MSA HW2

Problem 1

a.

$\mathbb{E}[x(n)] = \mathbb{E}[A_1]e^{j2\pi f_1 n} + \mathbb{E}[A_2]e^{j2\pi f_2 n} + \mathbb{E}[w(n)] = 0$

b.

$r(x) = \mathbb{E}[x(n)x^*(n-k)] = \mathbb{E}[(A_1 e^{j2\pi f_1 n} + A_2 e^{j2\pi f_2 n} + w(n))(A_1^* e^{-j2\pi f_1(n-k)} + A_2^* e^{-j2\pi f_2(n-k)} + w^*(n-k))]$
$= \mathbb{E}[(A_1 e^{j2\pi f_1 n} + A_2 e^{j2\pi f_2 n})(A_1^* e^{-j2\pi f_1(n-k)} + A_2^* e^{-j2\pi f_2(n-k)})] + \sigma_w^2 \delta(k) \quad (\because \mathbb{E}[A_p w^*(n)] = 0)$
$= \sigma^2 e^{j2\pi f_1 k} + \sigma^2 e^{j2\pi f_2 k} + \frac{\sigma^2}{3} e^{j2\pi(f_1 n - f_2 n + f_2 k)} + \frac{\sigma^2}{3} e^{j2\pi(f_2 n - f_1 n + f_1 k)} + \sigma_w^2 \delta(k)$

c.

Since the autocorrelation function relates not only to the time difference k, $x(n)$ is not WSS

Problem 2

a.

Since

$$\begin{bmatrix} x(n) \\ x(n+1) \\ \vdots \\ x(n+M-1) \end{bmatrix} = \begin{bmatrix} \sum_{p=1}^{P} \alpha_p e^{(\epsilon_p + j2\pi f_p)n} \\ \sum_{p=1}^{P} \alpha_p e^{(\epsilon_p + j2\pi f_p)(n+1)} \\ \vdots \\ \sum_{p=1}^{P} \alpha_p e^{(\epsilon_p + j2\pi f_p)(n+M-1)} \end{bmatrix} + \begin{bmatrix} w(n) \\ w(n+1) \\ \vdots \\ w(n+M-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{(\epsilon_1 + j2\pi f_1)} & e^{(\epsilon_2 + j2\pi f_2)} & \cdots & e^{(\epsilon_P + j2\pi f_P)} \\ \vdots & \vdots & \ddots & \vdots \\ e^{(\epsilon_1 + j2\pi f_1)(M-1)} & e^{(\epsilon_2 + j2\pi f_2)(M-1)} & \cdots & e^{(\epsilon_P + j2\pi f_P)(M-1)} \end{bmatrix} \begin{bmatrix} \alpha_1 e^{(\epsilon_1 + j2\pi f_1)n} \\ \alpha_2 e^{(\epsilon_2 + j2\pi f_2)n} \\ \vdots \\ \alpha_P e^{(\epsilon_P + j2\pi f_P)n} \end{bmatrix} + \begin{bmatrix} w(n) \\ w(n+1) \\ \vdots \\ w(n+M-1) \end{bmatrix}$$

We have

$$\mathbf{V}=\begin{bmatrix}1 & 1 & \dots & 1\\ e^{(\epsilon_1+j2\pi f_1)} & e^{(\epsilon_2+j2\pi f_2)} & \dots & e^{(\epsilon_P+j2\pi f_P)}\\ \vdots & \vdots & \ddots & \vdots\\ e^{(\epsilon_1+j2\pi f_1)(M-1)} & e^{(\epsilon_2+j2\pi f_2)(M-1)} & \dots & e^{(\epsilon_P+j2\pi f_P)(M-1)}\end{bmatrix}, \mathbf{s}(n)=\begin{bmatrix}\alpha_1 e^{(\epsilon_1+j2\pi f_1)n}\\ \alpha_2 e^{(\epsilon_2+j2\pi f_2)n}\\ \vdots\\ \alpha_P e^{(\epsilon_P+j2\pi f_P)n}\end{bmatrix}, \mathbf{w}(n)=\begin{bmatrix}w(n)\\ w(n+1)\\ \vdots\\ w(n+M-1)\end{bmatrix}$$

b.

Since

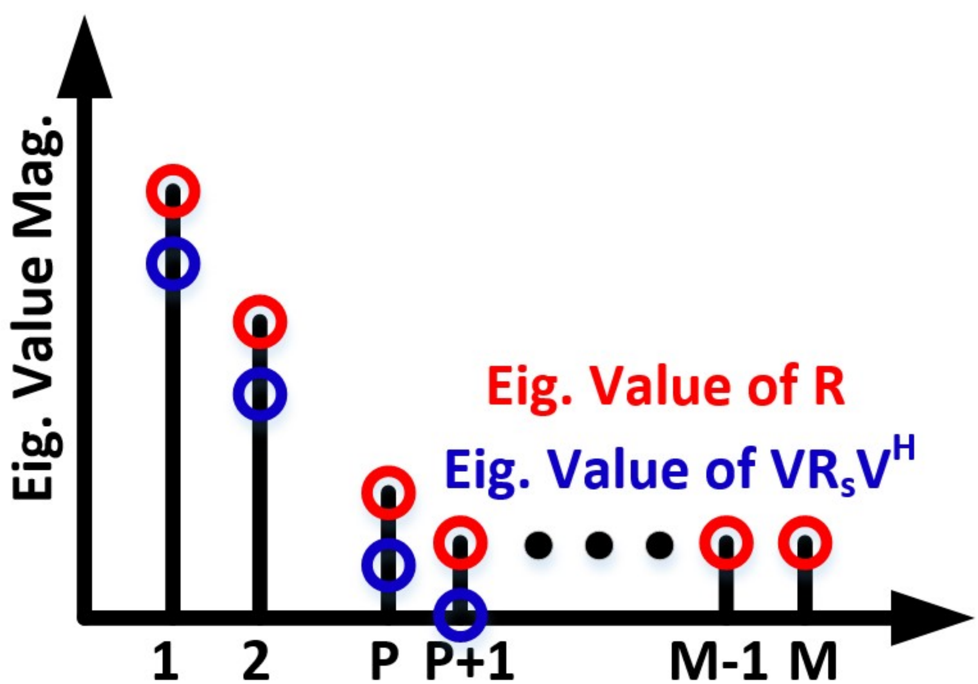
$\mathbf{R}(n)=\mathbb{E}[\mathbf{x}(n)\mathbf{x}^H(n)]=\mathbb{E}[(\mathbf{V}\mathbf{s}(n)+\mathbf{w}(n))(\mathbf{V}\mathbf{s}(n)+\mathbf{w}(n))^H]=\mathbf{V}\mathbf{R}_s(n)\mathbf{V}^H+\sigma_w^2\mathbf{I}$

where $\mathbf{R}_s(n)=\mathbf{E}[\mathbf{s}(n)\mathbf{s}^H(n)]=\begin{bmatrix}\sigma_1^2 e^{2\epsilon_1 n} & 0 & \dots & 0\\ 0 & \sigma_2^2 e^{2\epsilon_2 n} & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \sigma_P^2 e^{2\epsilon_P n}\end{bmatrix}$

Supposed the eigenvalue of $\mathbf{V}\mathbf{R}_s(n)\mathbf{V}^H$ are $\mu_1(n),\mu_2(n),\dots,\mu_P(n)$

Then the eigenvalues of $\mathbf{R}(n)$ are $(\mu_1(n)+\sigma_w^2),(\mu_2(n)+\sigma_w^2),\dots,(\mu_P(n)+\sigma_w^2),\sigma_w^2\dots,\sigma_w^2$

The eig. value magnitude is illustrated as follows



Problem 3

The frequency reponse of the system is $H(z)=\frac{1}{1+a_1 z^{-1}}$; therfore, the impulse reponse is $h(n)=(-a_1)^n u(n)$ where $u(n)$ is the step response.

if $k \leq 0$, we have
$r_x(k) = h(k) * r_w(k) * h^*(-k) = h(k) * h^*(-k) = \sum_{t=-\infty}^{\infty} h(t)h^*(-k+t) = \sum_{t=-\infty}^{\infty} (-a_1)^t u(t)(-a_1^*)^{(-k+t)} u(-k+t) = \sum_{t=0}^{\infty} (-a_1)^t (-a_1^*)^{(-k+t)} = (-a_1^*)^{-k} \sum_{t=0}^{\infty} (|a_1|^2)^t = \frac{(-a_1^*)^{-k}}{1-|a_1|^2}$
therefore, $r_x(0) = \frac{1}{1-|a_1|^2}$ and $r_x(1) = r_x(-1)^* = \frac{-a_1}{1-|a_1|^2}$
The correlation matrix $\mathbf{R} = \frac{1}{1-|a_1|^2} \begin{bmatrix} 1 & -a_1^* \\ -a_1 & 1 \end{bmatrix}$ and its inverse is $\mathbf{R}^{-1} = \begin{bmatrix} 1 & a_1^* \\ a_1 & 1 \end{bmatrix}$
The MVDR spectrum is $\hat{S}_{x,MVDR}(e^{j2\pi f}) = \frac{2}{\begin{bmatrix} 1 & e^{-j2\pi f} \end{bmatrix} \mathbf{R}^{-1} \begin{bmatrix} 1 \\ e^{j2\pi f} \end{bmatrix}} = \frac{1}{1+\Re\{a_1 e^{-j2\pi f}\}}$

Problem 4

From the MVDR specturm formula, we have
$\frac{3-cos(2\pi f)}{4} = \frac{1}{det(\mathbf{R})} \begin{bmatrix} 1 & e^{-j2\pi f} \end{bmatrix} \begin{bmatrix} r_x(0) & -r_x(1)^* \\ -r_x(1) & r_x(0) \end{bmatrix} \begin{bmatrix} 1 \\ e^{j2\pi f} \end{bmatrix} = \frac{2r_x(0) - 2\Re\{r_x(1)e^{-j2\pi f}\}}{det(\mathbf{R})}$ where $det(\mathbf{R}) = r_x^2(0) - |r_x(1)|^2$
Assume $r_x(1) = \alpha e^{\phi}$ where α is real, then $\frac{3-cos(2\pi f)}{4} = \frac{2r_x(0) - 2\alpha\Re\{e^{-j2\pi f+\phi}\}}{det(\mathbf{R})} = \frac{2r_x(0) - 2\alpha cos(j2\pi f - \phi)}{r_x^2(0) - \alpha^2}$
therefore, $\phi = 0, \alpha = 1 \rightarrow r_x(1) = 1$ and $r_x(0) = 3$

Problem 5

a.

Since $y(n) = \sum_{t=0}^{M-1} c_t^* x(n+t) = \sum_{t=-(M-1)}^{0} c_{-t}^* x(n-t)$, we have $h(n) = \begin{cases} c_{-n}^*, \text{ when } (-M+1) \leq n \leq 0 \\ 0, \text{ otherwise} \end{cases}$

b.

For z zero $z_0 = e^{j2\pi f_2}$, we have
$0 = H(z_0) = \sum_{k=-\infty}^{\infty} h(k) z_0^{-k} = \sum_{k=-M+1}^{0} c_{-k}^* z_0^{-k} = \sum_{k=0}^{M-1} c_k^* z_0^k = \mathbf{c}^H \mathbf{v}(f_2)$
also when $\mathbf{c} = \mathbf{c}_{MVDR} = \frac{\mathbf{R}^{-1}\mathbf{v}(f)}{\mathbf{v}(f)^H \mathbf{R}^{-1} \mathbf{v}(f)}$ and $\mathbf{R}^{-H} = \frac{1}{\sigma_w^2}(\mathbf{I} - \frac{\sigma_1^2}{\sigma_w^2 + M\sigma_1^2} \mathbf{v}(f_1)\mathbf{v}^H(f_1))$
The goal is to find f_2 such that

$$0 = lim_{\sigma_w^2 \to 0} \mathbf{c}^H \mathbf{v}(f_2) = lim_{\sigma_w^2 \to 0} \frac{\mathbf{v}(f)^H \frac{1}{\sigma_w^2}(\mathbf{I} - \frac{\sigma_1^2}{\sigma_w^2 + M\sigma_1^2} \mathbf{v}(f_1)\mathbf{v}^H(f_1))\mathbf{v}(f_2)}{\mathbf{v}(f)^H \frac{1}{\sigma_w^2}(\mathbf{I} - \frac{\sigma_1^2}{\sigma_w^2 + M\sigma_1^2} \mathbf{v}(f_1)\mathbf{v}^H(f_1))\mathbf{v}(f)} =$$

$$lim_{\sigma_w^2 \to 0} \frac{\mathbf{v}(f)^H (\mathbf{I} - \frac{\sigma_1^2}{\sigma_w^2 + M\sigma_1^2} \mathbf{v}(f_1)\mathbf{v}^H(f_1))\mathbf{v}(f_2)}{\mathbf{v}(f)^H (\mathbf{I} - \frac{\sigma_1^2}{\sigma_w^2 + M\sigma_1^2} \mathbf{v}(f_1)\mathbf{v}^H(f_1))\mathbf{v}(f)} = lim_{\sigma_w^2 \to 0} \frac{\mathbf{v}(f)^H \mathbf{v}(f_2) - \frac{\sigma_1^2}{\sigma_w^2 + M\sigma_1^2} \mathbf{v}(f)^H \mathbf{v}(f_1)\mathbf{v}^H(f_1)\mathbf{v}(f_2)}{\mathbf{v}(f)^H \mathbf{v}(f) - \frac{\sigma_1^2}{\sigma_w^2 + M\sigma_1^2} \mathbf{v}(f)^H \mathbf{v}(f_1)\mathbf{v}^H(f_1)\mathbf{v}(f)} =$$

$$lim_{\sigma_w^2 \to 0} \frac{[di\hat{r}ic(f-f_2)] - \frac{\sigma_1^2}{\sigma_w^2 + M\sigma_1^2}[di\hat{r}ic(f-f_1)][di\hat{r}ic(f_1-f_2)]}{M - \frac{\sigma_1^2}{\sigma_w^2 + M\sigma_1^2}[di\hat{r}ic(f-f_1)]^2}$$

where $di\hat{r}ic(x) = M(-1)^{x(M-1)}diric(2\pi x)$ and $diric(.)$ denotes the diriclet function

We seperate into two cases:

(1) When $f = f_1 + \frac{k}{M}$, where k represents any integer

The above equation can be reformulated into

$$lim_{\sigma_w^2 \to 0} \frac{[di\hat{r}ic(f_1-f_2)] - \frac{\sigma_1^2}{\sigma_w^2 + M\sigma_1^2}[di\hat{r}ic(\frac{k}{M})][di\hat{r}ic(f_1-f_2)]}{M - \frac{\sigma_1^2}{\sigma_w^2 + M\sigma_1^2}[di\hat{r}ic(\frac{k}{M})]^2} =$$

$$lim_{\sigma_w^2 \to 0} \frac{[di\hat{r}ic(f_1-f_2)] - \frac{\sigma_1^2}{\sigma_w^2 + M\sigma_1^2}M[di\hat{r}ic(f_1-f_2)]}{M - \frac{\sigma_1^2}{\sigma_w^2 + M\sigma_1^2}M^2} = lim_{\sigma_w^2 \to 0} \frac{di\hat{r}ic(f_1-f_2)}{M} = \frac{di\hat{r}ic(f_1-f_2)}{M}$$

This is zero when $f_2 = f_1 + \frac{l}{M}$, where l represents all integer except 0.

The zero in this case are $e^{j2\pi\frac{l}{M}}, l \in Z, l \neq 0$

(2) When $f \neq f_1 + \frac{k}{M}$

The equation can be reformulate as

$0 = lim_{\sigma_w^2 \to 0} \frac{[di\hat{r}ic(f-f_2)] - \frac{1}{M}[di\hat{r}ic(f-f_1)][di\hat{r}ic(f_1-f_2)]}{M - \frac{1}{M}[di\hat{r}ic(f-f_1)]^2} \to [di\hat{r}ic(f-f_2)] - \frac{1}{M}[di\hat{r}ic(f-f_1)][di\hat{r}ic(f_1-f_2)] = 0 \to f_2 = f_1 \text{ is one solution}$

In this case, $e^{j2\pi f_1}$ is a zero.

Problem 6

a.

$$\mathbb{E}[\hat{\mathbf{R}}_{i,j}] = \frac{1}{K}\sum_{k=0}^{K-1}\mathbb{E}[x(i+K)x^*(j+K)]$$
$$= \frac{1}{K}\sum_{k=0}^{K-1}\mathbb{E}[x(i)x^*(j)] \quad (\text{since } x \text{ is WSS})$$
$$= \mathbb{E}[x(i)x^*(j)] = \mathbf{R}_{i,j}$$

therefore, the sample correlation matrix is un-baised.

b.

Utilizing the results in **a.**, we have

$$\mathbb{E}[\hat{\mathbf{R}}_{i,j}] = \mathbf{R}_{i,j} + \delta\mathbf{I} \neq \mathbf{R}_{i,j}$$

therefore, the sample correlation matrix is baised.

c.

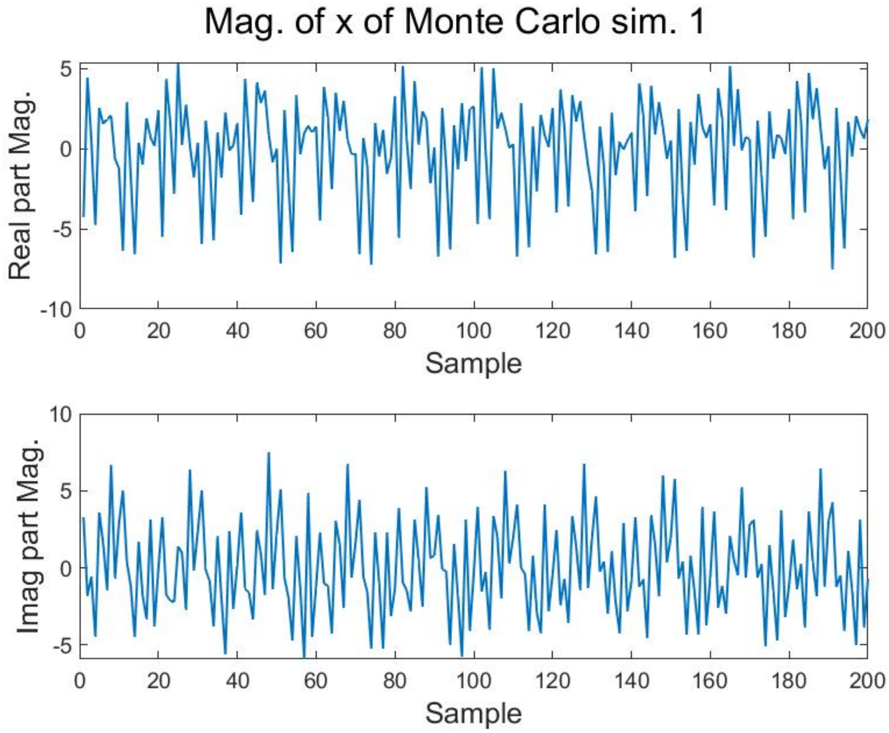
Utilizing the results in **a.**, we have

$$\mathbb{E}[\hat{\mathbf{R}}_{i,j}] = \sum_{k=0}^{K-1}\lambda^{K-k-1}\mathbf{R}_{i,j} + \delta\lambda^K\mathbf{I} = \frac{\lambda^{K-1}}{1-\lambda}\mathbf{R}_{i,j} + \delta\lambda^K\mathbf{I} \neq \mathbf{R}_{i,j}$$

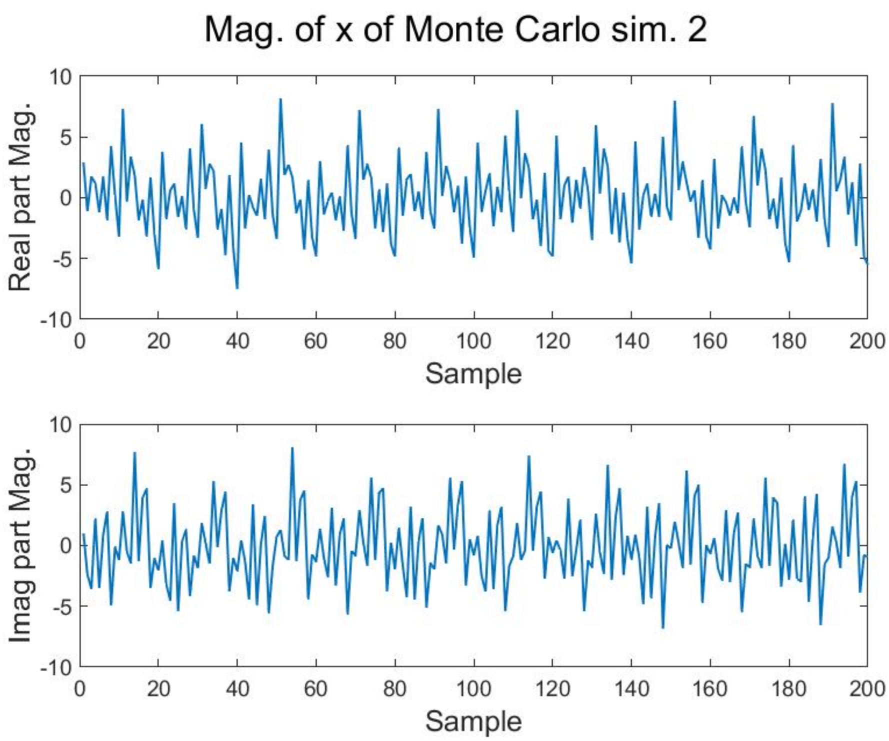
therefore, the sample correlation matrix is biased.

Problem 7

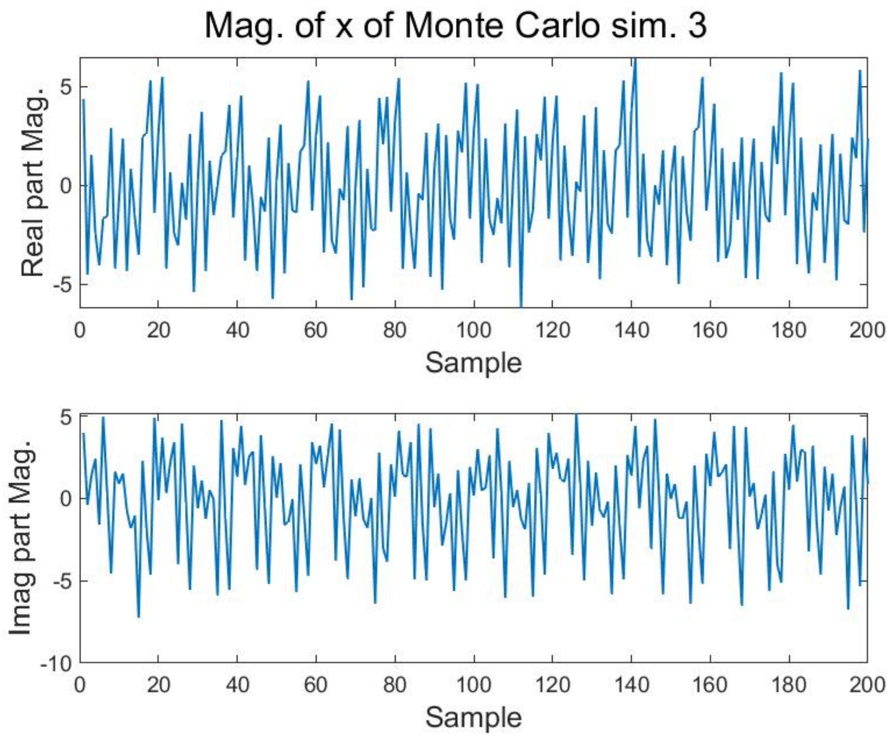
a.



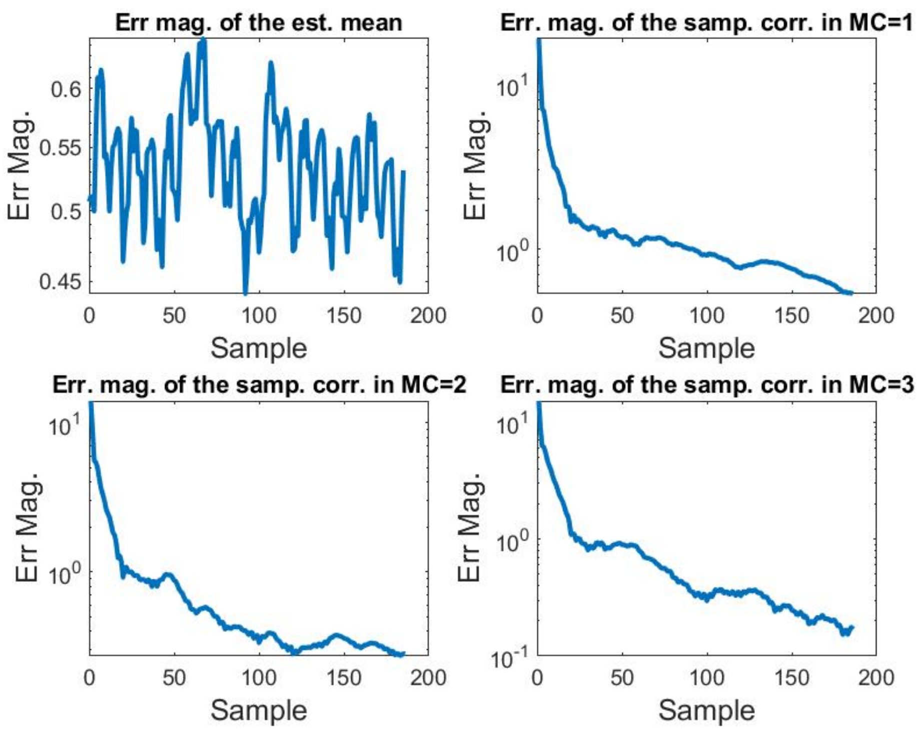
Mag. of x of Monte Carlo sim. 1
Real part Mag.
Sample
Imag part Mag.
Sample



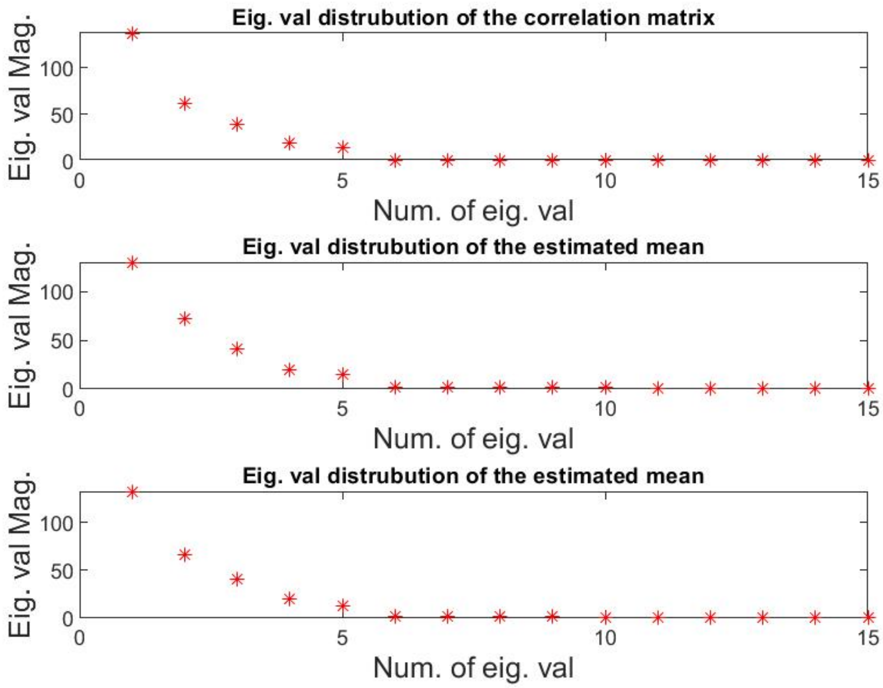
Mag. of x of Monte Carlo sim. 2
Real part Mag.
Sample
Imag part Mag.
Sample


Mag. of x of Monte Carlo sim. 3
Real part Mag.
Sample
Imag part Mag.
Sample

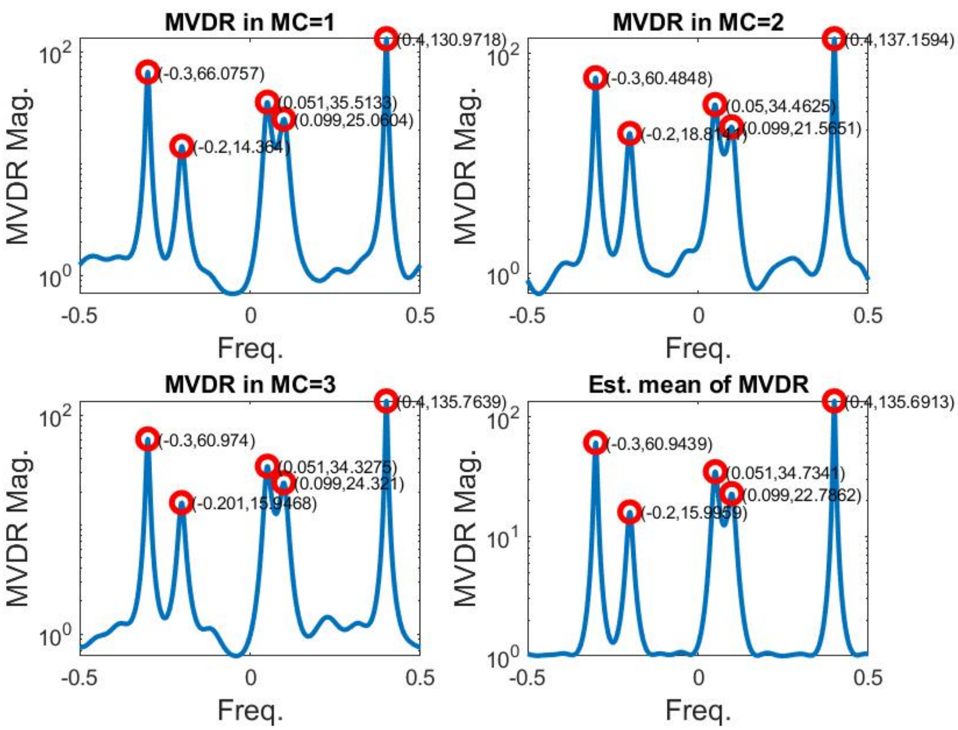
b.


Err mag. of the est. mean
Err Mag.
Sample
Err. mag. of the samp. corr. in MC=1
Err Mag.
Sample
Err. mag. of the samp. corr. in MC=2
Err Mag.
Sample
Err. mag. of the samp. corr. in MC=3
Err Mag.
Sample

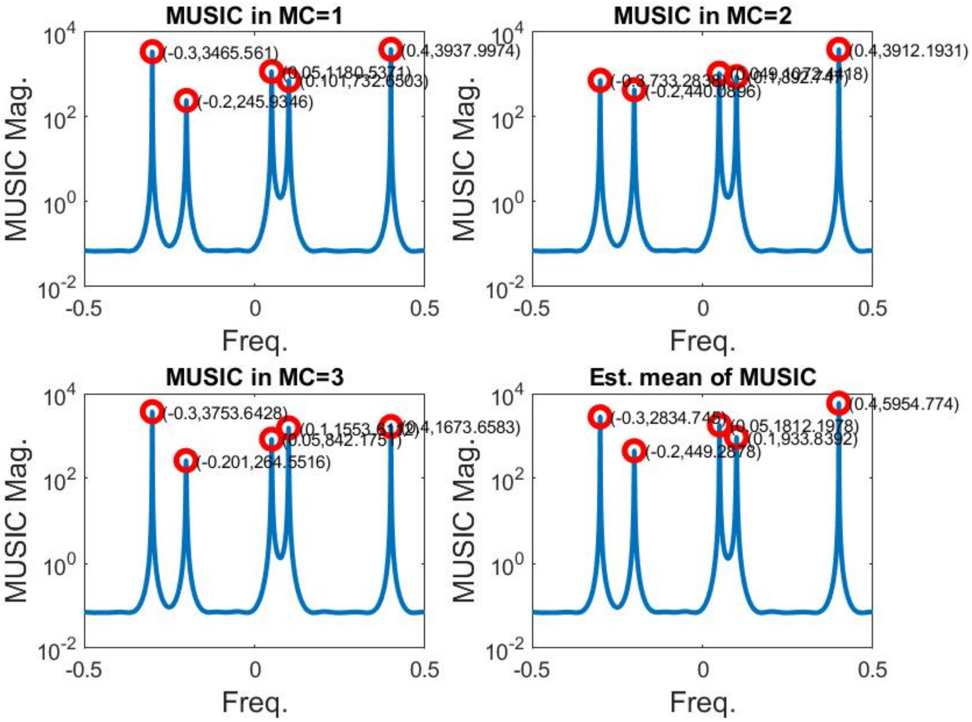
c.


Eig. val distrubution of the correlation matrix
Eig. val Mag.
Num. of eig. val
Eig. val distrubution of the estimated mean
Eig. val Mag.
Num. of eig. val
Eig. val distrubution of the estimated mean
Eig. val Mag.
Num. of eig. val

d.


MVDR in MC=1
(-0.3,66.0757)
(0.4,130.9718)
(0.099,25.0604)
MVDR Mag.
Freq.
MVDR in MC=2
(-0.3,60.4848)
(0.05,34.4625)
(0.099,21.5651)
(0.4,137.1594)
MVDR Mag.
Freq.
MVDR in MC=3
(-0.3,60.974)
(0.099,24.321)
(0.4,135.7639)
MVDR Mag.
Freq.
Est. mean of MVDR
(-0.3,60.9439)
(0.099,22.7862)
(0.4,135.6913)
MVDR Mag.
Freq.

e.



MUSIC in MC=1
MUSIC in MC=2
MUSIC in MC=3
Est. mean of MUSIC
MUSIC Mag.
Freq.
(-0.3,3465.561)
(-0.2,245.9346)
(0.101,732.6503)
(0.4,3937.9974)
(-0.2,440.0896)
(0.4,3912.1931)
(-0.3,3753.6428)
(-0.201,264.5516)
(0.05,842.1751)
(-0.3,2834.746)
(-0.2,449.2878)
(0.05,1812.1978)
(0.1,933.8392)
(0.4,5954.774)