Chapter 2: **Review** of Probability Theory

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2.1. Probability

2.1.1. Probability definition

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2.1.3. Bayes rule

2.1.4.Total Probability theorem

2.1.5. Independence between events

2.1.1. Probability definition

- Use *Measure Theory*
- Based on a triplet

$$(\Omega, F, P)$$

Where:

- Ω: sample space
  - Set of all possible outcomes
- F: σ−algebra
  - Set of all possible events or combinations of outcomes
- P: probability function
  - Any set function
    - Domain is Ω
    - Range is the closed unit interval $[0,1]$

2.1.1. Probability definition (Cont.)

- P must obey the following rules:
  - $P(\Omega) = 1$
  - Let A be any event in F, then $P(A) \geq 0$
  - Let A and B be two events in F such that $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$.
  - Probability of complement: $P(A) = 1 - P(A)$.
  - $P(A) \leq 1$.
  - $P(\emptyset) = 0$.
  - $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

2.1.2. Conditional probability (Cont.)

- Let A and B be two events, with $P(A) > 0$. The conditional probability of B given A is defined as:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

- Hence, $P(A \cap B) = P(B|A)P(A) = P(A|B)P(B)$
- If $A \cap B = \emptyset$ then $P(B|A) = 0$
- If $A \subset B$, then $P(B|A) = 1$

2.1.3.Bayes rule

- If A and B are events:

$$P(A|B) = \frac{P(B/A)P(A)}{P(B)}$$

2.1.4.Total Probability Theorem

- A set of B_i, $i = 1, \ldots, n$ of events is a partition of Ω when:
  - $\bigcup_{i=1}^{n} B_i = \Omega$.
  - $B_i \cap B_j = \emptyset$, if $i \neq j$.
- Theorem: If A is an event and B_i, $i = 1, \ldots, n$ of is a partition of Ω, then:

$$P(A) = \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(A|B_i)P(B_i)$$

2.1.5. Independence between Events

- Two events A and B are statistically independent when

$$P(A \cap B) = P(A)P(B)$$

- Supposing that both $P(A)$ and $P(B)$ are greater than zero, from the above definition we have that:

$$P(A|B) = P(A) \qquad P(B|A) = P(B)$$

2.1.5. Independence between Events (Cont.)

- N events are statistically independent if the intersection of the events contained in any subset of those N events have probability equal to the product of the individual probabilities
- Example: Three events A, B and C are independent if:

$$P(A \cap B) = P(A)P(B), \quad P(A \cap C) = P(A)P(C), \quad P(B \cap C) = P(B)P(C)$$

$$P(A \cap B \cap C) = P(A)P(B)P(C)$$

2.2. Random variable

2.1.1. What is random variables?

2.1.2. Cumulative Distribution Function (CDF)

2.2.3. Types of random variable

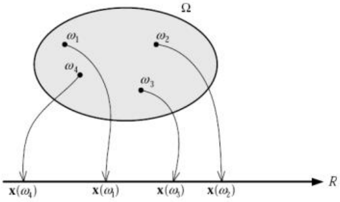
2.2.4. Probability Density Function (PDF)

2.2.1. What is random variables?

- A random variable (rv) is a function that maps each $\omega \in \Omega$ to a real number

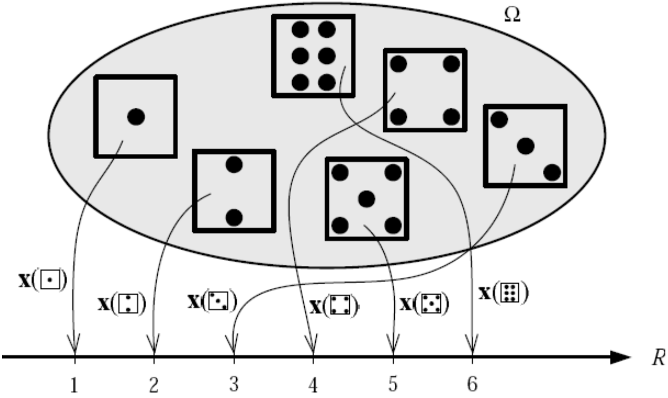
$$X \;:\; \Omega \to \mathrm{R}$$
$$\omega \to X(\omega)$$

E.g:



- A random variable is a *mapping* from the sample space Ω to the set of real numbers.

2.2.1. What is random variables? (cont.)



2.2.1. What is random variables? (cont.)

- Random variable is a representation in numeric form of sample space. Thus,
  - We can process events with digital computing devices
  - Values corresponding to the outcomes are sorted

2.2.1. What is random variables? (cont.)

- Through a random variable, subsets of Ω (w) are mapped as interval (I) of the real numbers. Hence:

$$P(X \in I) = P(\{w | X(\omega) \in I\})$$

Probability of the interval

Probability of the subset

2.2.1. What is random variables? (cont.)

Ex: Throw a dice with 6 faces

- Maps the sample space (1 dot, 2 dots…6 dots) to numbers (1,2…6).
    - The probability of each overcome is 1/6. The probability of each number is also 1/6
- The subset (1 dot, 2 dots, 3 dots) maps to interval from 1 to 3.
    - The probability of the subset will be 3/6. The probability of the interval will be also 3/6

2.2.2. Cumulative Distribution Function (CDF)

- CDF of a random variable X with a given value x ($F_X(x)$) is the probability for a random variable X has a value that does not exceed x

$$F_X(x) = P(X \leq x)$$

- $F_X(x)$ (∞) = 1
- $F_X(x)$ (-∞) = 0
- If $x_1 < x_2, F_X(x_2) \geq F_X(x_1)$

2.2.3. Types of random variable

- Discrete: Cumulative function is a step function (sum of unit step functions)

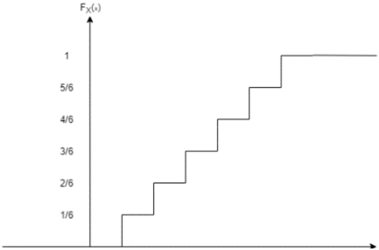
$$F_X(x)=\sum_i P\ (X = x_i)\ u\ (x - x_i)$$

where $u(x)$ is unit step function

2.2.3. Types of random variable (Cont.)

Example: X is the random variable that describes the outcome of the roll of a die $X \in \{1, 2, 3, 4, 5, 6\}$

P(x)=1/6 for all x



- Continous: Cumulative function is a continous function.
- Mixed: Neither discrete nor continous.

2.2.4. Probability Density Function (PDF)

- It is the derivative of the cumulative distribution function:

$$p_X(x) = \frac{d}{dx} F_X(x)$$

- $\int_{-\infty}^{x} p_X(x)dx = F_X(x)$.
- $p_X(x) \geq 0$.
- $\int_{-\infty}^{\infty} p_X(x)dx = 1$.
- $\int_{a}^{b} p_X(x)dx = F_X(b) - F_X(a) = P(a \leq X \leq b)$.

2.2.4. Probability Density Function (PDF) (Cont.)

- Discrete random variable:
  - P_X(x) = P(X=x)
  - The sum taken for every value of x of the probability density function must be equal to 1

$$\sum_{X} px(x) = 1$$

2.3. Function of random variable

- Random variable X = {x}, $x_{min} \leq$ x $\leq x_{min}$ with F_X(x)
- Y = G(X) is defined as function of random variable
  - Generate random variable Y ={y}, (y_{min} = G(x_{min}) $\leq$ y $\leq y_{max}$ = G(x_{max}))
  - Need to determine F_Y(y) from G(X) and F_X(x)
  - Example: Y = aX + b, a and b are constants, a > 0

$$F_Y(y) = P(Y \leq y) = P(aX + b \leq y) = P(X \leq \frac{y-b}{a}) =$$

$$\int_{-\infty}^{\frac{y-b}{a}} p_X(x)dx = F_X(\frac{y-b}{a})$$

2.3. Function of random variable (Cont.)

- Generalization:

$$p_Y(y) = \sum_{i=1}^{n} \frac{p_X(x_i)}{|g'(x_i)|}$$

Where x_i is the solution of equation g(x) = y

- Coding operation is to map a source to a new source which is suitable with transmission requirements
  - E.g. source coding is to map an information source to a code source that has uniform probability distribution
  - From random variable function, a mapping can be found to transfer from an arbitrary source to a source that has required probability distribution

2.4. Discrete Random vector

Let $Z = [X, Y]$ be a random vector with sample space $Z = X \times Y$
X and Y are random variables have sample space

The joint probability distribution function of Z is mapping

$$p_Z(z) : \mathcal{Z} \rightarrow [0,1] \text{ satisfying:}$$

$$\sum_{Z \in \mathcal{Z}} p_Z(z) = \sum_{x,y \times \mathcal{Y}} p_{XY}(x,y) = 1$$

2.4. Discrete Random vector (Cont.)

- Marginal Distributions:

$$p_X(x) = \sum_{y \in Y} p_{XY}(x, y)$$

$$p_Y(y) = \sum_{x \in X} p_{XY}(x, y)$$

2.4. Discrete Random vector (Cont.)

- Conditional Distributions:

$$p_{X|Y=y}(\mathrm{x}) = \frac{p_{XY}(x,y)}{p_Y(y)}$$

$$p_{Y|X=x}(\mathrm{y}) = \frac{p_{XY}(x,y)}{p_X(x)}$$

2.4. Discrete Random vector (Cont.)

- Random variables X and Y are independent if and only if

$$p_{XY}(x, y) = p_X(x)p_Y(y)$$

- Consequences:

$$p_{X|Y=y}(x) = p_X(x)$$

$$p_{Y|X=x}(y) = p_Y(y)$$

2.5. Moments of a Discrete Random Variable

- The $n-$th order moment of a discrete random variable X is defined as:

$$E[X^n] = \sum_{x \in \mathcal{X}} x^n p_X(x)$$

- if $n = 1$, we have the mean of X, $m_X = E[X]$.
- The $m-$th order central moment of a discrete random variable X is defined as:

$$E[(X - m_X)^m] = \sum_{x \in \mathcal{X}} (x - m_X)^m p_X(x)$$

- if $m = 2$, we have the variance of X, σ_X^2.

2.5. Moments of a Discrete Random Variable

- The joint moment $n-$th order with relation to X and $k-$th order with relation to Y:

$$m_{nk} = E[X^n Y^k] = \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} x^n y^k p_{XY}(x, y)$$

- The joint central $n-$th order with relation to X and $k-$th order with relation to Y:

$$\mu_{nk} = E[(X - m_X)^n (Y - m_Y)^k] == \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} (x - m_X)^n (y - m_Y)^k p_{XY}(x, y)$$

2.5. Moments of a Discrete Random Variable

- The correlation of two random variables X and Y is the expected value of their product (joint moment of order 1 in X and order 1 in Y):

$$Corr(X, Y) = m_{11} = E[XY]$$

- The covariance of two random variables X and Y is the joint central moment of order 1 in X and order 1 in Y:

$$Cov(X, Y) = \mu_{11} = E[(X - m_X)(Y - m_Y)]$$

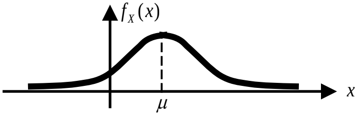
- $Cov(X, Y) = Corr(X, Y) - m_X m_Y$
- Correlation Coefficient:

$$\rho_{XY} = \frac{Cov(X, Y)}{\sigma_X \sigma_Y} \quad \rightarrow \quad -1 \leq \rho_{XY} \leq 1$$

2.6. Examples of Random variables

- Normal (Gaussian) random variables
  - Density function

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$



  - Bell shape curve
  - Distribution function

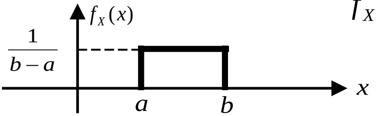
$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(y-\mu)^2/2\sigma^2} dy = G\left(\frac{x-\mu}{\sigma}\right),$$

$$G(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$

2.6. Examples of Random variables (Cont.)

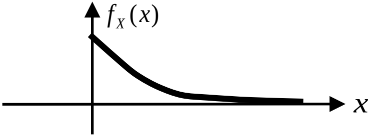
- Uniform random variables: $X \sim U(a,b),\ a < b,$
  - Density function

$$f_X(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b, \\ 0, & \text{otherwise.} \end{cases}$$



- Exponential random variables $X \sim \varepsilon(\lambda)$
  - Density function

$$f_X(x) = \begin{cases} \dfrac{1}{\lambda} e^{-x/\lambda}, & x \ge 0, \\ 0, & \text{otherwise.} \end{cases}$$



2.7. Sum of Gaussian random variable

- Gaussian independent random variables x_1, x_2,..., x_n have mean m_1, m_2,… m_n, variance σ_{x_1}, σ_2,…,σ_{x_n}
- Sum of all variables y =$\sum_1^n x_i$ have
  - Mean m = $\sum_1^n m_i$: constant one-dimensional component
  - Variance σ = $\sum_1^n \sigma_i$: Average power (alternating current power)
- E.g. A channel has input X, noise N, output Y = X+N
  - Noise and Input: independent
  - m_Y = m_X + m_N
  - P_Y = P_X + P_N with P(.) is average power

2.8. Random process

2.8.1 . Definition

2.8.2. Stationary random process

2.8.3. Statistical average or joint moments

2.8.4. Mean value or the first moment

2.8.5. Mean-squared value or the second moment

2.8.6. Correlation

2.8.7. Power spectral density of a random process

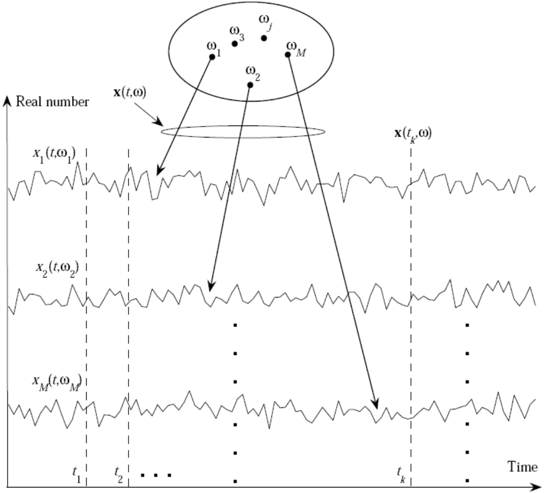
2.8.8. Time averaging and Egordicity

2.8.9. Random process and LTI system

2.8.1. Definition

- Random process is a set of potential random variables (realization, member) to describe a physical object
- Example:
  - To describe the movements of an animal, it is necessary to have different cameras
    - Sample space of each camera is a random variable

2.8.1. Definition (Cont.)



A mapping from a sample space to *a set of time functions.*

2.8.1 Definition (Cont.)

- To describe the random process, we need to have a set of values
- It is necessary to have the probability of appearing simultaneously at one time on a time function of the realization

$$f_{\mathbf{x}(t_1),\mathbf{x}(t_2)}(x_1, x_2; t_1, t_2)$$

2.8.2. Stationary random process

- Based on whether its statistics change with time: the process is *non-stationary* or *stationary*.
- Different levels of stationarity:
  - Strictly stationary: the joint pdf of any order is independent of a shift in time.
  - Nth-order stationarity: the joint pdf does not depend on the time shift

$$f_{\mathbf{x}(t_1),\mathbf{x}(t_2),\ldots\mathbf{x}(t_N)}(x_1, x_2, \ldots, x_N; t_1, t_2, \ldots, t_N) =$$
$$f_{\mathbf{x}(t_1+t),\mathbf{x}(t_2+t),\ldots\mathbf{x}(t_N+t)}(x_1, x_2, \ldots, x_N; t_1 + t, t_2 + t, \ldots, t_N + t).$$

- The first- and second-order stationarity:

$$f_{\mathbf{x}(t_1)}(x, t_1) = f_{\mathbf{x}(t_1+t)}(x; t_1 + t) = f_{\mathbf{x}(t)}(x)$$

$$f_{\mathbf{x}(t_1),\mathbf{x}(t_2)}(x_1, x_2; t_1, t_2) = f_{\mathbf{x}(t_1+t),\mathbf{x}(t_2+t)}(x_1, x_2; t_1 + t, t_2 + t)$$
$$= f_{\mathbf{x}(t_1),\mathbf{x}(t_2)}(x_1, x_2; \tau), \quad \tau = t_2 - t_1.$$

2.8.3. Statistical average or joint moments

- Consider N random variables $\mathbf{x}(t_1), \mathbf{x}(t_2), \ldots \mathbf{x}(t_N)$. The joint moments of these random variables is

$$E\{\mathbf{x}^{k_1}(t_1), \mathbf{x}^{k_2}(t_2), \ldots \mathbf{x}^{k_N}(t_N)\} = \int_{x_1=-\infty}^{\infty} \cdots \int_{x_N=-\infty}^{\infty} x_1^{k_1} x_2^{k_2} \cdots x_N^{k_N} f_{\mathbf{x}(t_1),\mathbf{x}(t_2),\ldots\mathbf{x}(t_N)}(x_1, x_2, \ldots, x_N; t_1, t_2, \ldots, t_N) \, \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_N,$$

  for all integers $k_j \geq 1$ and $N \geq 1$.
- Shall only consider the first- and second-order moments, i.e., $E\{\mathbf{x}(t)\}$, $E\{\mathbf{x}^2(t)\}$ and $E\{\mathbf{x}(t_1)\mathbf{x}(t_2)\}$. They are the *mean value*, *mean-squared value* and *(auto)correlation*.

2.8.4. Mean value or the first moment

- The mean value of the process at time t is

$$m_{\mathbf{x}}(t) = E\{\mathbf{x}(t)\} = \int_{-\infty}^{\infty} x f_{\mathbf{x}(t)}(x;t)\mathrm{d}x.$$

- The average is *across the ensemble* and if the pdf varies with time then the mean value is a (deterministic) function of time.
- If the process is stationary then the mean is independent of t or a constant:

$$m_{\mathbf{x}} = E\{\mathbf{x}(t)\} = \int_{-\infty}^{\infty} x f_{\mathbf{x}}(x)\mathrm{d}x.$$

2.8.5. Mean-squared value or the second moment

- This is defined as

$$\mathsf{MSV}_{\mathbf{x}}(t) = E\{\mathbf{x}^2(t)\} = \int_{-\infty}^{\infty} x^2 f_{\mathbf{x}(t)}(x;t)\mathrm{d}x \text{ (non-stationary)},$$

$$\mathsf{MSV}_{\mathbf{x}} = E\{\mathbf{x}^2(t)\} = \int_{-\infty}^{\infty} x^2 f_{\mathbf{x}}(x)\mathrm{d}x \text{ (stationary)}.$$

- The second central moment (or the variance) is:

$$\begin{aligned}
\sigma_{\mathbf{x}}^2(t) &= E\left\{[\mathbf{x}(t) - m_{\mathbf{x}}(t)]^2\right\} = \mathsf{MSV}_{\mathbf{x}}(t) - m_{\mathbf{x}}^2(t) \text{ (non-stationary)}, \\
\sigma_{\mathbf{x}}^2 &= E\left\{[\mathbf{x}(t) - m_{\mathbf{x}}]^2\right\} = \mathsf{MSV}_{\mathbf{x}} - m_{\mathbf{x}}^2 \text{ (stationary)}.
\end{aligned}$$

2.8.6. Correlation

- The autocorrelation function completely describes the power spectral density of the random process.
- Defined as the correlation between the two random variables $\mathbf{x}_1 = \mathbf{x}(t_1)$ and $\mathbf{x}_2 = \mathbf{x}(t_2)$:

$$\begin{aligned} R_{\mathbf{x}}(t_1, t_2) &= E\{\mathbf{x}(t_1)\mathbf{x}(t_2)\} \\ &= \int_{x_1=-\infty}^{\infty} \int_{x_2=-\infty}^{\infty} x_1 x_2 f_{\mathbf{x}_1,\mathbf{x}_2}(x_1, x_2; t_1, t_2) \mathrm{d}x_1 \mathrm{d}x_2. \end{aligned}$$

- For a stationary process:

$$\begin{aligned} R_{\mathbf{x}}(\tau) &= E\{\mathbf{x}(t)\mathbf{x}(t+\tau)\} \\ &= \int_{x_1=-\infty}^{\infty} \int_{x_2=-\infty}^{\infty} x_1 x_2 f_{\mathbf{x}_1,\mathbf{x}_2}(x_1, x_2; \tau) \mathrm{d}x_1 \mathrm{d}x_2. \end{aligned}$$

- *Wide-sense stationarity* (WSS) process: $E\{\mathbf{x}(t)\} = m_{\mathbf{x}}$ for any t, and $R_{\mathbf{x}}(t_1, t_2) = R_{\mathbf{x}}(\tau)$ for $\tau = t_2 - t_1$.

2.8.7. Power spectral density of a random process

- Taking the Fourier transform of the random process does not work
  - Each representation has a unique spectrum
- Need to determine how the average power of the process is distributed in frequency.

- It can be shown that *the power spectral density and the autocorrelation function are a Fourier transform pair*:

$$R_{\mathbf{x}}(\tau) \longleftrightarrow S_{\mathbf{x}}(f) = \int_{\tau=-\infty}^{\infty} R_{\mathbf{x}}(\tau) \mathrm{e}^{-j2\pi f\tau} \mathrm{d}\tau.$$

2.8.8. Time averaging and Egordicity

- Time averaging:

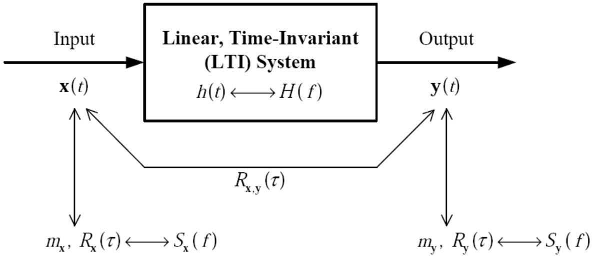
All time averages on a single ensemble member are equal to the corresponding ensemble average:

$$
\begin{aligned}
E\{\mathbf{x}^n(t))\} &= \int_{-\infty}^{\infty} x^n f_{\mathbf{x}}(x)\mathrm{d}x \\
&= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} [\mathbf{x}_k(t,\omega_k)]^n \mathrm{d}t, \ \forall\ n,\ k.
\end{aligned}
$$

- Ergodicity: must be stationary and time averaging = Statistical average

For an ergodic process: To measure various statistical averages, it is sufficient to look at only one realization of the process and find the corresponding time average.

2.8.9. Random process and LTI system



$$
\begin{aligned}
m_{\mathbf{y}} &= E\{\mathbf{y}(t)\} = E\left\{\int_{-\infty}^{\infty} h(\lambda)\mathbf{x}(t-\lambda)\mathrm{d}\lambda\right\} = m_{\mathbf{x}} H(0) \\
S_{\mathbf{y}}(f) &= |H(f)|^2 S_{\mathbf{x}}(f) \\
R_{\mathbf{y}}(\tau) &= h(\tau) * h(-\tau) * R_{\mathbf{x}}(\tau).
\end{aligned}
$$