STAT 583 Lecture 2

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Jan. 28th

Misc

- Average score: 6.4
- keep basic math concepts; remove some hard math concepts, and add a few applied examples
- Office Hours:
  - Instructor: 9:30-11:30am, 9:30pm-10pm every Tuesday,
  - TA: TBD

Last time

- The relationship between statistics and probability theory
  - Data and random variables
  - Sample and population
  - Estimates and parameters
- Sampling distributions
  - Central Limit Theorem
  - chi-squared distribution and t-distribution

Recap

Random variables / Data:

By data we mean the observed value of a random variable once some experiment has been performed.

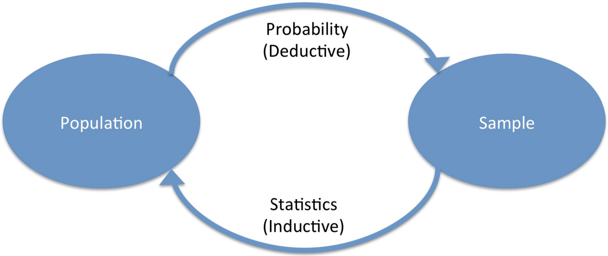
- Random variables: before the experiment
- Data: after the experiment

We typically write X_1 , . . . , X_n for hypothetical random sample, and the lower case $x_1, ..., x_n$ for their observed values.

Recap

Population / Sample

- Population: the complete collection of units about which info is sought
- Sample: a subset of a population that is actually observed



Recap

Parameter / Estimate

- Parameter: a numerical characteristic of a population
- Estimate: a numerical function of the sample data, used to make inference about the unknown parameter

Example:

- "true" kidney cancer death rate in a county (e.g., if we were to keep observing for an infinitely long time)

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- A company wishes to estimate the mean service time for customers.

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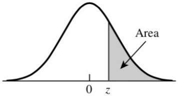
- observed kidney cancer death rate in a county in any given year
- A company wishes to estimate the mean service time for customers.
- A manufacturer wishes to estimate the standard deviation of the diameters of a part produced in a factory.

Warm-up

If $X \sim N(3, 2^2)$

- Find the probability $\mathbb{P}(X \leq 4)$.

Normal Curve Areas
Standard normal probability in right-hand tail (for negative values of z, areas are found by symmetry)



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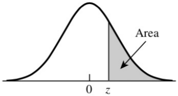
- Find the probability $\mathbb{P}(X \leq 4)$.
- $\mathbb{P}(X \leq 4) = \mathbb{P}(\frac{X-3}{2} \leq \frac{4-3}{2}) = \mathbb{P}(Z \leq 0.5) = 1 - 0.3085 = 0.6915.$

Normal Curve Areas
Standard normal probability in right-hand tail (for negative values of z, areas are found by symmetry)



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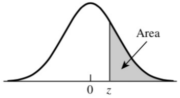

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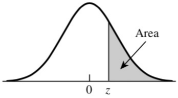
Normal Curve Areas
Standard normal probability in right-hand tail (for negative values of *z*, areas are found by symmetry)



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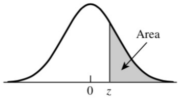

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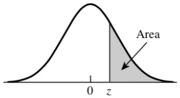

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- In R: qnorm and pnorm

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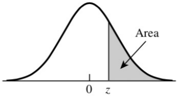

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Example

Roll a die 100 times. Find the chance that the sample mean is less than 3.7 and more than 3.3.

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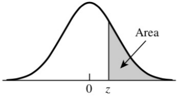

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1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455

Example

Answer

- Define X as the number to turn up. We then have $\mathbb{P}(X = 0) = \mathbb{P}(X = 1) = \mathbb{P}(X = 2) = ... = \mathbb{P}(X = 6) = \frac{1}{6}$.

Example

Answer

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- We have $\mathbb{E}(X)=3.5$, $Var(X)=\frac{1}{6}\cdot(1^2+\ldots+6^2)-3.5^2=2.917$.

Example

Answer

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- We have $\mathbb{E}(X)=3.5$, $Var(X)=\frac{1}{6}\cdot(1^2+\ldots+6^2)-3.5^2=2.917$.
- By CLT, $\bar{X}=\frac{X_1+\ldots+X_{100}}{100}\sim N(3.5,\frac{2.917}{100})$.

Example

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- We have $\mathbb{E}(X)=3.5$, $Var(X)=\frac{1}{6}\cdot(1^2+\ldots+6^2)-3.5^2=2.917$.
- By CLT, $\bar{X}=\frac{X_1+\ldots+X_{100}}{100}\sim N(3.5,\frac{2.917}{100})$.
- To find $\mathbb{P}(3.3<\bar{X}<3.7)$.

Example

Answer

- Define X as the number to turn up. We then have $\mathbb{P}(X = 0) = \mathbb{P}(X = 1) = \mathbb{P}(X = 2) = ... = \mathbb{P}(X = 6) = \frac{1}{6}$.
- We have $\mathbb{E}(X) = 3.5$, $Var(X) = \frac{1}{6} \cdot (1^2 + ... + 6^2) - 3.5^2 = 2.917$.
- By CLT, $\bar{X} = \frac{X_1 + ... + X_{100}}{100} \sim N(3.5, \frac{2.917}{100})$.
- To find $\mathbb{P}(3.3 < \bar{X} < 3.7)$.
- $\mathbb{P}(3.3 < \bar{X} < 3.7) = \mathbb{P}(\frac{3.3-3.5}{\sqrt{\frac{2.917}{100}}} < Z < \frac{3.3-3.5}{\sqrt{\frac{2.917}{100}}}) = \mathbb{P}(-1.17 < Z < 1.17) \approx \mathbb{P}(|Z| \leq 1.2) = 0.7699$.

Example

Suppose a sample is drawn *i.i.d* from a population whose mean is μ and whose standard deviation is σ. What is the approximate chance that the sample mean is within $2 \cdot \frac{\sigma}{\sqrt{n}}$ of the population mean μ?

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Answer

Answer: Since the sample mean $\bar{X}$ is approximately normal $N(\mu, \frac{\sigma^2}{n})$:

$$\mathbb{P}(\mu - 2\frac{\sigma}{\sqrt{n}} \leq \bar{X} \leq \mu + 2\frac{\sigma}{\sqrt{n}}) \approx 0.95.$$

This example is very important for the following lectures.

Overview

- Parameter Estimation
- Confidence Interval

An Overview of Statistical Inference Problems

- Making probabilistic statements about an unknown population parameter based on a random sample from the population
- Estimation
  - Point estimation: estimate the value of the unknown parameter
  - Confidence interval (CI): estimate an interval in which the parameter lies (how accurate is the estimate)
- Hypothesis Testing
  - Make decision (yes/no) on a hypothetical statement about the parameter

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- Basic setup: a random sample from a population with unknown parameter θ:

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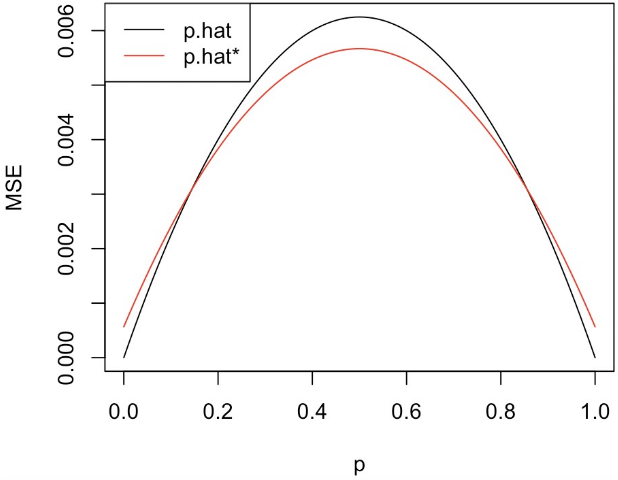
Comparison

For a given $n = 40$, plot $MSE(\hat{p})$ and $MSE(\hat{p}^*)$ against p.



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- $\sigma_{\hat{\theta}} = \frac{5}{7}$
- The 95% bound is $(0, \frac{10}{7})$.

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A comparison of the durability of two types of automobile tires was obtained by road testing samples of $n_1 = n_2 = 100$ tires of each type. The number of miles until wear-out was recorded, where wear-out was defined as the number of miles until the amount of remaining tread reached a prespecified small value. The measurements for the two types of tires were obtained independently, and the following means and variances were computed:

$$\bar{y}_1 = 26,400 \text{ miles}, \bar{y}_2 = 25,100 \text{ miles},$$

$$s_1^2 = 1,440,000, s_2^2 = 1,960,000.$$

Estimate the difference in mean miles to wear-out and place a 2-standard-error bound on the error of estimation.

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- Let $\frac{1}{k^2} = 0.05$ (i.e. $k = \sqrt{20} = 4.47$), a 95% bound is then $\epsilon \in (0, 4.47\sigma_{\hat{\theta}})$.

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- Or, you could give a range of numbers with a statement which conveys your confidence in the interval itself.

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- How to construct these intervals?

Estimation of the binomial parameter p

Let's consider a special example of population mean estimation. If we have an $i.i.d.$ sample $X_1, ..., X_n$ with mean μ and variance σ^2, how to estimate μ?

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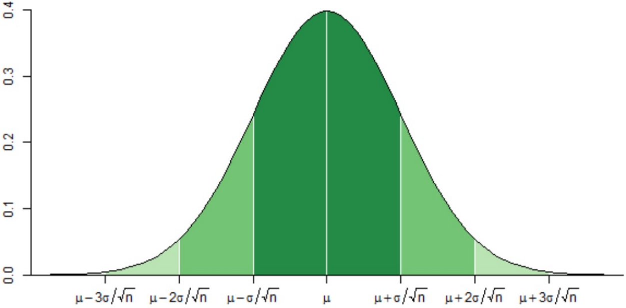
- How to construct an interval that contains μ with large probability?

The sampling distribution of $\bar{X}$

Since $\bar{X} \approx N(\mu, \frac{\sigma^2}{n})$, CLT enables us to make the following statements

1. $\mathbb{P}(\mu - \frac{\sigma}{\sqrt{n}} \leq \bar{X} \leq \mu + \frac{\sigma}{\sqrt{n}}) \approx 0.68$
2. $\mathbb{P}(\mu - 2\frac{\sigma}{\sqrt{n}} \leq \bar{X} \leq \mu + 2\frac{\sigma}{\sqrt{n}}) \approx 0.95$
3. $\mathbb{P}(\mu - 3\frac{\sigma}{\sqrt{n}} \leq \bar{X} \leq \mu + 3\frac{\sigma}{\sqrt{n}}) \approx 0.997$



The key idea

- Assuming that n is large enough. By the CLT and the Empirical Rule, we believe there is a 95% chance that a new $\bar{X}$ will be within $2\sigma/\sqrt{n}$ from μ:

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- Confidence Intervals do not have to be 95%. 95% is just a convention. If you want something else (say 90%) then switch the 2 to 1.645.

Confidence interval

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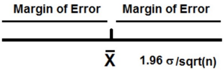

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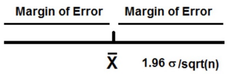

- Remark: Keep this in mind, the MoE is the SD of the estimate multiplied by some constant, where $sd(\bar{X}) = \frac{\sigma}{\sqrt{n}}$.

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- An illustration applet: `http://www.rossmanchance.com/applets/ConfSim.html`

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Example: Interpreting a confidence interval

- KEY FACT: since the sample mean is random, the confidence interval is random.
- If two people draw two independent samples, the resulting confidence intervals will not be the same.

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- If we are going to construct a 99% confidence interval for μ, then we need to look for $z_{0.005}$, such that $\mathbb{P}(-z_{0.005} \leq Z \leq z_{0.005}) = 0.99$.
- Since we have $\mathbb{P}(-2.576 < Z < +2.576) = 0.99$, $z_{0.005} = 2.576$

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A $(1-\alpha)$ level confidence interval for μ is $[\bar{X} - z_{\alpha/2} \cdot \sigma/\sqrt{n},\ \bar{X} + z_{\alpha/2} \cdot \sigma/\sqrt{n}]$.

Examples:

- A more accurate 95% confidence interval for μ is $[\bar{X} - 1.96 \cdot \sigma/\sqrt{n},\ \bar{X} + 1.96 \cdot \sigma/\sqrt{n}]$, since $\mathbb{P}(-1.96 \leq Z \leq 1.96) = 0.95$.
- If we are going to construct a 99% confidence interval for μ, then we need to look for $z_{0.005}$, such that $\mathbb{P}(-z_{0.005} \leq Z \leq z_{0.005}) = 0.99$.
- Since we have $\mathbb{P}(-2.576 < Z < +2.576) = 0.99$, $z_{0.005} = 2.576$
- A 99% level confidence interval for μ is $[\bar{X} - 2.576 \cdot \sigma/\sqrt{n},\ \bar{X} + 2.576 \cdot \sigma/\sqrt{n}]$.

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The SD of a normal population is 2 and the mean is unknown. A sample of 100 observations is drawn and the sample mean is 12.8. What is a 99% CI for the true mean?

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- $[\bar{X} - 2.576\frac{\sigma}{\sqrt{n}}, \bar{X} + 2.576\frac{\sigma}{\sqrt{n}}]$
- $12.8 \pm 2.576 \cdot 2/\sqrt{100} = [12.3, 13.3]$.

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- We will discuss the case where the sample size n is small next week.

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- Thus, we can use $U = \frac{Y}{\theta}$ as a pivotal quantity

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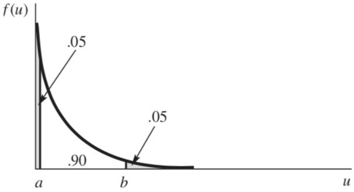

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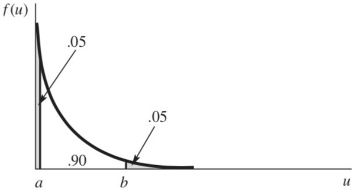

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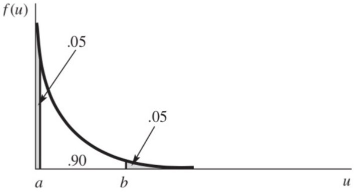

- $1 - e^{-a} = .05$ and $e^{-b} = .05$, equivalently, $a = 0.051$, $b = 2.996$
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Suppose that we take a sample Y from a uniform distribution defined on the interval $[0, \theta]$, where θ is unknown. Find a 95% lower confidence bound for θ, that is, find $L(Y)$, such that $\mathbb{P}(\theta \geq L(Y)) = 0.95$.

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- The interval on the transformed scale is: $(12 \times 9 + 32, 14 \times 9 + 32)°\text{F} = (140°\text{F}, 158°\text{F})$.

Manipulating confidence intervals

- In many medical settings clinicians like to report results in terms of odds ratios rather than probabilities.
- If the probability is given by p then the odds ratio is $p/(1-p)$. A study has found the 95% confidence interval for p to be (0.35, 0.38).
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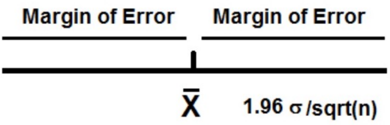
- Find the 95% CI for the odds ratio.

Answer

$(0.35/0.65,\ 0.38/0.62) = (0.538,\ 0.613)$.

Sample Size Choice

- How large a sample size do I need?
- Think about three things:
  1. What confidence level you want (say 95%).
  2. What Margin of Error (MoE) you want. $(2\sigma/\sqrt{n})$
  3. If σ is unknown, then you need an estimate for σ.
- Recall that the MoE is the distance from the center to the edge of the interval.



The sample size formula for a population mean, .

- As the $MoE = 1.96\sigma/\sqrt{n}$ in the 95% confidence interval for μ, it follows that:

$$n = (\frac{1.96\sigma}{MoE})^2 \approx (\frac{2\sigma}{MoE})^2 \approx (\frac{2s}{MoE})^2.$$

- There are different formulas for different statistical questions (not just the mean), but sample size is always a legitimate question.

Example

- An insurance company is being sued because it has paid bills late and then failed to pay interest on the late payments.
- Lawyers need to estimate the average amount of unpaid interest on the late bills.
- The population of bills is 300,000, far too many to review individually.
- How large a sample size do I need to make a 95% confidence interval for the mean amount of unpaid interest on bills paid late?
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- Answer:

$$n = \left(\frac{2 \times 25.25}{2}\right)^2 = 638.$$

Misc.

- Ideally, we would like a narrow 99.9999% confidence interval. Such interval conveys that we have a precise estimate of the unknown population parameter. However, this is not possible, because $z_{\alpha/2}$ is too large in this case. The statisticians often settle for a 95% confidence interval.
- What happens when you increase sample size n? Standard deviation becomes smaller and therefore CI becomes more narrow.

Summary

- Point Estimation
- Confidence intervals.
- Confidence interval for a population mean (μ).

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- Confidence interval for a population mean (μ).
- More confidence intervals based on the pivotal quantity

Homework

- Suppose $Y \sim Bin(n, p)$, calculate $MSE(\frac{Y+2}{n+3})$ and plot the MSE against p
- Prove that S^2 is an unbiased estimator of σ^2
- 7.52, 8.6(a), 8.10(a)
- Sample size calculations and the Margin of Error.