WHEN ARE THERE CONTINUOUS CHOICES FOR THE MEAN VALUE ABSCISSA?

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ABSTRACT. The mean value theorem of calculus states that, given a differentiable function f on an interval [a,b], there exists at least one mean value abscissa c such that the slope of the tangent line at c is equal to the slope of the secant line through (a,f(a)) and (b,f(b)). In this article, we study how the choices of c relate to varying the right endpoint b. In particular, we ask: When we can write c as a continuous function of b in some interval?

Drawing inspiration from graphed examples, we first investigate this question by proving and using a simplified implicit function theorem. To handle certain edge cases, we then build on this analysis to prove and use a simplified Morse's lemma. Finally, further developing the tools proved so far, we conclude that if f is analytic, then it is always possible to choose mean value abscissae so that c is a continuous function of b, at least locally.

1. Introduction and Statement of the Problem.

The mean value theorem is one of the truly fundamental theorems of calculus. It says that if f is a differentiable function defined on a closed interval [a, b], then there is at least one c in the open interval (a, b) such that

(1)
$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

We call c a mean value abscissa for f on [a,b]. Looking at a graph y=f(x) as in Figure 1, the left hand side of (1) is the slope of the secant line from (a,f(a)) to (b,f(b)), while the right hand side is the slope of the tangent line passing through (c,f(c)). Observe that there can be multiple choices of c; in the first graph in Figure 1 we could have chosen either c or c'.

In this article we are interested in how the set of mean value abscissae c changes as we vary one of the endpoints of the interval, say the right endpoint b. In particular, we are interested in the following problem: Suppose $c = c_0$ is our favorite mean value abscissa for $a = a_0$ and $b = b_0$. If b changes slightly, can we also change c slightly so that (1) is still satisfied? In other words, is there a locally continuous choice c = C(b) of the mean value abscissa? For example, in the right-hand graph in Figure 1, we consider the new value b_{new} . Here, it appears that the small change from b to b_{new} corresponds to small changes from c to c_{new} and c' to c'_{new} is this always possible?

2. Some examples.

To get a better feel for the problem we have set out for ourselves, let's graph some functions and their mean value abscissae. To make our life simpler, we will stick to examples with $a = a_0 = 0$ and $f(a_0) = f(b_0) = 0$. Then the left hand side of (1) is zero when $a = a_0$ and $b = b_0$, and so any corresponding mean value

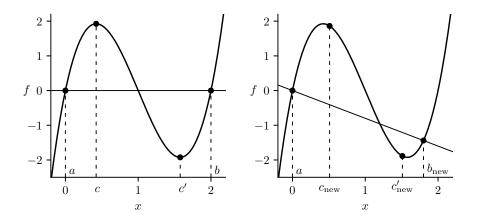


FIGURE 1. An illustration of the mean value theorem on the function $f(x) = x^3 - 3x^2 + 2x$. The straight lines are the secant lines. In each graph, the end-points of the secant line and two mean value abscissae are indicated by points on the curve. Dashed lines from each point indicate corresponding x-values.

abscissa $c = c_0$ has to be a critical point where $f'(c_0) = 0$. This may seem like a lot of assumptions, but in fact if someone hands us a more general function f we can always consider the related function

$$g(x) = f(x) - \left(\frac{f(b_0) - f(a_0)}{b_0 - a_0}(x - a_0) + f(a_0)\right),$$

which satisfies $g(a_0) = g(b_0) = 0$. Both f and g have the same solutions to the mean value condition (1).

Consider the parabola at the left of Figure 2. There is only one choice of c_0 : the vertex. When we slightly increase b_0 to b_1 , we have to slightly increase c_0 to c_1 . Similarly if we decrease b_0 to b_2 , then we have to decrease c_0 to c_2 . Plotting the mean value abscissae c for each b, we get the picture at the right of Figure 2. Looking at the figure, c seems to be a continuous function of b, and indeed in this case we can solve (1) explicitly to get c = b/2. In particular, the ratio c/b is constant; for more on the class of functions with this property, see [1].

Next consider the more complicated graph at left in Figure 3. There are now two critical points. One is a local maximum, and the behavior near this point is very similar to the behavior near the vertex of the parabola. The second one, which we have labeled as c_0 , is a non-extremal critical point (it is neither a local maximum nor a local minimum). Suppose that b_1 is just a bit bigger than b_0 . Then the slope of the secant line which appears on the right hand side of (1) is < 0. When c is close to c_0 , on the other hand, the right hand side f'(c) of (1) is ≥ 0 . There is no solution to (1) without choosing c far away from c_0 .

3. The implicit function theorem.

3.1. **Implicit equations.** In the last section, we saw that the set of solutions (b, c) of (1) can look quite complicated. On the right of Figure 3, for instance, c is not a function of b (the curve fails the "vertical line test") and b is also not a function

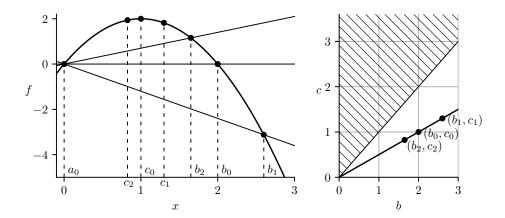


FIGURE 2. Left: the parabola $y = -x^2 + 2x$. Three pairs of points are shown: (b_0, c_0) , (b_1, c_1) , and (b_2, c_2) . For each b_i , the corresponding c_i is a mean value abscissa on the interval $[a_0, b_i]$. Right: the graph of *all* mean value abscissae as a function of b (where $a_0 = 0$ is fixed); b is on the horizontal axis, and c = b/2 is on the vertical axis. Points corresponding to the three pairs on the left are noted. In the mean value theorem, b > c, and we represent this by shading the region where $c \ge b$.

of c (the curve fails the "horizontal line test"). This is possible because (1) is an implicit equation.

Implicit equations show up all over the place in mathematics, for instance in geometry; $x^2 + y^2 = 1$ is the equation for the circle with radius one centered at the origin, while $x^2 + 4xy + y^2 = 2$ is the equation for a certain hyperbola. By subtracting off the right hand side, we can write any implicit equation in two variables as

$$(2) F(x,y) = 0$$

for some function F. It's often tempting to try and solve an implicit equation for one of the variables, and indeed that's indeed how we got the formula c=b/2 for the example on the right of Figure 2. When the equation gets more complex, though, like it is in Figure 3, this method becomes very tedious and is quite often impossible!

3.2. The implicit function theorem. Thankfully, calculus offers us a powerful tool, called the *implicit function theorem*, to help us understand implicit equations. The intuition behind the theorem is the following: Suppose that we have found one solution (x_0, y_0) to (2), and are only interested in solutions of (2) nearby this initial solution. If the function F is differentiable, then for $(x, y) \approx (x_0, y_0)$ we can approximate F as

(3)
$$F(x,y) \approx F(x_0,y_0) + F_x(x_0,y_0)(x-x_0) + F_y(x_0,y_0)(y-y_0),$$

where the subscripts on F are partial derivatives. This is a first-order Taylor approximation of F, a two-variable version of the tangent-line approximation for

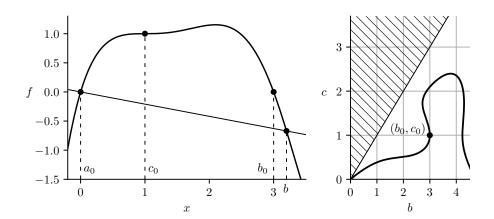


FIGURE 3. Left: the graph of a function with an inflection point. The point (1,1) is a mean value abscissa on the interval [0,3], but there is no continuous extension of this solution to a neighborhood of b_0 . The straight line is the secant line corresponding to the interval $[a_0,b_1]$. Right: the graph of all mean value abscissae as a function of b, as in the previous Figure. The behavior is substantially more complicated. At the point (b_0,c_0) , we observe that c is not a function of b.

functions of a single variable. Plugging (3) into the equation (2) that we are trying to solve and using $F(x_0, y_0) = 0$, we get

(4)
$$F_x(x_0, y_0)(x - x_0) + F_y(x_0, y_0)(y - y_0) \approx 0.$$

While (4) is only approximately true, it's advantage is that it's a *linear* equation. In particular, if $F_y(x_0, y_0) \neq 0$, then we can try to "solve for y", giving the approximation

(5)
$$y = Y(x) \approx y_0 - \frac{F_x(x_0, y_0)}{F_y(x_0, y_0)} (x - x_0).$$

The implicit function theorem says that the conclusion of this intuitive argument is nearly correct: as long as $F(x_0, y_0) = 0$ and $F_y(x_0, y_0) \neq 0$, we can indeed solve F(x, y) = 0 for y when (x, y) is close to (x_0, y_0) .

Theorem 1 (Implicit Function Theorem). Suppose that F = F(x, y) is a continuously differentiable function and that at some point (x_0, y_0) we have

(6)
$$F(x_0, y_0) = 0$$
 and $F_y(x_0, y_0) \neq 0$.

Then there exist $\varepsilon > 0$, $\delta > 0$, and a continuously differentiable function Y(x) such that the implicit equation F(x,y) = 0 is equivalent to the explicit equation y = Y(x) whenever $|x - x_0| < \delta$ and $|y - y_0| < \varepsilon$.

3.3. Proof of the implicit function theorem. The implicit function theorem above is an *existence* theorem: it says there *exists* a function Y = Y(x) with some special properties. Like many of the existence theorems in calculus, the implicit function theorem has a nice proof using the contraction mapping principle. The

implicit function theorem presented here is a simplified version, but the proofs of more general versions share have the same basic outline; see, for instance, [6, §13].

The contraction mapping principle, which is also called Banach's fixed-point theorem, concerns equations of the form

(7)
$$y = K(y), \qquad y \text{ in } I$$

where I = [A, B] is a closed interval. We call (7) a "fixed-point equation" because it says that the point y is "fixed" (or unchanged) when we apply the function K to it. The theorem assumes that the function K satisfies

(8)
$$|K(y) - K(y')| \le \rho |y - y'| \quad \text{for all } y, y' \text{ in } I$$

for some constant $\rho < 1$. If y, y' are two points a distance d apart, then (8) says that the distance between their images K(y), K(y') is at most ρd . Since $\rho d < d$, the points are closer together after we apply K, and so we call K a contraction.

Theorem 2 (Contraction Mapping Principle). Suppose that the function K is defined on a closed interval I where it satisfies (8) for some constant $\rho < 1$. If K(y) lies in I for every y in I, then the fixed-point equation (7) has a unique solution y^* .

Proof. First we show that solutions of (7) are unique. Suppose that y and y' both solve (7), so that they satisfy y = K(y) and y = K(y'). Then by (8) we see that

$$|y - y'| = |K(y) - K(y')| \le \rho |y - y'|.$$

Since $\rho < 1$, this is only possible if y = y'.

Having shown uniqueness of a potential solution, we now show that (7) has a solution y^* . Choose any point y_0 in I and define the sequence y_1, y_2, y_3, \ldots recursively by

$$(9) y_{n+1} = K(y_n).$$

Since K sends points in I to points in I, this definition makes sense and we can prove by induction that y_n lies in I for all n. We will show that $\lim_{n\to\infty} y_n$ exists, and that it is the fixed point y^* we are looking for.

By repeatedly using (8) and (9), we can estimate the distance between successive terms y_{n+1} and y_n in our sequence:

$$|y_{n+1} - y_n| = |K(y_n) - K(y_{n-1})|$$

$$\leq \rho |y_n - y_{n-1}|$$

$$= \rho |K(y_{n-1}) - K(y_{n-2})|$$

$$\leq \rho^2 |y_{n-1} - y_{n-2}|$$

$$\leq \cdots \leq \rho^{n-1} |y_1 - y_0|.$$

Since $\rho < 1$, the right hand side converges to zero very quickly as $n \to \infty$. If n > m, we can then repeatedly use (10) to estimate the difference between y_n and y_m :

$$|y_n - y_m| = |(y_n - y_{n-1}) + (y_{n-1} - y_{n-2}) + \dots + (y_{m+1} - y_m)|$$

$$\leq |y_n - y_{n-1}| + |y_{n-1} - y_{n-2}| + \dots + |y_{m+1} - y_m|$$

$$\leq |y_1 - y_0|(\rho^{n-1} + \rho^{n-2} + \dots + \rho^m)$$

$$= |y_1 - y_0|\rho^m \frac{1 - \rho^n}{1 - \rho}$$

$$< |y_1 - y_0| \frac{\rho^m}{1 - \rho}.$$

Here in the second step we have used the triangle inequality, and in the second-to-last step we have used the formula for the (partial) sum of a geometric series. As before the right hand side converges to 0 as $m \to 0$, which now shows that the sequence $\{y_n\}$ is Cauchy. In particular, the limit $y^* = \lim_{n \to \infty} y_n$ exists. Since each y_n lies in the closed interval I, the same is true for the limit y^* .

Finally, we note that (8) implies that K is continuous. Taking the limit of the recurrence (9) as $n \to \infty$, we therefore get $y^* = K(y^*)$, i.e. that y^* solves (7).

To use Theorem 2 to prove Theorem 1, we first rewrite F(x,y) = 0 as a fixed-point equation y = K(y;x) for y. When x is close to x_0 and I is a small interval centered at y_0 , we will then show that this K is a contraction mapping satisfying the hypotheses of Theorem 2. Here the notation K(y;x) is to remind us that y is the main variable while x is a parameter.

To keep things simple, we only prove the implicit function theorem for the special case $x_0 = y_0 = 0$. It's easy to prove the general case from this specific one by considering the shifted function $G(x,y) = F(x_0 + x, y_0 + y)$. The inspiration for the proof is our informal argument which lead to (5). This argument started with the Taylor expansion (3), but to make it rigorous we start with an exact version of that formula:

(11)
$$F(x,y) = F_x(0,0)x + F_y(0,0)y + r(x,y).$$

Here r(x, y) is the remainder term, which is small when (x, y) is close to (0, 0), and we have used that $x_0 = y_0 = 0$ and $F(x_0, y_0) = F(0, 0) = 0$. To devise a mapping K, we set F(x, y) = 0 and do some algebra to bring the y to the left hand side to get

(12)
$$y = -F_y(0,0)^{-1} \Big(F_x(0,0)x + r(x,y) \Big).$$

Solving (11) for r(x,y) and plugging into (12), things simplify a bit and we get

(13)
$$y = y - F_y(0,0)^{-1}F(x,y).$$

We see that (13) is true if and only if F(x, y) = 0. What we have gained, though, is that (13) gives a fixed-point equation for y, and so we can hope to solve it for y by applying Theorem 2 with

(14)
$$K = K(y;x) = y - F_y(0,0)^{-1}F(x,y).$$

We apply Theorem 2 with $\rho=1/2$ and $I=[-\varepsilon,\varepsilon]$ for some small number $\varepsilon>0$ which we still have to determine. This ensures that we are only considering y-values which are close to $y_0=0$. We will also restrict ourselves to x-values which are close

to $x_0 = 0$, say x in $[-\delta, \delta]$ for some other small number $\delta > 0$. The hypotheses of Theorem 2 will therefore be met as long as

(15)
$$|K(y;x)| \le \varepsilon \qquad \text{for } |x| \le \delta, \, |y| \le \varepsilon,$$

(16)
$$|K(y;x) - K(y';x)| \le \frac{1}{2}|y - y'|$$
 for $|x| \le \delta$, $|y|, |y'| \le \varepsilon$.

The second inequality (16) is the contraction condition (8), while the first (15) guarantees that K(y;x) lies in I whenever y does.

We'll prove (16) using, of all things, the mean value theorem. Differentiating with respect to y we get

$$K_{y}(y;x) = 1 - F_{y}(0,0)^{-1}F_{y}(x,y).$$

It's clear that at (0,0), the right hand side is 0. Since F_y is continuous, we can pick $\varepsilon > 0$ small enough that the right hand side is bounded by 1/2 whenever $|x|, |y| \le \varepsilon$. Now suppose that $|x|, |y|, |y'| \le \varepsilon$. Applying the mean value theorem to K_x on the interval between y and y', we get that

(17)
$$|K(y';x) - K(y;x)| = |K_y(c;x)||y' - y| \le \frac{1}{2}|y' - y|$$

for some point c between y and y'. This shows (16).

We still need to show (15). As long as $|x|, |y| \le \varepsilon$, we can use (16) to estimate

(18)
$$|K(y;x)| \le |K(y;x) - K(0;x)| + |K(0;x)|$$
$$\le \frac{1}{2}|y| + |K(0;x)|$$
$$\le \frac{1}{2}\varepsilon + |K(0;x)|.$$

Since K(0;x) is a continuous function of x and K(0;0)=0, there exists a $\delta>0$ so that $|K(0;x)|\leq \varepsilon/2$ whenever $|x|\leq \delta$. Picking δ smaller if necessary so that $\delta\leq \varepsilon$, (18) finally implies that $|K(y;x)|\leq \frac{1}{2}\varepsilon+\frac{1}{2}\varepsilon=\varepsilon$ whenever $|x|\leq \delta$ and $|y|\leq \varepsilon$.

Now that we have finished proving (15) and (16), we can apply Theorem 2 to guarantee that the fixed point equation y = K(y; x) has a unique solution y = Y(x) in $[-\varepsilon, \varepsilon]$ for each x in $[-\delta, \delta]$. Since the fixed-point equation y = K(y; x) is equivalent to F(x, y) = 0, this completes the proof of the theorem except for the claim that Y is a continuously differentiable function.

To see that Y(x) is continuous, we write

$$|Y(x') - Y(x)| = |K(Y(x'); x') - K_x(Y(x); x)|$$

$$\leq |K(Y(x'); x') - K(Y(x); x')| + |K(Y(x); x') - K(Y(x); x)|.$$

By (17), the first term on the right hand side is bounded by $\frac{1}{2}|Y(x')-Y(x)|$. Rearranging, this shows that

$$|Y(x') - Y(x)| \le 2|K(Y(x); x') - K(Y(x); x)|.$$

The continuity of Y then follows from the continuity of K(y;x) as a function of x. Next we show that Y is continuously differentiable and calculate its derivative. If we knew ahead of time that Y was differentiable, then we could solve for Y'(x) by differentiating F(x,Y(x))=0 using the chain rule. Since we do not know yet that Y is differentiable, we instead look at the difference quotient

$$0 = \frac{F(x+h, Y(x+h)) - F(x, Y(x))}{h}.$$

Using the fundamental theorem of calculus in a clever way, we rewrite the numerator of this difference quotient as

$$0 = F(x+h, Y(x+h)) - F(x, Y(x))$$

$$= \int_0^1 \frac{d}{dt} F(x+th, tY(x+h) + (1-t)Y(x)) dt$$

$$= h \int_0^1 F_x dt + (Y(x+h) - Y(x)) \int_0^1 F_y dt,$$

where the arguments of both F_x and F_y are (x+ty,tY(x+h)+(1-t)Y(x)). Notice that the chain rule has caused a (Y(x+h)-Y(x)) to appear. We rearrange this into an expression for the difference quotient

$$\frac{Y(x+h) - Y(x)}{h} = -\left[\int_{0}^{1} F_{y} dt\right]^{-1} \int_{0}^{1} F_{x} dt.$$

Since $F_y(0,0) \neq 0$, for h small enough the integral of F_y will not vanish, and so it is valid to divide by it.

Now we take the limit as $h \to 0$. On the left hand side, this gives Y'(x) directly. On the right hand side, we have to pass the limit inside the integrals. Since (x + th, tY(x + h) + (1 - t)Y(x)) converges uniformly to (x, Y(x)) as $h \to 0$, this is justified and we get

(19)
$$Y'(x) = -\frac{F_x(x, Y(x))}{F_y(x, Y(x))}.$$

This proves that Y is differentiable. Since Y is continuous and F is continuously differentiable, the right hand side of (19) is continuous, and so Y is in fact continuously differentiable. Looking at (19), we also see that this justifies the approximate formula (5) as we had hoped!

By repeatedly differentiating (19), we discover that if F is k-times continuously differentiable, then so is Y. We record this observation as a corollary to the proof of Theorem 1.

Corollary 3. In Theorem 1, the derivative Y' is given by (19). If F is k-times continuously differentiable, then Y is k-times continuously differentiable.

3.4. Application to the Mean Value Abscissa. With the implicit function theorem in hand, we are now ready to investigate the possibility of determining when there exist locally continuous choices of the mean value abscissa c in (1). The first step is to rewrite (1) as F(b,c) = 0 where

(20)
$$F(b,c) = \frac{f(b) - f(a)}{b - a} - f'(c).$$

From now on we assume that f is twice continuously differentiable, in which case F is once continuously differentiable.

Suppose that c_0 is a mean value abscissa corresponding to b_0 , i.e. that $F(b_0, c_0) = 0$. A quick computation shows that

$$F_b(b_0, c_0) = \frac{f'(b_0) - f'(c_0)}{b_0 - a}, \qquad F_c(b_0, c_0) = -f''(c_0).$$

Thus $F_c(b_0, c_0) \neq 0$ is true exactly when $f''(c_0) \neq 0$. And if $f''(c_0) \neq 0$, then by Theorem 1 there exists $\varepsilon > 0$, $\delta > 0$, and a continuously differentiable function C(b) so that F(b, c) = 0 is equivalent to c = C(b) whenever $|c - c_0| < \varepsilon$ and $|b - b_0| < \delta$.

Although we have focused on the question of when the mean value abscissa c can be written as a continuous function of the right endpoint b, we also have the data for the converse question: When is the right endpoint b a function of the mean value abscissa c? By Theorem 1, b can be written as a function of c near (b_0, c_0) when $F_b(b_0, c_0) \neq 0$, or equivalently when $f'(b_0) \neq f'(c_0)$.

In total we have proved the following theorem.

Theorem 4. Let f be a twice continuously differentiable function, fix an interval $[a_0, b_0]$, and let c_0 be a mean value abscissa for f on $[a_0, b_0]$.

(a) Suppose that $f''(c_0) \neq 0$. Then there is a continuously differentiable function C(b) so that

$$\frac{f(b) - f(a)}{b - a} = f'(C(b))$$

for all b close to b_0 . There are no other solutions (b,c) of (1) close to (b_0,c_0) .

(b) Suppose that $f'(b_0) \neq f'(c_0)$. Then there is a continuously differentiable function B(c) so that

$$\frac{f(B(c)) - f(a)}{B(c) - a} = f'(c)$$

for all c close to c_0 . There are no other solutions (b,c) of (1) close to (b_0,c_0) .

Remark. Given an initial solution (b_0, c_0) , our proof of Theorem 4 is *constructive* in that iterating the contraction map from the proof Theorem 1 actually gives us an algorithm for approximating B(c) or C(b) to any order of accuracy.

This theorem gives perspective on Figure 3. The mean value abscissa in that figure was at an inflection point, where $f''(c_0) = 0$, and so Theorem 4a is inconclusive about whether we can write c = C(b). Looking at the figure it appears that we cannot. On the other hand $f'(b_0) < f'(c_0) = 0$, and so Theorem 4 implies that we can write b = B(c).

4. The Morse Lemma.

We have now shown that there exist continuous choices of c = C(b) around those mean value abscissae such that $f''(c_0) \neq 0$. Conversely, we've shown that when there is a mean value abscissa c such that $f'(b_0) \neq f'(c_0)$, then b can be written as a continuous function in a neighborhood of c_0 .

But what if both $f''(c_0) = 0$ and $f'(b_0) = f'(c_0)$? As before, we return to pictorial investigation. Fortunately, these are two strong constraints and we quickly identify interesting aspects from graphs.

For ease, we suppose again that $f(a_0) = f(b_0) = 0$, and we now suppose that $f'(c_0) = f'(b_0) = f''(c_0) = 0$. In Figure 4, we examine three different functions f: each satisfies $f''(c_0) = 0$, but f''(b) is positive on the left, zero in the middle, and negative on the right. We've named these three values of b as b_m , b_i , and b_M (according to whether b is a minimum, an inflection point, or a maximum, respectively).

Examining the top left graph of Figure 4, we observe that in a small neighborhood of c_0 , all tangent lines have nonnegative slope. Similarly, for all b in a small

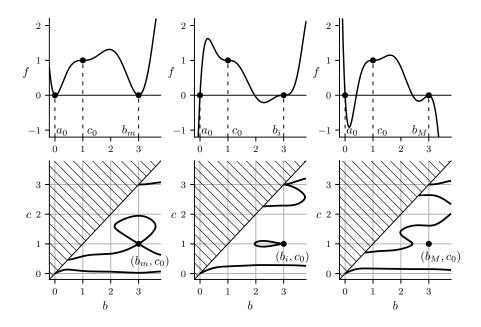


FIGURE 4. **Top**: Three graphs of functions f with an interval and corresponding mean value abscissa indicated. In each graph, $f''(c_0) = 0$. In the left graph, $f''(b_m) > 0$. In the middle graph, $f''(b_i) = 0$. In the right graph, $f''(b_M) < 0$. **Bottom**: Below each graph is a plot of all mean value abscissae as a function of b, as in previous figures. In the first graph, there appear to be multiple choices of continuous function c(b). In the second graph, there is a continuous extension on an interval with b_i as an endpoint. In the third graph, the initial solution is completely isolated.

neighborhood around b_m , the secant lines from (0,0) to (b,f(b)) have nonnegative slope. Qualitatively, it appears that for b' just a little less than b_m , we could vary c to match slopes. But which direction should c be moved? We can see this apparent choice of direction in the mean value abscissa graph at bottom left: near (b_m, c_0) , the graph resembles an X.

This reveals a key difference to the situation when $f''(c) \neq 0$. In both the implicit function theorem and Theorem 4, the resulting implicitly defined functions are unique. This is due to the uniqueness of the fixed points in the contraction mapping principle. But here, it appears that sometimes there are multiple different continuous choices of c(b) — that is, if there are any at all.

In the top right graph of Figure 4, we see that in a small neighborhood of c_0 , all tangent lines again have nonnegative slope. But in a small neighborhood around b_M , the secant lines from (0,0) to (b,f(b)) all have nonpositive slope. Thus there is no hope to extend c to a function to a larger neighborhood at all. We recognize this in the mean value abscissa graph below by seeing that (b_M,c_0) is an isolated point.

The behavior in the top center graph, near b_i , is a bit more delicate. Here, in a small neighborhood of c_0 , all tangent lines have nonpositive slope. For b just to the left of b_i , the secant lines from (0,0) to (b,f(b)) have nonpositive slope, and so it qualitatively appears that it might be possible to associate points near c with matching slopes. But for b just to the right of b_i , the secant lines all have positive slope, which cannot be matched to slopes of points in a neighborhood of c.

These examples indicate a wider variety of behavior, and it's not at all obvious what the general rule should be. We cannot hope to directly apply an implicit function theorem without some significant changes.

As with our investigation of the implicit function theorem, let us start with the Taylor expansion of F(b,c) at (b_0,c_0) . As $F(b_0,c_0)=F_b(b_0,c_0)=F_c(b_0,c_0)=0$, all the terms in this expansion are at least quadratic. For simplicity, let's assume that two of the quadratic terms are nonzero, more specifically that the partial derivatives $F_{bb}(b_0,c_0)\neq 0$ and $F_{cc}(b_0,c_0)\neq 0$. In this case, there is a result called the Morse lemma which is perfectly tailored for our situation! A simple version of the Morse lemma is the following.

Lemma 5 (Morse lemma). Let G = G(x,y) be a three-times continuously differentiable function and suppose that $G(0,0) = G_x(0,0) = G_y(0,0) = 0$ but that

(21)
$$G_{xx}(0,0)G_{yy}(0,0) - (G_{xy}(0,0))^2 \neq 0.$$

Then in a neighborhood of the origin there is a change of coordinates $(x,y)\mapsto (u,v)$ so that

(22)
$$G(x,y) = \pm u^2 \pm v^2.$$

The number of minus signs on the right hand side of (22) is called the *Morse index* of G at 0. It is independent of the particular choice of coordinates (u,v), and is one of the basic ingredients in Morse theory [3]. By a "change of coordinates" $(x,y)\mapsto (u,v)$, we mean that u and v can be written as continuously differentiable functions of (x,y), while at the same time x and y can be written as continuously differentiable functions of (u,v). We also require that $(0,0)\mapsto (0,0)$.

Remark. Those familiar with multivariable calculus might recognize the conditions of the Morse lemma as an alternate way of saying that the gradient of G vanishes at the origin, but the Hessian matrix is invertible there. A full proof of the Morse lemma involves the implicit function theorem in higher dimensions (or its close cousin the inverse function theorem). But we will see below that in our special case, Theorem 1 is sufficient.

We consider the function $G(x,y)=F(b_0+x,c_0+y)$, which effectively translates our focus to the origin. Just as a solution to F(b,c)=0 corresponds to a mean value abscissa, a solution to G(x,y)=0 also corresponds to a mean value abscissa; in particular, G(0,0)=0 corresponds to the given mean value abscissa c_0 on the interval $[a_0,b_0]$. Our assumptions on the partial derivatives of F at (b_0,c_0) similarly translate $G_x(0,0)=G_y(0,0)=0$ while $G_{xx}(0,0)\neq 0$ and $G_{yy}(0,0)\neq 0$. A quick calculation shows that $G_{xy}(0,0)=0$ —indeed $G_{xy}(x,y)$ is always zero! Thus (21) is satisfied, and, if f is four-time continuously differentiable so that G is three-times continuously differentiable, we can apply the Morse lemma.

In fact, our situation is a bit simpler than the one covered by the Morse lemma, and so we will only prove the special case of the lemma that we need. What's

special is that G naturally splits into a function depending only on x and a function depending only on y: We can write $G(x, y) = g_1(x) - g_2(y)$, where

(23)
$$g_1(x) = \frac{f(b_0 + x) - f(a_0)}{(b_0 + x) - a_0} - f'(c_0), \qquad g_2(y) = f'(c_0 + y) - f'(c_0).$$

Thus G(x,y) = 0 is equivalent to $g_1(x) = g_2(y)$. We include the $f'(c_0)$ terms in (23) so that $g_1(0) = g_2(0) = 0$, so we can continue to focus our attention on the origin. Our assumptions $G_x(0,0) = G_y(0,0)$ now translate into $g'_1(0) = g'_2(0)$, while our assumptions $G_{xx}(0,0) \neq 0$ and $G_{xy}(0,0) \neq 0$ translate into $g'_1(0) \neq 0$ and $g'_2(0) \neq 0$.

Taylor expanding g_1 and g_2 gives the approximations

$$g_1(x) \approx \frac{g_1''(0)}{2!}x^2 = \frac{1}{b_0 - a_0} \frac{f''(b_0)}{2!}x^2, \qquad g_2(y) \approx \frac{g_2''(0)}{2!}y^2 = \frac{f'''(c_0)}{2!}y^2,$$

and hence the approximation $G(x,y) \approx \alpha x^2 + \beta y^2$ with $\alpha = g_1''(0)/2$ and $\beta = g_2''(0)/2$. We will show that we can choose coordinates u and v to make this approximation *exact* while at the same time taking the constants to be ± 1 .

To do this, we use Taylor's theorem to write q_1 and q_2 exactly as

$$g_1(x) = \alpha x^2 + r_1(x)x^2 = x^2(\alpha + r_1(x))$$

$$g_2(y) = \beta y^2 + r_2(y)y^2 = y^2(\beta + r_2(y))$$

where $r_1(x)$ and $r_2(y)$ are remainder terms. Thus $r_1(x)$ is small when x is near 0 and $r_2(y)$ is small when y is near 0. Ideally, we would like to take coordinates like $u = x\sqrt{\alpha + r_1(x)}$ and $v = y\sqrt{\beta + r_2(y)}$, so that $G(x,y) = u^2 - v^2$ (whose zeros are very easy to study). But if, for instance, $\alpha < 0$ and $r_1(x) \approx 0$, then we would be trying to take the square root of a negative number!

To get around this, we multiply $g_1(x)$ by $\sigma_1 = \operatorname{sgn}(\alpha)$, which is 1 if $\alpha > 0$ and -1 if $\alpha < 0$. Then $\sigma_1 g_1(x) = x^2 (\sigma_1 \alpha + \sigma_1 r_1(x))$. As $r_1(x)$ is the remainder term, we can choose $\delta > 0$ such that $|r_1(x)| < |\alpha|$ for all x satisfying $|x| < \delta$. In this interval, $\sigma_1 \alpha + \sigma_1 r_1(x)$ is always positive. Similarly we multiply $g_2(x)$ by $\sigma_2 = \operatorname{sgn}(\beta)$, so that $\sigma_2 \beta > 0$. As with r_1 , in a sufficiently small neighborhood around 0 we have that $|r_2(y)| < |\beta|$, and in this neighborhood $\sigma_2 \beta + \sigma_2 r_2(y)$ is always positive.

This allows us to write $u = x\sqrt{\sigma_1(\alpha + r_1(x))}$ and $v = y\sqrt{\sigma_2(\beta + r_2(y))}$, which is nearly the ideal choices described above. But to be proper coordinates we require these maps to be invertible. To study these potential coordinates, we again use the implicit function theorem 1. Namely, we study the zeroes of the two functions

(24)
$$F_1(x,u) = x\sqrt{\sigma_1(\alpha + r_1(x))} - u, \qquad F_2(y,v) = y\sqrt{\sigma_2(\beta + r_2(y))} - v$$

in small neighborhoods of the origin (small enough so that the arguments of the square roots are always positive). We calculate that $F_1(0,0) = F_2(0,0) = 0$ while

$$\frac{\partial F_1}{\partial x}(0,0) = \sqrt{\sigma_1 \alpha}, \quad \frac{\partial F_2}{\partial y}(0,0) = \sqrt{\sigma_2 \beta}, \quad \frac{\partial F_1}{\partial y}(0,0) = -1, \quad \frac{\partial F_2}{\partial y}(0,0) = -1,$$

all of which are nonzero. By the implicit function theorem, the first two equalities show that x = X(u) and y = Y(v) in a neighborhood of (x, y, u, v) = (0, 0, 0, 0). The second two equalities confirm that u = U(x) and v = V(y) also holds in a neighborhood of the origin. Further, each of these coordinate maps is continuously differentiable. Thus $(x, y) \mapsto (u, v)$ is valid change of coordinates.

We have proved that $G(x,y) = g_1(x) - g_2(y) = \sigma_1 u^2 - \sigma_2 v^2$ around a neighborhood of the origin, i.e. that the conclusion (22) of the Morse lemma holds. To continue our investigation of mean value abscissae, we examine

$$G(x,y) = \sigma_1 u^2 - \sigma_2 v^2 = 0.$$

There are a few different possibilities depending on the combinations of the signs σ_1 and σ_2 . In terms of the original function f, we note that σ_1 is the sign of $f''(b_0)$ and σ_2 is the sign of $f'''(c_0)$.

- (i) If σ_1 and σ_2 have opposite signs, then G(x,y) = 0 is equivalent to $u^2 = -v^2$. The only solution is (u,v) = (0,0) and this solution is isolated.
- (ii) If σ_1 and σ_2 have the same sign, G(x,y)=0 is equivalent to $u^2=v^2$. This has two solutions $u=\pm v$, and no other nearby solutions.

Looking again at Figure 4, we see that case (i) corresponds to the right graph, and case (ii) corresponds to the left graph.

We summarize the results of our exploration in the following theorem.

Theorem 6. Let f be a four-times continuously differentiable function and fix an interval $[a_0,b_0] \subset \mathbb{R}$. Suppose c_0 is a mean value abscissa for f on the interval $[a_0,b_0]$, and suppose that $f''(c_0) = 0$ and $f'(b_0) = f'(c_0)$. Finally, suppose that both $f''(b_0)$ and $f'''(c_0)$ are nonzero. Then

- If $f''(b_0)$ and $f'''(c_0)$ have opposite signs, then c_0 cannot be extended to a continuous function c = C(b) near b_0 .
- If $f''(b_0)$ and $f'''(c_0)$ have the same sign, then there are two continuously differentiable functions $c = C_1(b)$ and $c = C_2(b)$ solving (1) for b near b_0 . There are no other nearby solutions.

5. Analytic Functions.

Looking back, if $f''(c_0) \neq 0$ we can use Theorem 4, while if $f''(c_0) = 0$ but $f'''(c_0) \neq 0$ and $f''(b_0) \neq 0$, then we can use Theorem 6. What if $f''(c_0) = f'''(c_0) = 0$ but $f^{(4)}(c_0) = f''''(c_0) \neq 0$? Or, even more ambitiously, what if

(25)
$$f'(c_0) = f''(c_0) = \dots = f^{(k)}(c_0) = 0$$
 but $f^{(k+1)}(c_0) \neq 0$

for k = 10 or k = 200? Ideally we do not want to have to prove a new theorem for each of these cases. There is also the possibility that *all* of the derivatives of f vanish at c_0 . For instance this is what happens at $c_0 = 1$ for the classic "bump function", defined to be $\exp(-1/(1-x^2))$ for -1 < x < 1 and 0 otherwise.

We can rule out this last possibility by restricting to analytic functions. Recall that a function f is analytic if the Taylor series for f centered at each point x_0 converges to f in a neighborhood of x_0 . That is, for each x_0 , we have the equality

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

for all x in a neighborhood of x_0 . One of the nice things about analytic functions is that we can only have $f^{(k)}(x_0) = 0$ for all $k \ge 1$ if f is a constant function. For the rest of this section we will assume that f is a non-constant analytic function, in which case we can always find a k so that (25) holds.

With this assumption in mind, let us return to the function $G(x,y) = F(x + b_0, y + c_0)$, where F is the implicit function (20) whose zeros represent solutions to the mean value theorem relation (1). As in the previous section, we will write

 $G(x,y) = g_1(x) - g_2(y)$ where g_1 and g_2 are as in (23). The given mean value abscissa implies that $g_1(0) = g_2(0) = 0$. But unlike before, the first nonzero term in the Taylor expansion of g_2 is when $f^{(k+1)}(c_0) \neq 0$, yielding the approximation

$$g_2(y) \approx \beta_0 y^k$$
.

where $\beta_0 = f^{(k+1)}(c_0)/k!$ is a nonzero constant. Similarly picking ℓ so that

(26)
$$f'(b_0) = f''(b_0) = \dots = f^{(\ell-1)}(b_0) = 0$$
 but $f^{(\ell)}(b_0) \neq 0$,

a slightly more involved calculation shows that

$$g_1(x) \approx \alpha_0 x^{\ell}$$

where this time the nonzero constant is $\alpha_0 = f^{(\ell)}(b_0)/(\ell!(b-a))$. We call ℓ and k the order of vanishing at the origin for g_1 and g_2 , respectively. Our equation G(x,y) = 0 now seems to be approximately

$$\alpha_0 x^{\ell} \approx \beta_0 y^k$$
.

As in our proof of a special case of the Morse lemma, we will make this precise by finding new coordinates u, v so that G(x, y) = 0 is exactly either $v^k = u^\ell$ or $v^k = -u^\ell$.

As f is analytic, we can see that both g_1 and g_2 are analytic. Representing g_1 and g_2 by their Taylor expansion near 0, we can write them as

$$g_1(x) = x^{\ell} \sum_{m=0}^{\infty} \alpha_m x^m, \qquad g_2(y) = y^k \sum_{m=0}^{\infty} \beta_m y^m,$$

where $\alpha_0 \neq 0$ and $\beta_0 \neq 0$ were defined above. As with our special case of the Morse lemma, we now multiply g_1 by $\sigma_1 = \operatorname{sgn}\alpha_0$ and g_2 by $\sigma_2 = \operatorname{sgn}\beta_2$, enabling us to take roots in a neighborhood of 0.

Taking these roots, we can define two smooth (in fact analytic) functions by

$$F_1(x,u) = x \Big(\sigma_1 \sum_{m=0}^{\infty} \alpha_m x^m \Big)^{1/\ell} - u, \qquad F_2(y,u) = y \Big(\sigma_2 \sum_{m=0}^{\infty} \beta_m y^m \Big)^{1/k} - v,$$

in a neighborhood of the origin. Suppose for the moment that the equations $F_1(x,u)=0$ and $F_2(y,v)=0$ defined a smooth change of coordinates $(x,y)\mapsto (u,v)$. Then, in the (u,v) coordinates, we would have $g_1=\sigma_1u^k$ and $g_2=\sigma_2u^\ell$ so that G(x,y)=0 was equivalent to

(27)
$$u^k = \frac{\sigma_2}{\sigma_1} v^\ell = \pm v^\ell,$$

analogous to the Morse lemma but with higher powers.

Checking that $F_1(x,u) = 0$ and $F_2(y,v) = 0$ define an invertible change of coordinates x = X(u) and y = Y(v) can be proved from the implicit function theorem Theorem 1, and in this case the proof is almost identical to the proof for (24), the change of coordinates from our consideration of Morse's lemma. Thus $(x,y) \mapsto (u,v)$ is a valid change of coordinates, and we can study G(x,y) = 0 by studying solutions to (27).

Thinking about the graph of (27) for different values of k and ℓ , and different combinations of signs of σ_1 and σ_2 , we find that

(i) If k is odd, then there is one continuous solution $v = V_1(u)$ of (27) in a neighborhood of the origin, and no other nearby solutions.

- (ii) If k and ℓ are both even and $\sigma_2/\sigma_1 = +1$ then there are two continuous solutions $v = V_1(u)$ and $v = V_2(u)$ of (27) in a neighborhood of the origin, and no other nearby solutions.
- (iii) If k and ℓ are both even and $\sigma_2/\sigma_1 = -1$, then the origin is an isolated solution of (27).
- (iv) If k is even and ℓ is odd, then there are two continuous solutions $v = V_1(u)$ and $v = V_2(u)$ of (27), but they are only defined in a one-sided neighborhood of the origin where $(\sigma_2/\sigma_1)u \geq 0$.

Since we can always write y = Y(v) and u = U(x), continuously solving for v in terms of u is equivalent to continuously solving for y in terms of x. As we're solving $G(x,y) = F(x+b_0,y+c_0) = 0$, this is equivalent to continuously solving (1) for the abscissa $c = c_0 + y$ in terms of the endpoint $b = b_0 + x$ for x in a neighborhood of $(b,c) = (b_0,c_0)$.

Thus we can find a continuous choice of mean value abscissa near any point where (i) or (ii) hold. The following lemma tells us that there there is *always* at least one such point (i) holds.

Lemma 7. Let f be a non-constant analytic function satisfying $f(a_0) = f(b_0) = 0$. Then there is a mean value abscissa c_0 of f on $[a_0, b_0]$ such that $f'(c_0) = 0$ and such that the smallest $k \ge 1$ such that $f^{(k)}(c_0) \ne 0$ is even, i.e. such that the order of vanishing of f' at c_0 is odd.

Proof. Since f is non-constant, f takes an absolute maximum or an absolute minimum at a point c within the interval (a_0, b_0) . Notice that c is not an endpoint of the interval since $f(a_0) = f(b_0) = 0$ and the function is not constant. We choose c_0 to be this point c.

As c_0 is an extremum, $f'(c_0) = 0$. From its Taylor expansion, we see that near c_0 , f is very closely approximated by $f(c_0) + a_k(x - c)^k$, where $a_k = f^{(k)}(c_0)/(k!)$. If k were odd, then f would be strictly increasing or decreasing at c_0 , contradicting the fact that it has a local extremum there.

With this lemma, we are now ready to complete our study of when there exist continuous choices of mean value abscissae c = C(b). Recall that we assume without loss of generality that $f(a_0) = f(b_0) = 0$. Let c_0 be a mean value abscissa for f on $[a_0, b_0]$ such that the order of vanishing of f' at c_0 is odd, as guaranteed by the lemma. The order of vanishing of f' at c_0 is the same as the order of vanishing of g_2 at 0 by construction, and thus the lemma indicates that the k appearing in (27) is odd. Thus we are in case (i), and we can uniquely solve (1) for c = C(b) in a neighborhood of b_0 . This completes the proof of the following theorem.

Theorem 8. Let f be real analytic on the interval $[a_0,b_0]$. Then there exists at least one mean value abscissa $c_0 \in (a_0,b_0)$ such that c_0 is a mean value abscissa for f on $[a_0,b_0]$, and for which there exists a continuous function c=C(b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(C(b))$$

for all b in a neighborhood of b_0 . There are no other solutions near (b_0, c_0) .

Remark. As a final note, we note that being "merely" infinitely differentiable is not strong enough to guarantee that there is always a choice of a mean value abscissa with a continuous dependence on the right endpoint. For a counterexample, see

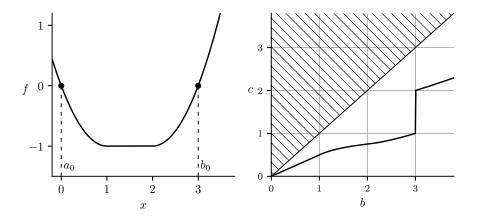


FIGURE 5. A smooth function where no continuous choice of mean value abscissa exists.

Figure 5. One can construct a smooth function of this shape from bump functions. On the indicated interval [0,3], every value c_0 with $1 \le c_0 \le 2$ is a valid mean value abscissa; this is reflected in the mean value abscissa plot on the right by a vertical line segment from (3,1) to (3,2). For b just to the left of b_0 , the slope of the secant line is negative, and for b just to the right of b_0 , the slope of the secant line is positive. But any value c_0 is either at least distance 1/2 away from a point c where f'(c) < 0 or a point c where f'(c) > 0. There is no continuous choice of mean value abscissa for this function.

6. Reflection and Further Questions.

The major selling-point of Theorem 8 is that it gives us a continuous choice c = C(b) of mean value abscissa without any assumptions on f other than analyticity. On the other hand, in cases where we do know more about our favorite abscissa c_0 , the classification (i)–(iv) of the curves (27) gives much more information about nearby solutions. And for the local picture, we should expect "most" points for "most" intervals to not be degenerate enough that Theorem 4 and Theorem 6 both fail.

There are many more questions that one could ask about the set of solutions to (1). Firstly, what if you allow the left endpoint a to vary as well as b and look for continuous choices c = C(a, b)? The techniques we have used will still be very powerful, but for instance the decomposition $G(x, y) = g_1(x) - g_2(y)$ in the above section will no longer be as simple.

One could also study the *global* structure of the solution sets shown in Figures 3 and 4. How many different connected components are there? In what ways can they "begin" and "end"? Answering such questions will require very different techniques.

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http://davidlowryduda.com/choosing-functions-for-mvt-abscissa/.

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