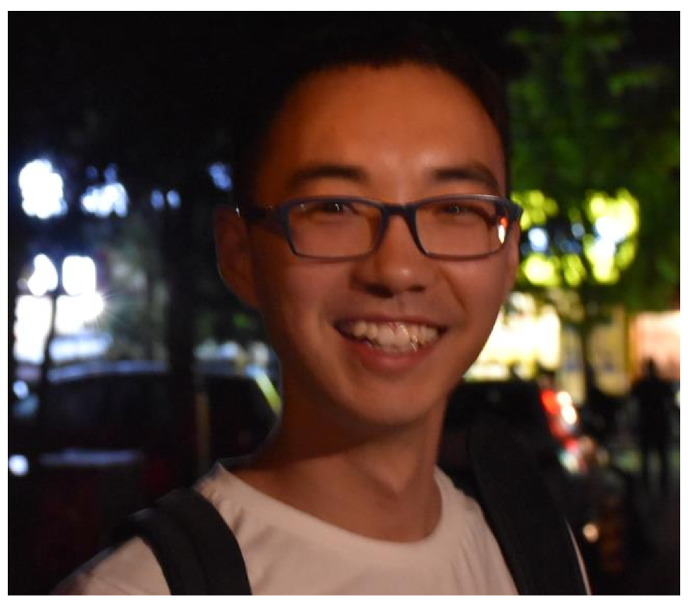
Hydrodynamics of the Binary Contact Path Process



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Contact Process

- Introduced by Harris [6] to describe the spread of a disease on $\mathbb{Z}^d$.
- A popular model in the area of interacting particle systems (see Liggett's two monographs [8, 9]).
- Appeared independently in the high-energy physics literature, and is equivalent to the reggeon spin model.
- Simple but exhibits a phase transition.

Regard each $x \in \mathbb{Z}^d$ as an individual. The state $\eta(x)$ of x takes value in $\{0,1\}$. The individual x is healthy if $\eta(x)=0$, while is infected if $\eta(x)=1$. The dynamics is as follows:
1. Each infected individual is recovered at rate 1, i.e., $1 \to 0$ at rate 1.
2. An individual x is infected by an infected neighbor y at rate λ, i.e., $0 \to 1$ at rate $\lambda \sum_{y \sim x} \eta(y)$.

Critical Value. It is easy to see that the probability that the disease survives is increasing as λ increases. Therefore, there exists a critical value λ_c such that

$$\text{The disease} \begin{cases} \text{is extinct with probability one,} & \text{if } \lambda < \lambda_c, \\ \text{survives with positive probability,} & \text{if} \lambda > \lambda_c. \end{cases}$$

Binary Contact Path Process

On top of the contact process, we also consider the *seriousness* of the disease. The state $\eta(x)$ of x takes value in $[0,\infty)$. The individual x is healthy if $\eta(x)=0$, while is infected if $\eta(x)>0$ and the value of $\eta(x)$ is the seriousness of the disease on x. The dynamics is as follows:

(I) Each infected individual is recovered at rate 1.

(II) An individual x is infected by a given neighbor y at rate λ. When the infection occurs, the seriousness of the disease on x is added with that of y.

(III) When there is no infection occurs for x during some time interval, then $\eta_t(x)$ evolves according to the deterministic ODE

$$\frac{d}{dt}\eta_t(x) = (1-2\lambda d)\eta_t(x).$$

- Introduced by Griffeath [5] as an auxiliary model to study the critical value of the contact process: If $d \geq 3$, then
$$\lambda_c \leq \frac{1}{2d(2\gamma_d - 1)},$$
where γ_d is the probability that the simple random walk on $\mathbb{Z}^d$ starting at O never return to O again.
- Let
$$\xi_t(x) = \begin{cases} 1 & \text{if } \eta_t(x) > 0, \\ 0 & \text{if } \eta_t(x) = 0, \end{cases}$$
for each $x \in \mathbb{Z}^d$, then $\{\xi_t\}$ is a version of the contact process.

Hydrodynamics

The theory says that the microscopic density field of the concerned model, after properly space time scaling, is dominated macroscopically be some PDE (*hydrodynamic equation*). We refer the reader to [7].

For each $N \geq 1$, let $\{\eta_t^N\}$ be the binary contact path process with **time speeded up by** N^2. We are concerned about the random empirical measure π_t^N of the process defined by

$$\pi_t^N := \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \eta_t^N(x) \delta_{\frac{x}{N}}(du).$$

	Macroscopic	Microscopic
Time	t	tN^2
Space	x/N	x
Space	$\mathbb{R}^d$	$\mathbb{Z}^d$

Main Result

We proved the law of large numbers for the empirical measure π_t^N. The hydrodynamic equation turns out to be the heat equation.

Theorem. *Let $\rho_0 : \mathbb{R}^d \to [0,\infty)$ be bounded and integrable. Initially, $\eta_0^N(x) = \rho_0(x/N)$. Suppose $d \geq 3$ and*

$$\lambda > \frac{1}{2d(2\gamma_d - 1)},$$

then for all $t \geq 0$, as $N \to \infty$,

$$\pi_t^N(du) \to \rho(t,u)du \ \textit{ in probability,} \tag{1}$$

where $\rho(t,u)$ is the unique solution of the heat equation

$$\begin{cases} \partial_t \rho(t,u) = \lambda \Delta \rho(t,u), \\ \rho(0,u) = \rho_0(u). \end{cases} \tag{2}$$

Remarks

- The favorite models in hydrodynamic theory are generally mass conserved, such as exclusion processes and zero range processes. However, the binary contact path process lacks this property.
- We are able to calculate the hydrodynamics of the binary contact path process due to the following two facts:
  - On average, the total mass of the system is conserved, see Equation (4). This fact is due to the dynamics (III) of the process.
  - The binary contact path process belongs to linear systems, which allows explicit calculations.
- A central limit theorem for the density of particles in the context of binary context path processes was proved in [10].
- It remains open to study the hydrodynamics of the process for small λ or in lower dimensions $d \leq 2$.

Sketch of the Proof

The proof is divided into four steps:

Step 1. Let Q^N be the law of $\{\pi_t^N, 0 \leq t \leq T\}$. Then $\{Q^N, N \geq 1\}$ is tight. (Prohorov's Theorem and Aldous' Criterion.)

Step 2. Let Q^* be any limit of Q^N. Then Q^* is concentrated on absolutely continuous trajectories.

Step 3. Show that the density of the trajectory is the solution of the heat equation (2) with probability one with respect to Q^*. (Martingale Approach.)

Step 4. The result follows from the uniqueness of the solution of the heat equation.

The **difficulty** is to prove the **absolute continuity (Step 2)** since the value at each site is not bounded. We overcome this difficulty by proving the convergence to zero of the variance: given $G \in C_c^2(\mathbb{R}^d)$ and $t > 0$,

$$\lim_{N \to +\infty} \mathrm{Var}\big(\langle \pi_t^N, G \rangle\big) = 0. \tag{3}$$

This implies that

$$\langle \pi_t, G \rangle = \mathbf{E}_{Q^*}\left[\langle \pi_t, G \rangle\right] \quad Q^*\text{-a.s.}$$

Then (B) follows from the following fact:

$$\mathbb{E}\left[\langle \pi_t^N, G \rangle\right] = \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \rho_0\left(\frac{x}{N}\right) G\left(\frac{x}{N}\right), \tag{4}$$

The proof of Equation (3) is based on *direct calculation*. We first write $\mathrm{Var}\big(\langle \pi_t^N, G \rangle\big)$ as

$$\frac{1}{N^{2d}} \sum_{x \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d} G\left(\frac{x}{N}\right) G\left(\frac{y}{N}\right) \Big(\mathbb{E}\big(\eta_t^N(x)\eta_t^N(y)\big) - \mathbb{E}\big(\eta_t^N(x)\big)\mathbb{E}\big(\eta_t^N(y)\big)\Big).$$

By Hille-Yosida Theorem and infinitesimal generator calculations,

$$\mathbb{E}\big(\eta_t(x)\big) = \sum_{y \in \mathbb{Z}^d} p_t(x,y) \mathbb{E}\big(\eta_0(y)\big),$$

where $\{p_t(x,y)\}$ is the transition probability of the continuous time simple random walk on $\mathbb{Z}^d$ with rate $2d\lambda$, and there exists a $(\mathbb{Z}^d \times \mathbb{Z}^d) \times (\mathbb{Z}^d \times \mathbb{Z}^d)$ matrix M_λ such that

$$\mathbb{E}\Big(\eta_t(x)\eta_t(y)\Big) = \sum_{u \in \mathbb{Z}^d} \sum_{v \in \mathbb{Z}^d} e^{tM_\lambda}\Big((x,y),(u,v)\Big) \mathbb{E}\Big[\eta_0(u)\eta_0(v)\Big].$$

Finally, Equation (3) follows from the following result: For any $\lambda > \frac{1}{2d(2\gamma_d - 1)}$, there exists $h_\lambda > 0$ such that

$$\sum_{u \in \mathbb{Z}^d} \sum_{v \in \mathbb{Z}^d} e^{tM_\lambda}\big((x,y),(u,v)\big) \leq \frac{k(y-x) + h_\lambda}{h_\lambda},$$

where $k(x)$ is the probability that the simple random walk hits site x finally when starting from the origin.

References

[1] De Masi, A. and Presutti, E. (1991). Mathematical methods for hydrodynamic limits. *Lecture Notes in Mathematics*.

[2] Deimling, K. (1977). *Ordinary Differential Equations in Banach Spaces.* Springer, Berlin.

[3] Dobrushin Moscow, R. O. and Siegmund-Schultze, R. (1982). The hydrodynamic limit for systems of particles with independent evolution. *Mathematische Nachrichten*, **105**, 199-224.

[4] Galves, A., Kipnis, C., Marchioro, C. and Presutti, E. (1981). Nonequilibrium measures which exhibit a temperature gradient: study of a model. *Communications in Mathematical Physics*, **81**, 127-147.

[5] Griffeath, D. (1983). The binary contact path process. *The Annals of Probability* **11**, 692-705.

[6] Harris, T. E. (1974). Contact interactions on a lattice. *The Annals of Probability* **2**, 969-988.

[7] Kipnis, C. and Landim, C. (1999) *Scaling limits of interacting particle systems.* Springer-Verlag, Berlin.

[8] Liggett, T. M. (1985). *Interacting Particle Systems.* Springer, New York.

[9] Liggett, T. M. (2013). *Stochastic interacting systems: contact, voter and exclusion processes* (Vol. 324). springer science & Business Media.

[10] Nagahata, Y. and Yoshida, N. (2009). Central limit theorem for a class of linear systems. *Electronic Journal of Probability*, **14**, 960-977.

[11] Presutti, E. and Spohn, H. (1983). Hydrodynamics of the voter model. *The Annals of Probability*, 867-875.

[12] Rost, H. (1981) Nonequilibrium behaviour of a many particle process: density profile and local equilibria. *Z. Wahrsch. Verw. Gebiete*, **58**, 41-53.