MSA HW1

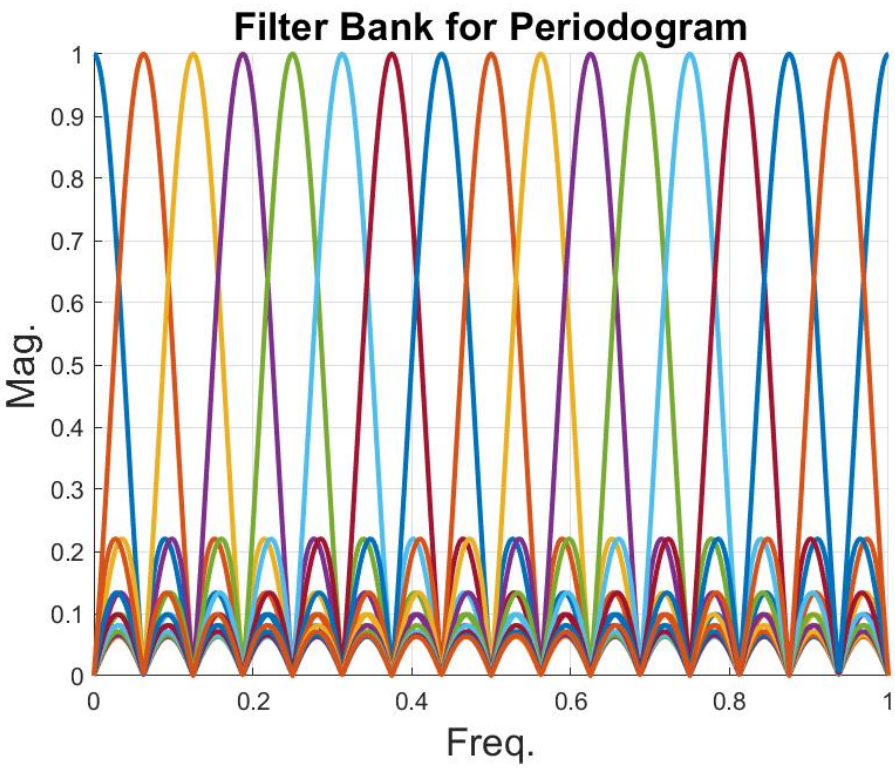
Problem 1

For simplicity, assuming $\alpha = \hat{S}_{x,per}(e^{j2\pi f})$, $\beta = S_x(e^{j2\pi f})$ and $\mu = \mathbb{E}[\alpha]$. Notice that α is a random variable and β and μ are deterministic. Also, α, β , μ are are real.

(1) By definition $\mathrm{MSE}(\alpha,\beta) = \mathbb{E}[|\alpha-\beta|^2] = \mathbb{E}[\alpha^2] - 2\mathbb{E}[\alpha\beta] + \mathbb{E}[\beta^2] = \mathbb{E}[\alpha^2] - 2\mu\beta + \beta^2$
(2) By definition $\mathrm{Bias}(\alpha)^2 = (\mu-\beta)^2 = \mu^2 - 2\mu\beta + \beta^2$
(3) By definition $\mathrm{Var}(\alpha) = \mathbb{E}[|\alpha-\mu|^2] = \mathbb{E}[\alpha^2] - 2\mu^2 + \mu^2 = \mathbb{E}[\alpha^2] - \mu^2$
(4) From (1), (2), (3), it is clear that $\mathrm{MSE}(\alpha,\beta) = \mathrm{Bias}(\alpha)^2 + \mathrm{Var}(\alpha)$

Problem 2

The magnitude response for the filter is as follows:



Problem 3

$$\mathbb{E}[\frac{1}{NU}|\sum_{n_1=-\infty}^{\infty} x(n)w(n)e^{-j2\pi fn}|^2]$$
$$= \frac{1}{NU}\mathbb{E}[(\sum_{n_1=-\infty}^{\infty} x(n_1)w(n_1)e^{-j2\pi fn_1})(\sum_{n_2=-\infty}^{\infty} x^*(n_2)w^*(n_2)e^{j2\pi fn_2})]$$
$$= \frac{1}{NU}\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\mathbb{E}[x(n_1)x^*(n_2)]w(n_1)w^*(n_2)e^{-j2\pi fn_1}e^{j2\pi fn_2}$$

Let $k = n_1 - n_2$
$$= \frac{1}{NU}\sum_{n_1=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} r_x(k)w(n_1)w^*(n_1-k)e^{-j2\pi fk}$$
$$= \frac{1}{NU}\sum_{k=-\infty}^{\infty} r_x(k)[\sum_{n_1=-\infty}^{\infty} w(n_1)w^*(n_1-k)]e^{-j2\pi fk}$$

Let $q(n) = w^*(-n) * w(n) \to q(k) = \sum_{n=-\infty}^{\infty} w^*(n-k)w(n)$
$$= \frac{1}{NU}\sum_{k=-\infty}^{\infty} r_x(k)q(k)e^{-j2\pi fk}$$
$$= \frac{1}{NU}S_x(e^{j2\pi f}) * Q(e^{j2\pi f})$$
$$= \frac{1}{NU}S_x(e^{j2\pi f}) * |W(e^{j2\pi f})|^2$$
Since $w^*(-n) \xrightarrow{\mathrm{DTFT}} W^*(e^{j2\pi f})$,
$q(n) = w^*(-n) * w(n) \xrightarrow{\mathrm{DTFT}} W^*(e^{j2\pi f})W(e^{j2\pi f}) = |W(e^{j2\pi f})|^2$

Problem 4

a.

(1) We prove that the parzen window is 4-times self-convolution of the rect window.

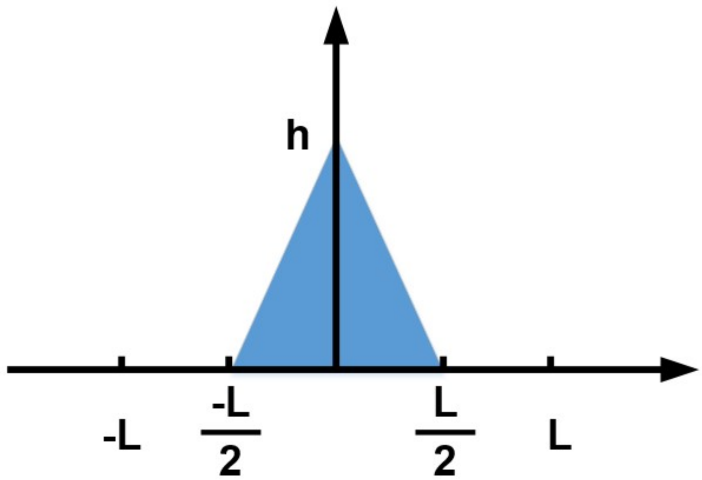
Let

$$w_{\text{rect}}(x) = \begin{cases} \frac{2}{L}\sqrt{\frac{6}{L}}, & \text{if } |x| \le \frac{L}{4} \\ 0, & \text{otherwise} \end{cases}$$

and let

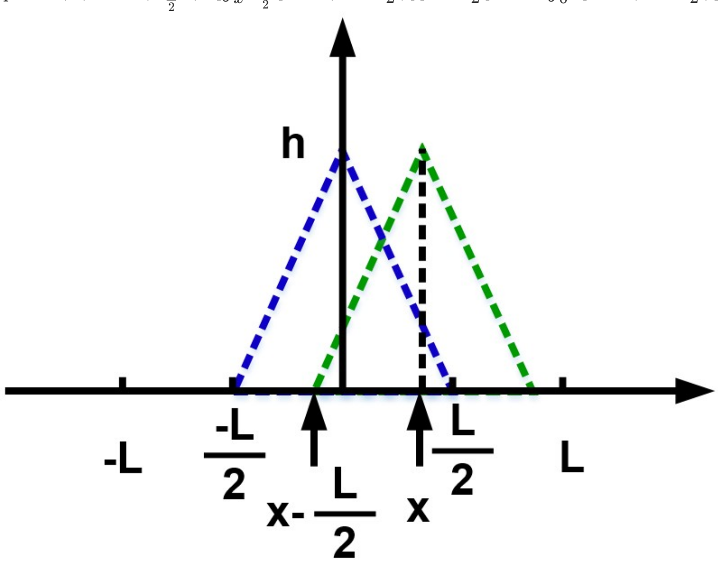
$$w_{\text{tri}}(x) = \text{rect}(x) * \text{rect}(x) = \begin{cases} \sqrt{\frac{6}{L}}, & \text{if } |x| \le \frac{L}{2} \\ 0, & \text{otherwise} \end{cases},$$

We define $h = \sqrt{\frac{6}{L}}$,



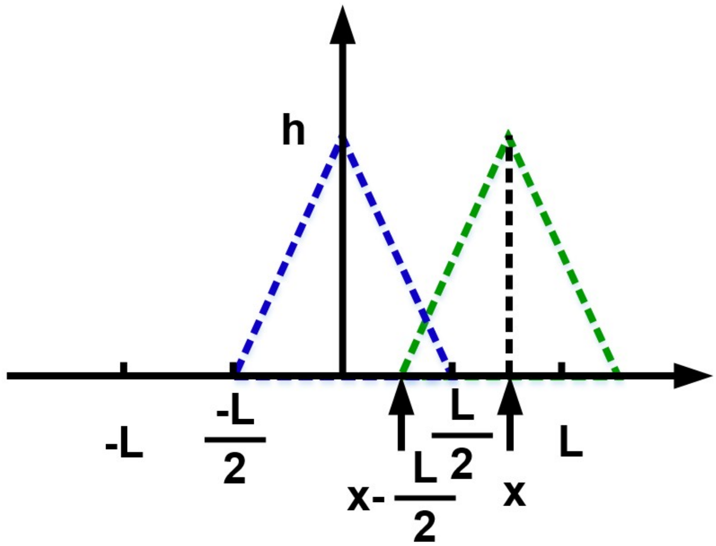
For $|x| \le \frac{L}{2}$,we have

$$w_{\text{parzen}}(x) = 2\left(\frac{\sqrt{\frac{6}{L}}}{\frac{L}{2}}\right)^2\left[\int_{x-\frac{L}{2}}^{0}[t-(x-\tfrac{L}{2})][t+\tfrac{L}{2}]dt + \int_{0}^{\frac{x}{2}}[t-(x-\tfrac{L}{2})][\tfrac{L}{2}-t]dt\right] = 1 - 6\frac{x^2}{L^2} + 6\frac{x^3}{L^3}$$



For $\frac{L}{2} \le |x| \le L$, we have

$$w_{\text{parzen}}(x) = \left(\frac{\sqrt{\frac{6}{L}}}{\frac{L}{2}}\right)^2\left[\int_{x-\frac{L}{2}}^{\frac{L}{2}}[t-(x-\tfrac{L}{2})][\tfrac{L}{2}-t]dt = 2\left(1-\frac{x}{L}\right)^3\right.$$



(2) Since $w_{\text{parzen}}(x) = (w_{\text{rect}} * w_{\text{rect}} * w_{\text{rect}} * w_{\text{rect}})(x)$, we have the DTFT of parzen window as
$W_{\text{parzen}}(e^{j\omega}) = |W_{\text{rect}}(e^{j\omega})|^4 \sim [\frac{\sin(\frac{\omega L}{4})}{\sin(\frac{\omega}{4})}]^4 \geq 0,$
since $W_{\text{rect}}(e^{j\omega}) \sim \frac{\sin(\frac{\omega L}{4})}{\sin(\frac{\omega}{4})}$

b.
Let $x = \frac{l}{L} \to Ldx = dl$, we have
$E_w = \int_0^{\frac{1}{2}} 2w_p(l)dl + \int_{\frac{1}{2}}^1 2w_p(l)dl$
$= \int_0^{\frac{1}{2}} 2(1-6x^2+6x^3)^2 Ldx + \int_{\frac{1}{2}}^1 2(2(1-x)^3)^2 Ldx = 0.539286L \sim 0.539L$
Therefore, the variance redunction ratio is $\frac{E_w}{N} \sim 0.539\frac{L}{N}$

Problem 5

$\frac{1}{N}\mathbb{E}[|\sum_{n=0}^{N-1} e^{-j2\pi n}x(n)|^2]$
$= \frac{1}{N}\mathbb{E}[(\sum_{n_1=0}^{N-1} e^{-j2\pi n_1}x(n_1))(\sum_{n_2=0}^{N-1} x(n_2)e^{-j2\pi n_2})^*]$
$= \frac{1}{N}\mathbb{E}[\sum_{n_1=0}^{N-1}\sum_{n_2=0}^{N-1} e^{-j2\pi n_1}x(n_1)x^*(n_2)e^{j2\pi n_2}]$
$=$

$$\frac{1}{N}\mathbb{E}[\begin{bmatrix}1 & e^{-j2\pi f} & e^{-j4\pi f} & \dots & e^{-j2(N-1)\pi f}\end{bmatrix}\begin{bmatrix} x(0)x^*(0) & x(0)x^*(1) & x(0)x^*(2) & \dots & x(0)x^*(N-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x(N-1)x^*(0) & x(N-1)x^*(1) & x(N-1)x^*(2) & \dots & x(N-1)x^*(N-1)\end{bmatrix}\begin{bmatrix}1 \\ e^{j2\pi f} \\ e^{j4\pi f} \\ \vdots \\ e^{j2(N-1)\pi}\end{bmatrix}$$

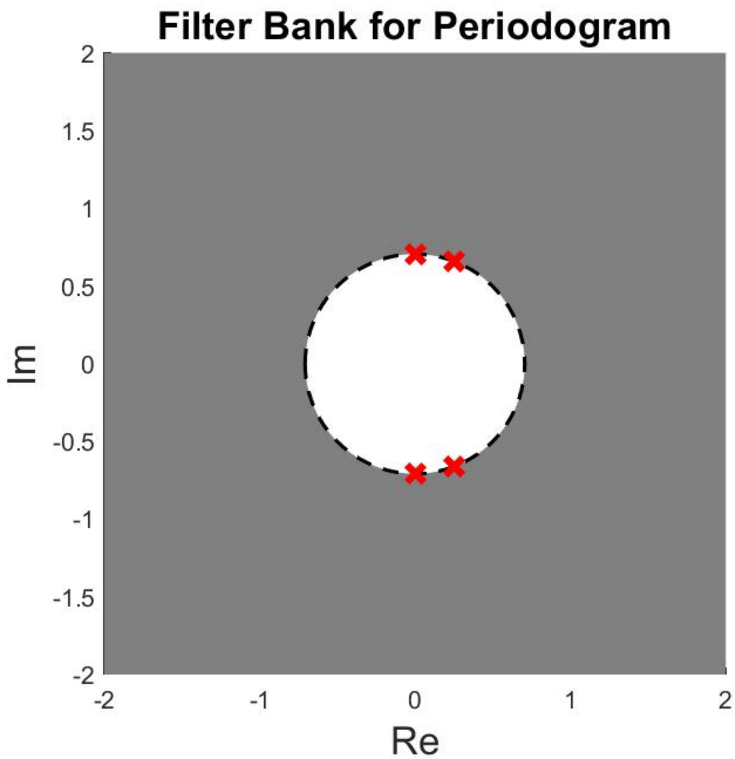
$$= \mathbf{v}^H(f)\frac{1}{N}\begin{bmatrix} r(0) & r(-1) & r(-2) & \dots & r(-(N-1)) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r(N-1) & r(N-2) & r(N-3) & \dots & r(0)\end{bmatrix}\mathbf{v}(f)$$

Therefore,

$$\mathbf{P} = \frac{1}{N}\begin{bmatrix} r(0) & r(-1) & r(-2) & \dots & r(-(N-1)) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r(N-1) & r(N-2) & r(N-3) & \dots & r(0)\end{bmatrix}$$

Problem 6

a.
$H(z) = \frac{1}{(1-\frac{1}{2}z^{-1}+\frac{1}{2}z^{-2})(1+\frac{1}{2}z^{-2})} = \frac{1}{(1-\frac{1+\sqrt{7}j}{4}z^{-1})(1-\frac{1-\sqrt{7}j}{4}z^{-1})(1+\frac{1}{\sqrt{2}}jz^{-1})(1-\frac{1}{\sqrt{2}}jz^{-1})}$
The z-plane plot is as follows, and the ROC is $|z| > \frac{1}{\sqrt{2}}$



b.

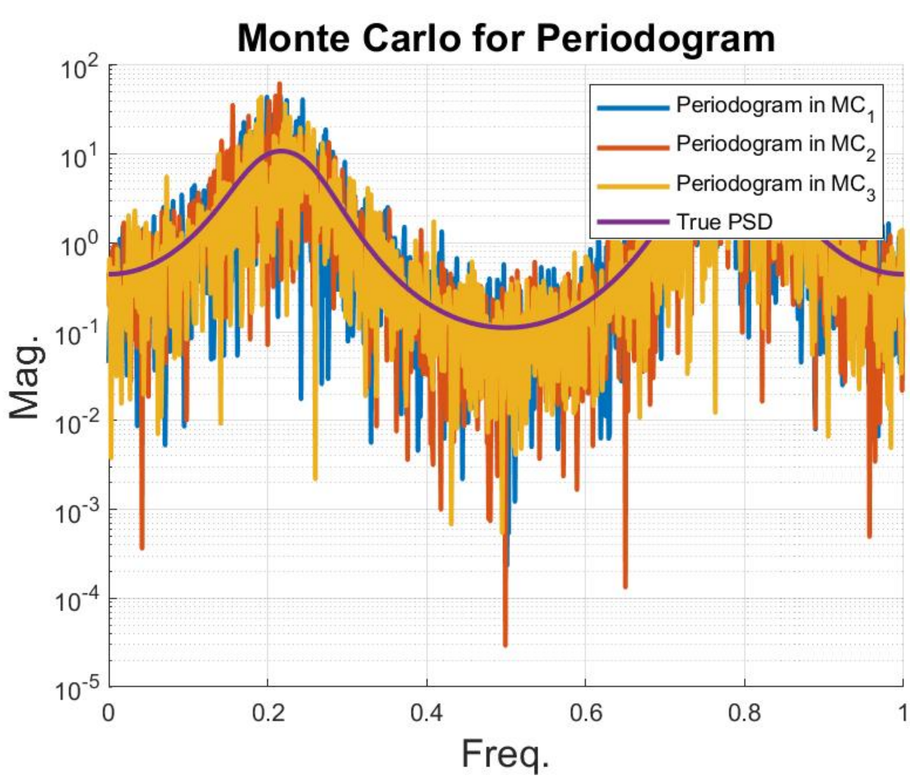
$\frac{X(z)}{V(z)} = H(z) = \frac{1}{(1-\frac{1}{2}z^{-1}+\frac{1}{2}z^{-2})(1+\frac{1}{2}z^{-2})}$
$\rightarrow X(z)(1-\frac{1}{2}z^{-1}+\frac{1}{2}z^{-2})(1+\frac{1}{2}z^{-2}) = V(z)$
$\rightarrow X(z)(1-\frac{1}{2}z^{-1}+z^{-2}-\frac{1}{4}z^{-3}+\frac{1}{4}z^{-4}) = V(z)$
$\rightarrow x(n)-\frac{1}{2}x(n-1)+x(n-2)-\frac{1}{4}x(n-3)+\frac{1}{4}x(n-4) = v(n)$

c.

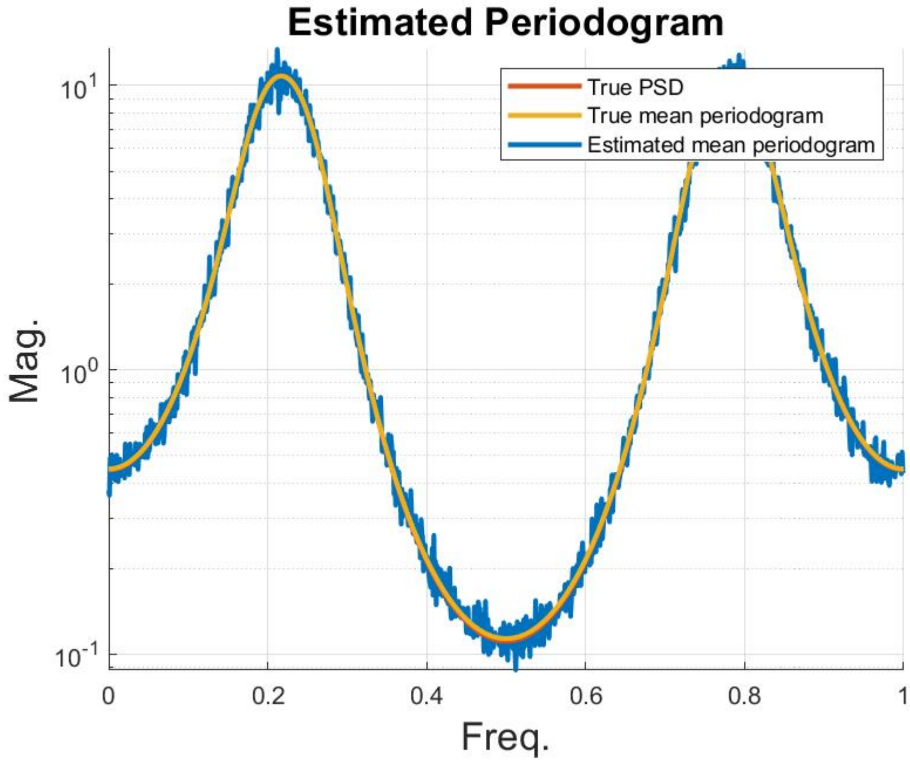
$S_x(e^{j2\pi f}) = |H(e^{j2\pi f})|^2 S_v(e^{j2\pi f}) = |\frac{1}{(1-\frac{1}{2}e^{-j2\pi f}+\frac{1}{2}e^{-j4\pi f})(1+\frac{1}{2}e^{-j4\pi f})}|^2 = \frac{1}{[\frac{3}{2}-\frac{3}{2}cos(2\pi f)+cos(4\pi f)][\frac{5}{4}+cos(4\pi f)]}$

Problem 7

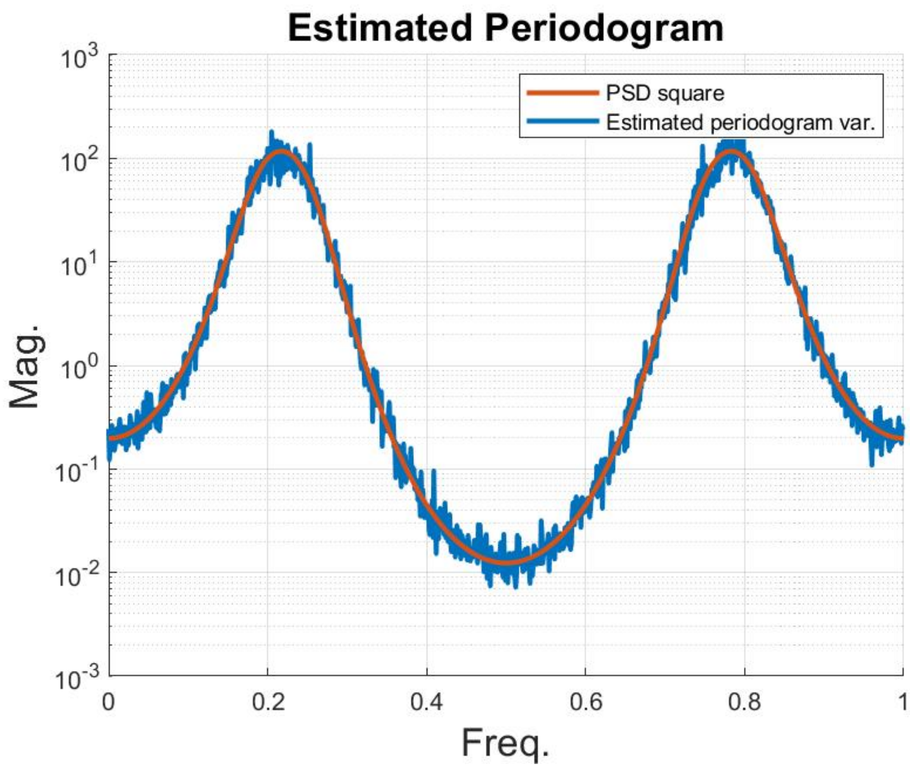
a.



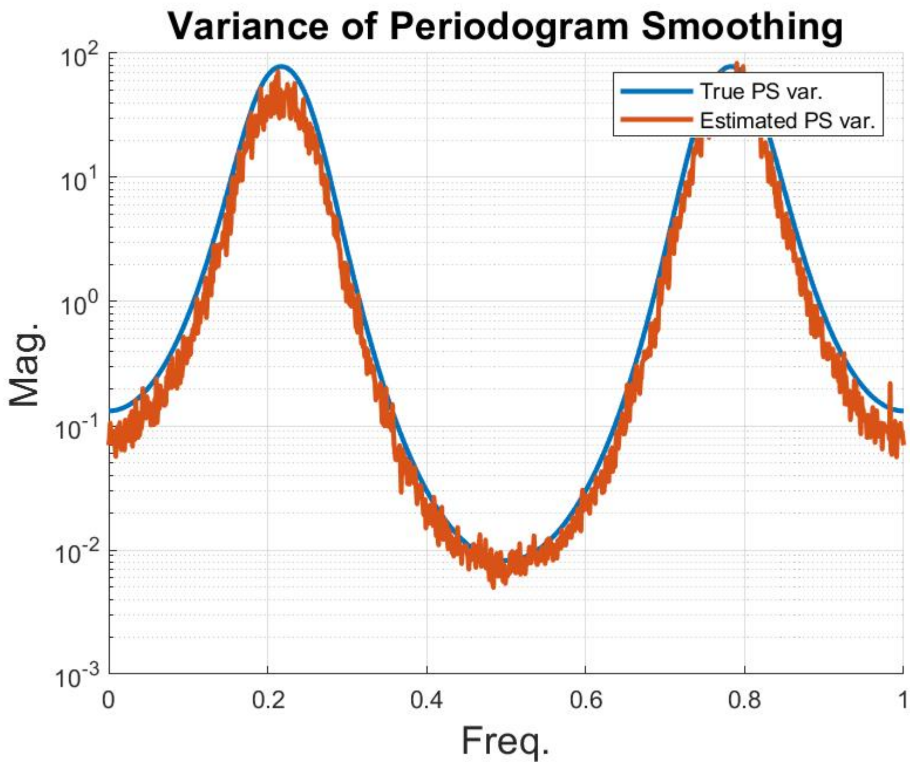
b.


Estimated Periodogram
True PSD
True mean periodogram
Estimated mean periodogram
Mag.
Freq.
10^{1}
10^{0}
10^{-1}
0
0.2
0.4
0.6
0.8
1

c.


Estimated Periodogram
PSD square
Estimated periodogram var.
Mag.
Freq.
10^{3}
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
0
0.2
0.4
0.6
0.8
1

d.



e.

Notice that the triangular is normalized for un-biased estimation.