EE5147 Modern Spectral Analysis
Homework Assignment #2

Notice

- **Due at 9:00pm, April 12, 2022 (Tuesday)** $= T_d$ for the electronic copy of your solution.
- Please submit your solution to NTU COOL (`https://cool.ntu.edu.tw/courses/11382`)
- ***Please justify your answers***. If the problem begins with "Show that" or "Prove", we expect a **rigorous proof**.
- All the figures should include labels for the horizontal and vertical axes, a title for a short description, and grid lines. Add legends and different line styles if there are multiple curves in one plot.
- No extensions, unless granted by the instructor one day before T_d.
- [MIK2005]: D. G. Manolakis, V. K. Ingle, and S. M. Kogon, *Statistical and Adaptive Signal Processing Spectral Estimation, Signal Modeling, Adaptive Filtering, and Array Processing*, Artech House, 2005.
- [SM2005]: P. Stoica and R. Moses, *Spectral Analysis of Signals*, Upper Saddle River, N.J.: Pearson/Prentice Hall, 2005.

Problems

1. (10 points) Consider the following random process

$$x(n) = A_1 e^{j2\pi f_1 n} + A_2 e^{j2\pi f_2 n} + w(n), \tag{1}$$

where A_1, A_2, and $w(n)$ are jointly complex circularly-symmetric Gaussian distributed with zero mean. Furthermore, we assume

$$\mathbb{E}\left[A_p A_q^*\right] = \begin{cases} \sigma^2, & \text{if } p = q, \\ \frac{1}{3}\sigma^2, & \text{if } p \neq q. \end{cases}, \quad \mathbb{E}\left[w(n)w^*(n-k)\right] = \sigma_w^2 \delta(k), \quad \mathbb{E}\left[A_p w^*(n)\right] = 0. \tag{2}$$

The frequencies f_1 and f_2 are deterministic. We assume that $f_1 \neq f_2$.

   (a) (3 points) Find the mean-value function of $x(n)$.

   (b) (5 points) Find the autocorrelation function of $x(n)$.

   (c) (2 points) Determine whether $x(n)$ is wide-sense stationary.

2. (10 points) We consider the exponentially damped sinusoidal components plus noise of the following form

$$x(n) = \sum_{p=1}^{P} \alpha_p e^{(\xi_p + j2\pi f_p)n} + w(n). \tag{3}$$

In this model, the quantities P, α_p, f_p, and $w(n)$ follow the same assumptions on page 9 of `04_Harmonic_MVDR.pdf`. Furthermore, the damping parameter ξ_p is deterministic and negative ($\xi_p < 0$). Let

$$\mathbf{x}(n) = \begin{bmatrix} x(n) \\ x(n+1) \\ \vdots \\ x(n+M-1) \end{bmatrix}, \tag{4}$$

be the vector form of (3). We assume that $P < M$.

(a) (5 points) Express $\mathbf{x}(n)$ in the following form

$$\mathbf{x}(n) = \boldsymbol{\mathcal{V}} \boldsymbol{s}(n) + \boldsymbol{w}(n), \tag{5}$$

where the matrix $\boldsymbol{\mathcal{V}}$ is deterministic and the entries of the first row of $\boldsymbol{\mathcal{V}}$ are 1. Specify $\boldsymbol{\mathcal{V}}$, $\boldsymbol{s}(n)$, and $\boldsymbol{w}(n)$.

(b) (5 points) Illustrate the eigenvalues of $\mathbf{R} \triangleq \mathbb{E}\left[\mathbf{x}(n)\mathbf{x}^H(n)\right]$.

3. (10 points) Consider an AR process $x(n)$ that is produced by filtering unit variance white circularly-symmetric Gaussian noise $w(n)$ with the system equation.

$$x(n) + a_1 x(n-1) = w(n), \tag{6}$$

where $|a_1| < 1$. Find the MVDR spectrum of $x(n)$ for $M = 2$.

4. (10 points) Suppose the MVDR spectrum of a WSS random process $x(n)$ is given by

$$\widehat{S}_{x,\text{MVDR}}(e^{j2\pi f}) = \frac{8}{3 - \cos(2\pi f)}. \tag{7}$$

We assume that $M = 2$. Find the autocorrelations $r_x(0)$ and $r_x(1)$ that produces $\widehat{S}_{x,\text{MVDR}}(e^{j2\pi f})$.

5. (15 points) Let $\mathbf{c} = [c_0, c_1, \ldots, c_{M-1}]^T$ be the linear combination coefficients for the MVDR spectrum estimation. The output $y(n) = \mathbf{c}^H \mathbf{x}(n)$, where

$$\mathbf{x}(n) = \begin{bmatrix} x(n) \\ x(n+1) \\ \vdots \\ x(n+M-1) \end{bmatrix}. \tag{8}$$

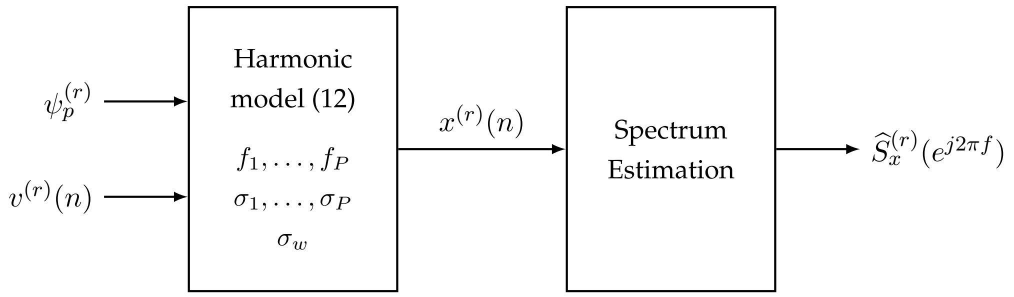

Figure 1: The system diagram for spectrum estimation driven by the *realization* $v^{(r)}(n)$ in the rth Monte-Carlo experiment.

(a) (5 points) Express the output $y(n)$ into the convolution $h(n) * x(n)$. Relate the impulse response $h(n)$ to the coefficients $c_0, c_1, \ldots, c_{M-1}$.

(b) (10 points) Assume that $x(n)$ is generated from a harmonic model with $P = 1$, frequency f_1, magnitude σ_1, and noise variance σ_w^2. Let $H(z)$ be the z-transform of $h(n)$. Find a zero of $H(z)$ if $\mathbf{c} = \mathbf{c}_{\text{MVDR}}$ and $\sigma_w^2 \to 0$.

6. (15 points) Let $x(n)$ be a WSS random process. The vector form of $x(n)$ is given by $\mathbf{x}(n) = [x(n),\ x(n+1),\ \ldots,\ x(n+M-1)]^T$. The correlation matrix $\mathbf{R}$ is defined as $\mathbb{E}[\mathbf{x}(n)\mathbf{x}^H(n)]$. Determine whether the following estimators of $\mathbf{R}$ are unbiased or not. Why or why not?

   (a) (5 points) The sample correlation matrix:

$$\widehat{\mathbf{R}} \triangleq \frac{1}{K}\sum_{k=1}^{K} \mathbf{x}(k)\mathbf{x}^H(k). \tag{9}$$

   (b) (5 points) The sample correlation matrix with diagonal loading ($\delta > 0$):

$$\widehat{\mathbf{R}} \triangleq \frac{1}{K}\sum_{k=1}^{K} \mathbf{x}(k)\mathbf{x}^H(k) + \delta\mathbf{I}. \tag{10}$$

   (c) (5 points) The exponentially-weighted and diagonally-loaded sample correlation matrix ($0 < \lambda \leq 1$ and $\delta > 0$):

$$\widehat{\mathbf{R}} \triangleq \sum_{k=1}^{K} \lambda^{K-k}\mathbf{x}(k)\mathbf{x}^H(k) + \delta \cdot \lambda^K \mathbf{I}. \tag{11}$$

7. (Spectrum estimation on harmonic models, 30 points) This problem aims to compare the performance of spectrum estimators through Monte-Carlo experiments. The system model is depicted in Figure 1, where the harmonic model is characterized by

$$x(n) = \sum_{p=1}^{P} \left(\sigma_p e^{j\psi_p}\right) e^{j2\pi f_p n} + \sigma_w v(n), \tag{12}$$

where the phase ψ_p and the noise $v(n)$ are random. We assume that $\psi_1, \ldots, \psi_P$ are independent and uniformly distributed over $[0, 2\pi]$. Furthermore, $v(n)$ is a complex circularly-symmetric white Gaussian noise with zero mean and unit variance.

To conduct Monte-Carlo experiments, please read the file `MSA_HW2_Problem_7.mat` for the realizations $\psi_p^{(r)}$ and $v^{(r)}(n)$ in the rth Monte-Carlo experiment. You will find two matrices in this mat file in the following layout:

$$\texttt{Psi} \triangleq \begin{bmatrix} \psi_1^{(1)} & \psi_2^{(1)} & \ldots & \psi_P^{(1)} \\ \psi_1^{(2)} & \psi_2^{(2)} & \ldots & \psi_P^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_1^{(R)} & \psi_2^{(R)} & \ldots & \psi_P^{(R)} \end{bmatrix}, \quad \texttt{V} \triangleq \begin{bmatrix} v^{(1)}(0) & v^{(1)}(1) & \ldots & v^{(1)}(N-1) \\ v^{(2)}(0) & v^{(2)}(1) & \ldots & v^{(2)}(N-1) \\ \vdots & \vdots & \ddots & \vdots \\ v^{(R)}(0) & v^{(R)}(1) & \ldots & v^{(R)}(N-1) \end{bmatrix}. \tag{13}$$

where R is the number of Monte-Carlo experiments. We set the following parameters

$$P = 5, \tag{14}$$

$$f_1 = -0.3, \quad f_2 = -0.2, \quad f_3 = 0.05, \quad f_4 = 0.1, \quad f_5 = 0.4, \tag{15}$$

$$\sigma_1 = 2, \quad \sigma_2 = 1, \quad \sigma_3 = 1.5, \quad \sigma_4 = 1.2, \quad \sigma_5 = 3, \tag{16}$$

$$\sigma_w = 1, \tag{17}$$

$$M = 15. \tag{18}$$

Save all the plots in Problem 7 to `fig` files. Include these `fig` files in your submission.

(a) (3 points) Plot the following curves The horizontal axis is time index n. The vertical axis is the real and imaginary parts of these signals.

- The realization $x^{(1)}(n)$.
- The realization $x^{(2)}(n)$.
- The realization $x^{(3)}(n)$.

(b) (4 points) Let $\mathbf{R}$ be the true correlation matrix based on (12) and the statistical assumptions of ψ_p and $v(n)$. We have two estimators of $\mathbf{R}$:

i. The estimated mean $\widehat{\mathbf{R}}(n)$ over R realizations at time index n. Namely,

$$\widehat{\mathbf{R}}(n) \triangleq \frac{1}{R} \sum_{r=1}^{R} \left(\mathbf{x}^{(r)}(n)\right) \left(\mathbf{x}^{(r)}(n)\right)^H. \tag{19}$$

ii. The sample correlation matrix in the rth Monte-Carlo trial. More specifically,

$$\widehat{\mathbf{R}}_L^{(r)} \triangleq \frac{1}{L} \sum_{n=0}^{L-1} \left(\mathbf{x}^{(r)}(n)\right) \left(\mathbf{x}^{(r)}(n)\right)^H. \tag{20}$$

To assess the estimation performance, we define the following error metric

$$\mathscr{E}(\widehat{\mathbf{R}}, \mathbf{R}) \triangleq \sqrt{\frac{1}{M^2} \left\| \widehat{\mathbf{R}} - \mathbf{R} \right\|_F^2} = \frac{1}{M} \left\| \widehat{\mathbf{R}} - \mathbf{R} \right\|_F, \tag{21}$$

where $\|\cdot\|_F$ denotes the Frobenius norm of a matrix. Next, we plot the following curves. The vertical axis is in the logarithmic scale.

- $\mathscr{E}(\widehat{\mathbf{R}}(n), \mathbf{R})$ over n, where $n = 0, 1, \ldots, N-M$.
- $\mathscr{E}(\widehat{\mathbf{R}}_L^{(1)}, \mathbf{R})$ over L, where $L = 1, 2, \ldots, N-M+1$.
- $\mathscr{E}(\widehat{\mathbf{R}}_L^{(2)}, \mathbf{R})$ over L, where $L = 1, 2, \ldots, N-M+1$.
- $\mathscr{E}(\widehat{\mathbf{R}}_L^{(3)}, \mathbf{R})$ over L, where $L = 1, 2, \ldots, N-M+1$.

(c) (3 points) Let $\lambda_\ell(\mathbf{A})$ be the eigenvalues of a Hermitian matrix $\mathbf{A} \in \mathbb{C}^{M\times M}$. These eigenvalues are sorted in the descending order $\lambda_1(\mathbf{A}) \geq \lambda_2(\mathbf{A}) \cdots \geq \lambda_M(\mathbf{A})$. Plot the following curves.

- $\lambda_\ell(\mathbf{R})$ over ℓ, where $\ell = 1, 2, \ldots, M$.
- $\lambda_\ell(\widehat{\mathbf{R}}_L^{(1)})$ over ℓ, where $L = \lfloor 0.5(N-M+1) \rfloor$ and $\ell = 1, 2, \ldots, M$.
- $\lambda_\ell(\widehat{\mathbf{R}}_L^{(1)})$ over ℓ, where $L = N-M+1$ and $\ell = 1, 2, \ldots, M$.

(d) (10 points) Let $\widehat{S}_{x,\mathrm{MVDR}}^{(r)}(e^{j2\pi f})$ be the MVDR spectrum associated with $\widehat{\mathbf{R}}_L^{(r)}$ and $L = N-M+1$. Plot the following curves in one plot for $-\frac{1}{2} \leq f \leq \frac{1}{2}$. The vertical axis is in the logarithmic scale.

- The MVDR spectrum $\widehat{S}_{x,\mathrm{MVDR}}^{(1)}(e^{j2\pi f})$. Mark the first P dominant peaks and specify their coordinates.
- The MVDR spectrum $\widehat{S}_{x,\mathrm{MVDR}}^{(2)}(e^{j2\pi f})$. Mark the first P dominant peaks and specify their coordinates.
- The MVDR spectrum $\widehat{S}_{x,\mathrm{MVDR}}^{(3)}(e^{j2\pi f})$. Mark the first P dominant peaks and specify their coordinates.
- The estimated mean of the MVDR spectrum over R Monte-Carlo trials. More specifically,

$$\widehat{\mathcal{M}}_{\mathrm{MVDR}}(e^{j2\pi f}) \triangleq \frac{1}{R}\sum_{r=1}^{R} \widehat{S}_{x,\mathrm{MVDR}}^{(r)}(e^{j2\pi f}). \tag{22}$$

(e) (10 points) Let $\widehat{P}_{x,\mathrm{MUSIC}}^{(r)}(e^{j2\pi f})$ be the MUSIC pseudospectrum associated with $\widehat{\mathbf{R}}_L^{(r)}$ and $L = N-M+1$. Plot the following curves in one plot for $-\frac{1}{2} \leq f \leq \frac{1}{2}$. The vertical axis is in the logarithmic scale.

- The MUSIC pseudospectrum $\widehat{P}_{x,\mathrm{MUSIC}}^{(1)}(e^{j2\pi f})$. Mark the first P dominant peaks and specify their coordinates.
- The MUSIC pseudospectrum $\widehat{P}_{x,\mathrm{MUSIC}}^{(2)}(e^{j2\pi f})$. Mark the first P dominant peaks and specify their coordinates.
- The MUSIC pseudospectrum $\widehat{P}_{x,\mathrm{MUSIC}}^{(3)}(e^{j2\pi f})$. Mark the first P dominant peaks and specify their coordinates.
- The estimated mean of the MUSIC pseudospectrum over R Monte-Carlo trials. More specifically,

$$\widehat{\mathcal{M}}_{\mathrm{MUSIC}}(e^{j2\pi f}) \triangleq \frac{1}{R}\sum_{r=1}^{R} \widehat{P}_{x,\mathrm{MUSIC}}^{(r)}(e^{j2\pi f}). \tag{23}$$

Last updated April 7, 2022.