3.4 多维 r.v. 函数的分布

回忆一维 r.v. 函数的分布

多维例子：已知平面上点 (X,Y) 分布，求其到原点距离 $\sqrt{X^2+Y^2}$ 的分布

3.4.1 多维离散情形

二项分布可加性

<u>例</u>. 若 X 与 Y 独立，$X\sim B(m,p)$，$Y\sim B(n,p)$，证 $Z=X+Y\sim B(m+n,p)$.

直观解释

<u>证</u>. $\forall\ 0\le k\le m+n$,

$$
\begin{aligned}
P(Z=k) &= \sum_{l=0}^{k} P(X=l, Y=k-l) \\
&\overset{\text{独立}}{=} \sum_{l=0}^{k} P(X=l)P(Y=k-l) \\
&= \sum_{l=0}^{k} C_m^l p^l (1-p)^{m-l}\, C_n^{k-l} p^{k-l} (1-p)^{n-(k-l)} \\
&= \sum_{l=0}^{k} C_m^l C_n^{k-l} p^k (1-p)^{n+m-k} \\
&= C_{m+n}^k p^k (1-p)^{n+m-k}
\end{aligned}
$$

证毕 ▥

<u>例</u>. X,Y 独立，$X\sim P(\lambda_1)$，$Y\sim P(\lambda_2)$ 证 $X+Y\sim P(\lambda_1+\lambda_2)$.

泊松分布可加性

<u>证</u>：$\forall k\ge 0$,

$$
\begin{aligned}
P(X+Y=k) &= \sum_{l=0}^{k} P(X=l, Y=k-l) \\
&= \sum_{l=0}^{k} P(X=l)P(Y=k-l) \\
&= \sum_{l=0}^{k} \frac{\lambda_1^l}{l!} e^{-\lambda_1} \frac{\lambda_2^{k-l}}{(k-l)!} e^{-\lambda_2} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{k!} \sum_{l=0}^{k} C_k^l \lambda_1^l \lambda_2^{k-l} \\
&= \frac{(\lambda_1+\lambda_2)^k}{k!} e^{-(\lambda_1+\lambda_2)}
\end{aligned}
$$

证毕 ▥

例. 设 X_1, X_2, X_3, X_4 独立同分布于 $B(1,p)$. 求

$$X = \begin{vmatrix} X_1 & X_2 \\ X_3 & X_4 \end{vmatrix}$$

的概率分布

解: $X = X_1X_4 - X_2X_3 \in \{-1,0,1\}$

$$P(X=1) = P(X_1=1, X_4=1, X_2=0) + P(X_1=1, X_4=1, X_2=1, X_3=0)$$

$$= p^2(1-p) + p^3(1-p) = p^2(1-p^2)$$

同理, $P(X=-1) = p^2(1-p^2)$.

$\therefore \quad P(X=0) = 1-2p^2(1-p^2)$

X 分布列为

X	-1	0	1
P	$p^2(1-p^2)$	$1-2p^2(1-p^2)$	$p^2(1-p^2)$

$\square$

例. 设 X 与 Y 独立同分布, 且服从几何分布, 即

$$P(X=k) = p(1-p)^{k-1}, \quad k=1,2,\cdots$$

证明: 1) $Z=\min\{X,Y\}$ 与 $W=X-Y$ 独立.

2) Z 与 $U=\max\{X,Y\}-Z$ 独立.

证: 1) $\forall k \geq 0, \ \forall l,$

$$P(Z=l, W=k) \overset{(X\geq Y)}{=} P(Y=l, X=k+l)$$

$$\overset{独立}{=} P(X=k+l)\,P(Y=l) = p(1-p)^{k+l-1}\,p(1-p)^{l-1}$$

$$= p^2(1-p)^k(1-p)^{2(l-1)}$$

同理, $\forall k \geq 0, \ \forall l$

$$P(Z=l, W=-k) = P(X=l, Y=l+k)$$

$$= p^2(1-p)^{2(l-1)}(1-p)^k$$

又 $\because$ $P(Z > l) = P(X > l, Y > l) = (1-p)^{2l}$, $\forall\ l \geq 0$

$\therefore$ $P(Z = l) = P(Z > l-1) - P(Z > l) = [1-(1-p)^2](1-p)^{2(l-1)}$

$\forall\ k \geq 0$, 由对称性,

$$P(W=k) = P(W=-k) = \sum_{l=1}^{\infty} P(Y=l, X=k+l)$$

$$= \sum_{l=1}^{\infty} p^2 (1-p)^{l-1+k+l-1} = \frac{p^2(1-p)^k}{1-(1-p)^2}$$

$\therefore$ $P(Z=l, W=k) = P(Z=l)P(W=k)$, $l \geq 1$, $k \in \mathbb{Z}$. 证毕.

2) $U = \max\{X, Y\} - \min\{X, Y\} = |X-Y| = |W|$.

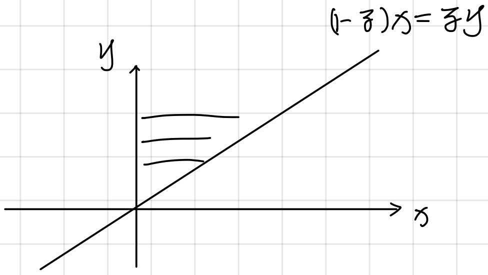
$\because$ Z 与 W 独立

$\therefore$ Z 与 U 独立. $\square$

3.4.2. 多维连续情形

例. 若 X, Y 独立同分布于 $E(\lambda)$, $Z = \frac{X}{X+Y}$, 求 Z 的分布.

解. 显然, $0 \leq Z \leq 1$. $\forall\ 0 < z < 1$,



$$P(Z \leq z) = P\left(\frac{X}{X+Y} \leq z\right)$$

确定积分区域

$$= \iint_{\substack{\frac{x}{x+y} \leq z, \\ x>0,\ y>0}} \lambda^2 e^{-\lambda x} e^{-\lambda y}\, dx\, dy$$

$$= \int_0^{+\infty} dy \int_0^{\frac{z}{1-z}y} dx\ \lambda^2 e^{-\lambda x} e^{-\lambda y}$$

$$= \int_0^{+\infty} dy\ \lambda e^{-\lambda y}\left(1 - e^{-\lambda \frac{z}{1-z} y}\right)$$

$$= \int_0^{+\infty} dy\ \lambda e^{-\lambda y} - \lambda e^{-\lambda \frac{1}{1-z} y}$$

$$= 1 - (1-z) = z$$

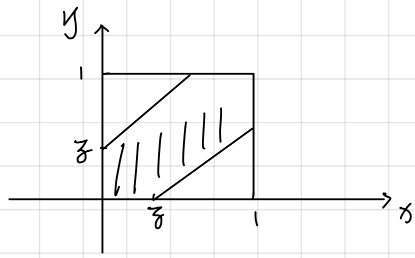
$\therefore$ $Z \sim U(0,1)$. $\square$

例. 在 $(0,1)$ 内任取两点，求两点间距离的分布函数与概率密度.

解. $X, Y \sim U(0,1)$ 且独立. 令 $Z = |X-Y|$. 显然，$0 \le Z \le 1$.

$\forall z \in (0,1)$,

$$P(Z \le z) = \iint_{\substack{|x-y| \le z \\ 0<x<1 \\ 0<y<1}} dx\,dy$$



$$= 1 - (1-z)^2 = 2z - z^2$$

$$\therefore \quad F_Z(z) = \begin{cases} 0, & z<0 \\ 2z - z^2, & 0 \le z < 1 \\ 1, & z \ge 1 \end{cases}$$

$$f_Z(z) = \begin{cases} 2-2z, & 0 \le z < 1 \\ 0, & \text{其它.} \end{cases}$$

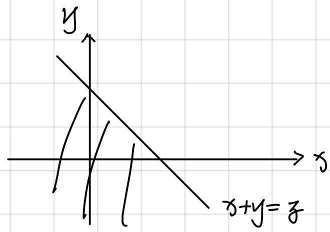
$\boxed{/\!/}$

和的分布

设 (X,Y) 概率密度为 $f(x,y)$. $Z = X+Y$, 则

$$P(Z \le z) = \iint_{x+y \le z} f(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{+\infty} dx \int_{-\infty}^{z-x} f(x,y)\,dy$$



$$\therefore \quad f_Z(z) = \int_{-\infty}^{+\infty} f(x, z-x)\,dx = \int_{-\infty}^{+\infty} f(z-y, y)\,dy.$$

若 X 与 Y 独立，则

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(x) f_Y(z-x)\,dx =: \underbrace{f_X * f_Y(z)}_{\text{卷积}}$$

例. $X, Y \sim N(0,1)$ 且独立，证 $X+Y \sim N(0,2)$.

证.
$$f_{X+Y}(z) = \int_{-\infty}^{+\infty} f_X(x) f_Y(z-x)\,dx$$
$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\frac{x^2+(z-x)^2}{2}}\,dx$$
$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-(x-\frac{z}{2})^2} e^{-\frac{z^2}{4}}\,dx$$
$$= \frac{1}{2\sqrt{\pi}} e^{-\frac{z^2}{4}}$$

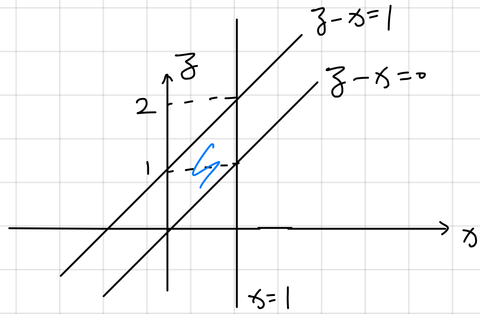
证毕. □

一般地，若 $X_i \sim N(\mu_i, \sigma_i^2)$, $1 \le i \le n$ 且独立，则
$$\sum_{i=1}^{n} a_i X_i \sim N\left(\sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} a_i^2 \sigma_i^2\right).$$

例. X, Y 独立，$X \sim U(0,1)$，$f_Y(y) = \begin{cases} 2y, & 0 \le y \le 1 \\ 0, & \text{其它} \end{cases}$

求 $Z = X+Y$ 的概率密度.

解.
$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(x) f_Y(z-x)\,dx$$
$$= \int_0^1 f_Y(z-x)\,dx$$



$0<z<1$，$f_Z(z) = \int_0^z 2y\,dy = z^2$

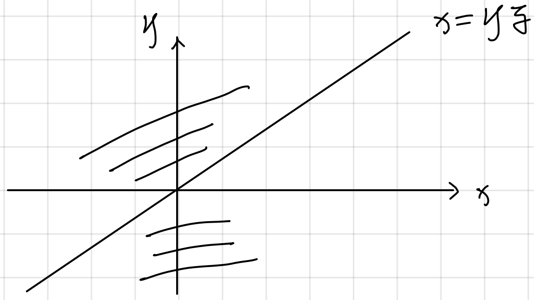
$1<z<2$，$f_Z(z) = \int_{z-1}^{1} 2y\,dy = 1-(z-1)^2 = 2z - z^2$

$f_Z(z) = 0$ 其它.

□

商的分布

设 (X,Y) 联合密度为 $f(x,y)$, $Z=\frac{X}{Y}$, 则



$$F_Z(z) = P\left(\frac{X}{Y} \le z\right)$$

$$= \iint_{\frac{x}{y}\le z} f(x,y)\,dx\,dy$$

$$= \iint_{y>0,\ x\le yz} f(x,y)\,dx\,dy + \iint_{y<0,\ x>yz} f(x,y)\,dx\,dy$$

$$= \int_0^{+\infty} dy \int_{-\infty}^{yz} f(x,y)\,dx + \int_{-\infty}^{0} dy \int_{yz}^{+\infty} f(x,y)\,dx$$

$$\Rightarrow\ f_Z(z) = F_Z'(z) = \int_0^{+\infty} y\, f(yz,y)\,dy + \int_{-\infty}^{0} (-y)\, f(yz,y)\,dy$$

$$= \int_{-\infty}^{+\infty} |y|\, f(yz,y)\,dy.$$

例 设 X,Y 独立同分布于 $E(1)$. 求 $Z=\frac{X}{Y}$ 概率密度.

解: $f(x,y) = e^{-(x+y)}$, $x,y>0$

则 $z>0$ 时,

$$f_Z(z) = \int_{-\infty}^{+\infty} f(yz,y)\,|y|\,dy$$

$$= \int_0^{+\infty} e^{-(yz+y)}\, y\,dy = (1+z)^{-2}.$$

$f_Z(z)=0$, 其它. □

最大最小值分布

设 $X_1, \cdots, X_n$ i.i.d.（独立同分布），概率密度为 $f(x)$，分布函数为 $F(x)$. $Y = \max\limits_{1\le i\le n} X_i$ ， $Z = \min\limits_{1\le i\le n} X_i$.

则 $F_Y(y) = P(Y \le y) = P(X_i \le y, \forall\ 1\le i\le n)$

$$= P(X_i \le y)^n = F_X(y)^n$$

$\therefore f_Y(y) = F_Y'(y) = n F_X(y)^{n-1} f_X(y)$.

$F_Z(z) = P(Z \le z) = 1 - P(\min\limits_{1\le i\le n} X_i > z)$

$$= 1 - P(X_i > z)^n = 1 - (1 - F_X(z))^n$$

$\therefore f_Z(z) = n(1 - F_X(z))^{n-1} f_X(z)$.

例. n 个独立工作的电子元件寿命 $\sim E(\lambda)$. 就下列工作方式求系统寿命：1) 串联　　2) 并联

解：1) $X_1, \cdots, X_n$ i.i.d. $\sim E(\lambda)$.

$Y = \min\limits_{1\le i\le n} X_i$

$P(Y > y) = P(X_1 > y)^n = e^{-n\lambda y}$, $y > 0$

即 $Y \sim E(n\lambda)$.

2) $Z = \max\limits_{1\le i\le n} X_i$

$\forall z > 0$, $P(Z \le z) = P(X_1 \le z)^n = (1 - e^{-\lambda z})^n$

$\therefore f_Z(z) = n\lambda(1 - e^{-\lambda z})^{n-1} e^{-\lambda z}$, $z > 0$.

一般情形

例. 若 $X \sim N(\mu, \sigma^2)$，Y 分布列为

Y	-1	1
P	$\frac{1}{3}$	$\frac{2}{3}$

且 X, Y 独立，求 $Z = XY$ 的分布

解: $\forall z \in \mathbb{R}$,

$$P(Z \le z) = P(Y=1, X \le z) + P(Y=-1, -X \le z)$$
$$= \frac{2}{3} P(X \le z) + \frac{1}{3} P(X \ge -z)$$
$$= \frac{2}{3} \Phi\left(\frac{z-\mu}{\sigma}\right) + \frac{1}{3}\left(1 - \Phi\left(\frac{-z-\mu}{\sigma}\right)\right)$$
$$= \frac{2}{3} \Phi\left(\frac{z-\mu}{\sigma}\right) + \frac{1}{3} \Phi\left(\frac{z+\mu}{\sigma}\right)$$

$$\therefore f_Z(z) = \frac{2}{3} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}} + \frac{1}{3} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z+\mu)^2}{2\sigma^2}}$$ ▨

例. 设 $X_1, X_2, \cdots, X_n$ i.i.d., $F(x)$, $f(x)$.

$Y = \max\limits_{1 \le i \le n} X_i$, $Z = \min\limits_{1 \le i \le n} X_i$.

求 (Y, Z) 联合密度.

解. $\forall y > z$,

$$F(y,z) = P(Y \le y, Z \le z) = P(Y \le y) - P(Y \le y, Z > z)$$
$$= F(y)^n - P(z < X_i \le y, \forall 1 \le i \le n)$$
$$= F(y)^n - [F(y) - F(z)]^n$$
$$\therefore f(y,z) = \frac{\partial^2 F}{\partial y \partial z}(y,z)$$
$$= n(n-1)[F(y) - F(z)]^{n-2} f(y) f(z).$$ ▨

例. 设 X 与 Y 独立, $X \sim U(0,1)$, Y 分布列为

Y	-1	0	1
P	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

求 $Z = X+Y$ 的分布

解. $-1 \le Z \le 2$. $\forall -1 < z < 2$,

$$F_Z(z) = P(X+Y \le z)$$

$$= \frac{1}{2}P(X \le z+1) + \frac{1}{3}P(X \le z) + \frac{1}{6}P(X \le z-1).$$

$-1 < z < 0$ 时,

$$F_Z(z) = \frac{1}{2}(z+1)$$

$0 < z < 1$ 时,

$$F_Z(z) = \frac{1}{2} + \frac{1}{3}z$$

$1 < z < 2$ 时,

$$F_Z(z) = \frac{5}{6} + \frac{1}{6}(z-1) = \frac{2}{3} + \frac{z}{6}$$

$$\therefore F_Z(z) = \begin{cases} 0, & z < -1 \\ \frac{z+1}{2}, & -1 \le z < 0 \\ \frac{1}{2} + \frac{z}{3}, & 0 \le z < 1 \\ \frac{2}{3} + \frac{z}{6}, & 1 \le z < 2 \\ 1, & z \ge 2. \end{cases}$$

四

总结: 1) 联合分布函数、分布列、概率密度
边缘分布函数、分布列、概率密度

2) 独立性

3) 求多维 r.v. 的函数的分布