Equilibria in a large production economy with an infinite dimensional commodity space and price dependent preferences*

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Abstract

We prove the existence of a competitive equilibrium in a production economy with infinitely many commodities and a measure space of agents whose preferences are price dependent. We employ a saturated measure space for the set of agents and apply recent results for an infinite dimensional separable Banach space such as Lyapunov's convexity theorem and an exact Fatou's lemma to obtain the result.

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1 Introduction

The purpose of this paper is to prove the existence of a competitive equilibrium in a production economy with infinitely many commodities and a measure space of agents whose preferences are price dependent. In a seminal paper, Aumann [3] demonstrated the existence of a competitive equilibrium for an exchange economy with a finite dimensional commodity space and a continuum of agents modeled as an atomless finite measure space by utilizing Lyapunov's convexity theorem to dispense with convex preferences. Aumann's model in [3] was generalized to allow incomplete preferences by Schmeidler [35] and to include production by Hildenbrand [15].

As Shafer[36]¹ and Balasko [5] pointed out, price dependent preferences have been traditionally explained by consumers taking relative prices as an indication of quality. In addition, we

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¹This was drawn to our attention by an anonymous referee.

see other applications of price dependent preferences in the literature: Shafer [36] showed the possibility of relating price dependent preferences to non-transitive preferences and Balasko [6] demonstrated the equivalence of a temporary financial equilibrium model with an Arrow-Debreu economy where preferences are price dependent.

Greenberg et al. [12] first proved the existence of a competitive equilibrium in a large economy with price dependent preferences and a finite number of commodities. In [12], the authors considered a large production economy with non-convex preferences. They reformulated the production economy as a three-person game and applied Debreu's social equilibrium existence theorem to obtain a Walrasian equilibrium. In their proof, they applied Lyapunov's convexity theorem and Fatou's Lemma in several dimensions. In order to utilize Fatou's lemma, Greenberg et al. [12] assumed the compactness of the consumption sets, which differs from Aumann's original model. Liu [29] dealt with a coalition production economy based on Greenberg et al. [12].

For infinite dimensional commodity spaces, Khan and Yannelis [27] considered a large exchange economy and showed the existence of a competitive equilibrium. In [27], the commodity space is an ordered separable Banach space whose positive cone has a non-empty interior. Until recently, Lyapunov's convexity theorem and an exact Fatou's lemma for an infinite dimensional separable Banach space were not available. Therefore, the authors had to impose the assumption of convex preferences. They relied on the weak compactness of feasible allocations to extract a convergent subsequence of competitive equilibria for truncated subeconomies to obtain the existence of a Walrasian equilibrium. Now that the necessary mathematical tools are at hand, it is natural to ask as to whether equilibrium existence results for a large economy with an infinite dimensional commodity space, non-convex preferences and price externalities are available. We give a positive answer in this paper.

Saturated or super-atomless measure spaces have played an important role in recent mathematical economics. Podczeck [33] and Sun and Yannelis [37] successfully proved the convexity of Bochner integrals of an infinite dimensional separable Banach space valued correspondence on a saturated measure space. Based on saturated measure spaces, Khan and Sagara [21] proved Lyapunov's convexity theorem for vector measures taking values in an infinite dimensional separable Banach space and Greinecker and Podczeck [13] also showed it. Khan and Sagara [22] established an exact Fatou's lemma for an infinite dimensional separable Banach space. Khan et al. [25] proved an exact Fatou lemma for Gelfand integrals which was also established via Young measures by Greinecker and Podczeck [14]. These results have already been applied to general equilibrium theory in several papers; see Khan and Sagara [23, 24], Khan and Suzuki [26] and Lee [28]. In [24], the authors emphasized the importance of saturated measures by saying that "the significance of the saturation property lies in the fact that it is necessary and sufficient for the weak/weak* compactness and the convexity of the Bochner/Gelfand integral

of a multifunction as well as the Lyapunov convexity theorem in separable Banach spaces/their dual spaces."

In this paper, we consider a large production economy whose commodity space is that of Khan and Yannelis [27] and whose agents have non-convex and price dependent preferences, similar to Greenberg et al. [12]. We employ a saturated measure space of agents and hence, we can utilize the convexity of a Bochner integral of a Banach space valued correspondence, Lyapunov's convexity theorem, and the exact Fatou's lemma for an infinite dimensional Banach space. With these new results, we are able to relax the convexity of preferences and production sets, and apply Debreu's social equilibrium existence theorem. Moreover, we can obtain a competitive equilibrium as the limit of a sequence of competitive equilibria for truncated subeconomies. We dispense with the uniform compactness assumption on the consumption sets and production sets, which was used in [12] and in [29].

The paper proceeds as follows: Section 2 contains notations and definitions. We present our model in Section 3, and our main and auxiliary results are in Section 4. The proof of the auxiliary result is in Section 5 followed by the proof of the main theorem in Section 6. Section 7 concludes the paper with our remarks.

2 Notation and Definitions

Let X, Y be topological spaces. A set-valued function or a correspondence F from Y to the family of non-empty subsets of Y is called *upper semicontinuous* if the set $\{x: X: F(x) \subset V\}$ is open in X and said to be *lower semicontinuous* if the set $\{x: X: F(x) \cap V \neq \emptyset\}$ is open in X for every V of Y. When Y is a Banach space, F is norm upper semicontinuous if the set $\{x: X: F(x) \subset V\}$ is open in X for every norm open subset V of Y. And F is called weakly upper semicontinuous if the set $\{x: X: F(x) \subset V\}$ is open in X for every weakly open subset V of Y. We say that F is norm lower semicontinuous if the set $\{x: X: F(x) \cap V \neq \emptyset\}$ is open in X for every norm open subset Y of Y and Y is said to be weakly lower semicontinuous if the set $\{x: X: F(x) \cap V \neq \emptyset\}$ is open in X for every weakly open subset Y of Y.

Let (T, \mathcal{T}, μ) be a finite measure space and E be a Banach space. A measurable function $f: (T, \mathcal{T}, \mu) \to E$ is said to be *Bochner integrable* if there exists a sequence of simple functions $\{f_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{n \to \infty} \int_{T} \|f_n(t) - f(t)\| d\mu = 0$$
 (2.1)

where \mathbb{N} denotes the set of natural numbers. For each $S \in \mathcal{T}$ the integral is defined to be $\int_S f(t)d\mu = \lim_{n\to\infty} \int_S f_n(t)d\mu$. Denote by $L^1(\mu, E)$ the space of (the equivalence classes of) E-valued Bochner integrable functions $f: T \to E$ normed by $||f||_1 = \int_T ||f(t)|| d\mu$.

The weak upper limit of a sequence $\{S_n\}$ of subsets in E is defined by

$$w-\operatorname{Ls} S_n = \{x \in E : \exists \{x_{n_k}\} \text{ such that } x = w-\lim x_{n_k}, x_{n_k} \in S_{n_k}, \text{ for all } k \in \mathbb{N}\}$$
 (2.2)

where $\{x_{n_k}\}$ is a subsequence of a sequence $\{x_n\}$ and w- $\lim_n x_{n_k}$ denotes the weak limit point of $\{x_{n_k}\}$.

A correspondence $F: T \to 2^E$ is said to be measurable if for every open subset V of E, the set $\{t \in T: F(t) \cap V \neq \emptyset\} \in \mathcal{T}$. The correspondence F is said to have a measurable graph if its graph $G_F = \{(t,x) \in T \times E: x \in F(t)\}$ belongs to the product σ -algebra $\mathcal{T} \otimes \mathcal{B}(E,w)$, where $\mathcal{B}(E,w)$ denotes the Borel σ -algebra of E generated by the weak topology. If correspondences from T to E are closed valued, measurability and graph measurability are equivalent when (T,\mathcal{T},μ) is complete and E is separable.² A measurable correspondence $F: T \to 2^E$ is integrably bounded if there exists a real-valued integrable function E on E on E is integrably bounded if or almost all E is a real-valued integrable function E on E is a real-valued integrable function E or E is E or E.

A measurable function f from (T, \mathcal{T}, μ) to E is called a measurable selection of the correspondence F if $f(t) \in F(t)$ for almost all $t \in T$. By Aumann's measurable selection theorem in [4], if (T, \mathcal{T}, μ) is a complete finite measure space, F has a measurable graph, and E is separable, then F has a measurable selection. We denote by \mathcal{S}_F^1 the set of all E-valued Bochner integrable selections for the correspondence F, i.e., $\mathcal{S}_F^1 = \{f \in L^1(\mu, E) : f(t) \in F(t) \text{ a.e. } t \in T\}$. When F is also integrably bounded, it admits a Bochner integrable selection so that \mathcal{S}_F^1 is non-empty. The integral of the correspondence F is defined by

$$\int_{T} F(t)d\mu = \{ \int_{T} f(t)d\mu : f \in \mathcal{S}_{F}^{1} \}.$$

$$(2.3)$$

A sequence of correspondences $\{F_n\}$ from T to E is said to be well-dominated if there exists an integrably bounded and weakly compact-valued correspondence $\phi: T \to 2^E$ such that $F_n(t) \subset \phi(t)$ a.e. $t \in T$ for each n.

Let E be an ordered Banach space equipped with ordering \geq such that the positive cone $E_+ = \{x \in E : x \geq 0\}$ of E is closed. For $x, y \in E$, x > y means $x - y \in E_+$ and $x \neq y$. We denote by E^* the dual space of E, i.e., the space of all continuous linear functionals from E into \mathbb{R} . For $x \in E, p \in E^*$, we write $p \cdot x$ for the value of p at x. We denote by E_+^* the dual cone of E_+ , i.e., $E_+^* = \{p \in E^* : p \cdot x \geq 0 \text{ for all } x \in E_+\}$. We denote by $\mathcal{B}(E^*, w^*)$ the Borel σ -algebra of E^* generated by the weak* topology. For any set E in E, E is an and E are the positive cone E and E and E and E are the positive cone E and E are the positive con

Let (T, \mathcal{T}, μ) be a finite measure space. Denote by $L^1(\mu)$ the the space of $(\mu$ -equivalence classes of) real valued integrable functions on T. Let $\mathcal{T}_S = \{A \cap S | A \in \mathcal{T}\}$ be the sub- σ -algebra of \mathcal{T} restricted to $S \in \mathcal{T}$ and μ_S be a restriction of μ to \mathcal{T}_S . We write $L^1_S(\mu)$ for the vector

 $^{^{2}}$ See Theorem 8.1.4 in [2].

subspace of $L^1(\mu)$ which consists of each function in $L^1(\mu)$ restricted to S.

Definition 1. A finite measure space (T, \mathcal{T}, μ) is saturated if $L_S^1(\mu)$ is non-separable for every $S \in \mathcal{T}$ with $\mu(S) > 0$.

A saturated measure space is also called "super-atomless" in Podczeck [33]. Other equivalent definitions for saturation are available in the literature; see [10], [11], [17], [20], and [33]. As mentioned in Khan and Sagara [24], "a germinal notion of saturation already appeared in [19, 30]," and Kakutani [19] constructed a non-separable extension of the Lebesgue measure space which can be seen as a saturated extension of the Lebesgue interval.³ Examples of saturated measure spaces include the product spaces of the form $[0,1]^{\kappa}$ and $\{0,1\}^{\kappa}$, where κ is an uncountable cardinal, [0,1] is endowed with the Lebesgue measure and $\{0,1\}$ the fair coin flipping measure. The cardinalities of these two examples are greater than the continuum. Podczeck [33] constructed a saturated measure structure on the unit interval by "enriching" the Lebesgue σ -algebra. Thus, as is pointed out in [33], when we have a saturated measure space of agents, the cardinality of the set of agents is not necessarily larger than the continuum.

3 The Model

The commodity space E is an ordered separable Banach Space with an interior point v in E_+ . For the space of agents, we employ a complete probability space (T, \mathcal{T}, μ) which is saturated. Let X be a correspondence from T to E_+ . The consumption set of agent $t \in T$ is given by $X(t) \subset E_+$. The initial endowment of each agent is given by a Bochner integrable function $e: T \to E$ where $e(t) \in X(t)$ for all $t \in T$. The aggregate initial endowment is $\int_T e(t)d\mu$. Let Y be a correspondence from T to E. The production set of agent t is given by $Y(t) \subset E$. A price is $p \in E_+^* \setminus \{0\}$. Let $\Delta = \{p \in E_+^* \setminus \{0\} : p \cdot v = 1\}$ be the price space. Then by Alaoglu's theorem, Δ is weak* compact. Let $\mathcal{E} = [(T, \mathcal{T}, \mu), (X(t), Y(t), U_t, e(t))_{t \in T}]$ be a production economy where $U_t: X(t) \times \Delta \to \mathbb{R}$ represents agent t's utility function. We also write $U(t, x, p) = U_t(x, p)$ for $t \in T$, $x \in X(t)$ and $p \in \Delta$. An allocation for \mathcal{E} is a Bochner integrable function $f: T \to E_+$ such that $f \in \mathcal{S}_X^1$ and a production plan is a Bochner integrable function $g: T \to E$ such that $g \in \mathcal{S}_Y^1$. The budget set of agent t at a price $p \in \Delta$ is $B(t, p) = \{x \in X(t) : p \cdot x \leq p \cdot e(t) + \max p \cdot Y(t)\}$.

A competitive equilibrium for \mathcal{E} is a triple of a price p, an allocation f and a production plan g such that

- 1. $p \cdot f(t) \leq p \cdot e(t) + p \cdot g(t)$ for almost all $t \in T$,
- 2. $\int_T f(t)d\mu \leq \int_T e(t)d\mu + \int_T g(t)d\mu$,

³We are grateful to an anonymous referee for drawing our attention to Kakutani [19].

⁴The examples of this space include C(K), the set of bounded continuous functions on a Hausdorff compact metric space K equipped with sup norm and a weakly compact subset of $L_{\infty}(\mu)$ where μ is a finite measure.

- 3. for any $x \in X(t)$, $U_t(x, p) > U_t(f(t), p)$ implies that $p \cdot x > p \cdot e(t) + p \cdot g(t)$ for almost all $t \in T$,
- 4. $p \cdot g(t) = \max p \cdot Y(t)$ for almost all $t \in T$.

We assume that the production economy \mathcal{E} satisfies the following assumptions:

- A.1 X(t) is non-empty, closed, convex, integrably bounded and weakly compact for all $t \in T$.
- A.2 Y(t) is non-empty, closed, integrably bounded and weakly compact for all $t \in T$.
- A.3 There is an element $\eta(t) \in X(t)$ such that $e(t) \eta(t)$ is in the norm interior of E_+ for all $t \in T$.
- A.4 (i) $U_t: X(t) \times \Delta \to \mathbb{R}$ is a jointly continuous function on $X(t) \times \Delta$ for all $t \in T$ where X(t) is equipped with the weak topology and Δ with the weak* topology. (ii) If $x \in X(t)$ is a satiation point for $U_t(\cdot, p)$, then $x \geq e(t) + y$ for any $y \in Y(t)$; if $x \in X(t)$ is not a satiation point for $U_t(\cdot, p)$, then x belongs to the weak closure of the set $\{x' \in X(t) : U_t(x', p) > U_t(x, p)\}$ for every $p \in \Delta$.
- A.5 *U* is jointly measurable with respect to $\mathcal{T} \otimes \mathcal{B}(E, w) \otimes \mathcal{B}(E^*, w^*)$.
- A.6 the correspondence $X: T \to 2^E$ has a measurable graph, i.e., $G_X \in \mathcal{T} \otimes \mathcal{B}(E, w)$.
- A.7 the correspondence $Y: T \to 2^E$ has a measurable graph, i.e., $G_Y = \{(t, y) \in T \times E : y \in Y(t)\} \in \mathcal{T} \otimes \mathcal{B}(E, w)$.
- A.8 $\mathbf{0} \in Y(t)$ for all $t \in T$ where $\mathbf{0}$ is the zero vector of E.

In A.1 and A.2, we assume that both the consumption sets and the production sets are weakly compact. Although these assumptions seem strong, the weakly compact consumption set assumption was employed in Khan and Yannelis [27], Podczeck [32] and Khan and Sagara [24].⁵ With this assumption, Khan and Yannelis [27] made the set of feasible allocations weakly compact, Podczeck [32] obtained a weakly compact-valued demand correspondence, and Khan and Sagara [24] were able to invoke the exact Fatou's lemma for an infinite dimensional separable Banach space. We use the weak compactness assumption to apply the exact Fatou's lemma for our results. A.4 (ii) is imposed in [24, 28, 32] and the second part plays a similar role to the "local nonsatiation" assumption.

⁵Since these three papers dealt with exchange economies, the production sets are irrelevant.

4 Results

The following theorem is our main result:

Main Theorem. Suppose that the production economy \mathcal{E} satisfies A.1-A.8. Then there exists a competitive equilibrium for \mathcal{E} .

The proof of the Main Theorem is provided in Section 6. As is well known, for $x \in E$ and $p \in \Delta$ the bilinear map $(p, x) \mapsto p \cdot x$ is not jointly continuous if E is equipped with the weak topology and Δ with the weak* topology. But when E is equipped with the norm topology, the bilinear map is continuous.⁶ To utilize this property, we modify A.1 and A.2:

A.1' X(t) is non-empty, closed, convex, integrably bounded and norm compact for all $t \in T$.

A.2' Y(t) is non-empty, closed, integrably bounded and norm compact for all $t \in T$.

We now introduce the following auxiliary result:

Auxiliary Theorem. Suppose that the production economy \mathcal{E} satisfies A.1', A.2' and A.3-A.8. Then there exists a competitive equilibrium for \mathcal{E} .

We provide the proof of the Auxiliary Theorem in Section 5. We follow the idea of [12] for the proof of the Auxiliary Theorem. Greenberg et al. [12] applied Debreu's [8] social equilibrium result to prove the existence of a competitive equilibrium.

We introduce a 3-person game Γ which consists of three sets K_1, K_2, K_3 , and three correspondence $A_1: K_2 \times K_3 \to 2^{K_1}, A_2: K_1 \times K_3 \to 2^{K_2}, A_3: K_1 \times K_2 \to 2^{K_3}$, and three functions $u_i: K_1 \times K_2 \times K_3 \to \mathbb{R}$ (i = 1, 2, 3). Let $I = \{1, 2, 3\}$ and let $K_{-i} = \prod_{j \neq i} K_j$ $(i, j \in I)$. We write k_i for an element in K_i and k_{-i} for K_{-i} .

An equilibrium for Γ is $k^* \in K_1 \times K_2 \times K_3$ such that for all $i \in I$

$$k_i^* \in \operatorname{argmax}_{k_i \in A_i(k_{-i}^*)} u_i(k_i, k_{-i}^*).$$
 (4.1)

The following lemma is Debreu's [8] social equilibrium theorem for a Banach space.

Lemma 1. Let Γ be a 3-person game and suppose Γ satisfies, for $i \in I$,

- (i) K_i is a non-empty, convex, and compact subset of a Banach space;
- (ii) A_i is continuous, non-empty, closed and convex valued;
- (iii) u_i is continuous and quasi-concave on K_i .

Then Γ has an equilibrium.

⁶See Aliprantis and Border [1] pp. 241-242.

Proof. By applying a standard argument to our Banach space, we can have the result. \Box

Based on Lemma 1, we will prove the Auxiliary Theorem. Toward this end, we specify our Γ . Without loss of generality, we assume the values of U_t are contained in [0,1] for all $t \in T$. Let $K_1 = \Delta$, $K_2 = \int_T X(t) d\mu \times [0,1]$, and $K_3 = \int_T Y(t) d\mu$. For $p \in K_1$, $(x,\alpha) \in K_2$ and $y \in K_3$, let $A_1((x,\alpha),y) = K_1$, $A_2(p,y) = \{(x,\alpha) \in K_2 : \exists f \in \mathcal{S}_X^1 \text{ such that } x = \int_T f(t) d\mu$, $f(t) \in B(t,p)$ a.e $t \in T$, $\alpha = \int_T U_t(f(t),p) d\mu$, $A_3(p,(x,\alpha)) = K_3$, and

$$u_1(p,(x,\alpha),y) = p \cdot (x - \int_T e(t)d\mu - y), \quad u_2(p,(x,\alpha),y) = \alpha, \quad u_3(p,(x,\alpha),y) = p \cdot y.$$
 (4.2)

Lemma 2. Under A.1' and A.2', $\int_T X(t) d\mu$ and $\int_T Y(t) d\mu$ are norm compact and convex.

Proof. By appealing to Proposition 1 in Sun and Yannelis [37], we have the results. \Box

Lemma 3. B(t,p) is a non-empty and continuous correspondence in p when X(t) and Y(t) are norm compact and Δ is weak* compact.

Proof. By A.8, it is clear that max $p \cdot Y(t) \geq 0$. Then $\eta(t) \in B(t,p)$ for any $p \in \Delta$. Therefore, B(t,p) is non-empty.

Let $\psi_t : \Delta \to \mathbb{R}$ be a function defined by $\psi_t(p) = \max_{y \in Y(t)} p \cdot y$. By Berge's theorem, $\psi_t(p)$ is continuous in p. We define a function $z_t : \Delta \to \mathbb{R}$ by

$$z_t(p) = p \cdot e(t) + \max p \cdot Y(t) = p \cdot e(t) + \psi_t(p). \tag{4.3}$$

Clearly, $z_t(p)$ is continuous in p. The budget correspondence can be rewritten as $B(t,p) = \{x \in X(t) : p \cdot x \leq z_t(p)\}$. By A.3 and A.8, $z_t(p) > 0$ for all $p \in \Delta$. Then a standard argument can be adopted to show that B(t,p) is continuous in p.

The following is the exact Fatou's lemma for Banach spaces proved by Khan and Sagara [22].

Lemma 4 (Theorem 3.5 in [22]). Let (T, \mathcal{T}, μ) be a complete saturated finite measure space and E be a Banach space. If $\{f_n\}$ is a well-dominated sequence in $L^1(\mu, E)$, then there exists $f \in L^1(\mu, E)$ such that

- (i) $f(t) \in w\text{-}Ls\{f_n(t)\}\ a.e.\ t \in T$,
- (ii) $\int f d\mu \in w\text{-}Ls\{\int f_n d\mu\}.$

Lemma 5. Under A.1' and A.2', A_i is continuous, non-empty, closed and convex valued for i = 1, 2, 3.

Proof. We adopt the idea of the proof from [12]. It is clear that $K_1 = \Delta$ is non-empty and convex. By Alaoglu's theorem, it is weak* compact and thus, weak* closed. It follows that A_1 is non-empty, closed and convex valued. From A.8, $\mathbf{0} \in \int_T Y(t) d\mu$ and thus $K_3 = \int_T Y(t) d\mu$ is non-empty. By Lemma 2, $\int_T Y(t) d\mu$ is convex and norm compact and thus, norm closed. Hence, A_3 is non-empty, closed and convex valued. Clearly, A_1 and A_3 are continuous.

We now turn to A_2 . Since the initial endowment map $e(t) \in B(t,p)$, A_2 is non-empty. Note $\int_T e(t)d\mu \in \int_T X(t)d\mu$ for all $t \in T$ and $\int_T U_t(e(t),p)d\mu \in [0,1]$. Thus, $K_2 = \int_T X(t)d\mu \times [0,1]$ is non-empty. By Lemma 2, $\int_T X(t)d\mu$ is norm compact and convex. It follows that K_2 is compact and convex.

We show the value of A_2 is closed. We need to show $(x,\alpha) \in A_2(p,y)$ when $x_n \to x$ in norm and $\alpha_n \to \alpha$ such that $(x_n,\alpha_n) \in A_2(p,y)$ for all n. Then there exists a sequence $\{f_n\} \subset \mathcal{S}_X^1$ such that $x_n = \int_T f_n(t) d\mu$ and $\alpha_n = \int_T U_t(f_n(t),p) d\mu$ with $f_n(t) \in B(t,p)$ for all n. By virtue of A.1', $\{f_n\}$ is well-dominated. We can appeal to Lemma 4 to have $f \in L^1(\mu,E)$ such that $f(t) \in X(t)$, $f(t) \in w$ -Ls $\{f_n(t)\}$ for a.e. $t \in T$, and $\int_T f(t) d\mu \in w$ -Ls $\{\int f_n d\mu\}$. Thus we can extract a subsequence from $\{f_n\}$ (which we do not relabel) such that $f_n(t) \to f(t)$ weakly for a.e. $t \in T$ and $\int_T f_n(t) d\mu \to \int_T f(t) d\mu$ weakly. Since B(t,p) is norm compact and $f_n(t) \in B(t,p)$ for all n, it follows $f(t) \in B(t,p)$. The weak limit $\int_T f(t) d\mu$ of the subsequence of $\{x_n\}$ must be equal to the norm limit x of the whole sequence $\{x_n\}$. Because $U_t(\cdot,p)$ is weakly continuous, $U_t(f_n(t),p) \to U_t(f(t),p)$ for a.e. $t \in T$. On the other hand, let $g_n(t) = U_t(f_n(t),p)$. Then from the boundedness of U, the sequence of functions $\{g_n\}$ is well-dominated. Lemma 4 implies that there exists $g \in L^1(\mu)$ such that $g_n(t) \to g(t)$ for a.e. $t \in T$ and $\alpha_n = \int_T g_n(t) d\mu \to \int_T g(t) d\mu$ up to subsequence. Hence $g(t) = U_t(f(t),p)$ for a.e. $t \in T$ and $\alpha = \lim_n \alpha_n = \int_T g(t) d\mu = \int_T U_t(f(t),p) d\mu$.

Next, we show the upper semicontinuity of A_2 . Since K_2 is compact, in order to prove A_2 is upper semicontinuous, it is sufficient to show that the graph of A_2 is closed. Let $p_n \to p$ in the weak* topology and $y_n \to y$ in the norm topology. We want to show that $(x, \alpha) \in A_2(p, y)$ when $x_n \to x$ in norm and $\alpha_n \to \alpha$ with $(x_n, \alpha_n) \in A_2(p_n, y_n)$ for all n. There exists $\{f_n\}$ such that $x_n = \int_T f_n(t) d\mu$ and $\alpha_n = \int_T U_t(f_n(t), p_n) d\mu$ with $f_n(t) \in B(t, p_n)$ for a.e $t \in T$ for all n. Clearly $\{f_n\}$ is well-dominated.

Let $g_n(t) = U_t(f_n(t), p_n)$ and $\phi_n(t) = (f_n(t), g_n(t))$. Then it is clear that $\{g_n\}$ and $\{\phi_n\}$ are both well-dominated. Consequently, there exists an integrable function ϕ on T such that $\phi(t) \in w$ -Ls $\{\phi_n(t)\}$ a.e. $t \in T$ and $\int_T \phi d\mu \in w$ -Ls $\{\int_T \phi_n d\mu\}$ by Lemma 4, where $\phi(t) = (f(t), g(t))$ for some $f \in L^1(\mu, E)$ and $g \in L^1(\mu)$ with $f(t) \in X(t)$ and $g(t) \in \mathbb{R}$. Then we can extract a convergent subsequence $\{\phi_n\}$ (we do not relabel) such that $\phi_n(t) \to \phi(t)$ weakly for a.e. $t \in T$ and $\int_T \phi_n(t) d\mu \to \int_T \phi(t) d\mu$ weakly. So we have $f_n(t) \to f_n(t)$ weakly for a.e. $t \in T$, $g_n(t) \to g(t)$ weakly for a.e. $t \in T$, $f_n(t) d\mu \to f_T(t) d\mu$ weakly and $\alpha_n = \int_T g_n(t) d\mu \to \int_T g(t) d\mu$.

Because $x_n = \int_T f_n(t) d\mu$ converges to x in norm, $\int_T f(t) d\mu = x$. By the joint continuity of U_t , $U_t(f_n(t), p_n) \to U_t(f(t), p)$ for a.e. $t \in T$. Hence, we have $g(t) = U_t(f(t), p)$ a.e. $t \in T$ and $\int_T U_t(f(t), p) d\mu = \int_T g(t) d\mu = \lim_n \alpha_n = \alpha$.

Now it remains to show $f(t) \in B(t,p)$. Because X(t) is norm compact, $f_n(t)$ converges up to subsequence to some limit in norm, which must be equal to f(t). It follows that for a.e. $t \in T$, $p_n \cdot f_n(t) \to p \cdot f(t)$. Since $p_n \cdot f_n(t) \le p_n \cdot e(t) + \max p_n \cdot Y(t)$, we have

$$p \cdot f(t) \le p \cdot e(t) + \max p \cdot Y(t). \tag{4.4}$$

Therefore, $f(t) \in B(t, p)$ for almost all $t \in T$. In sum, we showed that A_2 is norm upper semicontinuous.

We now prove the lower semicontinuity of A_2 . Suppose $(x,\alpha) \in A_2(p,y)$. In order to show A_2 is lower semicontinuous, it suffices to find a sequence (x_n,α_n) such that $(x_n,\alpha_n) \in A_2(p_n,y_n)$ converging to (x,α) in norm. Since $(x,\alpha) \in A_2(p,y)$, there exists a function f such that $x = \int_T f(t) d\mu$ and $\alpha = \int_T U_t(f(t),p)$. Notice that since for any $p \in \Delta$, B(t,p) is a norm closed subset of X(t), it is norm compact. Clearly it is convex.

Consider $p_n \to p$ in the weak* topology and, $y_n \to y$ in the norm topology. Note that $B(t, p_n)$ is convex and norm compact. Thus one can choose $f_n(t)$ from $B(t, p_n)$ such that $f_n(t)$ is the closest to f(t), i.e.,

$$||f_n(t) - f(t)|| \le ||z - f(t)|| \text{ for all } z \in B(t, p_n).$$
 (4.5)

We will show that f_n is measurable. Note that $B(\cdot, p)$ has a measurable graph. To see this, we adopt [27]. For $p \in \Delta$, define $\xi_p : T \times E \to [-\infty, \infty]$ by $\xi_p(t, x) = p \cdot x - p \cdot e(t) - \max p \cdot Y(t)$. By Proposition 3 in [16] (p.60), max $p \cdot Y(t)$ is measurable in t. Then ξ_p is measurable in t and continuous in x. By Proposition 3.1 in [39], $\xi_p(\cdot, \cdot)$ is jointly measurable. Notice that

$$G_{B(\cdot,p)} = \{(t,x) \in T \times X(t) : p \cdot x \le p \cdot e(t) + \max \ p \cdot Y(t)\} = \xi_p^{-1}([-\infty,0]) \cap G_X$$
 (4.6)

and thus the budget correspondence $B(\cdot, p)$ is graph measurable given p.

By Castaing's Representation Theorem in [39], there exists $\{h_m^n(t): m \in \mathbb{N}\}$ whose norm closure is $B(t, p_n)$. Let

$$\Psi_m^n(t) = \{ z \in B(t, p_n) : ||z - f(t)|| \le ||h_m^n(t) - f(t)|| \}$$

$$(4.7)$$

and

$$\Psi^n(t) \equiv \bigcap_{m=1}^{\infty} \Psi_m^n(t). \tag{4.8}$$

From the fact that B(t,p) is norm compact and the continuity of $\|\cdot\|$, it follows that $\Psi_m^n(t)$ is a non-empty measurable correspondence. Then the correspondence $\Psi^n: T \to 2^E$ has a measurable

graph. Since the set $\{h_m^n(t): m \in \mathbb{N}\}$ is dense in $B(t, p_n)$, only the closest point $f_n(t)$ to f(t) belongs to $\Psi^n(t)$. Therefore Ψ^n is a measurable function which is equal to f_n for μ -almost all $t \in T$. Hence, f_n is measurable for all n. It is now clear that $f_n \in \mathcal{S}_X^1$ for all n.

We will show that $\int_T f_n(t)d\mu \to \int_T f(t)d\mu$ in norm. Let $\varepsilon > 0$. Pick $b \in B(t,p) \cap N_{\varepsilon}(f(t))$ where $N_{\varepsilon}(f(t))$ is a neighborhood of f(t) with the radius ε . Suppose $b \notin B(t,p_n)$ for infinitely many n. Then

$$p_n \cdot b > p_n \cdot e(t) + \max p_n \cdot Y(t). \tag{4.9}$$

For some $\varepsilon \in (0,1)$, we have

$$p_n \cdot \varepsilon b > p_n \cdot e(t) + \max p_n \cdot Y(t).$$
 (4.10)

As $n \to \infty$, it follows

$$p \cdot \varepsilon b \ge p \cdot e(t) + \max p \cdot Y(t) \tag{4.11}$$

which contradicts $b \in B(t, p)$.

Thus, there is a \bar{n} such that $b \in B(t, p_n)$ for all $n \geq \bar{n}$. Because of the minimizing property (4.5) of $f_n(t)$ in $B(t, p_n)$, we have $||f_n(t) - f(t)|| < \varepsilon$. So $\lim_{n \to \infty} \int_T U_t(f_n(t), p_n) d\mu = \int_T U_t(f(t), p) d\mu$. And the Dominated Convergence Theorem ⁷ in [9] says

$$\lim_{n \to \infty} \int_{T} \|f_n(t) - f(t)\| \, d\mu = 0. \tag{4.12}$$

Let $x_n = \int_T f_n(t) d\mu$ and $\alpha_n = \int_T U_t(f_n(t), p_n) d\mu$. Then $(x_n, \alpha_n) \in A_2(p_n, y_n)$ for all $n \geq \bar{n}$. Moreover,

$$||x_n - x|| = \left\| \int_T f_n(t) d\mu - \int_T f(t) d\mu \right\| \le \int_T ||f_n(t) - f(t)|| d\mu \to 0.$$
 (4.13)

The last inequality comes from Theorem 4 in [9] (p.46). Hence, $x_n \to x$ in norm and $\alpha_n \to \alpha$. It follows that A_2 is norm lower semicontinuous.

We will show that A_2 is convex valued. Pick $(x,\alpha) \in A_2(p,y)$ and $(x',\alpha') \in A_2(p,y)$. Then there is a function $f: T \to E$ such that $\int_T f(t) d\mu = x$ and $\int_T U_t(f(t),p) d\mu = \alpha$ with $f(t) \in B(t,p)$ a.e. t and a function $f': T \to E$ such that $\int_T f'(t) d\mu = x'$ and $\int_T U_t(f'(t),p) d\mu = \alpha'$ with $f'(t) \in B(t,p)$ a.e t. Let $Z = E \times \mathbb{R}$ and we define a function $h: T \to Z$ by $h(t) = (f(t), U_t(f(t), p))$ and a function $h': T \to Z$ by $h'(t) = (f'(t), U_t(f'(t), p))$. It is clear that $h, h' \in L^1(\mu, Z)$. Let ν be a measure defined by

$$\nu(S) = \left(\int_{S} h(t)d\mu, \int_{S} h'(t)d\mu\right) \tag{4.14}$$

for $S \in \mathcal{T}$. Notice that $\nu(\emptyset) = ((\mathbf{0}, 0), (\mathbf{0}, 0))$ and $\nu(T) = ((x, \alpha), (x', \alpha'))$. It follows from Theorem 4.1 in [21] (Lyapunov's convexity theorem) that the range of ν is convex. Thus there

⁷See Theorem 3 in [9] p. 45.

exists $S \in \mathcal{T}$ such that $\nu(S) = \lambda \nu(T) = ((\lambda x, \lambda \alpha), (\lambda x', \lambda \alpha'))$ for $\lambda \in (0, 1)$. Let $f_{\lambda} = f \chi_S + f' \chi_{T \setminus S}$. Then $\int_T f_{\lambda}(t) d\mu = \int_S f(t) d\mu + \int_{T \setminus S} f'(t) d\mu = \lambda x + (1 - \lambda) x'$ and $\int_S U_t(f(t), p) d\mu + \int_{T \setminus S} U_t(f'(t), p) d\mu = \lambda \alpha + (1 - \lambda) \alpha'$. It is clear that $f_{\lambda}(t) \in B(t, p)$. Therefore, A_2 is a convex valued correspondence.

Lemma 6. Γ has an equilibrium.

Proof. As we proved in the proof of Lemma 5, K_1 , K_2 and K_3 are non-empty, convex and compact. Therefore, (i) of Lemma 1 is satisfied. Lemma 5 shows that A_i (i = 1, 2, 3) satisfies (ii) of Lemma 1. It is easy to see that u_i (i = 1, 2, 3) is continuous and quasi-concave on K_i . Hence, (iii) of Lemma 1 holds. Now we can appeal to Lemma 1 to have an equilibrium $(p^*, (x^*, \alpha^*), y^*)$ for Γ .

5 Proof of the Auxiliary Theorem

We are now ready to provide the proof of the Auxiliary Theorem.

Proof of the Auxiliary Theorem. We will prove that for an equilibrium for Γ , there is a competitive equilibrium for the economy.

Suppose that $(p^*, (x^*, \alpha^*), y^*)$ is an equilibrium for Γ . Hence there exist $f^* \in \mathcal{S}_X^1$ such that that $x^* = \int_T f^*(t) d\mu$ with $f^*(t) \in B(t, p^*)$ and $g^* \in \mathcal{S}_Y^1$ such that $y^* = \int_T g^*(t) d\mu$. We will show that (p^*, f^*, g^*) is a competitive equilibrium for the economy.

(i) We show that g^* is a profit maximization production plan.

By the definition of u_3 , $p^* \cdot y^* = p^* \cdot \int_T g^*(t) d\mu \ge p^* \cdot y$ for any $y \in \int_T Y(t) d\mu$. Therefore, $p^* \cdot \int_T g^*(t) d\mu = \max p^* \cdot \int_T Y(t) d\mu$. By Proposition 6 in [16] (p.63), we have $\max p^* \cdot \int_T Y(t) d\mu = \int_T \max p^* \cdot Y(t) d\mu$. Thus $p^* \cdot g^*(t) = \max p^* \cdot Y(t)$ for almost all $t \in T$. Note that Proposition 6 in [16] works in our commodity space E.

(ii) Let us prove $p^* \cdot f^*(t) \le p^* \cdot e(t) + p^* \cdot g^*(t)$ a.e. $t \in T$.

Note that $f^*(t) \in B(t, p^*) = \{x \in X(t) : p^* \cdot x \leq p^* \cdot e(t) + \max p^* \cdot Y(t)\}$ for almost all $t \in T$. From $p^* \cdot g^*(t) = \max p^* \cdot Y(t)$ for a.e. $t \in T$, we have the desired result.

(iii) We show that $U_t(x, p^*) > U_t(f^*(t), p^*)$ implies $p^* \cdot x > p^* \cdot e(t) + p^* \cdot g^*(t)$ for almost all $t \in T$.

By way of contradiction, suppose there exists a non-empty subset $S \in \mathcal{T}$ which is of positive measure and let F be a correspondence from S to X(t) defined by $F(t) = \{x \in X(t) : U_t(x, p^*) > U_t(f(t), p^*) \text{ and } p^* \cdot x \leq p^* \cdot e(t) + p^* \cdot g^*(t)\}$ for all $t \in S$. Recall that $U_t(\cdot, p^*)$ is measurable on the graph of X. Recall also that $B(\cdot, p^*)$ and X have measurable graphs. Therefore, F has a measurable graph. Moreover, since X is integrably bounded, so is F. Hence, there is a Bochner integrable selection f' of F. We now define $f'' = f'\chi_S + f^*\chi_{T\setminus S}$. It is clear that

 $\int_T U_t(f''(t), p^*) d\mu = \int_S U_t(f'(t), p^*) d\mu + \int_{T \setminus S} U_t(f^*(t), p^*) d\mu > \int_T U_t(f^*(t), p^*) d\mu = \alpha^* \text{ which is a contradiction.}$

(iv) We prove that (f^*, g^*) is a feasible allocation and a production plan.

We know that $p^* \cdot f^*(t) \leq p^* \cdot e(t) + p^* \cdot g^*(t)$ a.e. $t \in T$. By aggregating over T, we have $p^* \cdot (\int_T f^*(t) d\mu - \int_T e(t) d\mu - \int_T g^*(t) d\mu) \leq 0$. From the definition of the equilibrium of Γ , it follows that for any $p \in \Delta$,

$$p \cdot (\int_T f^*(t) d\mu - \int_T e(t) d\mu - \int_T g^*(t)) d\mu \le p^* \cdot (\int_T f^*(t) d\mu - \int_T e(t) d\mu - \int_T g^*(t) d\mu) \le 0. \tag{5.1}$$

Therefore, $-(\int_T f^*(t)d\mu - \int_T e(t)d\mu - \int_T g^*(t)d\mu) \in E_+$ which leads to $\int_T f^*(t)d\mu \le \int_T e(t)d\mu + \int_T g^*(t)d\mu$.

6 Proof of the Main Theorem

We provide the proof of the Main Theorem. The proof follows Noguchi [31] by considering a net of truncated subeconomies, whose consumption and production sets are norm compact, which is in line with Toussaint [38] and Khan and Yannleis [27]. From the Auxiliary Theorem, we have a net of competitive equilibria for the subeconomies. We then construct a sequence of competitive equilibria. Finally, by invoking the exact Fatou's lemma for infinite dimensional separable Banach spaces, we obtain a competitive equilibrium for the original economy.

Proof of the Main Theorem. As in Noguchi [31], we construct the norm compact subsets of X(t) and Y(t). Let $\mathcal{F} = \{K : T \to 2^E | K = \operatorname{co}(K^X \cup K^Y) \text{ where } K^X = \operatorname{co}(\bigcup_{i=1}^m \varphi_i), K^Y = \operatorname{co}(\bigcup_{j=1}^l \psi_j) \text{ such that } \varphi_i : T \to E \text{ and } \psi_j : T \to E \text{ are measurable with } \varphi_i(t) \in X(t) \text{ and } \psi_j(t) \in Y(t) \text{ for all } t \in T; e(t), \eta(t) \in K^X(t) \text{ and } 0 \in K^Y(t) \text{ for all } t \in T\}.$

Consider $K = \operatorname{co}(K^X \cup K^Y)$ such that $K^X(t) = \operatorname{co}(e(t) \cup \eta(t))$ for all $t \in T$ and $K^Y(t) = 0$ for all $t \in T$. Then $K \in \mathcal{F}$ and thus \mathcal{F} is non-empty. Let $K_1, K_2 \in \mathcal{F}$. Then it is clear that $\operatorname{co}(K_1 \cup K_2) \in \mathcal{F}$, which implies that \mathcal{F} is directed under the inclusion. Notice that for every $t \in T$, $K^X(t) = \operatorname{co}(\bigcup_{i=1}^m \varphi_i(t))$ and $K^Y(t) = \operatorname{co}(\bigcup_{j=1}^l \psi_j(t))$ are norm compact and thus K(t) is also norm compact (see Jameson [18] p.208). Now it follows that for $K \in \mathcal{F}$, K^X and K^Y are non-empty, convex and norm compact valued, respectively. By Theorem III. 30 in [7], K^X and K^Y are graph measurable.

We define a truncated economy $\mathcal{E}^K = [(T, \mathcal{T}, \mu), (K^X(t), K^Y(t), U_t^K, e(t))_{t \in T}]$ where U_t^K is the utility function U_t whose first domain is restricted to $K^X(t)$. Since $K^X(t)$ is convex and norm closed, by the separation theorem it is weakly closed. Thus it belongs to the Borel σ -algebra generated by the weak topology of E. It is clear that U^K is measurable.

It is easy to see that \mathcal{E}^K satisfies all the assumptions of the Auxiliary Theorem. Therefore, we appeal to the Auxiliary Theorem to obtain a competitive equilibrium (p_K, f_K, g_K) for \mathcal{E}^K .

Notice that $\{(p_K, f_K, g_K) : K \in \mathcal{F}\}$ is a net directed by inclusion. For all $K \in \mathcal{F}$, $K^X(t) \subset X(t)$ and, by A.1, X is integrably bounded and weakly compact valued. Thus $\{f_K\}$ is well-dominated. We apply the same logic to K^Y and Y to see $\{g_K\}$ is also well-dominated.

Since X and Y are non-empty closed valued correspondences by A.1 and A.2, (T, \mathcal{T}, μ) a complete probability space, E a complete separable metric space, by Theorem III. 30 in [7] there are two sequences of measurable functions $\varphi_i: T \to E$ and $\psi_i: T \to E$ such that

$$\operatorname{cl}\{\varphi_i(t)\}_{i\in\mathbb{N}} = X(t) \text{ and } \operatorname{cl}\{\psi_j(t)\}_{j\in\mathbb{N}} = Y(t) \text{ for all } t\in T.$$
 (6.1)

We then construct $K_m^X(t)$ using $\{\varphi_i(t)\}_{i=1}^m$ and $K_l^Y(t)$ using $\{\psi_j(t)\}_{j=1}^l$. Let us define $n=\min\{m,l\}$ where m,l are the numbers of φ_i and of ψ_j in K, respectively. Then consider a sequence of truncated subeconomies $\{\mathcal{E}^n\}$ consisting of $K_n^X(t)$ and $K_n^Y(t)$ for all $t \in T$. By the Auxiliary Theorem, we now have a sequence of competitive equilibria (p_n, f_n, g_n) for \mathcal{E}^n .

We appeal to Lemma 4 to have $f \in L^1(\mu, E)$ and $g \in L^1(\mu, E)$ such that $f(t) \in X(t)$, $f(t) \in w$ -Ls $\{f_n(t)\}$ a.e. $t \in T$ and $\int_T f d\mu \in w$ -Ls $\{\int_T f_n d\mu\}$ as well as $g(t) \in Y(t)$, $g(t) \in w$ -Ls $\{g_n(t)\}$ a.e. $t \in T$ and $\int_T g d\mu \in w$ -Ls $\{\int_T g_n d\mu\}$. Therefore, f is an allocation and g is a production plan. Since p_n belongs to Δ which is weak* compact, it has a subsequence still denoted by p_n weak* converging to p.

We will now show that (p, f, g) is a competitive equilibrium for \mathcal{E} .

Step 1: Let us show that for $x \in X(t)$,

$$U_t(x,p) > U_t(f(t),p)$$
 implies $p \cdot x > p \cdot e(t) + \max p \cdot Y(t)$ for almost all $t \in T$. (6.2)

We follow Khan and Sagara [24] for this proof. By method of contradiction, suppose that there exists $S \in \mathcal{T}$ of positive measure with the following property: for every $t \in S$ there exists $\hat{x} \in X(t)$ such that $U_t(\hat{x}, p) > U_t(f(t), p)$ and $p \cdot \hat{x} \leq p \cdot e(t) + \max p \cdot Y(t)$. Since $p \cdot e(t) + \max p \cdot Y(t) > 0$ by A.3 and A.8, it follows from the joint continuity of U_t that $U_t(\hat{x}, p) > U_t(f(t), p)$ and $p \cdot \hat{x} for some <math>\varepsilon \in (0, 1)$. Thus, we can assume without loss of generality that for every $t \in S$ there exists $\hat{x} \in X(t)$ such that $U_t(\hat{x}, p) > U_t(f(t), p)$ and $p \cdot \hat{x} . Let us define the correspondence <math>\Lambda: S \to 2^E$ by

$$\Lambda(t) = \{ x \in X(t) | U_t(x, p) > U_t(f(t), p), p \cdot x$$

 Λ is an integrably bounded correspondence and $\hat{x} \in \Lambda(t)$. We now show that Λ is graph measurable. Let $\Lambda_1(t) := \{x \in X(t) | U_t(x,p) > U_t(f(t),p)\}$ and $\Lambda_2(t) := \{x \in E | p \cdot x . Then <math>\Lambda(t) = \Lambda_1(t) \cap \Lambda_2(t)$. We need to prove that Λ_1 and Λ_2 are graph measurable. In the proof of Lemma 5, we already showed the joint measurability of the function given by $(t,x) \to p \cdot x - p \cdot e(t) - \max p \cdot Y(t)$. Therefore, Λ_2 is graph measurable.

We turn to Λ_1 . Let $\zeta: T \times E \to T \times E \times E$ be a mapping defined by $\zeta(t,x) = (t,x,f(t))$ and $\operatorname{proj}_{T \times E \times E}$ be a projection of $(T \times E) \times (T \times E \times E)$ onto the range space $T \times E \times E$ of ζ . By the projection theorem,⁸ $\operatorname{proj}_{T \times E \times E}(G_{\zeta})$ belongs to $\mathcal{T} \otimes \mathcal{B}(E,w) \otimes \mathcal{B}(E,w)$. We define a set H by

$$H := \{(t, x, x') \in T \times E \times E | U_t(x, p) > U_t(x', p) \} \cap ((G_X) \times E) \cap \operatorname{proj}_{T \times E \times E}(G_{\zeta}).$$

Then in view of A.5 and A.6, H belongs to $\mathcal{T} \otimes \mathcal{B}(E, w) \otimes \mathcal{B}(E, w)$. Let $\operatorname{proj}_{T \times E}$ be the projection of $(T \times E) \times E$ onto $T \times E$. Since $G_{\Lambda_1} = \operatorname{proj}_{T \times E}(H)$, we again appeal to the projection theorem to argue that G_{Λ_1} belongs to $\mathcal{T} \otimes \mathcal{B}(E, w)$.

Therefore, Λ has a measurable selection by Aumann's measurable selection theorem in [4]. Let $h: S \to E$ be a measurable selection from Λ . By Theorem III. 30 in [7], we can choose a sequence of measurable selections $h_n: S \to E$ such that $h_n(t) \in X(t)$ converges to h(t) in norm for all $t \in S$. By Lemma 4, there exists a Bochner integrable function $\hat{h}: S \to E$ such that $\hat{h}(t) \in \text{w-Ls}\{h_n(t)\}$ and $\hat{h}(t) \in X(t)$ a.e. $t \in S$. Hence, there is a subsequence of $\{h_n(t)\}$ in E converging weakly to $\hat{h}(t)$ for a.e. $t \in S$. It is clear that $\hat{h}(t) = h(t)$ a.e. $t \in S$ and we also have $(f(t), h(t)) \in \text{w-Ls}\{(f_n(t), h_n(t))\}$ a.e. $t \in S$.

Suppose now that the following set defined by

$$\bigcup_{n \in \mathbb{N}} \{ t \in S | U_t^n(h_n(t), p_n) > U_t^n(f_n(t), p_n), p_n \cdot h_n(t) < p_n \cdot e(t) + \max p_n \cdot K_n^Y(t) \}$$

is of measure zero. Then for each n, $U_t^n(f_n(t), p_n) \geq U_t^n(h_n(t), p_n)$ or $p_n \cdot h_n(t) \geq p_n \cdot e(t) + \max p_n \cdot K_n^Y(t)$ a.e. $t \in S$. Notice for any $y \in Y(t)$, there is a sequence of measurable functions $y_n : T \to E$ such that $y_n(t) \in K_n^Y(t)$ converges to y in norm for all $t \in T$ by Theorem III. 30 in [7]. When p_n converges to p in the weak* topology, we have $p_n \cdot h_n(t) \to p \cdot h(t)$ and $p_n \cdot y_n(t) \to p \cdot y$. Passing to the limit yields $U_t(f(t), p) \geq U_t(h(t), p)$ or $p \cdot h(t) \geq p \cdot e(t) + \max p \cdot Y(t)$ a.e. $t \in S$. But this is a contradiction to the fact that h is a measurable selection from Λ . Hence, there exists p_n such that $p_n \in S$ is a contradiction to the fact that $p_n \in S$ is of positive measure. However, this contradicts the fact that p_n and $p_n \in S$ are a price and an allocation of a Walrasian equilibrium for \mathcal{E}^n . We proved (6.2).

Indeed, we can further show that

$$p \cdot f(t) \ge p \cdot e(t) + \max p \cdot Y(t) \tag{6.3}$$

for almost all $t \in T$.

By A.4 (ii), if f(t) is a satiation point, (6.3) follows. If f(t) is not a satiation point, f(t) belongs to the weak closure of the upper contour set $\{x' \in X(t) : U_t(x', p) > U_t(f(t), p)\}$ for any

⁸see Theorem III.23 in [7].

⁹Note that for $x \in K_n^X(t)$, $U_t^n(x,p) = U_t(x,p)$ for any $p \in \Delta$.

 $p \in \Delta$. Thus, (6.2) implies (6.3).

Step 2: We show that f is a feasible allocation and g is a feasible production plan.

Since (p_n, f_n, g_n) is a competitive equilibrium for \mathcal{E}^n , it is clear that $\int_T f_n(t) d\mu \leq \int_T e(t) d\mu + \int_T g_n(t) d\mu$. Recall that for $\{f_n\}$ and $\{g_n\}$ Lemma 4 holds. Thus we can extract subsequences (which we do not relabel) from $\{f_n\}$ and $\{g_n\}$ such that $\int_T f_n(t) d\mu \to \int_T f(t) d\mu$ weakly and $\int_T g_n(t) d\mu \to \int_T g(t) d\mu$ weakly. Now from $\int_T f_n(t) d\mu \leq \int_T e(t) d\mu + \int_T g_n(t) d\mu$ we obtain

$$\int_{T} f(t)d\mu \le \int_{T} e(t)d\mu + \int_{T} g(t)d\mu. \tag{6.4}$$

Step 3: We prove that $p \cdot f(t) \leq p \cdot e(t) + p \cdot g(t)$ for almost all $t \in T$.

From (6.3), we have

$$p \cdot f(t) \ge p \cdot e(t) + p \cdot g(t) \tag{6.5}$$

for almost all $t \in T$. By integrating (6.5) over T,

$$\int_{T} [p \cdot f(t) - p \cdot e(t) - p \cdot g(t)] d\mu = p \cdot \int_{T} [f(t) - e(t) - g(t)] d\mu \ge 0.$$
 (6.6)

But from (6.4) it follows that

$$p \cdot \int_{T} [f(t) - e(t) - g(t)] d\mu = \int_{T} [p \cdot f(t) - p \cdot e(t) - p \cdot g(t)] \le 0.$$
 (6.7)

Hence, we can conclude $\int_T [p \cdot f(t) - p \cdot e(t) - p \cdot g(t)] = 0$. Therefore, we have

$$p \cdot f(t) = p \cdot e(t) + p \cdot g(t) \tag{6.8}$$

for almost all $t \in T$.

Step 4: Let us prove $p \cdot g(t) = \max p \cdot Y(t)$ a.e. $t \in T$.

From (6.3) and (6.8), we have the following inequality:

$$\max p \cdot Y(t) \le p \cdot f(t) - p \cdot e(t) = p \cdot g(t) \tag{6.9}$$

for almost all $t \in T$. Obviously, we have max $p \cdot Y(t) \ge p \cdot g(t)$. Hence, the conclusion follows. \square

7 Concluding Remarks

Remark 1 We can appeal to Galerkin approximations, as suggested by Khan and Sagara [24], to construct a sequence of truncated subeconomies $\{\mathcal{E}^n\}$ with finite dimensional commodity

spaces. Each \mathcal{E}^n can have a competitive equilibrium (p_n, f_n, g_n) due to Greenberg et al. [12]. We then apply the exact Fatou's lemma to the sequence of $\{f_n, g_n\}$ to obtain f and g. By the weak* compactness of Δ , we can extract a subsequence from $\{p_n\}$ that weak* converges to $p \in \Delta$. Applying similar arguments as in Khan and Sagara [24], we can show that (p, f, g) satisfies the properties of a competitive equilibrium.¹⁰

Remark 2 The Auxiliary theorem can be seen as a direct proof of Greenberg et al. [12] for infinite dimensional commodity spaces without taking the approximation approach.

Remark 3 We can replace the weak compactness of production sets by the following condition: Let A_Y be a set defined by $A_Y = \{g' \in \mathcal{S}_Y^1 : \exists f' \in \mathcal{S}_X^1 \text{ s.t. } \int_T f'(t)d\mu \leq \int_T e(t)d\mu + \int_T g'(t)d\mu \}$. We assume A_Y is weakly compact.

Applying our approach in the proof of the main theorem, we construct a sequence of truncated subeconomies $\{\mathcal{E}^n\}$ and obtain a sequence of competitive equilibria $\{p_n, f_n, g_n\}$.¹¹ Since X(t) is integrably bounded and weakly compact, we apply the exact Fatou's lemma to $\{f_n\}$ to have f and since A_Y is weakly compact, we have $g_n \to g$ in the weak topology. Also $p_n \to p \in \Delta$ in the weak* topology. We are then able to prove that (p, f, g) is a competitive equilibrium.

For the sequence $\{f_n\}$, we need two results: (by passing to a subsequence) $f_n \to f$ and $f_n(t) \to f(t)$ for almost all $t \in T$. These results make f a feasible allocation and f(t) a maximal element for the agent t. To our best knowledge, there are two ways to obtain these results: invoking the exact Fatou's Lemma or appealing to Theorem 5.1 in Khan and Yannelis [27]. Both approaches require weak compact subsets of the consumption sets. Therefore, even when we relax the weak compactness assumption of the consumption sets, we still need some weak compact subsets of the consumption sets which contain the set of maximal elements.

¹⁰We are grateful to an anonymous referee for drawing our attention to Khan and Sagara [24].

¹¹Remark 3 in [12] provided an equilibrium existence result with non-compact consumption and production sets for their economy.

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