Please show your work to earn credit.
Homework should be submitted to Gradescope by 11:59 pm on **Thursday, Oct 9**.
You can access the LaTeX source here:

(Unless otherwise stated, all homology groups and boundary matrices are taken with coefficients in $\mathbb{Z}_2$.)

1. (6 pt.) Consider the persistence diagrams

$$D_1 = \{(0,3),(2,5)\}, \qquad D_2 = \{(0,4),(0,4)\}.$$

   (a) List all partial matchings between D_1 and D_2.

   (b) Compute the cost of each partial matching and then determine the bottleneck distance $d_B(D_1, D_2)$.

2. (4 pt.) Show that two persistence modules are 0-interleaved if and only if they are isomorphic.

3. (6 pt.) Let M be the persistence module supported at a single point $a \in \mathbb{R}$, i.e.

$$M_t = \begin{cases} \mathbb{F} & t = a, \\ 0 & \text{otherwise.} \end{cases}$$

   Let 0 denote the zero persistence module.

   (a) Prove that M and 0 are not 0-interleaved.

   (b) Show that for every $\varepsilon > 0$, M and 0 are ε-interleaved.

   (c) Conclude that $d_I(M, 0) = 0$ even though $M \ncong 0$.

   (This exercise illustrates that having interleaving distance zero is strictly weaker than being 0*-interleaved.)*

4. (4 pt.) Let X be a simplicial complex with nontrivial first homology, i.e. $H_1(X) \neq 0$. Let Y be the simplicial complex consisting of a single vertex. Show that for any simplicial maps

$$f : X \to Y \quad \text{and} \quad g : Y \to X,$$

   the composition $g \circ f : X \to X$ is *not* contiguous to the identity map id_X.

5. (10 pt.) Recall from class the two ways of defining ε-interleavings between simplicial filtrations. Let $X_\bullet$ and $Y_\bullet$ be simplicial filtrations. An *ε-interleaving* consists of simplicial maps

$$f_t : X_t \to Y_{t+\varepsilon}, \qquad g_t : Y_t \to X_{t+\varepsilon}, \quad t \geq 0,$$

   such that for all $t \leq t'$ we have the following commutative diagrams

$$\begin{tikzcd} X_t \arrow[r, hook, "v_{t,t'}"] \arrow[dr, "f_t"'] & X_{t'} \arrow[dr, "f_{t'}"] & \\ & Y_{t+\varepsilon} \arrow[r, hook, "u_{t+\varepsilon,t'+\varepsilon}"'] & Y_{t'+\varepsilon} \end{tikzcd} \qquad \begin{tikzcd} & X_{t+\varepsilon} \arrow[r, hook, "v_{t+\varepsilon,t'+\varepsilon}"] & X_{t'+\varepsilon} \\ Y_t \arrow[ur, "g_t"] \arrow[r, hook, "u_{t,t'}"'] & Y_{t'} \arrow[ur, "g_{t'}"'] & \end{tikzcd}$$

   and

$$\begin{tikzcd} X_t \arrow[rr, hook, "v_{t,t+2\varepsilon}"] \arrow[dr, "f_t"'] & & X_{t+2\varepsilon} \\ & Y_{t+\varepsilon} \arrow[ur, "g_{t+\varepsilon}"'] & \end{tikzcd} \qquad \begin{tikzcd} & X_{t+\varepsilon} \arrow[dr, "f_{t+\varepsilon}"] & \\ Y_t \arrow[ur, "g_t"] \arrow[rr, hook, "u_{t,t+2\varepsilon}"'] & & Y_{t+2\varepsilon} \end{tikzcd}$$

   Depending on how these diagrams commute, we distinguish two notions:

   - If they commute *strictly* (on the nose), then $X_\bullet$ and $Y_\bullet$ are **strictly ε-interleaved**.

- If they commute *up to contiguity*, i.e. in each diagram the two composites are contiguous simplicial maps, then $X_\bullet$ and $Y_\bullet$ are **contiguously ε-interleaved**.

Consider two simplicial filtrations:

$$X_r = \begin{cases} \{a,b,c,ab,bc,ac\} & \text{if } r \in [0,2) \\ \{a,b,c,ab,bc,ac,abc\} & \text{if } r \in [2,\infty) \end{cases} \text{ and } Y_r = \{d\}, \forall r \geq 0.$$

(a) Prove that $X_\bullet$ and $Y_\bullet$ are not strictly ϵ-interleaved for any finite $\epsilon > 0$.

(b) Prove that $X_\bullet$ and $Y_\bullet$ are contiguously ε-interleaved for any $\varepsilon \geq 1$.

(c) (Bonus: 2 pt.) Prove that $X_\bullet$ and $Y_\bullet$ are not contiguously ε-interleaved for any $\varepsilon < 1$. *(Hint: use Problem 4.)*

6. (10 pt.) Let $X = S^1 \subset \mathbb{R}^2$ be the unit circle. Define two functions $f, g : X \to \mathbb{R}$ by

$$f(x,y) = y, \qquad g(x,y) = \tfrac{1}{2}y.$$

In other words, f is the height function and g is half the height. For each function, consider the associated sublevel set filtration: for any real number $r \in \mathbb{R}$,

$$X_f^r = \{(x,y) \in S^1 : f(x,y) \leq r\}, \qquad X_g^r = \{(x,y) \in S^1 : g(x,y) \leq r\}.$$

(a) Sketch how the filtrations $X_f^\bullet$ and $X_g^\bullet$ evolve as r increases. In particular, indicate the *critical values* of r where the homotopy type of the sublevel set changes.

(b) Compute the function distance $\|f - g\|_\infty$.

(c) Compute the bottleneck distance

$$d_B\big(\mathrm{dgm}_p(X_f^\bullet), \mathrm{dgm}_p(X_g^\bullet)\big)$$

between the persistence diagrams of the two filtrations for $p = 0, 1$.

(d) Compare your results from parts (b) and (c), and use them to verify that the stability property of persistent homology

$$d_I\big(H_p(X_f^\bullet), H_p(X_g^\bullet)\big) \;\leq\; \|f - g\|_\infty$$

holds in this example.