MATH412/COMPSCI434/MATH713
Fall 2025

Topological Data Analysis

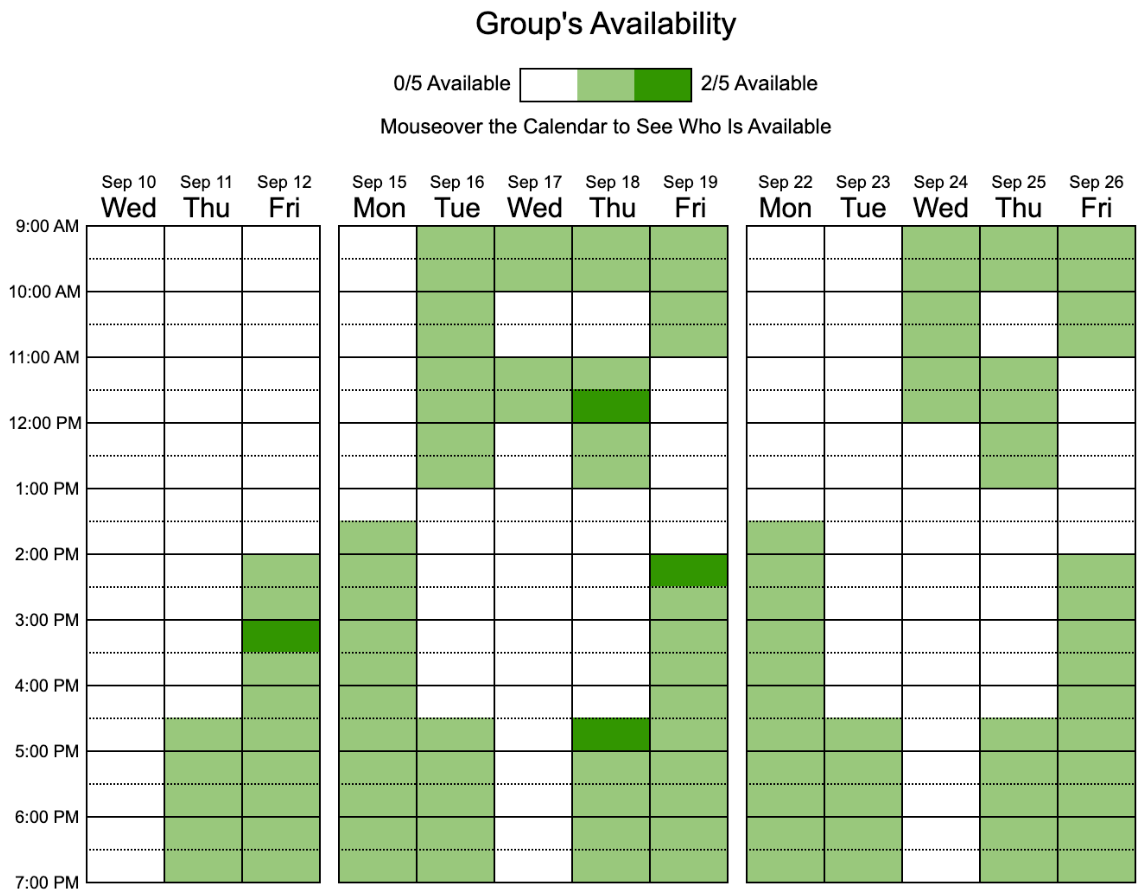
Topic 2: Simplicial Complexes - Part 3
Topic 3: Simplicial Homology - Part 1

Instructor: Ling Zhou

Announcement

Book appointment (Weeks 3-5) with me to discuss project ideas

https://www.when2meet.com/?32189665-Lb3QT (check Canvas announcement)



Simplicial Complexes: Common Choices

Review

(1) Nerve complex $\left|Nrv(\{U_1, \cdots, U_m\})\right| \xrightarrow[\text{Nerve Lemma}]{\simeq} X$ (assuming $\{U_i\}$ is a "good cover" of X)

- Vertices $= \{1, \cdots, m\}$
- Simplices $= \left\{ (i_0, \cdots, i_k) \;\middle|\; \bigcap_{j=0}^{k} U_{i_j} \neq \phi \right\}$

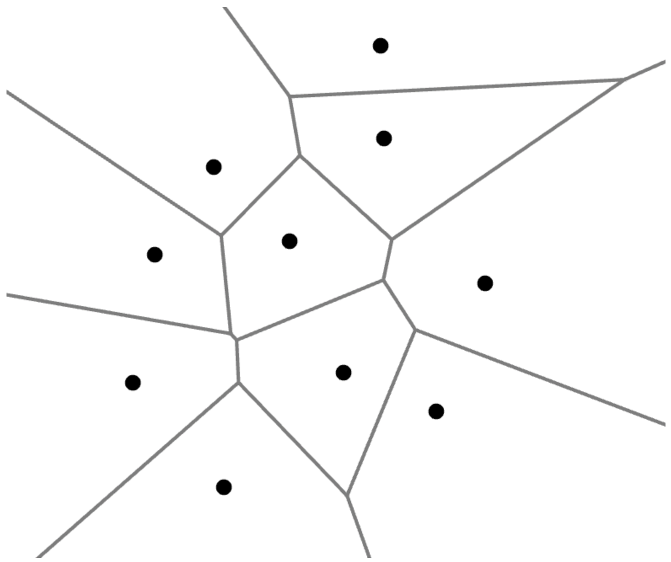
(2) Čech complex $\left|C^r(P)\right| := \left|Nrv(\{B(p_i, r) \mid p_i \in P\})\right| \xrightarrow[\text{Nerve Lem.}]{\simeq} \bigcup_{p_i \in P} B(p_i, r)$

$P = \{p_1, \cdots, p_m\} \subset \mathbb{R}^d$

(3) Alpha complex $Del^r(P) := Nrv(\{B(p_i, r) \cap Vor(p_i) \mid p_i \in P\})$

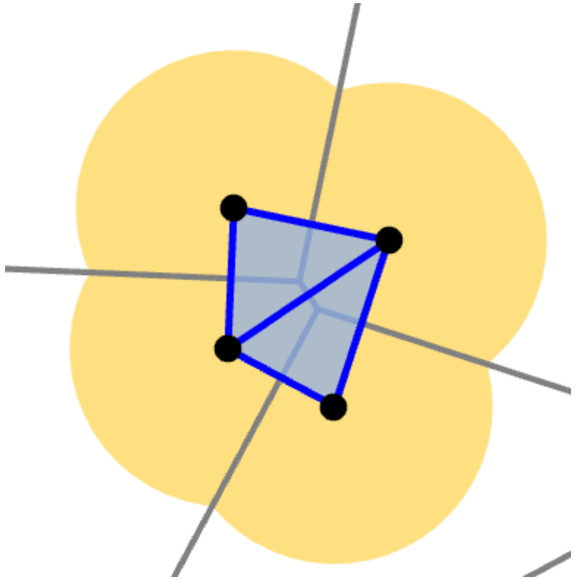
Recall: Voronoi Diagram

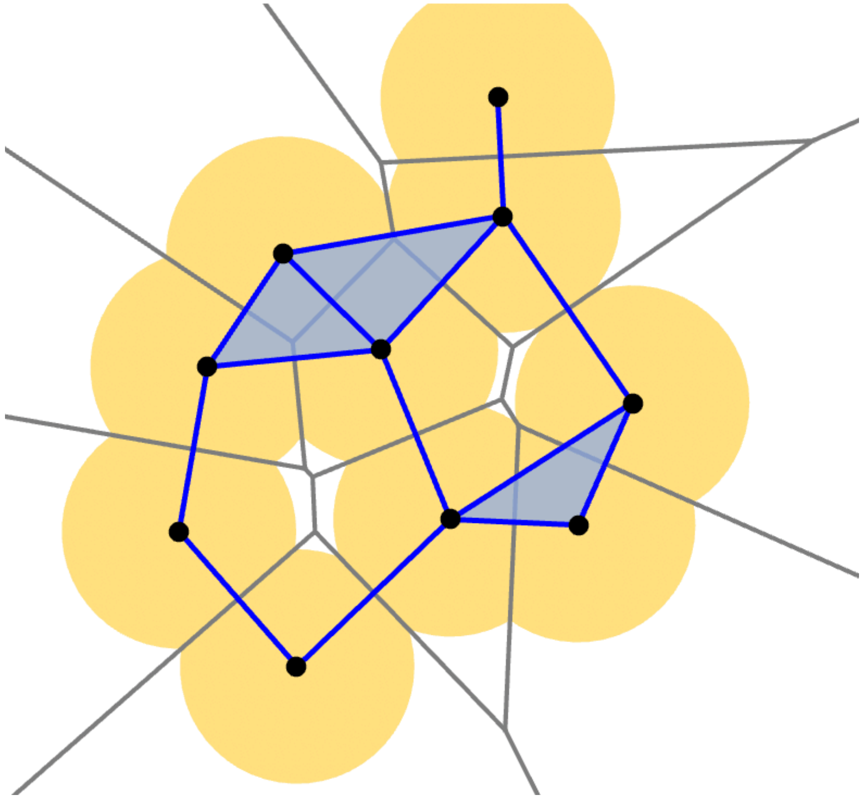
- Given a finite set $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^d$, the **Voronoi cell** of p_i is
  - $Vor(p_i) = \{x \in \mathbb{R}^d \,|\, \|x - p_i\| \leq \|x - p_j\|, \forall j \neq i\}$
- The **Voronoi Diagram** of P is the collection of all Voronoi cells.



Recall: Alpha Complex

- Given a set of points $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^d$
- Given a real value $r > 0$, the *Alpha complex* $Del^r(P)$ is the nerve of the set $\{B(p_i, r) \cap Vor(p_i)\}_{i=1}^n$



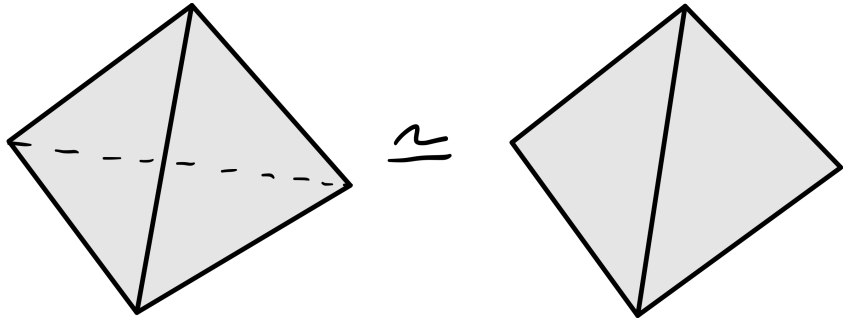


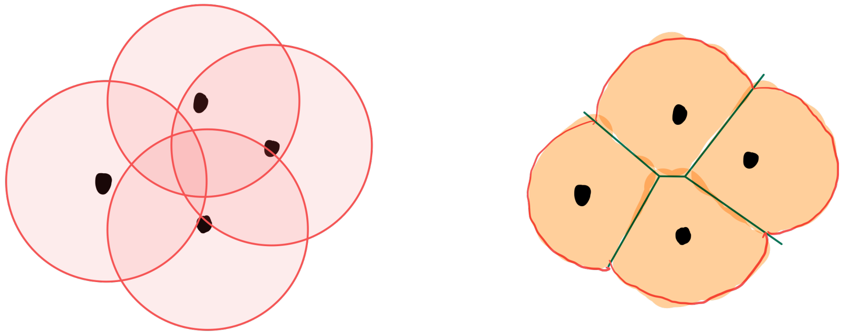
Alpha complex vs Čech complex

- $Del^r(P) \subset C^r(P)$

$$Del^r(P) := Nrv\left(\{B(p_i, r) \cap Vor(p_i) \mid p_i \in P\}\right)$$

$$C^r(P) := Nrv\left(\{B(p_i, r) : p_i \in P\}\right)$$





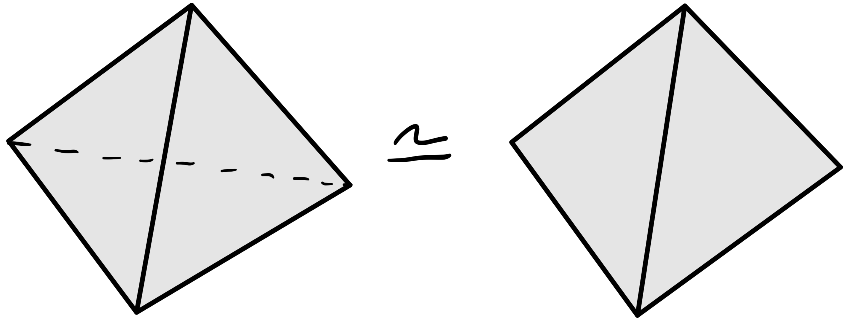
Čech Alpha

- Proposition: $Del^r(P) \simeq C^r(P) \simeq \cup_p B(p, r)$, i.e, $C^r(P)$ and $Del^r(P)$ are homotopy equivalent.

proof: $Del^r(P) \simeq \bigcup_p \left(B(p,r) \cap Vor(p)\right) = \left(\bigcup_p B(p,r)\right) \cap \left(\bigcup_p Vor(p)\right)$

$C^r(P) \cong \bigcup_p B(p,r)$ $\quad = \left(\bigcup_p B(p,r)\right) \cap \mathbb{R}^d = \bigcup_p B(p,r)$

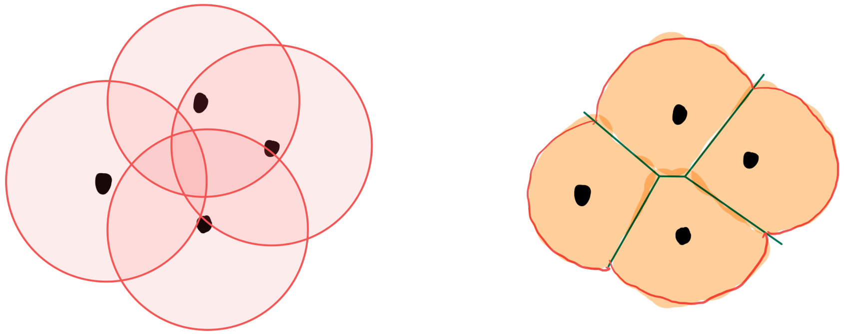
Alpha complex vs Čech complex



- $Del^r(P) \subset C^r(P)$

$Del^r(P) := Nrv\left(\{B(p_i, r) \cap Vor(p_i) \mid p_i \in P\}\right)$

$C^r(P) := Nrv\left(\{B(p_i, r) : p_i \in P\}\right)$



Čech

Alpha

- Proposition: $Del^r(P) \simeq C^r(P) \simeq \cup_p B(p, r)$, i.e, $C^r(P)$ and $Del^r(P)$ are homotopy equivalent.

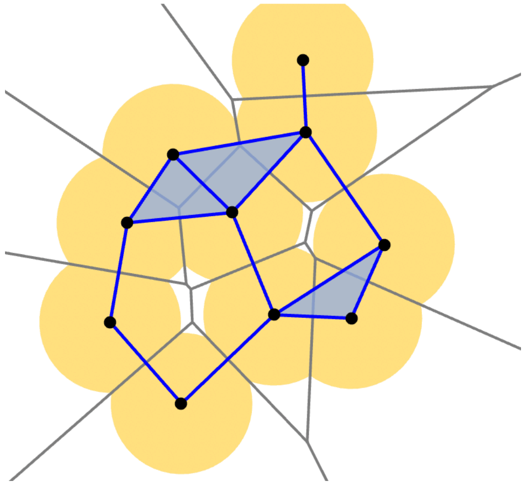
- $|Del^\infty(P)| = O(n^{\frac{d}{2}})$ whereas $|C^\infty(P)| = O(2^n)$
- $\dim Del^r(P) \leq d$ whereas $\dim C^r(P) \leq n$

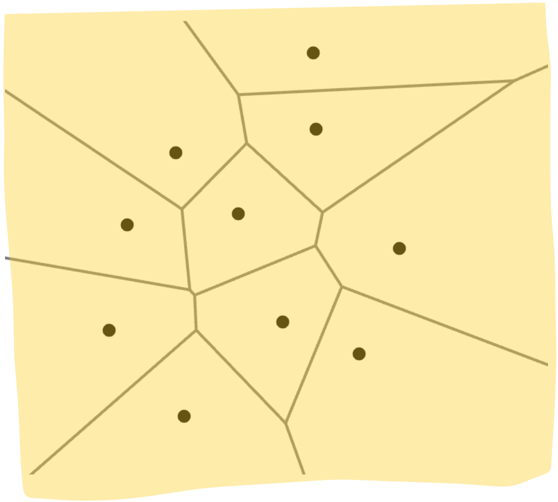
dimention of $P \subset \mathbb{R}^d$

number of points in P.

Delaunay Complex

- The **Delaunay complex** of P, denote by $Del(P)$
  - $Del(P) = Nrv(\left\{Vor(p) \mid p \in P\right\})$





$$\left(\begin{array}{c} Del^r(P) \\ = Nrv\left(\{B(p,r) \cap Vor(p) \mid p \in P\}\right) \end{array}\right) \overset{r \to \infty}{\rightsquigarrow} \left(\begin{array}{c} Del^\infty(P) \\ = Nrv\left(\{Vor(p) \mid p \in P\}\right) \end{array}\right)$$

(1) Nerve complex $\left|\mathrm{Nrv}(\{U_1, \cdots, U_m\})\right| \xrightarrow[\text{Nerve Lemma}]{\simeq} X$ (assuming $\{U_i\}$ is a "good cover" of X)

- Vertices $= \{1, \cdots, m\}$
- Simplices $= \left\{ (i_0, \cdots, i_k) \;\middle|\; \bigcap_{j=0}^{k} U_{i_j} \neq \emptyset \right\}$

(2) Čech complex $\left|C^r(P)\right| := \left|\mathrm{Nrv}(\{B(p_i, r) \mid p_i \in P\})\right| \xrightarrow[\text{Nerve Lem.}]{\simeq} \bigcup_{p_i \in P} B(p_i, r)$

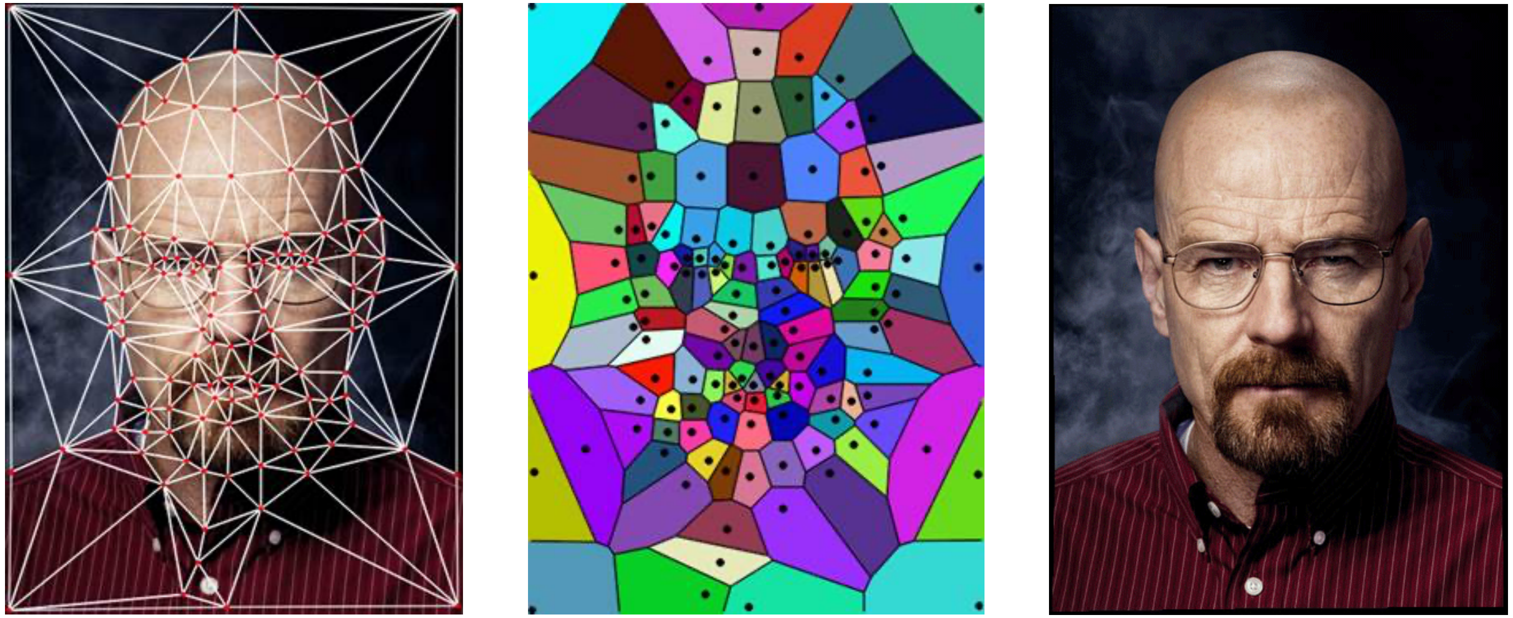
$P = \{p_1, \cdots, p_m\} \subset \mathbb{R}^d$

(3) Alpha complex $\left|\mathrm{Del}^r(P)\right| := \left|\mathrm{Nrv}(\{B(p_i, r) \cap \mathrm{Vor}(p_i) \mid p_i \in P\})\right| \xrightarrow[\text{N.L.}]{\simeq} \bigcup_{p_i \in P} B(p_i, r)$

(4) Delaunay complex $\left|\mathrm{Del}(P)\right| := \left|\mathrm{Nrv}(\{\mathrm{Vor}(p_i) \mid p_i \in P\})\right| \xrightarrow[\text{N.L.}]{\simeq} \bigcup_{p_i} \mathrm{Vor}(p_i) = \mathbb{R}^d$

Delaunay Complex

- Foundation for surface reconstruction and meshing in 3D
  - *[Dey, Curve and Surface Reconstruction, 2006],*
  - *[Cheng, Dey and Shewchuk, Delaunay Mesh Generation, 2012]*
- Face Morphing using Delaunay Triangulation: blog

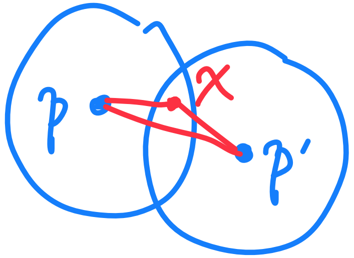


How to turn Walter White to Jesse Pinkman

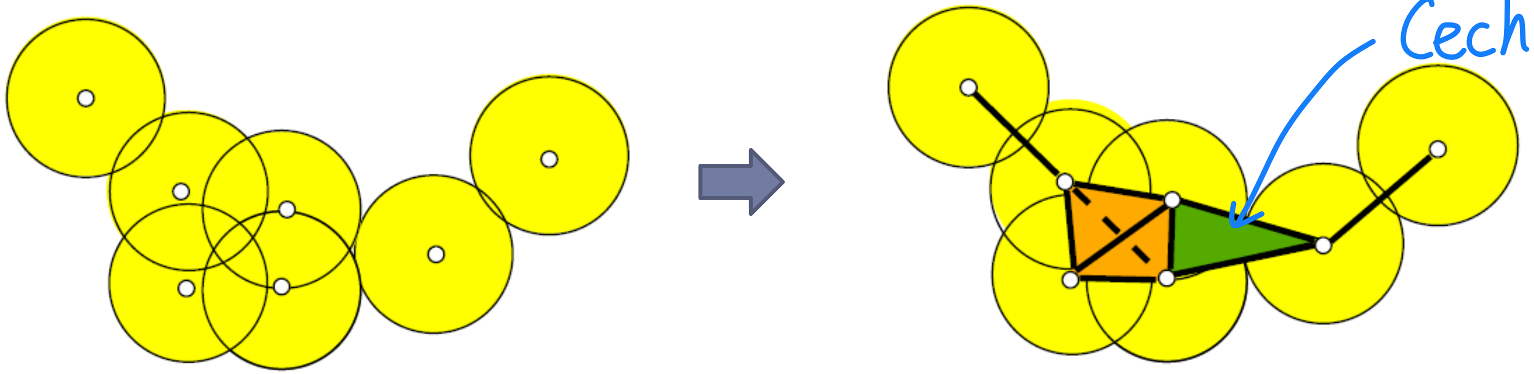
Vietoris-Rips (Rips) Complex

Čech cpx asks for $\bigcap_{j=0}^{k} B(p_{i_j}, r) \neq \emptyset$

- Given a set of points $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^d$
- Given a real value $r > 0$, the *Vietoris-Rips (Rips) complex* $Rips^r(P)$ is:
  - $\left\{ (p_{i_0}, \ldots, p_{i_k}) \mid B(p_{i_l}, r) \cap B(p_{i_j}, r) \neq \emptyset, \forall l, j = 0, \ldots, k \right\}$.
  - Equivalently, $Rips^r(P) = \left\{ (p_{i_0}, \ldots, p_{i_k}) \mid d(p_{i_l}, p_{i_j}) \leq 2r, \forall l, j = 0, \ldots, k \right\}$.



by Δ-inequality: $d(p, p') \leq d(p, x) + d(x, p') \leq 2r$



Čech cpx does NOT have this ▶

Vietoris-Rips (Rips) Complex

- The 1-skeleton of $Rips^r(P)$ is the same as the 1-skeleton of $C^r(P)$

$$\begin{cases} \text{Vertices} = P \\ \text{Edges} = \{ p_i p_j \mid \underbrace{d(p_i, p_j) \leq 2r}_{\Updownarrow} \} \end{cases}$$

1-skeleton of $C^r(P)$

$$\begin{cases} \text{Edges} = \{ p_i p_j \mid B(p_i, r) \cap B(p_j, r) \neq \phi \} \\ \text{Vertices} = P \end{cases}$$

Vietoris-Rips (Rips) Complex

- $Rips^r(P)$ is the **clique complex** of its 1-skeleton
  - If $\{p_{i_k} p_{i_l}\}_{k \neq l \in 0,\ldots,m}$ are edges,
  - then $d(p_{i_k}, p_{i_l}) \leq 2r$ for $k \neq l \in 0,\ldots,m$
  - Hence $\{p_{i_0}, \ldots, p_{i_m}\} \in Rips^r(P)$

makes Rips easier to compute, since it is determined by a graph

Recall

$$\text{graph} \underset{\text{clique}}{\overset{\text{1-skeleton}}{\rightleftarrows}} \text{simplicial complex}$$

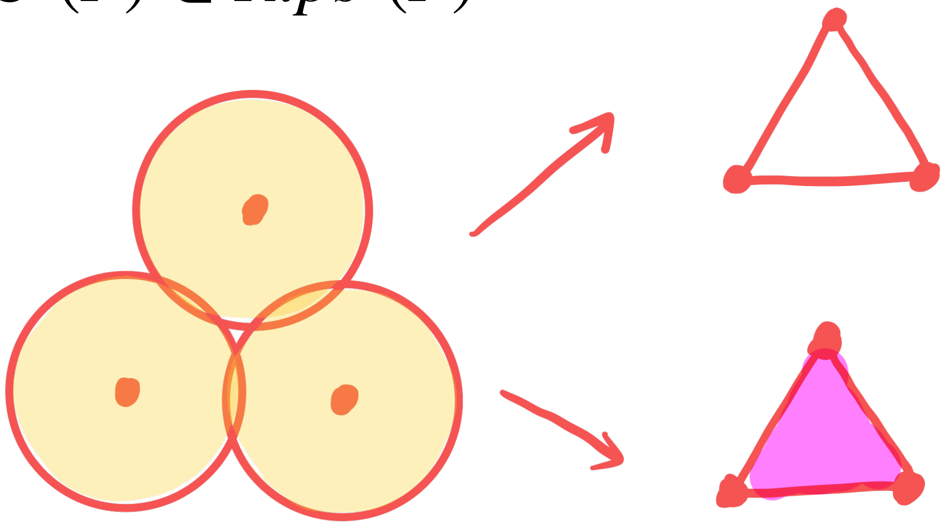
$$G = (V, E) \xmapsto{\text{clique}} K = (V, \Sigma)$$

$\sigma = (x_0, \cdots, x_k) \in \Sigma$ iff $x_i x_j \in E$, $\forall i \neq j$.

Rips vs Čech

- $C^r(P) \subset Rips^r(P)$

$\sigma = x_0 \cdots x_k$



Čech $\quad B(x_0, r) \cap \cdots \cap B(x_k, r) \neq \phi$

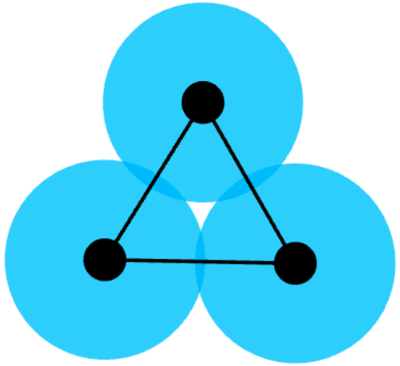
Rips $\quad B(x_i, r) \cap B(x_j, r) \neq \phi, \ \forall i \neq j$

- Cech preserves homotopy type of the union of balls but Rips doesn't
- Rips is the clique complex of its 1-skeleton but Cech is not, making Rips easier to compute
- $C^r(P) \subset Rips^r(P) \subset C^{2r}(P)$
  - $C^r(P) \subseteq Rips^r(P) \subseteq C^{\sqrt{2}r}(P)$ when $P \subseteq \mathbb{R}^d$

$\}$ exercise.

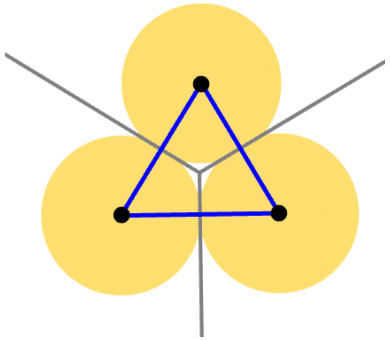
Summary

- A Čech complex is the nerve complex of a union of balls

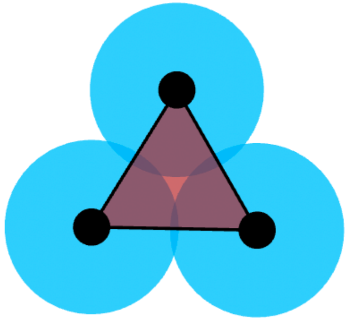


- Nerve lemma says the Čech complex is the **ideal complex** which has the same homotopy type as the union of balls
- Major problems
  - Hard to compute in high dimension
  - Too many simplices

- **Reduce the number of simplices**
  - α complex is the nerve of union of intersections of balls and Voronoi cells



- **Reduce the computation complexity**
  - Rips complex builds a clique complex on the neighborhood graph



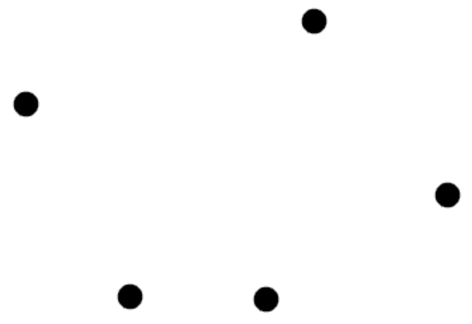
Topic 3: Simplicial Homology

Overview

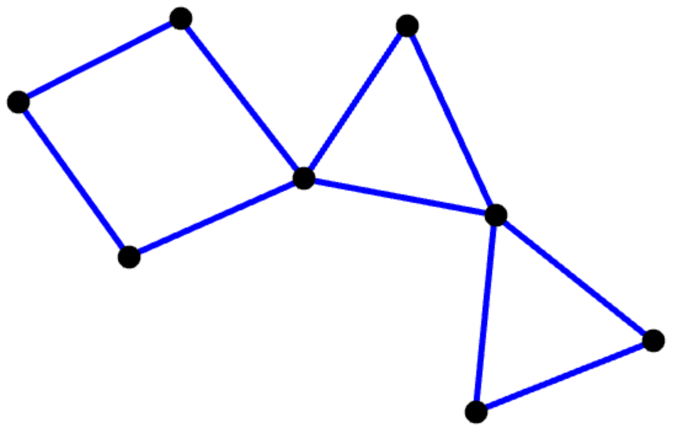
- Motivation
- Chains, boundary maps and chain complex
- Cycles, boundaries and homology groups
- Matrix view
  - Matrix reduction algorithm

Motivating examples

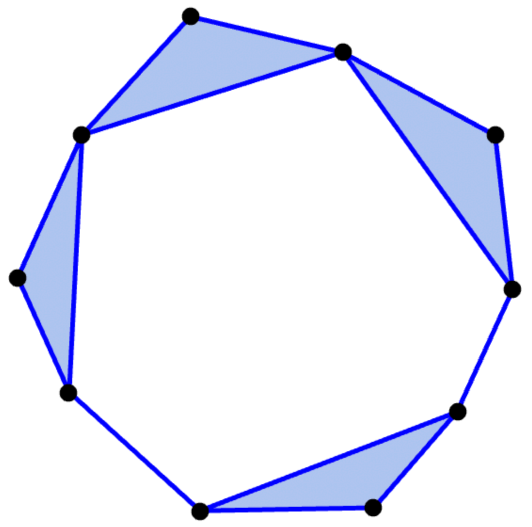
- ith homology "counts the number of i dimensional holes" in a topological space



$\dim H_0 = 5$
$\dim H_1 = 0$



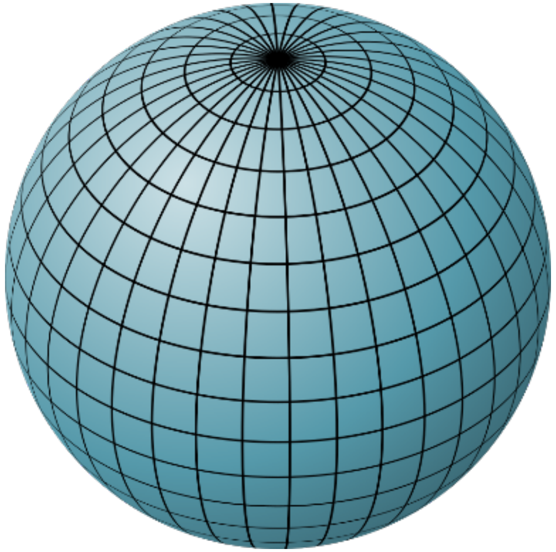
$\dim H_0 = 1$
$\dim H_1 = 3$



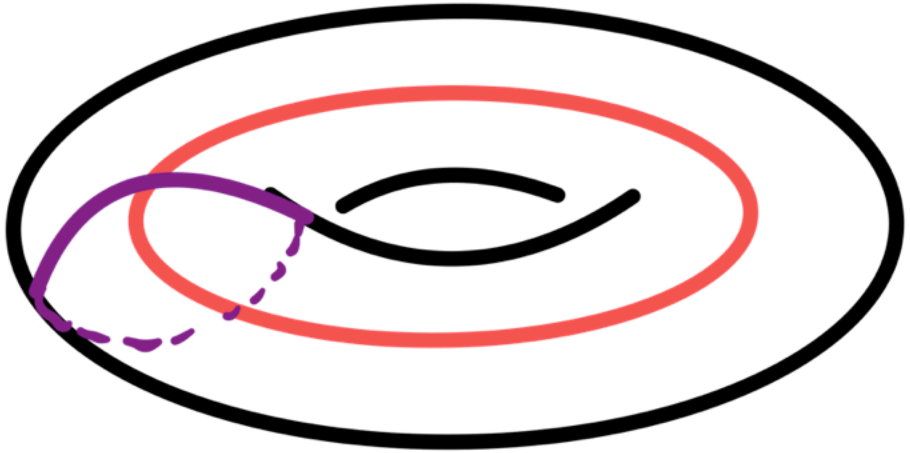
$\dim H_0 = 1$
$\dim H_1 = 1$

Motivating examples

- ith homology “counts the number of i dimensional holes” in a topological space

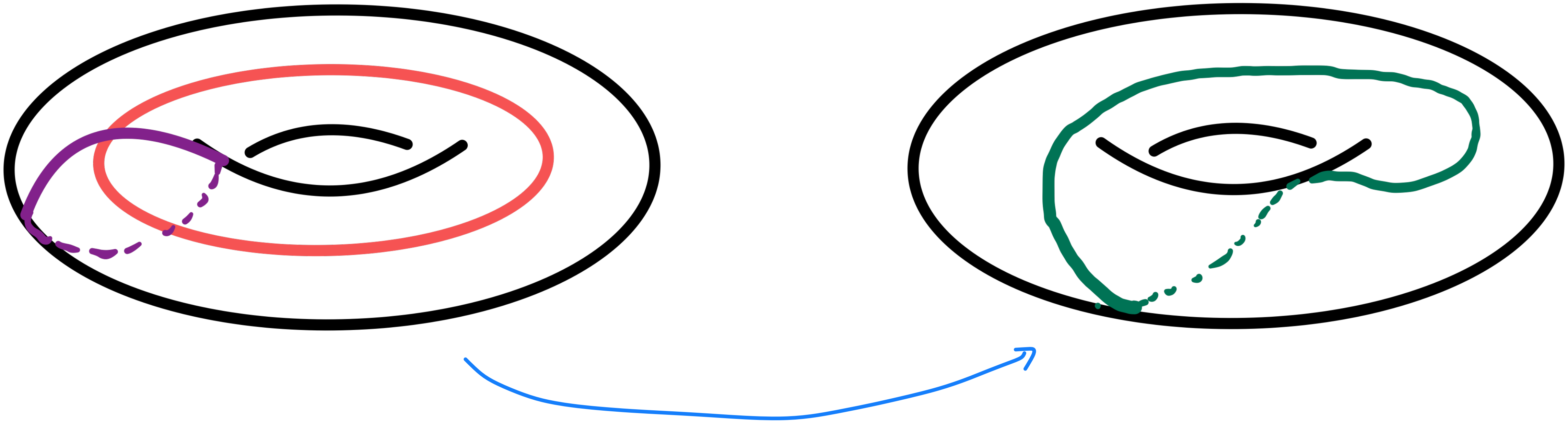


$\dim H_0 = 1$
$\dim H_1 = 0$
$\dim H_2 = 1$



$\dim H_0 = 1$
$\dim H_1 = 2$
$\dim H_2 = 1$

- ith homology has a vector space structure!



We can "add" holes

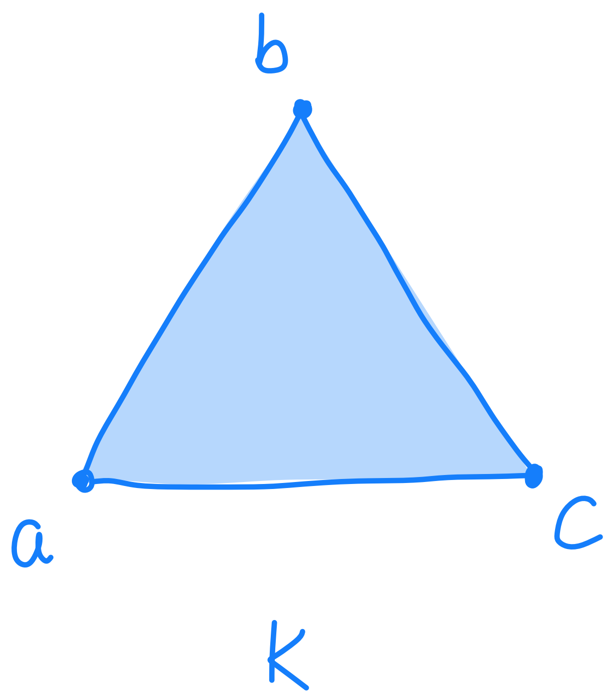
Chains and Boundary Operator

Review of Linear Algebra: Basis and Dimension

- Let V be a vector space over F (e.g. $F = \mathbb{R}$)
- A finite subset $W = \{w_1, \ldots, w_n\} \subset V$ is **linearly independent** if
- $$\lambda_1 w_1 + \cdots + \lambda_n w_n = 0 \text{ iff } \lambda_1 = \cdots = \lambda_n = 0$$
- W **spans** V if for any $v \in V$, there exist $\lambda_1, \ldots, \lambda_n \in F$ such that
- $$\lambda_1 w_1 + \cdots + \lambda_n w_n = v$$
- W is a **basis** for V if it is linearly independent and it spans V. In this case, we call n the dimension of V, denoted by $\dim V$

Chains

- Given a simplicial complex K, a **p-chain** is a formal linear sum of p-simplices $c = \sum c_i \sigma_i$, with coefficients c_i in a given field F.



0-simplices = {a, b, c}

1-simplices = {ab, bc, ac}

2-simplices = {abc}

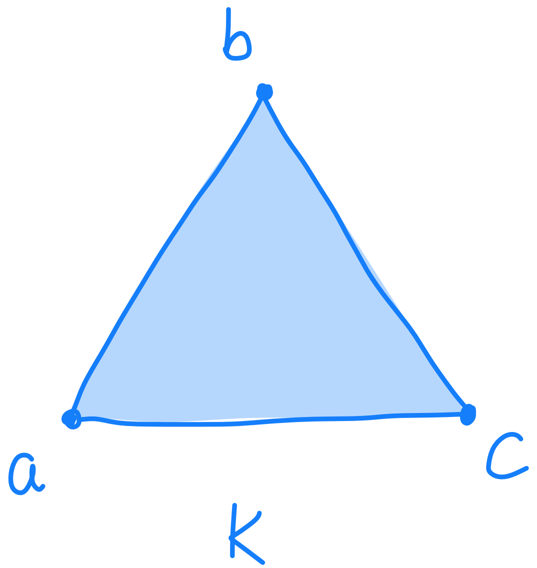
Example
0-chain : a - b

1-chain : ab - bc

2-chain : abc

Chains

- Given a simplicial complex K, a **p-chain** is a formal linear sum of p-simplices $c = \sum c_i \sigma_i$, with coefficients c_i in a given field F.
- The **p-th chain space** of K is the linear space of p-chains in K, denoted $C_p(K)$. Equivalently, $C_p(K)$ is the linear space spanned by p-simplices.



$C_0(K) = \langle a, b, c \rangle$

$C_1(K) = \langle ab, ac, bc \rangle$

$C_2(K) = \langle abc \rangle$

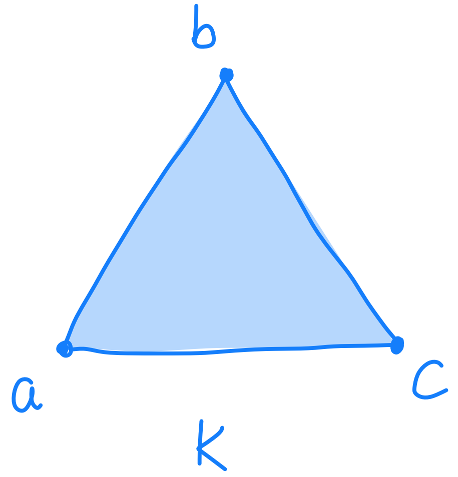
$C_p(K) = 0$, for $p > 2$.

Chains

- Given a simplicial complex K, a **p-chain** is a formal linear sum of p -simplices $c = \sum c_i \sigma_i$, with coefficients c_i in a given field F.
- The **p-th chain space** of K is the linear space of p-chains in K, denoted $C_p(K)$. Equivalently, $C_p(K)$ is the linear space spanned by p-simplices.
- If there are no simplices in dimension p, $C_p(K)$ is zero.
- There is a chain space for each dimension:
  - $C_0(K),\ C_1(K),\ \ldots C_n(K),\ \ldots$
  - The *boundary operators* connect them!

Boundary operator

- Recall: if $\tau \subset \sigma$, τ is called a *face* of σ.
- If $\dim(\tau) = \dim(\sigma) - 1$, τ is called a **facet** of σ.
- Boundary operator sends a simplex to a **suitable** linear combination of its facets.



$$\partial(ab) = \underset{\sim}{?}\, b + \underset{\sim}{?}\, a$$

$$\partial(abc) = \underset{\sim}{?}\, ab + \underset{\sim}{?}\, ac + \underset{\sim}{?}\, bc$$

ensure: if $a<b$ & $b<c$, then $a<c$

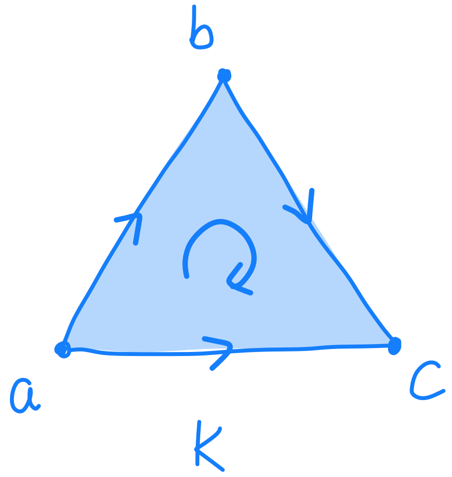
- From now on, assume an **ordering** on the vertices of the simplicial complex
  - Simplices are written as **ordered lists** of vertices: $\sigma = \{v_0, \ldots, v_p\}$

Boundary operator

- The p-th **boundary operator** (*a linear map*) $\partial_p : C_p \to C_{p-1}$
  - For a simplex $\sigma = \{v_0, \ldots, v_p\}$, define

    means removing the vertex v_i

    - $\partial_p(\sigma) := \sum_{i=0}^{p} (-1)^i \{v_0, \ldots, \boxed{\hat{v}_i}, \ldots, v_p\}$, a linear combination of $(p-1)$-faces (facets) of σ



Assume $a < b < c$.

$$\partial(ab) = \partial(v_0 v_1) = (-1)^0 \hat{v}_0 v_1 + (-1)^1 v_0 \hat{v}_1$$
$$= 1 \cdot v_1 + (-1) \cdot v_0 = v_1 - v_0 = b - a$$

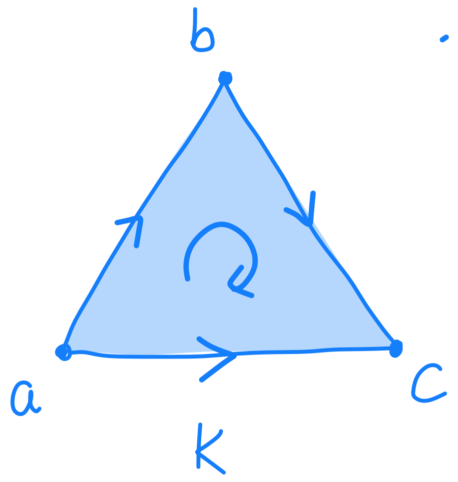
$$\partial(abc) = \hat{a} bc + (-1) a \hat{b} c + ab\hat{c} = bc - ac + ab$$

- For a general chain $c = \sum_j \sigma_j$, define $\partial_p(c) := \sum_j c_j \partial_p(\sigma_j)$

Chain complex

- Chain spaces & boundary operators together give the **chain complex**:
  - a sequence of vector spaces connected by linear maps

$$\cdots \xrightarrow{\partial_{p+2}} \mathrm{C}_{p+1} \xrightarrow{\partial_{p+1}} \mathrm{C}_p \xrightarrow{\partial_p} \mathrm{C}_{p-1} \xrightarrow{\partial_{p-1}} \cdots .$$



$$\begin{array}{ccccccccc} \cdots \to & C_3 & \xrightarrow{\partial_3} & C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{\partial_0} 0 \\ & \| & & \| & & \| & & \| & \\ & 0 & & \langle abc \rangle & & \langle ab, bc, ac \rangle & & \langle a, b, c \rangle & \end{array}$$

$$abc \longmapsto ab - ac + bc$$

$$\begin{aligned} ab &\longmapsto b - a \\ bc &\longmapsto c - b \\ ac &\longmapsto c - a \end{aligned}$$

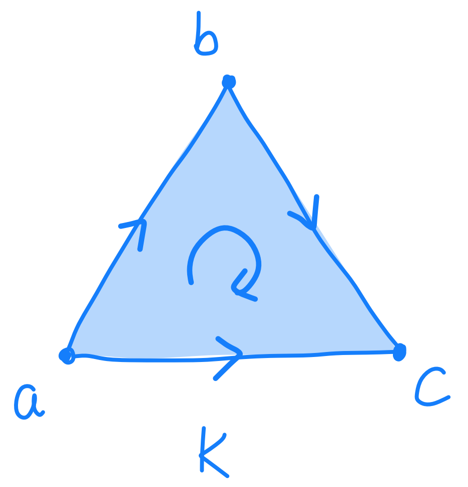
- We define $C_{-1}(K) := 0$

Chain complex

Theorem (Fundamental Boundary Property):
$\partial_p \circ \partial_{p+1} = 0$

- A **p-chain** is a formal linear sum of p-simplices $c = \sum c_i \sigma_i$.
- The **p-th chain space** of K is the linear space of p-chains in K, denoted $C_p(K)$.
- The p-th **boundary operator** (*a linear map*) $\partial_p : C_p \to C_{p-1}$
  - For a simplex $\sigma = \{v_0, \ldots, v_p\}$, define
    - $\partial_p(\sigma) := \sum_{i=0}^{p} (-1)^i \{v_0, \ldots, \hat{v}_i, \ldots, v_p\}$
  - For a general chain $c = \sum_j c_j \sigma_j$, define $\partial_p(c) := \sum_j c_j \partial_p(\sigma_j)$
- **Chain complex:** $\cdots \xrightarrow{\partial_{p+2}} C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \xrightarrow{\partial_{p-1}} \cdots .$

Verify $\partial_p \circ \partial_{p+1} = 0$



$$\begin{tikzcd}
\cdots \arrow[r] & C_3 \arrow[r, "\partial_3"] \arrow[d, equal] & C_2 \arrow[r, "\partial_2"] \arrow[d, equal] & C_1 \arrow[r, "\partial_1"] \arrow[d, equal] & C_0 \arrow[r, "\partial_0"] \arrow[d, equal] & 0 \\
& 0 & \langle abc \rangle & \langle ab, bc, ac \rangle & \langle a, b, c \rangle &
\end{tikzcd}$$

$$abc \longmapsto ab - ac + bc$$

$$ab \longmapsto b - a$$
$$bc \longmapsto c - b$$
$$ac \longmapsto c - a$$

$\boxed{p = 0}$ $\underset{\substack{\| \\ 0}}{}\!\!\overset{\substack{0 \\ \|}}{\partial_0} \circ \partial_1 = 0$

$\boxed{p \geq 2}$ $\partial_p \circ \underset{\substack{\| \\ 0}}{\partial_{p+1}} = 0$

$\boxed{p = 1}$

$$\begin{aligned}
&\partial_1 \circ \partial_2 (abc) \\
&= \partial_1 (ab - ac + bc) \\
&= \partial_1 (ab) - \partial_1 (ac) + \partial_1 (bc) \\
&= (\cancel{b} - \cancel{a}) - (\cancel{c} - \cancel{a}) + (\cancel{c} - \cancel{b}) = 0
\end{aligned}$$