STAT 583 Lecture 4

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Feb. 11th

Misc

- Midterm is scheduled at 6:50-8:50pm on March 3rd.
- TA: Youmeng Jiang
  - Office hours: 2:00-3:30pm every Monday.

Last time

- Parameter estimation for population mean μ and binomial parameter p
- CI for μ
  - Large n: z-interval
  - Small n: we need to assume the sample is normally distributed
    - when σ is unknown, use Student's t-distribution
    - when σ is known, use normal distribution
- CI for p
  - Large n: z-interval
- CI for $p_1 - p_2$, $\mu_X - \mu_Y$

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- Sometimes it appears we have data from two samples with the further feature that there is a natural "pairing" of the data between the two samples.
- For example, suppose that the data consists on n brother-sister pairs, with blood pressures $X_1, ..., X_n$ for the n sisters and blood pressures $Y_1, ..., Y_n$ for their respective brothers.

CI for the difference between two means

Let's first consider the case where n is small.

- If $X_1, X_2, ..., X_n$ i.i.d. $\sim N(\mu_X, \sigma_X^2)$, $Y_1, Y_2, ..., Y_n$ i.i.d. $\sim N(\mu_Y, \sigma_Y^2)$. In the paired samples' case, it's not realistic to assume that $\{X_i\}$ and $\{Y_i\}$ are independent.

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- We estimate $\mu_X - \mu_Y$ by $\bar{D} = \bar{X} - \bar{Y}$
- A $(1-\alpha)$ confidence interval for $\mu_X - \mu_Y$ is

$$[\bar{D} - t_{\alpha/2, n-1}\frac{s_D}{\sqrt{n}}, \quad \bar{D} + t_{\alpha/2, n-1}\frac{s_D}{\sqrt{n}}],$$

where $s_D = \sqrt{\frac{(D_1 - \bar{D})^2 + \ldots + (D_n - \bar{D})^2}{n}}$.

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  where $s_D = \sqrt{\frac{(D_1-\bar{D})^2+\ldots+(D_n-\bar{D})^2}{n}}$.
- If n is large, we can simply replace $t_{\alpha/2,n-1}$ with $z_{\alpha/2}$ and remove the normality assumptions.

Example

Suppose that we take the blood pressures of $n = 12$ women and their brothers, and get the following blood pressure reading:

Family	1	2	3	4	5	6	7	8	9	10	11	12
Sister	107	134	111	141	121	118	145	110	164	126	148	132
Brother	110	136	115	140	124	119	148	113	168	129	148	137

Construct the 95% CI for the difference of the mean blood pressure of men and women. ($s^2_{sister} = 307$, $s^2_{brother} = 299$, $s^2_{diff} = 3$, $t_{0.025,11} = 2.306$, $t_{0.05,11} = 1.860$, $t_{0.025,12} = 2.262$, $t_{0.05,12} = 1.833$)

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Answer

Family	1	2	3	4	5	6	7	8	9	10	11	12
Difference	3	2	4	-1	3	1	3	3	4	3	0	5

- $\bar{D} = 2.5,\ s^2_D = 3,\ t_{0.025,11} = 2.306$

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- $\bar{D} = 2.5,\ s^2_D = 3,\ t_{0.025,11} = 2.306$
- $2.5 \pm 2.306 \cdot \frac{\sqrt{3}}{\sqrt{12}} = [1.35, 3.65]$.

Overview

- General confidence intervals construction based on pivotal quantities
- General confidence intervals construction based on maximum likelihood estimators

Other confidence intervals

- We have now talked about CI for mean parameters: $\mu, p, \mu_1 - \mu_2, p_1 - p_2$

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  - Let $U = \frac{Y}{\theta}$, we have $f_U(u) = f_Y(u \cdot \theta) \cdot \frac{dy}{du} = \frac{1}{\theta} e^{-u} \cdot \theta = e^{-u}$ for $u > 0$, that is, $U \sim Exp(1)$

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  - Thus, we can use $U = \frac{Y}{\theta}$ as a pivotal quantity

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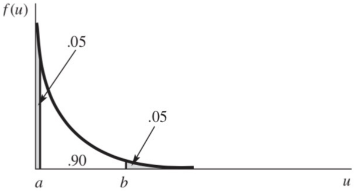

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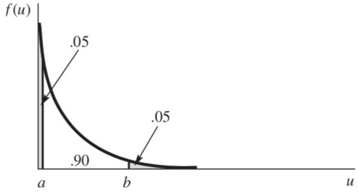

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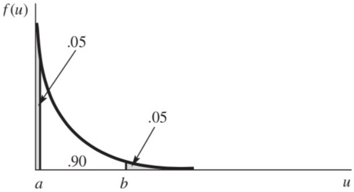

- $1 - e^{-a} = .05$ and $e^{-b} = .05$, equivalently, $a = 0.051$, $b = 2.996$
- $0.9 = \mathbb{P}(0.051 \leq \frac{Y}{\theta} \leq 2.996) = \mathbb{P}(\frac{Y}{2.996} \leq \theta \leq \frac{Y}{0.051})$

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- $U = \min\{\frac{X_1}{\theta}, \ldots \frac{X_{10}}{\theta}\} \sim Exp(\frac{1}{10})$

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- Find a, b such that $\mathbb{P}(U < a) = 1 - e^{-10a} = .05$ and $\mathbb{P}(U > b) = e^{-10b} = .05$, equivalently, $a = 0.005,\ b = 0.300$
- $0.9 = \mathbb{P}(0.005 < U < 0.300) = \mathbb{P}(0.005 < \frac{\min\{X_1,...,X_{10}\}}{\theta} < 0.300) = \mathbb{P}(\frac{\min\{X_1,...,X_{10}\}}{0.300} < \theta < \frac{\min\{X_1,...,X_{10}\}}{0.005})$

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Suppose that we take a sample Y from a uniform distribution defined on the interval $[0, \theta]$, where θ is unknown. Find a 95% lower confidence bound for θ, that is, find $L(Y)$, such that $\mathbb{P}(\theta \geq L(Y)) = 0.95$.

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- $U \sim U(0, 1)$
- $\mathbb{P}(U < 0.95) = 0.95$

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- $U \sim U(0, 1)$
- $\mathbb{P}(U < 0.95) = 0.95$
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Let $Y_1, ..., Y_n$ denote a random sample of size n from $U(0, \theta)$. Let $Y_{(n)} = \max\{Y_1, Y_2, ..., Y_n\}$. Find a 95% lower confidence bound for θ.

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- Consider $U = \frac{Y_{(n)}}{\theta}$
- U has the cdf
  $F(u) = \mathbb{P}(\frac{\max\{Y_1, Y_2, \ldots, Y_n\}}{\theta} \leq u) = \mathbb{P}(\frac{Y_1}{\theta} \leq u, ..., \frac{Y_n}{\theta} \leq u) = \mathbb{P}(\frac{Y_1}{\theta} \leq u)^n$

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- $F(u) = u^n$ for $u \in [0, 1]$.
- $0.95 = \mathbb{P}(U < (0.95)^{1/n}) = \mathbb{P}(\frac{Y_{(n)}}{\theta} < (0.95)^{1/n}) = \mathbb{P}(\frac{Y_{(n)}}{(0.95)^{1/n}} \leq \theta)$

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- $\frac{Y_{(n)}}{(0.95)^{1/n}} = \frac{5.7}{(0.95)^{1/10}} = 5.73$

Confidence interval for the variance

- Suppose $Y_1, ..., Y_n \overset{i.i.d.}{\sim} N(\mu, \sigma^2)$ with μ and σ^2 unknown. We seek a $100(1-\alpha)\%$ CI for σ^2

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- Recall that, for $S^2 = \frac{1}{n-1}\sum_{i=1}^n (Y_i - \bar{Y})^2$, we have the pivotal quantity
$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}.$$

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$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}.$$
- As a result,
$$\mathbb{P}(\chi^2_{1-(\alpha/2),n-1} \leq \frac{(n-1)S^2}{\sigma^2} \leq \chi^2_{\alpha/2,n-1}) = 1-\alpha$$

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- Recall that, for $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$, we have the pivotal quantity
$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}.$$
- As a result,
$$\mathbb{P}(\chi^2_{1-(\alpha/2),n-1} \leq \frac{(n-1)S^2}{\sigma^2} \leq \chi^2_{\alpha/2,n-1}) = 1-\alpha$$
- A $100(1-\alpha)\%$ Confidence Interval for σ^2:
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- The 90% CI is $[6.11, 24.55]$.

Maximum likelihood estimator

- Another general and important method to construct point estimation and confidence intervals are maximum likelihood estimator (MLE)

Likelihood function

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- By the i.i.d. assumption, we have

$$f(y_1, y_2, ..., y_n|\theta) = \Pi_{i=1}^{n} f(y_i \mid \theta)$$

Likelihood function

- We could also view $f(y_1, y_2, ..., y_n|\theta)$ as a function of θ for a given data set $y_1, ..., y_n$. In that case we write

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- Note: the joint density and likelihood function are the same function, but the first is treated as a function of the data, and the second as a function of the parameter θ.

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- Note 2: In many cases, it is easier to maximize $\log L(\theta)$ than to maximize $L(\theta)$ itself, since log is a strictly increasing function. We call $\log L(\theta)$ the log likelihood.

Maximum likelihood estimator

Often, the MLE is found by

1. Writing out the (log) likelihood asa function of the parameter (say, θ)
2. Taking the derivative with respect to θ
3. Setting the derivative equal to 0 and solving for $\hat{\theta}$
4. Checking that the second derivative is negative at $\hat{\theta}$ to ensure the solution is a maximum.

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  - Hence, the MLE of p is actually the intuitive estimator for p that we used before

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  - Hence, the MLE of θ is not unique

Example of maximum likelihood estimator

Problem

Suppose that $Y_1, ..., Y_n \overset{i.i.d.}{\sim} Exp(\theta)$. Find the MLE of θ.

Solution

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- Taking derivatives:

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- Setting these derivatives equal to zero, we get $\mu = \frac{1}{n} \sum_{i=1}^n y_i$ and $\hat{\sigma^2} = \frac{1}{n} \sum_{i=1}^n (y_i - \mu)^2$.

Properties of MLEs

- We are often interested in estimating a function of a parameter $g(\theta)$
- The invariance property of MLEs states that if $\hat{\theta}$ is a MLE of θ, and $g(\cdot)$ is any function, then $g(\hat{\theta})$ is a MLE of $g(\theta)$.

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Suppose that $Y_1, ..., Y_n \overset{i.i.d.}{\sim} Ber(p)$. Find the MLE of $Var(\sum_{i=1}^n Y_i)$.

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- $Var(\sum_{i=1}^n y_i) = nVar(y_1) = np(1-p)$

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- The MLE of $np(1-p)$ is $n\hat{p}(1-\hat{p}) = \sum_{i=1}^n Y_i(1 - \frac{1}{n}\sum_{i=1}^n Y_i)$

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- Intuitively, we would like our estimator to "get closer" to the target parameter as $n \to \infty$
- Definition: An estimator $\hat{\theta}_n$ is a consistent estimator of θ if, for any $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}(|\hat{\theta}_n - \theta| \leq \epsilon) = 1,$$

also denote as $\hat{\theta}_n \xrightarrow{p} \theta$ (converge in probability).

Asymptotic properties of MLE

Theorem

If $Y_1, ..., Y_n \overset{i.i.d.}{\sim} f_Y(y;\theta)$ and $\hat{\theta}$ is the MLE of θ, then assuming certain regularity conditions:

$$\hat{\theta}_n \xrightarrow{p} \theta \text{ as } n \to \infty.$$

$$\frac{\hat{\theta}_n - \theta}{\sigma_{\hat{\theta}}} \to N(0,1),$$

where $\sigma^2_{\hat{\theta}} = 1/(n\mathbb{E}[-\frac{\partial^2 \log f(Y;\theta)}{\partial \theta^2}])$.

Remark: this implies that $\hat{\theta}$ is consistent and asymptotically normal. The term $n\mathbb{E}[-\frac{\partial^2 \log f(Y;\theta)}{\partial \theta^2}]$ is called Fisher information.

CI based on MLE

We can then obtain the following approximate large-sample $100(1-\alpha)\%$ confidence interval for θ:

$$\hat{\theta}_n \pm z_{\alpha/2}\sqrt{1/(n\mathbb{E}[-\frac{\partial^2 \log f(Y;\theta)}{\partial\theta^2}])} \,|_{\theta=\hat{\theta}_n} .$$

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- For random variable with a Bernoulli distribution with parameter p. If $Y_1, Y_2, ..., Y_n$ denote a random sample of size n from this distribution, derive a $100(1-\alpha)\%$ confidence interval for p.

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Summary

- General confidence intervals construction based on
  - pivotal quantities
  - MLE (likelihood function, Fisher information)

Homework

8.44, 8.102, 9.82 (b), 9.84 (a, d), 9.97, 9.102