EE5147 Modern Spectral Analysis
Homework Assignment #3

Notice

- **Due at 9:00pm, May 31, 2022 (Tuesday)** $= T_d$ for the electronic copy of your solution.
- Please submit your solution to NTU COOL (`https://cool.ntu.edu.tw/courses/11382`)
- ***Please justify your answers***. If the problem begins with "Show that" or "Prove", we expect a **rigorous proof**.
- All the figures should include labels for the horizontal and vertical axes, a title for a short description, and grid lines. Add legends and different line styles if there are multiple curves in one plot.
- No extensions, unless granted by the instructor one day before T_d.
- [FR2013]: S. Foucart and H. Rauhut, *A Mathematical Introduction to Compressive Sensing*, New York, NY: Springer, 2013.

Problems

1. (10 points) We consider the vector

$$\mathbf{w} = \begin{bmatrix} 2 \\ 3 \\ -1 \\ 0 \end{bmatrix}. \tag{1}$$

   (a) (2 points) Find $\|\mathbf{w}\|_1$.

   (b) (2 points) Find $\|\mathbf{w}\|_2$.

   (c) (2 points) Find $\|\mathbf{w}\|_\infty$.

   (d) (2 points) Find $\mathrm{supp}(\mathbf{w})$.

   (e) (2 points) Find the ℓ_0 function $\mathrm{card}(\mathrm{supp}(\mathbf{w}))$.

2. (10 points) Show that the ℓ_0 function $\mathrm{card}(\mathrm{supp}(\mathbf{x}))$ is not a norm.

3. (20 points) Let $\mathbf{u} = [u_1, u_2, \ldots, u_N]^T \in \mathbb{C}^N$ and $\mathbf{v} = [v_1, v_2, \ldots, v_N]^T \in \mathbb{C}^N$. Assume that $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

   (a) (10 points) Show that

$$\left|\mathbf{u}^H \mathbf{v}\right| \leq \|\mathbf{u}\|_p \|\mathbf{v}\|_q. \tag{2}$$

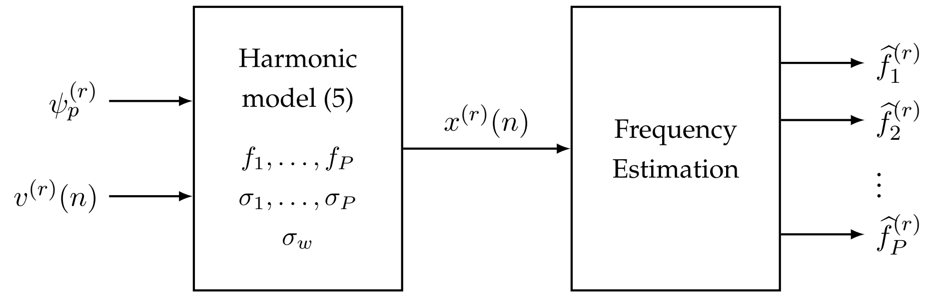

Figure 1: The system diagram for frequency estimation driven by the *realization* $v^{(r)}(n)$ in the rth Monte-Carlo experiment.

(b) (10 points) Show that the equality in (2) is attained if

$$u_i = \begin{cases} 0, & \text{if } v_i = 0, \\ |v_i|^q / \left(v_i^* \, \|\mathbf{v}\|_q^{q-1} \right), & \text{if } v_i \neq 0, \end{cases} \tag{3}$$

for $i = 1, 2, \ldots, N$.

4. (20 points) Consider the matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} 2 & -2 & 6 & -3 & 3 & 0 \\ -9 & -3 & 2 & 1 & -4 & 10 \\ -7 & -7 & -7 & -7 & 5 & -17 \\ 3 & 0 & -3 & -2 & 3 & -9 \end{bmatrix}. \tag{4}$$

(a) (10 points) Find the spark of $\mathbf{A}$. Identify a linearly dependent set of columns of $\mathbf{A}$ corresponding the spark.

(b) (10 points) Find the Kruskal rank of $\mathbf{A}$. Identify the linearly-independent columns of $\mathbf{A}$ achieving the Kruskal rank.

5. (10 points) Find all the solutions to the following optimization problem.

$$\min_{\mathbf{z} \in \mathbb{C}^N} \|\mathbf{z}\|_0$$

$$\text{subject to} \quad \begin{bmatrix} -2 & -4 & 4 & 9 & 8 \\ -7 & 9 & 4 & -4 & 5 \\ 5 & -3 & 0 & 6 & 3 \end{bmatrix} \mathbf{z} = \begin{bmatrix} -22 \\ 17 \\ -15 \end{bmatrix}.$$

Hint: Consider all possible support sets of $\mathbf{z}$.

6. (Parametric methods for harmonic models, 30 points) This problem aims to study the frequency estimators through *Monte-Carlo experiments*. The system model is depicted in Figure 1, where the harmonic model is characterized by

$$x(n) = \sum_{p=1}^{P} \left(\sigma_p e^{j\psi_p} \right) e^{j2\pi f_p n} + \sigma_w v(n), \tag{5}$$

where the phase ψ_p and the noise $v(n)$ are random. We assume that $\psi_1, \ldots, \psi_P$ are independent and uniformly distributed over $[0, 2\pi]$. Furthermore, $v(n)$ is a complex circularly-symmetric white Gaussian noise with zero mean and unit variance.

To conduct Monte-Carlo experiments, please read the file `MSA_HW3_Problem_6.mat` for the realizations $\psi_p^{(r)}$ and $v^{(r)}(n)$ in the rth Monte-Carlo experiment. You will find two matrices in this mat file in the following layout:

$$\texttt{Psi} \triangleq \begin{bmatrix} \psi_1^{(1)} & \psi_2^{(1)} & \ldots & \psi_P^{(1)} \\ \psi_1^{(2)} & \psi_2^{(2)} & \ldots & \psi_P^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_1^{(R)} & \psi_2^{(R)} & \ldots & \psi_P^{(R)} \end{bmatrix}, \quad \texttt{V} \triangleq \begin{bmatrix} v^{(1)}(0) & v^{(1)}(1) & \ldots & v^{(1)}(N-1) \\ v^{(2)}(0) & v^{(2)}(1) & \ldots & v^{(2)}(N-1) \\ \vdots & \vdots & \ddots & \vdots \\ v^{(R)}(0) & v^{(R)}(1) & \ldots & v^{(R)}(N-1) \end{bmatrix}. \tag{6}$$

where R is the number of Monte-Carlo experiments. We set the following parameters

$$P = 6, \tag{7}$$

$$f_1 = -0.25, \quad f_2 = -0.15, \quad f_3 = 0.01, \quad f_4 = 0.1, \quad f_5 = 0.2, \quad f_6 = 0.3, \tag{8}$$

$$\sigma_1 = 1, \quad \sigma_2 = 3, \quad \sigma_3 = 1, \quad \sigma_4 = 1, \quad \sigma_5 = 1.5, \quad \sigma_6 = 0.5, \tag{9}$$

$$\sigma_w = 1, \quad M = 10. \tag{10}$$

Let $x^{(r)}(n)$ be the realization of $x(n)$ in the rth Monte-Carlo experiment. The sample correlation matrix $\widehat{\mathbf{R}}_L^{(r)}$ in the rth Monte-Carlo trial is given by

$$\widehat{\mathbf{R}}_L^{(r)} \triangleq \frac{1}{L} \sum_{n=0}^{L-1} \left(\mathbf{x}^{(r)}(n)\right) \left(\mathbf{x}^{(r)}(n)\right)^H, \tag{11}$$

where the parameter L is selected such that the samples $x^{(r)}(0), \ldots, x^{(r)}(N-1)$ are included in (11).

(a) (15 points) Based on $\widehat{\mathbf{R}}_L^{(r)}$, the frequency estimates of the LS-ESPRIT algorithm are denoted by $\widehat{f}_{1,\mathrm{LS}}^{(r)}, \widehat{f}_{2,\mathrm{LS}}^{(r)}, \ldots, \widehat{f}_{P,\mathrm{LS}}^{(r)}$, where $-\frac{1}{2} \leq \widehat{f}_{1,\mathrm{LS}}^{(r)} \leq \cdots \leq \widehat{f}_{P,\mathrm{LS}}^{(r)} \leq \frac{1}{2}$.

   i. Plot the histogram based on $\left\{\widehat{f}_{1,\mathrm{LS}}^{(1)}, \widehat{f}_{1,\mathrm{LS}}^{(2)}, \ldots, \widehat{f}_{1,\mathrm{LS}}^{(R)}\right\}$. Specify the definition of these statistics and list their numerical values.
      - The sample mean of $\left\{\widehat{f}_{1,\mathrm{LS}}^{(1)}, \widehat{f}_{1,\mathrm{LS}}^{(2)}, \ldots, \widehat{f}_{1,\mathrm{LS}}^{(R)}\right\}$.
      - The sample variance of $\left\{\widehat{f}_{1,\mathrm{LS}}^{(1)}, \widehat{f}_{1,\mathrm{LS}}^{(2)}, \ldots, \widehat{f}_{1,\mathrm{LS}}^{(R)}\right\}$.
      - The sample mean-square error (MSE) of $\left\{\widehat{f}_{1,\mathrm{LS}}^{(1)}, \widehat{f}_{1,\mathrm{LS}}^{(2)}, \ldots, \widehat{f}_{1,\mathrm{LS}}^{(R)}\right\}$.

   ii. Repeat Problem 6(a)i for $\left\{\widehat{f}_{2,\mathrm{LS}}^{(1)}, \widehat{f}_{2,\mathrm{LS}}^{(2)}, \ldots, \widehat{f}_{2,\mathrm{LS}}^{(R)}\right\}$.

   iii. Repeat Problem 6(a)i for $\left\{\widehat{f}_{P,\mathrm{LS}}^{(1)}, \widehat{f}_{P,\mathrm{LS}}^{(2)}, \ldots, \widehat{f}_{P,\mathrm{LS}}^{(R)}\right\}$.

(b) (15 points) Repeat Problem 6a for the TLS-ESPRIT algorithm.

Last updated May 26, 2022.