MATH412/COMPSCI434/MATH713
Fall 2025
Topological Data Analysis

Topic 5: Stability

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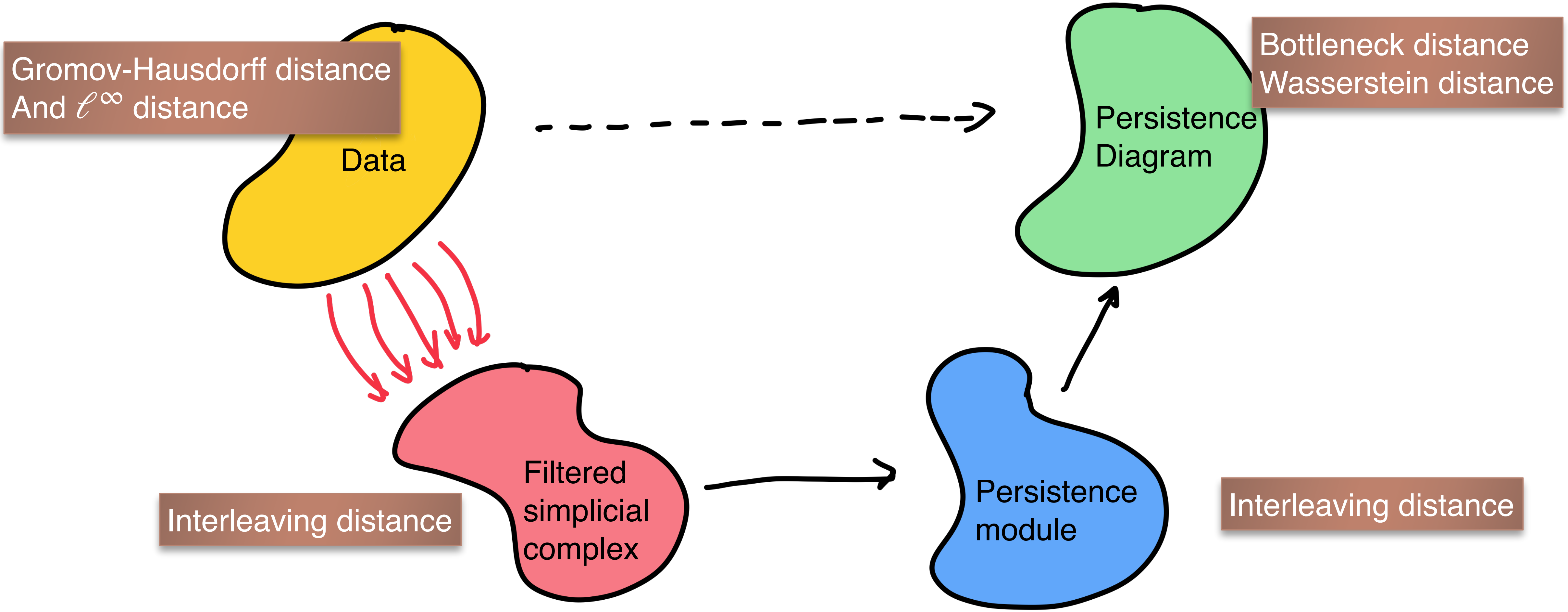
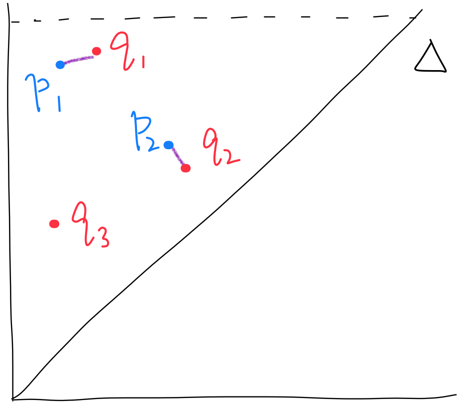
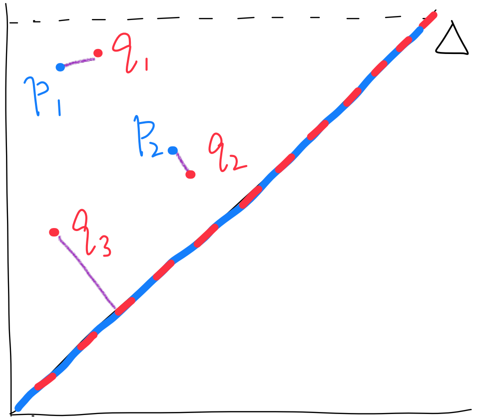
Using metrics to measure perturbations



Review: Bottleneck Distance



- The **bottleneck distance** between D_1 and D_2 is
  - $d_B(D_1, D_2) := \min\{cost(M) \mid M \subset D_1 \times D_2 \text{ a partial matching}\}$
- The cost of a partial matching $M \subset D_1 \times D_2$ is



$$cost(M) = \max\left(\max_{(p,q)\in M} \|p-q\|_\infty, \max_{p \text{ unmatched}} \|p-\Delta\|_\infty\right)$$

- The bottleneck distance between D_1 and D_2 can also be defined as

$$d_B(D_1, D_2) = \min\{cost(\bar{M}) \mid \bar{M} \subset \bar{D}_1 \times \bar{D}_2 \text{ a bijection}\} = \min_{\bar{M}} \max_{(p,q)\in\bar{M}} \|p-q\|_\infty,$$

where $\bar{D}_1 := D_1 \cup \Delta^\infty$ and $\bar{D}_2 := D_2 \cup \Delta^\infty$

p-th Wasserstein distance

- Given two persistence-diagrams (multiset of points in $(\mathbb{R} \cup \{\infty\})^2$)
  - $D_1 = \{p_1, p_2, \ldots, p_s\}$ and $D_2 = \{q_1, q_2, \ldots, q_t\}$
- Augment $\bar{D}_1 := D_1 \cup \Delta^\infty$ and $\bar{D}_2 := D_2 \cup \Delta^\infty$
  - where Δ^∞ is the diagonal points each with **infinite multiplicity**
- The ***p*-th Wasserstein distance** distance between D_1 and D_2
  - $d_{W,p}(D_1, D_2) := \inf_{\bar{M}} \left[\sum_{(x,y)\in\bar{M}} ||x - y||_\infty^p \right]^{\frac{1}{p}}$
  - $d_{W,\infty}(D_1, D_2) = d_B(D_1, D_2)$

Review: ϵ-Interleaving

- U and V are ϵ-**interleaved** if there exists maps
  - $\varphi_a : U_a \to V_{a+\epsilon}$ and $\phi_a : V_a \to U_{a+\epsilon}$ for any $a \in \mathbb{R}$
  - s.t. these maps commute with horizontal maps u's and v's
    - To verify commutativity of maps, only need to check four configurations):

$$\begin{tikzcd} U_a \arrow[rr, "u_{a,a'}"] \arrow[dr, "\varphi_a"'] & & U_{a'} \arrow[dr, "\varphi_{a'}"] & \\ & V_{a+\varepsilon} \arrow[rr, "v_{a+\varepsilon,a'+\varepsilon}"'] & & V_{a'+\varepsilon} \end{tikzcd} \qquad \begin{tikzcd} & U_{a+\varepsilon} \arrow[rr, "u_{a+\varepsilon,a'+\varepsilon}"] & & V_{a'+\varepsilon} \\ V_a \arrow[ur, "\psi_a"] \arrow[rr, "v_{a,a'}"'] & & V_{a'} \arrow[ur, "\psi_{a'}"'] & \end{tikzcd}$$

$$\begin{tikzcd} U_a \arrow[rr, "u_{a,a+2\varepsilon}"] \arrow[dr, "\varphi_a"'] & & U_{a+2\varepsilon} \\ & V_{a+\varepsilon} \arrow[ur, "\psi_{a+\varepsilon}"'] & \end{tikzcd} \qquad \begin{tikzcd} & U_{a+\varepsilon} \arrow[dr, "\varphi_{a+\varepsilon}"] & \\ V_a \arrow[ur, "\psi_a"] \arrow[rr, "v_{a,a+2\varepsilon}"'] & & V_{a+2\varepsilon} \end{tikzcd}$$

- The **interleaving distance** between two persistence modules V and U is

$$d_I(V, U) := \inf\{\epsilon > 0 \mid V \text{ and } U \text{ is } \epsilon\text{-interleaved}\}$$

Recall: Finitely presented filtration

- A filtration $(K_t)_{t \in [0,\infty)}$ is called **finitely represented** if
  - There exist $0 = t_0 < t_1 < \cdots < t_n$ such that
  - $K_t = K_{t'}, \quad \forall t_i \leq t < t' < t_{i+1}$ and $i = 0,\ldots,n$ $(t_{n+1} := \infty)$

- Both Čech and Rips filtrations are finitely represented

Stability of PD (v.s. persistence module)

- A persistence module $V = \{V_t\}$ is called **finitely represented** if
  - There exist $0 = t_0 < t_1 < \cdots < t_n$ such that
  - $\varphi : V_t \to V_{t'}$ is an isomorphism, $\quad \forall t_i \leq t < t' < t_{i+1}$ and $i = 0, \ldots, n$ $(t_{n+1} := \infty)$

Stability Theorem 1 [Chazal, Cohen-Steiner, Gliss, Guibas, Oudot 2009]

Given two finitely represented persistence modules U and V, let D_U and D_V be their corresponding persistence diagrams. We then have:

$$d_B(D_U, D_V) \leq d_I(U, V)$$

$(\text{PM} / \text{isomorphism}, d_I) \longrightarrow (\text{PD}, d_B)$ is isometry.

$U \longmapsto D_U$

Isometry Theorem

Isometry Theorem [Lesnick 2015], [Chazal, de Silva, Gliss and Oudot, 2016]

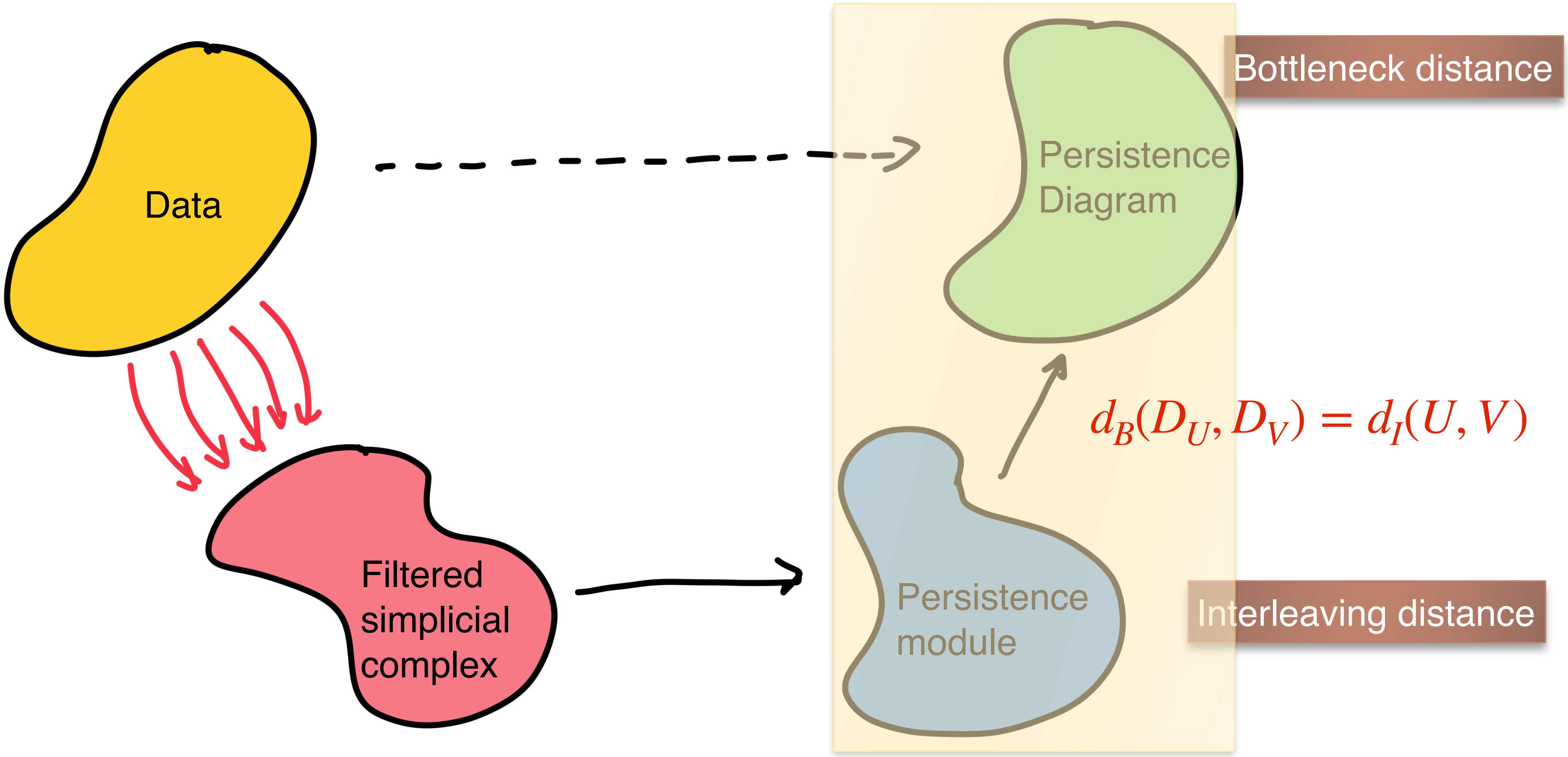
Given two finitely represented persistence modules U and V, let D_U and D_V be their corresponding persistence diagrams. We then have:

$$d_B(D_U, D_V) = d_I(U, V)$$

Holds for more general persistence modules

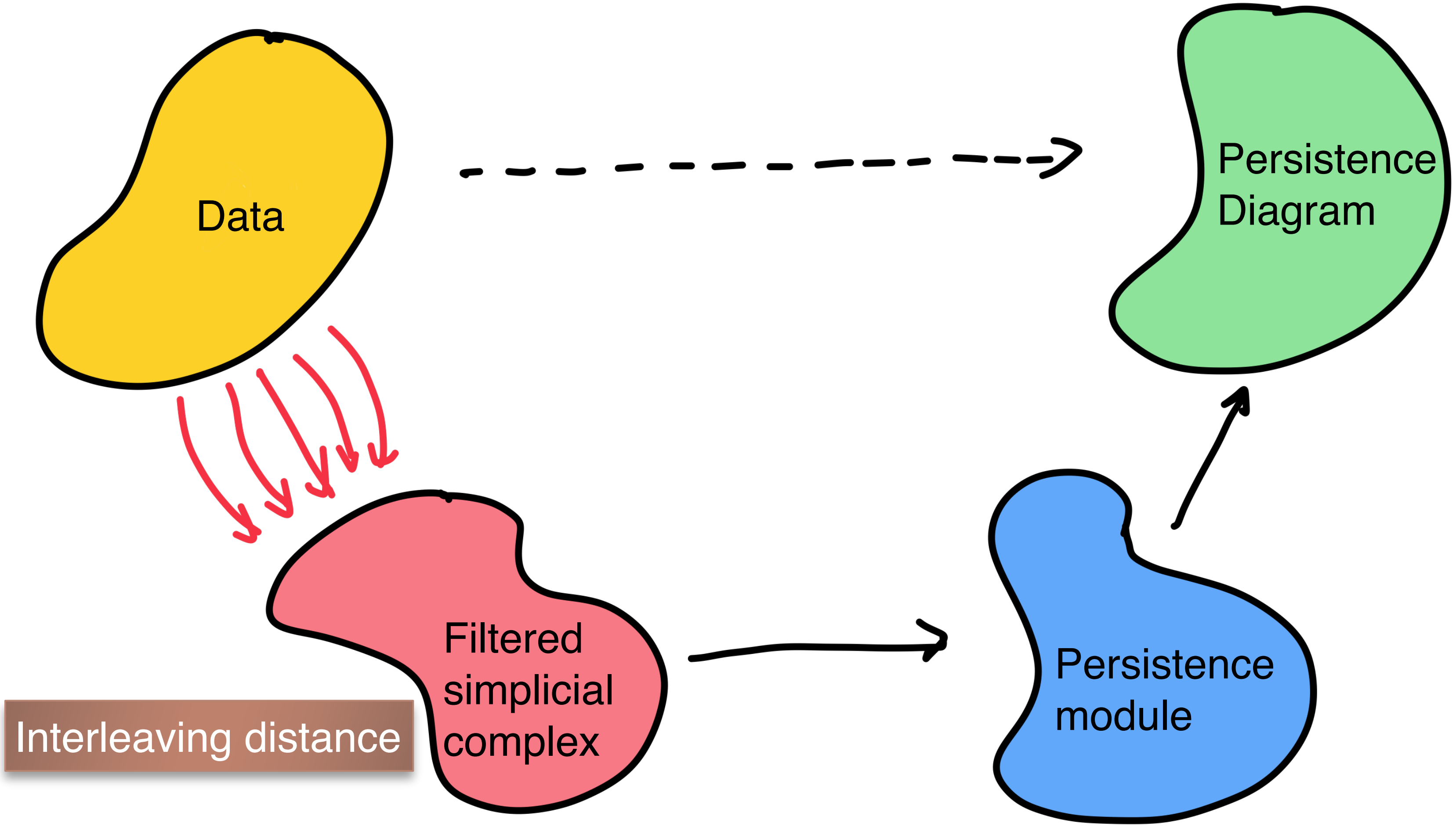
- A map $f : (X, d_X) \to (Y, d_Y)$ is **distance-preserving** if $d_Y(f(x), f(x')) = d_X(x, x'), \forall x, x' \in X$.
- A distance-preserving bijection is called an **isometry**.
- We say (X, d_X) and (Y, d_Y) are **isometric**, if there is an isometry between them.

Bottleneck distance vs interleaving distance



Interleaving distance between filtrations

Bottleneck distance vs interleaving distance



An educated guess

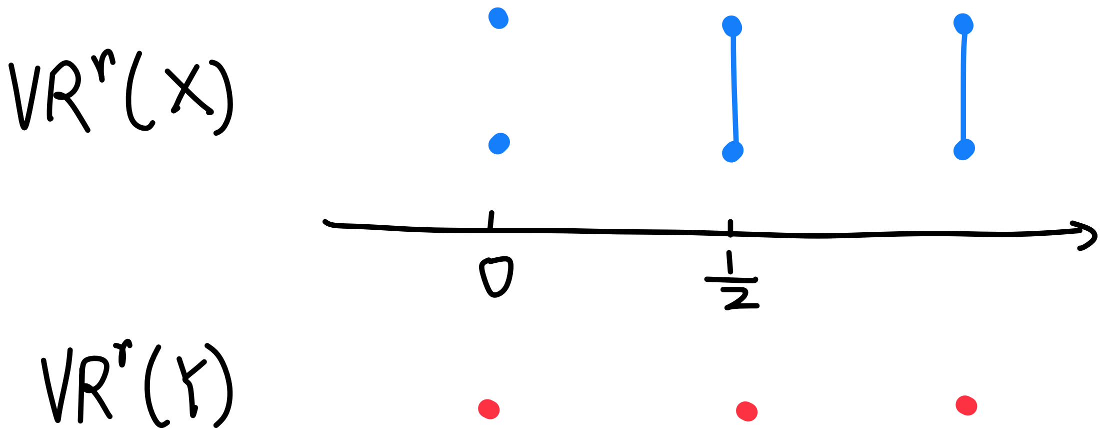
- Given any two simplicial filtrations $\mathscr{X}$ and $\mathscr{Y}$
- **Guess:** We say they are ϵ-interleaved if there exist simplicial maps $\varphi_a : X_a \hookrightarrow Y_{a+\epsilon}$ and $\phi_{a'} : Y_{a'} \hookrightarrow X_{a'+\epsilon}$ such that the following diagram commutes

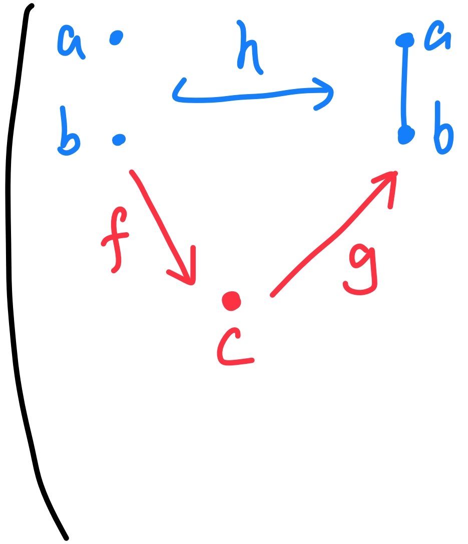
$$
\begin{tikzcd}
\mathscr{X}: & \cdots \subseteq & X_a \arrow[ddrr, "\varphi_a"] & \subseteq \cdots \subseteq & X_{a+\epsilon} \arrow[ddrr, "\varphi_{a+\epsilon}"] & \subseteq \cdots \subseteq & X_{a+2\epsilon} & \subseteq \cdots \\
 & \cdots & & & \cdots & & & \cdots \\
\mathscr{Y}: & \cdots \subseteq & Y_a \arrow[uurr, "\phi_a"'] & \subseteq \cdots \subseteq & Y_{a+\epsilon} \arrow[uurr, "\phi_{a+\epsilon}"'] & \subseteq \cdots \subseteq & Y_{a+2\epsilon} & \subseteq \cdots
\end{tikzcd}
$$

- $d_I(\mathscr{X}, \mathscr{Y}) = \inf_{\epsilon \geq 0} \{\mathscr{X} \text{ and } \mathscr{Y} \text{ are } \epsilon\text{-interleaved}\}$ (also written as $d_{\mathcal{I}}^{\text{strict}}$)

$VR^r(X)$ & $VR^r(Y)$ is NOT ε-interleaved for any finite $\varepsilon > 0$.

- Vietoris-Rips filtration





f can NOT be inj.

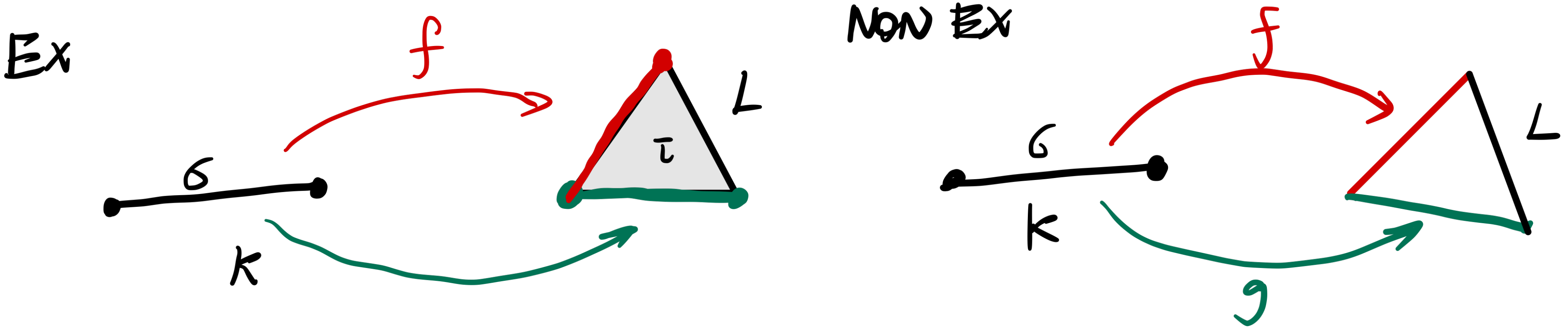
$\Rightarrow$ $g \circ f$ can NOT be inj.

$\Rightarrow$ $g \circ f \neq \underbrace{h}_{\text{(inj.)}}$

- $d_I(VR(X), VR(Y)) = \infty$ with this definition.
- This distance is definitely larger than any reasonable distance between the data sets X and Y. This makes Data $\to$ filtration unstable!
- Thus, we need to correct the definition by relaxing the interleaving constraint.

Contiguity

- Two simplicial maps $f, g : K \to L$ are **contiguous** if for any $\sigma \in \Sigma_K$ there exists a simplex $\tau \in \Sigma_L$ such that $f(\sigma) \cup g(\sigma) \subseteq \tau$



- Proposition 1: if simplicial maps $f, g : K \to L$ are contiguous, then
  - the induced maps $f, g : |K| \to |L|$ are homotopic.
  - $f_* : H_*(K) \to H_*(L)$ is the same map as $g_* : H_*(K) \to H_*(L)$

General filtered simplicial complexes

- Two simplicial filtrations $\mathscr{X}$ and $\mathscr{Y}$ are **ϵ-interleaved** if there exist **simplicial** maps $\varphi_a : X_a \hookrightarrow Y_{a+\epsilon}$ and $\phi_{a'} : Y_{a'} \hookrightarrow X_{a'+\epsilon}$ such that the following diagram commutes up to **contiguity** → All four pairs are contiguous:

① $\phi_{a+\varepsilon} \circ \varphi_a$ & $u_{a,a+2\varepsilon}$ ② $\varphi_{a+\varepsilon} \circ \phi_a$ & $v_{a,a+2\varepsilon}$ ③ $u_{a+\varepsilon,a+2\varepsilon} \circ \phi_a$ & $\phi_{a+\varepsilon} \circ v_{a,a+\varepsilon}$

④ $v_{a+\varepsilon,a+2\varepsilon} \circ \varphi_a$ & $\varphi_{a+\varepsilon} \circ u_{a,a+\varepsilon}$

$$
\begin{tikzcd}
\mathscr{X}: & \cdots \subseteq X_a \arrow[r, phantom, "\subseteq \cdots \subseteq"] \arrow[dr, "\varphi_a"] & X_{a+\epsilon} \arrow[r, phantom, "\subseteq \cdots \subseteq"] \arrow[dr, "\varphi_{a+\epsilon}"] & X_{a+2\epsilon} \subseteq \cdots \\
\mathscr{Y}: & \cdots \subseteq Y_a \arrow[r, phantom, "\subseteq \cdots \subseteq"] \arrow[ur, "\phi_a"'] & Y_{a+\epsilon} \arrow[r, phantom, "\subseteq \cdots \subseteq"] \arrow[ur, "\phi_{a+\epsilon}"'] & Y_{a+2\epsilon} \subseteq \cdots
\end{tikzcd}
$$

(Let $u_{t,s} : X_t \hookrightarrow X_s$
$v_{t,s} : Y_t \hookrightarrow Y_s$)

- $d_I(\mathscr{X}, \mathscr{Y}) = \inf_{\epsilon \geq 0} \{\mathscr{X} \text{ and } \mathscr{Y} \text{ are } \epsilon\text{-interleaved}\}$ (also written as $d_I^{\text{cont.}}$)

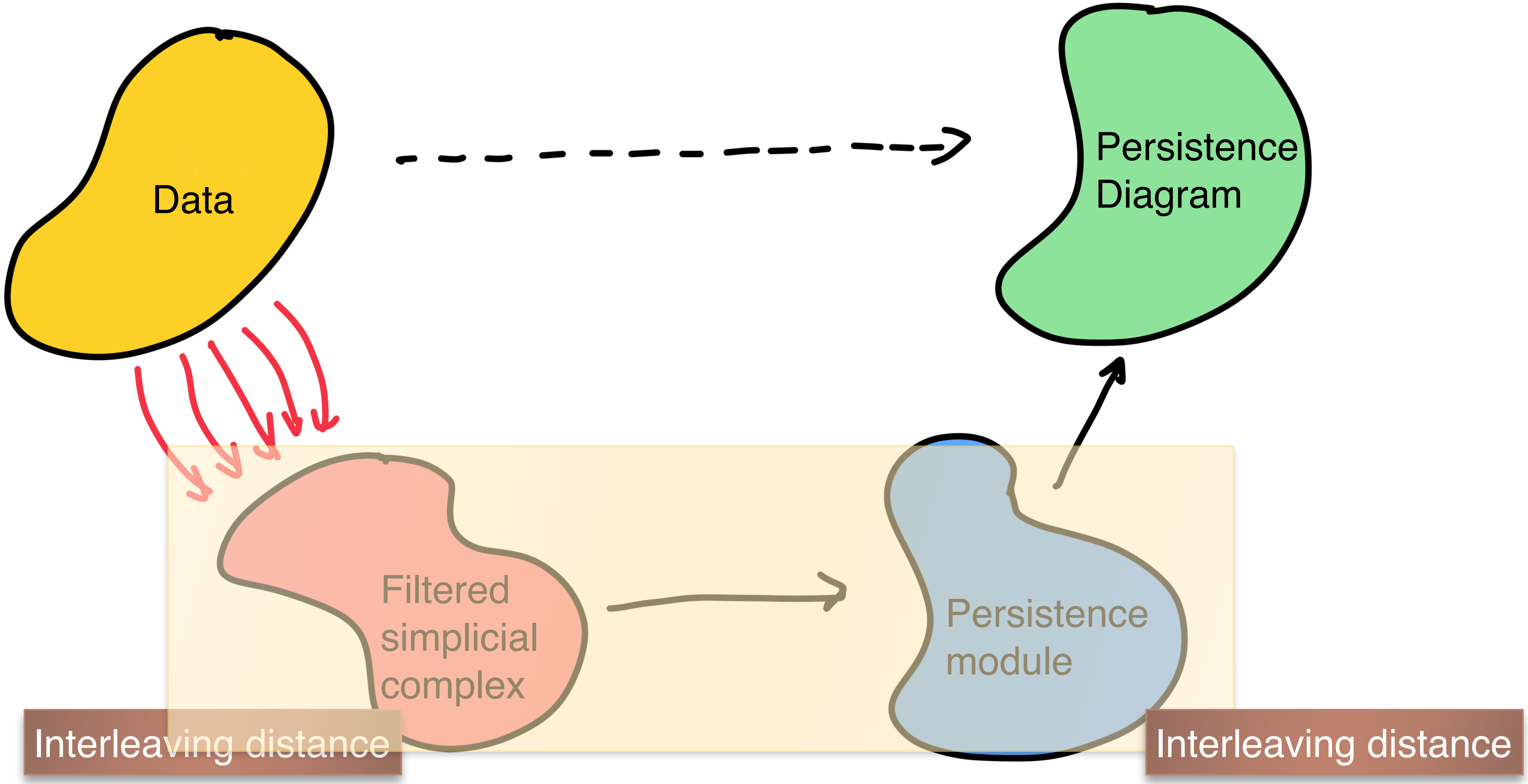
A generalization to simplicial towers

- A **simplicial tower** $\mathscr{X} : \cdots \to X_1 \to X_2 \to \cdots \to X_r \to \cdots$ is a sequence of simplicial complexes connected by simplicial maps (not necessarily inclusions)
- Two simplicial towers $\mathscr{X}$ and $\mathscr{Y}$ are **ϵ-interleaved** if there exist **simplicial** maps $\varphi_a : X_a \to Y_{a+\epsilon}$ and $\phi_{a'} : Y_{a'} \to X_{a'+\epsilon}$ such that the following diagram commutes up to **contiguity**

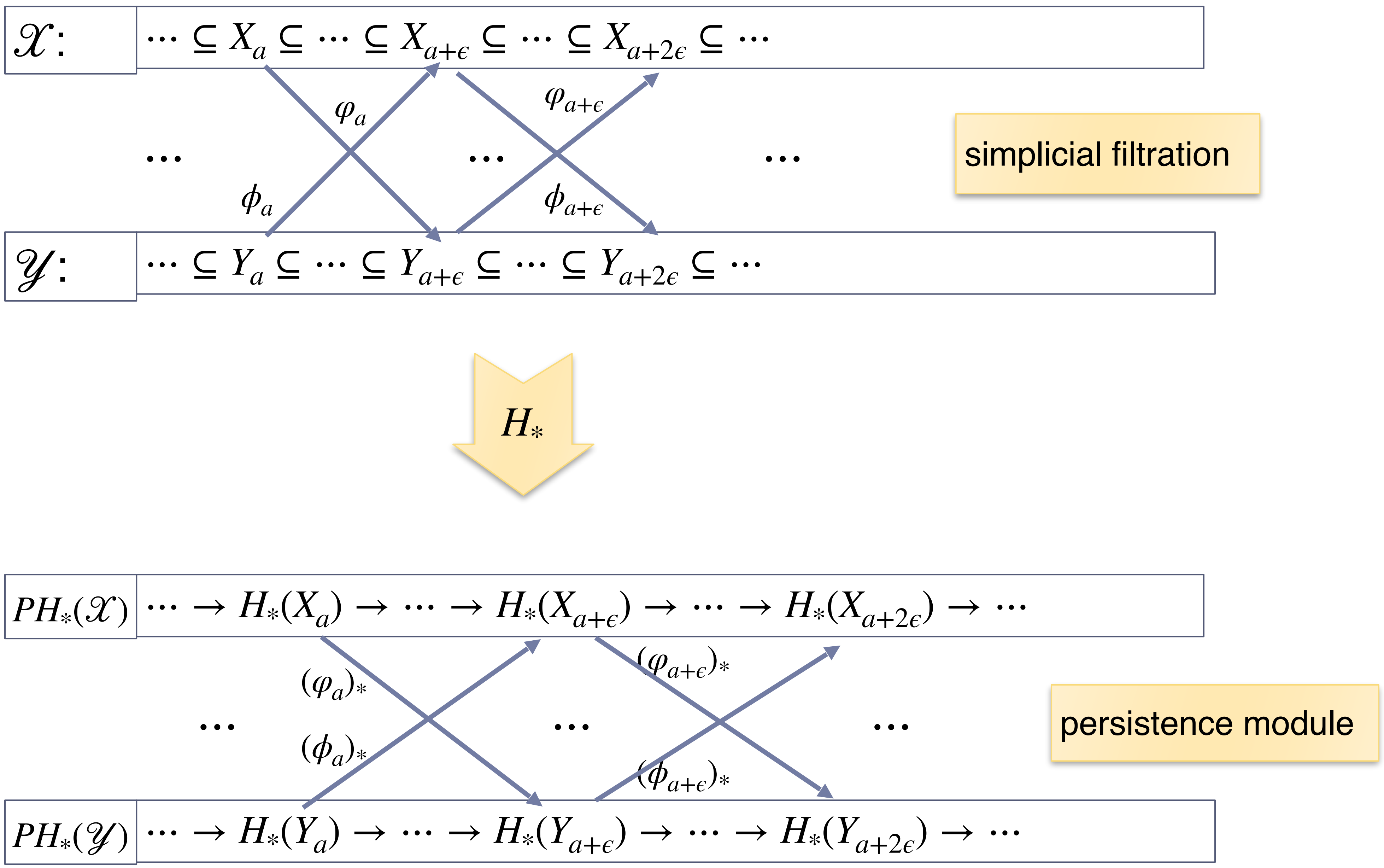
$$
\begin{tikzcd}
\mathscr{X}: & \cdots \arrow[r] & X_a \arrow[r] \arrow[ddrr, "\varphi_a"] & \cdots \arrow[r] & X_{a+\epsilon} \arrow[r] \arrow[ddrr, "\varphi_{a+\epsilon}"] & \cdots \arrow[r] & X_{a+2\epsilon} \arrow[r] & \cdots \\
& \cdots & & \cdots & & \cdots & & \\
\mathscr{Y}: & \cdots \arrow[r] & Y_a \arrow[r] \arrow[uurr, "\phi_a"] & \cdots \arrow[r] & Y_{a+\epsilon} \arrow[r] \arrow[uurr, "\phi_{a+\epsilon}"] & \cdots \arrow[r] & Y_{a+2\epsilon} \arrow[r] & \cdots
\end{tikzcd}
$$

- $d_I(\mathscr{X}, \mathscr{Y}) = \inf_{\epsilon \geq 0} \{\mathscr{X} \text{ and } \mathscr{Y} \text{ are } \epsilon\text{-interleaved}\}$

Interleaving distance vs interleaving distance



$$d_I(PH_p(\mathscr{X}), PH_p(\mathscr{X})) \leq d_I(\mathscr{X}, \mathscr{Y})$$


$\mathscr{X}$: $\cdots \subseteq X_a \subseteq \cdots \subseteq X_{a+\epsilon} \subseteq \cdots \subseteq X_{a+2\epsilon} \subseteq \cdots$
φ_a
ϕ_a
$\varphi_{a+\epsilon}$
$\phi_{a+\epsilon}$
$\cdots$ $\cdots$ $\cdots$
simplicial filtration
$\mathscr{Y}$: $\cdots \subseteq Y_a \subseteq \cdots \subseteq Y_{a+\epsilon} \subseteq \cdots \subseteq Y_{a+2\epsilon} \subseteq \cdots$
H_*
$PH_*(\mathscr{X})$ $\cdots \to H_*(X_a) \to \cdots \to H_*(X_{a+\epsilon}) \to \cdots \to H_*(X_{a+2\epsilon}) \to \cdots$
$(\varphi_a)_*$
$(\phi_a)_*$
$(\varphi_{a+\epsilon})_*$
$(\phi_{a+\epsilon})_*$
$\cdots$ $\cdots$ $\cdots$
persistence module
$PH_*(\mathscr{Y})$ $\cdots \to H_*(Y_a) \to \cdots \to H_*(Y_{a+\epsilon}) \to \cdots \to H_*(Y_{a+2\epsilon}) \to \cdots$

Stability of PH (v.s. simplicial filtration)

- An ϵ-interleaving between simplicial filtrations induces an ϵ-interleaving between persistence homology!

Theorem 2

Given two simplicial filtrations $\mathcal{X}$ and $\mathcal{Y}$, let $PH_p(\mathcal{X})$ and $PH_p(\mathcal{Y})$ be the corresponding p-th persistence homology induced by them. We then have:

$$d_I(PH_p(\mathcal{X}), PH_p(\mathcal{X})) \leq d_I(\mathcal{X}, \mathcal{Y})$$

Theorem holds for simplicial **towers** as well.

Remark

- This is not an isometry
- Show example for $d_I(PH_p(X_\cdot), PH_p(Y_\cdot)) < \overbrace{d_I^{\text{strict}}}^{\text{defined using stricted commutativity}}(X_\cdot, Y_\cdot)$

Let $X_\cdot = VR(\vdots)$, $Y_\cdot = VR(\cdot)$. Let $dgm_p(X_\cdot) = PD(PH_p(X_\cdot))$

(1) $dgm_0(X_\cdot) = \{[0,\infty), [0,\frac{1}{2})\}$ $\quad dgm_0(Y_\cdot) = \{[0,\infty)\}$

$$d_I(PH_0(X_\cdot), PH_0(Y_\cdot)) = d_B(dgm_0(X_\cdot), dgm_0(Y_\cdot)) = \frac{\frac{1}{2}-0}{2} = \frac{1}{4}$$

(2) We have seen $d_I^{\text{strict}}(X_\cdot, Y_\cdot) = \infty > \frac{1}{4}$.

Remark

defined using commutativity up to contiguity.

- This is not an isometry
- Show example for $d_I(PH_p(X_\bullet), PH_p(Y_\bullet)) < \underbrace{d_I^{\text{cont.}}}(X_\bullet, Y_\bullet)$

Attempt 1 $X_\bullet = VR(\vdots)$, $Y_\bullet = VR(\cdot)$. Let $dgm_p(X_\bullet) = PD(PH_p(X_\bullet))$

(1) $dgm_0(X_\bullet) = \{[0,\infty), [0,\frac{1}{2})\}$ $dgm_0(Y_\bullet) = \{[0,\infty)\}$

$$d_I(PH_0(X_\bullet), PH_0(Y_\bullet)) = d_B(dgm_0(X_\bullet), dgm_0(Y_\bullet)) = \frac{\frac{1}{2}-0}{2} = \frac{1}{4}$$

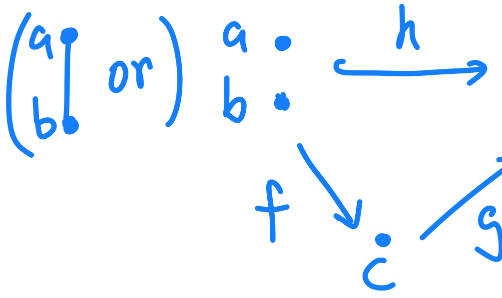
(2) Claim: $X_\bullet$ & $Y_\bullet$ are $\frac{1}{4}$-interleaved. (see next page)

$\Rightarrow d_I(X_\bullet, Y_\bullet) \le \frac{1}{4} \Rightarrow d_I(X_\bullet, Y_\bullet) = \frac{1}{4}$

$\underset{\frac{1}{4}}{\vee|}$ by (1)

(1) & (2) $\Rightarrow$ NOT the desired example.

(2) Claim: $X_\bullet$ & $Y_\bullet$ are $\frac{1}{4}$-interleaved.



$g: c \mapsto b$

$g \circ f: a \mapsto c \mapsto b$

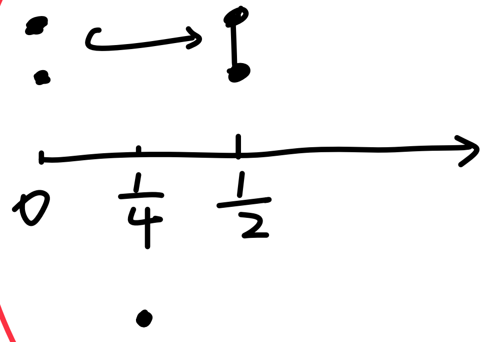
$\qquad\quad b \mapsto c \mapsto b$

$h: a \mapsto a$

$\qquad b \mapsto b$

$g \circ f$ & h are contiguous, because $g\circ f(a) \cup h(a) = \{a, b\} \in$ [edge]

$g\circ f(b) \cup h(b) = \{b\} \in$ [edge]

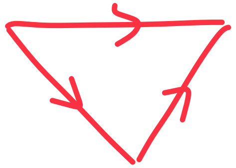


For any r, define

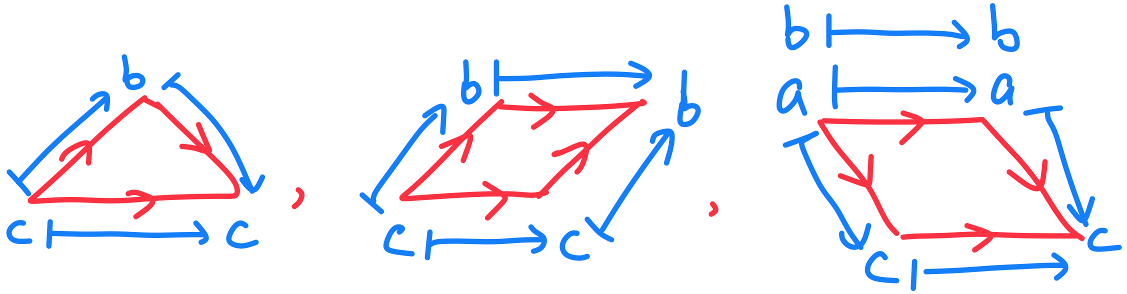
$$\varphi_r: VR^r(X) \longrightarrow VR^{r+\frac{1}{4}}(Y), \quad \psi_r: VR^r(Y) \longrightarrow VR^{r+\frac{1}{4}}(X)$$

$a \longmapsto c \qquad\qquad c \longmapsto b$

$b \longmapsto c$



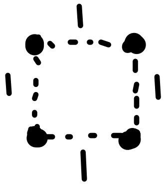
commutes up to contiguity.

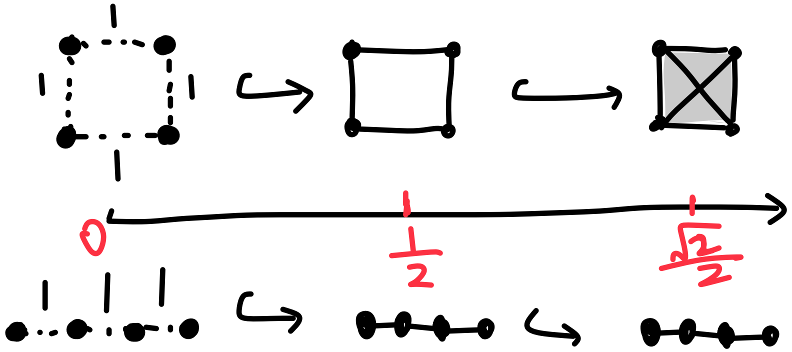
For 

we have strict commutativity.

Remark

- This is not an isometry
- Show example for $d_I\left(PH_p(X_\bullet), PH_p(Y_\bullet)\right) < \overbrace{d_I^{\text{cont.}}(X_\bullet, Y_\bullet)}^{\text{defined using commutativity up to contiguity.}}$

<u>Working example</u> Let $X =$  , $Y =$  , $X_\bullet = VR(X)$, $Y_\bullet = VR(Y)$.



$$\begin{aligned} dgm_0(X_\bullet) &= \left\{[0,\infty), [0,\tfrac{1}{2}), [0,\tfrac{1}{2}), [0,\tfrac{1}{2})\right\} \\ &= dgm_0(Y_\bullet) \end{aligned}$$

$$\Rightarrow d_I\left(PH_0(X_\bullet), PH_0(Y)\right) = 0$$

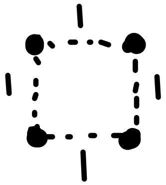
Claim: $X_\bullet$ & $Y_\bullet$ can NOT be ε-contiguous for any $\varepsilon < \frac{\sqrt{2}}{2} - \frac{1}{2}$.

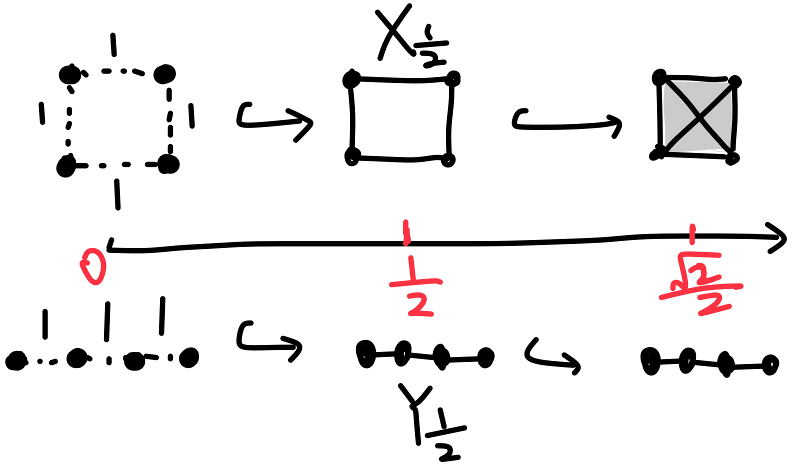
If claim holds, then $d_I^{\text{cont.}}(X_\bullet, Y_\bullet) \geq \frac{1}{2}(\sqrt{2}-1) > 0$

Remark

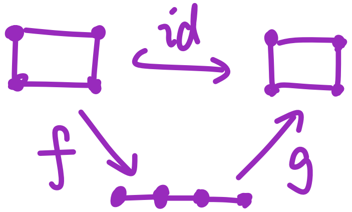
defined using commutativity up to contiguity.

- This is not an isometry
- Show example for $d_I(PH_p(X_\cdot), PH_p(Y_\cdot)) < d_I^{cont.}(X_\cdot, Y_\cdot)$

Working example: Let $X =$  , $Y =$  , $X_\cdot = VR(X)$, $Y_\cdot = VR(Y)$.



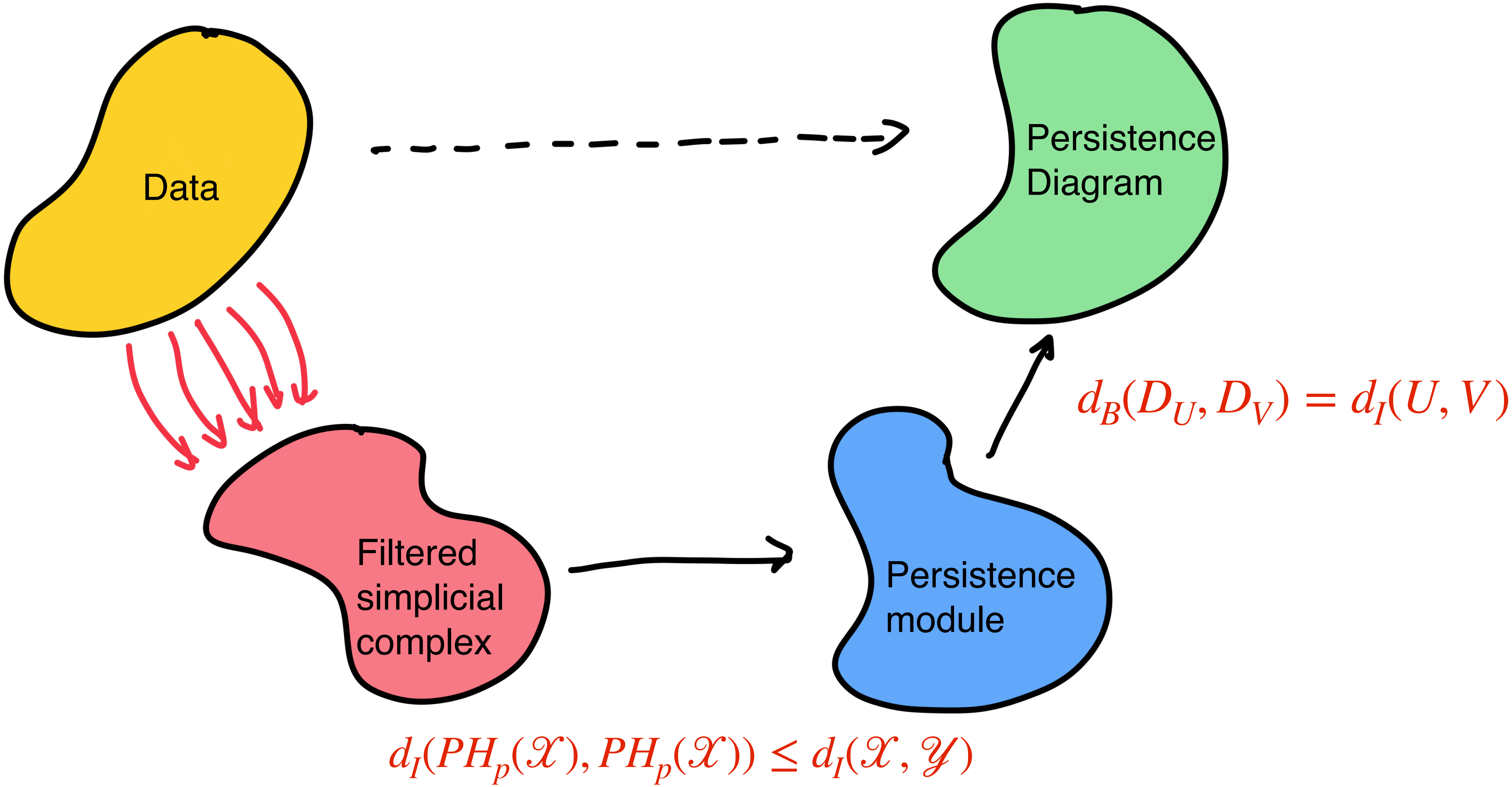
Claim: $X_\cdot$ & $Y_\cdot$ can NOT be ε-contiguous for any $\varepsilon < \frac{\sqrt{2}}{2} - \frac{1}{2}$.

proof of claim: Otherwise, we have 

$g \circ f$ & id contiguous $\Rightarrow$ they induce same map on H_1

$(id)_* = id : H_1(X_{\frac{1}{2}}) \to H_1(X_{\frac{1}{2}})$ but

$(g\circ f)_* : H_1(X_{\frac{1}{2}}) \longrightarrow H_1(Y_{\frac{1}{2}}) = 0 \longrightarrow H_1(X_{\frac{1}{2}})$ can only be $0 \neq (id)_*$ } contradiction.

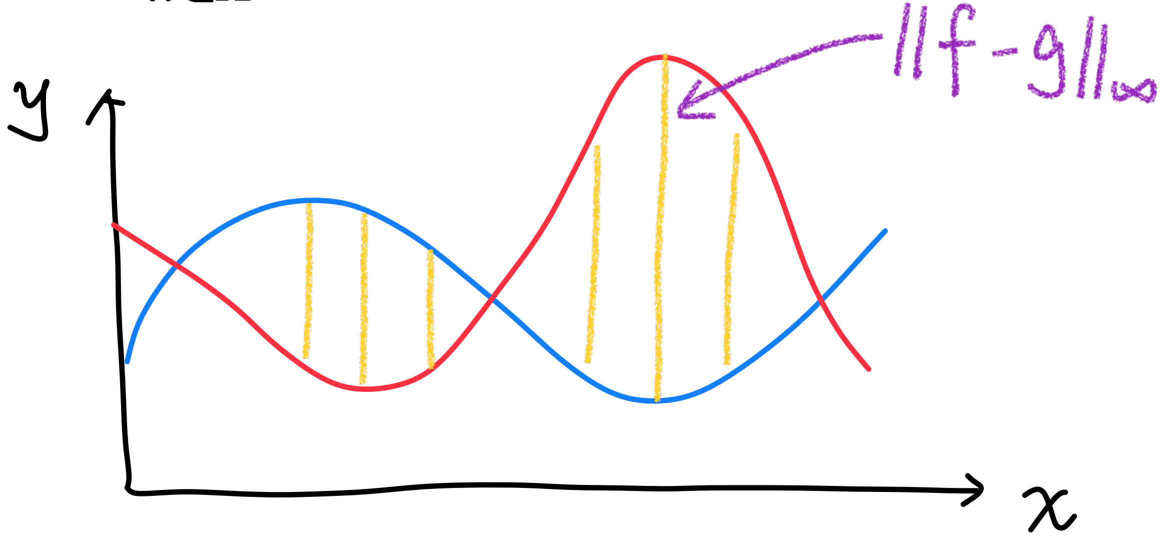


Data
Persistence Diagram
Filtered simplicial complex
Persistence module
$d_B(D_U, D_V) = d_I(U, V)$
$d_I(PH_p(\mathscr{X}), PH_p(\mathscr{X})) \leq d_I(\mathscr{X}, \mathscr{Y})$

Stability for function-induced persistence

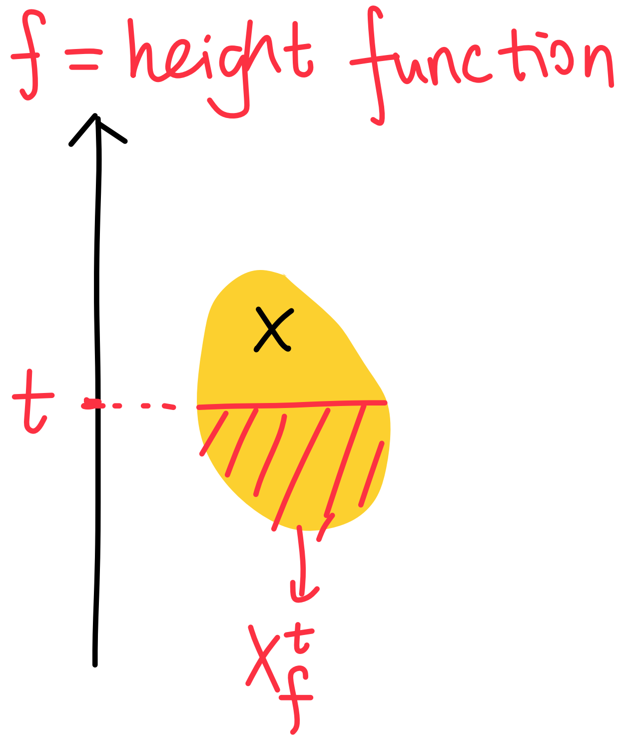
Functions on a given space

- Let X be a set (e.g., X is a manifold or a subset in $\mathbb{R}^d$)
- Consider the collection of functions $f : X \to \mathbb{R}$
- A natural distance between $f, g : X \to \mathbb{R}$ is the ℓ^∞ distance
  - $\|f - g\|_\infty := \sup_{x \in X} |f(x) - g(x)|$



Sublevel set filtration

- Given a topological space X and a function $f : X \to \mathbb{R}$, for any t, let $X_f^t := f^{-1}(-\infty, t]$.



- The **sublevel set filtration** is $X_f = \{X_f^t\}_t$.

X_f is a well-defined filtration,

because: $\forall\, t \le s,\ \underbrace{f^{-1}(-\infty, t]}_{= X_f^t} \subset \underbrace{f^{-1}(-\infty, s]}_{= X_f^s}$

- The **super-level set filtration** is analogously defined as $\{f^{-1}[t, \infty)\}_t$.

(filtration direction is reversed.)

Sublevel set filtration

- Given a topological space X and two functions $f, g : X \to \mathbb{R}$
- Proposition 2: Let $\epsilon = \|f - g\|_\infty$. Then the two sub level set filtrations $X_f = \{X_f^t\}_t$ and $X_g = \{X_g^t\}_t$ are ϵ-interleaved. Thus, $d_I(X_f, X_g) \leq \|f - g\|_\infty$

proof Claim: $\boxed{X_f^a \subseteq X_g^{a+\varepsilon}}$ (similarly, $X_g^a \subseteq X_f^{a+\varepsilon}$)

$$\left.\begin{aligned}
&\forall x \in X_f^a = f^{-1}(-\infty, a], \text{ we have } f(x) \leq a \\
&|f(x) - g(x)| \leq \varepsilon \Longrightarrow g(x) \leq f(x) + \varepsilon \leq a + \varepsilon \\
&\qquad\qquad\qquad \Longrightarrow x \in g^{-1}(-\infty, a+\varepsilon] = X_g^{a+\varepsilon}
\end{aligned}\right\} \Rightarrow X_f^a \subseteq X_g^{a+\varepsilon}$$

Sublevel set filtration

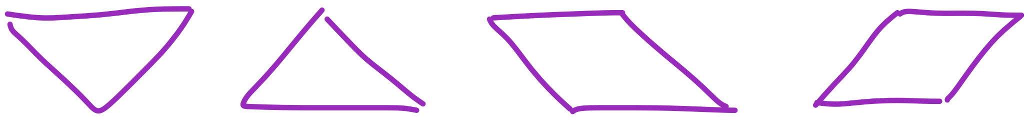
- Given a topological space X and two functions $f, g : X \to \mathbb{R}$
- Proposition 2: Let $\epsilon = \|f - g\|_\infty$. Then the two sub level set filtrations $X_f = \{X_f^t\}_t$ and $X_g = \{X_g^t\}_t$ are ϵ-interleaved. Thus, $d_I(X_f, X_g) \le \|f - g\|_\infty$

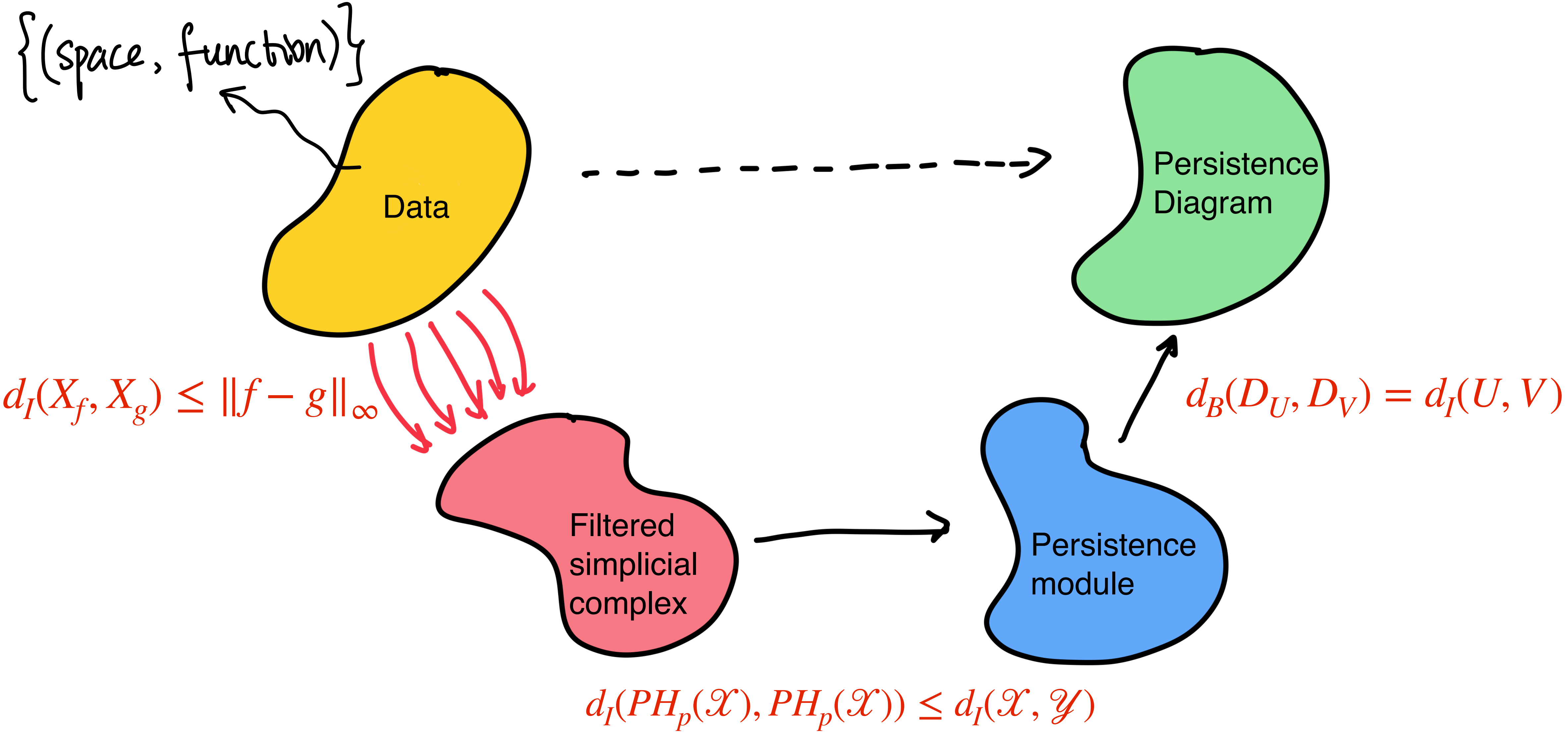
proof Claim: $X_f^a \overset{\varphi_a}{\subseteq} X_g^{a+\varepsilon}$ (similarly, $X_g^a \overset{\phi_a}{\subseteq} X_f^{a+\varepsilon}$)

$$
\begin{tikzcd}
X_f: & \cdots \subseteq X_f^a \arrow[dr, "\varphi_a"] \subseteq \cdots & \subseteq X_f^{a+\epsilon} \arrow[dr, "\varphi_{a+\epsilon}"] \subseteq \cdots & \subseteq X_f^{a+2\epsilon} \subseteq \cdots \\
\cdots & & \cdots & \cdots \\
X_g: & \cdots \subseteq X_g^a \arrow[ur, "\phi_a"] \subseteq \cdots & \subseteq X_g^{a+\epsilon} \arrow[ur, "\phi_{a+\epsilon}"] \subseteq \cdots & \subseteq X_g^{a+2\epsilon} \subseteq \cdots
\end{tikzcd}
$$

Define φ_a & ϕ_a as inclusions.

All diagrams commute:




{(space, function)}
Data
Persistence Diagram
$d_I(X_f, X_g) \leq \|f - g\|_\infty$
Filtered simplicial complex
Persistence module
$d_B(D_U, D_V) = d_I(U, V)$
$d_I(PH_p(\mathscr{X}), PH_p(\mathscr{X})) \leq d_I(\mathscr{X}, \mathscr{Y})$

Stability - Function induced persistence

(want $PH(X_f)$ to be finitely presented.)

Stability Theorem [Cohen-Steiner et al 2007]

Given two "nice" functions $f,\ g: X \to R$, let D^*_f and D^*_g be the persistence diagrams for the persistence modules induced by the sub-level set (resp. super-level set) filtrations w.r.t f and g, respectively. We then have:

$$d_B(D^*_f, D^*_g) = d_I(PH_*(X_f), PH_*(X_g)) \leq \|f - g\|_\infty$$

isometry theorem between PD & persistence modules

Proposition 2 $\Rightarrow$ $d_I(X_f, X_g) \leq \|f-g\|_\infty$

Theorem 2 $\Rightarrow$ $d_I(PH(X_f), PH(X_g)) \leq d_I(X_f, X_g)$