EE5147 Modern Spectral Analysis
Homework Assignment #1

Notice

- **Due at 9:00pm, March 22, 2022 (Tuesday)** $= T_d$ for the electronic copy of your solution.
- Please submit your solution to NTU COOL (`https://cool.ntu.edu.tw/courses/11382`)
- ***Please justify your answers***. If the problem begins with "Show that" or "Prove", we expect a **rigorous proof**.
- All the figures should include labels for the horizontal and vertical axes, a title for a short description, and grid lines. Add legends and different line styles if there are multiple curves in one plot.
- No extensions, unless granted by the instructor one day before T_d.
- [MIK2005]: D. G. Manolakis, V. K. Ingle, and S. M. Kogon, *Statistical and Adaptive Signal Processing Spectral Estimation, Signal Modeling, Adaptive Filtering, and Array Processing*, Artech House, 2005.
- [SM2005]: P. Stoica and R. Moses, *Spectral Analysis of Signals*, Upper Saddle River, N.J.: Pearson/Prentice Hall, 2005.

Problems

1. (10 points) Prove the bias-variance decomposition of the mean square error for the periodogram. Namely, show that

$$\mathrm{MSE}(\widehat{S}_{x,\mathrm{per}}(e^{j2\pi f}), S_x(e^{j2\pi f})) = \left|\mathrm{Bias}(\widehat{S}_{x,\mathrm{per}}(e^{j2\pi f}))\right|^2 + \mathrm{Var}(\widehat{S}_{x,\mathrm{per}}(e^{j2\pi f})). \tag{1}$$

2. (10 points) Assume that $N = 16$. Using MATLAB, plot the magnitude responses of the filters $H_k(e^{j2\pi f})$ for $k = 0, 1, \ldots, N-1$. Here $H_k(e^{j2\pi f})$ is associated with the filter bank implementation in the periodogram.

3. (10 points) The modified periodogram is defined as

$$\widehat{S}_{x,\mathrm{mod}}(e^{j2\pi f}) \triangleq \frac{1}{NU}\left|\sum_{n=-\infty}^{\infty} x(n)w(n)e^{-j2\pi fn}\right|^2, \tag{2}$$

where the window function $w(n)$ is defined over $n = 0, 1, \ldots, N-1$. For $n < 0$ or $n > N-1$, the weight function $w(n) = 0$. The parameter U is given by

$$U \triangleq \frac{1}{N}\sum_{n=0}^{N-1} |w(n)|^2. \tag{3}$$

Let $W(e^{j2\pi f})$ be the DTFT of $w(n)$. Show that

$$\mathbb{E}\left[\widehat{S}_{x,\mathrm{mod}}(e^{j2\pi f})\right] = \frac{1}{NU} S_x(e^{j2\pi f}) * \left|W(e^{j2\pi f})\right|^2. \tag{4}$$

4. (15 points)

   (a) (10 points) Problem 5.11(a) in [MIK2005].

   (b) (5 points) Show that the variance reduction ratio of the Parzen window for $L \gg 1$ is approximately $0.539L/N$.

5. (10 points) Show that the mean of the periodogram can be expressed as

$$\mathbb{E}\left[\widehat{S}_{x,\mathrm{per}}(e^{j2\pi f})\right] = \mathbf{v}^H(f)\mathbf{P}\mathbf{v}(f), \tag{5}$$

   where the vector $\mathbf{v}(f)$ is given by

$$\mathbf{v}(f) \triangleq \begin{bmatrix} 1 \\ e^{j2\pi f} \\ \vdots \\ e^{j2\pi f(N-1)} \end{bmatrix}. \tag{6}$$

   Express the matrix $\mathbf{P}$ in terms of the autocorrelation function $r_x(k)$ and the number of samples N.

6. (15 points) Consider an AR process $x(n)$ that is produced by filtering unit variance white Gaussian noise $v(n)$ with the causal and stable filter

$$H(z) = \frac{1}{\left(1 - \frac{1}{2}z^{-1} + \frac{1}{2}z^{-2}\right)\left(1 + \frac{1}{2}z^{-2}\right)}. \tag{7}$$

   (a) (5 points) Find the region of convergence of $H(z)$.

   (b) (5 points) Write a system equation for this AR process.
   Hints: The left-hand side of the system equations depends purely on $x(n)$. The right-hand side of the system equation is associated with $v(n)$.

   (c) (5 points) Find the power spectral density $S_x(e^{j2\pi f})$.

7. (MATLAB assignment, 30 points) In this problem, we will conduct *Monte-Carlo experiments* to study the estimation performance of nonparametric methods, as depicted in Figure 1. You can read the file `MSA_HW1_Problem_7.mat` for the realization $v^{(r)}(n)$ in the rth Monte-Carlo experiment. In this mat file, the matrix `V` is defined as

$$\mathtt{V} \triangleq \begin{bmatrix} v^{(1)}\left(-\frac{L}{2}\right) & v^{(1)}\left(-\frac{L}{2}+1\right) & \dots & v^{(1)}(0) & v^{(1)}(1) & \dots & v^{(1)}\left(\frac{L}{2}-1\right) \\ v^{(2)}\left(-\frac{L}{2}\right) & v^{(2)}\left(-\frac{L}{2}+1\right) & \dots & v^{(2)}(0) & v^{(2)}(1) & \dots & v^{(2)}\left(\frac{L}{2}-1\right) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ v^{(R)}\left(-\frac{L}{2}\right) & v^{(R)}\left(-\frac{L}{2}+1\right) & \dots & v^{(R)}(0) & v^{(R)}(1) & \dots & v^{(R)}\left(\frac{L}{2}-1\right) \end{bmatrix}, \tag{8}$$

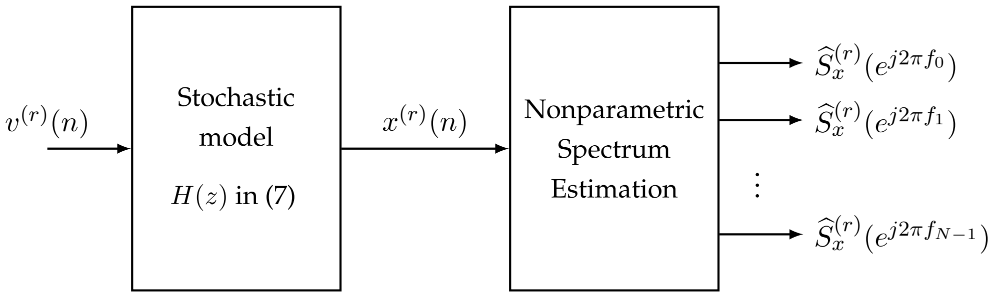

Figure 1: The system diagram for spectrum estimation driven by the *realization* $v^{(r)}(n)$ in the rth Monte-Carlo experiment.

where R is the number of Monte-Carlo experiments and L is the length of the realizations. We set $v^{(r)}(n) = 0$ if $n < -\frac{L}{2}$ or $n > \frac{L}{2} - 1$.

From the system equation in Problem 6b, we can compute the input signal $x^{(r)}(n)$ for $n = 0, 1, \ldots, N-1$. Then the nonparametric methods take these input signals to produce the spectrum estimates $\widehat{S}_x^{(r)}(e^{j2\pi f_0}), \widehat{S}_x^{(r)}(e^{j2\pi f_1}), \ldots, \widehat{S}_x^{(r)}(e^{j2\pi f_{N-1}})$.

(a) (6 points) Plot the following curves in one plot. The horizontal axis is the frequency $0 \leq f \leq 1$. The vertical axis is in the logarithmic scale.

- The power spectral density $S_x(e^{j2\pi f})$ in Problem 6c.
- The periodogram $\widehat{S}_{x,\mathrm{per}}^{(1)}(e^{j2\pi f_k})$ for $k = 0, 1, \ldots, N-1$. We assume $N = \frac{L}{2}$.
- The periodogram $\widehat{S}_{x,\mathrm{per}}^{(2)}(e^{j2\pi f_k})$ for $k = 0, 1, \ldots, N-1$. We assume $N = \frac{L}{2}$.
- The periodogram $\widehat{S}_{x,\mathrm{per}}^{(3)}(e^{j2\pi f_k})$ for $k = 0, 1, \ldots, N-1$. We assume $N = \frac{L}{2}$.

(b) (6 points) Plot the following curves in one plot for $0 \leq f \leq 1$. The horizontal axis is the frequency $0 \leq f \leq 1$. The vertical axis is in the logarithmic scale.

- The power spectral density $S_x(e^{j2\pi f})$ in Problem 6c.
- The mean of the periodogram at frequency f_k. Namely,

$$\mathcal{M}_{\mathrm{per}}(e^{j2\pi f_k}) \triangleq \mathbb{E}\left[\widehat{S}_{x,\mathrm{per}}(e^{j2\pi f_k})\right] = W_B(e^{j2\pi f}) * S_x(e^{j2\pi f})\Big|_{f=f_k}. \tag{9}$$

- The estimated mean of the periodogram over R Monte-Carlo trials. More specifically,

$$\widehat{\mathcal{M}}_{\mathrm{per}}(e^{j2\pi f_k}) \triangleq \frac{1}{R}\sum_{r=1}^{R} \widehat{S}_{x,\mathrm{per}}^{(r)}(e^{j2\pi f_k}). \tag{10}$$

  We assume $N = \frac{L}{2}$ in the periodogram.

(c) (6 points) Plot the following curves in one plot for $0 \leq f \leq 1$. The horizontal axis is the frequency $0 \leq f \leq 1$. The vertical axis is in the logarithmic scale.

- The function $S_x^2(e^{j2\pi f})$.

- The estimated variance of the periodogram over R Monte-Carlo trials. More specifically,

$$\widehat{\mathcal{V}}_{\mathrm{per}}(e^{j2\pi f_k}) \triangleq \frac{1}{R-1}\sum_{r=1}^{R}\left(\widehat{S}_{x,\mathrm{per}}^{(r)}(e^{j2\pi f_k}) - \widehat{\mathcal{M}}_{\mathrm{per}}(e^{j2\pi f_k})\right)^2, \tag{11}$$

where $\widehat{\mathcal{M}}_{\mathrm{per}}(e^{j2\pi f_k})$ is the estimated mean in (10). We assume $N = \frac{L}{2}$ in the periodogram.

(d) (6 points) Plot the following curves in one plot for $0 \leq f \leq 1$. The horizontal axis is the frequency $0 \leq f \leq 1$. The vertical axis is in the logarithmic scale.

- The function $S_x^2(e^{j2\pi f}) \times \frac{E_w}{N}$.
- The estimated variance of the Blackman-Tukey method (periodogram smoothing) $\widehat{S}_{x,\mathrm{PS}}(e^{j2\pi f_k})$ over R Monte-Carlo trials. More specifically,

$$\widehat{\mathcal{V}}_{\mathrm{PS}}(e^{j2\pi f_k}) \triangleq \frac{1}{R-1}\sum_{r=1}^{R}\left(\widehat{S}_{x,\mathrm{PS}}^{(r)}(e^{j2\pi f_k}) - \widehat{\mathcal{M}}_{\mathrm{PS}}(e^{j2\pi f_k})\right)^2, \tag{12}$$

where $\widehat{\mathcal{M}}_{\mathrm{PS}}(e^{j2\pi f_k})$ is the estimated mean. We assume $N = \frac{L}{2}$. The window function is the triangular window function, defined in (2.4.6) of [SM2005].

(e) (6 points) Plot the following curves in one plot for $0 \leq f \leq 1$. The horizontal axis is the frequency $0 \leq f \leq 1$. The vertical axis is in the logarithmic scale.

- The function $S_x^2(e^{j2\pi f}) \times \frac{1}{K}$.
- The estimated variance of the Bartlett-Welch method (periodogram averaging) $\widehat{S}_{x,\mathrm{PA}}(e^{j2\pi f_k})$ over R Monte-Carlo trials. More specifically,

$$\widehat{\mathcal{V}}_{\mathrm{PA}}(e^{j2\pi f_k}) \triangleq \frac{1}{R-1}\sum_{r=1}^{R}\left(\widehat{S}_{x,\mathrm{PA}}^{(r)}(e^{j2\pi f_k}) - \widehat{\mathcal{M}}_{\mathrm{PA}}(e^{j2\pi f_k})\right)^2, \tag{13}$$

where $\widehat{\mathcal{M}}_{\mathrm{PA}}(e^{j2\pi f_k})$ is the estimated mean. We assume $N = \frac{L}{2}$. The window function is the triangular window function. There are $K = 20$ segments, The segment length is $\mathscr{L}$. The offset distance $D = \mathscr{L}$. The frequency grid is $f_k = \frac{k}{\mathscr{L}}$ for $k = 0, 1, \ldots, \mathscr{L} - 1$.

Last updated March 2, 2022.