Scaling Limits of the SSEP with a Slow Site

T. Franco[1], P. Gonçalves[2], R. Marinho[2] and **L. Zhao**[3]

1 Background

One of the biggest problems in *interacting particle systems* is the derivation of macroscopic equations as large-scale limits from these models. For example, the Navier-Stokes equation is recovered from the evolution of the momentum in the incompressible lattices gas model described in [2]. The attainment of these partial differential equations from particle systems is known as *hydrodynamic limit*. The analysis of the hydrodynamic behaviour of interacting particle systems gives validity to many partial differential equations in the literature [18].

Obtaining hydrodynamic equations becomes even more difficult when the desired ones have boundary conditions. It is well known that the hydrodynamics of symmetric simple exclusion processes (**SSEP**) is driven by the heat equation [18]. In order to obtain the heat equation with boundary conditions, the SSEP in different settings was considered. We mention here the SSEP with slow bonds [6, 8–10], the SSEP with slow boundary [1, 7, 13] and the SSEP with a slow site [11].

For the SSEP with a slow site, the asymmetry in the jump rates that a particle enters and leaves the origin makes the usual arguments much more difficult. In [11], they only proved the supercritical case. We solved the remaining cases recently.

2 Model and Main Results

The model is defined on the one dimensional discrete torus $\mathbb{T}_n$ with n sites or on the integer lattice $\mathbb{Z}$. There is at most one particle per site. A particle at site x jumps to one of its neighbors $y \sim x$ at rate ξ_x^n provided that the target site is empty, where

$$\xi_x^n = \begin{cases} \alpha\, n^{-\beta}\,, & \text{if} \quad x = 0\,,\\ 1\,, & \text{if} \quad x \neq 0\,.\end{cases}$$

Here $\alpha > 0$ and $\beta \geq 0$.

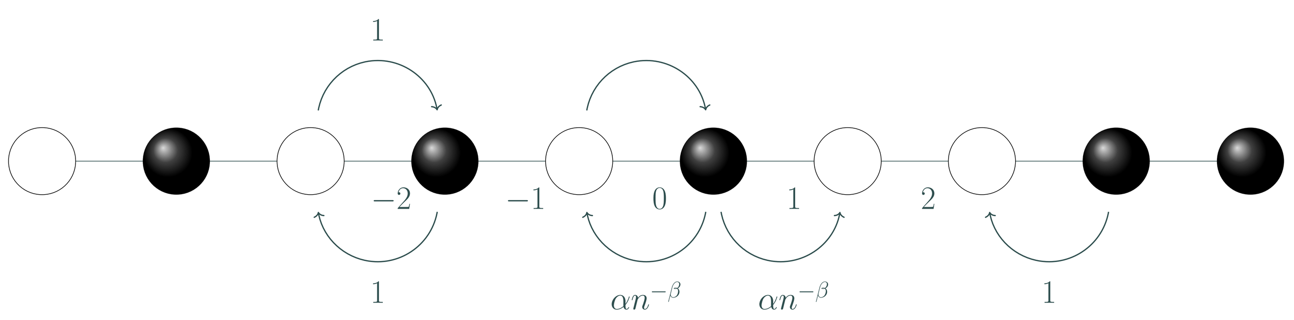

Figure 1: At most one particle per site. A particle at the origin jumps at a lower rate $\alpha n^{-\beta}$.

In order to see the process on its diffusive time scaling, we will consider the process $\{\eta_t : t \geq 0\}$ with *time speeded up by n^2*. We recall from [11, Proposition 2.1] that for any $p \in [0, 1]$, the Bernoulli product measure ν_p defined by

$$\nu_p(\eta;\ \eta(x) = 1) \ =:\ m_p(x) \ = \ \begin{cases} \frac{p/(\alpha n^{-\beta})}{(1-p)+p/(\alpha n^{-\beta})}, & \text{if } x = 0,\\ p, & \text{if } x \neq 0,\end{cases} \tag{1}$$

is *reversible* for the process $\{\eta_t : t \geq 0\}$.

2.1 Hydrodynamics

The process is defined on $\mathbb{T}_n$. Fix a profile $\rho_0 : \mathbb{T} \to [0, 1]$, representing the initial density of particles. To avoid uninteresting technical complications, we assume that ρ_0 is continuous at all $x \in \mathbb{T}\backslash\{0\}$ and bounded from below by a positive constant,

$$\zeta \ :=\ \inf_{x\in\mathbb{T}} \rho_0(x) \ >\ 0\,. \tag{2}$$

The *empirical measure* is defined as

$$\pi_t^n(du) = \frac{1}{n}\sum_x \eta_t(x)\delta_{x/n}(du). \tag{3}$$

Theorem 1 (Law of Large Numbers for the Empirical Measure)

Let the initial measure $\{\mu_n; n \in \mathbb{N}\}$ be a sequence of product measures on $\{0, 1\}^{\mathbb{T}_n}$ with marginals given by

$$\mu_n\{\eta : \eta(x) = 1\} = \rho_0(x/n). \tag{4}$$

Then, for every $t > 0$, $\pi_t^n(du)$ converges in the weak topology to $\rho(t, u)du$ in probability as $n \to \infty$, where $\rho(t, u)$ stands for the unique weak solution of

- the heat equation with *periodic boundary conditions* if $\beta \in [0, 1)$;
- the heat equation with *Robin boundary conditions* if $\beta = 1$;
- and the heat equation with *Neumann boundary conditions* if $\beta > 1$.

2.2 Equilibrium Fluctuations

The process is defined on $\mathbb{Z}$. The *fluctuation field*, which is the linear functional acting on test functions H, is defined as

$$\mathcal{Y}_t^n(H) \ =\ \frac{1}{\sqrt{n}}\sum_{x\in\mathbb{Z}} H\big(\tfrac{x}{n}\big)\overline{\eta}_t(x)\,,$$

where $\overline{\eta}_t(x) = \eta_t(x) - m_p(x)$.

Domain of the Test Functions in Different Regimes

We will set $\mathcal{S}_\beta$

- for $\beta \in [0, 1)$, as the usual Schwartz space;
- for $\beta = 1$, as the space of functions $H : \mathbb{R} \to \mathbb{R}$ such that

(1) H is continuous and infinitely differentiable, except possibly at $x = 0$.

(2) $H(0) = \frac{1}{2}\big[H(0^+) + H(0^-)\big]$.

(3) For all integers $k, \ell \geq 0$,

$$\|H\|_{k,\ell} \ :=\ \sup_{x\neq 0}\Big|(1 + |x|^\ell)\frac{d^kH}{dx^k}(x)\Big| \ <\ \infty\,.$$

(4) For any integer $k \geq 0$,

$$\frac{d^{2k+1}H}{dx^{2k+1}}(0^+) \ =\ \frac{d^{2k+1}H}{dx^{2k+1}}(0^-) \ =\ \alpha\Big(\frac{d^{2k}H}{dx^{2k}}(0^+) \ -\ \frac{d^{2k}H}{dx^{2k}}(0^-)\Big)\,.$$

- for $\beta > 1$, as the space of functions $H : \mathbb{R} \to \mathbb{R}$ that satisfy $(1), (2')$ below, $(3), (4)$ with $\alpha = 0$,

$(2')$ H is continuous from the right at zero.

Denote by $\mathcal{S}'_\beta(\mathbb{R})$ the dual space of $\mathcal{S}_\beta(\mathbb{R})$. We will define ∇_β and Δ_β

- for $\beta \in [0, 1)$, by the usual first and second space derivatives;
- for $\beta \geq 1$, by

$$\nabla_\beta H(u) = \begin{cases}\frac{dH}{du}(u), & \text{if} \ \ u \neq 0\,,\\ \lim_{u\to 0^+}\frac{dH}{du}(u), & \text{if} \ \ u = 0\,,\end{cases} \qquad \Delta_\beta H(u) = \begin{cases}\frac{d^2H}{du^2}(u), & \text{if} \ \ u \neq 0\,,\\ \lim_{u\to 0^+}\frac{d^2H}{du^2}(u), & \text{if} \ \ u = 0\,.\end{cases}$$

Theorem 2 (Central Limit Theorem for the Density of Particles)

Fix a horizonal time $T > 0$. Consider the Markov process $\{\eta_t : t \geq 0\}$ starting from the invariant state ν_p. Then, the sequence of processes $\{\mathcal{Y}_t^n, 0 \leq t \leq T\}_{n\in\mathbb{N}}$ converges in distribution, as $n \to +\infty$, with respect to the Skorohod topology of $\mathcal{D}([0, T], \mathcal{S}'_\beta(\mathbb{R}))$ to $\mathcal{Y}_t$ in $\mathcal{C}([0, T], \mathcal{S}'_\beta(\mathbb{R}))$, the generalized Ornstein-Uhlenbeck process which is the formal solution of the SPDE

$$d\mathcal{Y}_t = \Delta_\beta\mathcal{Y}_t dt + \sqrt{2\chi(p)}\nabla_\beta d\mathcal{W}_t \tag{5}$$

where $\mathcal{W}_t$ is a Brownian motion on the space of tempered distributions and $\chi(p) = p(1 - p)$ is the *compressibility* of the system.

Remark

(i) As we said before, the case $\beta > 1$ was proved earlier in [11].

(ii) The rigorous meaning of (5) is given in terms of martingale problems, see [9,11] for similar definitions.

3 Strategy of the Proof

- To prove the hydrodynamic limit when $\beta \in [0, 1)$, we use the *entropy method* of Guo, Papanicolaou and Varadhan [15], after a smart trick partitioning the space of configurations.
- To prove the hydrodynamic limit when $\beta = 1$, we appeal to *Gordin's martingale approximation method* [14]. The method was introduced by Gordin in 1969 as an approach to obtain central limit theorems for Markov processes. It basically consists in approximating an additive functional $\int_0^t F(X_s)\,ds$ of a Markov process to a martingale. The idea is the following: given a Markov process $\{X_t; t \geq 0\}$ with generator $\mathcal{L}$ and a function f in the domain of $\mathcal{L}$, the process

$$f(X_t) - f(X_0) - \int_0^t \mathfrak{L}\, f(X_s)ds \tag{6}$$

is a mean-zero martingale with respect to the law of $\{X_t; t \geq 0\}$ (see [18, Appendix 1, Lemma 5.1]). Therefore, if we solve the Poisson equation

$$\mathfrak{L}\, f = F \tag{7}$$

for an *appropriate* f, we can express the functional $\int_0^t F(X_s)ds$ as the sum of a martingale and a boundary term. This strategy will be used to prove the *replacement lemmas* for the case $\beta = 1$.

- To prove equilibrium fluctuations for both cases, we maintain the usual *entropy method*, which now works owing to the fact that since we are starting at the stationary measure, we do not need to pay an entropy price.

References

[1] Baldasso, R., Menezes, O., Neumann, A., and Souza, R. R. *Exclusion process with slow boundary.* Journal of Statistical Physics 167.5 (2017): 1112-1142.

[2] Beltrán, J., and C. Landim. *A lattice gas model for the incompressible Navier-Stokes equation.* Annales de l'IHP Probabilités et statistiques. Vol. 44. No. 5. 2008.

[3] De Masi, A., et al. *Current reservoirs in the simple exclusion process.* Journal of Statistical Physics 144.6 (2011): 1151.

[4] De Masi, A., et al. *Truncated correlations in the stirring process with births and deaths.* Electronic Journal of Probability 17 (2012).

[5] De Masi, A., Presutti, E. and Tsagkarogiannis, D., Vares, M.E., *Non-equilibrium stationary state for the SEP with births and deaths.* Journal of Statistical Physics (2012).

[6] Erhard, D., Franco, T., Gonçalves, P., Neumann, A., and Tavares, M. (2018). *Non-equilibrium fluctuations for the SSEP with a slow bond.* arXiv preprint arXiv:1809.04367.

[7] Franco, T., Gonçalves, P., and Neumann, A. *Equilibrium Fluctuations for the Slow Boundary Exclusion Process.* Meeting on Particle Systems and PDE's. Springer, Cham, 2015.

[8] Franco, T., Gonçalves, P., and Neumann, A. *Hydrodynamical behavior of symmetric exclusion with slow bonds.* Annales de l'IHP Probabilités et statistiques. Vol. 49. No. 2. 2013.

[9] Franco, T., Gonçalves, P., and Neumann, A. *Phase transition in equilibrium fluctuations of symmetric slowed exclusion.* Stochastic Processes and their Applications 123.12 (2013): 4156-4185.

[10] Franco, T., Gonçalves, P. and Neumann, A. *Phase transition of a heat equation with Robin's boundary conditions and exclusion process.* Transactions of the American Mathematical Society 367.9 (2015): 6131-6158.

[11] Franco, T., Gonçalves, P. and Schütz, G. M. *Scaling limits for the exclusion process with a slow site.* Stochastic Processes and their applications 126.3 (2016): 800-831.

[12] Franco, T., and Tavares, M. *Hydrodynamic Limit for the SSEP with a Slow Membrane.* arXiv preprint arXiv:1809.07911 (2018).

[13] Gonçalves, P., Jara, M., Menezes, O., and Neumann, A. (2018). *Non-equilibrium and stationary fluctuations for the SSEP with slow boundary.* arXiv preprint arXiv:1810.05015.

[14] Gordin, M. I. *The central limit theorem for stationary processes.* Doklady Akademii Nauk. Vol. 188. No. 4. Russian Academy of Sciences, 1969.

[15] Guo, M. Z., G. C. Papanicolaou, and S. R. S. Varadhan. *Nonlinear diffusion limit for a system with nearest neighbor interactions.* Communications in Mathematical Physics 118.1 (1988): 31-59.

[16] Jacod, J., and Shiryaev, A. *Limit theorems for stochastic processes.* Springer, 2003.

[17] Holley, R. A., & Stroock, D. W. (1978). Generalized Ornstein-Uhlenbeck processes and infinite particle branching Brownian motions. Publications of the Research Institute for Mathematical Sciences, 14(3), 741-788.

[18] Kipnis, C., and Landim, C. *Scaling Limits of Interacting Particle Systems.* Vol. 320. Springer Science and Business Media, 1998.

[19] Kipnis, C., and Varadhan, S. *Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions.* Communications in Mathematical Physics 104.1 (1986): 1-19.

[20] Liggett, T.M. *Interacting particle systems.* Vol. 276. Springer Science and Business Media, 2012.

[21] Mitoma, I. (1983). Tightness of Probabilities On $C([0,1];\mathscr{S}')$ and $D([0,1];\mathscr{S}')$. The Annals of Probability, 11(4), 989-999.

[22] M. Reed, B. Simon, Functional Analysis, Vol. 1, first ed., in: Methods of Modern Mathematical Physics, Academic Press, 1981.

[1]Universidade Federal da Bahia, Brazil, [2]Instituto Superior Técnico, Portugal, [3]Peking University, China

Email: tertu@ufba.br, patricia.goncalves@math.tecnico.ulisboa.pt, rodrigo.marinho@tecnico.ulisboa.pt and zhaolinjie@pku.edu.cn