ASIAN OPTION PRICING UNDER UNCERTAIN VOLATILITY MODEL

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ABSTRACT. In this paper, we study the asymptotic behavior of Asian option prices in the worst case scenario under an uncertain volatility model. We give a procedure to approximate the Asian option prices with a small volatility interval. By imposing additional conditions on the boundary condition and cutting the obtained Black-Scholes-Barenblatt equation into two Black-Scholes-like equations, we obtain an approximation method to solve the fully nonlinear PDE.

Key Words: Asian option, nonlinear Black-Scholes-Barenblatt PDE, uncertain volatility model, stochastic control

1. Introduction

An option on a traded account is a financial contract which allows the buyer of the contract obtains the right to trade an underlying asset for a specified price, called strike price, during the life of the option. There are varieties of options, such as European option, American option, Asian option and barrier option. As the foundations for the modern analysis of options, the Black-Scholes-Merton pricing formula for European option was introduced by Black, Scholes [1] and Merton [2]. In Blcak-Scholes-Merton model, the volatility is assumed to be constant. However, constant volatility cannot explain observed market prices for options.

After Black, Scholes and Merton's work, some scholars studied option pricing models with stochastic volatility. In a series of papers, several models for stochastic volatility were introduced, such as Hull-White stochastic volatility model [3] and Heston stochastic volatility model [4].

Uncertain volatility model is another approach to describe the non-constant volatility. In 1995, uncertain volatility model was introduced by Lyons [5] and Avellaneda et al. [6]. In these models, volatility is assumed to lie within a range of values. So the prices are no longer unique. We can only get the best-case scenario prices and the worst-case scenario prices. Several problems about uncertain volatility have been studied. We can see these results in Lyons [5], Avellaneda et al. [6], Dokuchaev, Savkin [7], Forsyth, Vetzal [8] and Vorbrink [9]. Pricing in uncertain volatility

models involving nonlinear partial differential equations have been showed in their paper. Some numerical methods have been proposed in Pooley, Forsyth, Vetzal[10], and Avellaneda et al. [6].

In 2014, Fouque and Ren [11] studied the price of European derivatives in the worst case scenario under the uncertain volatility model. They provided an approximate method of pricing the derivatives with a small volatility interval. In addition, the paper also presented that the solution reduces to a constant volatility problem when it comes to simple options with convex payoffs.

In this paper, the pricing problem of Asian options is studied. The payoff function is path-dependent on risky asset price processes. Another variable is given to solve the problem. In the process of finding the estimation of the worst case scenario Asian option prices, the first problem that we meet is obtaining the Hamilton-Jacobi-Bellman (HJB) equation of the prices. The HJB equation is called Black-Scholes-Barenblatt (BSB) equation in the financial mathematics. We can get the BSB equation by the stochastic control theory. The next difficulty is to proof the convergence of the estimation. To control the error term, we obtain its expectation form by Dynkin formula and find what conditions should we impose on the payoff function by proving and deducing. Finally, we get the approximation procedure for the prices. Compare with Fouque and Ren's paper [11], we add an equation in the stochastic control system and it can also be reflected in the BSB equation. In terms of the dynamic of the risky asset price process, we give an equation to describe the path-dependence. When estimate the expectation form, We use the relationship between the two processes. In section 4.4, we fix one of two variables first to simplify the problem. Another method we used to manage the two variables is changing the form of the BSB equation.

The organization of this paper is as follows. In section 2, we briefly describe Asian options under uncertain volatility model and give the Black-Scholes-Barenblatt (BSB) equations of option prices. In section 3, we find the estimation of Asian option prices in the worst case scenario and the estimation is relied on two Black-Scholes-like PDEs. Next, we propose the main result of this paper which shows the rationality of estimation. In section 4, we give the proof of the main result. Through the conditions imposed on the payoff function, we get the convergence of the error term. In the process, we obtain the expectation form of the error term and it is cut into three parts. The controls of the three parts are given by the stochastic control theory and

the properties of the worst-case scenario Asian option price process. Finally, we give the conclusion of this paper.

2. Asian options under uncertain volatility model

In this section, we introduce the Asian options under uncertain volatility model. Then we give the Black-Scholes-Barenblatt (BSB) equation of the Asian options' prices. Suppose that \mathcal{X} is an Asian option written on the risky asset with maturity T and payoff $\varphi(\cdot)$. $\varphi(\cdot)$ is a non-convex function and the result is identical to Black-Scholes result under convex condition. That is to say, the results of this article cover generalized Asian options.

Assume that the price process of the risky asset X_t solves the following stochastic differential equation

$$(2.1) dX_t = rX_t dt + \sigma_t X_t dW_t,$$

where r is the constant risk-free interest rate, W_t is a standard Brownian motion on the probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ and the volatility process $\sigma_t \in \mathcal{A}[\underline{\sigma}, \overline{\sigma}]$ for each $t \in [0, T]$, which is a family of progressively measurable and $[\underline{\sigma}, \overline{\sigma}]$ -valued processes. By the definition as above, we know that the volatility in an uncertain volatility model is not a stochastic process with a probability distribution, but a family of stochastic processes with unknown prior information. Thus, what can we use to distinguish the difference between uncertain volatility models is the model ambiguity.

Due to the path-dependence on risky asset price processes, we assume that $Y_{t,T}$ satisfies the expression as follows

(2.2)
$$Y_{t,T} = \frac{Y_T - Y_t}{T - t},$$

where
$$Y_t = \int_0^t X_u du$$
.

Then we can get Asian option prices in the worst case scenario price at time t < T as follows

(2.3)
$$V(t, X_t, Y_t) = \exp\left(-r(T - t)\right) \underset{\sigma \in \mathcal{A}[\underline{\sigma}, \overline{\sigma}]}{\operatorname{ess sup}} E[\varphi(Y_{0,T}) | \mathfrak{F}_t],$$

where ess sup is essential supremum. By the ambiguity of the uncertain volatility model, we obtain the definition of price as equation (2.3). Obviously, the worst case scenario price is for the seller of options. It is related to coherent risk measure which quantifies the model risk induced by volatility uncertainty (see [12]). Moreover, the model ambiguity in mathematical finance has captured the attention of many. Therefore, we should pay attention to the importance of the worst case prices.

Through the stochastic control theory (see [13]), $V(t, X_t, Y_t)$ satisfies the Hamilton-Jacobi-Bellman (HJB) equation (Black-Scholes-Barenblatt (BSB) equation).

Lemma 2.1. $V(t, X_t, Y_t)$ satisfies the following Black-Scholes-Barenblatt equation

(2.4)
$$\begin{cases} \partial_t V + r(x\partial_x V - V) + x\partial_y V + \sup_{\sigma \in \mathcal{A}[\underline{\sigma}, \overline{\sigma}]} \left[\frac{1}{2} x^2 \sigma^2 \partial_{xx}^2 V \right] = 0, \\ 0 \le t \le T, \ x \ge 0, \ y \ge 0, \\ V(T, x, y) = \varphi(\frac{y}{T}), \ x \ge 0, \ y \ge 0. \end{cases}$$

Proof. Notice that the stochastic control system is

$$\begin{cases} dX_t = rX_t dt + \sigma_t X_t dW_t, & \sigma_t \in \mathcal{A}[\underline{\sigma}, \overline{\sigma}], \\ dY_t = X_t dt. \end{cases}$$

Then for all $(s, x, y) \in [0, T] \times \mathbb{R}^+ \times \mathbb{R}^+$, we first establish the dynamic program frame

(2.5)
$$\begin{cases} dX_t = rX_t dt + \sigma_t X_t dW_t, \\ dY_t = X_t dt, \\ X_s = x, \\ Y_s = y. \end{cases}$$

The cost function is

$$J(s, x, y; \sigma) = E_s \left[e^{-r(T-s)} \varphi(Y_{0,T}) \right],$$

where $E_s[\cdot] = E[\cdot|\mathfrak{F}_s]$. The value function is

$$V(s, x, y) = \underset{\sigma \in \mathcal{A}[\sigma, \overline{\sigma}]}{\operatorname{ess sup}} J(s, x, y; \sigma).$$

For all $0 \le s \le \hat{s} \le T$, $\sigma \in \mathcal{A}[\underline{\sigma}, \overline{\sigma}]$, we have

$$V(s,x,y) \geq E_s \left[e^{-r(T-s)} \varphi(Y_{0,T}) \right]$$
$$= E_s \left[\int_{\hat{s}}^{\hat{s}} -re^{-r(T-t)} \varphi dt + e^{-r(T-\hat{s})} \varphi \right].$$

Then we obtain

$$0 \ge E_s \left[\int_s^{\hat{s}} -re^{-r(T-t)} \varphi dt \right] + V(\hat{s}, x, y) - V(s, x, y).$$

Divided by $\hat{s} - s$ on both sides of the inequality, we have that

$$0 \ge E_s \left[\frac{\int_s^{\hat{s}} -re^{-r(T-t)} \varphi dt}{\hat{s} - s} \right] + \frac{V(\hat{s}, x, y) - V(s, x, y)}{\hat{s} - s}.$$

Here, assume that φ is Lipschitz continuous. Then according to Itô formula and equations (2.5), we obtain

$$dV = V_t dt + V_x dX_t + V_y dY_t + \frac{1}{2} V_{xx} dX_t dX_t + \frac{1}{2} V_{yy} dY_t dY_t + \frac{1}{2} V_{xy} dX_t dY_t$$

= $(V_t + rX_t V_x + X_t V_y + \frac{1}{2} \sigma_t^2 X_t^2 V_{xx}) dt + \sigma_t X_t V_x dW_t.$

Let $\hat{s} \to s$. For all $\sigma \in \mathcal{A}[\underline{\sigma}, \overline{\sigma}]$, we have that

$$0 \geq -rE_{s}[e^{-r(T-s)}\varphi] + V_{t} + rX_{s}V_{x} + X_{s}V_{y} + \frac{1}{2}\sigma_{s}^{2}X_{s}^{2}V_{xx}$$

$$\geq -rV(s,x,y) + V_{t}(s,x,y) + rxV_{x}(s,x,y) + xV_{y}(s,x,y) + \frac{1}{2}\sigma_{s}^{2}X_{s}^{2}V_{xx}(s,x,y),$$

which is

$$(2.6) 0 \ge -rV + V_t + rxV_x + xV_y + \sup_{\sigma \in \mathcal{A}[\underline{\sigma}, \overline{\sigma}]} \frac{1}{2} \sigma^2 x^2 V_{xx}.$$

On the other hand, for any $\varepsilon > 0$, there is $\sigma(\varepsilon) \in \mathcal{A}[\underline{\sigma}, \overline{\sigma}]$ such that

$$V(s, x, y) - \varepsilon(\hat{s} - s) \leq E_s \left[e^{-r(T - s)} \varphi \right]$$

$$= E_s \left[\int_s^{\hat{s}} -re^{-r(T - t)} \varphi dt \right] + E_s \left[e^{-r(T - \hat{s})} \varphi \right].$$

So we have that

$$-\varepsilon \leq E_s \left[\frac{\int_s^{\hat{s}} -re^{-r(T-t)} \varphi dt}{\hat{s} - s} \right] + \frac{V(\hat{s}, x, y) - V(s, x, y)}{\hat{s} - s}.$$

Argument as above, we obtain

$$(2.7) 0 \le -rV + V_t + rxV_x + xV_y + \sup_{\sigma \in \mathcal{A}[\underline{\sigma}, \overline{\sigma}]} \frac{1}{2} \sigma^2 x^2 V_{xx}.$$

Combining (2.6) with (2.7), we have

$$0 = -rV + V_t + rxV_x + xV_y + \sup_{\sigma \in \mathcal{A}[\underline{\sigma}, \overline{\sigma}]} \frac{1}{2} \sigma^2 x^2 V_{xx}.$$

Remark 2.2. Here, adding variable Y into dynamical system leads to a more complex stochastic control system, which adds the dimensionality of the BSB equation.

Remark 2.3. Notice that, (2.4) is a fully nonlinear PDE which doesn't have a solution like Black-Scholes equation. Thus, we decide to solve the problem by reducing it to solving two Black-Scholes-like PDEs.

3. Black-Scholes-like PDEs and Main Result

In this section, we first reparameterize the uncertain volatility model to study the prices in the worst case scenario. Assume that the risky asset price process X_t^{ε} has a dynamic

(3.1)
$$\begin{cases} dX_t^{\varepsilon} = rX_t^{\varepsilon}dt + \sigma_t X_t^{\varepsilon}dW_t, \\ dY_t^{\varepsilon} = X_t^{\varepsilon}dt, \end{cases}$$

where $\sigma_t \in \mathcal{A}^{\varepsilon} = {\{\sigma_t | \sigma_t \text{ is a } [\sigma_0, \sigma_0 + \varepsilon] - \text{valued processively measurable process}\}}$ and $\sigma_0 \in [\underline{\sigma}, \overline{\sigma}].$

The cost function is

$$J^{\varepsilon}(t, x, y; \sigma) = e^{-r(T-t)} E_{txy} \left[\varphi(Y_{0,T}^{\varepsilon}) \right],$$

where $E_{txy}[\cdot]$ means the conditional expectation taken with respect to $X_t^{\varepsilon} = x$, $Y_t^{\varepsilon} = y$. The value function is

$$V^{\varepsilon}(t, x, y; \sigma) = \underset{\sigma \in \mathcal{A}^{\varepsilon}}{\operatorname{ess sup}} [J^{\varepsilon}(t, x, y; \sigma)].$$

By Lemma 2.1 we get the following Black-Scholes-Barenblatt equation of V^{ε} .

(3.2)
$$\begin{cases} \partial_t V^{\varepsilon} + r(x\partial_x V^{\varepsilon} - V^{\varepsilon}) + x\partial_y V^{\varepsilon} + \sup_{\sigma \in \mathcal{A}^{\varepsilon}} \frac{1}{2} x^2 \sigma^2 \partial_{xx}^2 V^{\varepsilon} = 0, \\ 0 \le t \le T, \ x \ge 0, \ y \ge 0, \\ V^{\varepsilon}(T, x, y) = \varphi(\frac{y}{T}), \ x \ge 0, \ y \ge 0, \end{cases}$$

which is equivalent to

(3.3)
$$\begin{cases} \partial_t V^{\varepsilon} + r(x\partial_x V^{\varepsilon} - V^{\varepsilon}) + x\partial_y V^{\varepsilon} + \sup_{\gamma \in \mathcal{A}[0,1]} \frac{1}{2}x^2(\sigma_0 + \varepsilon\gamma)^2 \partial_{xx}^2 V^{\varepsilon} = 0, \\ 0 \le t \le T, \ x \ge 0, \ y \ge 0, \\ V^{\varepsilon}(T, x, y) = \varphi(\frac{y}{T}), \ x \ge 0, \ y \ge 0, \end{cases}$$

where $A[0,1] = {\gamma_t | \gamma_t \text{ is a } [0,1] - \text{valued processively measurable process}}.$

It is obvious that the worst case scenario price is larger than any Black-Scholes price with a constant volatility $\sigma_0 \in [\underline{\sigma}, \overline{\sigma}]$. We will show that the worst case scenario price of Asian option converges to its Black-Scholes price with constant volatility σ_0 in following section. In addition, the rate of convergence of the Asian option prices as the volatility interval shrinks to a single point can be obtained. Then we can get the estimation of the prices through this result when the interval is sufficiently small.

Let V_0 be the Black-Scholes prices, $V^0 = V^{\varepsilon}|_{\varepsilon=0}$, $V_1 = \partial_{\varepsilon} V^{\varepsilon}|_{\varepsilon=0}$. Now, we suppose that V^{ε} is continuous with respect to ε . Then, by the continuity of V^{ε} and equation (2.3), we have $V_0 = V^0 = V^{\varepsilon}|_{\varepsilon=0}$. It's well known that V_0 satisfies the following partial differential equation.

(3.4)
$$\begin{cases} \partial_t V_0 + r(x\partial_x V_0 - V_0) + x\partial_y V_0 + \frac{1}{2}\sigma_0^2 x^2 \partial_{xx}^2 V_0 = 0, \\ 0 \le t \le T, \ x \ge 0, \ y \ge 0, \\ V_0(T, x, y) = \varphi(\frac{y}{T}), \ x \ge 0, \ y \ge 0. \end{cases}$$

On the other hand, we have $V_1 = \partial_{\varepsilon} V^{\varepsilon}|_{\varepsilon=0}$, which is the rate of convergence of the Asian option prices as ε approaches 0. To obtain the equation characterizing V_1 , we differentiate both sides of equations (3.3) with respect to ε and let $\varepsilon = 0$, then

we have that

$$(3.5) \begin{cases} \partial_t V_1 + r(x\partial_x V_1 - V_1) + x\partial_y V_1 + \frac{1}{2}\sigma_0^2 x^2 \partial_{xx}^2 V_1 + \sup_{\gamma \in \mathcal{A}[0,1]} \gamma \sigma_0 x^2 \partial_{xx}^2 V_0 = 0, \\ 0 \le t \le T, \ x \ge 0, \ y \ge 0, \\ V_1(T,x,y) = 0, \ x \ge 0, \ y \ge 0. \end{cases}$$

Now, we have two Black-Scholes-like PDEs as above. Next, we want to find the connection between V^{ε} and V_0 , V_1 . Then we try to prove if we can impose additional conditions on the payoff function to make the error term $V^{\varepsilon} - (V_0 + \varepsilon V_1)$ be of order $\circ(\varepsilon)$. That is to say, the estimation of the worst case scenario Asian option prices will approach the truth-value as the model ambiguity vanishes. It will also show us a method to estimate the worst case Asian option prices. By the deducing in the next section, we get following theorem which is the main result of this paper.

Theorem 3.1. Assume that $\varphi \in \mathcal{C}_p^2(\mathbb{R}^+)$ is Lipschitz continuous, and the second derivative of φ is continuous. Then

$$\lim_{\varepsilon \downarrow 0} \frac{V^{\varepsilon} - (V_0 + \varepsilon V_1)}{\varepsilon} = 0.$$

Here $\varphi \in \mathcal{C}_p^2(\mathbb{R}^+)$ means that its derivatives up to order 2 have polynomial growth.

Remark 3.2. To prove the theorem 3.1, there are some difficulties. The first one is how to convert the error term into an estimable form. We get its expectation form and cut it into three parts in next section. The second difficulty is how to estimate the three parts. We will use the stochastic control theory, the zero set property of the equation (4.1), the properties of the sublinear expectation [14] and the properties of the worst case scenario Asian option price processes.

Remark 3.3. By theorem 3.1, we can compute Asian option price $V^{\varepsilon}(t, X_t^{\varepsilon}, Y_t^{\varepsilon})$ with its approximation, $V_0(t, X_t^{\varepsilon}, Y_t^{\varepsilon}) + \varepsilon V_1(t, X_t^{\varepsilon}, Y_t^{\varepsilon})$, where $V_0(t, X_t^{\varepsilon}, Y_t^{\varepsilon})$ is the Black-Scholes price of Asian option and $V_1(t, X_t^{\varepsilon}, Y_t^{\varepsilon})$ can be numerically computed by a simple difference scheme according to (3.5).(see [10])

Remark 3.4. Notice that (3.4) and (3.5) are independent of ε . So when we compute V^{ε} with different ε , we just need to compute V_0 and V_1 once for all small values of ε by Theorem 3.1.

4. The proof of the main result

In this section, we try to control the error term to prove that we can compute V^{ε} with its estimation $V_0 + \varepsilon V_1$. As the conditions imposed on φ which mentioned in

Theorem 3.1, we have following process of proof. On the other hand, our thinking process is also reflected in the next parts.

4.1. The Lipschitz continuity of payoff function. From section 3 we know that only with the continuity of V^{ε} can we obtain the PDEs of V_0 (= $V^{\varepsilon}|_{\varepsilon=0}$) and V_1 (= $\partial_{\varepsilon}V^{\varepsilon}|_{\varepsilon=0}$). Thus, to get the continuity of V^{ε} , we suppose that φ is Lipschitz continuous. Then, there exists a constant K_1 such that

$$|\varphi(x) - \varphi(y)| \le K_1|x - y|$$
, for all $x \ne y$, $x, y \in \mathbb{R}^+$.

Thus, we have Lemma as follows.

Lemma 4.1. Assume that φ is Lipschitz continuous. Then V^{ε} is continuous with respect to ε .

Proof. Let $0 \le \varepsilon_0 \le \varepsilon < 1$. Notice that

$$V^{\varepsilon}(t,x,y;\sigma) = \underset{\sigma \in \mathcal{A}^{\varepsilon}}{\operatorname{ess\,sup}} \; \left\{ e^{-r(T-t)} E_{txy} \left[\varphi(Y_{0,T}^{\varepsilon}) \right] \right\}.$$

We have that

$$e^{r(T-t)}V^{\varepsilon_0}(t,x,y;\sigma) = \underset{\sigma \in \mathcal{A}^{\varepsilon_0}}{\operatorname{ess \, sup}} E_{txy} \left[\varphi(Y_{0,T}^{\varepsilon_0}(\sigma)) \right]$$

$$= \underset{\sigma \in \mathcal{A}^{\varepsilon}}{\operatorname{ess \, sup}} E_{txy} \left[\varphi(Y_{0,T}^{\varepsilon}(\sigma \wedge (\sigma_0 + \varepsilon_0))) \right].$$

By the Lipschitz continuity of φ and equation (2.1), there is a constant K_1 such that

$$e^{r(T-t)} |V^{\varepsilon}(t, x, y; \sigma) - V^{\varepsilon_{0}}(t, x, y; \sigma)|$$

$$\leq \operatorname{ess\,sup}_{\sigma \in \mathcal{A}^{\varepsilon}} |E_{txy} \left[\varphi \left(Y_{0,T}^{\varepsilon}(\sigma) \right) \right] - E_{txy} \left[\varphi \left(Y_{0,T}^{\varepsilon}(\sigma \wedge (\sigma_{0} + \varepsilon_{0})) \right) \right] |$$

$$\leq K_{1} \operatorname{ess\,sup}_{\sigma \in \mathcal{A}^{\varepsilon}} \left(E_{txy} \left| Y_{0,T}^{\varepsilon}(\sigma) - Y_{0,T}^{\varepsilon}(\sigma \wedge (\sigma_{0} + \varepsilon_{0})) \right|^{2} \right)^{1/2}$$

$$\leq (K_{1}/T) \operatorname{ess\,sup}_{\sigma \in \mathcal{A}^{\varepsilon}} \left(E_{txy} \int_{0}^{T} |X_{u}^{\varepsilon}(\sigma) - X_{u}^{\varepsilon}(\sigma \wedge (\sigma_{0} + \varepsilon_{0}))|^{2} du \right)^{1/2}.$$

With the estimates of the moments of solutions of stochastic differential equations (Theorem 9 in Section 2.9 and Corollary 12 in section 2.5 of [15]), there are constants $N = N(q, r, \sigma_0), N' = N'(q, r, \sigma_0), \text{ and } C = \max\{NN', N+N'\}$ such that

$$E_{txy} \quad \left[\sup_{s \le u} |X_s^{\varepsilon}(\sigma) - X_s^{\varepsilon}(\sigma \wedge (\sigma_0 + \varepsilon_0))|^{2q} \right]$$

$$\leq Nu^{q-1} e^{Nu} E_{txy} \left[\int_0^u |X_s^{\varepsilon}(\sigma)|^{2q} \cdot |\sigma_s - \sigma_s \wedge (\sigma_s + \varepsilon_0)|^{2q} ds \right]$$

$$\leq Nu^{q-1} e^{Nu} N' e^{N'u} u(1 + x^{2q}) |\varepsilon - \varepsilon_0|^{2q}$$

$$= Cu^q e^{Cu} (1 + x^{2q}) |\varepsilon - \varepsilon_0|^{2q}.$$

Thus we have that

$$e^{r(T-t)}|V^{\varepsilon}(t,x,y) - V^{\varepsilon_{0}}(t,x,y)|$$

$$\leq (K_{1}/T) \operatorname{ess\,sup}_{\sigma \in \mathcal{A}^{\varepsilon}} \left(\int_{0}^{T} E_{txy} \sup_{s \in [0,u]} |X_{u}^{\varepsilon}(\sigma) - X_{u}^{\varepsilon}(\sigma \wedge (\sigma_{0} + \varepsilon_{0}))|^{2} du \right)^{1/2}$$

$$\leq (K_{1}/T) \operatorname{ess\,sup}_{\sigma \in \mathcal{A}^{\varepsilon}} \left(\int_{0}^{T} Cue^{Cu}(1+x^{2})|\varepsilon - \varepsilon_{0}|^{2} du \right)^{1/2}$$

$$\leq K'_{1}(1+x^{2})^{1/2}|\varepsilon - \varepsilon_{0}|,$$

where $K'_1 = K'_1(K_1, C, T)$.

Let $\varepsilon \to \varepsilon_0$. We have that $|V^{\varepsilon}(t,x,y) - V^{\varepsilon_0}(t,x,y)| \to 0$.

The continuity of V^{ε} with respect to ε can be proved similarly when $\varepsilon \leq \varepsilon_0$.

4.2. Expectation form of the error term. In this part, we analyze the error term and give its expectation form as preparation work before we prove the convergence of $V_0 + \varepsilon V_1$.

Let $\hat{\sigma}_t$ be the worst case scenario volatility process and \hat{X}_t^{ε} be the worst case scenario risky asset process. Then equations (3.1) can be rewritten as follows.

(4.1)
$$\begin{cases} d\hat{X}_t^{\varepsilon} = r\hat{X}_t^{\varepsilon}dt + \hat{\sigma}_t\hat{X}_t^{\varepsilon}dW_t, \\ d\hat{Y}_t^{\varepsilon} = \hat{X}_t^{\varepsilon}dt. \end{cases}$$

We can get the expression of $\hat{\sigma}$ by equations (3.3) and $\hat{\sigma}(\varepsilon) = \sigma_0 + \varepsilon \hat{\gamma}$, where

$$\hat{\gamma}(t, x, y; \varepsilon) = \begin{cases} 1, & \partial_{xx}^2 V^{\varepsilon}(t, x, y) \ge 0, \\ 0, & \partial_{xx}^2 V^{\varepsilon}(t, x, y) < 0. \end{cases}$$

Similarly, by solving equations (3.5) of V_1 , we have the volatility process: $\bar{\sigma}(\varepsilon) = \sigma_0 + \varepsilon \bar{\gamma}$, where

(4.3)
$$\bar{\gamma}(t,x,y) = \begin{cases} 1, & \partial_{xx}^2 V_0(t,x,y) \ge 0, \\ 0, & \partial_{xx}^2 V_0(t,x,y) < 0. \end{cases}$$

Here, we use the short notation $\hat{\gamma}_t$ and $\bar{\gamma}_t$ for $\hat{\gamma}(t, x, y; \varepsilon)$ and $\bar{\gamma}(t, x, y)$.

Let $Z^{\varepsilon} = V^{\varepsilon} - (V_0 + \varepsilon V_1)$. To estimate the error term Z^{ε} , we define the operator $L(\sigma) = \partial_t + rx\partial_x - r + \frac{1}{2}\sigma^2x^2\partial_{xx}^2 + x\partial_y$. According to partial differential equations

(3.2), (3.4) and (3.5), we have that

$$L(\hat{\sigma}_t)Z^{\varepsilon} = L(\hat{\sigma}_t)(V^{\varepsilon} - (V_0 + \varepsilon V_1))$$

$$= 0 - L(\hat{\sigma}_t)(V_0 + \varepsilon V_1)$$

$$= -(L(\hat{\sigma}_t) - L(\sigma_0))V_0 - L(\sigma_0)V_0 - \varepsilon(L(\hat{\sigma}_t) - L(\sigma_0))V_1 - \varepsilon L(\sigma_0)V_1$$

$$= \varepsilon(\bar{\gamma}_t - \hat{\gamma}_t)\sigma_0 x^2 \partial_{xx}^2 V_0 - (\varepsilon^2/2)((\hat{\gamma}_t)^2 x^2 \partial_{xx}^2 V_0 + 2\sigma_0 \hat{\gamma}_t x^2 \partial_{xx}^2 V_1)$$

$$- (\varepsilon^3/2)(\hat{\gamma}_t)^2 x^2 \partial_{xx}^2 V_1$$

$$= -f^{\varepsilon}(t, x, y),$$

with the boundary condition $Z^{\varepsilon}(T) = V^{\varepsilon}(T) - V_0(T) - \varepsilon V_1(T) = 0$. We have the following expectation form of Z^{ε} by Dynkin formula.

$$Z^{\varepsilon} = E_{txy} \left[\int_{t}^{T} f^{\varepsilon}(s, x, y) ds \right]$$

$$= \varepsilon E_{txy} \left[\int_{t}^{T} (\hat{\gamma}_{s} - \bar{\gamma}_{s}) \cdot \sigma_{0} \cdot (\hat{X}_{s}^{\varepsilon})^{2} \partial_{xx}^{2} V_{0}(s, \hat{X}_{s}^{\varepsilon}, \hat{Y}_{s}^{\varepsilon}) ds \right]$$

$$+ \varepsilon^{2} E_{txy} \left[\int_{t}^{T} \left\{ \frac{1}{2} (\hat{\gamma}_{s})^{2} (\hat{X}_{s}^{\varepsilon})^{2} \partial_{xx}^{2} V_{0}(s, \hat{X}_{s}^{\varepsilon}, \hat{Y}_{s}^{\varepsilon}) ds \right] \right.$$

$$+ \sigma_{0} (\hat{\gamma}_{s}) (\hat{X}_{s}^{\varepsilon})^{2} \partial_{xx}^{2} V_{1}(s, \hat{X}_{s}^{\varepsilon}, \hat{Y}_{s}^{\varepsilon}) \right\} ds \right]$$

$$+ \varepsilon^{3} E_{txy} \left[\int_{t}^{T} \frac{1}{2} (\hat{\gamma}_{s})^{2} (\hat{X}_{s}^{\varepsilon})^{2} \partial_{xx}^{2} V_{1}(s, \hat{X}_{s}^{\varepsilon}, \hat{Y}_{s}^{\varepsilon}) ds \right]$$

$$= \varepsilon I_{1} + \varepsilon^{2} I_{2} + \varepsilon^{3} I_{3},$$

where

$$(4.4) I_{1} = E_{txy} \left[\int_{t}^{T} (\hat{\gamma}_{s} - \bar{\gamma}_{s}) \cdot \sigma_{0} \cdot (\hat{X}_{s}^{\varepsilon})^{2} \partial_{xx}^{2} V_{0}(s, \hat{X}_{s}^{\varepsilon}, \hat{Y}_{s}^{\varepsilon}) ds \right],$$

$$I_{2} = E_{txy} \left[\int_{t}^{T} \left\{ \frac{1}{2} (\hat{\gamma}_{s})^{2} (\hat{X}_{s}^{\varepsilon})^{2} \partial_{xx}^{2} V_{0}(s, \hat{X}_{s}^{\varepsilon}, \hat{Y}_{s}^{\varepsilon}) + \sigma_{0} (\hat{\gamma}_{s}) (\hat{X}_{s}^{\varepsilon})^{2} \partial_{xx}^{2} V_{1}(s, \hat{X}_{s}^{\varepsilon}, \hat{Y}_{s}^{\varepsilon}) \right\} ds \right],$$

$$(4.5) I_{3} = E_{txy} \left[\int_{t}^{T} \frac{1}{2} (\hat{\gamma}_{s})^{2} (\hat{X}_{s}^{\varepsilon})^{2} \partial_{xx}^{2} V_{1}(s, \hat{X}_{s}^{\varepsilon}, \hat{Y}_{s}^{\varepsilon}) ds \right].$$

Thus we have that

$$(4.7) |Z^{\varepsilon}| \le \varepsilon |I_1| + \varepsilon^2 |I_2| + \varepsilon^3 |I_3|.$$

So we can estimate Z^{ε} by controlling $|I_1|$, $|I_2|$ and $|I_3|$.

4.3. The polynomial growth condition of payoff function. From section 4.2, we know that to control the error term, we need to analyze the three parts. By (4.7), we have

$$\left|\frac{Z^{\varepsilon}}{\varepsilon}\right| \leq |I_1| + \varepsilon(|I_2| + \varepsilon|I_3|).$$

Therefore, it is sufficient to prove

$$\lim_{\varepsilon \downarrow 0} |I_1| + \varepsilon(|I_2| + \varepsilon|I_3|) = 0.$$

Obviously, it is necessary to give controls of $|I_2|$ and $|I_3|$. When it comes to $|I_1|$, we need to prove the convergence of it. Now, let us consider controlling $|I_2|$ and $|I_3|$ first.

By the expressions of I_2 and I_3 , we can see that partial derivatives of V_0 and V_1 are involved. Thus, we should consider to estimate them before giving the controls of I_2 and I_3 .

Next, we can obtain the expectation form of V_0 and V^{ε} by the classical results. When $\varepsilon = 0$, we have

$$X(u) = x \exp\{(r - \frac{\sigma_0^2}{2})(u - t) + \sigma_0(W_u - W_t)\}.$$

Thus

$$V_{0}(t, x, y) = e^{-r(T-t)} E_{txy} [\varphi(Y_{0,T})]$$

$$= e^{-r(T-t)} E_{txy} \left[\varphi\left(\frac{1}{T} \int_{0}^{T} X(u) du\right) \right]$$

$$= e^{-r(T-t)} E_{txy} \left[\varphi\left(\frac{1}{T} \cdot x \cdot \left(\int_{0}^{T} e^{(r - \frac{\sigma_{0}^{2}}{2})(u-t) + \sigma_{0}(W_{u} - W_{t})} du\right) \right) \right]$$

$$= e^{-r(T-t)} E_{txy} [\varphi(x \cdot H)],$$

where $H(=(1/T)\int_0^T \exp\{(r-\sigma_0^2/2)(u-t)+\sigma_0(W_u-W_t)\}du$ is a random variable for fixed $t \in [0,T]$.

Similarly, there is

(4.9)
$$V^{\varepsilon}(t, x, y) = e^{-r(T-t)} \operatorname{ess \, sup}_{\sigma \in \mathcal{A}^{\varepsilon}} \left\{ E_{txy} \left[\varphi(Y_{0,T}^{\varepsilon}) \right] \right\}$$
$$= e^{-r(T-t)} E_{txy} \left[\varphi(x \cdot G) \right],$$

where $G(=(1/T)\int_0^T \exp\{(r-(\hat{\sigma}_u)^2/2)(u-t)-\hat{\sigma}_u(W_u-W_t)\}du)$ is a random variable for fixed $t \in [0,T]$.

By equation (4.8) and equation (4.9), we notice that it is necessary to impose polynomial growth conditions on φ to control $\partial_{xx}^2 V_0$ and $\partial_{xx}^2 V^{\varepsilon}$. Then we give the estimations of $\partial_{xx}^2 V_0(t,x,y)$ and $\partial_{xx}^2 V^{\varepsilon}(t,x,y)$ in following Lemma.

Lemma 4.2. Suppose that the second derivative of payoff function satisfies the polynomial growth condition, i.e. there are constants K_2 and m such that $\varphi''(x) \leq K_2(1+|x|^m)$. Then, we have constant K_3 such that

$$\left|\partial_{xx}^{2} V_{0}(t, x, y)\right| \leq K_{3} \left(1 + |x|^{m}\right),$$

where K_3 depends on T, t, $E_{txy}[|H|^2]$, $E_{txy}[|H|^{m+2}]$ and K_2 . Moreover, there is constant K_4 such that

$$\left|\partial_{xx}^{2}V^{\varepsilon}(t,x,y)\right| \leq K_{4}(1+|x|^{m}).$$

where K_4 depends on T, t, $E_{txy}\left[|G|^2\right]$, $E_{txy}\left[|G|^{m+2}\right]$ and K_2 .

Proof. As the assumption of φ in the lemma, we have

$$\begin{aligned} \left| \partial_{xx}^{2} V_{0}(t, x, y) \right| &= e^{-r(T-t)} E_{txy} \left[\varphi''(xH) H^{2} \right] \\ &\leq e^{-r(T-t)} E_{txy} \left[K_{2} (1 + |xH|^{m}) H^{2} \right] \\ &\leq K_{3} \left(1 + |x|^{m} \right). \end{aligned}$$

Here K_3 depends on T, t, $E_{txy}[|H|^2]$, $E_{txy}[|H|^{m+2}]$ and K_2 . Indeed, for a constant m > 0, we have that

$$EH^{m} = (1/(T))^{m} E \left(\int_{-t}^{T-t} \exp\{(r - \sigma_{0}^{2}/2)u + \sigma_{0}W_{u}\}du \right)^{m}$$

$$\leq \left(\frac{1}{T} \right)^{m} E \left(\int_{-t}^{T-t} e^{|r - \sigma_{0}^{2}/2|(T-t) + \sigma_{0}W_{u}}du \right)^{m}$$

$$\leq \left(\frac{1}{T} \right)^{m} e^{m|r - \sigma_{0}^{2}/2|(T-t)} E \left(\sup_{s \in (-t, T-t)} \left\{ e^{\sigma_{0}W_{s}} \right\} \right)^{m} < +\infty.$$

On the other hand, we get the controls of $\partial_{xx}^2 V^{\varepsilon}$ similarly. Then there is a constant K_4 which depends on T, t, $E_{txy}[|G|^2]$, $E_{txy}[|G|^{m+2}]$ and K_2 such that

$$|\partial_{xx}^2 V^{\varepsilon}(t, x, y)| \le K_4(1 + |x|^m).$$

Now, by following proposition, we can get the control of I_2 and I_3 .

Proposition 4.3. Assume that $\varphi \in C_p^2(\mathbb{R}^+)$ and satisfies Lipschitz continuity condition. Then there exist constants C_1 and p_1 such that I_2, I_3 in equation (4.5) and equations (4.6) satisfy

$$|I_2| + |I_3| \le C_1(1 + |x|^{p_1}).$$

Proof. By Lemma 4.2, we have the following inequality from (3.3) and (4.11).

$$\left| \partial_t V^{\varepsilon} + r(x \partial_x V^{\varepsilon} - V^{\varepsilon}) + x \partial_y V^{\varepsilon} \right| \leq \left| \frac{1}{2} (\sigma_0 + \varepsilon)^2 x^2 \partial_{xx}^2 V^{\varepsilon} \right| \\ \leq \left| (K_4/2) (\sigma_0 + \varepsilon)^2 \left(|x|^2 + |x|^{m+2} \right) \right|.$$

By the expression of V_1 , it is true that

$$\left| \partial_t V_1 + r(x \partial_x V_1 - V_1) + x \partial_y V_1 \right| \leq \left| K_4 \sigma_0 \left(|x|^2 + |x|^{m+2} \right) \right|.$$

By (3.5) and (4.10), we get the controls of $x^2 \partial_{xx}^2 V_1$,

$$|x^{2}\partial_{xx}^{2}V_{1}| = |\partial_{t}V_{1} + r(x\partial_{x}V_{1} - V_{1}) + x\partial_{y}V_{1} + \bar{g}_{t}\sigma_{0}x^{2}\partial_{xx}^{2}V_{0}| \cdot (2/\sigma_{0}^{2})$$

$$\leq (|\partial_{t}V_{1} + r(x\partial_{x}V_{1} - V_{1}) + x\partial_{y}V_{1}| + |\sigma_{0}x^{2}\partial_{xx}^{2}V_{0}|) \cdot (2/\sigma_{0}^{2})$$

$$\leq M_{1}(|x|^{2} + |x|^{m+2}),$$

where M_1 depends on K_3 , K_4 and σ_0 .

We can obtain the existence and uniqueness of \hat{X}_t^{ε} from Theorem 5.2.1 in [16]. Then, by the estimates of the moments of solutions of stochastic differential equations (Corollary 12 in Section 2.5 of [15]), there is a constant $N_1(q)$ for fixed q > 0 such that

(4.13)
$$E_{txy} \left[\sup_{s \in [t,T]} \left| \hat{X}_s^{\varepsilon} \right|^q \right] \le N_1(q) e^{N_1(q)(T-t)} \left(1 + |x|^q \right).$$

By (4.6), (4.12) and (4.13), we have the following inequality.

$$(4.14) |I_3| = \left| E_{txy} \left[\int_t^T \frac{1}{2} (\hat{\gamma}_s)^2 (\hat{X}_s^{\varepsilon})^2 \partial_{xx}^2 V_1(s, \hat{X}_s^{\varepsilon}, \hat{Y}_s^{\varepsilon}) ds \right] \right|$$

$$\leq (M_1/2) E_{txy} \left[\int_t^T (|\hat{X}_s^{\varepsilon}|^2 + |\hat{X}_s^{\varepsilon}|^{m+2}) ds \right] \leq M_1' \left(1 + |x|^{m+2} \right).$$

Here M'_1 depends on T, t, $N_1(2)$, $N_1(m+2)$ and M_1 .

By (4.5), (4.10), (4.12) and (4.13), we obtain the controls of $|I_2|$.

$$|I_{2}| = \left| E_{txy} \left[\int_{t}^{T} \frac{1}{2} (\hat{\gamma}_{s})^{2} (\hat{X}_{s}^{\varepsilon})^{2} \partial_{xx}^{2} V_{0} + \sigma_{0} (\hat{\gamma}_{s}) (\hat{X}_{s}^{\varepsilon})^{2} \partial_{xx}^{2} V_{1} ds \right] \right|$$

$$\leq (K_{3}/2) E_{txy} \left[\int_{t}^{T} (\hat{X}_{s}^{\varepsilon})^{2} + (\hat{X}_{s}^{\varepsilon})^{m+2} ds \right]$$

$$+ M_{1} E_{txy} \left[\int_{t}^{T} (\hat{X}_{s}^{\varepsilon})^{2} + (\hat{X}_{s}^{\varepsilon})^{m+2} ds \right]$$

$$\leq M_{2} (1 + |x|^{p_{1}}),$$

where M_2 depends on T, t, M_1 , K_3 , $N_1(2)$ and $N_1(m+2)$, $p_1 \ge m+2$. Combine (4.14) and (4.15), there is a constant C_1 such that

$$|I_2| + |I_3| \le C_1(1 + |x|^{p_1}).$$

4.4. The continuity of the second derivative of payoff function. By Proposition 4.3, we obtain the controls of I_2 and I_3 . Next, for fixed point $(t, x, y) \in [0, T] \times R^+ \times R^+$, it suffices to prove that

$$\lim_{\varepsilon \downarrow 0} |I_1| = 0.$$

Notice that, if $\varphi \in \mathcal{C}_p^2(\mathbb{R}^+)$ (i.e. its derivatives up to order 2 have polynomial growth), we can get following inequality by (4.4), (4.10), (4.13) and Hölder inequality,

$$|I_{1}| \leq \left[E_{txy} \left[\int_{t}^{T} (\sigma_{0}(\hat{X}_{s}^{\varepsilon})^{2} \partial_{xx}^{2} V_{0})^{2} ds \right] \right]^{1/2} \left[E_{txy} \left[\int_{t}^{T} (\hat{\gamma}_{s} - \bar{\gamma}_{s})^{2} ds \right] \right]^{1/2}$$

$$\leq M_{3} (1 + |x|^{p_{2}})^{1/2} \left[E_{txy} \left[\int_{t}^{T} |\hat{\gamma}_{s} - \bar{\gamma}_{s}| ds \right] \right]^{1/2}.$$

Here, M_3 depends on K_3, T, t, σ_0 , and $p_2 \geq 4 + 2m$. Moreover, M_3 is independent of ε .

Let $h^{\varepsilon}(t,x,y) = \hat{\gamma}(t,x,y;\varepsilon) - \bar{\gamma}(t,x,y)$. By (4.2) and (4.3), we have

$$|h^{\varepsilon}(t, x, y)| = \begin{cases} 1, & \partial_{xx}^{2} V^{\varepsilon} \partial_{xx}^{2} V_{0} < 0, \\ 0, & \partial_{xx}^{2} V^{\varepsilon} \partial_{xx}^{2} V_{0} \ge 0. \end{cases}$$

Thus, to prove $|I_1| \to 0$ as $\varepsilon \to 0$, it suffices to prove that

(4.17)
$$\lim_{\varepsilon \downarrow 0} E_{txy} \left[\int_{t}^{T} |h^{\varepsilon}(s, \hat{X}_{s}^{\varepsilon}, \hat{Y}_{s}^{\varepsilon})| ds \right] = 0.$$

By the expression of h^{ε} , we should analyze the derivatives of V_0 and V^{ε} . Here, we find that the continuity of φ'' is necessary.

Lemma 4.4. Assume that φ'' is continuous. Then $\partial_{xx}^2 V_0$ and $\partial_{xx}^2 V^{\varepsilon}$ are continuous with respect to (x,y).

Proof. By (4.8), we have $V_0(t,x,y)=e^{-r(T-t)}E_{txy}[\varphi(xH)]$ and $\partial_{xx}^2V_0(t,x,y)=e^{-r(T-t)}E_{txy}[\varphi''(xH)H^2]$. If φ'' is continuous, then for all $x_0\in R^+$, $\delta>0$, there is a constant $\xi=\xi(\delta,x_0)$ such that

$$|\varphi''(xH) - \varphi''(x_0H)| \le \delta$$

for all $xH \in (x_0H - \xi, x_0H + \xi)$.

So for all $(x_0, y_0) \in R^+ \times R^+$, $xH \in (x_0H - \xi, x_0H + \xi)$, $y \in (y_0 - \xi, y_0 + \xi)$, we have

$$\begin{aligned} |\partial_{xx}^{2}V_{0}(t,x,y) - \partial_{xx}^{2}V_{0}(t,x_{0},y_{0})| &= e^{-r(T-t)}|E_{txy}[\varphi''(xH)H^{2} - \varphi''(x_{0}H)H^{2}]| \\ &\leq e^{-r(T-t)}E_{txy}[H^{2}|\varphi''(xH) - \varphi''(x_{0}H)|] \\ &\leq e^{-r(T-t)}\delta E_{txy}[H^{2}]. \end{aligned}$$

Thus we obtain

$$\lim_{(x,y)\to(x_0,y_0)} \partial_{xx}^2 V_0(t,x,y) = \partial_{xx}^2 V_0(t,x_0,y_0).$$

Similarly, we can get the continuity of $\partial_{xx}^2 V^{\varepsilon}$.

Remark 4.5. It is rational to think that V^{ε} and its derivatives converge to V_0 and its corresponding derivatives as ε approaching 0 by Lemma 4.1.

Remark 4.6. To simplify the complexity brought by the variable Y, which is called path-dependence and to study the behavior of h^{ε} , we define

$$D_{ty}^{\lambda} = \{ x \in R^+ \mid \partial_{xx}^2 V^{\varepsilon_0} \partial_{xx}^2 V_0 \le 0, \ \exists \varepsilon_0 > \lambda \}.$$

Let $D_{ty}^0 = \lim_{\lambda \downarrow 0} D_{ty}^{\lambda}$. Then we can get following equation when $\partial_{xx}^2 V^{\varepsilon}$ is continuous,

$$D_{ty}^0 = \{ x \in R^+ \mid \partial_{xx}^2 V_0(t, x, y) = 0 \}.$$

Remark 4.7. To control h^{ε} , we divide D_{ty}^{λ} into two parts. Let $\alpha(\rho) = [-\rho, \rho]$, we will disscuss the characters of $D_{ty}^{\lambda} \cap \alpha(\rho)$ and $D_{ty}^{\lambda} \cap \alpha(\rho)^{c}$.

Lemma 4.8. Assume that φ'' is continuous. Then we have

$$P_{txy}(D_{ty}^0 \cap \alpha(\rho)) = 0.$$

Here, $P_{txy}(\cdot)$ means the conditional probability taken with respect to $X_t^{\varepsilon} = x$, $Y_t^{\varepsilon} = y$.

Proof. By (4.1) and (3.4), we can get following equation

(4.18)
$$\begin{cases} 2\partial_t V_0 + r(x\partial_x V_0 - V_0) + \frac{1}{2}\sigma_0^2 x^2 \partial_{xx}^2 V_0 = 0, \\ V_0(T) = \varphi(xH). \end{cases}$$

Let $Q = \partial_{xx}^2 V_0$. Then by (4.18) we have

$$\begin{cases} 2\partial_t Q + (r + \sigma_0^2)Q + (r + 2\sigma_0^2)x\partial_x Q + \frac{1}{2}\sigma_0^2 x^2 \partial_{xx}^2 Q & = 0, \\ Q(T) & = \varphi''(xH)H^2. \end{cases}$$

Let $x = \log k$. Then we have that

$$(4.19) \begin{array}{lcl} 2\partial_t Q + (r + \sigma_0^2)Q + (r + 2\sigma_0^2)\partial_k Q + \frac{1}{2}\sigma_0^2\partial_k^2 Q & = & 0, \\ Q(T) & = & \varphi''((\log k)H)H^2. \end{array}$$

Notice that the coefficients in equations (4.19) are constants and Q is bounded on $D_{ty}^0 \cap \alpha(\rho)$ by the continuity of φ'' and Lemma 4.4. Moreover, by equations (4.19),

we find that y is not related to the equations. Then by Theorem A of [17] and the remark below it, we have that the number of zero points of Q is

no more than countable for all $(s, y) \in [t, T] \times R$.

Thus, $\partial_{xx}^2 V_0$ has no more than countable zero points.

Hence we have $P_{txy}(D_{ty}^0 \cap \alpha(\rho)) = 0$ by Lemma 4.10 of [11]. Then the proof of Lemma 4.8 is completed.

On the basis of previous analysis, we will prove (4.17) now. We're going to split the expectation into two parts. By proving the convergence of each part, we can get the convergence of the expectation.

Proposition 4.9. Assume that $\varphi \in \mathcal{C}_p^2(\mathbb{R}^+)$ and φ'' is continuous. Then we obtain equation (4.17).

Proof. Let \bar{D}_{ty}^{λ} is closure of D_{ty}^{λ} , $\bar{D}_{ty}^{0} = \lim_{\lambda \downarrow 0} \bar{D}_{ty}^{\lambda}$ and $0 \leq \lambda < \varepsilon < 1$.

By the definition of D_{ty}^{λ} , we have

$$E_{txy} \left[\int_{t}^{T} |h^{\varepsilon}(s, \hat{X}_{s}^{\varepsilon}, \hat{Y}_{s}^{\varepsilon})| ds \right]$$

$$(4.20) \leq E_{txy} \left[\int_{t}^{T} \mathbb{I}_{\bar{D}_{(s\hat{Y}_{s}^{\varepsilon})}^{\lambda}}(\hat{X}_{s}^{\varepsilon}) ds \right]$$

$$= E_{txy} \left[\int_{t}^{T} \mathbb{I}_{\bar{D}_{(s\hat{Y}_{s}^{\varepsilon})}^{\lambda} \cap \alpha(\rho)}(\hat{X}_{s}^{\varepsilon}) ds \right] + E_{txy} \left[\int_{t}^{T} \mathbb{I}_{\bar{D}_{(s\hat{Y}_{s}^{\varepsilon})}^{\lambda} \cap \alpha(\rho)^{c}}(\hat{X}_{s}^{\varepsilon}) ds \right]$$

$$= \Phi_{1} + \Phi_{2}.$$

Now, we consider the second part of (4.20) first. By (4.13) and Chebyshev's inequality, there is

$$\Phi_{2} \leq E_{txy} \left[\int_{t}^{T} \mathbb{I}_{\alpha(\rho)^{c}}(\hat{X}_{s}^{\varepsilon}) ds \right]$$

$$\leq \int_{t}^{T} P_{txy} \left(\sup_{s \in [t,T]} |\hat{X}_{s}^{\varepsilon}| \geq \rho \right) ds$$

$$\leq ((T-t)/\rho) E_{txy} \left[\sup_{s \in [t,T]} |\hat{X}_{s}^{\varepsilon}| \right]$$

$$\leq \frac{(T-t)N_{1}(1)}{\rho} e^{N_{1}(1)(T-t)} (1+|x|).$$

Thus, we have

$$\lim_{\rho \to \infty} \Phi_2 = 0.$$

When it comes to the first part, we note that

$$\Phi_1 = \int_t^T P_{txy} \left(\hat{X}_s^{\varepsilon} \in \bar{D}_{(s\hat{Y}_s^{\varepsilon})}^{\lambda} \cap \alpha(\rho) \right) ds.$$

Let $\theta(\Omega) = \sup_{\lambda \in [0,1]} P_{txy}(\Omega)$, then there is

$$(4.22) P_{txy}\left(\hat{X}_s^{\varepsilon} \in \bar{D}_{(s\hat{Y}_s^{\varepsilon})}^{\lambda} \cap \alpha(\rho)\right) \leq \theta\left(\bar{D}_{(s\hat{Y}_s^{\varepsilon})}^{\lambda} \cap \alpha(\rho)\right).$$

Notice that $\lambda < \varepsilon$. Then \bar{D}_{sy}^{λ} is a sequence of decreasing closed sets as $\varepsilon \downarrow 0$. Obviously, $\hat{X}_{s}^{\varepsilon}$ converges weakly to X_{s} . Thus $\{X_{s}\}$ is a weakly compact. By the Lemma 8 of [14], it can be seen that $\theta\left(\bar{D}_{(s\hat{Y}_{s}^{\varepsilon})}^{\lambda}\cap\alpha(\rho)\right)\downarrow\theta\left(\bar{D}_{(s\hat{Y}_{s}^{\varepsilon})}^{0}\cap\alpha(\rho)\right)$ as $\varepsilon\downarrow 0$. By Lemma 4.4, we have $\bar{D}_{sy}^{0}=D_{sy}^{0}$. Hence, there is

$$P_{txy}\left(\hat{X}_s^{\varepsilon} \in \bar{D}_{(s\hat{Y}_s^{\varepsilon})}^0 \cap \alpha(\rho)\right) = 0.$$

Then by definition of $\theta(\Omega)$, we have

$$\lim_{\varepsilon \downarrow 0} \theta \left(\bar{D}_{(s\hat{Y}_s^{\varepsilon})}^{\lambda} \cap \alpha(\rho) \right) = \theta \left(\bar{D}_{(s\hat{Y}_s^{\varepsilon})}^{0} \cap \alpha(\rho) \right) = 0.$$

Thus, there is

$$\lim_{\varepsilon \downarrow 0} P_{txy} \left(\hat{X}_s^{\varepsilon} \in \bar{D}_{(s\hat{Y}_s^{\varepsilon})}^{\lambda} \cap \alpha(\rho) \right) = 0.$$

Then, we obtain

$$\lim_{\varepsilon \downarrow 0} \Phi_1 = 0.$$

By equation (4.21) and equation (4.23), for any $\delta > 0$, there is $\rho_0 = \rho_0(t, x, y, \delta) > 0$ such that

$$\Phi_2 < \delta/2$$
, for all $\rho > \rho_0$.

Next, for given ρ_0 and δ , there is $\varepsilon_0 = \varepsilon_0(t, x, y, \delta, \rho_0(t, x, y, \delta))$ such that

$$\Phi_1 < \delta/2$$
, for all $\varepsilon < \varepsilon_0$.

Therefore, for any $\delta > 0$, there is $\varepsilon_0 = \varepsilon_0(t, x, y, \delta)$ such that

$$\Phi_1 + \Phi_2 < \delta$$
, for all $\varepsilon < \varepsilon_0$,

i.e.

$$\lim_{\varepsilon \downarrow 0} E_{txy} \left[\int_t^T |h^{\varepsilon}(s, \hat{X}_s^{\varepsilon}, \hat{Y}_s^{\varepsilon})| ds \right] = 0.$$

4.5. **The proof of Main result.** Now, as the analysis above, we can give the brief proof of theorem 3.1.

The proof of main result. By inequality (4.16) and Proposition 4.9, we have

$$\lim_{\varepsilon \downarrow 0} |I_1| = 0.$$

By inequality (4.7), we have

$$\left|\frac{V^{\varepsilon} - (V_0 + \varepsilon V_1)}{\varepsilon}\right| \le |I_1| + \varepsilon(|I_2| + \varepsilon|I_3|).$$

By Proposition 4.3 and equation (4.24), we obtain the Theorem.

5. Conclusion

In this paper, we analyze the behavior of Asian option prices in the worst case scenario. The model studied in this paper is an uncertain volatility model with volatility interval $[\sigma_0, \sigma_0 + \varepsilon]$. As ε close to 0, the ambiguity of model vanishes. We can also see that the worst case scenario prices of Asian option converge to its Black-Scholes prices with constant volatility as the interval shrinks. And through the study, we get an approach of estimating the worst case scenario Asian option prices. At the same time, it means that we give an estimation method to solve a fully nonlinear PDE (3.2) by imposing additional conditions on the boundary condition and cutting it into two Black-Scholes-like equations.

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