1 Lecture 1

1.1 Set theory

A simple random experiment - roll a fair die,

$$\Omega = \mathcal{S} = \{1, 2, 3, 4, 5, 6\}.$$

Also, $\Omega, \mathcal{S}$ is <u>sample space</u>.

- Each event of interest can be describe by a subset of $\Omega = \{1, 2, 3, 4, 5, 6\}$.
  For example,

$$\begin{aligned} A_1 &= \text{ the outcome is odd } = \{1, 3, 5\}, \\ A_2 &= \text{ the outcome is divisible by 3 } = \{3, 6\}, \\ A_3 &= \text{ the outcome is prime } = \{2, 3, 5\}. \end{aligned}$$

- There are $C_6^0 + C_6^1 + C_6^2 + C_6^3 + C_6^4 + C_6^5 + C_6^6 = 2^6 = 64$ distinct subsets of Ω.
- In order to fully characterize a random experiment, we must know the probability of each of these sets.

1.2 Events

- Events are subsets of Ω.
- The collection of all events is called the <u>event space</u>,

$$\mathcal{F}(\Omega) = \{A_1, A_2, \ldots, A_{64}\}.$$

- Our random experiment is completely characterized by

$$\{\Omega, \mathcal{F}(\Omega), P(\cdot)\},$$

  where

$$P(\cdot) : \mathcal{F}(\Omega) \to [0, 1],$$

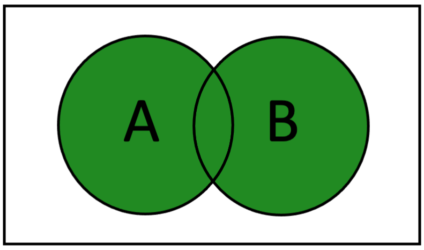
  and assigns probability to each of the events.

1.3 Basic Set Theory

A set is simply a collection of objects. In any given problem, the set containing all possible elements of interest is called the universe, universal set, or space. We typically denote the space by Ω or $\mathcal{S}$.

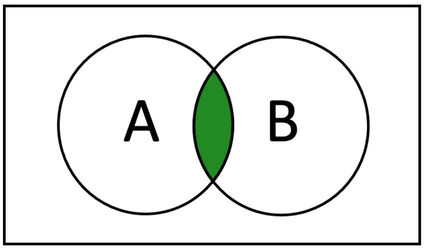
- The union of two sets A and B, denoted $A \cup B$, is defined as

$$A \cup B = \{\omega \in \Omega : \omega \in A \text{ or } \omega \in B\}.$$



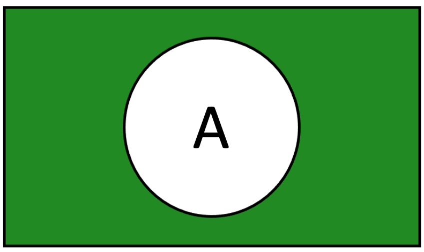
- The intersection of two sets A and B, denoted $A \cap B$, is defined as

$$A \cap B = \{\omega \in \Omega : \omega \in A \text{ and } \omega \in B\}.$$



- The complement of a set A (with respect to Ω), denoted $\overline{A}$, A'', A^c, is defined as

$$\overline{A} = \{\omega \in \Omega : \omega \notin A\}.$$

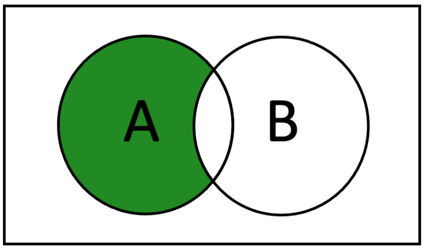


- The set containing no elements is called the empty set or null set, and is denoted by $\varnothing$ or $\{\}$. (n.b. $\{\varnothing\}$ is not correct notation.)

- If two sets A and B have no elements in common, then $A \cap B = \varnothing$, and A and B are said to be disjoint.

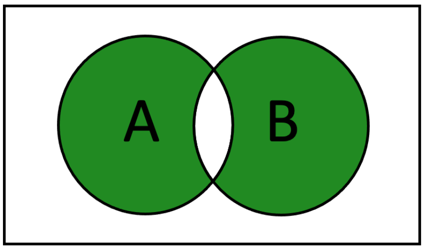
- The set difference of two sets A and B is defined as

$$\begin{aligned} A - B &= \{\omega \in \Omega : \omega \in A \text{ and } \omega \notin B\} \\ &= A \cap \overline{B}. \end{aligned}$$



- The symmetric difference of two sets A and B, is defined as

$$\begin{aligned} A\Delta B &= \{\omega \in \Omega : \omega \in A \text{ or } \omega \in B, \text{ but not both. }\} \\ &= (A \cup B) - (A \cap B) \\ &= \left(A \cap \overline{B}\right) \cup \left(\overline{A} \cap B\right). \end{aligned}$$



- Two sets A and B are equal if and only if they contain exactly the same elements.
  $\Longleftrightarrow A = B \Leftrightarrow A \subset B$ and $B \subset A$.
  Proof,
  $\Longleftarrow$
  If $B \subset A$, then $\omega \in B \Rightarrow \omega \in A$, and if $A \subset B$, then $\omega \in A \Rightarrow \omega \in B$.
  If both conditions hold, that means every element of B is an element of A, and every element of A is an element of B. (or say, A has no element that are not in B and B has no elements that are not in A.)
  A and B have exactly the same elements, $A = B$.
  $\Longrightarrow$
  From the definition of the subset, $A = \{\omega \in \Omega : \omega \in A\}$, $B = \{\omega \in \Omega : \omega \in B\}$.
  If $A = B$, every element of A is an element of B, $A \subset B$. Similarly, every element of B is an element of A, $B \subset A$.

1.4 Index sets $\mathcal{I}$

- Indexed collections of sets $\{A_i; i \in \mathcal{I}\}$, where $\mathcal{I}$ is an index set.
- So $\{A_i; i \in \mathcal{I}\}$ is a "set of sets", or a family of sets, or a collection of sets.
- There is one and only one set A_i, for each $i \in \mathcal{I}$.

  For example,

$$\begin{aligned} &\mathcal{I} = \{1, 2, 3\}, \\ &A_1 = [0, 1] = \{x \in \mathbb{R}; 0 \leq x \leq 1\}, \\ &A_2 = [1, 2], \\ &A_3 = [2, 3], \end{aligned}$$

$$\{A_i; i \in \mathcal{I}\} = \{A_1, A_2, A_3\} = \{[0,1],[1,2],[2,3]\},$$
$$\bigcup_{i\in\mathcal{I}} A_i = [0,1] \cup [1,2] \cup [2,3].$$

Typical Index sets:

$$\begin{aligned}
&\mathbb{N} = \{1,2,3,\ldots\} = \text{ natural numbers },\\
&\mathbb{Z} = \{\ldots -1,0,1,2,\ldots\} = \text{ integers },\\
&\mathbb{Z}_+ = \{0,1,2,\ldots\} = \text{ non-negative integers },\\
&\mathbb{R} = (-\infty,\infty),\\
&\mathbb{I}_n = \{0,1,2,\ldots,n-1\}.
\end{aligned}$$

For another example,
$$\mathcal{S} = \{1,2,3,4,5,6\}.$$

There are 64 subsets, $A_1, A_2, \cdots, A_{64}$,

$$\{A_1, A_2, \cdots, A_{64}\} \text{ is an indexed collection of sets.}$$

- An index set $\mathcal{I}$ is countable if it has an infinite number of elements and they can be put in one-to-one correspondance with the natural numbers $\mathbb{I}$.
- Given an indexed collection of sets $\{A_i; i \in \mathcal{I}\}$, the union of the family is

$$\bigcup_{i\in I} A_i \triangleq \{\omega \in \Omega : \omega \in A_i \text{ for at least one } i \in \mathcal{I}\}.$$

the intersection of the family is

$$\bigcap_{i\in I} A_i \triangleq \{\omega \in \Omega : \omega \in A_i \text{ for all } i \in \mathcal{I}\}.$$

For example,
Let $F_r = [0,1/r), r \in (0,1]$. Find

$$\bigcup_{r\in(0,1]} F_r \quad \text{and} \quad \bigcap_{r\in(0,1]} F_r.$$

Short answer,

$$\bigcup_{r\in(0,1]} F_r = \bigcup_{r\in(0,1]} [0,1/r) = \Big\{\omega : \omega \in [0,1/r) \text{ for at least one } r \in (0,1]\Big\} = [0,\infty).$$

$$\bigcap_{r\in(0,1]} F_r = \bigcap_{r\in(0,1]} [0,1/r) = \Big\{\omega : \omega \in [0,1/r) \text{ for all } r \in (0,1]\Big\} = [0,1).$$

1.5 Algebra of Set Theory

There are 16 rules,

$$\left.\begin{array}{l} 1.\ A \cup B = B \cup A\ .\\ 2.\ A \cap B = B \cap A\ . \end{array}\right\} \text{Commutative laws}$$
$$\left.\begin{array}{l} 3.\ A \cup (B \cup C) = (A \cup B) \cup C.\\ 4.\ A \cap (B \cap C) = (A \cap B) \cap C. \end{array}\right\} \text{Associative laws}$$

5. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
6. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

$\left.\begin{matrix} \\ \end{matrix}\right\}$ Distributive laws

7. $\overline{\overline{A}} = A$.
8. $\overline{A \cap B} = \overline{A} \cup \overline{B}$.
9. $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

$\left.\begin{matrix} \\ \end{matrix}\right\}$ De Morgan's Laws

10. $\overline{\mathcal{S}} = \varnothing$.
11. $A \cap \mathcal{S} = A$.
12. $A \cap \varnothing = \varnothing$.
13. $A \cup \mathcal{S} = \mathcal{S}$.
14. $A \cup \varnothing = A$.
15. $A \cup \bar{A} = \mathcal{S}$.
16. $A \cap \bar{A} = \varnothing$.

Proof:

1) Union is commutative.

$$\begin{aligned} A \cup B &\triangleq \{x \in \Omega : \quad x \in A \text{ or } x \in B\} \\ &= \{x \in \Omega : \quad x \in B \text{ or } x \in A\} \\ &= B \cup A. \end{aligned}$$

2) Intersection is commutative.

$$\begin{aligned} A \cap B &\triangleq \{x \in \Omega : \quad x \in A \text{ and } x \in B\} \\ &= \{x \in \Omega : \quad x \in B \text{ and } x \in A\} \\ &= B \cap A. \end{aligned}$$

3) Union is associative.

$$\begin{aligned} A \cup (B \cup C) &\triangleq \{x \in \Omega : \qquad x \in A \text{ or } x \in (B \cup C)\} \\ &= \{x \in \Omega : \qquad x \in A \text{ or } x \in B \text{ or } x \in C\} \\ &= \{x \in \Omega : \ x \in (A \cup B) \text{ or } x \in C)\} \\ &= (A \cup B) \cup C. \end{aligned}$$

4) Intersection is associative.

$$\begin{aligned} A \cap (B \cap C) &\triangleq \{x \in \Omega : \qquad x \in A \text{ and } x \in (B \cap C)\} \\ &= \{x \in \Omega : \qquad x \in A \text{ and } x \in B \text{ and } x \in C\} \\ &= \{x \in \Omega : \ x \in (A \cap B) \text{ and } x \in C)\} \\ &= (A \cap B) \cap C. \end{aligned}$$

5) Intersection is distributive over union.
Let $x \in A \cap (B \cup C)$. Then, $x \in A$ and $x \in (B \cup C)$,

$$\Rightarrow x \in A \text{ and at the same time, } x \in B \text{ or } x \in C, \text{ possibly both,}$$

$$\Rightarrow \text{ either } x \in A \text{ and } x \in B, \text{ or } x \in A \text{ and } x \in C \text{ (possibly both).}$$

Hence,

$$x \in (A \cap B) \text{ or } x \in (A \cap C),$$

i.e. $x \in (A \cap B) \cup (A \cap C)$. So we have that

$$\begin{aligned} x \in A \cap (B \cup C) &\Rightarrow x \in (A \cap B) \cup (A \cap C), \\ &\Rightarrow A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C). \end{aligned}$$

Next we assume that $x \in (A \cap B) \cup (A \cap C)$. Then, $x \in (A \cap B)$ or $x \in (A \cap C)$,

$$\begin{aligned} &\Rightarrow x \in A \text{ in addition to being in } B \text{ or } C\text{, or both,} \\ &\Rightarrow x \in A \cap (B \cup C). \end{aligned}$$

This gives us that

$$\begin{aligned} x \in (A \cap B) \cup (A \cap C) &\Rightarrow x \in A \cap (B \cup C), \\ &\Rightarrow (A \cap B) \cup (A \cap C) \subset A \cap (B \cup C). \end{aligned}$$

Combining the two results we have

$$A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C) \text{ and } (A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$$

$$\Rightarrow A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

6) Union is distributive over intersection.
Let $x \in A \cup (B \cap C)$. Then, $x \in A$ or $x \in (B \cap C)$,

$$\Rightarrow \text{ either } x \in A\text{, or } x \in B \text{ and } x \in C\text{, possibly both.}$$

If $x \in A$, we have $x \in A \cup B$ and $x \in A \cup C$ both satisfied. If $x \in B \cap C$, we have $x \in B$ and $x \in C$. Then $x \in A \cup B$ and $x \in A \cup C$ both hold,

$$\Rightarrow \text{ either } x \in A \text{ or } x \in B \text{ and at the same time, either } x \in A \text{ or } x \in C,$$

$$\Rightarrow x \in (A \cup B)\text{, and } x \in (A \cup C),$$

i.e. $x \in (A \cup B) \cap (A \cup C)$. So we have that

$$\begin{aligned} x \in A \cup (B \cap C) &\Rightarrow x \in (A \cup B) \cap (A \cup C), \\ &\Rightarrow A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C). \end{aligned}$$

Next we assume that $x \in (A \cup B) \cap (A \cup C)$. Then, $x \in (A \cup B)$ and $x \in (A \cup C)$. If $x \in A$, $x \in A \cup (B \cap C)$ holds. If $x \notin A$, since $x \in (A \cup B)$ and $x \in (A \cup C)$, we have $x \in B$ and $x \in C$. It gives $x \in (B \cap C)$, related on $x \in A \cup (B \cap C)$.

$$\begin{aligned} x \in (A \cup B) \cap (A \cup C) &\Rightarrow x \in A \cup (B \cap C), \\ &\Rightarrow (A \cup B) \cap (A \cup C) \subset A \cup (B \cap C). \end{aligned}$$

Combining the two results we have

$$A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C) \text{ and } (A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$$

$$\Rightarrow A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

7)

$$\begin{aligned} \overline{\overline{A}} &= \{x \in \Omega : \omega \notin \overline{A}\} \\ &= \{x \in \Omega : \omega \in A\} \\ &= A \end{aligned}$$

8) De Morgan's Laws

$$
\begin{aligned}
x \in \overline{A \cap B} &\Leftrightarrow x \notin A \cap B, && \text{By definition of complement.}\\
&\Leftrightarrow x \in \overline{A} \text{ or } x \in \overline{B}, && \text{By definition of intersection and complement.}\\
&\Leftrightarrow x \in \overline{A} \cup \overline{B}, && \text{By definition of union.}\\
\overline{A \cap B} &= \overline{A} \cup \overline{B}.
\end{aligned}
$$

9) De Morgan's Laws

$$
\begin{aligned}
x \in \overline{A \cup B} &\Leftrightarrow x \notin A \text{ and } x \notin B, && \text{By definition of union and complement.}\\
&\Leftrightarrow x \in \overline{A} \text{ and } x \in \overline{B}, && \text{By definition of complement.}\\
&\Leftrightarrow x \in \overline{A} \cap \overline{B}, && \text{By definition of intersection.}\\
\overline{A \cup B} &= \overline{A} \cap \overline{B}.
\end{aligned}
$$

10)Here $\mathcal{S}$ is the universal set. Based on the definition,

$$\overline{\mathcal{S}} = \{\omega \in \Omega(\Omega = \mathcal{S}) : \omega \notin \mathcal{S}\}.$$

The complement of the set $\mathcal{S}$ is the difference between the universal set $\mathcal{S}$ and set $\mathcal{S}$ itself. Obviously, no element satisfies this statement. The set containing no elements is empty set. Hence, $\overline{\mathcal{S}} = \varnothing$.

11) Here $\mathcal{S}$ is the universal set. First we show that $A \cap \mathcal{S} \subset A$. In general, the set resulting from the intersection of two sets is a subset of both of the sets. Let $x \in A \cap B$, x is an element of $A \cap B$.

$$
\begin{aligned}
x \in A \cap B &\Rightarrow x \in A \text{ and } x \in B, && \text{By definition of intersection.}\\
&\Rightarrow x \in A,\\
&\Rightarrow A \cap B \subset A.
\end{aligned}
$$

Next, we want to show that $A \subset A \cap \mathcal{S}$. Let $x \in A$, then $x \in \mathcal{S}$ such that $x \in A$. Therefore, $x \in A \Rightarrow x \in (A \cap \mathcal{S}) \Rightarrow A \subset A \cap \mathcal{S}$. Since $A \cap \mathcal{S} \subset$ A and A $\subset$ A $\cap \mathcal{S}$, we have that A $\cap \mathcal{S} =$ A.

12) First we show that $A \cap \varnothing \subset \varnothing$. We have shown in 11) that the set resulting from the intersection of two sets is a subset of both of the sets. Hence, $A \cap \varnothing \subset \varnothing$ holds.

Next, we want to show that $\varnothing \subset A \cap \varnothing$. Since every set contains the set itself and the empty set, the $\varnothing$ is subset of a set resulting from $A \cap \varnothing$, i.e., $\varnothing \subset A \cap \varnothing$.

Since $A \cap \varnothing \subset \varnothing$ and $\varnothing \subset A \cap \varnothing$, we have $A \cap \varnothing = \varnothing$.

13) Here $\mathcal{S}$ is the universal set. First we show that $\mathcal{S} \subset A \cup \mathcal{S}$. In general, both of the sets are subsets of the set resulting from the union of two sets. Let $x \in B$, x is an element of B.

$$
\begin{aligned}
x \in B &\Rightarrow x \in A \text{ or } x \in B,\\
&\Rightarrow x \in A \cup B, && \text{By definition of union.}\\
&\Rightarrow B \subset A \cup B.
\end{aligned}
$$

Next, we want to show that $A \cup \mathcal{S} \subset \mathcal{S}$. Let $x \in A \cup \mathcal{S}$, then $x \in \mathcal{S}$ such that $x \in A$ or $x \in \mathcal{S}$.

$$
\begin{aligned}
x \in A \cup \mathcal{S} &\Rightarrow x \in A \text{ or } x \in \mathcal{S}, && \text{By definition of union.}\\
&\Rightarrow x \in \mathcal{S}, && \text{By the case that } A \text{ is subset of } \mathcal{S}.\\
&\Rightarrow A \cup \mathcal{S} \subset \mathcal{S}.
\end{aligned}
$$

Since $\mathcal{S} \subset$ A $\cup\, \mathcal{S}$ and $A \cup \mathcal{S} \subset \mathcal{S}$, we have that A $\cup\, \mathcal{S} = \mathcal{S}$.

14) First we show that $A \subset A \cup \varnothing$. We have shown in 13) that both of the sets are subsets of the set resulting from the union of two sets. Hence, $A \subset A \cup \varnothing$ holds.

Next, we want to show that $A \cup \varnothing \subset A$. Let $x \in A \cup \varnothing$, we have $x \in A$ or $x \in \varnothing$. Since no element exists in empty set, $x \in \varnothing$ is logically false. Hence, only $x \in A$ holds. It results in $x \in A \cup \varnothing \Rightarrow x \in A$.

Since $A \subset A \cup \varnothing$ and $A \cup \varnothing \subset A$, we have $A \cup \varnothing = A$.

15) First we show that $A \cup \overline{A} \subset \mathcal{S}$. We have shown in 13) that both of the sets are subsets of the set resulting from the union of two sets. Hence, $A \subset A \cup \overline{A} \Rightarrow A \subset \mathcal{S}$, and $\overline{A} \subset A \cup \overline{A} \Rightarrow \overline{A} \subset \mathcal{S}$. Let $x \in A \cup \overline{A}$, $x \in A$ or $x \in \overline{A}$, we all have $x \in \mathcal{S}$. $A \cup \overline{A} \subset \mathcal{S}$ holds.

Next, let $x \in \mathcal{S}$, for the statements x is an element of A, and x is not an element of A, only one statement is true. Hence, either $x \in A$ is true or $x \notin A$ is true, i.e., $x \in \mathcal{S} \Rightarrow x \in A$ or $x \notin A$. Hence, $\mathcal{S} \subset A \cup \overline{A}$.

Since $A \cup \overline{A} \subset \mathcal{S}$ and $\mathcal{S} \subset A \cup \overline{A}$, we have $A \cup \overline{A} = \mathcal{S}$.

16) First we show that $A \cap \overline{A} \subset \varnothing$. Let $x \in A \cap \overline{A}$,

$$\begin{aligned} x \in A \cap \overline{A} &\Rightarrow x \in A \text{ and } x \in \overline{A} \\ &\Rightarrow x \in A \text{ and } x \notin A \\ &\Rightarrow \text{no element satisfies both statement.} \end{aligned}$$

Since no element exists in empty set, such statement result in a set contain no element, i.e., $A \cap \overline{A} \subset \varnothing$ holds.

Next, we want to show that $\varnothing \subset A \cap \overline{A}$. Since every set contains the set itself and the empty set, the $\varnothing$ is subset of a set resulting from $A \cap \overline{A}$, i.e., $\varnothing \subset A \cap \overline{A}$.

Since $A \cap \overline{A} \subset \varnothing$ and $\varnothing \subset A \cap \overline{A}$, we have $A \cap \overline{A} = \varnothing$.

[Ref]:

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