Machine Learning 1 - Week 2

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1 Problems

1. Proof that:
   (a) Gaussian distribution is normalized
   (b) Expectation of Gaussian distribution is μ (mean)
   (c) Variance of Gaussian distribution is σ^2 (variance)
   (d) Multivariate Gaussian distribution is normalized
2. Calculate:
   (a) The conditional of Gaussian distribution
   (b) The marginal of Gaussian distribution

2 Solutions

2.1 Gaussian distribution is normalized

The probability density function of Gaussian distribution is given by:

$$p(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

To prove that the above expression is normalized, we have to show that:

$$\int_{-\infty}^{\infty} p(x \mid \mu, \sigma^2) dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1$$

For the sake of simplicity let us assume that the mean is zero. We need to prove that:

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} dx = \sqrt{2\pi\sigma^2}$$

Let

$$I = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} dx$$

Squaring the above expression:

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2\sigma^2}} dxdy$$

To integrate this expression we make the transformation from Cartesian coordinates (x, y) to polar coordinates (r, θ), which is defined by

$$\begin{aligned} x &= rcos(\theta) \\ y &= rsin(\theta) \end{aligned}$$

Also the Jacobian of the change of variables is given by:

$$\begin{aligned} \frac{\partial(x,y)}{\partial(r,\theta)} &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} \\ &= \begin{vmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{vmatrix} \\ &= rcos^2\theta + rsin^2\theta \\ &= r \end{aligned}$$

After all the transformations, we end up with:

$$\begin{aligned} I^2 &= \int_0^{2\pi} \int_0^{\infty} e^{-\frac{r^2}{2\sigma^2}} rdrd\theta \\ &= 2\pi \int_0^{\infty} e^{-\frac{r^2}{2\sigma^2}} rdr \\ &= -2\pi\sigma^2 \int_0^{\infty} e^{-\frac{r^2}{2\sigma^2}} d\Big(-\frac{r^2}{2\sigma^2} \Big) \\ &= -2\pi\sigma^2 \Big(e^{-\frac{r^2}{2\sigma^2}} \Big) \Big|_0^{\infty} \\ &= 2\pi\sigma^2 \end{aligned}$$

Take the square root of both sides, we get:

$$I = \sqrt{2\pi\sigma^2}$$

To prove that $p(x \mid \mu, \sigma^2)$ is normalized, we make the tranformation $y = x - \mu$ so that:

$$\begin{aligned} \int_{-\infty}^{\infty} p(x \mid \mu, \sigma^2) dx &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} dy \\ &= \frac{I}{\sqrt{2\pi\sigma^2}} \\ &= 1 \end{aligned}$$

2.2 Expectation of Gaussian distribution is μ

The mean of Gaussian distribution is given by:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} xp(x \mid \mu, \sigma^2)dx = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Changing the variables using $y = x - \mu$, the above expression becomes:

$$\begin{aligned}
\mathbb{E}[X] &= \int_{-\infty}^{\infty} (y+\mu) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} dy \\
&= \int_{-\infty}^{\infty} \frac{y}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} dy + \int_{-\infty}^{\infty} \frac{\mu}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} dy
\end{aligned}$$

Let

$$I_1 = \int_{-\infty}^{\infty} \frac{y}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} dy$$

Using additivity, we have:

$$\begin{aligned}
I_1 &= \int_{-\infty}^{0} \frac{y}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} dy + \int_{0}^{\infty} \frac{y}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} dy \\
&= -\int_{0}^{-\infty} \frac{y}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} dy + \int_{0}^{\infty} \frac{y}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} dy \\
&= \int_{0}^{\infty} \frac{(-y)}{\sqrt{2\pi\sigma^2}} e^{-\frac{(-y)^2}{2\sigma^2}} dy + \int_{0}^{\infty} \frac{y}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} dy \\
&= -\int_{0}^{\infty} \frac{y}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} dy + \int_{0}^{\infty} \frac{y}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} dy \\
&= 0
\end{aligned}$$

So we have that:

$$\begin{aligned}
\mathbb{E}[X] &= \int_{-\infty}^{\infty} \frac{\mu}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} dy \\
&= \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} dy \\
&= \mu
\end{aligned}$$

2.3 Variance of Gaussian distribution is σ^2

The variance of Gaussian distribution is given by:

$$Var(X) = \int_{-\infty}^{\infty} (x-\mu)^2 p(x \mid \mu, \sigma^2)dx = \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

We have proved that:

$$\int_{-\infty}^{\infty} p(x \mid \mu, \sigma^2)dx = 1 \tag{1}$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1 \tag{2}$$

$$\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \sqrt{2\pi\sigma^2} \tag{3}$$

Differentiating both sides of (3) with respect to σ^2:

$$\int_{-\infty}^{\infty} \left[\frac{(x-\mu)^2}{2(\sigma^2)^2}\right] e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{\sqrt{2\pi}}{2\sqrt{\sigma^2}}$$

$$\int_{-\infty}^{\infty} (x-\mu)^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \sqrt{2\pi}\sigma^3$$

$$\int_{-\infty}^{\infty} \frac{(x-\mu)^2}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \sigma^2$$