ON THE DISTRIBUTION OF NUMBERS RELATED TO THE DIVISORS OF $x^n - 1$

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ABSTRACT. Let $n_1, \cdots, n_r$ be any finite sequence of integers and let S be the set of all natural numbers n for which there exists a divisor $d(x) = 1 + \sum_{i=1}^{deg(d)} c_i x^i$ of $x^n - 1$ such that $c_i = n_i$ for $1 \leq i \leq r$. In this paper we show that the set S has a natural density. Furthermore, we find the value of the natural density of S.

1. INTRODUCTION

Cyclotomic polynomials arise naturally as irreducible divisors of $x^n - 1$. The polynomial $x^n - 1$ can be factored in the following way

$$(1) \qquad x^n - 1 = \prod_{d|n} \phi_d(x).$$

Applying Mobius inversion we get

$$(2) \qquad \phi_n(x) = \prod_{d|n} (x^d - 1)^{\mu(\frac{n}{d})}.$$

The problem of determining size of maximum coefficient of cyclotomic polynomials has been the subject of the papers [4] and [1]. In [3] Pomerance and Ryan study the size of maximum coefficient of divisors of $x^n - 1$.

It has been proven that for every finite sequence of integers $(n_i)_{i=1}^r$, there exists $d(x) = 1 + \sum_{i=1}^{deg(d)} c_i x^i$, a divisor of $x^n - 1$ for some $n \in \mathbb{N}$, such that $c_i = n_i$ for $1 \leq i \leq r$. In this paper we investigate the following problem. For a given sequence $(n_i)_{i=1}^r$, let $S(n_1, \cdots, n_r)$ denote the set of all n such that $x^n - 1$ has a divisor $d(x)$ of the form $d(x) = 1 + \sum_{i=1}^r n_i x^i + \sum_{i=r+1}^{deg(d)} c_i x^i$. We prove that $S(n_1, \cdots, n_r)$ has a natural density. Observe that if $n \in S(n_1, \cdots, n_r)$ then every multiple of n is in $S(n_1, \cdots, n_r)$.

2. NOTATION

If $f(x)$ and $g(x)$ are two analytic functions in some neighborhood of 0, we denote $f(x) \equiv g(x) \mod x^{r+1}$ if the coefficients of x^i in the power series of $f(x)$ and $g(x)$ are equal for $0 \leq i \leq r$.

We denote $\omega(n)$ for number of distinct prime factors of n. Let $\delta(d)$ be 1 if $d \neq 1$ and $\delta(d)$ be -1 otherwise. Note that

$$(3) \qquad \phi_n(x) = \delta(n) \prod_{d|n} \left(1 - x^d\right)^{\mu\left(\frac{n}{d}\right)}.$$

3. Proof of Main Theorem

We require several lemmas in order to prove that $S(n_1, \cdots, n_r)$ has a natural density.

Lemma 3.1. *For every finite sequence of integers $n_1, \cdots, n_r$ there exists a unique sequence of integers $k_1, \cdots, k_r$ such that*

$$(4) \qquad \prod_{i=1}^{r}(1 - x^i)^{k_i} \equiv 1 + \sum_{i=1}^{r} n_i x^i \mod x^{r+1}.$$

Proof. The proof that there exists a sequence $k_1, \cdots, k_r$ is by induction on r. If $r = 1$ and $n_1 \in \mathbb{Z}$ then $(1 - x)^{-n_1} \equiv 1 + n_1 x \mod x^2$ hence the existence part of lemma is true for $r = 1$. If we assume that the existence part of lemma is true for r, then for any sequence of $r + 1$ integers $(n_i)_{i=1}^{r+1}$, there exist r integers $k_1, \cdots, k_r$ such that

$$\prod_{i=1}^{r}(1 - x^i)^{k_i} \equiv 1 + \sum_{i=1}^{r} n_i x^i \mod x^{r+1}.$$

Let n'_{r+1} be an integer such that

$$\prod_{i=1}^{r}(1 - x^i)^{k_i} \equiv 1 + \sum_{i=1}^{r} n_i x^i + n'_{r+1} x^{r+1} \mod x^{r+2}.$$

We have

$$\prod_{i=1}^{r}(1 - x^i)^{k_i}(1 - x^{r+1})^{n'_{r+1} - n_{r+1}} \equiv 1 + \sum_{i=1}^{r+1} n_i x^i \mod x^{r+2}.$$

Hence the existence part of the lemma is true for $r + 1$.

For the uniqueness part, if there are two finite sequences $k_1, \cdots, k_r$ and $k'_1, \cdots, k'_r$ such that

$$\prod_{i=1}^{r}(1 - x^i)^{k_i} \equiv \prod_{i=1}^{r}(1 - x^i)^{k'_i} \mod x^{r+1}.$$

If the two sequences are distinct then let i be the least index such that $k_i - k'_i \neq 0$ then we have

$$\prod_{j=i}^{r}(1 - x^i)^{k_i - k'_i} \equiv 1 \mod x^{i+1}$$

or

$$1 - (k_i - k'_i)x^i \equiv 1 \mod x^{i+1}$$

which implies $k_i - k'_i = 0$ contradicting the assumption that $k_i - k'_i \neq 0$. $\square$

For a given sequence $n_1, \cdots, n_r$ we proved that there exists a unique sequence $k_1(n_1, \cdots, n_r), \cdots, k_r(n_1, \cdots, n_r)$ such that equation (4) is true. Let $A(n_1, \cdots, n_r)$ be the set defined by $A(n_1, \cdots, n_r) := \{1 \leq i \leq r : k_i(n_1, \cdots, n_r) \neq 0\}$. If the set $A(n_1, \cdots, n_r)$ is non empty let $l(n_1, \cdots, n_r)$ be the least common multiple of elements of $A(n_1, \cdots, n_r)$, otherwise let $l(n_1, \cdots, n_r)$ be 1.

Lemma 3.2. *If $n \in S(n_1, \cdots, n_r)$ then n is a multiple $l(n_1, \cdots, n_r)$.*

Proof. We will prove that if $l(n_1, \cdots, n_r) \nmid n$ then $n \notin S(n_1, \cdots, n_r)$. If $l(n_1, \cdots, n_r)$ does not divide n then there exists an $i \in A(n_1, \cdots, n_r)$ such that $i \nmid n$. That is, $k_i(n_1, \cdots, n_r) \neq 0$ and $i \nmid n$.

Any divisor $d(x)$ of $x^n - 1$ such that $d(0) = 1$ will be of the form

$$d(x) = \prod_{d \in S} \delta(d)\phi_d(x),$$

where S is some subset of set of divisors of n. Hence

$$
\begin{aligned}
d(x) &= \prod_{d \in S} \delta(d)\phi_d(x) \\
&= \prod_{d \in S} \prod_{d'|d} \left(1 - x^{d'}\right)^{\mu\left(\frac{d}{d'}\right)} \\
&\equiv \prod_{1 \leq d' \leq r} \prod_{\substack{d \equiv 0 \!\!\!\mod d' \\ d \in S}} \left(1 - x^{d'}\right)^{\mu\left(\frac{d}{d'}\right)} \quad \mod x^{r+1} \\
&\equiv \prod_{j=1}^{r} (1 - x^j)^{l_j} \quad \mod x^{r+1},
\end{aligned}
$$

where $l_m = \sum_{\substack{d \in S \\ d \equiv 0 \!\!\!\mod m}} \mu\left(\frac{d}{m}\right)$ for $1 \leq m \leq r$. Therefore as $i \nmid n$, $l_i = 0$. Hence $l_i \neq k_i(n_1, \cdots, n_r)$ and from uniqueness part of Lemma 3.1 we have $d(x) \not\equiv 1 + \sum_{j=1}^{r} n_j x^j \mod x^{r+1}$. Hence $n \notin S(n_1, \cdots, n_r)$. □

Lemma 3.3. *If $p_1, \cdots, p_s$ are distinct primes greater than r not dividing d and $q_1, \cdots, q_s$ are distinct primes greater than r and not dividing d then for all natural numbers $e_1, \cdots, e_s$ we have $\phi_{dp_1^{e_1}\cdots p_s^{e_s}}(x) \equiv \phi_{dq_1^{e_1}\cdots q_s^{e_s}}(x) \mod x^{r+1}$.*

Proof. For every divisor d' of d we have $\mu\left(\frac{dp_1^{e_1}\cdots p_s^{e_s}}{d'}\right) = \mu\left(\frac{dq_1^{e_1}\cdots q_s^{e_s}}{d'}\right)$. From equation (2)

$$
\begin{aligned}
\phi_{dp_1^{e_1}\cdots p_s^{e_s}}(x) &\equiv \prod_{d'|d} \left(1 - x^{d'}\right)^{\mu\left(\frac{dp_1^{e_1}\cdots p_s^{e_s}}{d'}\right)} \quad \mod x^{r+1} \\
&\equiv \prod_{d'|d} \left(1 - x^{d'}\right)^{\mu\left(\frac{dq_1^{e_1}\cdots q_s^{e_s}}{d'}\right)} \quad \mod x^{r+1} \\
&\equiv \phi_{dq_1^{e_1}\cdots q_s^{e_s}}(x) \quad \mod x^{r+1}.
\end{aligned}
$$

□

Lemma 3.4. *If p_1 and p_2 are two distinct primes greater than r and if $d \leq r$ then $\phi_{dp_1p_2}(x) \equiv \delta(d)\phi_d \mod x^{r+1}$.*

Proof. From (2) we have

$$\phi_{dp_1p_2}(x) \equiv \prod_{d'|d} \left(1 - x^{d'}\right)^{\mu\left(\frac{dp_1p_2}{d'}\right)} \mod x^{r+1}$$

$$\equiv \prod_{d'|d} \left(1 - x^{d'}\right)^{\mu\left(\frac{d}{d'}\right)} \mod x^{r+1}$$

$$\equiv \delta(d)\phi_d(x) \mod x^{r+1}.$$

$$\square$$

Lemma 3.5. *For every finite sequence $n_1, \cdots, n_r$ there exist k distinct primes $q_1, \cdots, q_k$ greater than r such that $n = l(n_1, \cdots, n_r)q_1q_2 \cdots q_k \in S(n_1, \cdots, n_r)$.*

Proof. From Lemma 3.1 we have $\prod_{i=1}^{r}(1 - x^i)^{k_i} \equiv 1 + \sum_{i=1}^{r} n_i x^i \mod x^{r+1}$, where $k_i = k_i(n_1, \cdots, n_r)$. From the definition of $A(n_1, \cdots, n_r)$, $k_i \neq 0$ if and only if $i \in A(n_1, \cdots, n_r)$. Let $i_1, \cdots, i_p$ be the elements of $A(n_1, \cdots, n_r)$. We have

$$(5) \qquad 1 + \sum_{i=1}^{r} n_i x^i \equiv \prod_{j=1}^{p}(1 - x^{i_j})^{k_{i_j}} \mod x^{r+1}.$$

Let $r_1^{(j)}, \cdots, r_{|k_j|}^{(j)}$ for $1 \le j \le p$ be numbers such that for $1 \le a \le |k_{i_{j_1}}|$ and $1 \le b \le |k_{i_{j_2}}|$, $r_a^{(j_1)} = r_b^{(j_2)}$ if and only if $j_1 = j_2$ and $a = b$. If $k_{i_j} > 0$ then $r_a^{(j)}$ is a product of two distinct primes and each prime factor of $r_a^{(j)}$ is greater than r. If $k_{i_j} < 0$ then $r_a^{(j)}$ is a prime number greater than r.

If $k_{i_j} > 0$ then let

$$(6) \qquad d_j(x) = \prod_{m=1}^{k_{i_j}} \prod_{d|i_j} \phi_{dr_m^{(j)}}(x).$$

If $k_{i_j} > 0$ then as $r_m^{(j)}$ is a product two prime factors greater from Lemma 3.4 we have $\phi_{dr_m^{(j)}}(x) \equiv \delta(d)\phi_d(x) \mod x^{r+1}$. Therefore

$$\prod_{d|i_j} \phi_{dr_m^{(j)}}(x) \equiv \prod_{d|i_j} \delta(d)\phi_d(x) \mod x^{r+1}$$

$$\equiv (1 - x^{i_j}) \mod x^{r+1}.$$

Hence from (6) we have

$$(7) \qquad d_j(x) \equiv \prod_{m=1}^{k_{i_j}}(1 - x^{i_j}) \equiv (1 - x^{i_j})^{k_{i_j}} \mod x^{r+1}.$$

If $k_{i_j} < 0$ let

$$d_j(x) = \prod_{m=1}^{-k_{i_j}} \prod_{d|i_j} \phi_{dr_m^{(j)}}(x) \mod x^{r+1}.$$

As $k_{i_j} < 0$, $r_m^{(j)}$ is a prime number greater than r. Hence

$$\prod_{d|i_j} \phi_{dr_m^{(j)}}(x) = \frac{\prod_{d|i_j r_m^{(j)}} \phi_d(x)}{\prod_{d|i_j} \phi_d(x)}$$

$$\equiv \frac{\left(x^{i_j r_m^{(j)}} - 1\right)}{\left(x^{i_j} - 1\right)} \quad \mod x^{r+1}$$

$$\equiv \left(1 - x^{i_j}\right)^{-1} \quad \mod x^{r+1}.$$

Therefore

$$(8) \qquad d_j(x) \equiv \prod_{m=1}^{-k_{i_j}} \left(1 - x^{i_j}\right)^{-1} \equiv \left(1 - x^{i_j}\right)^{k_{i_j}} \quad \mod x^{r+1}.$$

From (5), (7) and (8) we have

$$(9) \qquad d(x) = \prod_{j=1}^{p} d_j(x) \equiv \prod_{j=1}^{p} (1 - x^{i_j})^{k_{i_j}} \equiv 1 + \sum_{i=1}^{r} n_i x^i \quad \mod x^{r+1}.$$

If the set $\{i_j r_m^{(j)} : 1 \leq j \leq p, 1 \leq m \leq |k_{i_j}|\}$ is non empty, let n be the least common multiple of the elements of the set and let $n = 1$ if the set is empty. Clearly $d(x)$ is a divisor of $x^n - 1$ and therefore $n \in S(n_1, \cdots, n_r)$. Observe that n is of the form $l(n_1, \cdots, n_r) q_1 q_2 \cdots q_k$ where q_i's are distinct prime factors greater than r. □

Theorem 3.6. *For every finite sequence* $n_1, \cdots, n_r$, *let* $N(n_1, \cdots, n_r, x)$ *denote number of* $n \leq x$ *such that* $n \in S(n_1, \cdots, n_r)$. *There exists a* $k \in \mathbb{N}$ *such that*

$$N(n_1, \cdots, n_r, x) = C(n_1, \cdots, n_r)x + O\left(\frac{x(\log\log x)^k}{\log x}\right),$$

where $C(n_1, \cdots, n_r) = \frac{1}{l(n_1, \cdots, n_r)}$.

Proof. For brevity, let $S(n_1, \cdots, n_r) = S$ and $l(n_1, \cdots, n_r) = l$. From Lemma 3.5 there exists an m_1 of the form $m_1 = lq_1 \cdots q_k$ and a divisor $d_1(x)$ of $x^{m_1} - 1$ such that

$$(10) \qquad\qquad d_1(x) \equiv 1 + \sum_{i=1}^{p} n_i x^i \quad \mod x^{r+1}.$$

For every m_2 of the form $m_2 = lp_1 \cdots p_k$ such that $p_1, \cdots, p_k$ are distinct primes greater than r. Let S_1 be the set of divisors of m_1 and S_2 be the set of divisors of m_2. Let $g : S_1 \to S_2$ be a map defined as follows. As l and $q_1 \cdots q_k$ are relatively prime, every divisor of d of $lq_1 \cdots q_k$ can be uniquely written in the form $d = d_1' q_{i_1} \cdots q_{i_s}$ where d_1' divides l. Define $g(d_1' q_{i_1} \cdots q_{i_s}) = d_1' p_{i_1} \cdots p_{i_s}$. From Lemma 3.3 it follows that $\phi_d(x) \equiv \phi_{g(d)}(x) \mod x^{r+1}$. As $d_1(x)$ is of the form $\prod_{d' \in R_1} \delta(d')\phi_{d'}(x)$ where R_1 is a subset of S_1 there will be $d_2(x) = \prod_{d' \in R_1} \delta(g(d'))\phi_{g(d')}(x)$, a divisor of $x^{m_2} - 1$, and $d_2(x) \equiv d_1(x) \equiv 1 + \sum_{i=1}^{r} n_i x^i \mod x^{r+1}$. Therefore every number of the form $lp_1 \cdots p_k$ where p_i's are distinct primes greater than r belongs to S which implies that every number lm belongs to S, if number of distinct prime factors of m greater than r is at least k. Hence if $\omega(m) \geq r + k$ then $lm \in S$ as $\omega(m) \geq r + k$

implies that number of prime factors of m greater than r is at least k. From 3.1. Lemma B of [2]

$$N(n_1, \cdots, n_r, x) \geq |\{lm \leq x : \omega(m) \geq r + k\}| = \frac{x}{l} + O\left(\frac{x(\log \log x)^{r+k-1}}{\log x}\right).$$

From Lemma 3.2, if $n \in S$ then $l|n$ which implies that $N(n_1, \cdots, n_r, x) \leq \frac{x}{l}$. Combining the two inequalities we get

$$N(n_1, \cdots, n_r, x) = \frac{x}{l} + O\left(\frac{x(\log \log x)^{r+k-1}}{\log x}\right)$$

which completes the proof of the theorem. $\square$

References

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