Mapping Hydrodynamics for the Facilitated Exclusion and Zero-range Processes

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Notation and Results

Formal Proof

Mapping

Introduction

- Hydrodynamic limit: understanding macroscopic behaviors, usually described by PDEs via a long-time and large-space scaling limit, of microscopic systems.
- Symmetric exclusion: heat equation. Asymmetric exclusion: conservation law.
- Goal: deriving Stefan problems from interacting particle systems.

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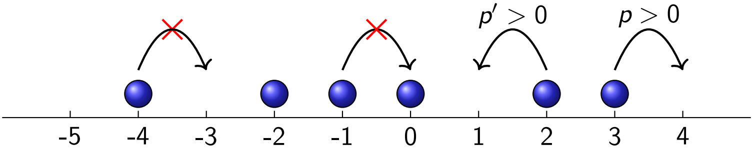
Facilitated Exclusion Process

The state space is $\Sigma_N := \{0,1\}^{\mathbb{L}_N}$, where $\mathbb{L}_N = \mathbb{Z}$ or $\mathbb{T}_N$. For a configuration $\eta = (\eta_x)_{x \in \mathbb{L}_N} \in \Sigma_N$, $\eta_x = 1$ if and only if x is occupied by a particle.



- Exclusion rule: there is at most one particle at each site.
- Facilitated rule: a particle has to be pushed forward in order to jump.
- Symmetric: $p = p'$, $\mathbb{L}_N = \mathbb{T}_N$. Asymmetric: $p > p'$, $\mathbb{L}_N = \mathbb{Z}$.

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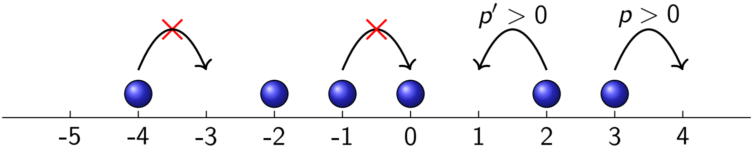

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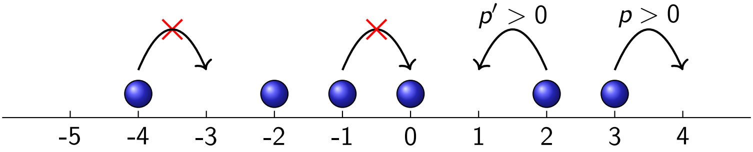

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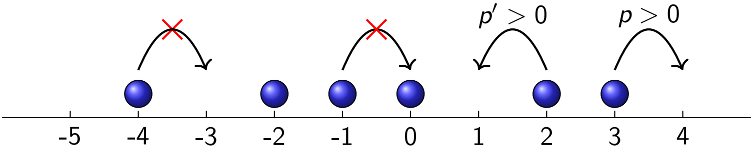

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Facilitated Exclusion Process

- For $x, y \in \mathbb{L}_N$, $\eta^{x,y}$denotes the configuration obtained from η by swapping the values at sites x and y.
- The jump rate is given by

$$c_{x,x+1}(\eta) = p\eta_{x-1}\eta_x(1-\eta_{x+1}) + p'(1-\eta_x)\eta_{x+1}\eta_{x+2}.$$

- For any local function $f\colon \Sigma_N \to \mathbb{R}$, the infinitesimal generator is

$$\mathscr{L}_N^{\mathrm{FEX}} f(\eta) := \sum_{x\in\mathbb{L}_N} c_{x,x+1}(\eta)\left(f\left(\eta^{x,x+1}\right) - f(\eta)\right).$$

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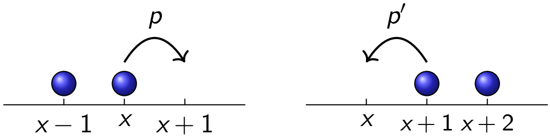

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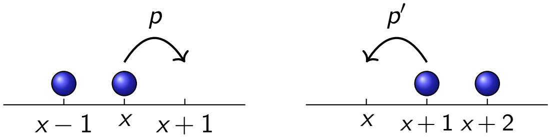

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Phase Transitions

The process displays a phase transition at the critical particle density $\rho_c = 1/2$.

- If initially $\rho > 1/2$, then the system evolves until there are no longer two neighboring empty sites.

$t = 0$

$t > 0$:ergodic

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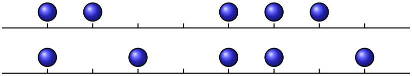

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  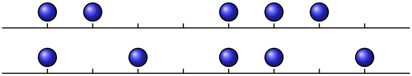

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  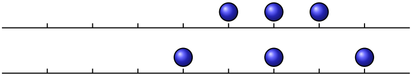

Invariant Measures π_ρ on $\{0,1\}^{\mathbb{Z}}$

- Fix $\rho > 1/2$. For some $\sigma = (\sigma_x, 1 \le x \le \ell)$ where $\sigma_x \in \{0,1\}$ and $(\sigma_x, \sigma_{x+1}) \neq (0,0)$, let $k = \sum_{x=1}^{\ell} \sigma_x$.

$$\pi_\rho\left(\eta : \eta_x = \sigma_x, \forall 1 \le x \le \ell\right) = (1-\rho)\left(\frac{1-\rho}{\rho}\right)^{\ell-1-k}\left(\frac{2\rho-1}{\rho}\right)^{2k-\ell+1-\sigma(1)-\sigma(\ell)}$$

- Put a hole in some position with probability $1-\rho$, then put a random geometric number of parameter $\frac{1-\rho}{\rho}$ particles to its right, then a hole, starts again and so on.

$$\pi_\rho(11) = \pi_\rho(1) - \pi_\rho(01) = \rho - (1-\rho)\times 1 = 2\rho - 1.$$

$$\pi_\rho(111) = \pi_\rho(11) - \pi_\rho(011) = (2\rho-1) - (1-\rho)\times 1 \times \frac{2\rho-1}{\rho} = \frac{(2\rho-1)^2}{\rho}.$$

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Hydrodynamic Limit

- If at the initial time, there exists $\rho^{\mathrm{ini}} : \mathbb{T}$ (or $\mathbb{R}$) $\to \mathbb{R}$

  $$\eta_x(0) \approx \rho^{\mathrm{ini}}(\tfrac{x}{N}),$$

  then, at any time $t > 0$,

  $$\eta_x(t\theta(N)) \approx \rho(t, \tfrac{x}{N})$$

  for some $\rho(t, u)$ with initial condition ρ^{ini}.

- $$\theta(N) = \begin{cases} N^2 \text{ (diffusive scaling)}, & \text{in the symmetric case;} \\ N \text{ (hyperbolic scaling)}, & \text{in the asymmetric case.} \end{cases}$$

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$p = p' = 1$

- Space/Time scaling:

	Macroscopic	Microscopic
Space	$u \approx \frac{x}{N}$	x
Time	t	tN^2

- Free boundary problem:

$$\begin{cases} \partial_t \rho = \partial_u^2 \left(\frac{2\rho - 1}{\rho} \mathbf{1}_{\{\rho > 1/2\}} \right) & \text{in } \{t > 0\} \times \mathbb{T}, \\ \rho(0, \cdot) = \rho^{\text{ini}} & \text{in } \mathbb{T}. \end{cases}$$

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- Define the empirical measure $\pi_t^N(du)$, which is a random measure on $\mathbb{T}$, as

$$\pi_t^N(du) = \frac{1}{N} \sum_{x \in \mathbb{T}_N} \eta_x(tN^2) \delta_{x/N}(du).$$

- For a sequence of measures $\{\mu_N\}_{N \geq 1}$, μ on $\mathbb{T}$ with total mass bounded by one,

$$\lim_{N \to \infty} \mu_N = \mu$$

if for any continuous function $f \colon \mathbb{T} \to \mathbb{R}$,

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Theorem [Blondel *et al.* '20, '21, Erignoux, Simon and Z. '22]

Under mild conditions on the initial density profile ρ^{ini}, if

$$\lim_{N\to\infty} \pi_0^N(du) = \rho^{\mathrm{ini}}(u)du \quad \text{in probability},$$

then, for any time $t > 0$,

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where $\rho(t,u)$ is the unique weak solution to

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$p = 1 - p' \in (1/2, 1]$

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- Hyperbolic free boundary problem:

$$\begin{cases} \partial_t \rho + (2p-1)\partial_u\left(\frac{(1-\rho)(2\rho-1)}{\rho}\mathbf{1}_{\{\rho>1/2\}}\right) = 0 & \text{in } \{t>0\}\times\mathbb{R}, \\ \rho(0,\cdot) = \rho^{\mathrm{ini}} & \text{in } \mathbb{R}. \end{cases}$$

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Entropy solution

Let $\mathfrak{H}(\rho) = \frac{(1-\rho)(2\rho-1)}{\rho}\mathbf{1}_{\{\rho>1/2\}}$. An entropy solution satisfies

- *(entropy inequality)* for any non-negative test function $\varphi \in C^{1,1}(\mathbb{R}_+ \times \mathbb{R})$ with compact support in $(0,\infty) \times \mathbb{R}$, for any $0 \le c \le 1$,

$$\int_0^\infty \langle |\rho_t - c|, \partial_t \varphi_t \rangle + (2p-1)\langle \mathfrak{q}(\rho_t; c, \mathfrak{H}), \partial_u \varphi_t \rangle dt \ge 0,$$

where $\mathfrak{q}(\rho; c, \mathfrak{H}) = \mathrm{sign}(\rho - c)(\mathfrak{H}(\rho) - \mathfrak{H}(c))$;

- *(initial condition)* for any $A > 0$,

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Formal Proof

Mapping

$p = p' = 1$

- First recall

$$\rho(t, \tfrac{x}{N}) \approx \mathbb{E}[\eta_x(tN^2)], \quad x \in \mathbb{T}_N,$$

or equivalently,

$$\rho(t, u) \approx \mathbb{E}[\eta_{uN}(tN^2)], \quad u \in \mathbb{T}.$$

- Conservation law

$$\frac{d}{dt}\mathbb{E}[\eta_{uN}(tN^2)] = N^2\Big(\mathbb{E}[j_{uN-1,uN}(tN^2)] - \mathbb{E}[j_{uN,uN+1}(tN^2)]\Big),$$

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- Gradient condition

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where

$$h_x(t) = \eta_{x-1}(t)\eta_x(t) + \eta_x(t)\eta_{x+1}(t) - \eta_{x-1}(t)\eta_x(t)\eta_{x+1}(t).$$

- Therefore,

$$\partial_t \rho(t,u) \approx N^2\Big(\mathbb{E}[h_{uN-1}(tN^2)] + \mathbb{E}[h_{uN+1}(tN^2)] - 2\mathbb{E}[h_{uN}(tN^2)]\Big).$$

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- Local ergodicity: the distribution of the process at time tN^2 around site uN is approximately $\pi_{\rho(t,u)}$.
- For $\rho(t,u) > 1/2$,

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Notation and Results

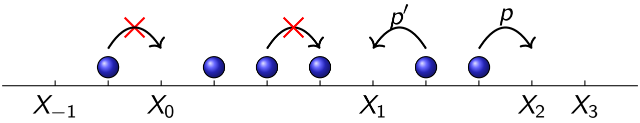
Formal Proof

Mapping

Facilitated Zero Range Process

- In the exclusion process, label the empty sites from the left to the right in an increasing order. Let $X_y(t)$ be the position of the y–th empty site at time t.

EP:



- Let $\omega_y(t) \in \{0, 1, 2, \ldots\}$ be the number of particles between the y–th and $(y+1)$–th empty sites at time t.

ZRP:

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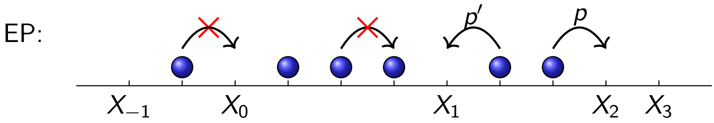

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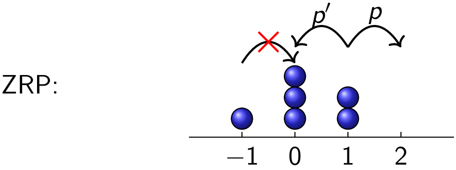
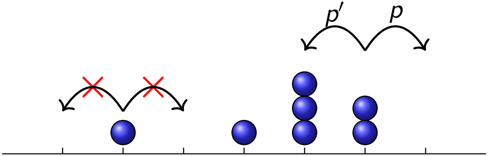


Facilitated Zero Range Process

- If there are at least two particles at a site, then one of them jumps to the right (resp. to the left) at rate p (resp. p').



- The critical particle density is $\alpha_c = 1$.
- The process has a family of product invariant measures when the particle density $\alpha > 1$.
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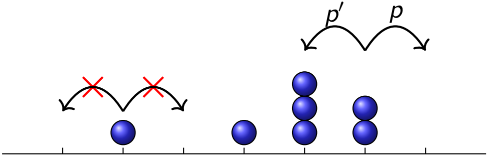

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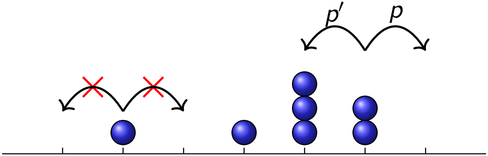

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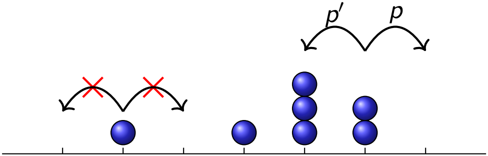

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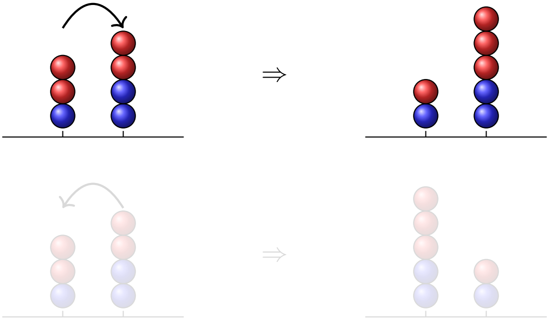

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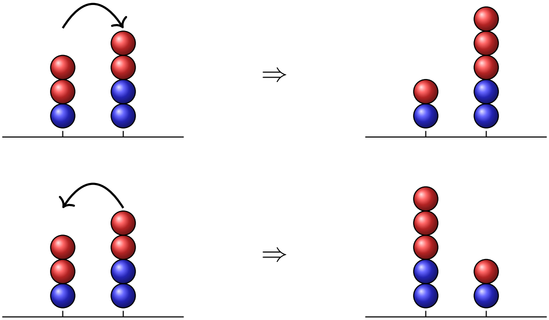

Proof Outline

Facilitated Exclusion $\xleftrightarrow{\text{microscopic mapping}}$ Facilitated Zero Range

HL???
non-ergodic.
non-attractive.
invariant measures
not product.

$$\mathcal{G}(r) = \tfrac{r-1}{r}\mathbf{1}_{\{r>1\}}$$
$$\mathcal{H}(r) = \tfrac{2r-1}{r}\mathbf{1}_{\{r>1/2\}}$$
$$\mathfrak{H}(r) = (1-r)\mathcal{H}(r)$$

HL ✓
by attractiveness

(S) $\partial_t\rho = \partial_u^2\mathcal{H}(\rho)$
(A) $\partial_t\rho + (2p-1)\partial_u\mathfrak{H}(\rho) = 0$

macroscopic mapping

(S) $\partial_t\alpha = \partial_v^2\mathcal{G}(\alpha)$
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Macroscopic Mapping

- Exclusion η_x, $\rho(t,u)$; Zero range ω_y, $\alpha(t,v)$.
- Consider the process on $\mathbb{Z}$. For any $\varphi \in C_c(\mathbb{R})$,

$$\frac{1}{N}\sum_{x\in\mathbb{Z}}[1-\eta_x(tN)]\varphi(x/N) = \frac{1}{N}\sum_{y\in\mathbb{Z}}\varphi(X_y(tN)/N).$$

- Note that

$$X_y = \sum_{y'=1}^{y}[X_{y'} - X_{y'-1}] + X_0 = \sum_{y'=1}^{y}[1+\omega_{y'-1}] + X_0.$$

- Let $\sigma_t = \lim_{N\to\infty} X_0(tN)/N$,

$$\int (1-\rho(t,u))\varphi(u)du = \int \varphi\Big(\int_0^v (1+\alpha(t,v'))dv' + \sigma_t\Big)dv$$

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- Problem: the solutions to the Stefan problem and hyperbolic equation are not smooth.
- For the Stefan problem

$$\partial_t \rho = \partial_u^2 \mathcal{H}(\rho),$$

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Thanks!

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