MATH412/COMPSCI434/MATH713
Fall 2025
Topological Data Analysis

Topic 1: Topology Basics

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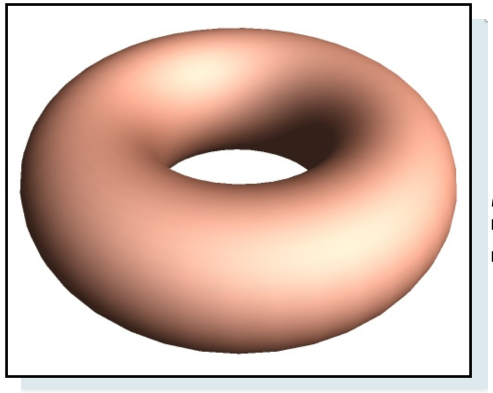
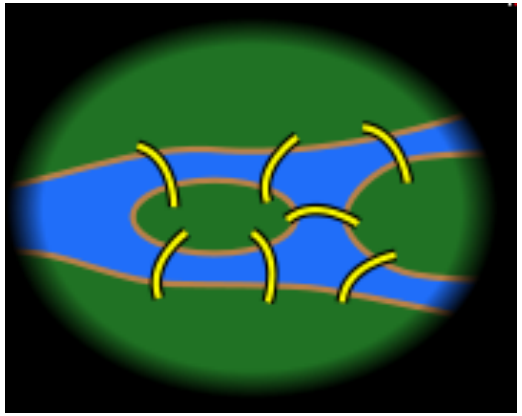
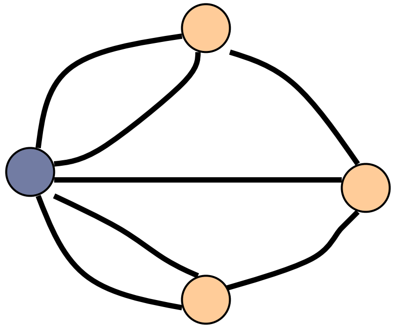
Goal

 $\cong$ 

 $\simeq$ 

- Fundamental Questions
  - What is a topological space?
  - What is a “continuous” way of turning one space to another?
  - When can we say two spaces are the “same”?

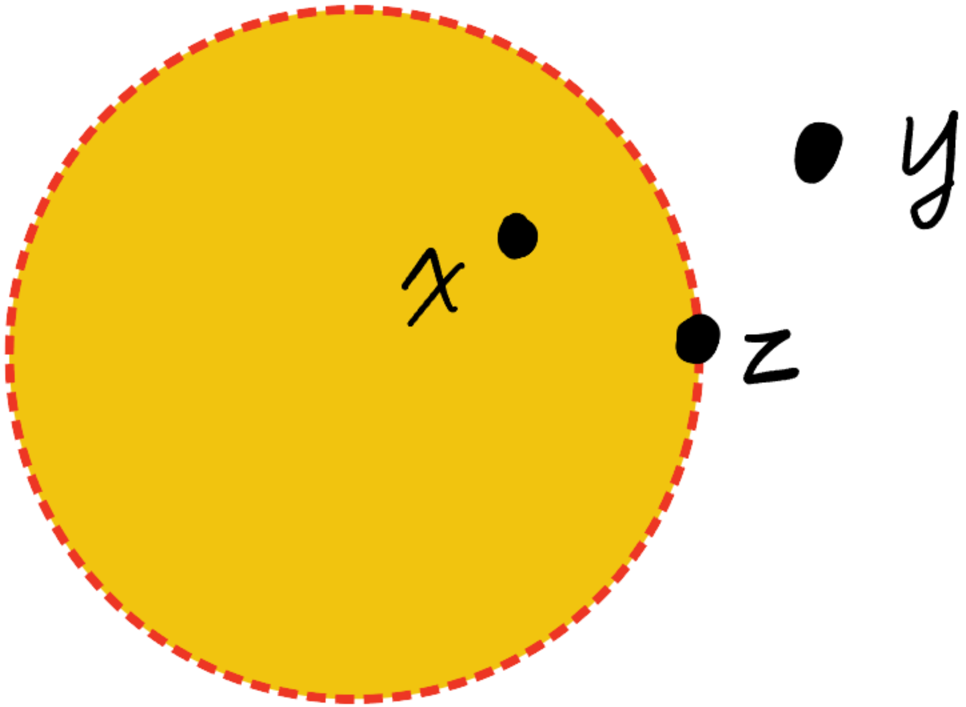
Overview

- Fundamental concepts
  - Topological space
  - Continuous maps
  - Homeomorphisms and homotopies
  - Manifolds

How we mathematically talk about space of interest

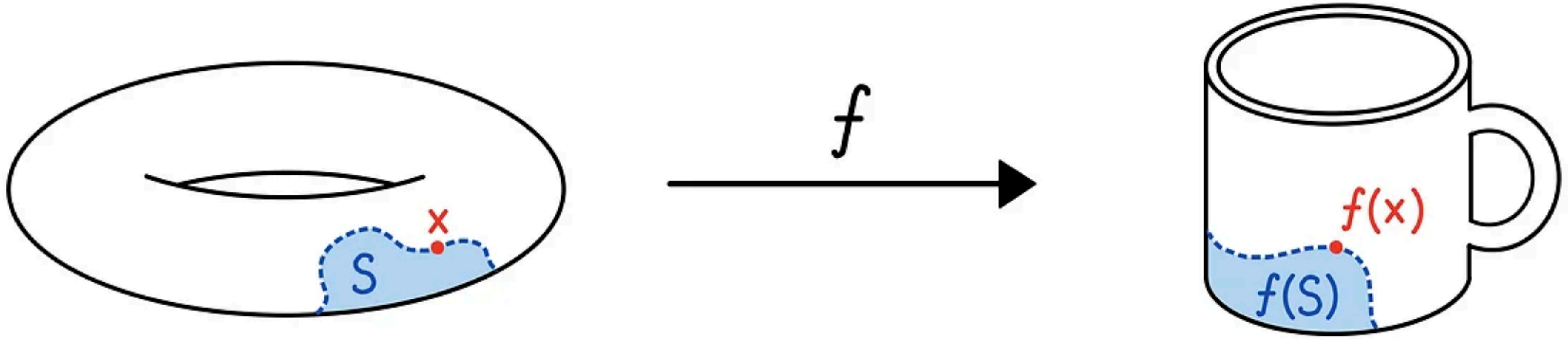
Set theory and beyond

- Given a disk D (without boundary)
  - $x \in D$
  - $y \notin D$ } What is the difference?
  - $z \notin D$
  - D **contacts** both x and z
    - x and z are in the "**closure**" of D



Why do we care?

- We want to rigorously define “continuous transformation”
  - A continuous map shouldn’t tear things apart
  - If S “contacts” x, under a continuous transformation, we want that $f(S)$ “contacts” $f(x)$



From https://wgyory.wixsite.com/toolatetopologize/post/post-1

Why do we care?

- We want to rigorously define "continuous transformation"
  - A continuous map shouldn't tear things apart
  - If S "contacts" x, under a continuous transformation, we want that $f(S)$ "contacts" $f(x)$

- We keep track of **ALL** the relations "S contacts x" to make the above intuition rigorous!

Topological space

Definition 1.1 (Topological space) *A* topological space *is a set* X *endowed with a* topological structure *(a* topology*)* $\mathcal{T}$ *such that the following conditions are satisfied:*

1. *Both the empty set and* X *are elements of* $\mathcal{T}$.
2. *Any union of arbitrarily many elements of* $\mathcal{T}$ *is an element of* $\mathcal{T}$.
3. *Any intersection of finitely many elements of* $\mathcal{T}$ *is an element of* $\mathcal{T}$.

1) $\emptyset, X \in \mathcal{T}$
2) $\bigcup_{i\in I} U_i \in \mathcal{T}$
3) $U \cap V \in \mathcal{T}$

any collection of U_i

- $\mathscr{T}$ is a system of subsets of X. It is called a **topology** on X .
- Examples:
  - Trivial topology $\{\emptyset, X\}$
  - Discrete topology $2^X =$ all subsets of X
  - **Metric space topology**

Topological space

- Examples:
  - Trivial topology $\{\varnothing, X\}$

1) $\varnothing, X \in \mathcal{T}$

2) $\bigcup_{i\in I} U_i \in \mathcal{T}$

3) $U \cap V \in \mathcal{T}$

X : a set

$\mathcal{T} := \{\phi, X\}$

1) $\phi \in X, \ \mathcal{T} \in X$ ✓

2) $\phi \cup X = X \in \mathcal{T}$ ✓

3) $\phi \cap X = \phi \in \mathcal{T}$ ✓

Topological space

- Examples:
  - Discrete topology 2^X = all subsets of X

1) $\emptyset, X \in \mathcal{T}$
2) $\bigcup_{i\in I} \mathcal{U}_i \in \mathcal{T}$
3) $\mathcal{U} \cap V \in \mathcal{T}$

X: a set

$\mathcal{T} := 2^X$

$= \{A \mid A \subset X\}$

1) $\phi \in \mathcal{T}$, $X \in \mathcal{T}$

2) $\bigcup_{i\in I} U_i \subset X \Rightarrow \bigcup_{i\in I} U_i \in \mathcal{T}$

3) $U \cap V \subset X \Rightarrow U \cap V \in \mathcal{T}$

Open / Closed sets

Definition 1.1 (Topological space) *A* topological space *is a set* X *endowed with a* topological structure *(a* topology*)* $\mathcal{T}$ *such that the following conditions are satisfied:*

1. *Both the empty set and* X *are elements of* $\mathcal{T}$.
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1) $\emptyset, X \in \mathcal{T}$
2) $\bigcup_{i \in I} \mathcal{U}_i \in \mathcal{T}$
3) $\mathcal{U} \cap V \in \mathcal{T}$

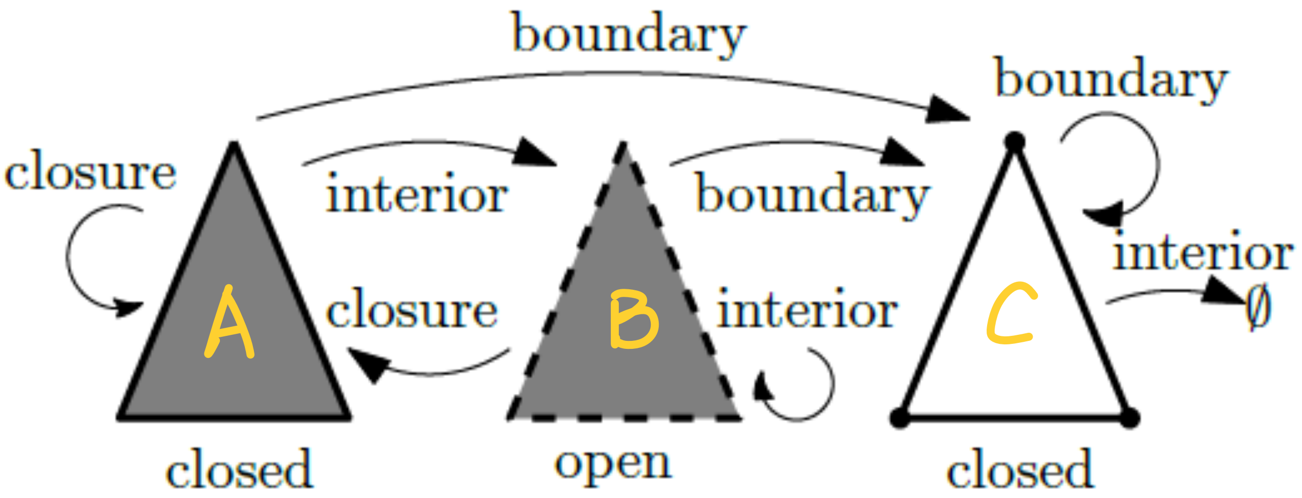
- $\mathscr{T}$ is a system of subsets of X. It is called a *topology* on X .
- Each set $A \in T$ is called an *open set*
- A set B is *closed* if its complement is open
  - i.e., there exists A such that $B = X \setminus A$

Closure, interior, boundary

- Given a topological space $(\mathsf{X}, \mathscr{T})$ and a subset $A \subseteq \mathsf{X}$:
  - the *closure* of A, denoted by $\bar{A}$, is the smallest closed set containing A.
    - $\bar{A} = \cap_{\text{closed } C \supset A} C$
  - its *interior* A^o is the union of all open subsets of A.
  - the *boundary* of A is $\partial A = \bar{A} \backslash A^o$

"S contacts x" can be formally defined as $x \in \bar{S}$

Closure, interior, boundary



$X = \mathbb{R}^2$

$\mathcal{T} = \{\text{open disks}\}$

1) A, C are closed ; B is open

2) $A = \overline{B} = \overline{A}$

$B = A^\circ = B^\circ$

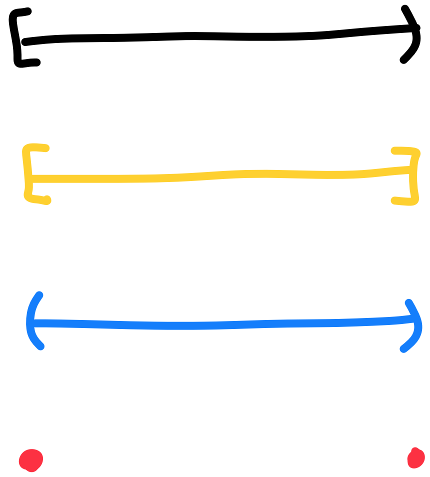
$C = \partial A = \partial B$

$\partial B = \overline{B} \setminus B^\circ$

$= A \setminus B = C$

Examples in $\mathbb{R}$

- Let $A = [1,2)$
  - $\bar{A} = [1,2]$
  - $A^o = (1,2)$
  - $\partial A = \{1,2\}$



- For any given set X, one can define different topologies on top of that. Some of them can be bizarre, such as the trivial topology

Recall : $\mathcal{T} = \{\phi, X\}$

- The most useful topology in this class is the **metric space topology**

Metric space

Definition 2 (Metric space). *A* metric space *is a pair* (X, d) *where* X *is a set and* d *is a distance function* $\mathsf{d} : \mathsf{X} \times \mathsf{X} \to \mathbb{R}$ *satisfying the following properties:*

- $\mathsf{d}(p, q) = 0$ *if and only if* $p = q$
- $\mathsf{d}(p, q) = \mathsf{d}(q, p), \forall p, q \in \mathsf{X}$;
- $\mathsf{d}(p, q) \leq \mathsf{d}(p, r) + \mathsf{d}(r, q), \forall p, q, r \in \mathsf{X}$.

- Examples:
  - $(\mathbb{R}^k, \|\cdot\|_2)$ k-dimensional Euclidean space, equipped with the standard Euclidean distance $d(p, q) = \|p - q\|_2$

$$= \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2 + \cdots + (p_k - q_k)^2}$$

Metric space

- More Examples:
  - "Curved" space (manifolds), equipped with geodesic distance
    - e.g, the surface of earth.
  - Space can also be discrete, as very often in data analysis
    - (P, d): a set of points with pairwise distance (or similarity) given.
    - or graphs, equipped with shortest path metric.

Metric space topology

- Open ball:
  - $B_o(c,\ r) = \{x \in X \mid d(c, x) < r\}$

Definition 3 (Metric space topology). *Given a metric space* X*, all metric balls* $\{B_o(c, r) \mid c \in \mathbb{T}$ *and* $0 < r \leq \infty\}$ *and their union constituting the open sets define a topology on* X.

  - Exercise: prove that this is a topology on X
- The set of metric balls is called a *basis* for this topology on X
  - it generates all open sets in this topology
- In general, when we refer to a common metric space, say Euclidean space, we refer to this metric space topology induced by standard metric.

Metric space topology

(X, d) : a metric space

$\mathcal{T}$: metric topology ($\mathcal{T} = \{$unions of open balls$\}$)

- A is open if $\forall x \in A$, $\exists$ an open ball $B_0(y, r) \subset A$
  & $x \in B_0(y, r)$
- A point p is a limit point of A if $\forall \varepsilon > 0$, $\exists q \in A$
  s.t. $d(p, q) < \varepsilon$
- A set B is close if it contains all its limit points

Metric space topology on $\mathbb{R}$

- Each open ball is an open interval $(c - r, c + r)$
- Each open set is a union of arbitrarily many open intervals (by definition)
- Each open set is a countable union of open intervals

Exercise

Subspace topology

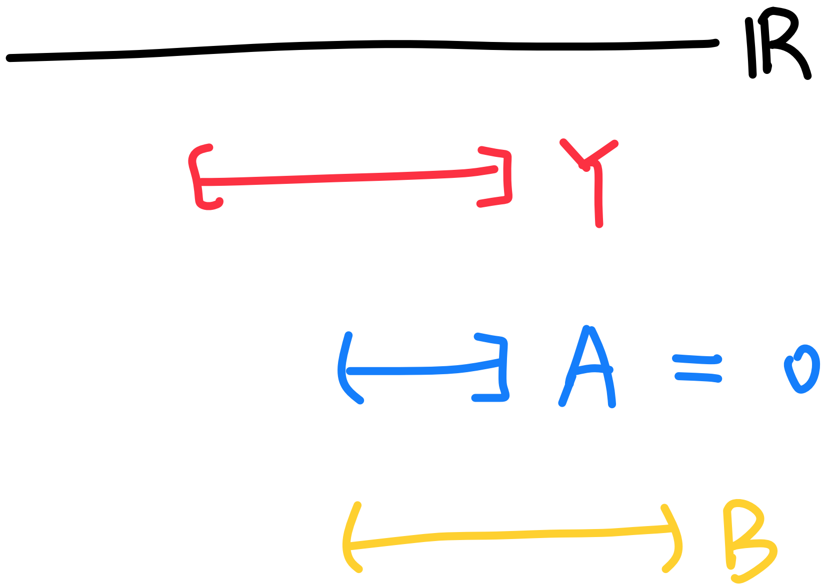
- A topological space $(\mathsf{X}, \mathcal{T})$, say the Euclidean space
- Given a subset $Y \subseteq \mathsf{X}$, the subspace topology $\left(Y, \mathcal{T}_Y\right)$, (inherited from $(\mathsf{X}, \mathcal{T})$), is such that $\mathcal{T}_Y$ consists of intersection between open sets in $\mathcal{T}$ and Y.

  $\mathcal{T}_Y = \{ A \cap Y \mid A \in \mathcal{T} \}$

- Common subspaces of Euclidean space
  - Euclidean d-ball: $\mathbb{B}^d = \left\{x \in \mathbb{R}^d \mid \|x\| \leq 1\right\}$
  - Open Euclidean d-ball: $\mathbb{B}_o^d = \left\{x \in \mathbb{R}^d \mid \|x\| < 1\right\}$
  - Euclidean d-sphere: $\mathbb{S}^d = \left\{x \in \mathbb{R}^{d+1} \mid \|x\| = 1\right\}$
  - Euclidean half-space: $\mathbb{H}^d = \left\{x \in \mathbb{R}^d \mid x_d \geq 0\right\}$

Subspace topology

- Example: $X = \mathbb{R}$ and $Y = [1,2]$. Then, $\underbrace{(1.5,2]}_{A} = \underbrace{(1.5,3)}_{B} \cap [1,2]$ is an open set in subspace topology.

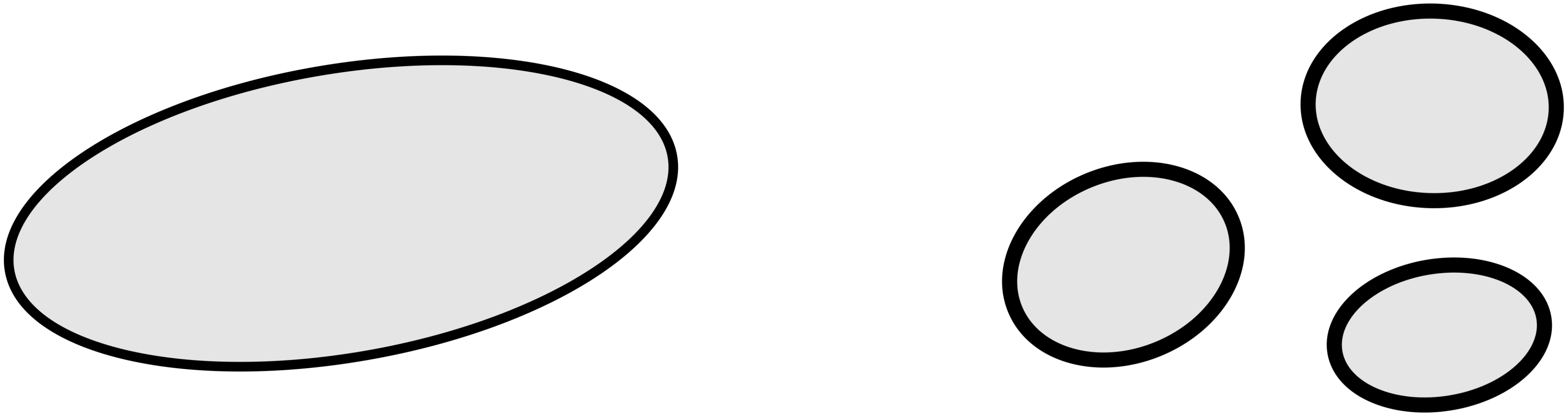


A = open set $\cap\, Y \Rightarrow A$ is open in Y

(However, A is NOT open in $\mathbb{R}$)

Connectivity

- Open sets determine connectivity.
- A topological space $(\mathsf{X}, \mathcal{T})$ is *disconnected* if there are two disjoint non-empty open sets A, $B \in \mathcal{T}$ so that $\mathsf{X} = A \cup B$.
- A topological space is *connected* if it is not disconnected.
- Any maximal connected subsets of X is called a **connected component**.



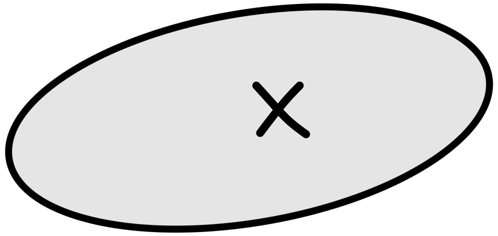
Compactness

- This generalizes the notion of **closed** and **bounded** sets in Euclidean space

$A = \bar{A}$

$\exists M > 0$ s.t. $\forall x \in A, \|x\|_2 \leq M$

- **Open cover:** $\mathscr{U} = \{U_\alpha\}_{\alpha \in A}$ is an open cover for $(X, \mathscr{T})$ if $U_\alpha \in \mathscr{T}$ and $X = \cup_{\alpha \in A} U_\alpha$



Compactness

- $(X, \mathcal{T})$ is called **compact** if for any open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ there exists a finite subcover, i.e., a finite set $A' \subseteq A$ such that $X = \cup_{\alpha \in A'} U_\alpha$

- In metric space topology, compact = closed + bounded.

- Example: $\underbrace{(0,1)}$ is not compact but $[0,1]$ is compact

  ↳ not closed ⇒ not compact

Check-in: Where are we?

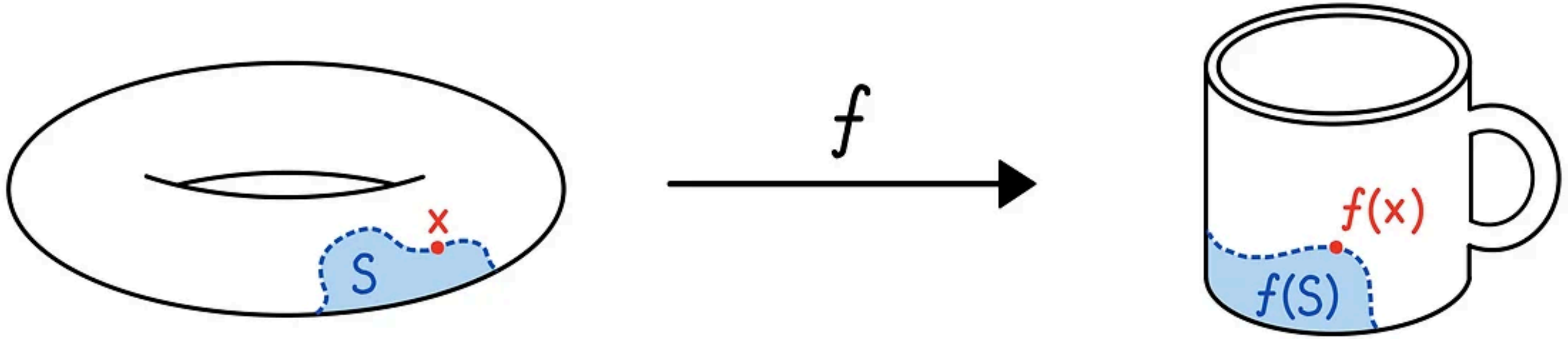
- Fundamental concepts
  - Topological space
  - Continuous maps
  - Homeomorphisms and homotopies
  - Manifolds

How we mathematically talk about space of interest

Now we need ways to connect different spaces!

Recall

- We want to rigorously define "continuous transformation"
  - A continuous map shouldn't tear things apart
  - If S "contacts" x, under a continuous transformation, we want that $f(S)$ "contacts" $f(x)$



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Continuous function

- A function $f : X \to Y$ between two topological spaces is called **continuous** if for any subset $S \subset X$ we have that
    - $f(\bar{S}) \subset \overline{f(S)}$
- A formal way describing "If S contacts x, then $f(S)$ contacts $f(x)$"

1. Closure of image contains image of closure
2. Continuous map does not tear things apart

From https://wgyory.wixsite.com/toolatetopologize/post/post-1

Continuous function: limit-preserving

- Prove the notion of continuity in calculus is compatible with the new definition of continuity
  - $\lim_{x \to x_0} f(x) = f(x_0)$ for all x_0 $\Longleftrightarrow$ $f(\bar{S}) \subseteq \overline{f(S)}$, $\forall\, S \subset \mathbb{R}$

"$\Rightarrow$": $\forall\, x_0 \in \bar{S}$, $\exists \{x_n\} \subset S$ s.t. $x_n \to x_0$

$$f(x_0) = \lim_{x_n \to x_0} f(x_n) \in \overline{f(S)}$$

Thus, $f(\bar{S}) \subseteq \overline{f(S)}$

Continuous function: limit-preserving

- Prove the notion of continuity in calculus is compatible with the new definition of continuity
  - $\lim_{x \to x_0} f(x) = f(x_0)$ for all x_0 $\Longleftrightarrow$ $f(\overline{S}) \subseteq \overline{f(S)}$, $\forall S \subset \mathbb{R}$

"$\Leftarrow$" Consider a sequence $x_n \to x_0$.

Take any subsequence $S := \{x_{n_k}\}$ of $\{x_n\}$

$x_n \to x_0 \Rightarrow x_{n_k} \to x_0 \Rightarrow x_0 \in \overline{S} \Rightarrow f(x_0) \in \overline{f(S)} = \overline{\{f(x_{n_k})\}}$

$\Rightarrow \exists$ a subsequence of $\{f(x_{n_k})\}$ converging to $f(x_0)$

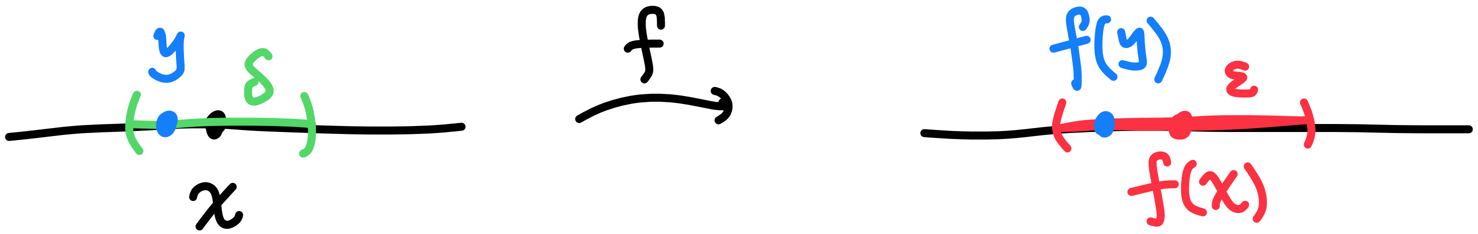
Any subsequence of $\{f(x_n)\}$ has a (further) subsequence converging to $f(x_0)$

(a non-trivial step) $\Downarrow$ (prove by contradiction)

$f(x_n) \longrightarrow f(x_0)$

Continuous function: epsilon-delta

- Recall the simple case $f: R \to R$
  - f is continuous at $x \in \mathbb{R}$ if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $y \in (x - \delta, x + \delta)$, $f(y) \in (f(x) - \epsilon, f(x) + \epsilon)$



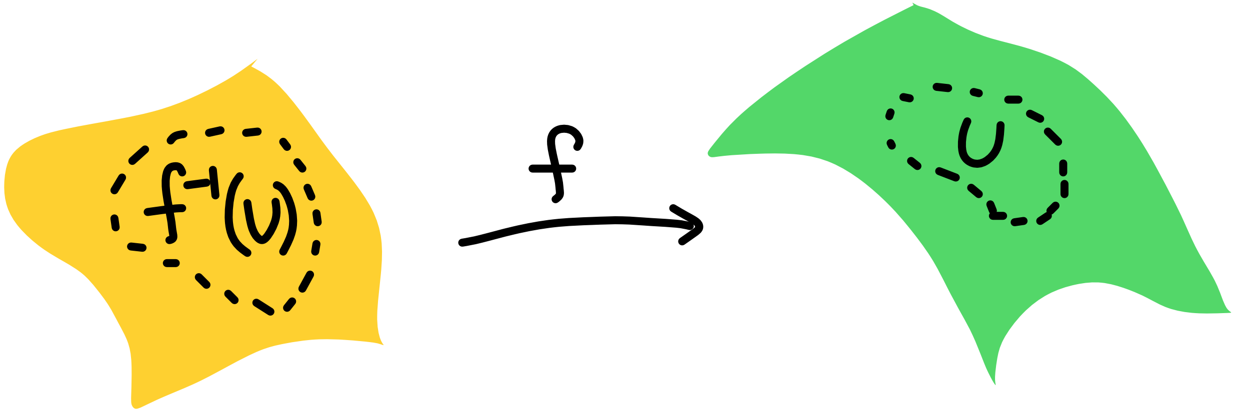
Exercise: f is continuous at $x \in \mathbb{R}$ if

$\forall \varepsilon > 0$, $f^{-1}\big((f(x)-\varepsilon, f(x)+\varepsilon)\big)$ is open.

Continuous function: preimage of open is open

Definition 1.15 (Continuous function; Map)**.** A function $f : \mathbb{T} \to \mathbb{U}$ from the topological space $\mathbb{T}$ to another topological space $\mathbb{U}$ is *continuous* if for every open set $Q \subseteq \mathbb{U}$, $f^{-1}(Q)$ is open. Continuous functions are also called *maps*.

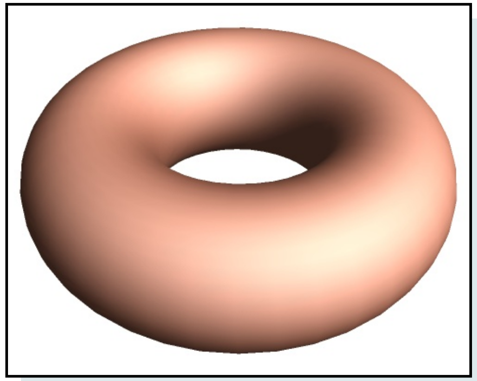
Definition 1.16 (Embedding)**.** A map $g : \mathbb{T} \to \mathbb{U}$ is an *embedding* of $\mathbb{T}$ into $\mathbb{U}$ if g is injective.



The Preimage of an open set is open

Check-in: Where are we?

- Fundamental concepts
  - Topological space → How we mathematically talk about space of interest
  - Continuous maps → Now we need ways to connect different spaces!
  - Homeomorphisms and homotopies → Describe relations of spaces
  - Manifolds



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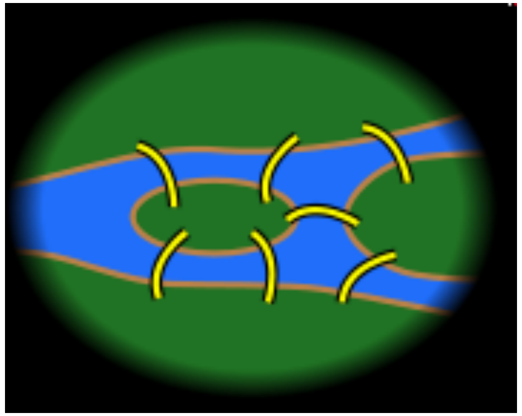

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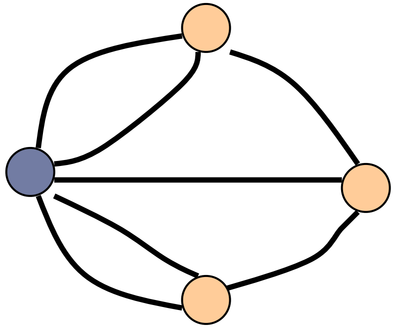


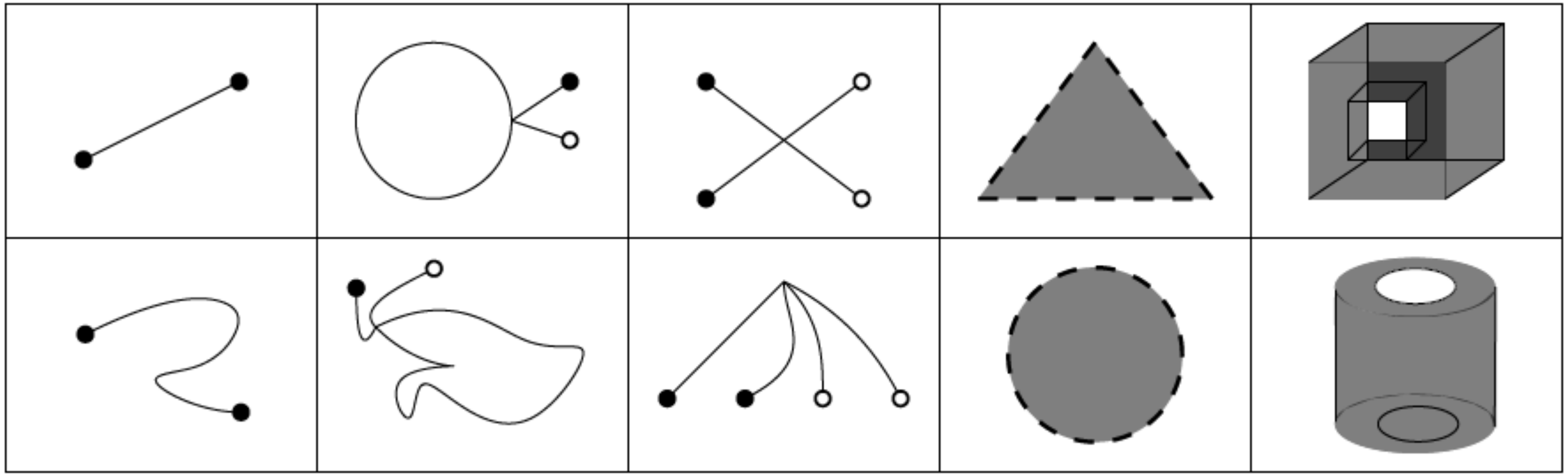
Homeomorphism = homoios + morphē = Similar shapes

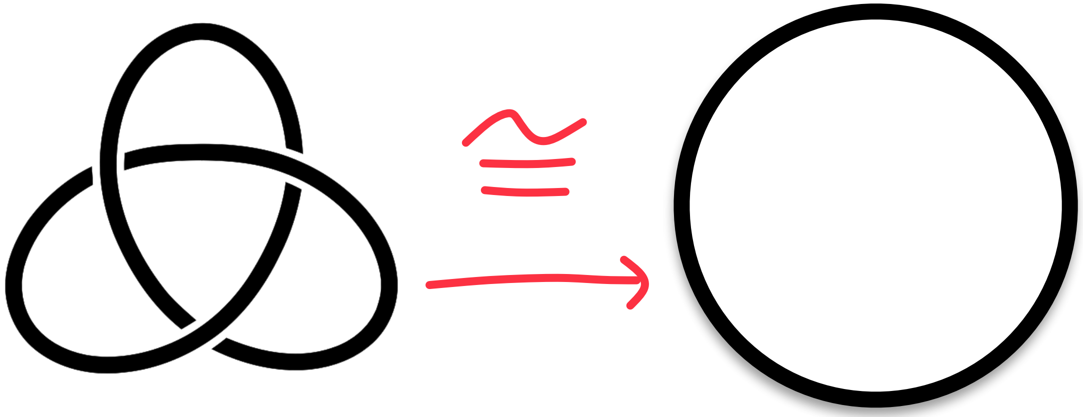
Definition 5 (Homeomorphism) *Given two topological spaces* X *and* Y*, a* homeomorphism *between them is a map* $h : \mathsf{X} \to \mathsf{Y}$ *such that* h *is bijection and the inverse of* h *is also continous.*

Two topological spaces are X *and* Y *are* homeomorphic*, denoted by* $\mathsf{X} \cong \mathsf{Y}$*, if there is a homeomorphism between them.*

- Homeomorphic spaces are called *topologically equivalent*
  - Note that equivalent relations are transitive.
  - $X \cong Y$ and $Y \cong Z$ implies $X \cong Z$
- Layman's terms: two spaces are homeomorphic if one can continuously deform (stretch, compress) into the other without ever breaking or stitching them
  - Caveat: not always true
- Homeomorphism preserves all topological quantities: dimension, number of connected components, number of holes, voids, etc.

Examples





Non-examples

$X \cong Y \Rightarrow X \setminus \{x\} \cong Y \setminus \{y\}$

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Two topological spaces are X *and* Y *are* homeomorphic*, denoted by* $\mathsf{X} \cong \mathsf{Y}$*, if there is a homeomorphism between them.*

- A trick: remove one point from each space and check if the remained spaces have the same topological properties.
  - Y and I are not homeomorphic; X and Y are not homeomorphic
  - $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic
  - What about $\mathbb{R}^2$ and $\mathbb{R}^3$?
- In general, hard to decide whether two spaces are homeomorphic or not!

More Examples and Non-Examples

- The Euclidean space $\mathbb{R}^d$ is homeomorphic to any open ball $\mathbb{B}_o(c, r)$
  - Exercise: try to construct the homeomorphism by yourself

- [0,1], (0,1], (0,1)