Exclusion Processes and Normalized Binary Contact Path Processes: Ergodic Properties and Large Scale Behaviors

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Chen, D. , & Zhao, L. (2019). The invariant measures and the limiting behaviors of the facilitated TASEP. *Statistics & Probability Letters*, 154, 108557.

Xue, X., & Zhao, L. (2019). Hydrodynamics of the Binary Contact Path Process. *arXiv preprint arXiv:1901.04660*.

Models

Invariant Measures

Convergence Theorems

Hydrodynamics

- General Theory
- FTASEP

Future Research

Exclusion Processes

- First introduced by Spitzer (1970).
- Extensively studied since then, see Liggett's monographs.

Spitzer, F. (1970) Interaction of Markov processes. *Advances in Math.* **5**, 246–290.

Liggett, T. M. (2012). *Interacting particle systems* (Vol. 276). Springer Science & Business Media.

Liggett, T. M. (2013). *Stochastic interacting systems: contact, voter and exclusion processes* (Vol. 324). springer science & Business Media.

Totally Asymmetric Simple Exclusion Processes (TASEP)

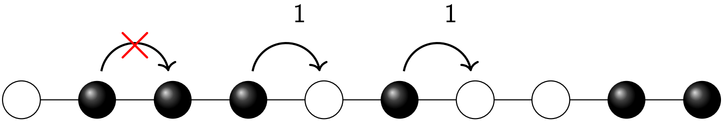

Figure: At most one particle per site.

Facilitated TASEP (FTASEP)

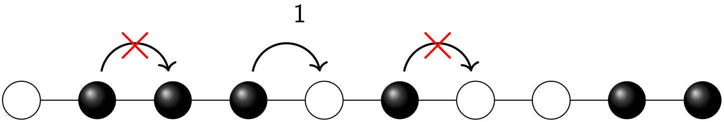

Figure: A particle jumps if its left neighbor is occupied.

Facilitate Exclusion Processes

- Phase transition in the presence of a conserved field.
- A new universality class of non-equilibrium phase transitions.
- Motion in glasses.

Basu, U., & Mohanty, P. K. (2009). Active–absorbing-state phase transition beyond directed percolation: A class of exactly solvable models. *Physical Review E*, 79(4), 041143.

Gabel, A., Krapivsky, P. L., & Redner, S. (2010). Facilitated asymmetric exclusion. *Physical review letters*, 105(21), 210603.

Questions

Consider the FTASEP on $\mathbb{Z}$.

- Ergodic properties: invariant measures, convergence theorems.
- Hydrodynamics: macroscopically, the evolution of the density profile.

TASEP

- Let ν_ρ, $\rho \in (0,1)$, be the product measure on $\{0,1\}^{\mathbb{Z}}$ with marginals given by

$$\nu_\rho\{\eta : \eta(x) = 1\} = \rho, \quad \text{for all } x \in \mathbb{Z}.$$

- Let δ_n, $n \in \mathbb{Z}$, be the Dirac measure on the configuration η such that $\eta(x) = 1$ if and only if $x \geq n$.

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TASEP

$\mathcal{I}$: the family of invariant measures.

$\mathcal{I}_e$: the family of the extreme points of $\mathcal{I}$.

Invariant measures of TASEP, Liggett (1976)

For TASEP, $\mathcal{I}_e = \{\nu_\rho, 0 \leq \rho \leq 1\} \bigcup \{\delta_n, -\infty < n < \infty\}$.

Liggett, T. M. (1976). Coupling the simple exclusion process. *The Annals of Probability*, 4(3), 339-356.

FTASEP

For $0 < \rho < 1$, ν_ρ is NOT invariant for the FTASEP!

Take $f(\eta) = (1 - \eta(1))(1 - \eta(0))$, then

$$\int \mathcal{L} f d\nu_\rho = -\rho^2(1-\rho)^2 \neq 0, \tag{1}$$

where $\mathcal{L}$ is the infinitesimal generator of the FTASEP.

FTASEP

Let μ be a probability measure on $\{0,1\}^{\mathbb{Z}}$.

- μ is (spatial) ergodic if $\{\tau_x \eta, x \in \mathbb{Z}\}$ is ergodic under μ, where $\tau_x \eta(y) = \eta(x+y)$ for any y.
- μ has density ρ if

$$\mu\{\eta : \eta(x) = 1\} = \rho \quad \text{for all } x.$$

- μ is degenerate if

$$\mu\{\eta : \eta(x) = \eta(x+1) = 1\} = 0 \quad \text{for all } x.$$

FTASEP

Chen and Z. [STAT PROBABIL LETT, 2019, 154, 108557].

- For $1/2 < \rho < 1$, the FTASEP has a family of (spatial) ergodic non-degenerate invariant measures having density ρ.
- For $0 < \rho \leq 1/2$, the FTASEP does not have (spatial) ergodic non-degenerate invariant measures having density ρ.

Existence

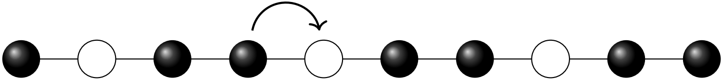


Figure: FTASEP

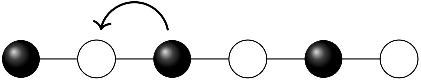


Figure: TASEP

Existence

- At most one hole between two consecutive particles in the FTASEP.
- View the FTASEP holes as the TASEP particles.
- The FTASEP holes evolve as the TASEP.

Existence

- Insert a hole between two consecutive FTASEP particles with probability p.
- If the FTASEP particle density is ρ, then

$$\rho = \frac{1}{1 - p + 2p}.$$

- The stationary state current

$$\langle 110 \rangle_\rho = \rho(1-p)p = \frac{(1-\rho)(2\rho-1)}{\rho}.$$

Bernoulli Product Initial Distributions

- Let $\boldsymbol{\eta}^1$ be the *alternate configuration* with particles placed on odd sites: $\boldsymbol{\eta}^1(x) = 1$ if and only if x is odd.
- Let $\boldsymbol{\eta}^0$ be the *alternate configuration* with particles placed on even sites: $\boldsymbol{\eta}^0(x) = 1$ if and only if x is even.

$$\ldots 101010101010101010101 \ldots$$

Bernoulli Product Initial Distributions

Chen and Z. [STAT PROBABIL LETT, 2019, 154, 108557].

For the FTASEP on $\mathbb{Z}$,

$$\lim_{t\to\infty} \nu_\rho S(t) = \begin{cases} \pi_\rho & \text{if } \rho < 1/2, \\ \frac{1}{2}\delta_{\boldsymbol{\eta}^1} + \frac{1}{2}\delta_{\boldsymbol{\eta}^0} & \text{if } \rho = 1/2. \end{cases} \tag{2}$$

General Theory

- Let η_t be some stochastic process. For each $n \geq 1$, define the empirical measure $\pi_t^n(du)$ as

$$\pi_t^n(du) = \frac{1}{a(n)} \sum_x \eta_{tb(n)}(x)\delta_{x/n}.$$

- Generally, $a(n) = n, b(n) = n$ for asymmetric systems (Euler scaling); $a(n) = n, b(n) = n^2$ for symmetric systems (diffusive scaling).

General Theory

➢ If initially,

$$\langle \pi_0^n, H \rangle \to \int \rho_0(u) H(u) du \quad \text{in probability as } n \to \infty,$$

for all $H \in C(\mathbb{R})$ (or resp. $H \in C_b(\mathbb{R})$), then for all $t > 0$,

$$\langle \pi_t^n, H \rangle \to \int \rho(t,u) H(u) du \quad \text{in probability as } n \to \infty,$$

for all $H \in C(\mathbb{R})$ (or resp. $H \in C_b(\mathbb{R})$).

General Theory

- If initially,

$$\pi_0^n(du) \to \rho_0(u)du \quad \text{in probability as } n \to \infty,$$

then for all $t > 0$,

$$\pi_t^n(du) \to \rho(t,u)du \quad \text{in probability as } n \to \infty.$$

- $\rho(t,u)$ evolves according to some partial differential equation (hydrodynamic equation).

General Theory

- De Masi, A. and Presutti, E. (1991). *Mathematical methods for hydrodynamic limits*. Lecture Notes in Mathematics.
- Kipnis, C. and Landim, C. (1999) *Scaling limits of interacting particle systems.* Springer-Verlag, Berlin.
- Spohn, H. (2012). *Large scale dynamics of interacting particles*. Springer Science & Business Media.

Step Initial Configuration

$$\ldots 111111000000 \ldots$$

LLN for FTASEP, Z. 2019+

Consider the FTASEP with step initial configuration.

$$\lim_{t\to\infty}\frac{1}{t}\sum_{x>ct}\eta_t(x)=\int_c^{\infty}f(u)du,\quad \forall c\in\mathbb{R}, \tag{3}$$

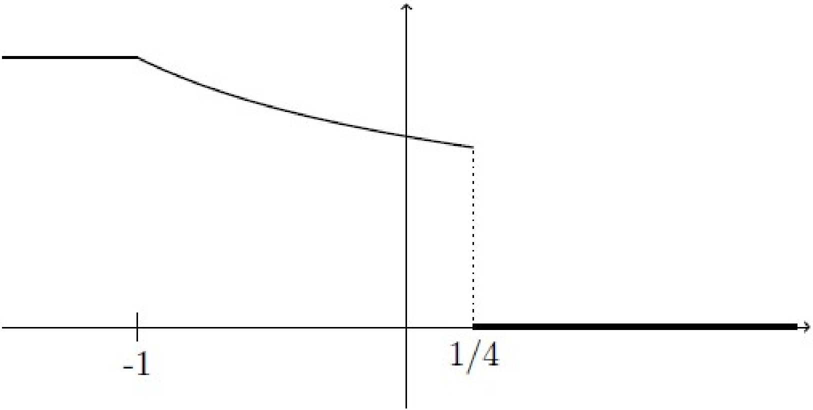
in L^1 and a.s., where

$$f(u)=\begin{cases}1 & \text{if } u<-1\\ \frac{1}{\sqrt{2+u}} & \text{if } -1\leqslant u\leqslant 1/4\\ 0 & \text{if } u>1/4\end{cases} \tag{4}$$

- If we take $a(n) = b(n) = n$ and $H = \mathbf{1}\{\cdot \in (c, \infty)\}$, then

$$\langle \pi_t^n, H \rangle = \frac{1}{n} \sum_{x > cn} \eta_{tn}(x).$$

- $f(u) =$ the density of the particles at the macroscopic time $t = 1$ at the macroscopic point u.


-1
1/4

Remarks

- The rightmost particle moves at speed $1/4$ [Baik *et al.* (2016)].
- The system develops a discontinuity at the point $1/4$.

Baik, J., Barraquand, G., Corwin, I., & Suidan, T. (2016). Facilitated exclusion process. In *The Abel Symposium* (pp. 1-35). Springer, Cham.

Hydrodynamic Equation

- The hydrodynamic equation should be

$$\partial_t \rho(t,u) + \partial_u J(\rho(t,u)) = 0.$$

- $J(\rho)$, $\rho > 1/2$, is the stationary state current

$$J(\rho) := \langle \eta(-1)\eta(0)(1-\eta(1)) \rangle_\rho = \frac{(1-\rho)(2\rho-1)}{\rho}.$$

- $f(u) = \rho(1,u)$ under the initial condition $\rho(0,u) = \mathbf{1}\{u \leq 0\}$.

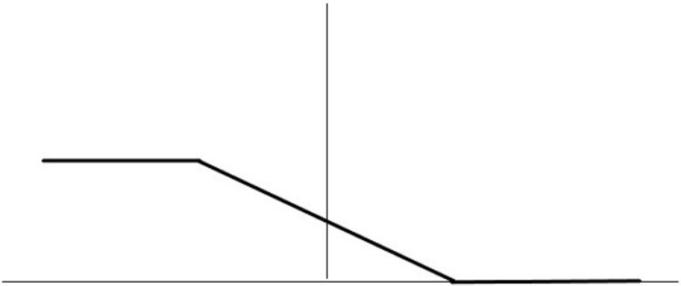
TASEP

For TASEP, it was proved by Rost (1981) that

$$f^{\mathrm{TASEP}}(u) = \begin{cases} \frac{1}{2}(1-u) & \text{for } |u| \leqq 1, \\ 1(0) & \text{for } u < -1 (u > 1). \end{cases}$$

Rost, H. (1981). Non-equilibrium behaviour of a many particle process: Density profile and local equilibria. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 58(1), 41-53.

TASEP



Facilitated Symmetric Exclusion

The hydrodynamics of the facilitated symmetric simple exclusion process was considered by Blondel *et al.* (2018).

Blondel, O., Erignoux, C., Sasada, M., & Simon, M. (2018). Hydrodynamic limit for a facilitated exclusion process. arXiv preprint arXiv:1805.09000.

FTASEP

CLT for FTASEP, Z. (2019+)

Consider the FTASEP with step initial configuration. For

$-1 < c < 1/4$,

$$\lim_{t\to\infty} P\left(\sum_{x>ct} \eta_t(x) \geq (3 - 2\sqrt{2+c})t - 2\sigma t^{1/3} x\right) = F(x), \qquad (5)$$

where $\sigma = (2 - \sqrt{2+c})^{5/3}(\sqrt{2+c} - 1)^{-1/3}$, and F is the Tracy-Widom GUE distribution.

F is defined by the Fredholm determinant,

$$F(s) = \det(I - A)|_{L^2[s,\infty)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{[s,\infty)^k} \det\left(A\left(x_i, x_j\right)\right)_{i,j=1}^{k} d^k x, \tag{6}$$

where $A(\cdot,\cdot)$ is the Airy kernel

$$A(x,y) = \frac{\mathrm{Ai}(x)\,\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\,\mathrm{Ai}(y)}{x - y}, \tag{7}$$

and $Ai(\cdot)$ is the Airy function,

$$\mathrm{Ai}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(t+is)^3/3 + ix(t+is)} dt. \tag{8}$$

CLT for TASEP

➢ Similar results for TASEP were obtained by Johansson (2000).

Johansson, K. (2000). Shape fluctuations and random matrices. *Communications in mathematical physics*, 209(2), 437-476.

Hydrodynamics of FTASEP with general initial distributions?

Rezakhanlou, F. (1991). Hydrodynamic limit for attractive particle systems on $\mathbb{Z}^d$. *Communications in mathematical physics*, **140(3)**, 417-448.

Thanks!