

# Tableau Theorem Prover for Intuitionistic Propositional Logic

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# Motivation

## Tableau for Classical Logic

- If  $\neg A$  is contradictory in all paths, then  $A \vee \neg A$  lets us conclude  $A$  is a *tautology*.
- For *satisfaction*, running tableau on  $A$  yield a (classical model) evaluation context  $\sigma$ .
- Tableau seems awfully tied to classical logic, is intuitionistic tableau doomed!?

# Classical vs Intuitionistic Logic

## Classical Logic

- The *meaning* of a proposition is its truth value.
- **Satisfaction:** Does evaluating it yield true?
- $A \vee \neg A$
- $\neg\neg A \supset A$
- $A \supset \neg\neg A$

## Intuitionistic Logic

- The *meaning* of a proposition is its constructive content.
- **Satisfaction:** Can you write it as a program?
- $A \supset \neg\neg A$

# Proof Theory for Intuitionistic Logic

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B} \supset_I \quad \frac{\Gamma \vdash A \supset B \quad \Gamma \vdash A}{\Gamma \vdash B} \supset_E$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge_I \quad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \wedge_{E_1} \quad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} \wedge_{E_2}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \vee_{I_1} \quad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \vee_{I_2} \quad \frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C} \vee_E$$

$$\frac{\Gamma, A \vdash \perp}{\Gamma \vdash \neg A} \neg_I \quad \frac{\Gamma \vdash A \quad \Gamma \vdash \neg A}{\Gamma \vdash \perp} \neg_E$$

$$\frac{}{\Gamma \vdash \top} \top_I \quad \frac{\Gamma \vdash \perp}{\Gamma \vdash A} \perp_E$$

... intuitionistic rules plus ...

$$\frac{\Gamma \vdash A}{\Gamma \vdash \neg\neg A} \neg\neg I \qquad \frac{\Gamma \vdash \neg\neg A}{\Gamma \vdash A} \neg\neg E$$

...or...

$$\overline{\Gamma \vdash A \vee \neg A}$$

# Model Theory for Classical Logic

**Boolean Algebra**  $\langle \mathbb{B}, \text{false}, \text{true}, \&\&, ||, ! \rangle$

Classical truth is a boolean value.

## Satisfaction

$$\sigma \models A \Leftrightarrow \sigma \triangleright A \equiv \text{true}$$

$$\sigma \not\models A \Leftrightarrow \sigma \triangleright A \equiv \text{false}$$

## Evaluation

$$\sigma \triangleright p \Leftrightarrow \sigma p$$

$$\sigma \triangleright A \wedge B \Leftrightarrow \sigma \triangleright A \&\& \sigma \triangleright B$$

$$\sigma \triangleright A \vee B \Leftrightarrow \sigma \triangleright A || \sigma \triangleright B$$

$$\sigma \triangleright A \supset B \Leftrightarrow !(\sigma \triangleright A) || \sigma \triangleright B$$

$$\sigma \triangleright \neg A \Leftrightarrow !(\sigma \triangleright A)$$

# Model Theory for Intuitionistic Logic

## Kripke Model $\langle \mathbb{C}, \leq, \emptyset, \Vdash \rangle$

Intuitionistic truth is constructive evidence, or a program.

### Forcing (intuitionistic satisfaction)

$$\Gamma \Vdash p \Leftrightarrow \Gamma \Vdash^p p$$

$$\Gamma \Vdash A \wedge B \Leftrightarrow \Gamma \Vdash A \times \Gamma \Vdash B$$

$$\Gamma \Vdash A \vee B \Leftrightarrow \Gamma \Vdash A \uplus \Gamma \Vdash B$$

$$\Gamma \Vdash A \supset B \Leftrightarrow \Gamma \leq \Delta \Rightarrow \Delta \Vdash A \Rightarrow \Delta \Vdash B$$

$$\Gamma \Vdash \neg A \Leftrightarrow \Gamma \leq \Delta \Rightarrow \Delta \Vdash A \Rightarrow \perp$$

$$\Gamma \nVdash A \Leftrightarrow \Gamma \Vdash \neg A$$

# Classical vs Intuitionistic Model Theory

Many more intuitionistic models than classical models because intuitionistic implication and negation allow arbitrary intrinsically distinct functions.

*Much bigger search space for an intuitionistic theorem prover!*

## Evaluation

$$\begin{aligned}\sigma \triangleright A \supset B &\Leftrightarrow !(\sigma \triangleright A) \parallel \sigma \triangleright B \\ \sigma \triangleright \neg A &\Leftrightarrow !(\sigma \triangleright A)\end{aligned}$$

## Forcing

$$\begin{aligned}\Gamma \Vdash A \supset B &\Leftrightarrow \Gamma \leq \Delta \Rightarrow \Delta \Vdash A \Rightarrow \Delta \Vdash B \\ \Gamma \Vdash \neg A &\Leftrightarrow \Gamma \leq \Delta \Rightarrow \Delta \Vdash A \Rightarrow \perp\end{aligned}$$



# Classical Tableau Calculus

$$\frac{S, T(A \wedge B)}{S, TA, TB} T\wedge \qquad \frac{S, F(A \wedge B)}{S, FA \mid S, FB} F\wedge$$

$$\frac{S, T(A \vee B)}{S, TA \mid S, TB} T\vee \qquad \frac{S, F(A \vee B)}{S, FA, FB} F\vee$$

$$\frac{S, T(A \supset B)}{S, FA \mid S, TB} T\supset \qquad \frac{S, F(A \supset B)}{S_T, TA, FB} F\supset$$

$$\frac{S, T(\neg A)}{S, FA} T\neg \qquad \frac{S, F(\neg A)}{S_T, TA} F\neg$$

# Intuitionistic Tableau Calculus

$$S_T \Leftrightarrow \{TA \mid TA \in S\}$$

$$\frac{S, T(A \wedge B)}{S, TA, TB} T\wedge \qquad \frac{S, F(A \wedge B)}{S, FA \mid S, FB} F\wedge$$

$$\frac{S, T(A \vee B)}{S, TA \mid S, TB} T\vee \qquad \frac{S, F(A \vee B)}{S, FA, FB} F\vee$$

$$\frac{S, T(A \supset B)}{S, FA \mid S, TB} T\supset \qquad \frac{S, F(A \supset B)}{\textcolor{red}{S}_T, TA, FB} F\supset$$

$$\frac{S, T(\neg A)}{S, FA} T\neg \qquad \frac{S, F(\neg A)}{\textcolor{red}{S}_T, TA} F\neg$$

# Classical Tableau Interpretation

Gradually build an evaluation context  $\sigma$  for  $A$  (such that  $\sigma \models A$ ), until tableau is finished or the model is contradictory.

## Judgments

- $TA$  means  $A$  is true in the model.
- $FA$  means  $A$  is false in the model.

## Inference Rules

If the premise is true, then the conclusion is true.

# Intuitionistic Tableau Interpretation

Gradually build a “proof” of  $A$  (an “element” of  $\Gamma \Vdash A$ ), until tableau is finished or the model is contradictory.

## Judgments

- $\text{TA}$  means we have a proof of  $A$ .
- $\text{FA}$  means  $A$  we do not (yet) have a proof of  $A$ .

## Inference Rules

- If the premise is true, then the conclusion **may** be true.
- The conclusion is logically consistent with the premise.

# Intuitionistic Tableau Calculus

$$S_T \Leftrightarrow \{TA \mid TA \in S\}$$

$$\frac{S, T(A \wedge B)}{S, TA, TB} T\wedge \qquad \frac{S, F(A \wedge B)}{S, FA \mid S, FB} F\wedge$$

$$\frac{S, T(A \vee B)}{S, TA \mid S, TB} T\vee \qquad \frac{S, F(A \vee B)}{S, FA, FB} F\vee$$

$$\frac{S, T(A \supset B)}{S, FA \mid S, TB} T\supset \qquad \frac{S, F(A \supset B)}{\textcolor{red}{S}_T, TA, FB} F\supset$$

$$\frac{S, T(\neg A)}{S, FA} T\neg \qquad \frac{S, F(\neg A)}{\textcolor{red}{S}_T, TA} F\neg$$

# Closed Example $A \supset A$

$[\{F(A \supset A)\}],$   
 $[\{TA, FA\}].$

$$\frac{S, T(A \wedge B)}{S, TA, TB} T\wedge$$

$$\frac{S, F(A \wedge B)}{S, FA \mid S, FB} F\wedge$$

$$\frac{S, T(A \vee B)}{S, TA \mid S, TB} T\vee$$

$$\frac{S, F(A \vee B)}{S, FA, FB} F\vee$$

$$\frac{S, T(A \supset B)}{S, FA \mid S, TB} T\supset$$

$$\frac{S, F(A \supset B)}{S_T, TA, FB} F\supset$$

$$\frac{S, T(\neg A)}{S, FA} T\neg$$

$$\frac{S, F(\neg A)}{S_T, TA} F\neg$$

# Closed Example $A \supset (A \wedge A)$

$[\{F(A \supset (A \wedge A))\},$   
 $[\{TA, F(A \wedge A))\},$   
 $[\{TA, FA\}, \{TA, FA\}].$

$$\frac{S, T(A \wedge B)}{S, TA, TB} T\wedge$$

$$\frac{S, F(A \wedge B)}{S, FA \mid S, FB} F\wedge$$

$$\frac{S, T(A \vee B)}{S, TA \mid S, TB} T\vee$$

$$\frac{S, F(A \vee B)}{S, FA, FB} F\vee$$

$$\frac{S, T(A \supset B)}{S, FA \mid S, TB} T\supset$$

$$\frac{S, F(A \supset B)}{S_T, TA, FB} F\supset$$

$$\frac{S, T(\neg A)}{S, FA} T\neg$$

$$\frac{S, F(\neg A)}{S_T, TA} F\neg$$

# Open Example $A \vee \neg A$

$[\{F(A \vee \neg A)\}],$   
 $[\{FA, F(\neg A)\}],$   
 $[\{TA\}].$

$$\frac{S, T(A \wedge B)}{S, TA, TB} T\wedge$$

$$\frac{S, F(A \wedge B)}{S, FA \mid S, FB} F\wedge$$

$$\frac{S, T(A \vee B)}{S, TA \mid S, TB} T\vee$$

$$\frac{S, F(A \vee B)}{S, FA, FB} F\vee$$

$$\frac{S, T(A \supset B)}{S, FA \mid S, TB} T\supset$$

$$\frac{S, F(A \supset B)}{S_T, TA, FB} F\supset$$

$$\frac{S, T(\neg A)}{S, FA} T\neg$$

$$\frac{S, F(\neg A)}{S_T, TA} F\neg$$



# Closed Example $A \supset (A \supset B) \supset B$

$$\begin{array}{l}
 [\{F(A \supset (A \supset B) \supset B)\}], \\
 [\{TA, F((A \supset B) \supset B)\}], \\
 [\{TA, T(A \supset B), FB\}],
 \end{array}
 \quad
 \frac{S, T(A \wedge B)}{S, TA, TB} T\wedge
 \quad
 \frac{S, F(A \wedge B)}{S, FA \mid S, FB} F\wedge$$

$$\begin{array}{l}
 [\{TA, FA, FB\}, \{TA, TB, FB\}].
 \end{array}
 \quad
 \frac{S, T(A \vee B)}{S, TA \mid S, TB} T\vee
 \quad
 \frac{S, F(A \vee B)}{S, FA, FB} F\vee$$

$$\frac{S, T(A \supset B)}{S, FA \mid S, TB} T\supset
 \quad
 \frac{S, F(A \supset B)}{S_T, TA, FB} F\supset$$

$$\frac{S, T(\neg A)}{S, FA} T\neg
 \quad
 \frac{S, F(\neg A)}{S_T, TA} F\neg$$

# Classical vs Intuitionistic Tableau Search

When looking for a closed tableau:

## Classical

You can prioritize **any** rule to apply to  $S$  to shrink the search space.

## Intuitionistic

You must try applying **all** rules to  $S$ , but can still prioritize some and backtrack if they fail.

# “Open” Example $\neg A \supset \neg A$

$[\{F(\neg A \supset \neg A)\}],$   
 $[\{T(\neg A), F(\neg A)\}],$   
 $[\{FA, F(\neg A)\}],$   
 $[\{TA\}].$

$$\frac{S, T(A \wedge B)}{S, TA, TB} T\wedge$$

$$\frac{S, F(A \wedge B)}{S, FA \mid S, FB} F\wedge$$

$$\frac{S, T(A \vee B)}{S, TA \mid S, TB} T\vee$$

$$\frac{S, F(A \vee B)}{S, FA, FB} F\vee$$

$$\frac{S, T(A \supset B)}{S, FA \mid S, TB} T\supset$$

$$\frac{S, F(A \supset B)}{S_T, TA, FB} F\supset$$

$$\frac{S, T(\neg A)}{S, FA} T\neg$$

$$\frac{S, F(\neg A)}{S_T, TA} F\neg$$

# Closed Example $\neg A \supset \neg A$

$[\{F(\neg A \supset \neg A)\}],$   
 $[\{T(\neg A), F(\neg A)\}],$   
 $[\{T(\neg A), TA\}],$   
 $[\{FA, TA\}].$

$$\frac{S, T(A \wedge B)}{S, TA, TB} T\wedge$$

$$\frac{S, F(A \wedge B)}{S, FA \mid S, FB} F\wedge$$

$$\frac{S, T(A \vee B)}{S, TA \mid S, TB} T\vee$$

$$\frac{S, F(A \vee B)}{S, FA, FB} F\vee$$

$$\frac{S, T(A \supset B)}{S, FA \mid S, TB} T\supset$$

$$\frac{S, F(A \supset B)}{S_T, TA, FB} F\supset$$

$$\frac{S, T(\neg A)}{S, FA} T\neg$$

$$\frac{S, F(\neg A)}{S_T, TA} F\neg$$

# Using Classical vs Intuitionistic Tableau

## Classical

To show that  $A$  is true:

- 1 Assume that  $A$  is false.
- 2 Build a tableau for  $\neg A$ .
- 3 If some sub-proposition is true and false,  $A$  must be true.

## Intuitionistic

To show that  $A$  is provable:

- 1 Assume that  $A$  has not been proven.
- 2 Build a tableau for  $\neg A$ .
- 3 If some sub-proposition is provable and not yet proven, it must be impossible that  $A$  has not been proven.

# Classical vs Intuitionistic Tableau Soundness

## Classical

**Have** a model  $\sigma$  from the tableau *conclusion*, so **check** that  $\sigma \models A$ .

## Intuitionistic

**Have** a tableau *derivation* of  $A$ , so **construct** an element of  $\Gamma \Vdash A$ .

# Intuitionistic Tableau Soundness

## Theorem

**Have** a tableau *derivation* of  $A$ , so **construct** an element of  $\Gamma \Vdash A$ .

## Fitting's Proof

By showing the contrapositive.

Sadly,  $(\neg B \supset \neg A) \not\Rightarrow (A \supset B)$  intuitionistically.

# References

Classical is to Intuitionistic as Smullyan is to Fitting

## Classical Tableau Book

*First Order Logic* - Smullyan'68

## Intuitionistic Tableau Book

*Intuitionistic Logic: Model Theory and Forcing* - Fitting'69

## Intuitionistic Tableau Optimization Papers

- *An  $O(n \log n)$ -Space Decision Procedure for Intuitionistic Propositional Logic* - Hudelmaier'93
- *A Tableau Decision Procedure for Propositional Intuitionistic Logic* - Avellone et. al.'06