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4 Level Sets

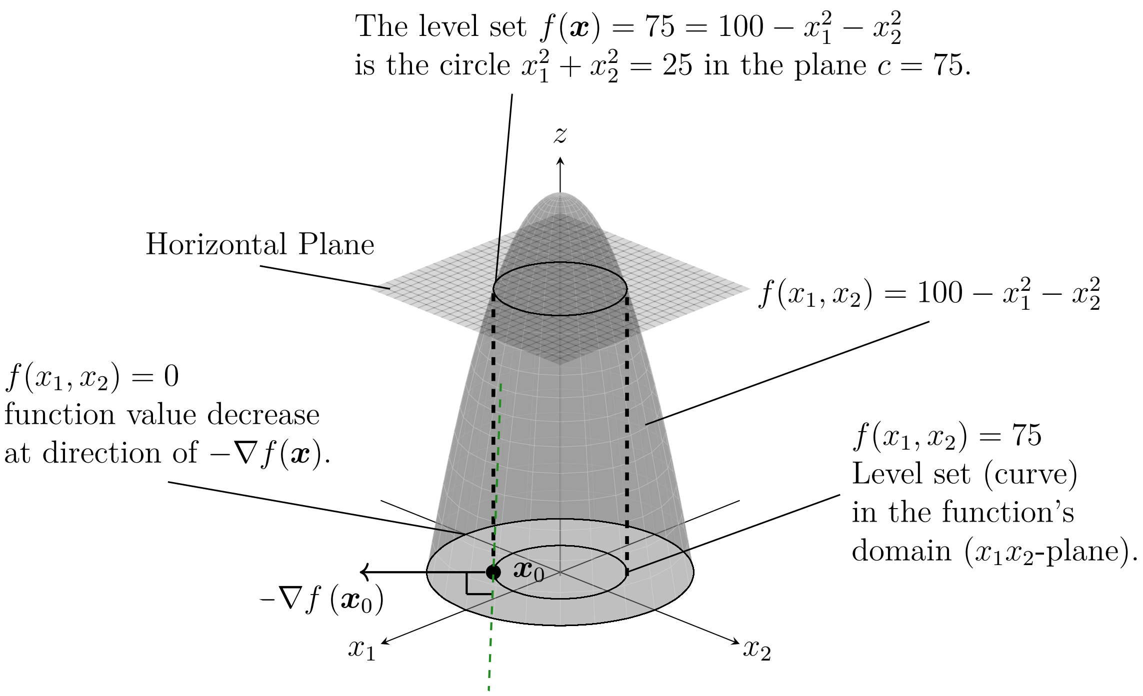
4.1 Level set

- The level set of a function $f : \mathbb{R}^n \to \mathbb{R}$ at level $c \in \mathbb{R}$ is the set of *points*

$$S_c = \{\boldsymbol{x} \in \mathbb{R}^n : f(\boldsymbol{x}) = c\}.$$

  If $n = 2$ then S_c is a curve [1]. If $n = 3$ then S_c is a surface.

- $\mathfrak{THEOREM}$ 5.7 For any $c, \nabla f(\boldsymbol{x})$ is *orthogonal* to the tangent of level set S_c at $\boldsymbol{x} \in S_c$.
- $-\nabla f(\boldsymbol{x})$ points in the direction of decreasing f.



- $\nabla f(\boldsymbol{x}_0)$ is the direction of maximum rate of increase of f at $\boldsymbol{x}_0$.
- $\nabla f(\boldsymbol{x}_0)$ is orthogonal to the level set through $\boldsymbol{x}_0$ determined by $f(\boldsymbol{x}) = f(\boldsymbol{x}_0)$,

$$\nabla f(\boldsymbol{x}_0)^\top (\boldsymbol{x} - \boldsymbol{x}_0) = 0 \quad \text{if} \quad \nabla f(\boldsymbol{x}_0) \neq 0.$$

  The direction of *maximum rate of increase* of a real-valued differentiable function at a point is orthogonal to the level set of the function through that point.

- If $\nabla f(\boldsymbol{x}) \neq 0$, $-\dfrac{\nabla f(\boldsymbol{x})}{\|\nabla f(\boldsymbol{x})\|}$ is the direction of fastest decrease (steepest descent direction) of f at $\boldsymbol{x}$.

[1] Ref: cis, "Draw a paraboloid and its contours in TikZ."

4.2 Neighborhood

- A neighborhood of a point $x \in \mathbb{R}^n$ is defined by

$$B_\varepsilon(\boldsymbol{x}) := \{\boldsymbol{y} \in \mathbb{R}^n : \|\boldsymbol{y} - \boldsymbol{x}\| < \varepsilon\}$$

  for some $\varepsilon > 0$. Note that $B_\varepsilon(\boldsymbol{x})$ is open.

- Let $S \subset \mathbb{R}^n$, then $\boldsymbol{x}$ is called an interior point of S if there exists $\varepsilon > 0$ such that $B_\varepsilon(\boldsymbol{x}) \subset S$. The set of interior points of S is called the interior of S, denoted by $\operatorname{int}(S)$.

- $\boldsymbol{x}$ is called a boundary point of S if any neighborhood of $\boldsymbol{x}$ contains a point in S and a point in S^c. A boundary point may or may not be in S. The set of boundary points of S is called the boundary of S.

- A set $S \subset \mathbb{R}^n$ is called open if all its point are an interior points. S is called closed if S^c is open.

- S is called bounded if $S \subset B_R(0)$ for some $R > 0.S$ is called compact if S is closed and bounded.

- Weierstrass theorem. Let $S \subset \mathbb{R}^n$ be compact and $f : S \to \mathbb{R}$ be continuous, then f attains maximum and minimum in S.

- The intersection of finitely many half-spaces is called a polytope. Note that a polytope is convex, since all half-spaces are convex.

- A nonempty bounded polytope is called a polyhedron.

4.3 Sequences and limits

- Let $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)}, \ldots$ be a sequence in $\mathbb{R}^n$, then we say $\boldsymbol{x}^{(k)}$ converges to $\boldsymbol{x}^*$ if for any $\varepsilon > 0$, there exists $K \in \mathbb{N}$ (depending on ε) such that

$$\|\boldsymbol{x}_k - \boldsymbol{x}^*\| < \varepsilon$$

  for all $k \geq K$. This is denoted by $\lim_{k\to\infty} \boldsymbol{x}^{(k)} = \boldsymbol{x}^*$ or $\boldsymbol{x}^{(k)} \to \boldsymbol{x}^*$. $\boldsymbol{x}^*$ is called the limit of the sequence $\left(\boldsymbol{x}^{(k)}\right)_{k=1}^\infty$. If a sequence is convergent, then the limit is unique. Note that $\boldsymbol{x}^{(k)} \to \boldsymbol{x}^*$ iff $x_i^{(k)} \to x_i^*$ for all $i = 1, \ldots, n$.

- Theorem. A convergent sequence is bounded. A bounded sequence has at least one convergent subsequence.

- Theorem. A sequence $\left(\boldsymbol{x}^{(k)}\right)_{k=1}^\infty$ converges to $\boldsymbol{x}^*$ iff every subsequence of $\left(\boldsymbol{x}^{(k)}\right)_{k=1}^\infty$ converges to $\boldsymbol{x}^*$.

- We say $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $\boldsymbol{x} \in \mathbb{R}^n$ if

$$f\left(\boldsymbol{x}^{(k)}\right) \to f(\boldsymbol{x})$$

  for any sequence $\boldsymbol{x}^{(k)} \to \boldsymbol{x}$.

- We say f is continuous on $S \subset \mathbb{R}^n$ if f is continuous at every point of S.

- Let $f : \mathbb{R}^n \to \mathbb{R}$ and $f \in \mathcal{C}^2$, and denote $\boldsymbol{h} = \boldsymbol{b} - \boldsymbol{a}$, then

$$f(\boldsymbol{b}) = f(\boldsymbol{a}) + Df(\boldsymbol{a})\boldsymbol{h} + \frac{1}{2}\boldsymbol{h}^\top D^2 f(\boldsymbol{a})\boldsymbol{h} + o\left(\|\boldsymbol{h}\|^2\right),$$

  where $\lim_{\|\boldsymbol{h}\|\to 0} o\left(\|\boldsymbol{h}\|^2\right) / \|\boldsymbol{h}\|^2 = 0$.

[Ref]: Edwin K.P. Chong, Stanislaw H. Żak, "PART I MATHEMATICAL REVIEW" in "An introduction to optimization", 4th Edition, John Wiley and Sons, Inc. 2013.