MATH412/COMPSCI434/MATH713
Fall 2025
Topological Data Analysis

Topic 3: Simplicial Homology - Part 2

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Chains and Boundary Operator

Chain complex

Theorem (Fundamental Boundary Property):
$$\partial_p \circ \partial_{p+1} = 0$$

- A **p-chain** is a formal linear sum of p-simplices $c = \sum c_i \sigma_i$, with coefficients c_i in a given field F. ← Need tools from abstract algebra to generalize
- The **p-th chain space** of K is the linear space of p-chains in K, denoted $C_p(K)$.
- The p-th **boundary operator** (*a linear map*) $\partial_p : C_p \to C_{p-1}$
  - For a simplex $\sigma = \{v_0, \ldots, v_p\}$, define
    - $\partial_p(\sigma) := \sum_{i=0}^{p} (-1)^i \{v_0, \ldots, \hat{v}_i, \ldots, v_p\}$
  - For a general chain $c = \sum_j \sigma_j$, define $\partial_p(c) := \sum_j c_j \partial_p(\sigma_j)$
- **Chain complex:** $\ldots \xrightarrow{\partial_{p+2}} C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \xrightarrow{\partial_{p-1}} \ldots$.

Review of algebraic tools - Group (Optional)

- A **group** is a tuple $(G, +)$ where G is a set and $+ : G \times G \to G$ is a binary operation
  - (Associativity) $a + (b + c) = (a + b) + c$
  - (Identity) There exist $0 \in G$ such that $a + 0 = 0 + a = a$
  - (Inverse) For any $a \in G$, there exist $-a \in G$ such that $a + (-a) = 0$
- If G further satisfies the following property, then we call $(G, +)$ an **abelian group**
  - (Commutativity) $a + b = b + a$
- Examples:
  - $(\mathbb{Z}, +)$ is an abelian group
  - $(\mathbb{R}, +)$ is an abelian group

Review of algebraic tools - Ring (Optional)

- A **ring** is a tuple $(F, +, \times)$ where $(F, +)$ is an abelian group and $\times : F \times F \to F$ is another binary operation such that
  - (Associativity) $a \times (b \times c) = (a \times b) \times c$
  - (Multiplicative identity)There exist 1 in F such that $a \times 1 = a$
  - (Distributivity) $a \times (b + c) = (a \times b) + (a \times c)$
- If $(F, +, \times)$ further satisfies the following property, then we call it an **commutative ring**
  - (Commutativity) $a \times b = b \times a$
- Examples:
  - $(\mathbb{Z}, +, \times)$ is a commutative ring
  - $(\mathbb{R}, +, \times)$ is a commutative ring

Review of algebraic tools - Field (Optional)

- A commutative ring $(F, +, \times)$ is called a **field** if the following satisfies:
  - (Multiplicative inverse) For any $a \neq 0$ in F, there exists $a^{-1} \in F$ such that $a \times a^{-1} = 1$
- In summary, a field $(F, +, \times)$ satisfies:
  - For addition: identity, inverse, associativity, commutativity
  - For multiplication: identity, inverse, associativity, commutativity
  - For both operations: distributivity
- Examples:
  - $(\mathbb{Q}, +, \times)$, $(\mathbb{R}, +, \times)$, $(\mathbb{C}, +, \times)$ are fields.
  - $(\mathbb{Z}, +, \times)$ is NOT a field.
  - Finite fields: For any prime number p, $\mathbb{Z}_p = \{0,1,\ldots,p-1\}$, with $+, \times$ modulo p

The most important example in this class: $\mathbb{Z}_2$

- $\mathbb{Z}_2 = \{0,1\}$ is the smallest field

$+$	0	1
0	0	1
1	1	$\boxed{0}$

$\times$	0	1
0	0	0
1	0	1

Chain complex

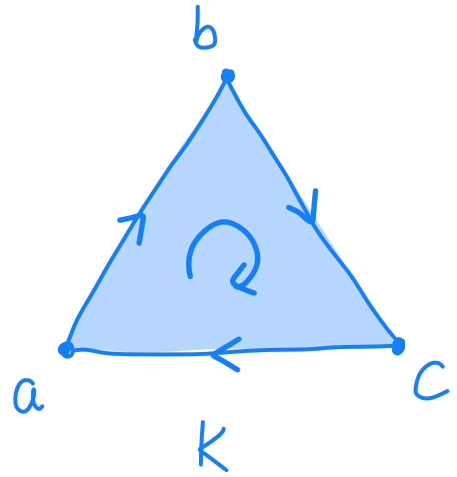
Theorem (Fundamental Boundary Property):
$$\partial_p \circ \partial_{p+1} = 0$$

- A **p-chain** is a formal linear sum of p-simplices $c = \sum c_i \sigma_i$, with coefficients c_i in a given field F.
- In the most general case, F can be any ring (not necessarily commutative).
- Commonly used choices for F are
  - $(\mathbb{Z}, +, \times)$, which is not a field: The resulting chain space is not a vector space. It is a group, called the **chain group.**
  - $(\mathbb{Z}_2 = \{0,1\}, +, \times)$: The coefficients in a chain is either 0 or 1. The p-th **boundary operator** simplifies: For a simplex $\sigma = \{v_0, \ldots, v_p\}$,
    - $\partial_p(\sigma) := \sum_{i=0}^{p} (-1)^i \{v_0, \ldots, \hat{v}_i, \ldots, v_p\} = \sum_{i=0}^{p} \{v_0, \ldots, \hat{v}_i, \ldots, v_p\}$

Cycles and Boundaries

Cycles (kernel of boundary operator)

- A **p-cycle** is a p-chain c such that $\partial_p(c) = 0$
- The **p-th cycle space** is $Z_p(K) := \ker(\partial_p) \subset C_p(K)$



$$\begin{tikzcd}
\cdots \arrow[r] & C_3 \arrow[r, "\partial_3"] \arrow[d, phantom, "\shortparallel"] & C_2 \arrow[r, "\partial_2"] \arrow[d, phantom, "\shortparallel"] & C_1 \arrow[r, "\partial_1"] \arrow[d, phantom, "\shortparallel"] & C_0 \arrow[r, "\partial_0"] \arrow[d, phantom, "\shortparallel"] & 0 \\
& 0 & \langle abc \rangle & \langle ab, bc, ac \rangle & \langle a, b, c \rangle &
\end{tikzcd}$$

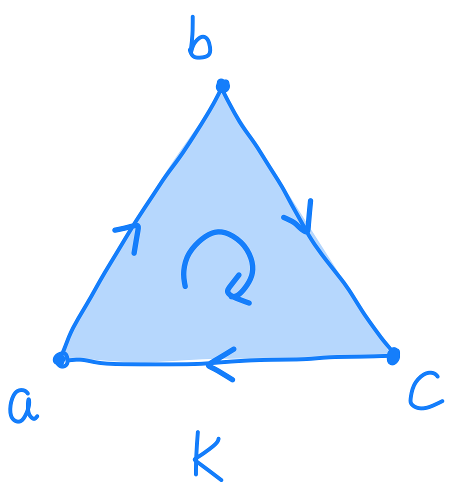
$\boxed{\text{Example}}$ 0-cycle: a, b, c

1-cycle: $ab + bc - ac$

2-cycle:

Cycles (kernel of boundary operator)

- A **p-cycle** is a p-chain c such that $\partial_p(c) = 0$
- The **p-th cycle space** is $Z_p(K) := \ker(\partial_p) \subset C_p(K)$



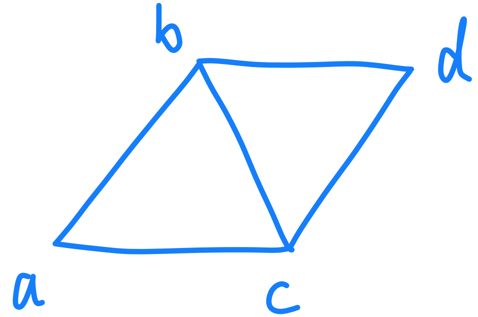
$$
\begin{tikzcd}
\cdots \arrow[r] & C_3 \arrow[r, "\partial_3"] \arrow[d, equal] & C_2 \arrow[r, "\partial_2"] \arrow[d, equal] & C_1 \arrow[r, "\partial_1"] \arrow[d, equal] & C_0 \arrow[r, "\partial_0"] \arrow[d, equal] & 0 \\
& 0 & \langle abc \rangle & \langle ab, bc, ac \rangle & \langle a, b, c \rangle &
\end{tikzcd}
$$

$$Z_1 = \langle ab + bc - ac \rangle$$

$$\dim Z_1 = 1$$

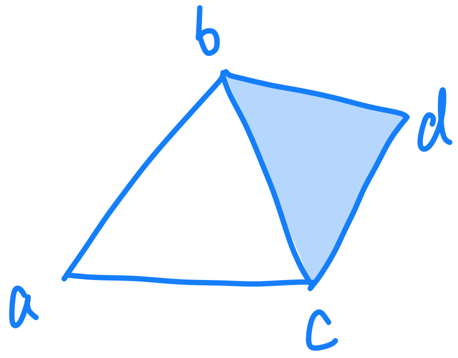
Cycles (kernel of boundary operator)

- A **p-cycle** is a p-chain c such that $\partial_p(c) = 0$
- The **p-th cycle space** is $Z_p(K) := \ker(\partial_p) \subset C_p(K)$



$\dim Z_1(K) = 2$

$Z_1(K) = \langle ab - ac + bc, bc - bd + cd \rangle$

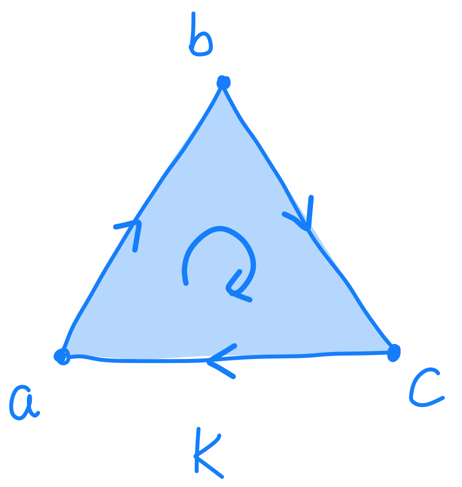


$\dim Z_1(K) = 2$

$Z_1(K) = \langle ab - ac + bc, bc - bd + cd \rangle$

Boundaries (image of boundary operator)

- A **p-boundary** is a p-chain c that is a boundary of some (p+1)-chain, i.e., there exists (p+1)-chain c' such that $c = \partial_{p+1}(c')$
- The **p-th boundary space** is $B_p(K) := \mathrm{im}(\partial_{p+1}) \subset C_p(K)$



$$\begin{tikzcd}
\cdots \arrow[r] & C_3 \arrow[r, "\partial_3"] \arrow[d, phantom, "\parallel"] & C_2 \arrow[r, "\partial_2"] \arrow[d, phantom, "\parallel"] & C_1 \arrow[r, "\partial_1"] \arrow[d, phantom, "\parallel"] & C_0 \arrow[r, "\partial_0"] \arrow[d, phantom, "\parallel"] & 0 \\
 & 0 & \langle abc \rangle & \langle ab, bc, ac \rangle & \langle a, b, c \rangle &
\end{tikzcd}$$

Example 0-boundary: a, b, c

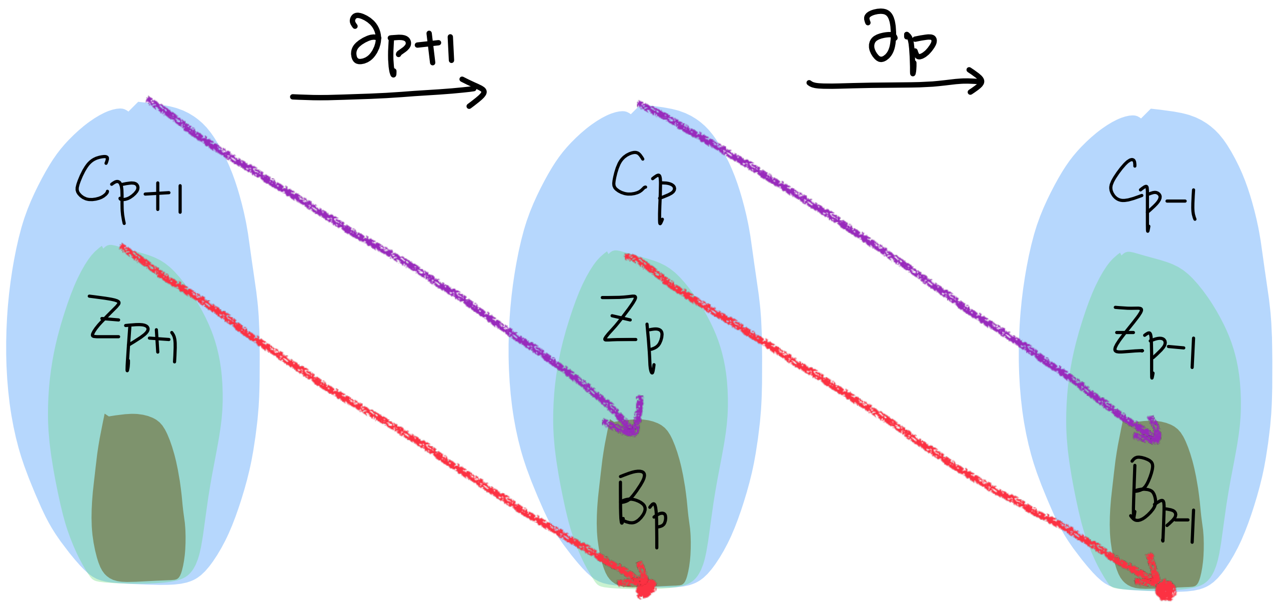
1-boundary: $ab + bc - ac$

Cycles and Boundaries

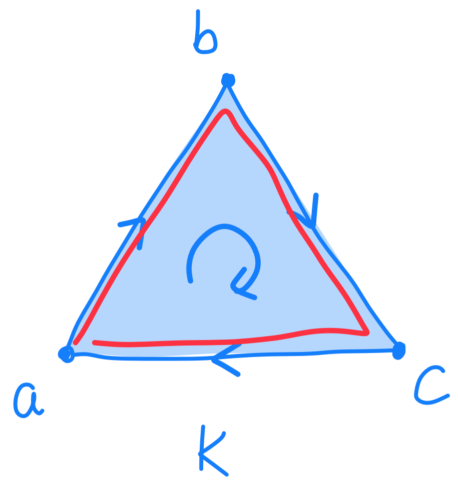
boudaries ⊆ cycles ⊆ chains

Under field (e.g. $\mathbb{Z}_2$) coefficients, B_p, Z_p, C_p are all vector spaces.

$$\partial_p \circ \partial_{p+1} = 0 \Rightarrow B_p \subseteq Z_p \subseteq C_P$$



Cycles and Boundaries



$c = ab + bc - ac$

(1) $\partial_1(c) = 0 \Rightarrow c$ is a cycle

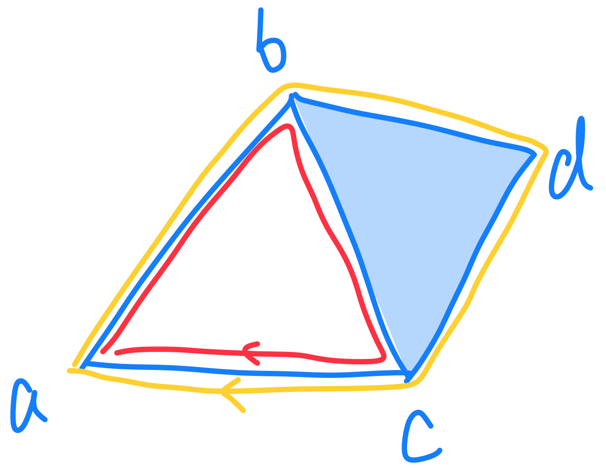
(c makes a hole)

(2) $c = \partial_2(abc) \Rightarrow c$ is a boundary

(but the hole is filled in)

"Actual holes" $\approx$ cycles / boundaries
(homology)

Cycles and Boundaries



$c_1 = ab + bc - ac$

$c_2 = ab + bd - cd - ac$

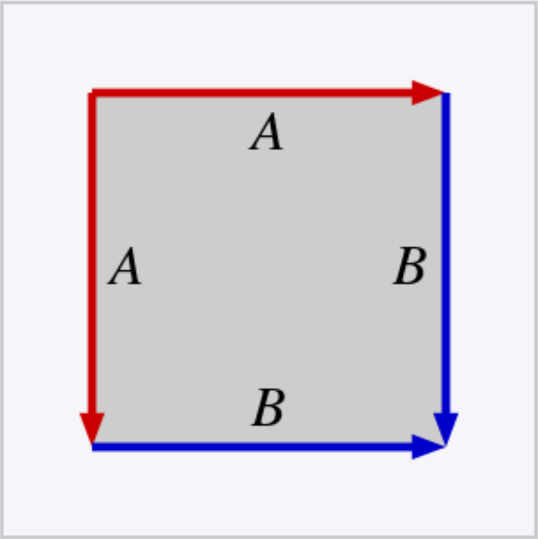
$$
\begin{aligned}
c_2 - c_1 &= (ab + bd - cd - ac) \\
&\quad - (ab + bc - ac) \\
&= bd - cd - bc \\
&= -(bc - bd + cd) \\
&= -\partial_2(bcd) = \partial_2(-bcd)
\end{aligned}
$$

homology: $c_2 \sim c_1$ because they differ by a boundary

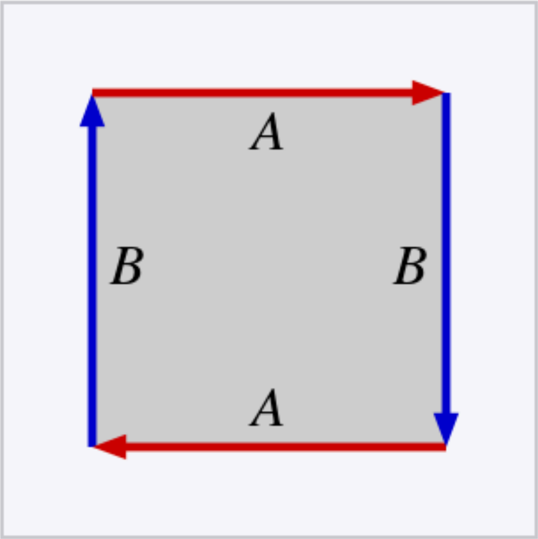
Homology Groups

Quotient is a way of identifying/collapsing things

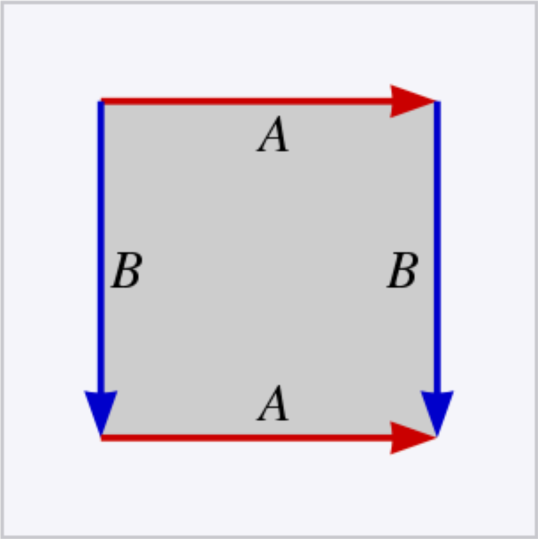
- Example: Quotient topological space



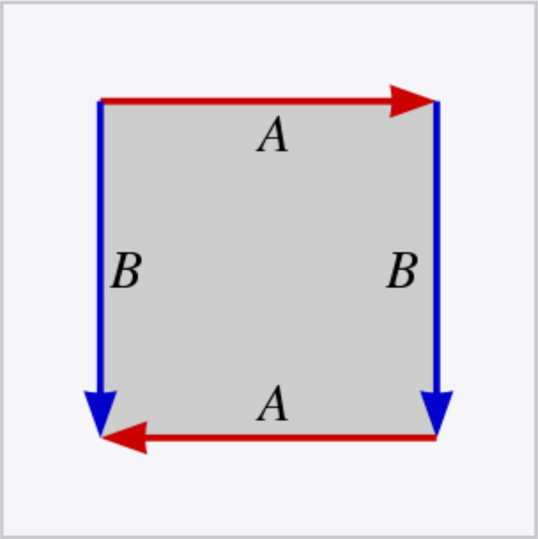
sphere



real projective plane



torus

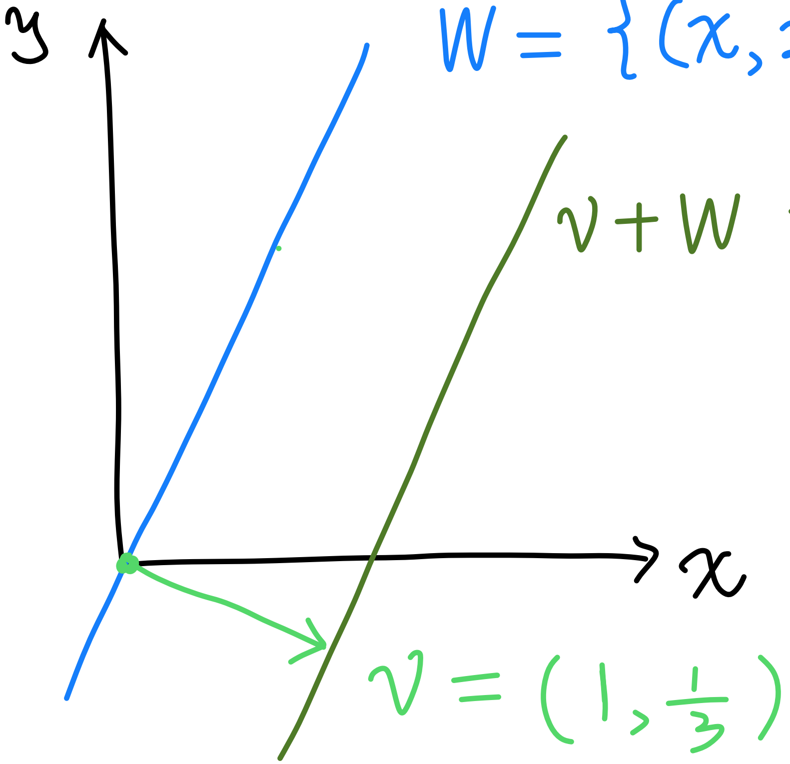


Klein bottle

Quotient

- Let V be a vector space and $W \subset V$ be a linear subspace.
- Define $[v] = v + W = \{u \in V \,|\, v - u \in W\}$

$$W = \{(x, 2x) \,|\, x \in \mathbb{R}\}$$



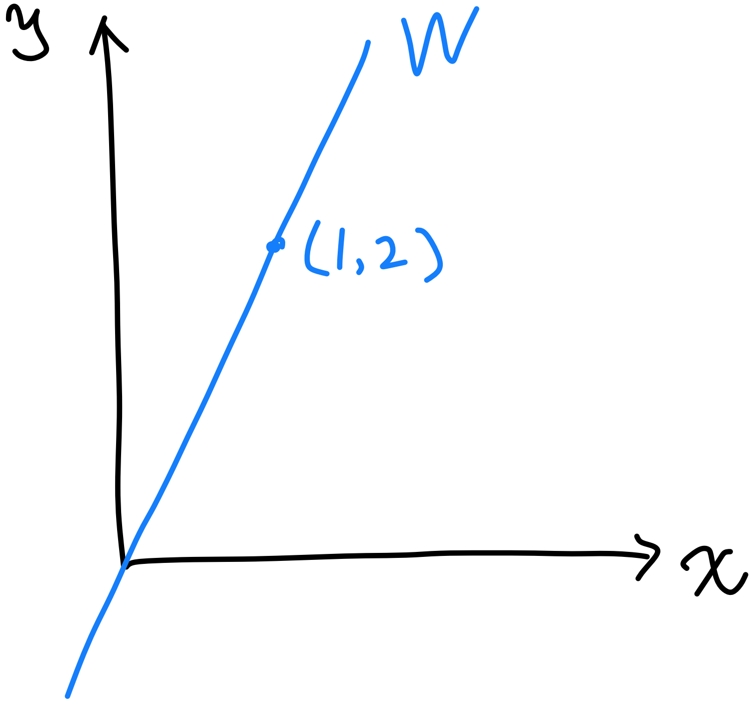
$$v + W = \{v + w \,|\, w \in W\}$$

$$= \{u \,|\, u - v \in W\}$$

$u := v + w$

Quotient

- Let V be a vector space and $W \subset V$ be a linear subspace.
- Define $[v] = v + W = \{u \in V \mid v - u \in W\}$
- The **quotient** of V by W is the set $V/W = \{[v] \mid v \in V\}$



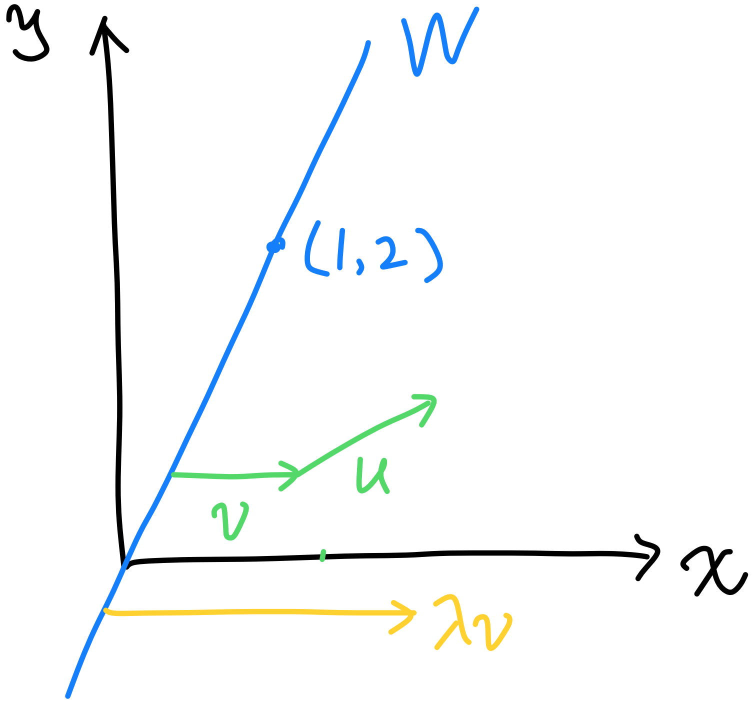
$V = \mathbb{R}^2$

$V/W = \{v + W\} = \{\text{lines} \mathbin{/\!/} W\}$

Quotient

- Let V be a vector space and $W \subset V$ be a linear subspace.
- Define $[v] = v + W = \{u \in V \,|\, v - u \in W\}$
- The **quotient** of V by W is the set $V/W = \{[v] \,|\, v \in V\}$ with
  - Vector addition $[v] + [u] := [v + u]$
  - Scalar multiplication $\lambda[v] := [\lambda v]$

V/W is a vector space.



Quotient

- $\dim V/W = \dim V - \dim W$
- If V has a basis $\{v_1, \ldots, v_n\}$, and W is spanned by $\{v_1, \ldots, v_k\}$ for $k \le n$, then V/W has a basis $\{[v_{k+1}], \ldots, [v_n]\}$.
- Examples: $\mathbb{R}^3/\mathbb{R} \cong \mathbb{R}^2$, $\mathbb{R}^3/\mathbb{R}^2 \cong \mathbb{R}$

$V := \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$

$W := \{(0, 0, z) \mid z \in \mathbb{R}\}$

$V/W = \{[v] \mid v \in V\}$

$e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$

$V = \langle e_1, e_2, e_3 \rangle$

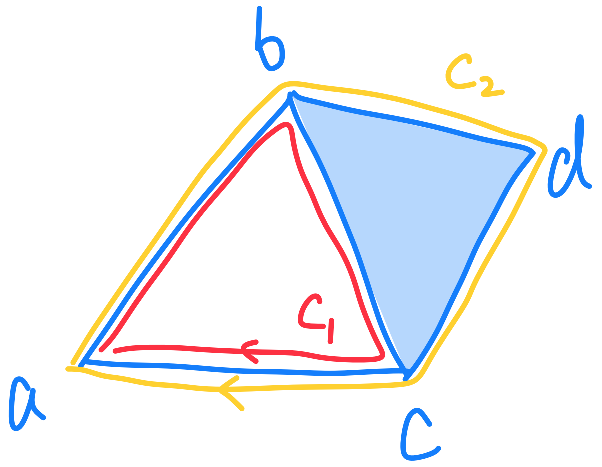
$W = \langle e_3 \rangle$

$\Rightarrow V/W = \langle [e_1], [e_2] \rangle$

Homology groups

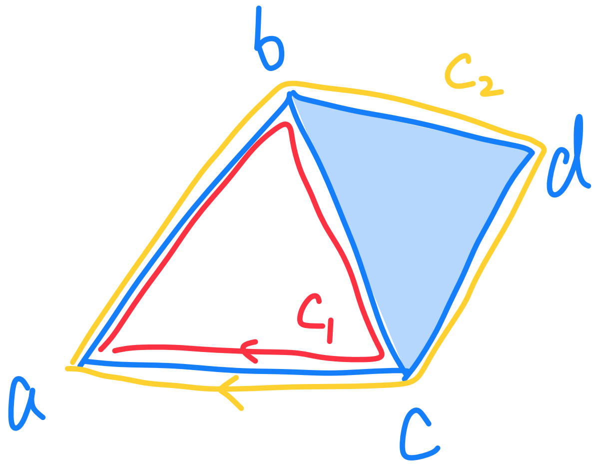
Under field coefficients, C_p, B_p, Z_p, H_p are all vector spaces.

- p-th cycle group $Z_p(K) = \ker(\partial_p)$
- p-th boundary group $B_p(K) = \mathrm{Im}(\partial_{p+1})$
- p-th **homology group** is $H_p(K) = Z_p / B_p$
  - c_1 is **homologous to** c_2 if
    - $c_1 - c_2 \in B_p$, i.e, $c_1 - c_2$ is a boundary cycle
  - $h = [c] \in H_p$:
    - the family p-cycles homologous to c
    - called a **homology class**
  - A cycle is **null-homologuous** if it is a boundary; we also say its homology class is trivial.



C_2 & C_1 are homologous

Homology



$$B_1 = \mathrm{im}(\partial_2 : C_2 \to C_1)$$
$$= \partial_2(C_2)$$
$$= \partial_2(\langle bcd \rangle)$$
$$= \langle \partial_2(bcd) \rangle$$

$$Z_1(K) = \langle \overbrace{ab + bc - ac}^{c_1}, \overbrace{ab + bd - cd - ac}^{c_2} \rangle$$

$$B_1(K) = \langle bc - bd + cd \rangle$$

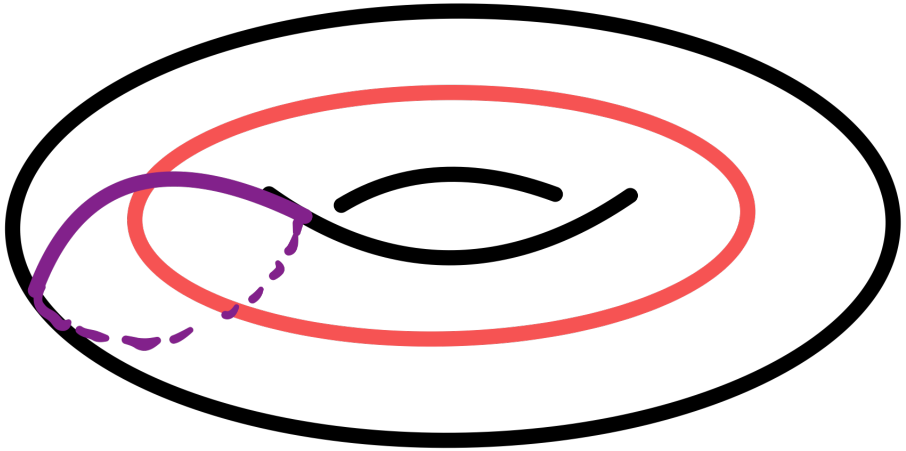
$$H_1(K) = \langle [ab + bc - ac] \rangle$$

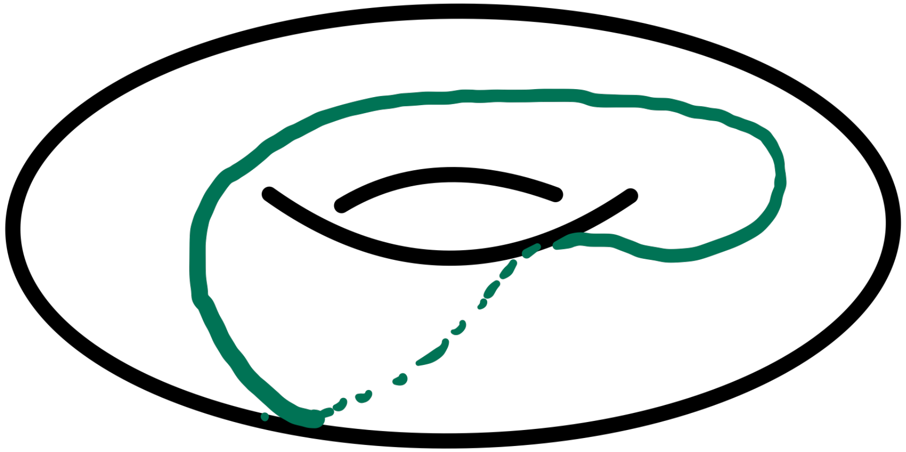
$$Z_1 = \langle c_1, c_2 \rangle = \langle c_1, c_1 - c_2 \rangle$$

$$B_1 = \langle c_1 - c_2 \rangle$$

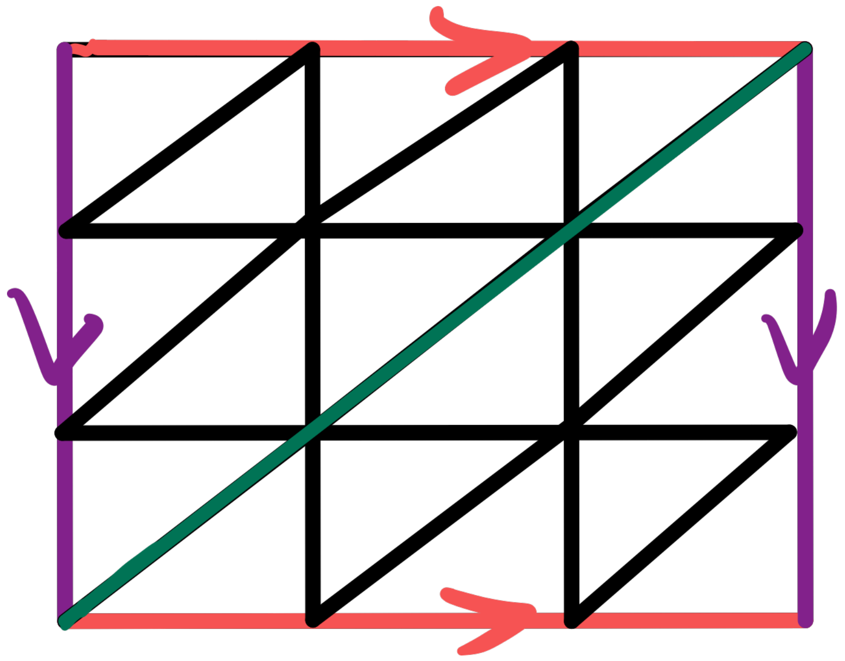
$$H_1 = \langle [c_1] \rangle$$

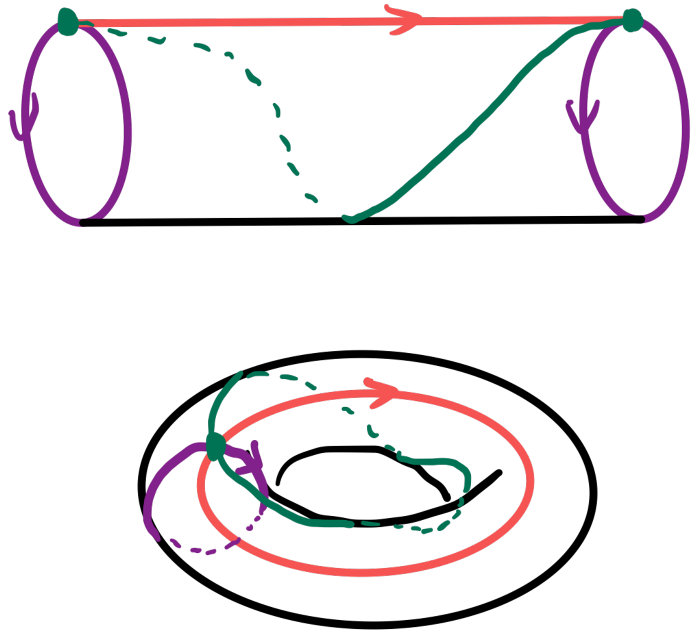
Torus example

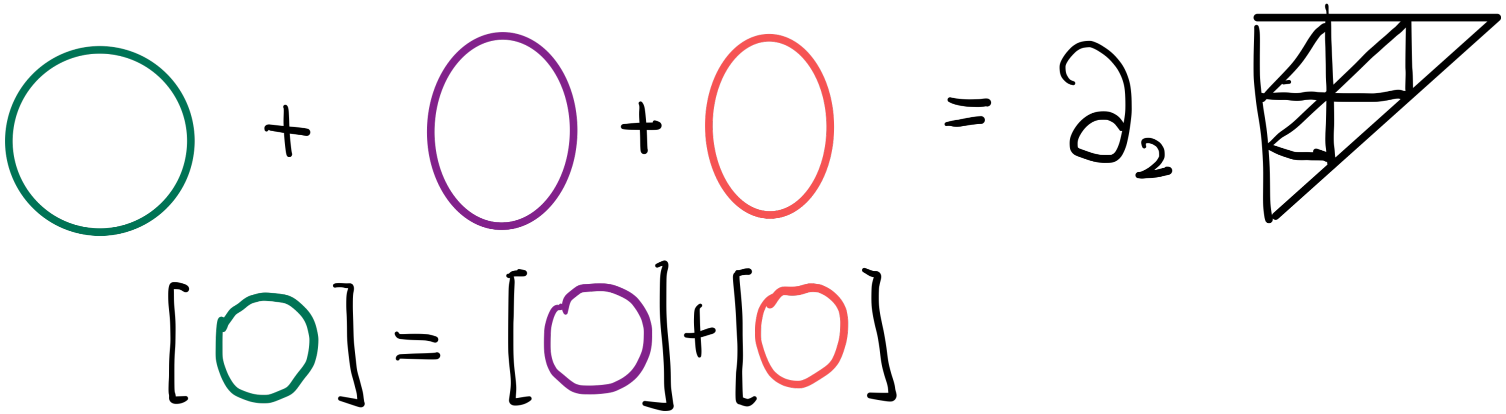




Torus example



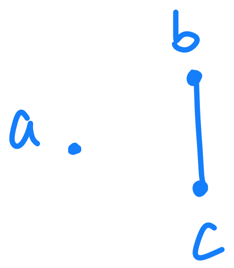




Homology is homotopy invariant

- Theorem: If X and Y are homotopy equivalent, then $H_n(X) \cong H_n(Y)$
- Hence one can define the homology groups of a manifold M through any triangulation K, since $H_n(M) \cong H_n(|K|)$
- Example:
  - $H_0(K) \cong F^k$ where k is the number of connected components
  - $H_n(\mathbb{S}^n) = F$ and $H_m(\mathbb{S}^n) = 0$ for $m \neq 0,n$

Examples



$$H_0 = Z_0 / B_0 = \ker \partial_0 / \operatorname{im} \partial_1 = C_0 / \partial_1(C_1)$$

$$= \langle a, b, c \rangle / \langle \partial_1(bc) \rangle = \langle a, b, c \rangle / \langle c-b \rangle$$

$$= \langle a, b, c-b \rangle / \langle c-b \rangle = \langle [a], [b] \rangle.$$

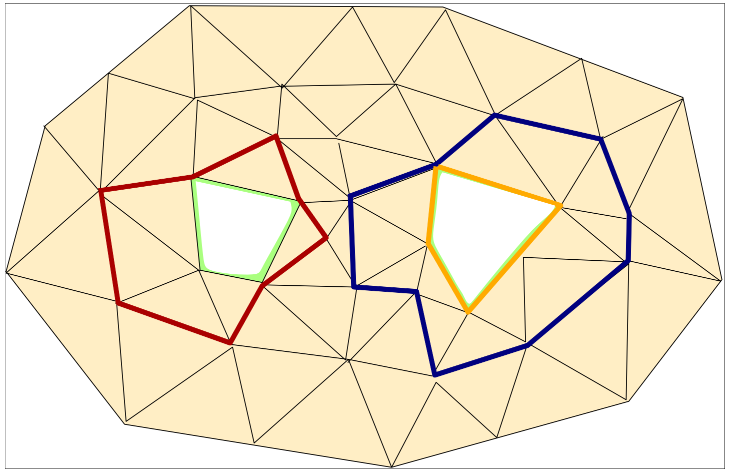
Note: (1) all vertices in a connected component are homologous to each other

(2) vertices in different connected components are NOT homologous.

Betti numbers

- Betti number: $\beta_p(K) = \dim H_p(K)$
- Theorem: $\beta_p(K) = \dim Z_p - \dim B_p$

- Examples:



$\beta_0(K) = ?$ $\beta_1(K) = ?$

Betti numbers are homotopy invariants

- Theorem: Two homotopy equivalent topological spaces have isomorphic homology groups and thus same Betti numbers.

- Sometimes in practice, one only cares about Betti numbers (dimensions) instead of explicit structures (bases) of homology groups

Another definition for Euler characteristic

- Recall the Euler characteristic of a simplicial complex: $\chi(K) = \sum_{p=0} (-1)^p n_p(K)$

- Theorem (Euler-Poincaré formula): $\chi(K) = \sum_{i=0} (-1)^i \beta_i(K)$

$$\cdots \longrightarrow C_{i+1} \xrightarrow{\partial_{i+1}} C_i \xrightarrow{\partial_i} C_{i-1} \longrightarrow \cdots$$

rank-nullity $\Rightarrow$ $n_i = \dim C_i = \mathrm{null}(\partial_i) + \mathrm{rank}(\partial_i)$

$= \dim \ker \partial_i + \dim \operatorname{im} \partial_i = \dim Z_i + \dim B_{i-1}$

$\Rightarrow \chi = n_0 - n_1 + n_2 - \cdots = \dim Z_0 - (\dim Z_1 + \dim B_1) + (\dim Z_2 + \dim B_2) - \cdots$

$= (\dim Z_0 - \dim B_1) - (\dim Z_1 - \dim B_2) + \cdots = \beta_0 - \beta_1 + \cdots$