

State-space modeling using first-principles

Lecture 2

Principles of Modeling for Cyber-Physical Systems

Instructor: Madhur Behl

But first..

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UVACollab : PoM-CPS 6501-6581

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Site Information Display

The collab site for this course will be used for course videos and submission.

All course materials are available on: https://linklab-uva.github.io/modeling_cps/index.html

Help Recent Announcements

(viewing announcements from the last 10 days)

Principles of Modeling for CPS - Fall 2018 (Madhur Behl - Aug 26, 2018 9:43 am)

Message Center Notifications

New in Forums

https://collab.its.virginia.edu/portal/site/9d9970a1-1811-44cb-b7d8-5c2225a60c73/page/4a2a1a14-013d-4ddd-aae2-5cd3c0c82fbc

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A tentative schedule is posted below. Detailed lecture by lecture schedule will be posted soon.

	Module	Topic	Assignment
Week 1		Course introduction	Lecture 1 
		First-principles and state-space modeling	Worksheet 1 out
Week 2	Energy CPS	Energy CPS and HVAC basics	
	Energy CPS	'RC' network building modeling	Worksheet 2 out Worksheet 1 due.
Week 3	Energy CPS	Introduction to EnergyPlus + Tutorial	
	Energy CPS	Parameter estimation algorithms	Worksheet 3 out

Teaching Assistants



Varundev Sukhil
PhD student, CpE
vss8sm@virginia.edu
Office Hours: Noon-2pm
Wednesdays
Olsson 204



Behnam Malmir
PhD student, SIE
bm3hz@virginia.edu
Office Hours: 4-5pm
Fridays
Location: TBA

Download Matlab

Campus-wide license for MATLAB, Simulink, and companion toolboxes

<http://its.virginia.edu/software/matlab.html>

Contact res-consult@virginia.edu for questions regarding access to Matlab licenses.

Assignment 1 is out

- Due in 1 week: **Tuesday, September 11, 2018 by 2pm (EST)**
- State-Space modeling w. Matlab
- Submission instructions:
 - Submit on collab.
 - Upload a single .zip file with the filename
<firstname_lastname_UVA_computing_ID>.zip
e.g. madhur_behl_mb2kg.zip

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Matlab

Several assignments require the use of Matlab products and toolboxes.

Campus-wide license for MATLAB, Simulink, and companion toolboxes

MathWorks has created a web portal for UVa faculty, staff, and students to download, install and activate the latest version of Matlab and the toolboxes included as part of the campus-wide license.

The portal also provides access to the MathWork's online course offerings.

The following URL starts the process of accessing the MathWorks portal.

<http://its.virginia.edu/software/matlab.html>

Contact res-consult@virginia.edu for questions regarding access to Matlab licenses.

Course assignments will be posted here.

Assignment 1 : State-Space Modeling in Matlab

Assignment 1 

Due Date: 11 September, 2018 at 2:00pm (EST) i.e. before the lecture

Submit on UVA collab. Upload a single .zip file with the filename .zip

0

In today's lecture we will learn about...

Prediction is very difficult, especially
if it's about the future.

— Niels Bohr —

In today's lecture we will learn about...

How to predict the future states and outputs of
systems using physics based mathematical modeling

In today's lecture we will learn about...

- Ordinary differential equations (ODEs).
- Linear dynamical systems
- State-space representation
- Elements of first-principles based modeling:
 - Mechanical and electrical modeling

What is a System ?

Cruise control system

Cardio-pulmonary system

Autopilot system

Economic system

Grading system

Governance system

Communication system

Complex system

Tropical storm system

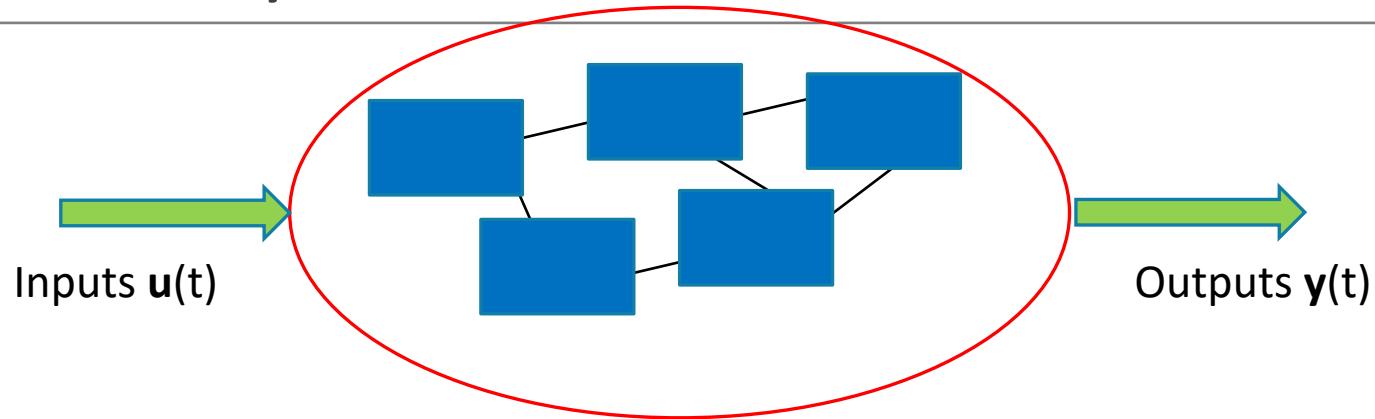
System of systems

Taxation system

Cyber-Physical systems

Healthcare system

What is a System ? Intuitive definition

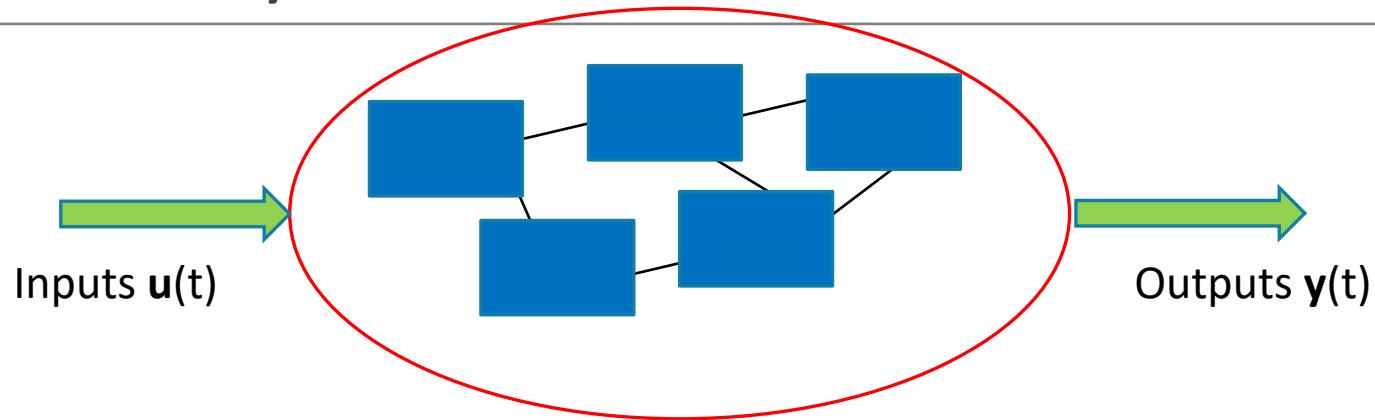


Collection of components

Non-trivial interactions

Well defined boundary
with the environment

What is a System ?



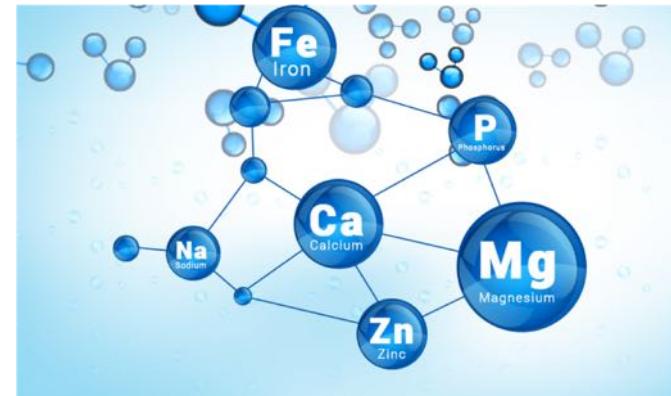
Mapping from time dependent inputs to time dependent outputs
(causal definition)

Differential equations

Many phenomena can be expressed by equations which involve the **rates of change** of quantities (position, population, concentration, temperature...) that describe the **state** of the phenomena.



Economics



Chemistry



Mechanics



Engineering



Social Science



Biology

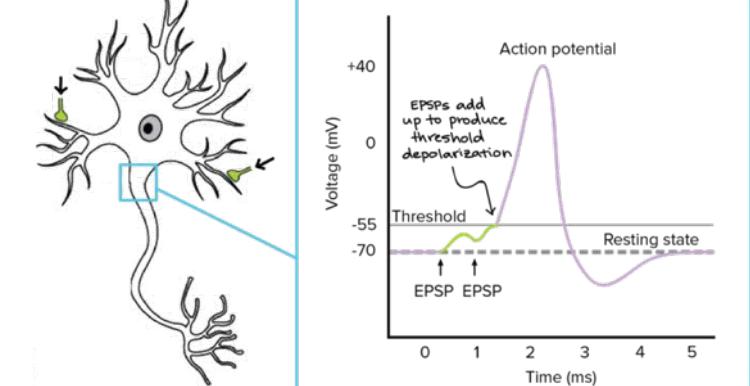
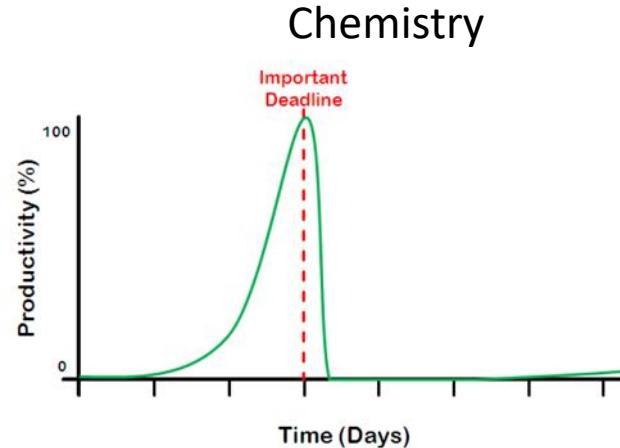
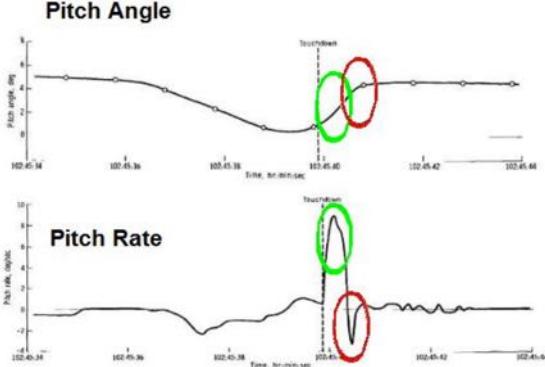
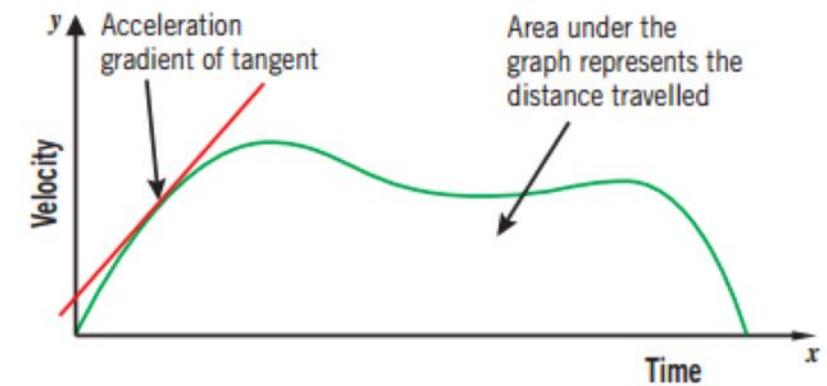
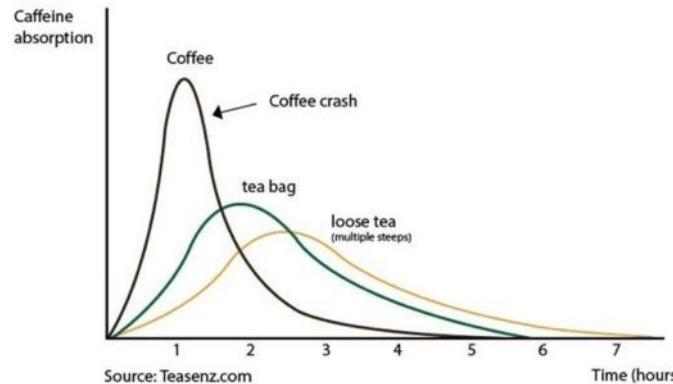
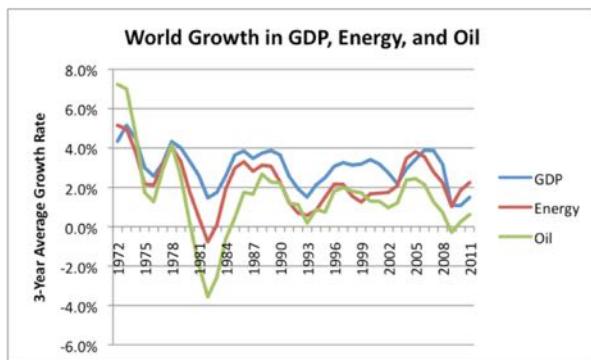
The ***state*** of a system describes enough information about the system to determine its future behavior in the absence of any external inputs affecting the system.

The set of possible combinations of state variable values is called the ***state space*** of the system.

Differential equations

The state of the system is characterized by **state variables**, which describe the system.

The rate of change is (usually) expressed with respect to time



Differential equations – A simple example

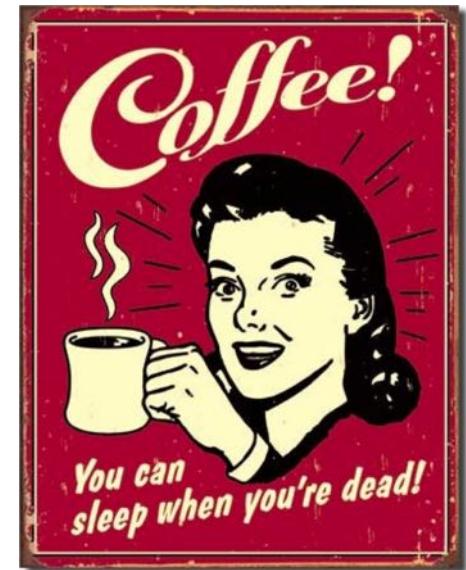
After drinking a cup of coffee, the amount C of caffeine in person's body follows the differential equation:

$$\frac{dC}{dt} = -\alpha C$$

Where the constant α has a value of 0.14 hour⁻¹

How many hours will it take to metabolize half of the initial amount of caffeine ?

$$\int \frac{dC}{C} = -\alpha \int dt ; \quad C(t) = C_0 e^{-\alpha t} ; \quad \text{if } C(t) = C_0/2, \quad t = \ln 2 / \alpha$$



Recall: Differential equations

- Ordinary differential equation (ODE): all derivatives are with respect to single independent variable, often representing time.
- **Order** of ODE is determined by highest-order derivative of state variable function appearing in ODE.
- ODE with higher-order derivatives can be transformed into equivalent first-order system.
- Most ODE software's are designed to solve only first-order equations.

Higher order ODE's

For k -th order ODE

$$y^{(k)}(t) = f(t, y, y', \dots, y^{(k-1)})$$

define k new unknown functions

$$u_1(t) = y(t), \ u_2(t) = y'(t), \ \dots, \ u_k(t) = y^{(k-1)}(t)$$

Then original ODE is equivalent to first-order system

$$\begin{bmatrix} u'_1(t) \\ u'_2(t) \\ \vdots \\ u'_{k-1}(t) \\ u'_k(t) \end{bmatrix} = \begin{bmatrix} u_2(t) \\ u_3(t) \\ \vdots \\ u_k(t) \\ f(t, u_1, u_2, \dots, u_k) \end{bmatrix}$$

What makes a system dynamic ?

Inputs change with time ?

Outputs change with time ?

USD

\$100



\$200



\$300



Currency Exchange
System

Euro

€85



€170



€255



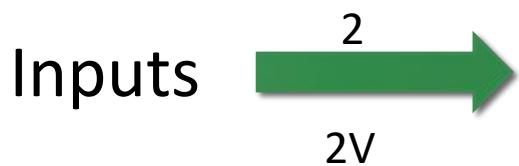
Static vs Dynamic Systems

Static System

Output is determined only by the current input, reacts instantaneously

Relationship between the inputs and outputs does not change (it is static!)

Relationship is represented by an algebraic equation



Motor

Dynamic System

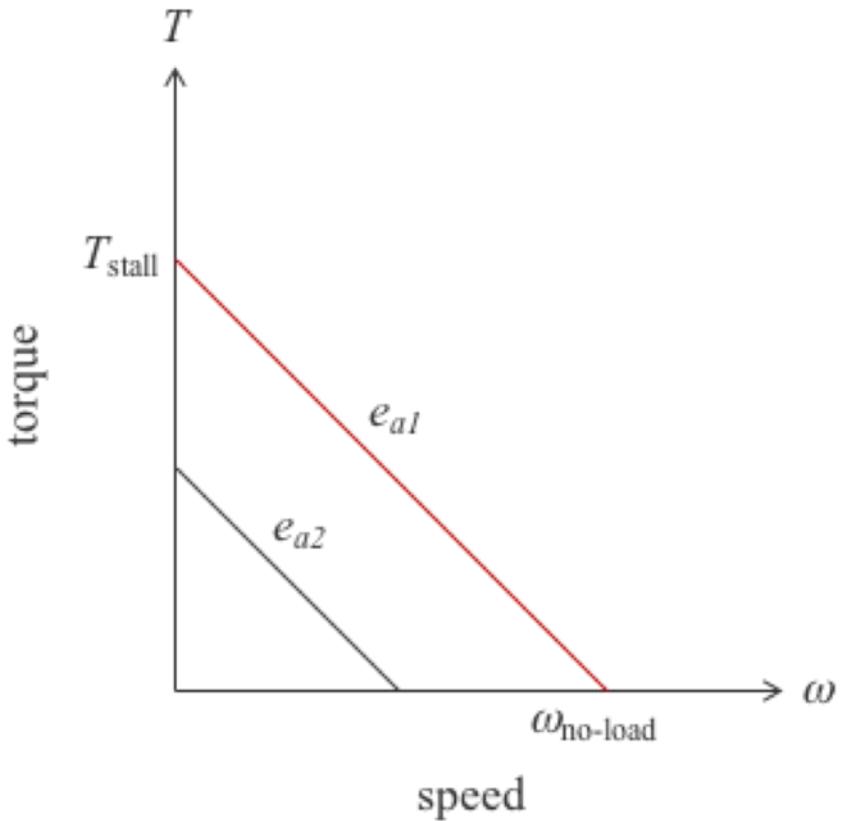
Output takes time to react

Relationship changes with time, depends on past inputs and initial conditions (it is dynamic!)

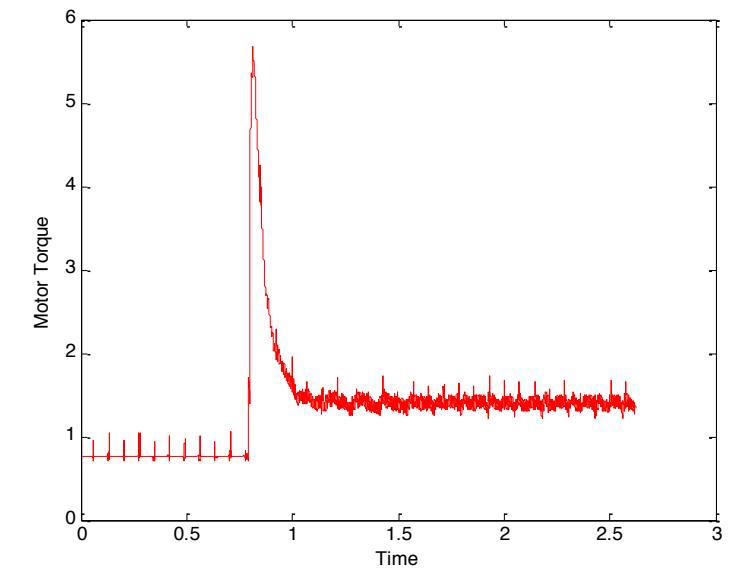
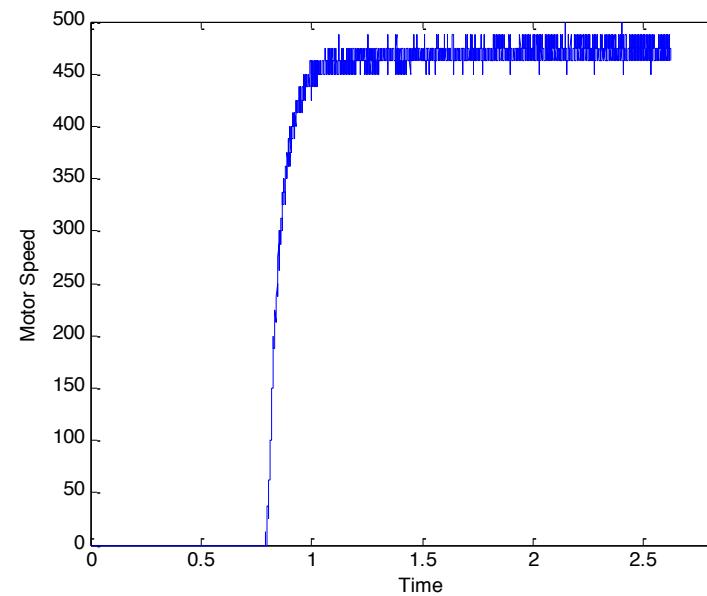
Relationship is represented by a differential equation

Static vs Dynamic Systems

Static System viewpoint



Dynamic System viewpoint



Dynamical System



$$\frac{dx}{dt} = \dot{x} = f(x(t), u(t), t)$$

Dynamical System

$$\frac{dx}{dt} = \dot{x} = f(x(t), u(t), t)$$

Possibly a non-linear function

Rate of change

The state $x(t_1)$ at any future time, may be determined exactly given knowledge of the initial state, $x(t_0)$ and the time history of the inputs, $u(t)$ between t_0 and t_1

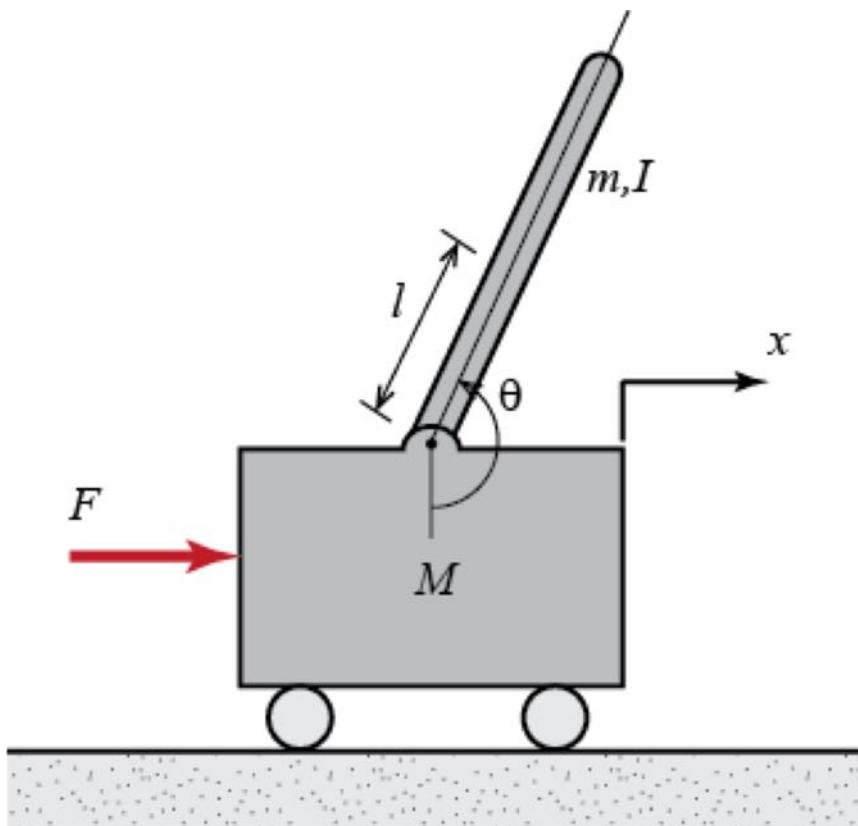
System order: n, min number of states required for the above statement to be true.

$$\frac{dx}{dt} = \dot{x} = f(x(t), u(t), t)$$

Possibly a non-linear function

Rate of change

Inverted pendulum



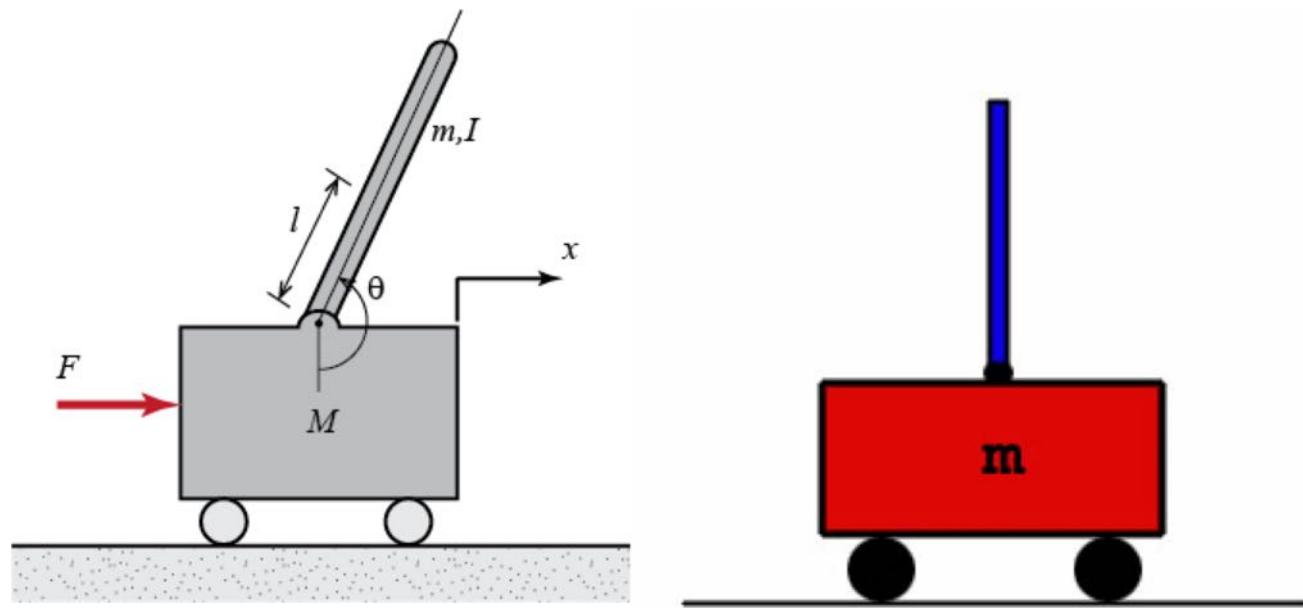
- Inverted pendulum mounted to a motorized cart.
- Unstable without control :
 - pendulum will simply fall over if the cart isn't moved to balance it.

Balance the inverted pendulum by applying a force to the cart on which the pendulum is attached.

Inverted pendulum

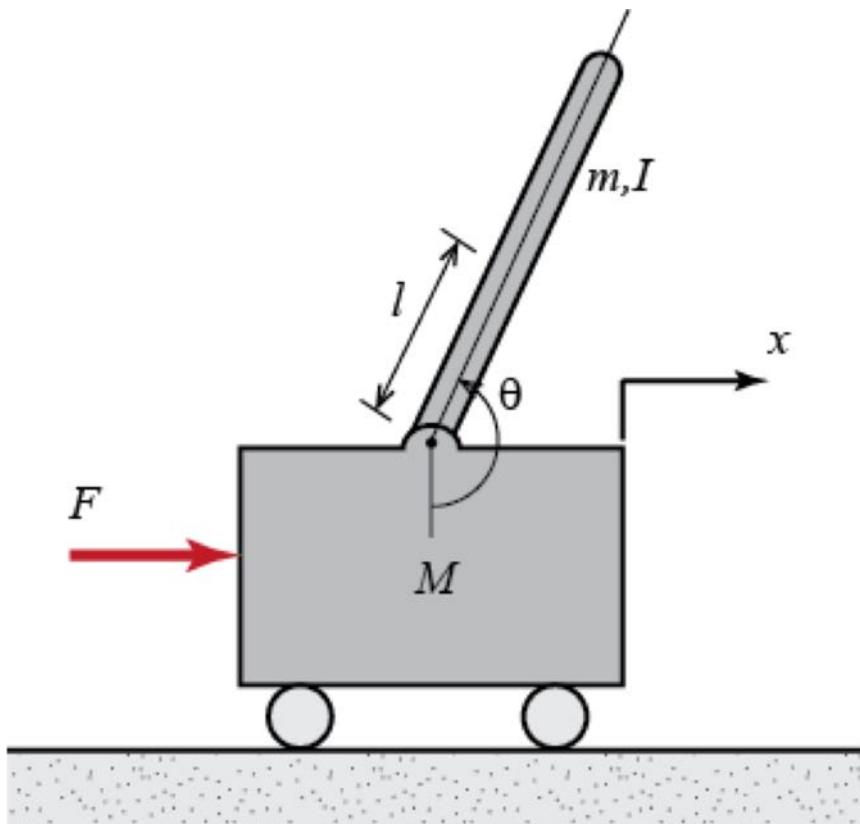


Inverted pendulum



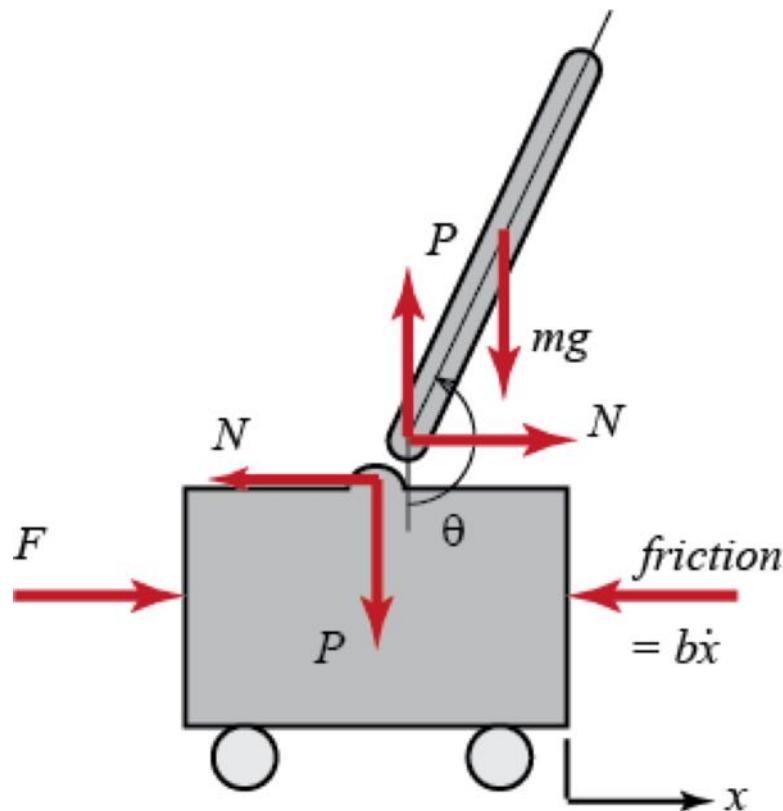
MakeAGIF.com

Inverted pendulum



- Initially pendulum begins with $\theta = \pi$
- Requirements:
 - Settling time for θ less than 5 secs.
 - Pendulum angle θ never exceeds 0.05 radians from the vertical.

Inverted pendulum – ODEs



Forces in the horizontal direction

$$M\ddot{x} + b\dot{x} + N = F$$

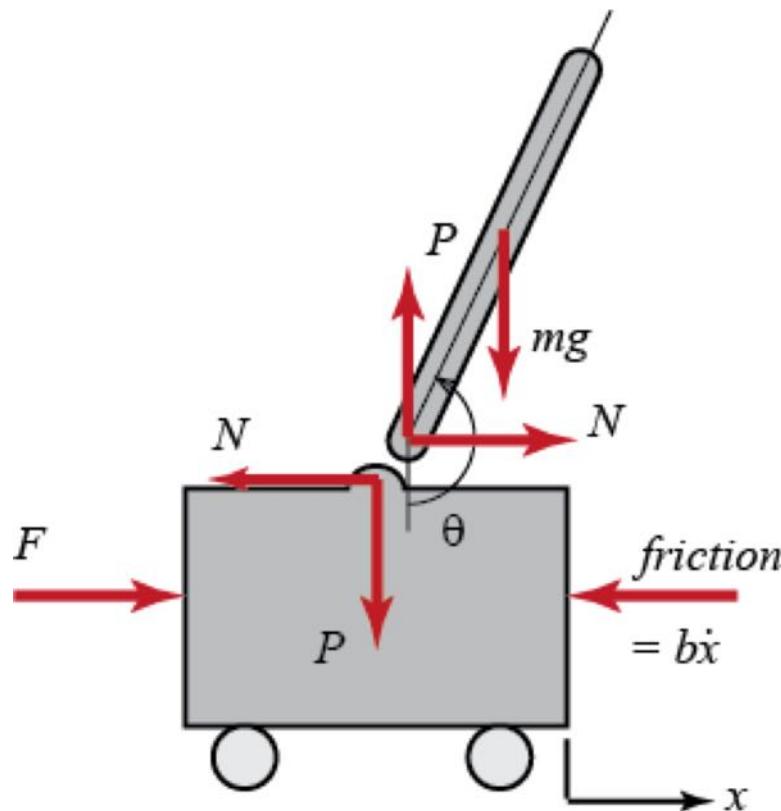
Reaction force N :

$$N = m\ddot{x} + ml\ddot{\theta} \cos \theta - ml^2\dot{\theta}^2 \sin \theta$$

Governing equation (1) of this system: Horizontal

$$(M + m)\ddot{x} + b\dot{x} + ml\ddot{\theta} \cos \theta - ml^2\dot{\theta}^2 \sin \theta = F$$

Inverted pendulum -- ODEs



Forces in the vertical direction:

$$P \sin \theta + N \cos \theta - mg \sin \theta = ml\ddot{\theta} + m\ddot{x} \cos \theta$$

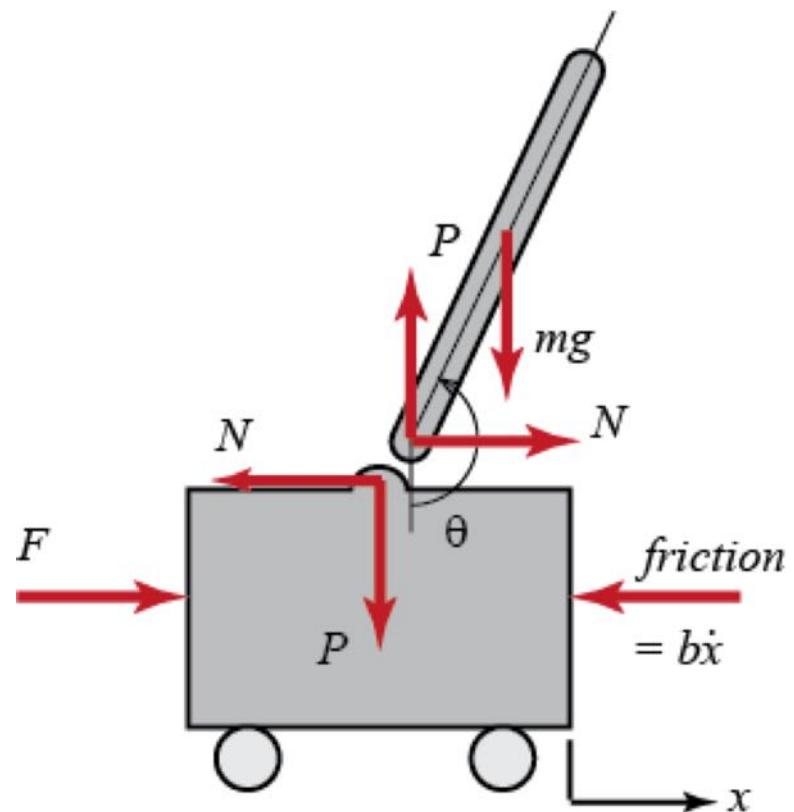
Get rid of the P and the N terms:
(moment balance equation)

$$-Pl \sin \theta - Nl \cos \theta = I\ddot{\theta}$$

Governing equation (2) of this system: Vertical

$$(I + ml^2)\ddot{\theta} + mgl \sin \theta = -ml\ddot{x} \cos \theta$$

Inverted pendulum



Assuming that the system remains within a small neighborhood of the equilibrium $\theta = \pi$

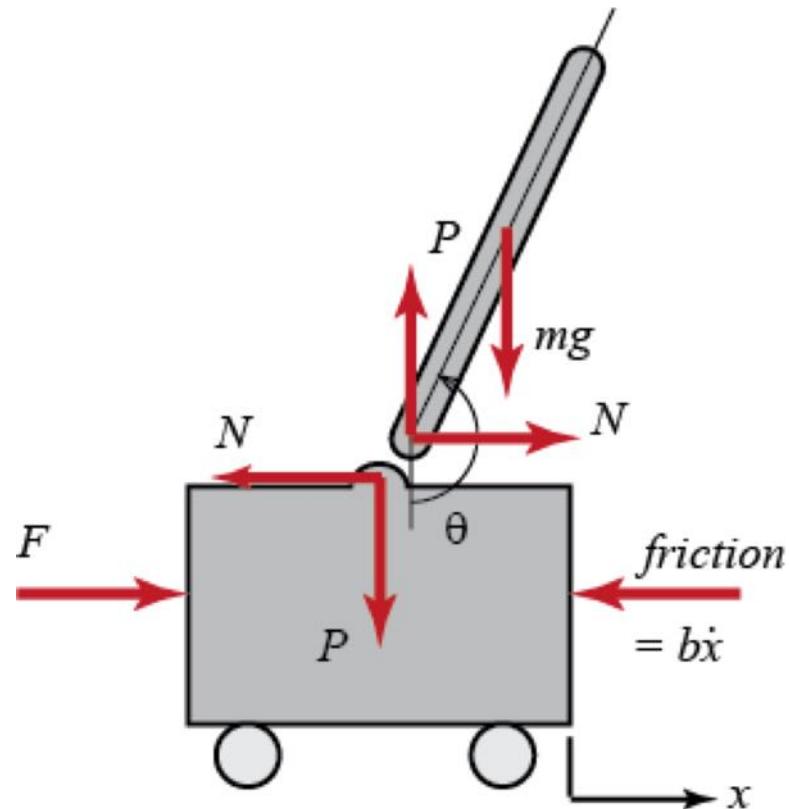
For small deviations ϕ :

$$\cos(\pi + \phi) \approx -1$$

$$\sin(\pi + \phi) \approx -\phi$$

$$\dot{\theta}^2 = \dot{\phi}^2 \approx 0$$

Inverted pendulum - Dynamics

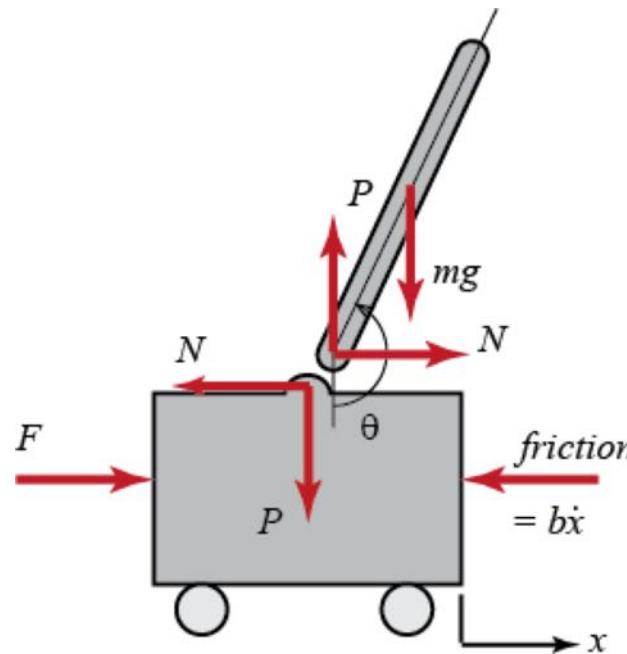


Equations of motion are:

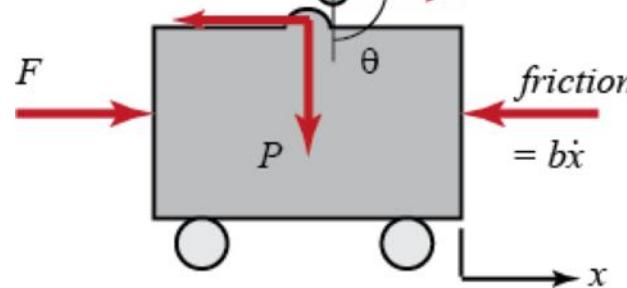
$$(I + ml^2)\ddot{\theta} + mgl\dot{\theta} = ml\ddot{x}$$

$$(M + m)\ddot{x} + b\dot{x} - ml\ddot{\theta} = F$$

Rearranging – State-Space representation

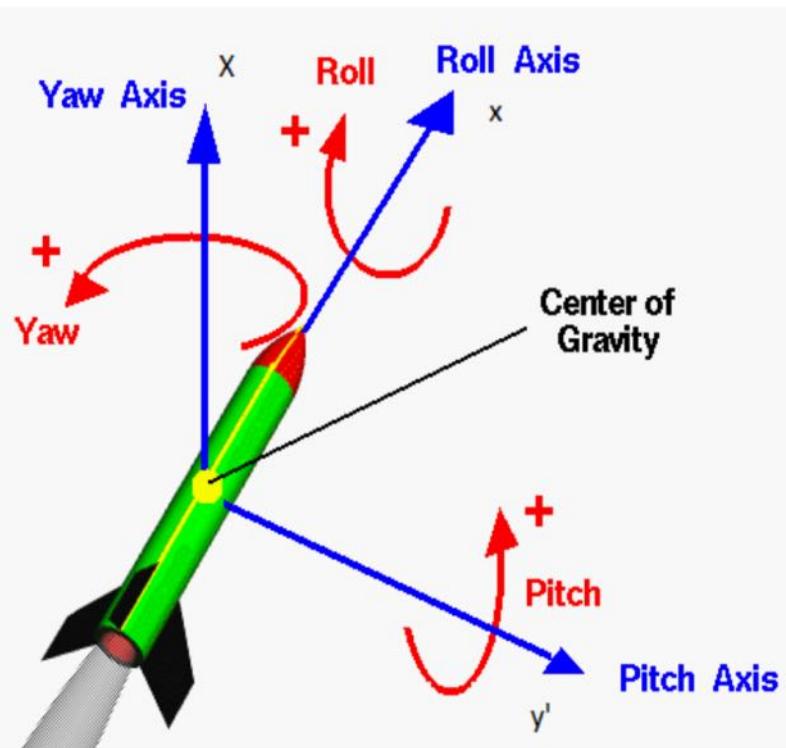


$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-(I+ml^2)\dot{b}}{I(M+m)+Mml^2} & \frac{m^2gl^2}{I(M+m)+Mml^2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-ml\dot{b}}{I(M+m)+Mml^2} & \frac{mg\dot{l}(M+m)}{I(M+m)+Mml^2} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{I+ml^2}{I(M+m)+Mml^2} \\ 0 \\ \frac{ml}{I(M+m)+Mml^2} \end{bmatrix} u$$



$$\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

From State-Space to Space..and back



$$\begin{aligned}\ddot{x} &= \frac{1}{m}(F_x c\psi c\theta + F_y(c\psi s\theta s\phi - s\psi c\phi) + F_z(s\psi s\phi + c\psi s\theta c\phi)) - g \\ \ddot{y} &= \frac{1}{m}(F_x s\psi c\theta + F_y(c\psi c\phi + s\psi s\theta s\phi) + F_z(s\psi s\theta c\phi - c\psi s\phi)) \\ \ddot{z} &= \frac{1}{m}(-F_x s\theta + F_y c\theta s\phi + F_z c\theta c\phi) \\ \dot{\phi} &= \frac{M_x}{I_a} + \dot{\psi}\dot{\theta}c\theta + \frac{s\theta}{I_t c\theta}(M_z c\phi + M_y s\phi + I_a(\dot{\phi}\dot{\theta} - \dot{\psi}\dot{\theta}s\theta) + 2I_t\dot{\psi}\dot{\theta}s\theta) \\ \ddot{\theta} &= \frac{1}{I_t}(0.5(I_a - I_t)\dot{\psi}^2 s2\theta - I_a\dot{\phi}\dot{\psi}c\theta + M_y c\phi - M_z s\phi) \\ \dot{\psi} &= \frac{1}{I_t c\theta}(M_z c\phi + M_y s\phi + I_a(\dot{\phi}\dot{\theta} - \dot{\psi}\dot{\theta}s\theta) + 2I_t\dot{\psi}\dot{\theta}s\theta)\end{aligned}$$



From state-space to Space



Dynamical System

$$\frac{dx}{dt} = \dot{x} = f(x(t), u(t), t)$$

Possibly a non-linear function

Rate of change

Time invariant system: Simplifying assumption #1

$$\frac{dx}{dt} = \dot{x} = f(x, u)$$

f does not depend on time

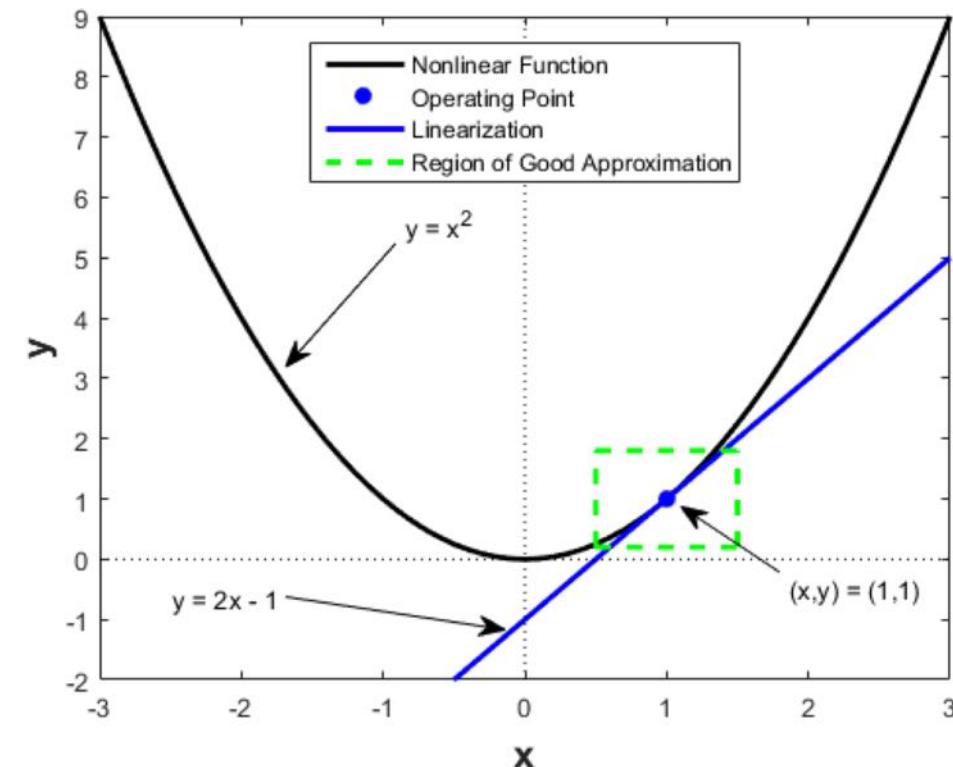
Rate of change

- The underlying physical laws themselves do not typically depend on time.
- Inputs $u(t)$ may be time dependent
- The parameters/constants which describe the function f remain the same.

Linearity: Simplifying assumption #2

Over a sufficiently small operating range (think tangent line near a curve), the dynamics of most systems are approximately **linear**

$$\dot{x} = Ax + Bu$$



State-Space representation

A **state-space model** represents a system by a series of **first-order differential state equations** and algebraic output equations.

Differential equations have been rearranged as a series of first order differential equations.

Example

Consider the following system where $u(t)$ is the input and $\dot{x}(t)$ is the output.

$$\ddot{x} + 5\ddot{x} + 3\dot{x} + 2x = u, y = \dot{x}$$

Can create a state-space model by pure mathematical manipulation through changing variables

$$x_1 = x, x_2 = \dot{x}, x_3 = \ddot{x}$$

Resulting in the following three first order differential equations (ODEs)

$$\dot{x}_1 = x_2,$$

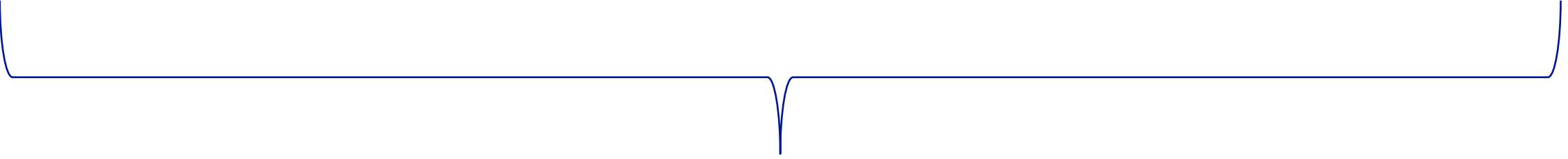
$$\dot{x}_2 = x_3,$$

$$\dot{x}_3 = -5x_3 - 3x_2 - 2x_1 + u$$

State Equations

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -5x_3 - 3x_2 - 2x_1 + u\end{aligned}$$

Output Equation

$$y = x_2$$


System has 1 input (u), 1 output (y), and 3 state variables (x_1, x_2, x_3)

In general state-space models have the following form (equations can be nonlinear and time varying)

State Equations

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2, \dots, x_n, u_1, \dots, u_m) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, x_2, \dots, x_n, u_1, \dots, u_m)\end{aligned}$$

Output Equations

$$\begin{aligned}y_1 &= h_1(x_1, x_2, \dots, x_n, u_1, \dots, u_m) \\ &\vdots \\ y_n &= h_n(x_1, x_2, \dots, x_n, u_1, \dots, u_m)\end{aligned}$$

State-space representation

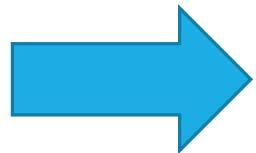
$$\dot{\vec{x}} = A\vec{x} + B\vec{u}$$

$$y = C\vec{x} + D\vec{u}$$

for linear systems

From our prior example

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -5x_3 - 3x_2 - 2x_1 + u\end{aligned}$$



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -5 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = x_2$$

$$[y] = [0 \ 1 \ 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0] u$$

The State-Space Modeling Process

- 1) Identify ***input*** variables (actuators and exogenous inputs).
- 2) Identify ***output*** variables (sensors and performance variables).
- 3) Identify ***state*** variables. (Hmmm...how ? – indep. energy storage)
- 4) Use first principles of physics to relate derivative of state variables to the input, state, and the output variables.

Why use state-space representations ?

State-space models:

- are numerically efficient to solve,
- can handle complex systems,
- allow for a more geometric understanding of dynamic systems, and
- form the basis for much of modern control theory

Linear dynamical system

Continuous-time linear dynamical system (CT LDS) has the form

$$\dot{x} = A(t)x(t) + B(t)u(t) \quad y = C(t)x(t) + D(t)u(t)$$

- $t \in \mathbf{R}$ denotes *time*
- $x(t) \in \mathbf{R}^n$ is the *state* (vector)
- $u(t) \in \mathbf{R}^m$ is the *input* or *control*
- $y(t) \in \mathbf{R}^p$ is the *output*

Continuous-time linear dynamical system (CT LDS)

$$\dot{x} = A(t)x(t) + B(t)u(t) \quad y = C(t)x(t) + D(t)u(t)$$

- $A(t) \in \mathbb{R}^{n \times n}$ is the *dynamics matrix*
- $B(t) \in \mathbb{R}^{n \times m}$ is the *input matrix*
- $C(t) \in \mathbb{R}^{p \times n}$ is the *output or sensor matrix*
- $D(t) \in \mathbb{R}^{p \times m}$ is the *feedthrough matrix*

Linear dynamical system

Some terminology

- most linear systems encountered are *time-invariant*: A, B, C, D are constant, i.e., don't depend on t

Linear dynamical system

Some terminology

- most linear systems encountered are *time-invariant*: A, B, C, D are constant, *i.e.*, don't depend on t
- when there is no input u (hence, no B or D) system is called *autonomous*

Linear dynamical system

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- very often there is no feedthrough, *i.e.*, $D = 0$

Linear dynamical system

Some terminology

- most linear systems encountered are *time-invariant*: A, B, C, D are constant, *i.e.*, don't depend on t
- when there is no input u (hence, no B or D) system is called *autonomous*
- very often there is no feedthrough, *i.e.*, $D = 0$
- when $u(t)$ and $y(t)$ are scalar, system is called *single-input, single-output* (SISO); when input & output signal dimensions are more than one, MIMO

Discrete-time linear dynamical system (DT LDS)

$$x(k+1) = A(k)x(k) + B(k)u(k)$$

$$y(k) = C(k)x(k) + D(k)u(k)$$

where

- $k \in \mathbf{Z} = \{0, \pm 1, \pm 2, \dots\}$
- (vector) signals x, u, y are *sequences*

Many dynamical systems are nonlinear (a fascinating topic) so why study linear systems?

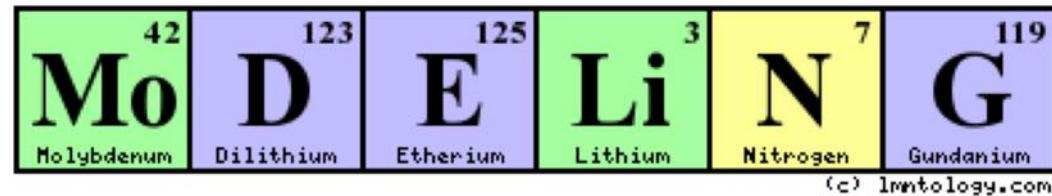
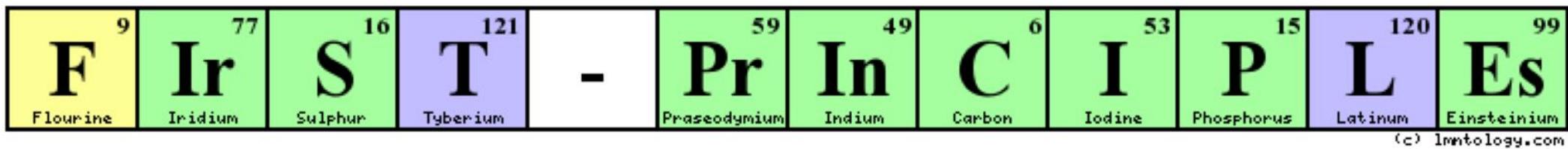
- Most techniques for nonlinear systems are based on linear systems.
- Methods for linear systems often work unreasonably well, in practice, for nonlinear systems.
- If you do not understand linear dynamical systems, you certainly cannot understand nonlinear dynamical systems.

Many dynamical systems are nonlinear (a fascinating topic) so why study linear systems?

“Finally, we make some remarks on why linear systems are so important. The answer is simple: because **we can solve them!**”

- Richard Feynman [Fey63, p. 25-4]

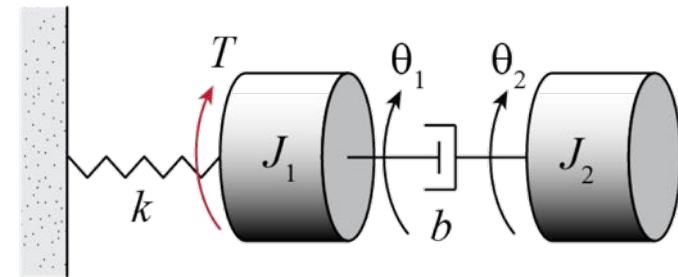
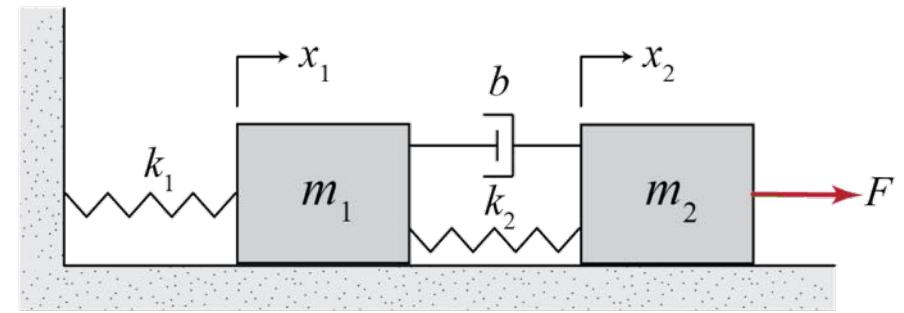
Elements of..



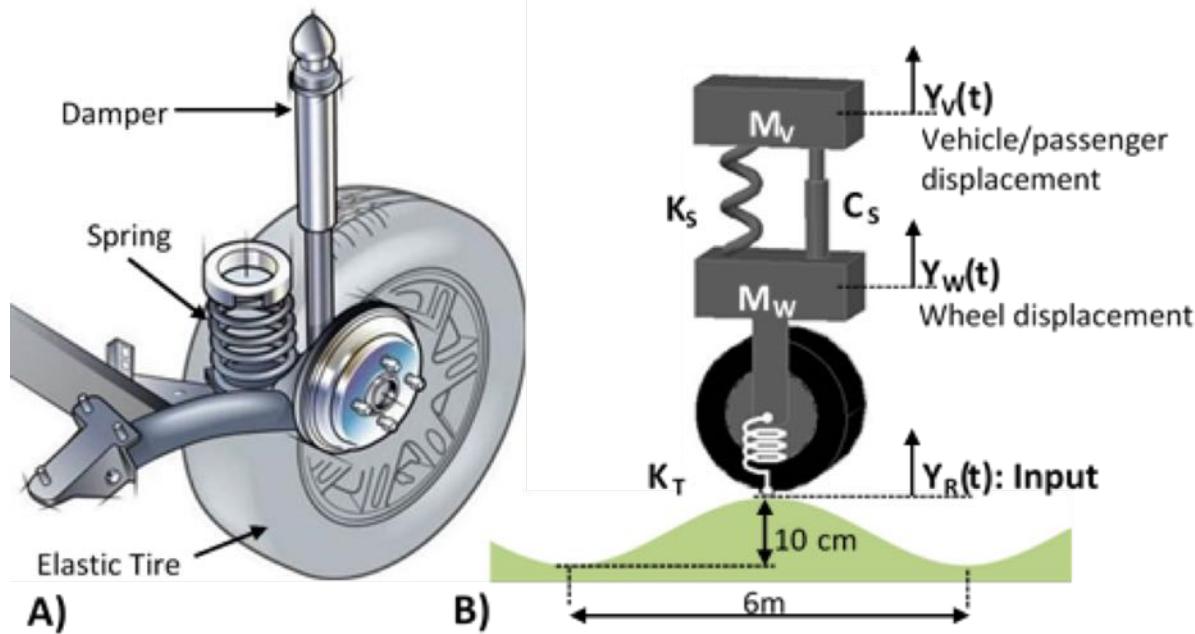
Modeling Mechanical Systems

Mechanical systems consist of three basic types of elements:

1. Inertia elements
2. Spring elements
3. Damper elements



Vehicle suspension – Mass-spring-damper



Inertia elements

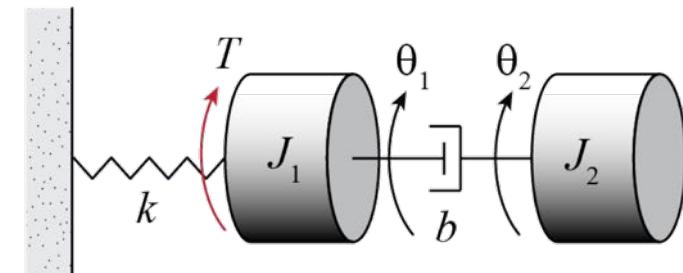
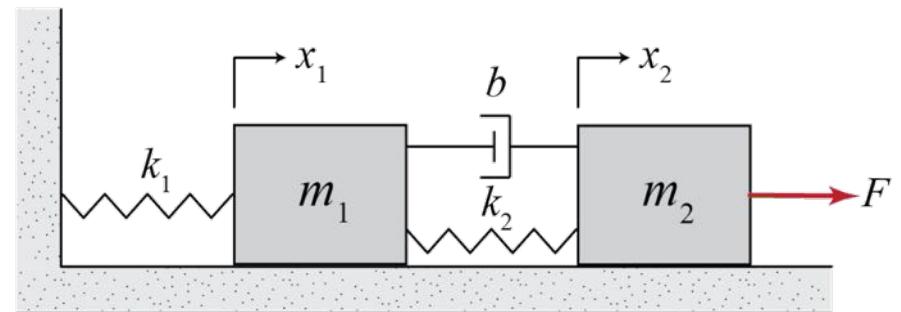
- Example: any mass in the system, or moment of inertia.
- Each inertia element with motion needs its own differential equation (Newton's 2nd Law, Euler's 2nd law)

$$\sum F = ma$$

$$\sum M = J\alpha$$

- **Inertia elements store kinetic energy**

$$E = \int Fv \, dt = \int m\dot{v}v \, dt = \frac{1}{2}mv^2$$

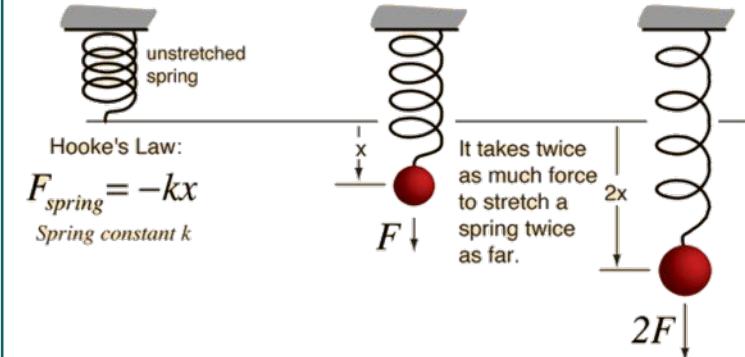
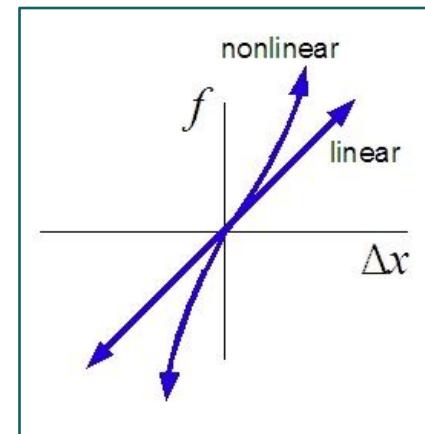
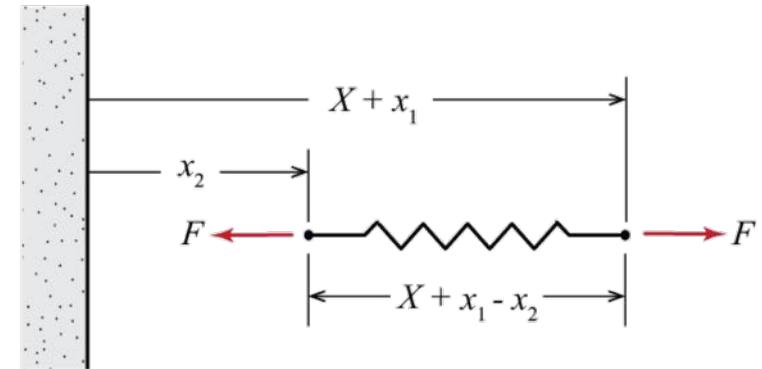


Spring elements

$$F = k(x_1 - x_2)$$

- Force is generated to resist deflection.
- Examples: translational and rotational springs
- **Spring elements store potential energy**

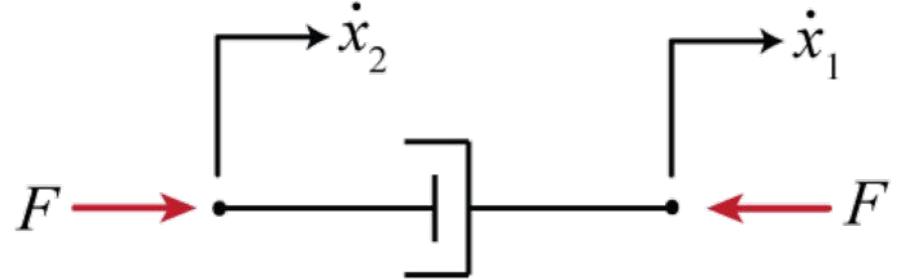
$$E = \int Fv \, dt = \int kx\dot{x} \, dt = \frac{1}{2}kx^2$$



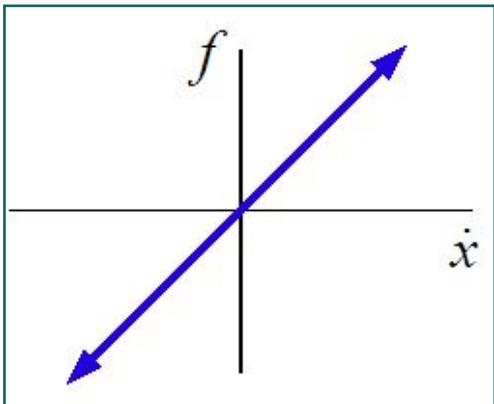
Damper elements

$$F = b(\dot{x}_1 - \dot{x}_2)$$

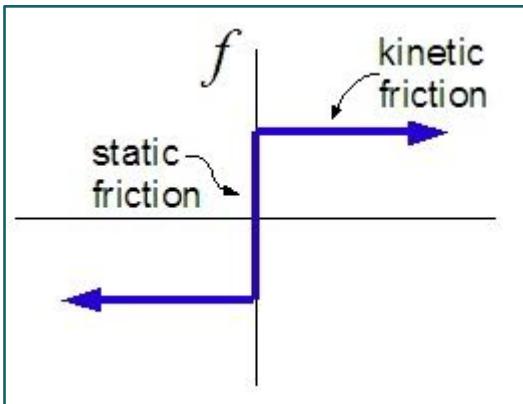
- Force is generated to resist motion.
- Examples: dashpots, friction, wind drag
- **Damper elements dissipate energy**



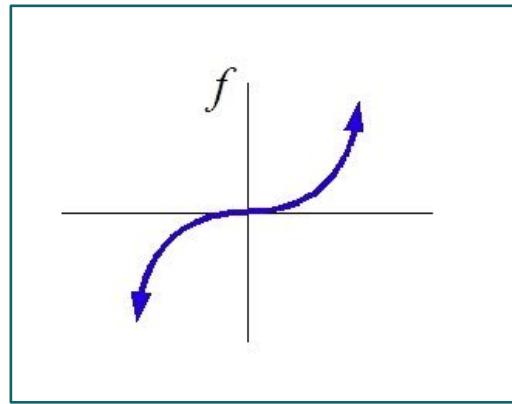
linear damping



friction



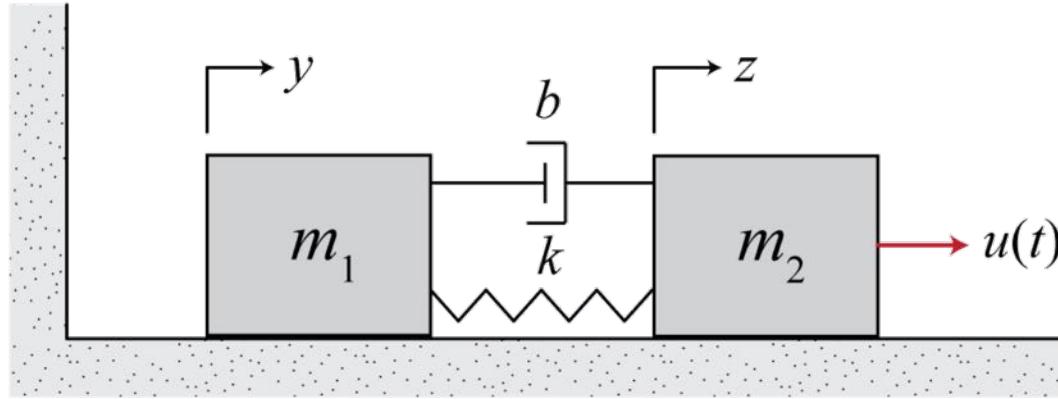
drag



How many state variables are required ?

- There is an intuitive way to find state-space models
- What initial conditions do I need to capture the system's state?
- Definition: the **state** of a dynamic system is the set of variables (called **state variables**) whose knowledge at $t = t_0$ along with knowledge of the inputs for $t \geq t_0$ completely determines the behavior of the system for $t \geq t_0$
- **# of state variables = # of independent energy storage elements**

Example



Equations of motion

$$m_1 \ddot{y} + b(\dot{y} - \dot{z}) + k(y - z) = 0$$

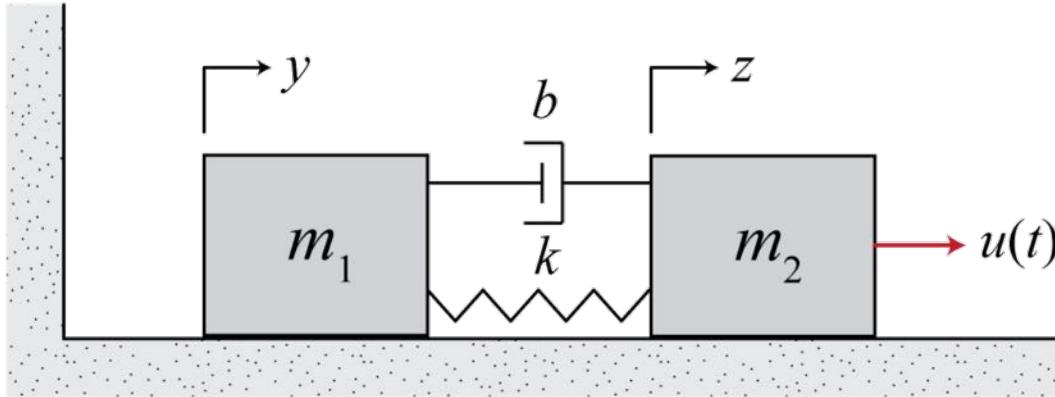
$$m_2 \ddot{z} + b(\dot{z} - \dot{y}) + k(z - y) = u$$

Choice of state variables

$$x_1 = y, x_2 = \dot{y}$$

$$x_3 = z, x_4 = \dot{z}$$

Example



Equations of motion

$$m_1 \dot{x}_2 + b(x_2 - x_4) + k(x_1 - x_3) = 0$$

$$m_2 \dot{x}_4 + b(x_4 - x_2) + k(x_3 - x_1) = u$$

Choice of state variables

$$x_1 = y, x_2 = \dot{y}$$

$$x_3 = z, x_4 = \dot{z}$$

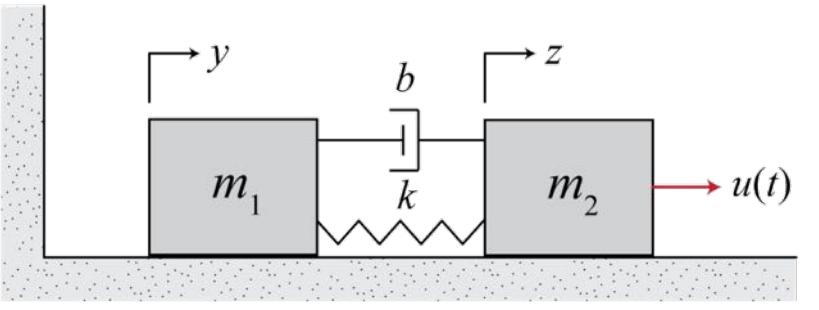
Example

$$\dot{x}_1 = x_2$$

$$x_2 = \frac{-b(x_2 - x_4) - k(x_1 - x_3)}{m_1}$$

$$\dot{x}_3 = x_4$$

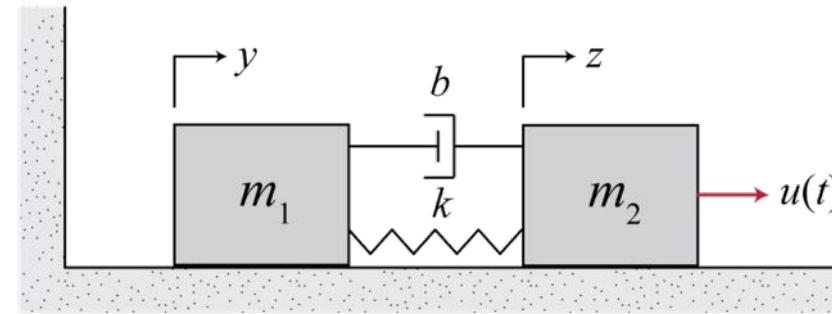
$$x_4 = \frac{u - b(x_4 - x_2) - k(x_3 - x_1)}{m_2}$$



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{-k}{m_1} & \frac{-b}{m_1} & \frac{k}{m_1} & \frac{b}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k}{m_2} & \frac{b}{m_2} & \frac{-k}{m_2} & \frac{-b}{m_2} \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m_2} \end{bmatrix} u$$

Is this the minimum set of states ?

Example



Look at where energy is stored

Energy Storage Element

spring (stores elastic PE)

mass 1 (stores KE)

mass 2 (stores KE)

State Variable

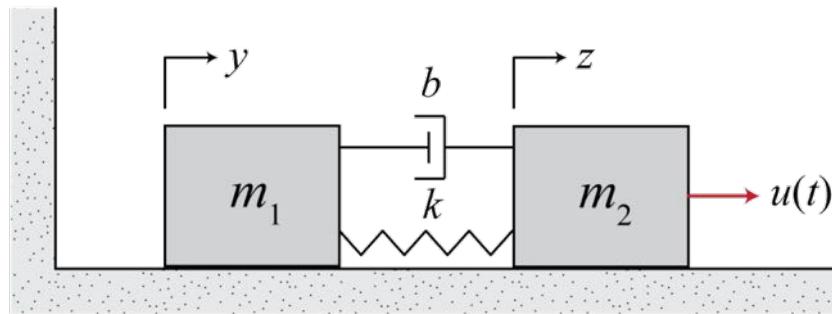
$$x_1 = (y - z)$$

$$x_2 = \dot{y}$$

$$x_3 = \dot{z}$$

damper does not store energy, it dissipates energy

Example



$$m_1 \ddot{y} + b(\dot{y} - \dot{z}) + k(y - z) = 0$$

$$m_2 \ddot{z} + b(\dot{z} - \dot{y}) + k(z - y) = u$$

$$x_1 = (y - z)$$

$$x_2 = \dot{y}$$

$$x_3 = \dot{z}$$



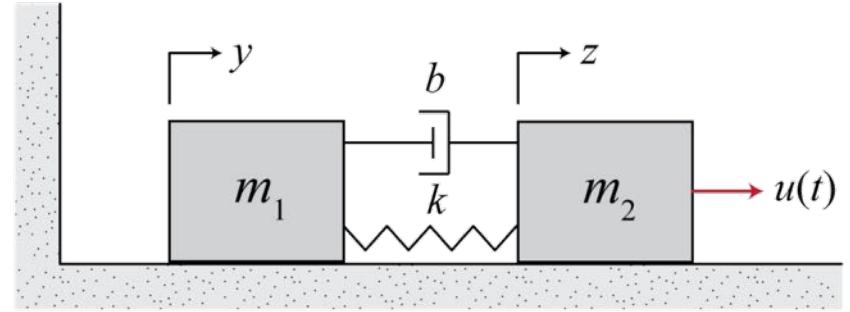
Rewriting in state-space representation

$$\dot{x}_1 = x_2 - x_3$$

$$\dot{x}_2 = \ddot{y} = \frac{1}{m_1} (-b(x_2 - x_3) - kx_1)$$

$$\dot{x}_3 = \ddot{z} = \frac{1}{m_2} (-b(x_3 - x_2) + kx_1 + u)$$

Example



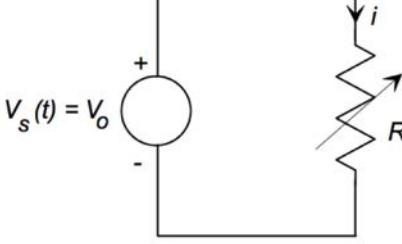
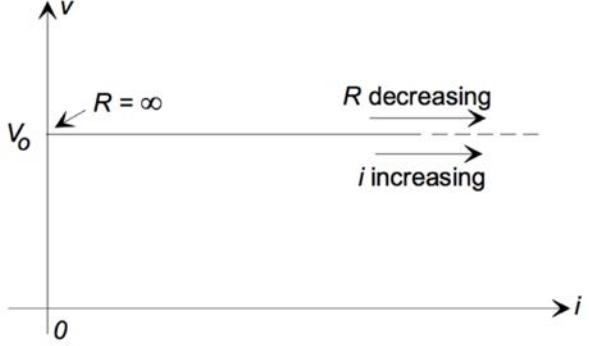
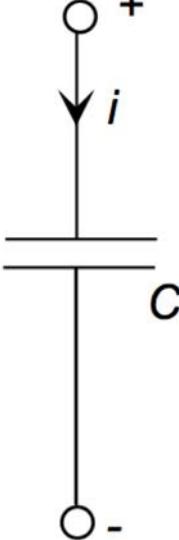
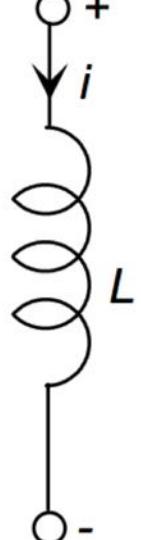
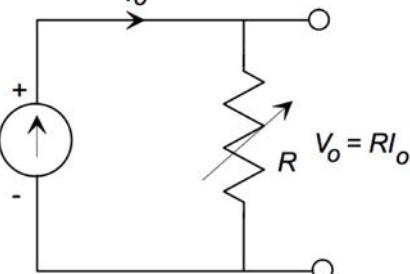
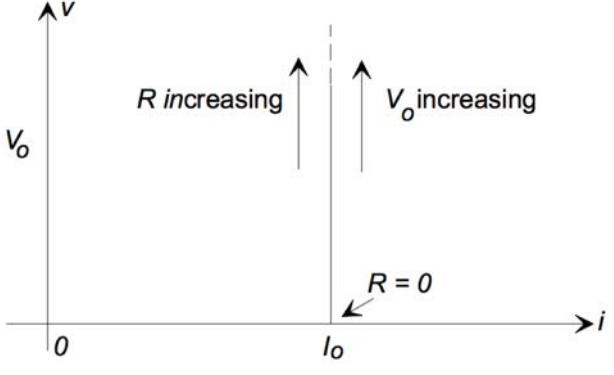
$$\dot{x}_1 = x_2 - x_3$$

$$\dot{x}_2 = \ddot{y} = \frac{1}{m_1} (-b(x_2 - x_3) - kx_1)$$

$$\dot{x}_3 = \ddot{z} = \frac{1}{m_2} (-b(x_3 - x_2) + kx_1 + u)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ \frac{-k}{m_1} & \frac{-b}{m_1} & \frac{b}{m_1} \\ \frac{k}{m_2} & \frac{b}{m_2} & \frac{-b}{m_2} \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_2} \end{bmatrix} u$$

Modeling electrical systems

Passive elements			Active elements	
Capacitor [storage]	Inductor [storage]	Resistor [dissipative]	 $V_s(t) = V_o$	
 $i = C \frac{dv}{dt}$	 $v = L \frac{di}{dt}$	 $v = Ri$	 $I_s(t) = I_o$ $V_o = RI_o$	

Mechanical – Electrical equivalency

We recognize a **common form** to the ODE describing each system and create analogs in the various energy domains, for example:

$\left\{ \begin{array}{l} \text{voltage} \\ \text{velocity} \\ \text{pressure} \end{array} \right\}$ and $\left\{ \begin{array}{l} \text{current} \\ \text{force} \\ \text{volume flow rate} \end{array} \right\}$ in the $\left\{ \begin{array}{l} \text{electrical} \\ \text{mechanical} \\ \text{fluidic} \end{array} \right\}$ domains.

Capacitor - Mass

Electrical Capacitance

$$q = CV$$



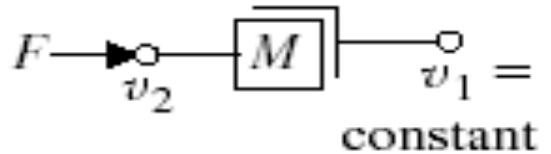
Describing Equation

$$i = C \cdot \frac{d}{dt} v_{21}$$

Energy

$$E = \frac{1}{2} \cdot M \cdot v_{21}^2$$

Translational Mass



$$F = M \cdot \frac{d}{dt} v_2$$

$$E = \frac{1}{2} \cdot M \cdot v_2^2$$

Inductor - Spring

Electrical Inductance



Describing Equation

$$v_{21} = L \cdot \frac{di}{dt}$$

Energy

$$E = \frac{1}{2} \cdot L \cdot i^2$$

Translational Spring

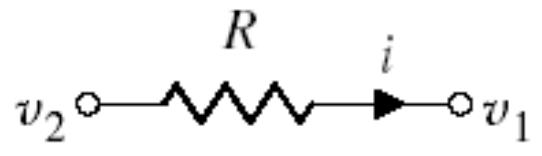


$$v_{21} = \frac{1}{k} \cdot \frac{dF}{dt}$$

$$E = \frac{1}{2} \cdot \frac{F^2}{k}$$

Resistor - Damper

Electrical Resistance



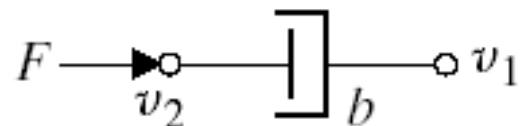
Describing Equation

$$i = \frac{1}{R} \cdot v_{21}$$

Energy

$$P = \frac{1}{R} \cdot v_{21}^2$$

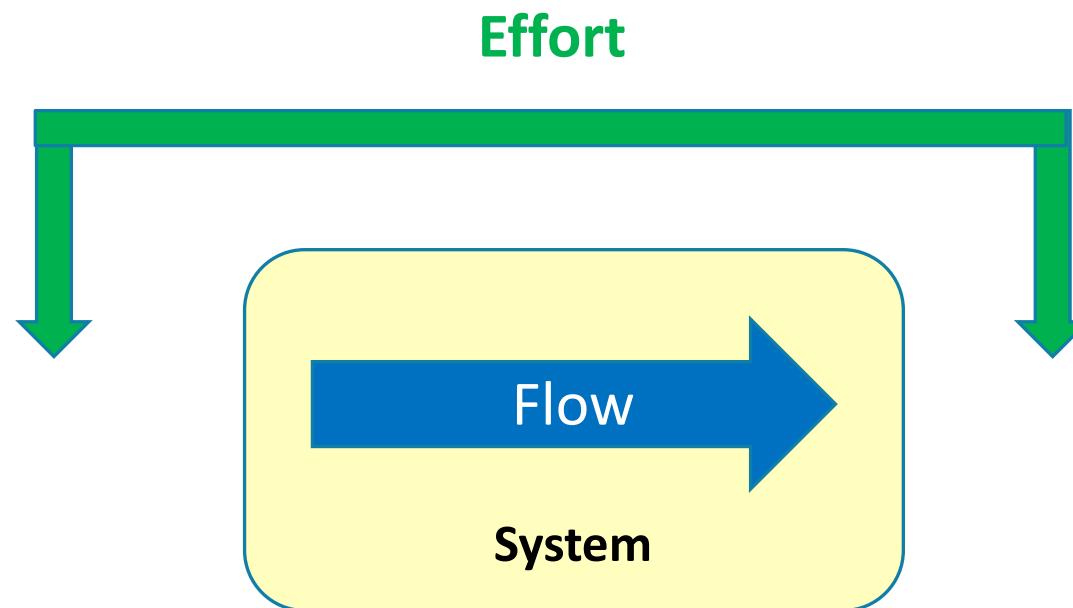
Translational Damper



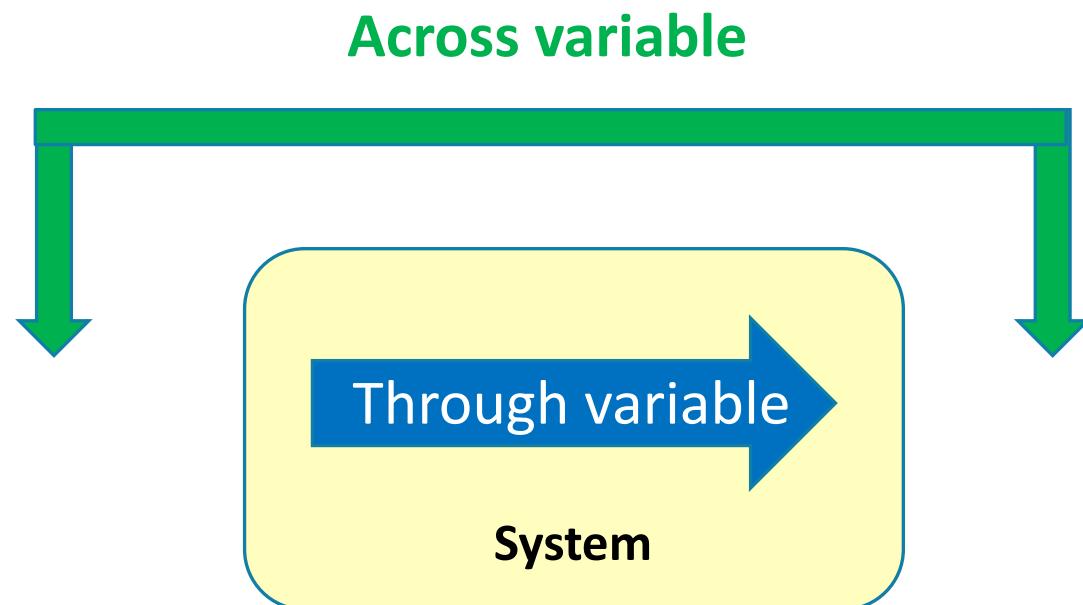
$$F = b \cdot v_{21}$$

$$P = b \cdot v_{21}^2$$

Generalized system representation.



$$Power = Through\ variable \times Across\ variable$$



Power = Through variable × Across variable

Power is voltage times current

$$P = i \times V$$

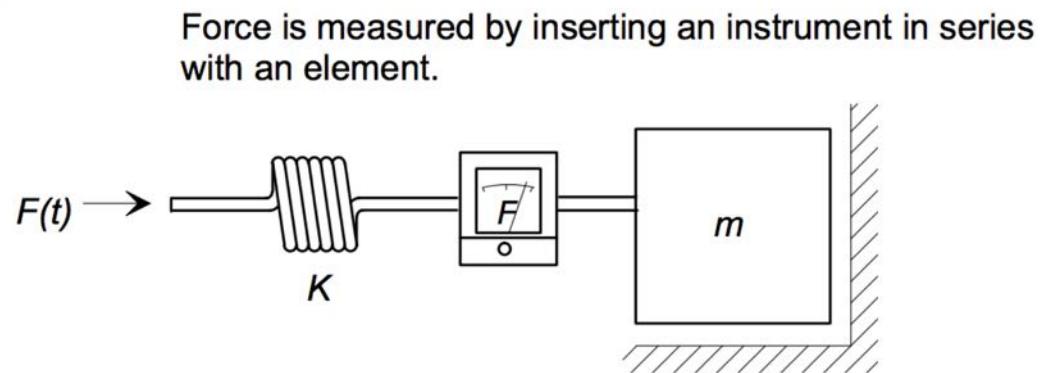
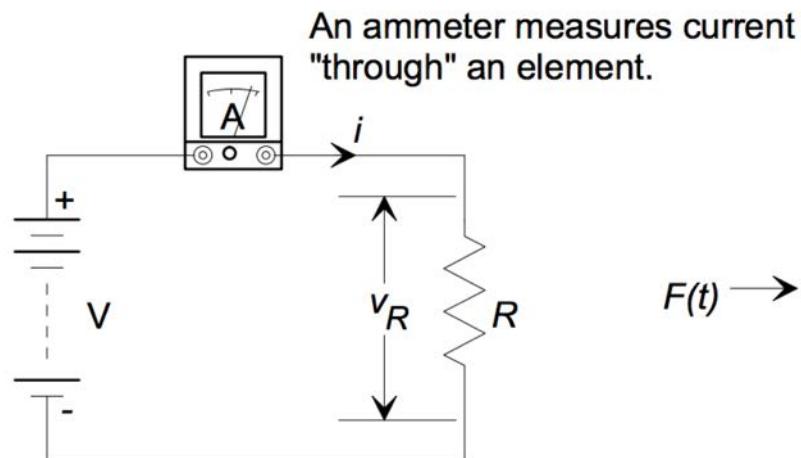
Power is velocity times force

$$P = F \times v$$

Power = Through variable × Across variable

Through variables:

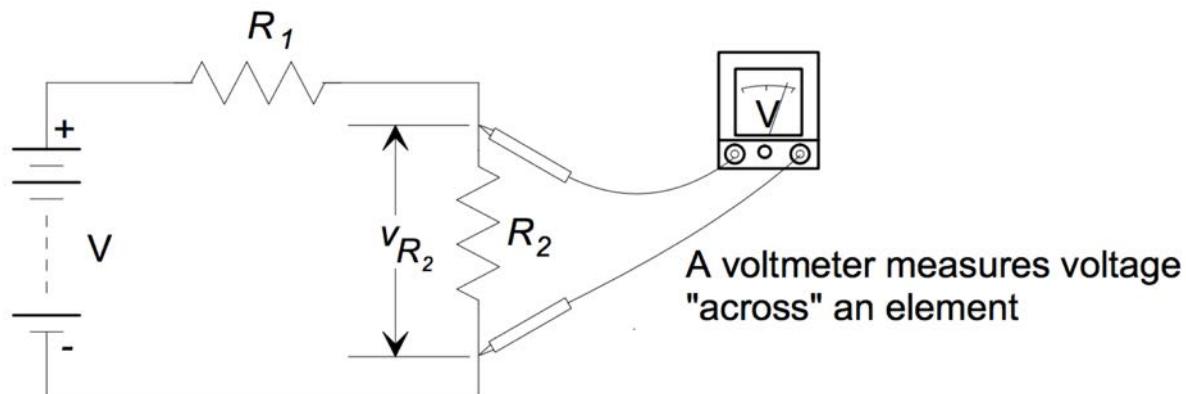
- Variables that are measured through an element.
- Variables sum to zero at the nodes on a graph/circuit/free body diagram.
- Variables that are measured with a gauge connected in series to an element.



Power = Through variable × Across variable

Across variables:

- Variables that are defined by measuring a difference, or drop, across an element, that is between nodes on a graph (across one or more branches).
- Variables sum to zero around any closed loop on the graph
- Variables that are measured with a gauge connected in parallel to an element.

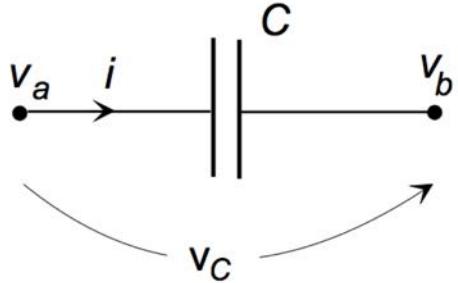


Power = Through variable × Across variable

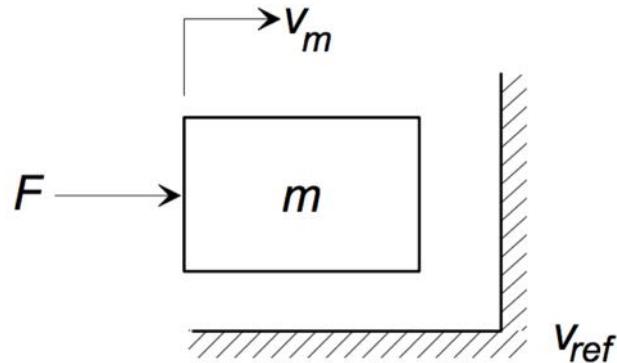
Physical Domain	Across Variable	Through Variable
Electrical	Voltage	Current
Hydraulic	Pressure	Flow rate
Magnetic	Magnetomotive force (mmf)	Flux
Mechanical rotational	Angular velocity	Torque
Mechanical translational	Translational velocity	Force
Gas	Pressure and temperature	Mass flow rate and energy flow rate
Thermal	Temperature	Heat flow
Thermal liquid	Pressure and temperature	Mass flow rate and energy flow rate
Two-phase fluid	Pressure and specific internal energy	Mass flow rate and energy flow rate

Energy storage : A-Type elements

Stored energy is a function of the Across-variable.



$$\begin{aligned} i &= C \frac{dv}{dt} \\ E &= \int_{-\infty}^t vi dt = \int_0^t Cv dv \\ &= \frac{1}{2} Cv^2 \end{aligned}$$



$$\begin{aligned} F &= m \frac{dv}{dt} \\ E &= \int_{-\infty}^t vF dt = \int_0^t mv dv \\ &= \frac{1}{2} mv^2 \end{aligned}$$

Generalized, Capacitance

The diagram illustrates the generalized form of the equation for capacitance. On the left, a wavy line with an arrow points to the variable v in the term $\frac{dv}{dt}$. A curved arrow from this wavy line points to the text "generalized through variable". On the right, a wavy line with an arrow points to the constant C in the term $C \frac{dv}{dt}$. A curved arrow from this wavy line points to the text "generalized capacitance". Above the term $\frac{dv}{dt}$, another wavy line with an arrow points to the entire expression $f = C \frac{dv}{dt}$. A curved arrow from this wavy line points to the text "generalized across variable".

$$f = C \frac{dv}{dt}$$

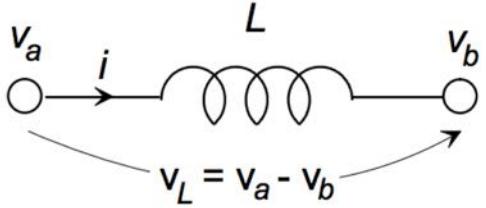
generalized through variable

generalized capacitance

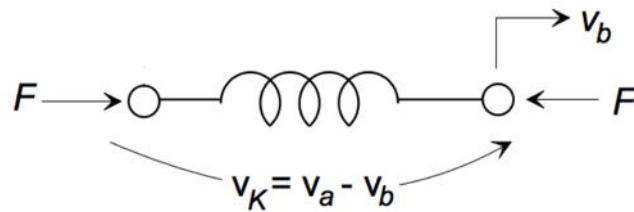
generalized across variable

Energy storage: T-Type elements

Stored energy is a function of the Through-variable.



$$\begin{aligned}v &= L \frac{di}{dt} \\E &= \int_{-\infty}^t vi dt = \int_0^t Li di \\&= \frac{1}{2}Li^2\end{aligned}$$



$$\begin{aligned}v &= \frac{1}{K} \frac{dF}{dt} \\E &= \int_{-\infty}^t vF dt = \frac{1}{K} \int_0^t F dF \\&= \frac{1}{2K}F^2\end{aligned}$$

Generalized inductance, L

$$V = L \frac{df}{dt}$$

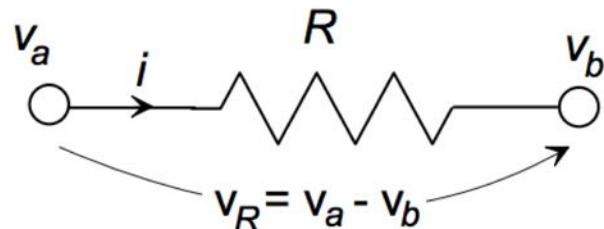
generalized across variable

generalized through variable

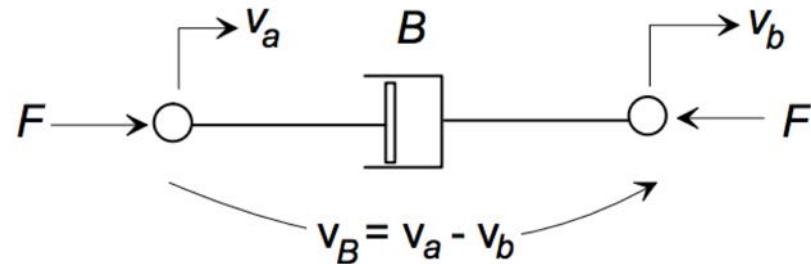
generalized inductance

Dissipative elements : D-Type

Dissipative elements (non-energy storage)

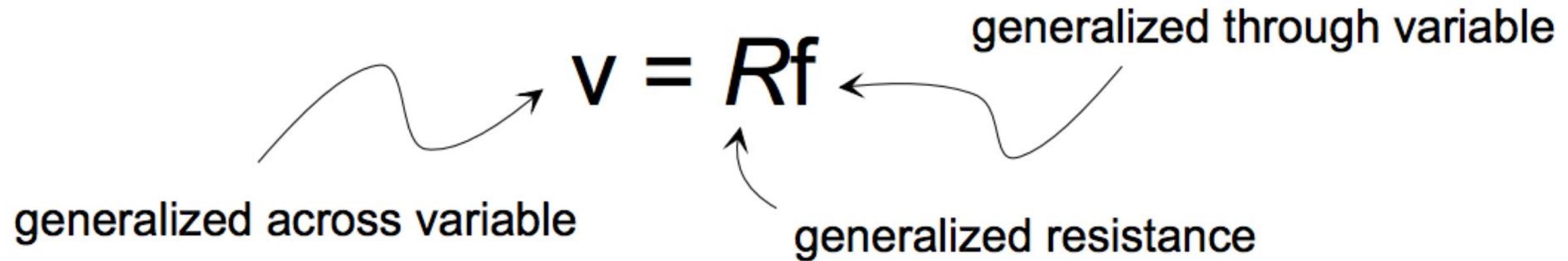


$$\begin{aligned}v &= iR \\P &= vi = i^2 R = v^2/R \\&\geq 0\end{aligned}$$



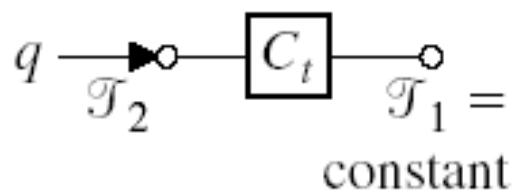
$$\begin{aligned}F &= Bv \\P &= vF = Bv^2 = F^2/B \\&\geq 0\end{aligned}$$

Generalized resistance, R



Cyber-Physical Energy Systems Modeling

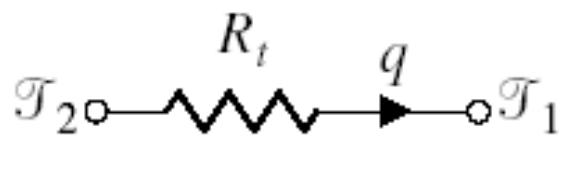
Thermal Capacitance



$$q = C_t \cdot \frac{d}{dt} T_2$$

$$E = C_t \cdot T_2$$

Thermal Resistance



$$q = \frac{1}{R_t} \cdot T_{21}$$

$$P = \frac{1}{R_t} \cdot T_{21}$$

Next lecture..

Learn how to get paid for doing nothing while saving the environment !

....the answer might have to do with drinking tea.

