v_3\}, \{v_2, v_3\},$$
$$\{v_0, v_1, v_2\}, \{v_0, v_1, v_3\}, \{v_0, v_2, v_3\}, \{v_1, v_2, v_3\}\Big\}.$$

Draw and describe its geometric realization $|K|$. Is $|K|$ a topological manifold? If yes, determine its dimension.

(c) Prove that the intersection of two simplicial complexes on the same vertex set is again a simplicial complex.

(d) Is the union of two simplicial complexes always a simplicial complex? Justify your answer with a proof or counterexample.

9. (6 pt.) Let (X, d) be a metric space and let $\epsilon > 0$. Define

$$K_\epsilon := \Big\{\{x_0, \ldots, x_k\} \subset X \;\Big|\; d(x_i, x_j) \leq \epsilon \text{ for all } 0 \leq i, j \leq k\Big\}.$$

(a) Prove that K_ϵ is an abstract simplicial complex.

(b) Give an explicit example of a finite subset $X \subseteq \mathbb{R}^2$ or $\mathbb{R}^3$ and a value of $\epsilon > 0$ such that the geometric realization $|K_\epsilon|$ is the boundary of a regular octahedron. State the coordinates of the points in X, the chosen ϵ, and draw the resulting complex.

(Remark: The simplicial complex K_ϵ is called the *Vietoris–Rips complex* of (X, d) at scale ϵ.)