Please show your work to earn credit.
Homework should be submitted to Gradescope by 11:59 pm on **Thursday, Oct 2**.
You can access the LaTeX source here:

(Unless otherwise stated, all homology groups and boundary matrices are taken with coefficients in $\mathbb{Z}_2$.)

1. (15 pt.) Let $X \subset \mathbb{R}^2$ be the six vertices of a regular hexagon of side length 1. Consider the Vietoris–Rips filtration $\{\mathrm{Rips}^r(X)\}_r$, where a simplex is included whenever all pairwise distances between its vertices are at most $2r$.
   (a) Identify the critical values of r at which the simplicial complex $\mathrm{Rips}^r(X)$ changes, and draw the complexes at these values.
   (b) Compute the persistent homology in dimensions $p = 0, 1$.
   (c) Sketch the barcode and the persistence diagram for H_0 and H_1.
   (d) Compute the persistent Betti numbers
   $$\beta_p^{i,j} := \dim \operatorname{im}\big(H_p(\mathrm{Rips}^i(X)) \to H_p(\mathrm{Rips}^j(X))\big),$$
   for every pair of critical values $i \leq j$ and for $p = 0, 1$.
   (e) Compute the persistent pairing function
   $$\mu_p : \text{Intervals} \to \mathbb{Z}, \text{ with } [i, j) \mapsto \mu_p^{i,j}$$
   which assigns to each interval in the barcode its multiplicity, for $p = 0, 1$.

2. (5 pt.) Fix a homological degree p. Let $M_\bullet = \{H_p(K_i)\}_{i=0}^3$ be the persistence module associated to the filtration $K_0 \subseteq K_1 \subseteq K_2$. You are given the persistent Betti numbers
   $$\beta_p^{i,j} = \begin{cases} 1 & \text{if } (i,j) \in \{(0,0), (0,1), (1,1)\}, \\ 0 & \text{otherwise.} \end{cases}$$
   Use the Möbius inversion formula to compute the barcode of $M_\bullet$.

3. (3 pt.) The following is the reduced boundary matrix R for a simplicial filtration $K_0 \hookrightarrow \cdots \hookrightarrow K_5$ with 6 simplices.
   $$R = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$
   Read off the barcodes of the filtration: list all intervals $[i, j)$ and $[i, \infty)$ corresponding to homology classes (across all dimensions).

4. (4 pt.) Let $\mathcal{F}$ be a simplex-wise filtration of a simplicial complex K given by an ordering of the simplices
   $$\sigma_1, \sigma_2, \ldots, \sigma_N.$$
   Form a new sequence $\mathcal{F}'$ by swapping two consecutive simplices σ_k and σ_{k+1}, under the assumption that σ_k is not a face of σ_{k+1}. That is,
   $$\sigma_1, \ldots, \sigma_{k-1}, \sigma_{k+1}, \sigma_k, \sigma_{k+2}, \ldots, \sigma_N.$$
   Determine the relation between the persistence diagrams $\mathrm{Dgm}(\mathcal{F})$ and $\mathrm{Dgm}(\mathcal{F}')$.

5. (6 pt.) Consider two persistence modules $U_\bullet$ and $V_\bullet$ with linear maps $f_i : U_i \to V_i$ so that all squares commute:

$$\begin{tikzcd} U_1 \arrow[r] \arrow[d, "f_1"'] & U_2 \arrow[r] \arrow[d, "f_2"'] & U_3 \arrow[r] \arrow[d, "f_3"'] & \cdots \arrow[r] & U_m \arrow[d, "f_m"'] \\ V_1 \arrow[r] & V_2 \arrow[r] & V_3 \arrow[r] & \cdots \arrow[r] & V_m \end{tikzcd}$$

   (a) Define the sequence

   $$\ker \mathcal{F} : \ker f_1 \longrightarrow \ker f_2 \longrightarrow \cdots \longrightarrow \ker f_m,$$

   where the maps are induced by the structure maps of $U_\bullet$. Prove that $\ker \mathcal{F}$ is a persistence module.

   (b) Define the sequence

   $$\operatorname{im} \mathcal{F} : \operatorname{im} f_1 \longrightarrow \operatorname{im} f_2 \longrightarrow \cdots \longrightarrow \operatorname{im} f_m,$$

   where the maps are induced by the structure maps of $V_\bullet$. Prove that $\operatorname{im} \mathcal{F}$ is a persistence module.

6. (10 pt. + (Bonus 2 pt.)) Read through the provided Jupyter notebook `persistent_homology.ipynb` and answer the questions in Section 2.