MSA HW2

Problem 1

a.

$||\mathbf{w}||_1 = |2| + |3| + |-1| + |0| = 6$

b.

$||\mathbf{w}||_2 = \sqrt{2^2 + 3^2 + (-1)^2 + 0^2} = \sqrt{14}$

c.

$||\mathbf{w}||_\infty = \max\{|2|, |3|, |-1|, |0|\} = 3$

d.

$\mathrm{supp}(\mathbf{w}) = \{1, 2, 3\}$

e.

$\mathrm{card}(\mathrm{supp}(\mathbf{w})) = 3$

Problem 2

According to [FR2013], a non-negative function is called a norm if
(a) $||\mathbf{x}|| = 0$ if and only if $\mathbf{x} = 0$
(b) $||\lambda\mathbf{x}|| = |\lambda| \times ||\mathbf{x}||$ for all scaler λ and $\mathbf{x}$
(c) $||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$
However, (b) does not hold for all pairs of $\mathbf{x}$ and $\mathbf{y}$. For instance, let $\lambda = 2$ and $\mathbf{y} = [3]$, then
$||\lambda\mathbf{y}|| = ||[6]|| = 1 \neq 2 = |2| \times ||[3]||$

Problem 3

a.

The Young's Inequality states that
$\gamma\delta \leq \frac{\gamma^p}{p} + \frac{\delta^q}{q}$
Let $\gamma_i = \frac{|u_i^*|}{||\mathbf{u}||_p}$ and $\delta_i = \frac{|v_i|}{||\mathbf{v}||_q}$, we have
$\frac{|u_i^*|}{||\mathbf{u}||_p}\frac{|v_i|}{||\mathbf{v}||_q} \leq \frac{1}{p}(\frac{|u_i^*|}{||\mathbf{u}||_p})^p + \frac{1}{q}(\frac{|v_i|}{||\mathbf{v}||_q})^q$
summing by i for both sides, we get
$\frac{\sum_i |u_i^* v_i|}{||\mathbf{u}||_p||\mathbf{v}||_q} \leq \frac{1}{p}\sum_i(\frac{u_i^*}{||\mathbf{u}||_p})^p + \frac{1}{q}\sum_i(\frac{v_i}{||\mathbf{v}||_q})^q = \frac{1}{p} + \frac{1}{q} = 1$
$\rightarrow \sum_i |u_i^* v_i| \leq ||\mathbf{u}||_p||\mathbf{v}||_q \ - (1)$

Also, by the triangular inequality, $\sum_i |u_i^* v_i| \geq |\sum_i u_i^* v_i| = |\mathbf{u}^H \mathbf{v}| \ - (2)$
The inequality $|\mathbf{u}^H \mathbf{v}| \leq ||\mathbf{u}||_p ||\mathbf{v}||_q$ then holds due to (1) and (2)

b.

$|\mathbf{u}^H \mathbf{v}| = |\sum_i u_i^* v_i| = |\sum_i \frac{|v_i|^q}{v_i ||\mathbf{v}||_q^{q-1}} v_i| = |\sum_i \frac{|v_i|^q}{||\mathbf{v}||_q^{q-1}}| = |\frac{\sum_i |v_i|^q}{||\mathbf{v}||_q^{q-1}}| = ||\mathbf{v}||_q$
Also,
$||\mathbf{u}||_p = \sqrt[p]{\sum_i |u_i|^p} = \sqrt[p]{\sum_i [\frac{|v_i|^q}{|v_i| ||\mathbf{v}||_q^{q-1}}]^p} = \sqrt[p]{\sum_i \frac{|v_i|^{pq}}{|v_i|^p ||\mathbf{v}||_q^{pq-p}}} = \sqrt[p]{\sum_i \frac{|v_i|^{p+q}}{|v_i|^p ||\mathbf{v}||_q^q}} =$
$\sqrt[p]{\sum_i \frac{|v_i|^q}{||\mathbf{v}||_q^q}} = \sqrt[p]{\frac{||\mathbf{v}||_q^q}{||\mathbf{v}||_q^q}} = 1$
Therefore,
$|\mathbf{u}^H \mathbf{v}| = ||\mathbf{u}||_p ||\mathbf{v}||_q$ when the condition holds.

Problem 4

a.

Assume that $\mathbf{A} = [\mathbf{a}_1 \ \mathbf{a}_2 \ \ldots \mathbf{a}_6]$
Since $\mathbf{a}_i \neq c\mathbf{a}_j$ for all pairs of i, j and scaling factor c, the spark is not 2.
Since $\mathbf{a}_3 - 2\mathbf{a}_5 - \mathbf{a}_6 = \mathbf{0}$, $\mathbf{a}_3, \mathbf{a}_5, \mathbf{a}_6$ is linearly dependent.
Therefore, the spark of the matrix is 3.

b.

The kruskal rank is therefore 2 due to **a.**.
The linearly dependent sets are:
$\mathcal{S} = \{\{\mathbf{a}_i, \mathbf{a}_j\} | i \neq j\}$

Problem 5

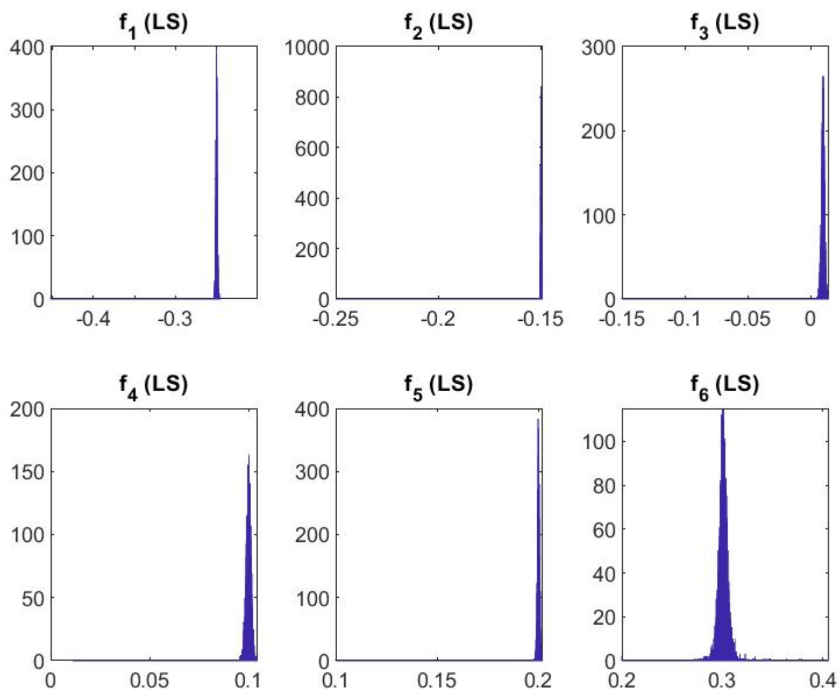
For $||\mathbf{z}||_0 = 0$, it is obviouly impossible.
For $||\mathbf{z}||_0 = 1$, it is clear that no column vectors in $\mathbf{A}$ is a scaled from $\mathbf{y}$.
For $||\mathbf{z}||_0 = 2$, we check if all pairs of column vectors in $\mathbf{A}$ are linear independent with $\mathbf{y}$.
MATLAB code:

```
A = [-2 -4 4 9 8; ...
-7 9 4 -4 5; ...
5 -3 0 6 3];
z = [-22; 17; -15];
for i = 1 : 4
    for j = i+1 : 5
        if rank([A(:,i) A(:,j) z]) < 3
            fprintf("a_%d & a_%d\n",i,j)
        end
    end
end
```

The results find that only $\mathbf{a}_2, \mathbf{a}_4, \mathbf{y}$ are linearly dependent and this forms a unique solution. Therefore, the solution set is:
$\mathcal{S}_{\mathbf{z}} = \{[0, 1, 0, -2, 0]^T\}$

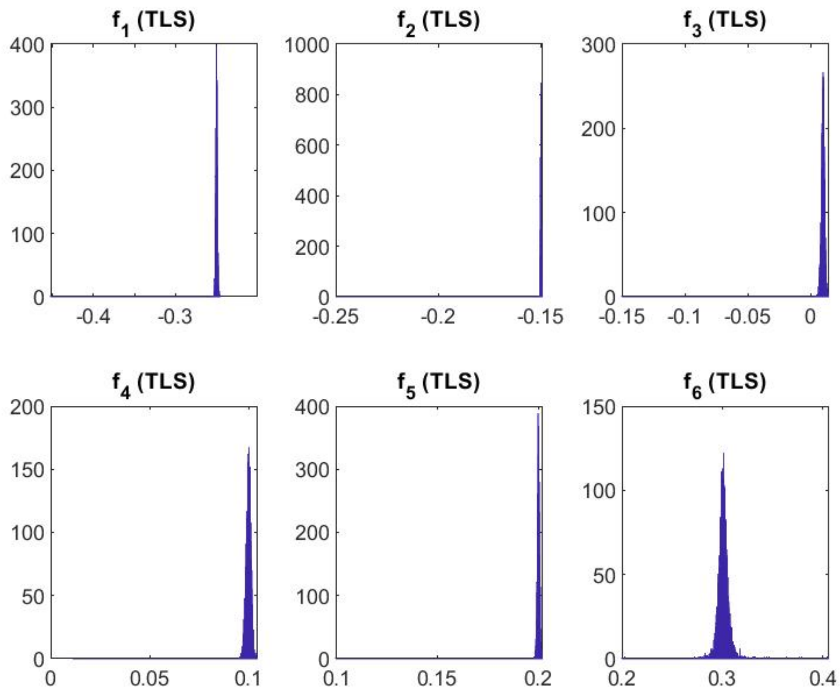
Problem 6

a.



$Var(f) : [9.38 \times 10^{-6}, 2.06 \times 10^{-6}, 6.87 \times 10^{-6}, 3.14 \times 10^{-6}, 2.35 \times 10^{-6}, 3.57 \times 10^{-5}]$
$Mean(f) : [-2.50 \times 10^{-1}, -1.50 \times 10^{-1}, 9.96 \times 10^{-3}, 1.00 \times 10^{-1}, 2.00 \times 10^{-1}, 3.01 \times 10^{-1}]$
$MSE(f) : [9.40 \times 10^{-6}, 2.06 \times 10^{-6}, 6.87 \times 10^{-6}, 3.14 \times 10^{-6}, 2.35 \times 10^{-6}, 3.65 \times 10^{-5}]$

b.



$Var(f) : [9.38 \times 10^{-6}, 2.06 \times 10^{-6}, 6.87 \times 10^{-6}, 3.14 \times 10^{-6}, 2.36 \times 10^{-6}, 3.56 \times 10^{-5}]$
$Mean(f) : [-2.50 \times 10^{-1}, -1.50 \times 10^{-1}, 9.95 \times 10^{-3}, 1.00 \times 10^{-1}, 2.00 \times 10^{-1}, 3.01 \times 10^{-1}]$
$MSE(f) : [9.40 \times 10^{-6}, 2.06 \times 10^{-6}, 6.87 \times 10^{-6}, 3.14 \times 10^{-6}, 2.36 \times 10^{-6}, 3.64 \times 10^{-5}]$