Please show your work to earn credit.
Homework should be submitted to Gradescope by 11:59 pm on **Thursday, Sep 18**.
You can access the LaTeX source here:

1. (3 pt.) Let K be the boundary of a 3-simplex (a hollow tetrahedron) with vertices $\{a, b, c, d\}$. Consider the edge $\sigma = abc$.

   (a) Describe the (open) star $\mathrm{St}(\sigma)$.
   (b) Describe the closed star $\mathrm{ClSt}(\sigma)$, and its geometric realization $|\mathrm{ClSt}(\sigma)|$.
   (c) Describe the link $\mathrm{Lk}(\sigma)$.

2. (10 pt.) Let K, L be simplicial complexes. Two simplicial maps $f, g : K \to L$ are called *contiguous* if for every simplex $\sigma \in K$, the union of vertex sets $f(\sigma) \cup g(\sigma)$ spans a simplex of L.

   (a) Give an explicit example of two contiguous simplicial maps $f, g : K \to L$ where K is the boundary of a triangle and L is the boundary of a square.
   (b) Show that if $f, g : K \to L$ are contiguous, then the induced maps on geometric realizations $|f|, |g| : |K| \to |L|$ are homotopic. (You may give an informal explanation rather than a full proof.)

3. (6 pt.) Suppose we have a collection of sets $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ where there exists an element $U \in \mathcal{U}$ that contains all other elements in $\mathcal{U}$. Show that the geometric realization of the nerve complex $\mathrm{Nrv}(\mathcal{U})$ is contractible to a point. *(You do not need to explicitly construct maps to show contractibility. A clear reasoning argument is sufficient.)*

4. (4 pt.) Let K_1 and K_2 be two subcomplexes of a larger simplicial complex K. Prove the inclusion-exclusion principle for the Euler characteristic:
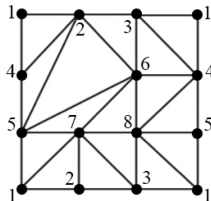$$\chi(K_1 \cup K_2) = \chi(K_1) + \chi(K_2) - \chi(K_1 \cap K_2).$$

5. (4 pt.) Let $A = \{x \in \mathbb{R}^2 : 1 < \|x\| < 2\}$ be an annulus. Define two open subsets of A as follows:

   - $U_1 = $ the subset of A with $x_1 > -\frac{1}{2}$
   - $U_2 = $ the subset of A with $x_1 < \frac{1}{2}$

   (a) Describe the nerve $\mathrm{Nrv}(\{U_1, U_2\})$. What is its homotopy type?
   (b) Is $|\mathrm{Nrv}(\{U_1, U_2\})|$ homotopy equivalent to A? If not, which condition of the Nerve Lemma is violated?

6. (6 pt.) Let P be a finite set of points and $r > 0$.

   (a) Prove $\mathrm{Rips}^r(P) \subseteq C^{2r}(P)$, where $C^\bullet(P)$ denotes the Čech complex.
   (b) Assume $P \subseteq \mathbb{R}^d$. Using Jung's theorem: any set of diameter $\leq 2r$ in $\mathbb{R}^d$ fits in a ball of radius $\sqrt{2}\, r$, deduce $\mathrm{Rips}^r(P) \subseteq C^{\sqrt{2}r}(P)$.

7. (4 pt.) Let $P \subset \mathbb{R}^d$ be a finite point set and let $r > 0$.

   (a) Prove that $\mathrm{Del}^r(P) \subseteq \mathrm{Del}(P) \cap C^r(P)$.
   (b) Does equality always hold, i.e. $\mathrm{Del}^r(P) = \mathrm{Del}(P) \cap C^r(P)$? If not, give an example or explain why not.

8. (5 pt.) Let K be a simplicial complex with boundary maps
$$\partial_n[v_0, \ldots, v_n] = \sum_{i=0}^{n} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_n].$$

   Verify that $\partial_{n-1} \circ \partial_n = 0$ for all n.

9. (8 pt.) The figure below shows the minimal triangulation of the torus $\mathbb{T}^2$ with 7 vertices. Opposite sides of the square are identified.

   (a) Find bases for the chain spaces $C_0(\mathbb{T}^2), C_1(\mathbb{T}^2), C_2(\mathbb{T}^2)$ over $\mathbb{R}$.

   (b) Compute the Euler characteristic $\chi(\mathbb{T}^2)$.

   (c) Find a non-zero 2-cycle, i.e. a non-zero element in $\ker(\partial_2)$ in this triangulation.

10. (5 pt.) *(This problem guides you through a proof by contradiction to show that it is impossible to triangulate the torus, $\mathbb{T}^2$, with only 6 vertices.)*

   It is known that in any triangulation of $\mathbb{T}^2$ with V vertices, E edges, and F triangular faces, we have $3F = 2E$.

   (a) Assume $V = 6$. Use the relation $3F = 2E$ and your answer to Question 9(b) to determine E and F.

   (b) The *degree* of a vertex is the number of edges incident to it. Compute the average degree in the hypothetical triangulation with 6 vertices. Can such an average degree occur in a simple graph on 6 vertices? Explain.