Zero Range Process with Particle Destruction at the Origin

Linjie Zhao
joint work with **Clément Erignoux** and **Marielle Simon**

Inria Lille-Nord Europe, France

October 2021

Hydrodynamic Limit

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- The PDEs macroscopically describe the evolution of densities, energies and momentum of the particle system.
- See [Kipnis and Landim'98].

Microscopic/Macroscopic Scale

Let N be the scaling parameter.

	Microscopic	Macroscopic
Space	$x \in \mathbb{Z}^d$	$x/N \in \mathbb{R}^d$
Time	$t\Theta(N)$	t
Mass	1	$1/N^d$

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The particles travel a distance of order N during a time interval of order $\Theta(N)$.

Diffusive Scaling

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- Entropy method (GPV method), Yau's relative entropy method...

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- Attractiveness method, Fritz's compensated compactness method...

PDEs with Boundary Conditions

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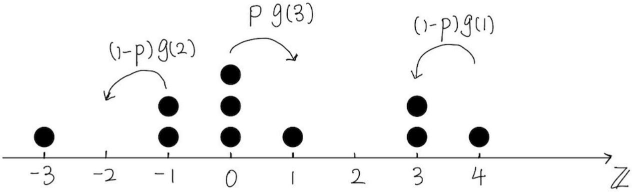
- Hyperbolic equation with boundary conditions: [Lamdim'96, Bahadoran'12, Xu'21].

Zero Range Process

Let $1/2 < p \leq 1$ and $g : \{0, 1, 2, \ldots\} \to \mathbb{R}_+$ such that $g(0) = 0$.



Zero Range Process

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- The infinitesimal generator is

$$L_{\mathrm{ZRP}} f(\eta) = \sum_{x \in Z} \Big\{ pg(\eta(x))[f(\eta^{x,x+1}) - f(\eta)] + (1-p)g(\eta(x+1))[f(\eta^{x+1,x}) - f(\eta)] \Big\},$$

where

$$\eta^{x,y}(z) = \begin{cases} \eta(x) - 1, & z = x \\ \eta(y) + 1, & z = y \\ \eta(z), & \text{otherwise} \end{cases}$$

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  This is needed to well define the process, cf. [Andjel'82].
- Attractiveness: $g(k+1) \ge g(k) \ge g(1) > 0,\ k \ge 1.$

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- We could couple processes together such that

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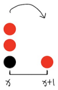
- Current state: 

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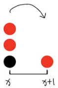
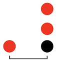
- Current state: 

- Jump together to the right at rate $pg(1)$: 

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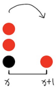
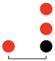
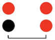
- Current state: 

- Jump together to the right at rate $pg(1)$: 
- The red particle jumps alone to the right at rate

  $p(g(2) - g(1)) \geq 0$: 

Hydrodynamic Limit

- The process has a family of invariant measures $\{\nu_\rho\}_{\rho \geq 0}$, where ρ is the particle density. The one-site marginal of ν_ρ is

$$\nu_\rho^1(k) = \frac{1}{Z(\Phi(\rho))} \frac{\Phi(\rho)^k}{g(k)!}, \quad k \geq 0.$$

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- Roughly speaking, if

$$\eta_0([uN]) \approx {}^{\exists}\rho_0(u), \quad u \in \mathbb{R},$$

  then

$$\eta_{tN}([uN]) \approx {}^{\exists}\rho(t,u).$$

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- By a simple calculation, $N L_{\mathrm{ZRP}} \eta_{tN}([uN])$ is equal to

$$N\left\{pg(\eta_{tN}([uN]) - 1) + (1-p)g(\eta_{tN}([uN]) + 1) - g(\eta_{tN}([uN]))\right\}.$$

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$$N \left\{ p g(\eta_{tN}([uN]) - 1) + (1-p) g(\eta_{tN}([uN]) + 1) - g(\eta_{tN}([uN])) \right\}.$$

- Replacing g with its average Φ, the last formula is approximate to

$$\begin{aligned} N(p\Phi\left(\rho(t,u-1/N)\right) + (1-p)\Phi(\rho(t,u+1/N)) - \Phi(\rho(t,u))) \\ \approx -(2p-1)\partial_u \Phi(\rho(t,u)). \end{aligned}$$

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- Whence, the hydrodynamic equation should be

$$\partial_t \rho(t,u) + (2p-1)\partial_u \Phi(\rho(t,u)) = 0.$$

Theorem [Rezakhanlou'91]

Fix an initial density profile $\rho_0 : \mathbb{R} \to \mathbb{R}_+$, which is bounded and locally integrable. Suppose initially, $\{\eta_0(x)\}_{x\in\mathbb{Z}}$ are independent and $\eta_0(x)$ has distribution $\nu^1_{\rho_0(x/N)}$. Then for every time $t > 0$, every $H \in C_c(\mathbb{R})$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{x\in\mathbb{Z}} \eta_{tN}(x) H\left(\tfrac{x}{N}\right) = \int_{\mathbb{R}} \rho(t,u)H(u)\,du \quad \text{in probability,}$$

where $\rho(t,u)$ is the unique entropy solution to

$$\begin{cases} \partial_t \rho(t,u) + (2p-1)\partial_u \Phi(\rho(t,u)) = 0, \\ \rho(0,u) = \rho_0(u). \end{cases}$$

Entropy Solution

An entropy solution means

(i) $\rho(t,u)$ is a weak solution;

(ii) $\rho(t,u)$ satisfies the entropy inequality: for every $c \geq 0$,

$$\partial_t|\rho(t,u)-c| + (2p-1)\partial_u|\Phi(\rho(t,u)) - \Phi(c)| \leq 0$$

in the sense of distributions;

(iii) for every constant $k>0$,

$$\lim_{t\to 0}\int_{|u|\leq k} |\rho(t,u)-\rho_0(u)|\,du = 0.$$

Example: Independent Random Walks

Let $g(k) = k$. Then $\Phi(\rho) = \rho$. The hydrodynamic equation is the simpler transport equation

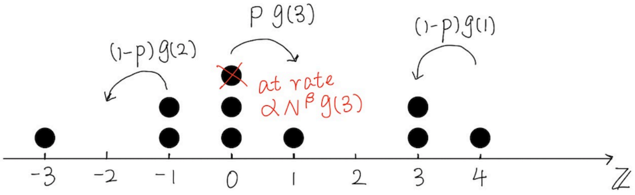
$$\begin{cases} \partial_t \rho(t,u) + (2p-1)\partial_u \rho(t,u) = 0, \\ \rho(0,u) = \rho_0(u). \end{cases}$$

The solution is explicitly given by

$$\rho(t,u) = \rho_0(u - (2p-1)t).$$

ZRP with Destruction

Let $\alpha > 0, \beta \in \mathbb{R}$. N is the scaling parameter.



ZRP with Destruction

- The generator is $L_{ZRP} + L_N^{\alpha,\beta}$, where

$$L_N^{\alpha,\beta} f(\eta) = \alpha N^{\beta} g(\eta(0))[f(\eta^{(0)}) - f(\eta)],$$

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$$\eta^{(0)}(z) = \begin{cases} \eta(0) - 1, & z = 0 \\ \eta(z), & \text{otherwise} \end{cases}$$

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- We assume g is Lipschitz and non-decreasing so that the attractiveness property holds.

Hydrodynamic Limit

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- Initially, $\{\eta_0(x)\}_{x\in\mathbb{Z}}$ are independent and $\eta_0(x)$ has distribution $\nu^1_{\rho_0(x/N)}$ for some bounded and locally integrable initial profile ρ_0. In particular, every $H \in C_c(\mathbb{R})$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{x\in\mathbb{Z}} \eta_0(x) H\left(\tfrac{x}{N}\right) = \int_{\mathbb{R}} \rho_0(u) H(u)\, du \quad \text{in probability.}$$

Hydrodynamic Limit

Theorem for $\beta < 0$, [Erignoux, Simon and Z.'21]

If $\beta < 0$, then the hydrodynamic equation is

$$\begin{cases} \partial_t \rho(t,u) + (2p-1)\partial_u \Phi(\rho(t,u)) = 0, \\ \rho(0,u) = \rho_0(u). \end{cases}$$

Theorem for $\beta > 0$, [Erignoux, Simon and Z.'21]

If $\beta > 0$, then the hydrodynamic equation is

$$\rho(t,u) = \rho_L(t,u)\mathbf{1}_{\{u<0\}} + \rho_R(t,u)\mathbf{1}_{\{u\geq 0\}}$$

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and ρ_R is the unique entropy solution to

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and ρ_R is the unique entropy solution to

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Entropy Solution

We have two equivalent definitions for

$$\begin{cases} \partial_t \rho(t,u) + (2p-1)\partial_u \Phi(\rho(t,u)) = 0, \quad u > 0, \\ \rho(t,0) = \varrho(t), \\ \rho(0,u) = \rho_0(u). \end{cases}$$

One is based on a weak formula, and the other is based on traces, cf. [Málek *et al.*'19]

Entropy Solution

Definition $(\beta = 0)$ [Málek *et al.*'19]

A bounded function ρ is said to be the entropy solution of the above PDE if there exists $M > 0$ such that for every $0 \leq H \in C_c^{1,1}(\mathbb{R} \times \mathbb{R})$ and for every $c \geq 0$,

$$\int_0^\infty \int_0^\infty \{\partial_t H(t,u)(\rho(t,u) - c)^\pm + (2p-1)\partial_u H(t,u)(\Phi(\rho(t,u)) - \Phi(c))^\pm\} \, du \, dt + M \int_0^\infty H(t,0)(\varrho(t) - c)^\pm \, dt + \int_0^\infty H(0,u)(\rho_0(u) - c)^\pm \geq 0.$$

Entropy Solution

Definition $(\beta > 0)$ [Málek *et al.*'19]

A bounded function ρ is said to be the entropy solution of the above PDE if

(i) the entropy inequality holds for every $0 \leq H \in C_c^{1,1}((0,\infty) \times (0,\infty))$;

(ii) for every $T > 0$,

$$\lim_{u \to 0} \int_0^T |\Phi(\rho(t,u)) - \Phi(\varrho(t))| dt = 0;$$

(iii) for every constant $k > 0$,

$$\lim_{t \to 0} \int_{|u| \leq k} |\rho(t,u) - \rho_0(u)| \, du = 0.$$

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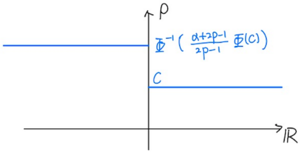

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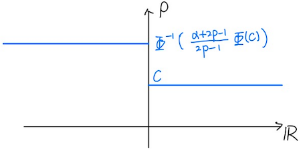

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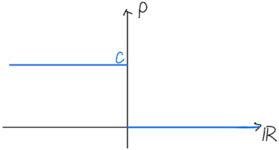

Independent Random Walks with Destruction

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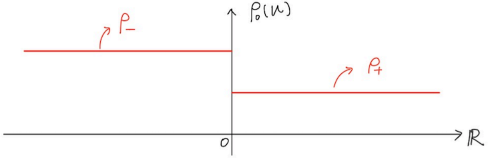

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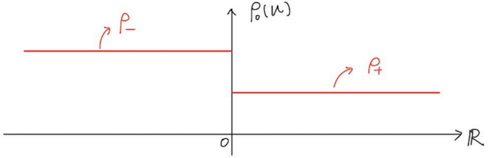
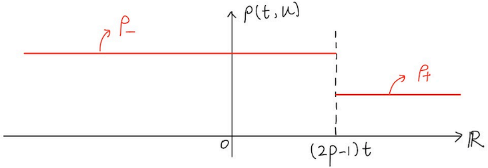


If without destruction or $\beta < 0$, then at time $t > 0$



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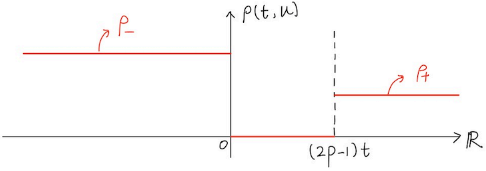

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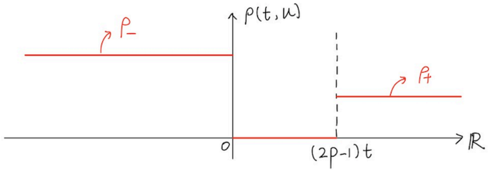

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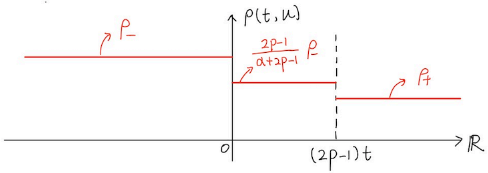

Independent Random Walks with Destruction

The solution $\rho(t,u)$ is explicitly given by

$$\rho(t,u) = \left(1 - \widetilde{\alpha}\, \mathbf{1}\{0 \leqslant u < (2p-1)t\}\right) \rho_0(u - (2p-1)t),$$

where

$$\widetilde{\alpha} = \begin{cases} 1 & \text{if } \beta > 0, \\ \alpha/(\alpha + 2p - 1) & \text{if } \beta = 0, \\ 0 & \text{if } \beta < 0. \end{cases}$$

Future Work

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- Joint work with Clément Erignoux and Lu Xu.

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Thanks!

Email: linjie.zhao@inria.fr