Introduction to the Hydrodynamic Limit of Interacting Particle Systems

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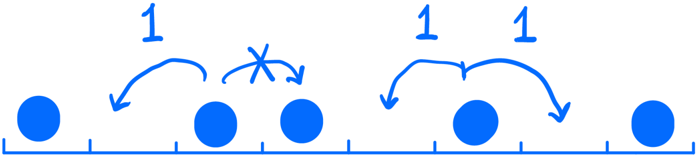
Hydrodynamic Limit

Rigorous Proof

Weakly Asymmetric Normalized Binary Contact Path Process

- Derive partial differential equations from microscopic systems.
- Those PDEs macroscopically describe the evolution of quantities (such as temperature, density, pressure) of the system.

Symmetric Simple Exclusion Process



- State space $\Omega_N = \{0,1\}^{\mathbb{T}_N^d}$. Here, $\mathbb{T}_N^d = \{1, \dots, N\}^d$ with the convention $N \equiv 0$.
- For a configuration $\eta \in \Omega_N$, $\eta(x) = 1$ means there is a particle at site x; otherwise site x is empty.
- The infinitesimal generator L_N of the process is

$$L_N f(\eta) = \sum_{x \in \mathbb{T}_N^d} \sum_{j=1}^{d} \{f(\eta^{x,x+e_j}) - f(\eta)\}, \tag{1}$$

  where $\eta^{x,y}(y) = \eta(x)$, $\eta^{x,y}(x) = \eta(y)$ and $\eta^{x,y}(z) = \eta(z)$ for $z \neq x,\ y$.
- Let $\{\eta_t^N; t \geq 0\}$ be the process with generator $N^2 L_N$.

- We are concerned with the empirical measure of the process defined as

$$\pi_t^N(du) := \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \eta_t^N(x) \delta_{x/N}(du).$$

	Macroscopic $\mathbb{T}^d$	Microscopic $\mathbb{T}_N^d$
Space	u	$[Nu]$
Time	t	tN^2
Mass	N^{-d}	1

Theorem. See Theorem 4.2.1 in Kipnis&Landim'98 for example.

If initially, there exists a density profile $\rho_0 : \mathbb{T}^d \to [0,1]$ such that for any $H \in C(\mathbb{T}^d)$

$$\lim_{N\to\infty} \langle \pi_0^N, H \rangle = \langle \rho_0, H \rangle,$$

then for any $t > 0$ and for any $H \in C(\mathbb{T}^d)$

$$\lim_{N\to\infty} \langle \pi_t^N, H \rangle = \langle \rho(t,\cdot), H \rangle,$$

where $\rho(t,u)$ is the unique solution to the heat equation

$$\begin{cases} \partial_t \rho = \Delta \rho, \\ \rho(0,\cdot) = \rho_0(\cdot). \end{cases}$$

Kipnis, C., & Landim, C. (1998). *Scaling limits of interacting particle systems* (Vol. 320). Springer Science & Business Media.

Why Diffusive Scaling?

- Central limit theorem. Let X_1, X_2, $\dots$, be i.i.d. random variables with mean zero and variance σ^2, then

$$\frac{1}{n}\sum_{i=1}^{n^2} X_i \Rightarrow \mathcal{N}(0,\sigma^2).$$

  A particle spends time N^2 moving a distance of order N.
- If the underlying random walk has non zero mean, we shall need Euler scaling, i.e., space divided by N and time speed up by N.

Why Heat Equation?

- We consider the one dimensional case $d = 1$.
- Let $\rho_N(t, u) = \mathbb{E}[\eta_{tN^2}([Nu])]$, then

$$\begin{aligned}\partial_t \rho_N(t, u) &= N^2 \Big(\mathbb{E}[\eta_{tN^2}([Nu] + 1)] - \mathbb{E}[\eta_{tN^2}([Nu])] \\ &\qquad + \mathbb{E}[\eta_{tN^2}([Nu] - 1)] - \mathbb{E}[\eta_{tN^2}([Nu])] \Big) \\ &= N^2 \big(\rho_N(t, u + 1/N) + \rho_N(t, u - 1/N) - 2\rho_N(t, u) \big).\end{aligned}$$

- Letting $N \to \infty$, we get the heat equation

$$\partial_t \rho(t, u) = \Delta \rho(t, u).$$

- Fix a time horizon $T > 0$. Let Q^N be the distribution of the empirical measure $\{\pi_t^N, 0 \leq t \leq T\}$.
- Prove that the measure Q^N is relatively compact. Prohorov's Theorem. See Theorem 4.1.3 in [Kipnis&Landim'98] for example.
- Once we have compactness, for any subsequence of $\{Q^N\}_{N \geq 1}$, there exists a further subsequence that converges. Denote the limit by Q^*.

- The limit Q^* concentrates on absolutely continuous trajectories,

$$Q^*\Big(\pi_t(du) = \rho(t,u)\,du,\ \forall 0 \le t \le T\Big) = 1.$$

To see this,

$$\Big|N^{-d} \sum_{x \in \mathbb{T}_N^d} H(x/N)\eta_t^N(x)\Big| \le N^{-d} \sum_{x \in \mathbb{T}_N^d} |H(x/N)| \to \int_{\mathbb{T}^d} |H(u)|\,du.$$

- The limit Q^* concentrates on trajectories whose densities are solutions to the heat equation.

- By Dynkin's martingale formula,

$$M_t^N(H) = \langle \pi_t^N, H \rangle - \langle \pi_0^N, H \rangle - \int_0^t N^2 L_N \langle \pi_s^N, H \rangle \, ds$$

is a martingale.

- The martingale vanishes as $N \to \infty$ in the sense

$$\lim_{N\to\infty} \mathbb{E}\Big[\sup_{0 \le t \le T} (M_t^N(H))^2 \Big] = 0.$$

We prove the last line by calculating the quadratic variation of the martingale.

- The term $N^2 L_N \langle \pi_s^N, H\rangle$ equals

$$
\begin{aligned}
&\frac{N^2}{N^d} \sum_{x\in\mathbb{T}_N^d} H(x/N) L_N \eta_s^N(x) \\
=&\frac{N^2}{N^d} \sum_{i=1}^{d} \sum_{x\in\mathbb{T}_N^d} H(x/N)\big(\eta_s^N(x+e_i) + \eta_s^N(x-e_i) - 2\eta_s^N(x)\big) \\
=&\frac{N^2}{N^d} \sum_{i=1}^{d} \sum_{x\in\mathbb{T}_N^d} \eta_s^N(x)\big(H((x+e_i)/N) + H((x-e_i)/N) - 2H(x/N)\big) \\
=&\langle \pi_s^N, \Delta_N H\rangle,
\end{aligned}
$$

where

$$
\Delta_N H(x/N) = N^2 \sum_{i=1}^{d} \big\{H((x+e_i)/N) + H((x-e_i)/N) - 2H(x/N)\big\}.
$$

- The above calculations permit us to close the martingale in the sense

$$M_t^N(H) = \langle \pi_t^N, H \rangle - \langle \pi_0^N, H \rangle - \int_0^t \langle \pi_s^N, \Delta_N H \rangle \, ds.$$

- Taking the limit $N \to \infty$,

$$0 = \langle \rho_t, H \rangle - \langle \rho_0, H \rangle - \int_0^t \langle \rho_s, \Delta H \rangle \, ds.$$

Step 3: Uniqueness

Uniqueness of weak solution to the heat equation $\Rightarrow$ Uniqueness of Q^*.

- In general, $L_N\langle \pi_s^N, H\rangle$ cannot be expressed as a function of the empirical measure. We need to prove extra replacement lemmas.
- GPV method or entropy method. Recall the entropy

$$H(\mu|\nu) = \int f \log f d\nu,$$

where $f = d\mu/d\nu$. See Chapter 5 in [Kipnis&Landim'98]. This method requires the uniqueness of the hydrodynamic equation.
- Yau's relative entropy method. See Chapter 6 in [Kipnis&Landim'98]. This method requires the existence and smoothness of the solution.

- State space $= [0,\infty)^{\mathbb{Z}^d}$. For $x \in \mathbb{Z}^d$, the value of $\eta(x)$ could be interpreted as the seriousness of disease of individual x.
- Fix $\lambda,\ \lambda_1,\ \lambda_2 \geq 0$. The dynamics is as follows:
  - (i) Recovery. $\eta_t(x) \to 0$ at rate 1;
  - (ii) Infection. $\eta_t(x) \to \eta_t(x) + \eta_t(x \pm e_i)$ at rate $\lambda \mp \lambda_1/N$.
  - (iii) ODE. If neither recovery nor infection,

$$\frac{d}{dt}\eta_t(x) = \big(1 - 2\lambda d + \lambda_2/N^2\big)\eta_t(x).$$

- Let $\lambda_1 = \lambda_2 = 0$. Consider the indicator function $\mathbf{1}\{\eta_t(x) > 0\}$. Then we obtain a version of the contact process.
- At site x, $1 \to 0$ at rate 1; $0 \to 1$ at rate $\lambda \sum_{|y-x|=1} \eta(y)$.
- There exists a critical value $\lambda_c(d)$ such that

$$\text{the disease} \begin{cases} \text{dies with probability one if } \lambda < \lambda_c; \\ \text{survives with positive probability if } \lambda > \lambda_c. \end{cases}$$

Griffeath, D. (1983). The binary contact path process. *Annals of Probability*, **11(3)**, 692-705.

- Diffusive scaling. Let $\eta_t^N = \eta_{tN^2}$. Consider the empirical measure

$$\pi_t^N(du) := \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \eta_t^N(x) \delta_{x/N}(du).$$

- Initial distribution. Let $\rho_0 : \mathbb{R}^d \to [0, \infty)$ be bounded and integrable. Assume the random variables $\{\eta_0^N(x),\, x \in \mathbb{Z}^d\}$ are independent. Moreover,

$$\mathbb{E}[\eta_0^N(x)] = \rho_0(x/N),\, \forall x, \quad \text{and} \quad \sup_{x \in \mathbb{Z}^d,\, N \geq 1} \mathbb{E}[\eta_0^N(x)^2] \leq +\infty.$$

- We have law of large numbers for the empirical measure at time zero.

Hydrodynamic Limit

Theorem, Xue&Z'20

Suppose $d \geq 3$ and $\lambda > \frac{1}{2d(2\gamma_d - 1)}$, where γ_d is the escape probability of a simple random walk on $\mathbb{Z}^d$, then the hydrodynamic equation is given by

$$\begin{cases} \partial_t \rho = \lambda \Delta \rho - 2\lambda_1 \sum_{i=1}^d \partial_{u_i} \rho + \lambda_2 \rho & \text{on } \{t > 0\} \times \mathbb{R}^d, \\ \rho(0, \cdot) = \rho_0(\cdot). \end{cases}$$

Remark.

Critical value of contact process $\lambda_c(d) < \frac{1}{2d(2\gamma_d - 1)}$. The best we could expect is $\lambda > \lambda_c(d)$.

Xue, X., & Zhao, L. (2020). Hydrodynamics of the weakly asymmetric normalized binary contact path process. *Stochastic Processes and their Applications*, **130(11)**, 6757-6782.

The coefficient $1 - 2\lambda d$ is important!

Remark

The system is NOT conserved. However, the expected number of total spins is roughly conserved.

Remark

(1) If we replace $1 - 2\lambda d$ by some constant $K_1 > 1 - 2\lambda d$, then the limit is $+\infty$;
(2)If we replace $1 - 2\lambda d$ by some constant $K_2 < 1 - 2\lambda d$, then the limit is zero.

- We consider the density fluctuation field

$$\mathcal{Y}_t^N(H) = \frac{1}{N^{1+d/2}} \sum_{x\in\mathbb{Z}^d} \Big(\eta_t^N(x) - \mathbb{E}[\eta_t^N(x)]\Big) H(x/N).$$

- Initial distribution. The random variables $\{\eta_0^N(x),\, x \in \mathbb{Z}^d\}$ are i.i.d., and

$$\sup_{N\geq 1} \mathbb{E}[\eta_0^N(x)^4] < +\infty.$$

Presutti, E., & Spohn, H. (1983). Hydrodynamics of the voter model. *The Annals of Probability*, 867-875.

Theorem, Xue&Z'21

Let $\lambda_2 = 0$. If d and λ are large enough, then the density fluctuation field, as a process, converges in distribution to a generalized Ornstein-Uhlenbeck process.

Holley, R. A., & Stroock, D. W. (1978). Generalized Ornstein-Uhlenbeck processes and infinite particle branching Brownian motions. *Publications of the Research Institute for Mathematical Sciences*, **14(3)**, 741-788.

Xue, X., & Zhao, L. (2021). Non-equilibrium fluctuations of the weakly asymmetric normalized binary contact path process. *Stochastic Processes and their Applications*, **135**, 227-253.

- Hydrodynamics. We use the theory of linear systems to prove the absolute continuity of the limiting path.
- Fluctuations. We need to bound the fourth moment. If we use linear systems' theory, we need to consider a $(\mathbb{Z}^d)^4 \times (\mathbb{Z}^d)^4$ matrix.
- Instead, we relate the fourth moment to a random walk.

Problems

- Large deviations of asymmetric systems.
- Fluctuations of asymmetric systems.
- Non equilibrium fluctuations.
- Universality of KPZ equation.

Chang, C. C., Landim, C., & Olla, S. (2001). Equilibrium fluctuations of asymmetric simple exclusion processes in dimension $d \geq 3$. *Probability Theory and Related Fields*, **119(3)**, 381-409.

Gonçalves, P., & Jara, M. (2014). Nonlinear fluctuations of weakly asymmetric interacting particle systems. *Archive for Rational Mechanics and Analysis*, **212(2)**, 597-644.

Jara, M., & Menezes, O. (2018). Non-equilibrium fluctuations of interacting particle systems. *arXiv preprint arXiv:1810.09526.*

Thanks!