LECTURE NOTE 2
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2 Lecture 2

2.1 Fundamental Set Operations

[Recall] There are 3 fundamental set operations:

- Union: $A \cup B = \{\omega \in \Omega : \omega \in A \text{ or } \omega \in B\}$.
- Intersection: $A \cap B = \{\omega \in \Omega : \omega \in A \text{ and } \omega \in B\}$.
- Complement: $\overline{A} = A^c = \{\omega \in \Omega : \omega \notin A\}$.

There are 2 other "set difference operations" that are sometimes used:

- $A - B = \{\omega \in A : \omega \notin B\}$
- $B - A = \{\omega \in B : \omega \notin A\}$

2.2 Index sets $\mathcal{I}$

[Recall]

- Indexed collections of sets $\{A_i; i \in \mathcal{I}\}$, where $\mathcal{I}$ is an index set.
- So $\{A_i; i \in \mathcal{I}\}$ is a "set of sets", or a family of sets, or a collection of sets.
- There is one and only one set A_i, for each $i \in \mathcal{I}$.

  Typical Index sets:

$$
\begin{aligned}
&\mathbb{N} = \{1, 2, 3, \ldots\} = \text{ natural numbers },\\
&\mathbb{Z} = \{\ldots - 1, 0, 1, 2, \ldots\} = \text{ integers },\\
&\mathbb{Z}_+ = \{0, 1, 2, \ldots\} = \text{ non-negative integers },\\
&\mathbb{R} = (-\infty, \infty),\\
&\mathbb{I}_n = \{0, 1, 2, \ldots, n-1\}.
\end{aligned}
$$

2.3 Cardinality of Sets

- A set is finite if it has a finite number of elements.

  (i.e., its elements can be put in one-to-one correspondence with the numbers $1, 2, \ldots, n$ for some natural number n.)

- A set is infinite if it is not finite.

2.4 Countable and Uncountable Sets

Infinite sets come in two varieties: *countable* and *uncountable*.

- An infinite set is countable if its elements can be put in one-to-one correspondence with the natural (counting) numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$.
- An infinite set is uncountable if it is not countable.
- The following are examples of uncountable sets:
  - $\mathbb{R} = (-\infty, \infty)$
  - $[0, 1]$ and $(0, 1)$
  - $[a, b]$, $[a, b)$, $(a, b]$, (a, b) for $a, b \in \mathbb{R}$ such that $a < b$.

2.5 Collectively Exhaustive and Disjoint Families

- Given an indexed family of sets $\{A_i; i \in I\}$, the union of the sets in the family is

$$\bigcup_{i\in I} A_i \triangleq \{\omega \in \Omega : \omega \in A_i \text{ for at least one } i \in \mathcal{I}\}.$$

- The intersection of the sets in the family is

$$\bigcap_{i\in I} A_i \triangleq \{\omega \in \Omega : \omega \in A_i \text{ for all } i \in \mathcal{I}\}.$$

- If $G \subseteq \Omega$ and $\{A_i; i \in \mathcal{I}\}$ is a family of sets, then if

$$\bigcup_{i\in\mathcal{I}} A_i = G$$

  we say that $\{A_i; i \in \mathcal{I}\}$ is collectively exhaustive over G.

- A family of sets $\{A_i; i \in \mathcal{I}\}$ is disjoint if

$$A_i \cap A_j = \varnothing, \forall i, j \in \mathcal{I} \text{ such that } i \neq j.$$

2.6 Partitioned Sets

- A family of sets $\{A_i; i \in \mathcal{I}\}$ is a partition of Ω if it is disjoint and collectively exhaustive over Ω, meaning:
  - *Disjointness:* $A_i \cap A_j = \varnothing$ for all $i \neq j$.
  - *Collectively exhaustive:* $\bigcup_{i\in I} A_i = \Omega$, every element of Ω is in at least one A_i.
- Let $\{A_i; i \in \mathcal{I}\}$ be a partition of Ω. Define $B_i \triangleq A_i \cap G$ for some $G \subseteq \Omega$ and for all $i \in \mathcal{I}$. Show that, $\{B_i; i \in \mathcal{I}\}$ is a partition of G.

  Need to prove,

  1. The family of sets $\{B_i; i \in \mathcal{I}\}$ are disjoint.

     Assume B_i and B_j are not disjoint for some $i \neq j$, meaning there exists an element x such that $x \in B_i$ and $x \in B_j$. By definition, $x \in A_i \cap G$ and $x \in A_j \cap G$, implying $x \in A_i$ and $x \in A_j$, which contradicts the assumption that $\{A_i\}$ are disjoint. Therefore,$\{B_i; i \in \mathcal{I}\}$ is disjoint.

2. The union of all $\{B_i; i \in \mathcal{I}\}$ is G.
   Since $B_i = A_i \cap G$, then $\bigcup_{i\in\mathcal{I}} B_i = \bigcup_{i\in\mathcal{I}}(A_i \cap G)$. Given $\bigcup_{i\in\mathcal{I}} A_i = \Omega$ and $G \subseteq \Omega$, it follows that $\bigcup_{i\in\mathcal{I}}(A_i \cap G) = G = \bigcup_{i\in\mathcal{I}} B_i$.
   Combine 1 and 2, $\{B_i; i \in \mathcal{I}\}$ forms a partition of G.

2.7 Probability Spaces $(\Omega, \mathcal{F}, P)$

A probability space $(\Omega, \mathcal{F}, P)$ is a triple consisting of:

- A sample space Ω (or hand written notation $\mathcal{S}$ in this course).
- A collection of events (subsets of Ω) $\mathcal{F}(\Omega)$.
- The probabilities $P(A)$ for each $A \in \mathcal{F}(\Omega)$,

$$P : \mathcal{F}(\Omega) \to [0, 1].$$

  For example, consider the random experiment of flipping a fair coin. The probability space for this experiment:

  - $\Omega = \{\text{“Heads”}, \text{“Tails”}\}$,
  - $\mathcal{F} = \{\varnothing, \{\text{“Heads”}\}, \{\text{“Tails”}\}, \Omega\}$,
  - $P(\{\text{“Heads”}\}) = 0.5$, $P(\{\text{“Tails”}\}) = 0.5$, $P(\Omega) = 1$, and $P(\varnothing) = 0$.

2.8 The Sample Space Ω

- The sample space Ω of a random experiment is a non-empty set of possible outcomes of the random experiment.

One and only one outcome from the sample space Ω occurs when we perform a random experiment.

2.9 The Event Space $\mathcal{F}(\Omega)$

The event space $\mathcal{F}(\Omega)$ is a non-empty collection of subsets of Ω satisfying the following closure properties:

1. If $A \in \mathcal{F}(\Omega)$, then $\overline{A} \in \mathcal{F}(\Omega)$.
2. For any finite n, if $A_i \in \mathcal{F}(\Omega)$ for $i = 1, 2, \ldots, n$, then $\bigcup_{i=1}^{n} A_i \in \mathcal{F}(\Omega)$.
3. If $A_i \in \mathcal{F}(\Omega)$ for $i = 1, 2, 3, \ldots$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}(\Omega)$.

A set of subsets of Ω satisfying these three properties is called a σ-field ($\mathcal{F}_\sigma$).
Note: If only 1 and 2 hold, we have a field of sets.

2.10 Intersections in the Event Space $\mathcal{F}(\Omega)$

- Suppose $A, B \in \mathcal{F}(\Omega)$, is $A \cap B \in \mathcal{F}(\Omega)$?

  Short answer, by the closure property 1, closed under complements:

  $$\overline{A}, \overline{B} \in \mathcal{F}(\Omega).$$

  Using the closure property 2, closed under unions:

  $$\overline{A} \cup \overline{B} \in \mathcal{F}(\Omega).$$

  Applying De Morgan's laws:

  $$A \cap B = \overline{\overline{A \cap B}} = \overline{\overline{A} \cup \overline{B}}.$$

  Since the event space is closed under complements, it follows that:

  $$\overline{\overline{A} \cup \overline{B}} \in \mathcal{F}(\Omega),$$

  thus proving $A \cap B \in \mathcal{F}(\Omega)$.

- It follows from the closure properties that $\varnothing, \Omega \in \mathcal{F}(\Omega)$.

  Proof:

  Suppose $A \in \mathcal{F}(\Omega)$ (non-empty), by closure property 1, closed under complements,

  $$\overline{A} \in \mathcal{F}(\Omega).$$

  Furthermore, by closure property 2, closed under unions,

  $$\Omega = A \cup \overline{A} \in \mathcal{F}(\Omega).$$

  By closure property 1 again, closed under complements,

  $$\Rightarrow \varnothing = \overline{\Omega} \in \mathcal{F}(\Omega)$$

  Hence, $\varnothing, \Omega \in \mathcal{F}(\Omega)$.

2.11 Examples

1. Show that the power set of any set Ω, is a σ-field ($\mathcal{F}_\sigma$).

   Note: the power set of Ω, is the set contains all subsets of Ω including Ω and $\varnothing$.

Short answer,

Let $\rho(\Omega)$ be the power set of Ω, then

i) $\Omega \in \rho(\Omega)$, it's non-empty.
(By definition, the power set contains all subsets of Ω including Ω.)

ii) **For any** $A \in \rho(\Omega)$, $\quad \overline{A} \in \rho(\Omega)$.
(as $\rho(\Omega)$ contains all subsets of Ω.)

iii) **For any** $A_1, A_2, A_3, \cdots \in \rho(\Omega)$, $\displaystyle\bigcup_{n=1}^{\infty} A_n$ is also a subset of Ω. Hence, $\displaystyle\bigcup_{n=1}^{\infty} A_n \in \rho(\Omega)$.

Thus, $\rho(\Omega)$ is a σ-field.

2. Let A and B be a two σ-fields of subsets of Ω. Show that $A \cap B$ is also a σ-fields of subsets of Ω.
Short answer,

Need to verify that the intersection of A and B satisfies the three properties:

i) The sample space Ω is in the intersection.

ii) The intersection is closed under complements.

iii) The intersection is closed under countable unions.

Given A is σ-field of **subsets of** Ω, and B is σ-field of **subsets of** Ω.
Let $D = A \cap B$, then,

i) A is σ-field of **subsets of** Ω, and B is σ-field of **subsets of** Ω.
$\Rightarrow \Omega \in A$ and $\Omega \in B$,
$\Rightarrow \Omega \in A \cap B$,
$\Rightarrow \Omega \in D$.

ii) **For any** $G \in D$, G is also in A ($G \in A$) & in B ($G \in B$). A is σ-field, and B is σ-field.
Thus, $\overline{G} \in A$ and $\overline{G} \in B$,
$\Rightarrow \overline{G} \in A \cap B$,
$\Rightarrow \overline{G} \in D$.

iii) **For any** $G_1, G_2, G_3, \dots \in D$, $G_1, G_2, \dots$ in A and $G_1, G_2, \dots$ in B. A is σ-field, and B is σ-field.
$\Rightarrow \bigcup_{n=1}^{\infty} G_n \in A$ and $\bigcup_{n=1}^{\infty} G_n \in B$,
$\Rightarrow \bigcup_{n=1}^{\infty} G_n \in A \cap B$,
$\Rightarrow \bigcup_{n=1}^{\infty} G_n \in D$.

Thus, $A \cap B$ is also a σ-field of **subsets of** Ω.

3. Let B be a σ-field of subsets of the real line $\mathbb{R}$. B is generated by intervals of the form (a,b) where $a < b, -\infty < a < \infty, -\infty < b < \infty$, that is, $B = \sigma$(all finite open intervals). Show that

a) B contains all sets of the form $(-\infty, a)$

b) B contains all sets of the form $(-\infty, a]$

Short answer,

a)

Because $B = \sigma$(all finite open intervals), **for any** positive integer n, $(a - n, a)$ is a subset of B.

$$(a - n, a) \in B$$

Because B is σ-field of real line, for σ-field property iii)

$$\begin{aligned} \bigcup_{n=1}^{\infty} (a - n, a) &\quad \in B \\ \Rightarrow \quad (-\infty, a) &\quad \in B. \end{aligned}$$

b)
Given $B = \sigma$(all finite open intervals),
we set G_n, for any positive integer n,the open interval $\left(-\infty, a+\frac{1}{n}\right)$ is a subset of B.
we set H_m, for any positive integer m,the open interval $\left(-\infty, a+\frac{1}{m}\right)$ is a subset of B.
$\overline{G}, \overline{H} \in B$ [by ii)], $\overline{G} \cup \overline{H} \in B$ [by iii)], $\overline{\overline{G} \cup \overline{H}} \in B$ [by ii)].

$$\Rightarrow \quad G \cap H \in B, \quad \text{for countable intersections.}$$

$$\bigcap_{n=1}^{\infty} G_n = \overline{\overline{\bigcap_{n=1}^{\infty} G_n}} = \overline{\bigcup_{n=1}^{\infty} \overline{G}_n} \in B$$

$$\text{that is, } \sum_{n=1}^{\infty} \left(-\infty, a+\frac{1}{n}\right) \in B$$

$$\Rightarrow \quad (-\infty, a] \in B$$

4. A class A of subsets of Ω is said to be monotone if the limit of any monotone increasing/decreasing sequence of sets in A is also in A. Show that a σ-field is monotone.
Short answer,
Let A be a monotone **class** of **subsets of** Ω.
i.e. for any **M**onotone **I**ncreasing **S**ubsets (MIS) of events,

$$B_n \uparrow B, \quad B \in A.$$

Similarly, for any **M**onotone **D**ecreasing **S**ubsets (MDS) of events,

$$D_n \downarrow D, \quad D \in A.$$

$\Longrightarrow$
Let $\mathcal{F}$ be any σ-field of **subsets of** Ω. Suppose $B_1 \subset B_2 \subset \cdots$ is an MIS sequence of events in $\mathcal{F}$. Becanse $\mathcal{F}$ is a σ-field, then

$$\bigcup_{n=1}^{\infty} B_n \in \mathcal{F} \quad \Rightarrow \quad \lim_{n\to\infty} B_n \in \mathcal{F}.$$

Similarly, for any MDS sequence of events in $\mathcal{F}$, $D_1 \supset D_2 \supset \dots$

$$\bigcap_{n=1}^{\infty} D_n \in \mathcal{F} \quad \Rightarrow \quad \lim_{n\to\infty} D_n \in \mathcal{F}.$$

Because $\mathcal{F}$ is a σ-field,

$$\Rightarrow \mathcal{F} \text{ is monotone.}$$

$\Longleftarrow$ conversely,
Suppose G is a **field** of **subsets of** Ω that is monotone, we can show that G is a σ-field.

i) Becanse G is a **field**,
$\Rightarrow \Omega \in G$.

ii) Becanse G is a **field**, **for any** $A \in G$, $\quad \overline{A} \in G$

iii) Now consider any arbitary sequence of events $A_1, A_2, A_3, \cdots$, We need to show that $\bigcup\limits_{n=1}^{\infty} A_n \in G$. Define a new sequence of events,

$$\begin{aligned}
B_1 &= A_1 && \in G \\
B_2 &= A_1 \cup A_2 && \in G \\
&\vdots \\
B_n &= \bigcup_{j=1}^{n} A_j && \in G
\end{aligned}$$

Any finite union is in the **field** G(field Definition). We know $B_1 \subset B_2 \subset B_3 \subset \cdots$, $\bigcup\limits_{j=1}^{n} B_j \in G$.

i.e. this is an **M**onotone **I**ncreasing **S**ubsets (MIS) of events.

Because G is monotone, by the definition, the limit of any monotone increasing/decreasing sequence of sets in G is also in G. That is,

$$\lim_{n\to\infty} \bigcup_{j=1}^{n} B_j \in G.$$

The monotone limits of sequence is still in G. Give that $\lim\limits_{n\to\infty} \bigcup\limits_{j=1}^{n} B_j = \bigcup\limits_{n=1}^{\infty} A_n$,

$$\Rightarrow \quad \bigcup_{n=1}^{\infty} A_n \in G.$$

Combine (i), (ii) and (iii), G is a σ-field.

5. Suppose $(\Omega, \mathcal{F}, P)$ is a probability space with $B \in \mathcal{F}$ such that $P(B) > 0$. Define a new set function $Q : \mathcal{F} \to [0,1]$ by $Q(A) = P(A \mid B)$ for any $A \in \mathcal{F}$. Show that

a) $(\Omega, \mathcal{F}, Q)$ is a probability space.

b) If $C \in \mathcal{F}$ such that $Q(C) > 0$, show that $Q(A \mid C) = P(A \mid B \cap C)$.

For any $A \in \mathcal{F}$,

$$Q(A) = P(A \mid B) = \frac{P(A \cap B)}{P(A)}$$

Short answer,

a)

To show $(\Omega, \mathcal{F}, Q)$ is probability space, we need to show,

1. $Q(A) \geqslant 0$.

   We know that $Q(A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{P(A \cap B) \geqslant 0}{P(A) > 0}$. Thus, $Q(A) \geqslant 0$

2. $Q(\Omega) = 1$.

$$Q(\Omega) = P(\Omega \mid B) = \frac{P(\Omega \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1.$$

3. For any disjoint sets $A_1 A_2, A_3 \cdots, Q\left(\bigcup_{n=1}^{\infty} A_n\right) \equiv \sum_{n=1}^{\infty} Q\left(A_n\right)$.

$$\begin{aligned} Q\left(\bigcup_{n=1}^{\infty} A_n\right) &= P\left(\bigcup_{n=1}^{\infty} A_n \mid B\right) \\ &= P\left(\left(\bigcup_{n=1}^{\infty} A_n\right) \cap B\right) / P(B) \quad (A_n \text{ are disjointed.}) \\ &= \sum_{n=1}^{\infty} \frac{P\left(A_n \cap B\right)}{P(B)} \\ &= \sum_{n=1}^{\infty} Q\left(A_n\right). \end{aligned}$$

Thus, $(\Omega, \mathcal{F}, Q)$ is a probability space.
b)
For any C, set $Q(C) > 0$.

$$Q(A \mid C) = \frac{Q(A \cap C)/P(B)}{Q(C)/P(B)} = \frac{P(A \cap C \mid B) P(B)}{P(C \mid B) P(B)} = \frac{P(A \cap C \cap B)}{P(C \cap B)} = P(A \mid B \cap C).$$

6. Prove the Continuity Theorem.
Short answer,

i) Check **M**onotone **i**ncreasing **s**equence of events.
If $A_1 \subset A_2 \subset A_3 \subset \cdots$ s.t. $A_n \uparrow A$, We need to show $P(A) = \lim_{n\to\infty} P\left(A_n\right)$. Because $A = \bigcup_{n=1}^{\infty} A_n$, We define $B_1 = A_1, \quad B_2 = A_2 \backslash A_1, \quad B_3 = A_3 \backslash A_2, \cdots,$

$$\begin{aligned} \Rightarrow A \quad &= \bigcup_{n=1}^{\infty} B_n, \\ \Rightarrow P(A) \quad &= P\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} P\left(B_n\right). \quad (B_n\text{'s are disjoint.}) \end{aligned}$$

Because $\quad B_n = A_n \backslash A_{n-1} \quad$ and $\quad A_{n-1} \backslash A_n$.

$$\begin{aligned} \Rightarrow P\left(B_n\right) \quad &= P\left(A_n\right) - P\left(A_{n-1}\right) \\ \Rightarrow \sum_{i=1}^{\infty} P\left(B_n\right) \quad &= \lim_{n\to\infty} P\left(A_n\right) - P\left(A_0\right) \end{aligned}$$

Assume $A_0 = \varnothing$,

$$P(A) = \lim_{n\to\infty} P\left(A_n\right).$$

ii) Check **M**onotone **d**ecreasing **s**equence of events.
Suppose $D_1 \supset D_2 \supset D_3 \cdots$, $D = \bigcap_{n=1}^{\infty} D_n, \quad D_n \downarrow D$.

$$\begin{aligned} \Rightarrow \quad & D_n^C && \uparrow \quad D^C, \\ \Rightarrow \quad & P\left(D^C\right) && = \lim_{n\to\infty} P\left(D_n^C\right), \\ \Rightarrow \quad & 1 - P(D) && = \lim_{n\to\infty} \left[1 - P\left(D_n\right)\right], \\ \Rightarrow \quad & P(D) && = \lim_{n\to\infty} P\left(D_n\right). \end{aligned}$$

[Ref]:

R. G. Bartle, D. R. Sherbert, “Sets and Functions” in “Introduction to Real Analysis”, 3rd Edition, John Wiley and Sons, Inc. 2000. ch 1, pp 3.

W. Rudin, “Basic Topology” in “Principles of Mathematical Analysis”, 3rd Edition, McGraw-Hill Inc. ch 2, pp 28.