

# MSA HW1

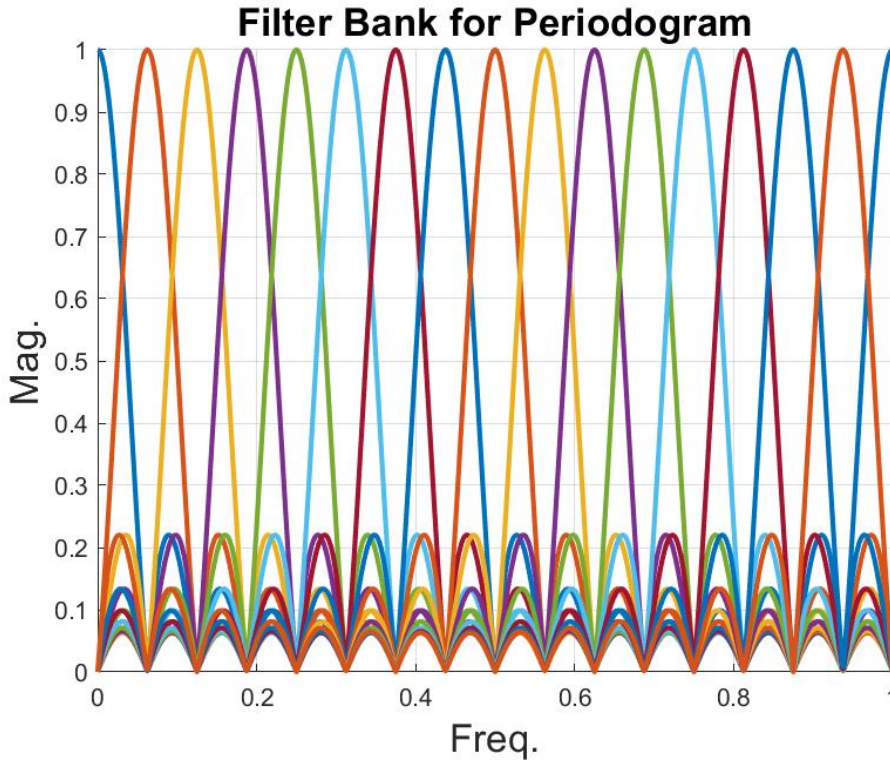
## Problem 1

For simplicity, assuming  $\alpha = \hat{S}_{x,per}(e^{j2\pi f})$ ,  $\beta = S_x(e^{j2\pi f})$  and  $\mu = \mathbb{E}[\alpha]$ . Notice that  $\alpha$  is a random variable and  $\beta$  and  $\mu$  are deterministic. Also,  $\alpha$ ,  $\beta$ ,  $\mu$  are real.

- (1) By definition  $\text{MSE}(\alpha, \beta) = \mathbb{E}[|\alpha - \beta|^2] = \mathbb{E}[\alpha^2] - 2\mathbb{E}[\alpha\beta] + \mathbb{E}[\beta^2] = \mathbb{E}[\alpha^2] - 2\mu\beta + \beta^2$
- (2) By definition  $\text{Bias}(\alpha)^2 = (\mu - \beta)^2 = \mu^2 - 2\mu\beta + \beta^2$
- (3) By definition  $\text{Var}(\alpha) = \mathbb{E}[|\alpha - \mu|^2] = \mathbb{E}[\alpha^2] - 2\mu^2 + \mu^2 = \mathbb{E}[\alpha^2] - \mu^2$
- (4) From (1), (2), (3), it is clear that  $\text{MSE}(\alpha, \beta) = \text{Bias}(\alpha)^2 + \text{Var}(\alpha)$

## Problem 2

The magnitude response for the filter is as follows:



## Problem 3

$$\begin{aligned} & \mathbb{E}\left[\frac{1}{NU} \left| \sum_{n_1=-\infty}^{\infty} x(n_1)w(n_1)e^{-j2\pi f n_1} \right|^2\right] \\ &= \frac{1}{NU} \mathbb{E}\left[\left(\sum_{n_1=-\infty}^{\infty} x(n_1)w(n_1)e^{-j2\pi f n_1}\right)\left(\sum_{n_2=-\infty}^{\infty} x^*(n_2)w^*(n_2)e^{j2\pi f n_2}\right)\right] \\ &= \frac{1}{NU} \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \mathbb{E}[x(n_1)x^*(n_2)]w(n_1)w^*(n_2)e^{-j2\pi f n_1}e^{j2\pi f n_2} \end{aligned}$$

$$\begin{aligned} & \text{Let } k = n_1 - n_2 \\ &= \frac{1}{NU} \sum_{n_1=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} r_x(k)w(n_1)w^*(n_1 - k)e^{-j2\pi f k} \\ &= \frac{1}{NU} \sum_{k=-\infty}^{\infty} r_x(k) \left[ \sum_{n_1=-\infty}^{\infty} w(n_1)w^*(n_1 - k) \right] e^{-j2\pi f k} \end{aligned}$$

$$\begin{aligned} & \text{Let } q(n) = w^*(-n) * w(n) \rightarrow q(k) = \sum_{n=-\infty}^{\infty} w^*(n - k)w(n) \\ &= \frac{1}{NU} \sum_{k=-\infty}^{\infty} r_x(k)q(k)e^{-j2\pi f k} \\ &= \frac{1}{NU} S_x(e^{j2\pi f}) * Q(e^{j2\pi f}) \\ &= \frac{1}{NU} S_x(e^{j2\pi f}) * |W(e^{j2\pi f})|^2 \end{aligned}$$

$$\text{Since } w^*(-n) \xrightarrow{\text{DTFT}} W^*(e^{j2\pi f}),$$

$$q(n) = w^*(-n) * w(n) \xrightarrow{\text{DTFT}} W^*(e^{j2\pi f})W(e^{j2\pi f}) = |W(e^{j2\pi f})|^2$$

## Problem 4

**a.**

(1) We prove that the parzen window is 4-times self-convolution of the rect window.

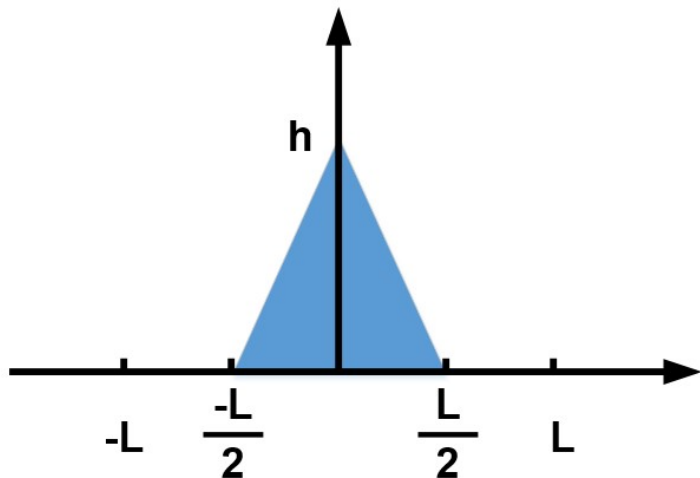
Let

$$w_{\text{rect}}(x) = \begin{cases} \frac{2}{L} \sqrt{\frac{6}{L}}, & \text{if } |x| \leq \frac{L}{4} \\ 0, & \text{otherwise} \end{cases}$$

and let

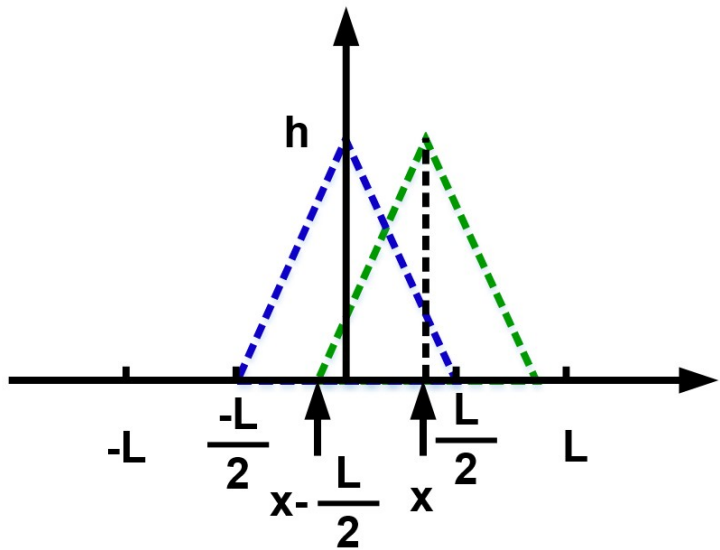
$$w_{\text{tri}}(x) = \text{rect}(x) * \text{rect}(x) = \begin{cases} \sqrt{\frac{6}{L}}, & \text{if } |x| \leq \frac{L}{2} \\ 0, & \text{otherwise} \end{cases},$$

We define  $h = \sqrt{\frac{6}{L}}$ ,



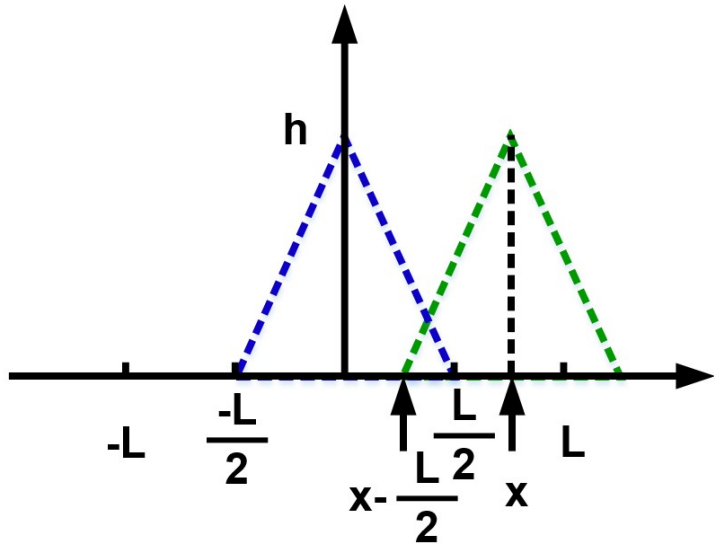
For  $|x| \leq \frac{L}{2}$ , we have

$$w_{\text{parzen}}(x) = 2\left(\frac{\sqrt{\frac{6}{L}}}{2}\right)^2 \left[ \int_{x-\frac{L}{2}}^0 [t - (x - \frac{L}{2})][t + \frac{L}{2}] dt + \int_0^{\frac{L}{2}} [t - (x - \frac{L}{2})][\frac{L}{2} - t] dt \right] = 1 - 6\frac{x^2}{L^2} + 6\frac{x^3}{L^3}$$



For  $\frac{L}{2} \leq |x| \leq L$ , we have

$$w_{\text{parzen}}(x) = \left(\frac{\sqrt{\frac{6}{L}}}{2}\right)^2 \left[ \int_{x-\frac{L}{2}}^{\frac{L}{2}} [t - (x - \frac{L}{2})][\frac{L}{2} - t] dt \right] = 2\left(1 - \frac{x}{L}\right)^3$$



(2) Since  $w_{\text{parzen}}(x) = (w_{\text{rect}} * w_{\text{rect}} * w_{\text{rect}} * w_{\text{rect}})(x)$ , we have the DTFT of parzen window as

$$W_{\text{parzen}}(e^{j\omega}) = |W_{\text{rect}}(e^{j\omega})|^4 \sim \left[ \frac{\sin(\frac{\omega L}{4})}{\sin(\frac{\omega}{4})} \right]^4 \geq 0,$$

$$\text{since } W_{\text{rect}}(e^{j\omega}) \sim \frac{\sin(\frac{\omega L}{4})}{\sin(\frac{\omega}{4})}$$

**b.**

Let  $x = \frac{l}{L} \rightarrow Ldx = dl$ , we have

$$\begin{aligned} E_w &= \int_0^{\frac{1}{2}} 2w_p(l)dl + \int_{\frac{1}{2}}^1 2w_p(l)dl \\ &= \int_0^{\frac{1}{2}} 2(1 - 6x^2 + 6x^3)^2 Ldx + \int_{\frac{1}{2}}^1 2(2(1 - x)^3)^2 Ldx = 0.539286L \sim 0.539L \end{aligned}$$

Therefore, the variance reduction ratio is  $\frac{E_w}{N} \sim 0.539 \frac{L}{N}$

## Problem 5

$$\begin{aligned} & \frac{1}{N} \mathbb{E} [ | \sum_{n=0}^{N-1} e^{-j2\pi n} x(n) |^2 ] \\ &= \frac{1}{N} \mathbb{E} [ (\sum_{n_1=0}^{N-1} e^{-j2\pi n_1} x(n_1)) (\sum_{n_2=0}^{N-1} x(n_2) e^{-j2\pi n_2})^* ] \\ &= \frac{1}{N} \mathbb{E} [ \sum_{n_1=0}^{N-1} \sum_{n_2=0}^{N-1} e^{-j2\pi n_1} x(n_1) x^*(n_2) e^{j2\pi n_2} ] \\ &= \end{aligned}$$

$$\begin{aligned} & \frac{1}{N} \mathbb{E} \left[ \begin{bmatrix} 1 & e^{-j2\pi f} & e^{-j4\pi f} & \dots & e^{-j2(N-1)\pi f} \end{bmatrix} \begin{bmatrix} x(0)x^*(0) & x(0)x^*(1) & x(0)x^*(2) & \dots & x(0)x^*(N-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x(N-1)x^*(0) & x(N-1)x^*(1) & x(N-1)x^*(2) & \dots & x(N-1)x^*(N-1) \end{bmatrix} \begin{bmatrix} 1 \\ e^{j2\pi f} \\ e^{j4\pi f} \\ \vdots \\ e^{j2(N-1)\pi f} \end{bmatrix} \right] \\ &= \mathbf{v}^H(f) \frac{1}{N} \begin{bmatrix} r(0) & r(-1) & r(-2) & \dots & r(-(N-1)) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r(N-1) & r(N-2) & r(N-3) & \dots & r(0) \end{bmatrix} \mathbf{v}(f) \end{aligned}$$

Therefore,

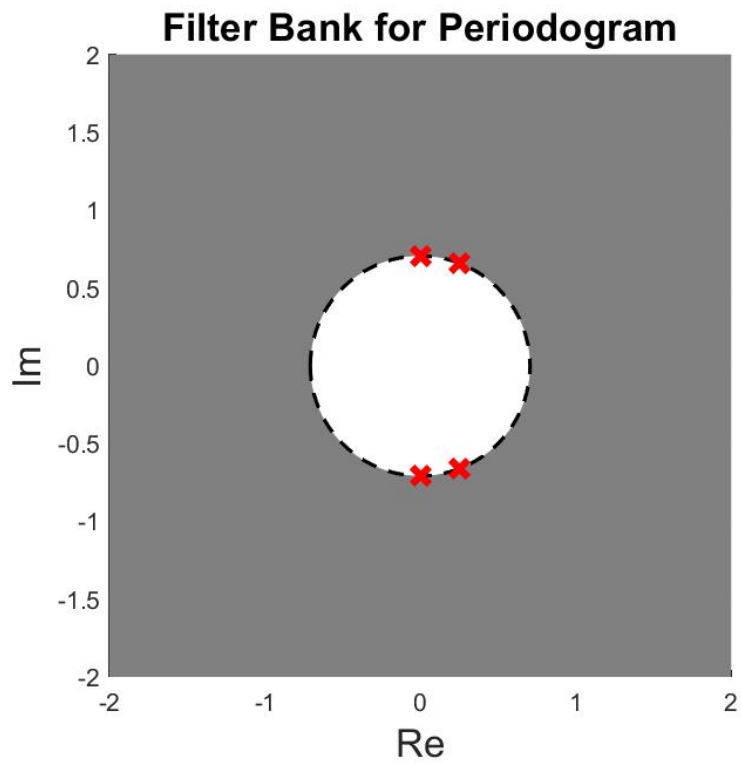
$$\mathbf{P} = \frac{1}{N} \begin{bmatrix} r(0) & r(-1) & r(-2) & \dots & r(-(N-1)) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r(N-1) & r(N-2) & r(N-3) & \dots & r(0) \end{bmatrix}$$

## Problem 6

**a.**

$$H(z) = \frac{1}{(1 - \frac{1}{2}z^{-1} + \frac{1}{2}z^{-2})(1 + \frac{1}{2}z^{-2})} = \frac{1}{(1 - \frac{1+\sqrt{7}j}{4}z^{-1})(1 - \frac{1-\sqrt{7}j}{4}z^{-1})(1 + \frac{1}{\sqrt{2}}jz^{-1})(1 - \frac{1}{\sqrt{2}}jz^{-1})}$$

The z-plane plot is as follows, and the ROC is  $|z| > \frac{1}{\sqrt{2}}$



b.

$$\frac{X(z)}{V(z)} = H(z) = \frac{1}{(1 - \frac{1}{2}z^{-1} + \frac{1}{2}z^{-2})(1 + \frac{1}{2}z^{-2})}$$

$$\rightarrow X(z)(1 - \frac{1}{2}z^{-1} + \frac{1}{2}z^{-2})(1 + \frac{1}{2}z^{-2}) = V(z)$$

$$\rightarrow X(z)(1 - \frac{1}{2}z^{-1} + z^{-2} - \frac{1}{4}z^{-3} + \frac{1}{4}z^{-4}) = V(z)$$

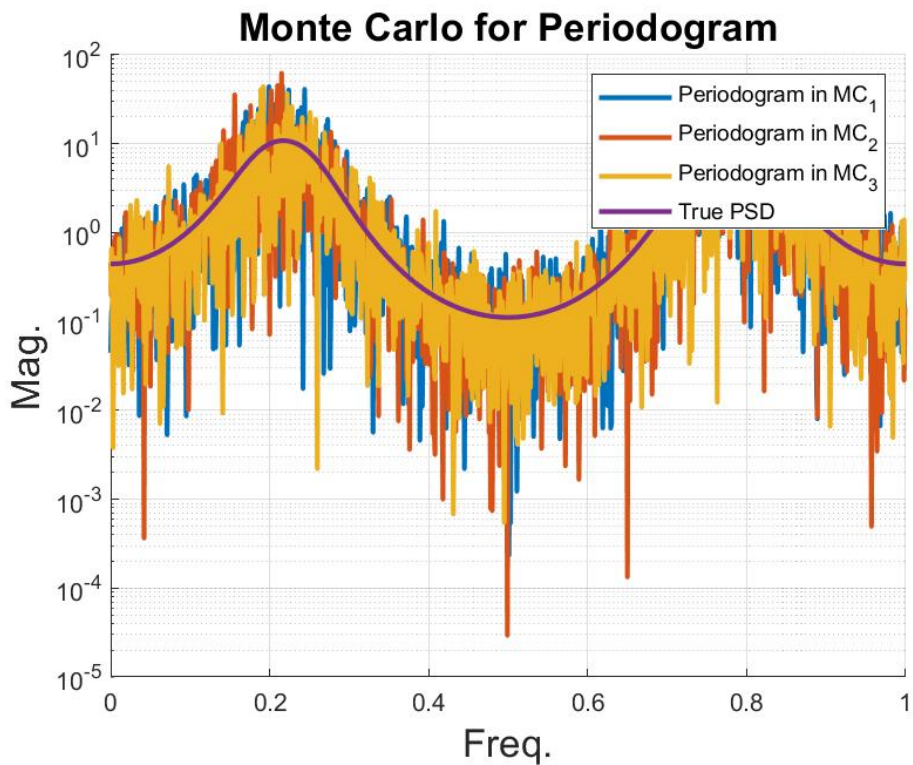
$$\rightarrow x(n) - \frac{1}{2}x(n-1) + x(n-2) - \frac{1}{4}x(n-3) + \frac{1}{4}x(n-4) = v(n)$$

c.

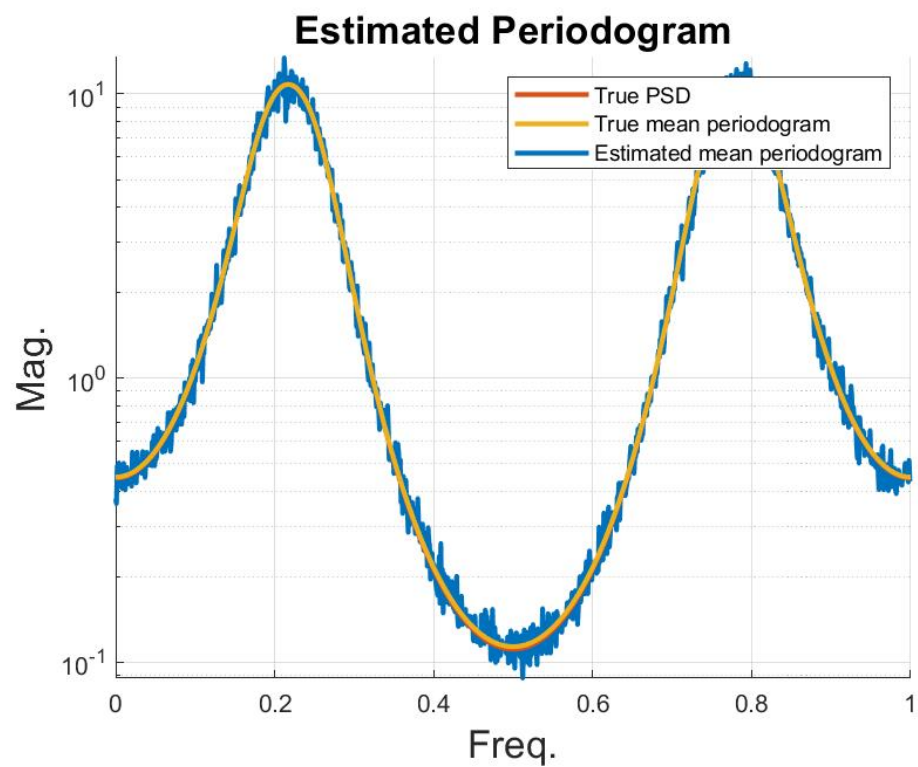
$$S_x(e^{j2\pi f}) = |H(e^{j2\pi f})|^2 S_v(e^{j2\pi f}) = \left| \frac{1}{(1 - \frac{1}{2}e^{-j2\pi f} + \frac{1}{2}e^{-j4\pi f})(1 + \frac{1}{2}e^{-j4\pi f})} \right|^2 = \frac{1}{[\frac{3}{2} - \frac{3}{2}\cos(2\pi f) + \cos(4\pi f)][\frac{5}{4} + \cos(4\pi f)]}$$

## Problem 7

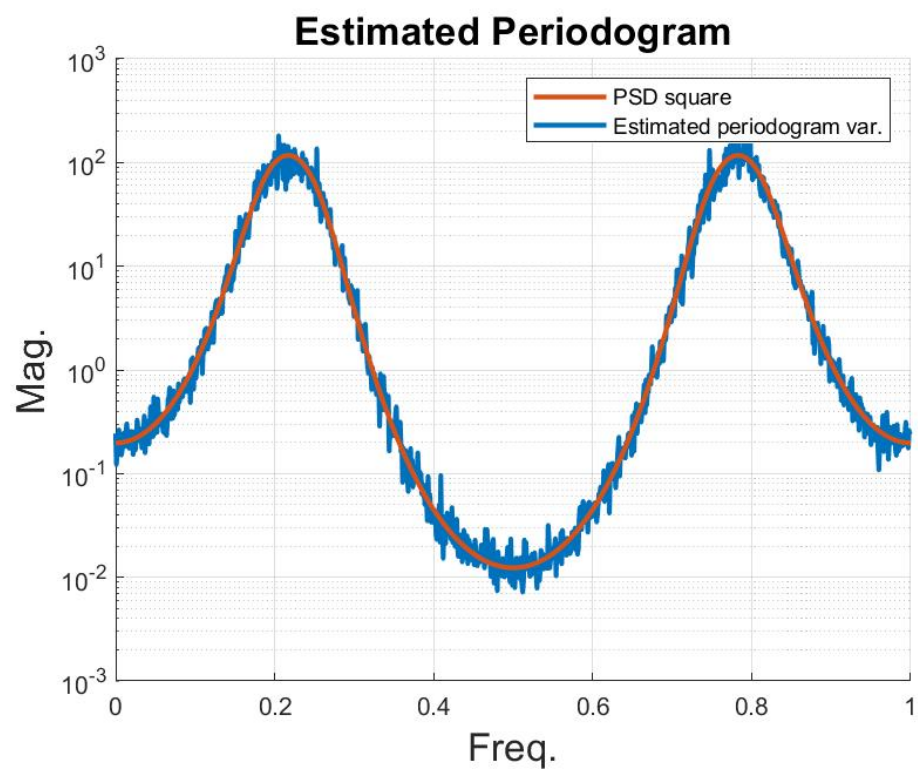
a.



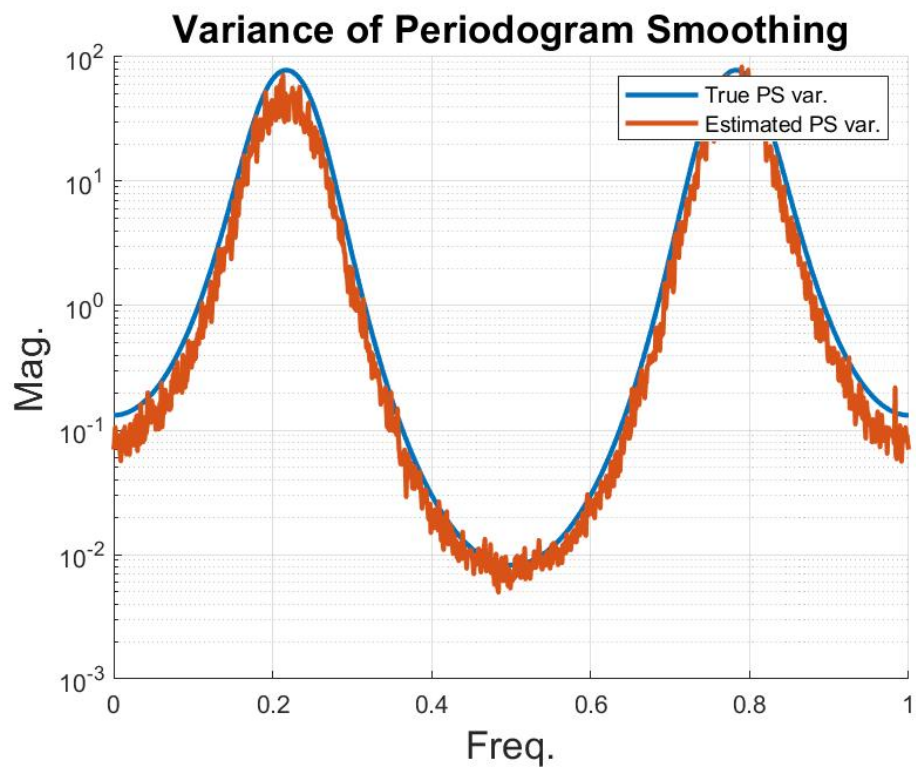
b.



c.



d.



e.

