Exclusion Processes and Normalized Binary Contact Path Processes: Ergodic Properties and Large Scale Behaviors

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Contact Processes

- For each $x \in \mathbb{Z}^d$, $\eta(x) \in \{0, 1\}$.
- $\eta(x) = 1$ means x is infected, and $\eta(x) = 0$ means x is healthy.
- At site x,

$$1 \to 0 \quad \text{at rate } 1, \quad \text{while} \quad 0 \to 1 \quad \text{at rate } \lambda \sum_{y \sim x} \eta(y).$$

Contact Processes

- Introduced by Harris (1974) to describe the spread of a disease.
- There exists a critical value λ_c such that

$$\mathbb{P}^O(\text{the disease survives}) \begin{cases} = 0, & \text{if} \quad \lambda < \lambda_c, \\ > 0, & \text{if} \quad \lambda > \lambda_c. \end{cases}$$

Harris, T. E. (1974). Contact interactions on a lattice. *The Annals of Probability* **2**, 969-988.

Normalized Binary Contact Path Processes

- Introduced by Griffeath (1983) as an auxiliary model to study the critical value.
- For each $x \in \mathbb{Z}^d$, $\eta(x) \in [0, \infty)$.
- Interpret $\eta(x)$ as the seriousness of the disease at site x.

Griffeath, D. (1983). The binary contact path process. *The Annals of Probability* **11**, 692-705.

Normalized Binary Contact Path Process

The dynamics is as follows:

- if $\eta_t(x) > 0$, then $\eta_t(x) \to 0$ at rate 1.
- if $y \sim x$, then $\eta_t(x) \to \eta_t(x) + \eta_t(y)$ at rate λ.
- when there is neither recovery nor infection for x during some time interval, then

$$\frac{d}{dt}\eta_t(x) = (1 - 2\lambda d)\eta_t(x).$$

Normalized Binary Contact Path Processes

- The indicator function $\left\{\mathbf{1}\{\eta_t(x) > 0\}, x \in \mathbb{Z}^d\right\}$ is a version of the contact process.
- Griffeath (1983) proved that if $d \geq 3$, then

$$\lambda_c \leq \frac{1}{2d(2\gamma_d - 1)},$$

where γ_d is the escape probability that the simple symmetric random walk on $\mathbb{Z}^d$ starting at O never returns to O again.

Normalized Binary Contact Path Processes

- Closely related to the contact process.
- Driven by the heat equation (symmetric exclusion processes, the voter model).
- Belong to the linear system.

Liggett, T. M. (1985). *Interacting Particle Systems.* Springer, New York.

Presutti, E. and Spohn, H. (1983). Hydrodynamics of the voter model. *The Annals of Probability*, 867-875.

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Hydrodynamics

- Diffusive scaling. For each $N \geq 1$, let $\{\eta_t^N, t \geq 0\}$ be a version of the normalized binary contact path process with time speed up by N^2.
- Define the empirical measure $\pi_t^N(du)$ as

$$\pi_t^N(du) = \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \eta_t^N(x) \delta_{x/N}(du).$$

- $\pi_t^N(du)$ is a random measure on $\mathbb{R}^d$.

Theorem 1 (LLN), Xue and Z. (2019)

Let $\rho_0 : \mathbb{R}^d \to [0,\infty)$ be bounded and integrable. Assume that for each $N \geq 1$, $\{\eta_0^N(x), x \in \mathbb{Z}^d\}$ are independent,

$$\mathbb{E}\left[\eta_0^N(x)\right] = \rho_0\left(\frac{x}{N}\right), \forall x, \quad \text{and} \quad \sup_{x\in\mathbb{Z}^d, N\geq 1} \mathbb{E}\left[\eta_0^N(x)^2\right] < \infty.$$

Furthermore, suppose $d \geq 3$ and

$$\lambda > \frac{1}{2d(2\gamma_d - 1)}.$$

Theorem 1 (LLN), Xue and Z. (2019)

Then for all $t \geq 0$, under the vague topology, as $N \to \infty$,

$$\pi_t^N(du) \to \rho(t,u)du \text{ in probability,} \tag{1}$$

where $\rho(t,u)$ is the unique solution of the heat equation

$$\begin{cases} \partial_t \rho(t,u) = \lambda \Delta \rho(t,u), \\ \rho(0,u) = \rho_0(u). \end{cases} \tag{2}$$

Theorem 2 (LLN), Xue and Z. (2019)

Under the initial conditions in Theorem 1, if $\lambda < \lambda_c(d)$, $d \geq 1$, then as $N \to \infty$,

$$\pi_t^N(du) \to 0du \quad \text{in probability.}$$

Bezuidenhout, C. and Grimmett, G. (1991). Exponential decay for subcritical contact and percolation processes. *The Annals of Probability* **19**, 984-1009.

Remark

(i) The initial assumption ensures that

$$\pi_0^N(du) \to \rho_0(u)du \quad \text{in probability as } N \to \infty.$$

(ii) Is there a $\tilde{\lambda}_c$ such that for $\lambda < \tilde{\lambda}_c$ the limit is zero and for $\lambda > \tilde{\lambda}_c$ nonzero? Furthermore, $\tilde{\lambda}_c = \lambda_c$?

(iii) The correct scaling for $\lambda < \lambda_c$?

(iv) $d \geq 3$, $\lambda_c(d) \leq \lambda \leq (2d(2\gamma_d - 1))^{-1}$ or $d \leq 2$, $\lambda \geq \lambda_c(d)$?

(v) Hydrodynamics of contact processes?

Fluctuations

- The density fluctuation filed $\mathcal{Y}_t^N$, $N \geq 1$, is an $\mathcal{S}'(\mathbb{R}^d)$-valued process defined as

$$\mathcal{Y}_t^N(H) = \frac{1}{N^{1+d/2}} \sum_{x \in \mathbb{Z}^d} \left(\eta_t^N(x) - \mathbb{E}\left[\eta_t^N(x) \right] \right) H\left(\frac{x}{N} \right),$$

where $H \in \mathcal{S}(\mathbb{R}^d)$.

- The $N^{1+d/2}$ scaling is also observed in the voter model.

Presutti, E. and Spohn, H. (1983). Hydrodynamics of the voter model. *The Annals of Probability*, 867-875.

Fluctuations

Under suitable initial conditions, we **guess** that if $d \geq 3$ and $\lambda > (2d(2\gamma_d - 1))^{-1}$, then the sequence of the processes $\{\mathcal{Y}_t^N, 0 \leq t \leq T\}_{N \geq 1}$ converges to the formal solution of

$$\begin{cases} d\mathcal{Y}_t = \lambda \Delta \mathcal{Y}_t dt + \sqrt{C(\lambda, d)} d\mathcal{W}_t, \\ \mathcal{Y}_0 = 0, \end{cases}$$

where $\mathcal{W}_t$ is a space time white noise of unit variance.

Fluctuations

➢ The problem is to prove the uniform boundedness of the fourth moment,

$$\sup_{N\geq 1,\,0\leq t\leq T} \mathbb{E}\left[\eta_t^N(x)^4\right] < \infty.$$

➢ Nagahata and Yoshida (2009) considered the fluctuations of the process when the initial configuration is finite.

Nagahata, Y. and Yoshida, N. (2009). Central limit theorem for a class of linear systems. *Electronic Journal of Probability*, **14**, 960-977.

Hydrodynamics

- Establish the partial differential equations that describe the evolution of the thermodynamic characteristics of a fluid.
- Hamiltonian systems where particles evolve deterministically according to Newton's equations (Hard).
- Two simplifications: (1) assume the evolution of the microscopic system to be stochastic, (2) or consider systems with low density of particles.

Hydrodynamics

- Entropy method: Guo, M. Z., Papanicolaou, G. C. and Varadhan, S. R. S. (1988).
- Relative entropy method: Yau, H. T. (1991).

Guo, M. Z., Papanicolaou, G. C. and Varadhan, S. R. S. (1988). Nonlinear diffusion limit for a system with nearest neighbor interactions. *Communications in Mathematical Physics*, **118**, 31-59.

Yau, H. T. (1991). Relative entropy and hydrodynamics of Ginzburg-Landau models. *Letters in Mathematical Physics*, **22**, 63-80.

Hydrodynamics

- De Masi, A. and Presutti, E. (1991). *Mathematical methods for hydrodynamic limits*. Lecture Notes in Mathematics.
- Kipnis, C. and Landim, C. (1999) *Scaling limits of interacting particle systems.* Springer-Verlag, Berlin.
- Spohn, H. (2012). *Large scale dynamics of interacting particles.* Springer Science & Business Media.

Outline

By Dynkin's martingale formula,

$$M_t^N(G) := \langle \pi_t^N, G \rangle - \langle \pi_0^N, G \rangle - \int_0^t N^2 \mathcal{L} \langle \pi_s^N, G \rangle ds$$

is a martingale. A direct calculation yields that

$$N^2 \mathcal{L} \langle \pi_s^N, G \rangle = \frac{\lambda}{N^d} \sum_{x \in \mathbb{Z}^d} \eta_s^N(x) \Delta_N G\left(\frac{x}{N}\right) = \lambda \langle \pi_s^N, \Delta_N G \rangle,$$

where $\Delta_N G(x/N) = N^2 \sum_{|y-x|=1} [G(y/N) - G(x/N)]$.

Outline

Finally, we have

$$M_t^N(G) := \langle \pi_t^N, G \rangle - \langle \pi_0^N, G \rangle - \int_0^t \lambda \langle \pi_s^N, \Delta_N G \rangle ds.$$

Suppose $\pi_t^N(du) \to \rho(t,u)du$ as $N \to \infty$, then

$$0 = \langle \rho_t, G \rangle - \langle \rho_0, G \rangle - \int_0^t \lambda \langle \rho_s, \Delta G \rangle ds.$$

Outline

- ➢ Vanish of the Martingale in probability.
- ➢ Tightness of the sequence $\{\pi_t^N, 0 \leq t \leq T\}_{N \geq 1}$.
- ➢ Absolute Continuity of the limit.
- ➢ Uniqueness of the solution to the heat equation.

Absolute Continuity

The absolute continuity in other models is either trivial (exclusion processes) or can be proved by using the entropy inequality (zero range processes).

Absolute Continuity

First, note that

$$\mathrm{E}\left[\langle \pi_t, G\rangle\right] \leq \liminf_{N\to\infty} \mathbb{E}\left[\left\langle \pi_t^N, |G|\right\rangle\right] \leq ||\rho_0||_\infty \int_{\mathbb{R}^d} |G(u)|\, du.$$

Next, we prove

$$\lim_{N\to+\infty} \mathrm{Var}\big(\langle \pi_t^N, G\rangle\big) = 0,$$

where we use properties of the linear system.

Totally Asymmetric Simple Exclusion Processes (TASEP)

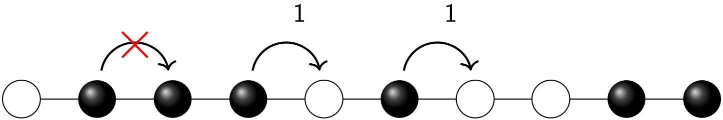


Figure: At most one particle per site.

Spitzer, F. (1970) Interaction of Markov processes. *Advances in Math*. **5**, 246–290.

Liggett, T. M. (1985). *Interacting particle systems*. Springer Science & Business Media.

Liggett, T. M. (1999). *Stochastic interacting systems: contact, voter and exclusion processes*. springer science & Business Media.

Facilitated TASEP (FTASEP)

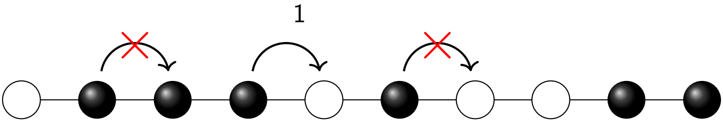


Figure: A particle jumps if its left neighbor is occupied.

Facilitate Exclusion Processes

- Phase transition in the presence of a conserved field.
- A new universality class of non-equilibrium phase transitions.
- Motion in glasses.

Basu, U., & Mohanty, P. K. (2009). Active–absorbing-state phase transition beyond directed percolation: A class of exactly solvable models. *Physical Review E*, 79(4), 041143.

Gabel, A., Krapivsky, P. L., & Redner, S. (2010). Facilitated asymmetric exclusion. *Physical review letters*, 105(21), 210603.

Theorem 3 (Invariant measures), Chen and Z. (2019)

Consider the FTASEP on $\mathbb{Z}$.

- For $1/2 < \rho < 1$, the FTASEP has a family of (spatial) ergodic non-degenerate invariant measures having density ρ.
- For $0 < \rho \leq 1/2$, the FTASEP does **NOT** have (spatial) ergodic non-degenerate invariant measures having density ρ.

Degenerate configurations: $\ldots 010001010100 \ldots$

Bernoulli Product Initial Distributions

Let ν_ρ, $\rho \in (0,1)$, be the product measure on $\{0,1\}^{\mathbb{Z}}$ with marginals given by

$$\nu_\rho\{\eta : \eta(x) = 1\} = \rho, \quad \text{for all } x \in \mathbb{Z}.$$

Theorem 4 (Convergence theorem), Chen and Z. (2019)

For the FTASEP on $\mathbb{Z}$,

$$\lim_{t\to\infty} \nu_\rho S(t) = \begin{cases} \pi_\rho & \text{if } \rho < 1/2, \\ \frac{1}{2}\delta_{\boldsymbol{\eta}^1} + \frac{1}{2}\delta_{\boldsymbol{\eta}^0} & \text{if } \rho = 1/2. \end{cases} \tag{3}$$

$\rho = 1/2, \ \ldots 10101010 \ldots$

$\rho < 1/2, \ \ldots 0000\,10101\,00000000\,1010101\,000 \ldots$

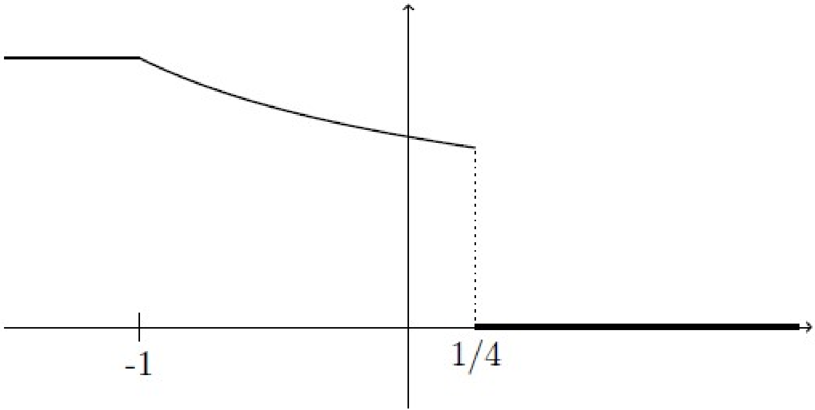
Theorem 5 (Hydrodynamics), Z. (2019+)

If the FTASEP starts from the **step configuration** $\ldots 11110000 \ldots$, then under the **Euler scaling**, i.e., with time speeded up by N and space divided by N, then the hydrodynamic equation satisfies the **conservation law**:

$$\partial_t \rho + \partial_u \left(\frac{(1-\rho)(2\rho-1)}{\rho} \right) = 0$$

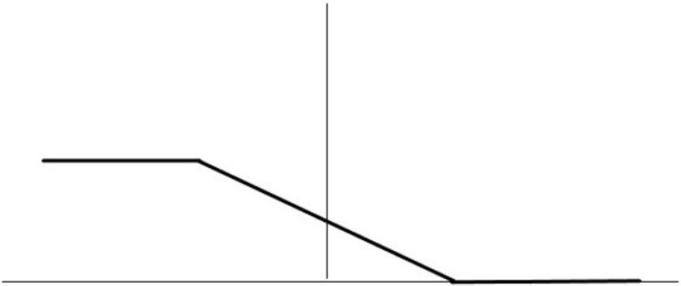
with initial condition $\rho(0,u) = \mathbf{1}\{u \leq 0\}$.

General initial distributions?



Baik, J., Barraquand, G., Corwin, I., & Suidan, T. (2016). Facilitated exclusion process. In *The Abel Symposium* (pp. 1-35). Springer, Cham.

TASEP



Rost, H. (1981). Non-equilibrium behaviour of a many particle process: Density profile and local equilibria. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 58(1), 41-53.

Future Research

- Kinetically constrained models (IPS with degenerate rates). Related to glassy dynamics.
- Asymmetric systems: fluctuations, large deviations...

Blondel, O., Erignoux, C., Sasada, M., & Simon, M. (2018). Hydrodynamic limit for a facilitated exclusion process. arXiv preprint arXiv:1805.09000.

Ritort, F., & Sollich, P. (2003). Glassy dynamics of kinetically constrained models. *Advances in Physics*, **52(4)**, 219-342.

Publication List

- Chen, D. , & Zhao, L. (2019). The invariant measures and the limiting behaviors of the facilitated TASEP. *Statistics & Probability Letters*, 154, 108557.

- Xue, X., & Zhao, L. (2019). Hydrodynamics of the Binary Contact Path Process. *arXiv preprint arXiv:1901.04660*.

- Franco, T., Gonçalves, P., Marinho, R. and & Zhao, L. (2019+). Scaling limits for the SSEP with a slow site. *In preparation.*

Thanks!