PRINCIPLES OF MODELING FOR CYBER PHYSICAL SYSTEMS

Assignment #1

State-space modeling in MATLAB

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1 DYNAMICAL SYSTEM EQUATIONS:

The first step in the control design process is to develop appropriate mathematical models of the system derived either from physical laws and experimental data. The goal of this worksheet is to review the state-space modeling approach to modeling mechanical, electrical, and thermal systems and learn how to enter these models into MATLAB for simulation and analysis.
Key `MATLAB` commands for this tutorial are: `ss`, `step`, and `tf`
Dynamic systems are systems that change or evolve in time according to a fixed rule. For many physical systems, this rule can be stated as a set of first-order differential equations:

$$\dot{x} = \frac{dx}{dt} = f[x(t), u(t), t] \tag{1.1}$$

where $x(t)$ is the state vector, a set of variables representing the configuration of the system at time t. For instance in a simple mechanical mass-spring-damper system, the two state variables could be the position and velocity of the mass. $\mathbf{u}(t)$ is the vector of control inputs at time t, representing the externally "forces" or inputs on the system, and $\mathbf{f}$ is a function giving the time derivative (rate of change) of the state vector, $d\mathbf{x}/dt$ for a particular state, input, and time.
The state at any future time, $\mathbf{x}(t_1)$, may be determined exactly given knowledge of the initial state, $\mathbf{x}(t_0)$, and the time history of the inputs, $\mathbf{u}(t)$, between t_0 and t_1. The minimum number of state variables, n, required in a given system for the above statement to hold true is referred to as the system order and determines the dimensionality of the state-space.

In many physical systems, the function, $\mathbf{f}$, does not depend explicitly on time, i.e. $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$. Such a system is said to be **time invariant.** This is often a very reasonable assumption, since the underlying physical laws themselves do not typically depend on time. For time invariant systems, the parameters or coefficients of the function, $\mathbf{f}$, are constant over time. The input, however, may still be time dependent, $\mathbf{u}(t)$.

Another common assumption is the linearity of the system. Nearly every physical system is nonlinear, i.e. $\mathbf{f}$ is typically some complicated function of the state and inputs. However, over a sufficiently small operating range (visualize a tangent line near a curve), the dynamics of most systems can be approximated as linear, that is $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$. Which is why it is worthwhile to study linear time invariant (LTI) systems.

1.1 State-space representation

For continuous linear time invariant (LTI) systems, the standard state-space representation is given below:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

where $\mathbf{x}$ is the vector of state variables (nx1), $\dot{\mathbf{x}}$ is the time derivative of state vector (nx1), $\mathbf{u}$ is the input or control vector (px1), $\mathbf{y}$ is the output vector (qx1), $\mathbf{A}$ is the system matrix (nxn), $\mathbf{B}$ is the input matrix (nxp), $\mathbf{C}$ is the output matrix (qxn), $\mathbf{D}$ is the feedforward matrix (qxp).

Often there are state variables which are not directly observed or are otherwise not of interest. The output matrix, $\mathbf{C}$, is used to specify which state variables (or combinations thereof) are available for use by the controller. Also often there is no direct feedforward in which case $\mathbf{D}$ is the zero matrix.

1.2 System 1: Automatic cruise control

Automatic cruise control is an example of a control system found in many modern vehicles. The purpose of the cruise control system is to maintain a constant vehicle speed despite external disturbances, such as changes in wind or road grade. This is accomplished by measuring the vehicle speed, comparing it to the desired or reference speed, and automatically adjusting the throttle according to a control law.

We consider here a simple model of the vehicle dynamics, shown in Figure 1.1 (referred to as the free-body diagram (FBD)). The vehicle, of mass m, is acted on by a control force, u. The force u represents the force generated at the road/tire interface. For this simplified model we will assume that we can control this force directly and will neglect the dynamics of the power train, tires, etc., that go into generating the force. The resistive forces, bv, due to rolling resistance and wind drag, are assumed to vary linearly with the vehicle velocity, v, and act in the direction opposite the vehicle's motion. b is the damping coefficient in this case.

With these assumptions we are left with a first-order system. Summing forces in the x-direction and applying Newton's laws of motion, we arrive at the following system equation:

$$m\dot{v} + bv = u \tag{1.2}$$

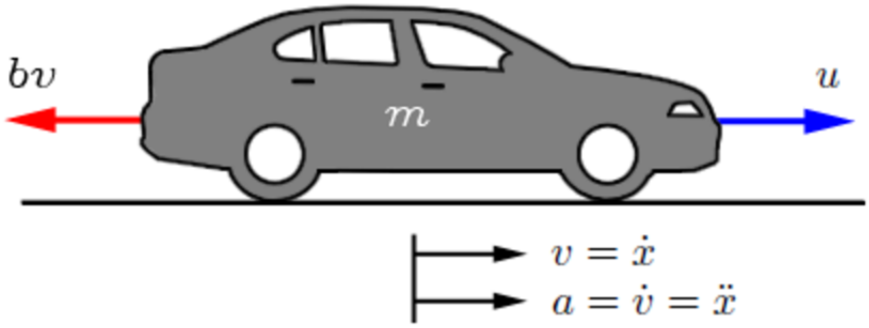

Figure 1.1: Simplified vehicle dynamics.

The output we are interested in is the speed of the vehicle v , therefore the output equation is:

$$y = v$$

Assume the following parameters of the system:

- vehicle mass (m) = 1000 kg;
- damping coefficient (b) = 50 N.s/m

Problem 1-1: Submit the following: (20 points)

- **[P1-1.a]** Write the dynamical equation of the system in state space representation.
- **[P1-1.b]** How many states are needed ?
- **[P1-1.c]** Is the system linear/non-linear in the state ?
- **[P1-1.d]** Is the system time-invariant ?
- **[P1-1.e]** What are the state variables (i.e. which physical quantities do they represent?)
- **[P1-1.f]** What are the dimensions of the A,B,C, and D matrices of the state-space model ?
- **[P1-1.g]** Write a script `cruise.m` to create a state-space model object for the vehicle in `MATLAB`. Submit your code.
- **[P1-1.h]** Is the model implemented in `MATLAB` a continuous-time or discrete-time model ?

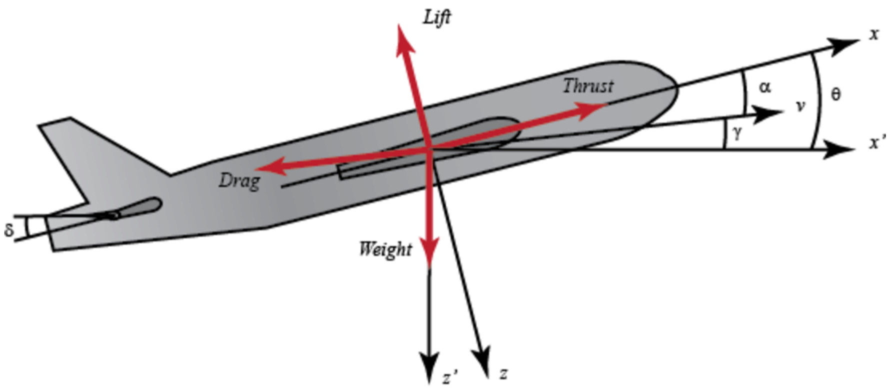

Figure 1.2: Simplified aircraft pitch dynamics.

- **[P1-1.i]** Using the `MATLAB` command `step`, plot and submit the step response of this dynamical system. What is the physical interpretation of the step-response plot ?
- **[P1-1.j]** Can you infer the order of the system from its step response ?

1.3 System 2: Aircraft pitch autopilot

The equations governing the motion of an aircraft are a very complicated set of nonlinear differential equations. However, under certain assumptions, they can be decoupled and linearized into longitudinal and lateral equations. Aircraft pitch is governed by the longitudinal dynamics. In this example we will design an autopilot that controls the pitch of an aircraft. The basic coordinate axes and forces acting on an aircraft are shown in the Figure 1.2.
We will assume that the aircraft is in steady-cruise at constant altitude and velocity; thus, the thrust, drag, weight and lift forces balance each other in the x- and y-directions. We will also assume that a change in pitch angle will not change the speed of the aircraft under any circumstance (unrealistic but simplifies the problem a bit). Under these assumptions, the longitudinal equations of motion for the aircraft can be written as follows:

$$\begin{aligned}
\dot{\alpha} &= \mu\Omega\sigma[-(C_L + C_D)\alpha + \frac{1}{(\mu - C_L)}q - (C_W \sin\gamma)\theta + C_L] \\
\dot{q} &= \frac{\mu\Omega}{2i_{yy}}[[C_M - \eta(C_L + C_D)]\alpha + [C_M + \sigma C_M(1 - \mu C_L)]q + (\eta C_W \sin\gamma)\delta] \\
\dot{\theta} &= \Omega q
\end{aligned}$$

For this system, the *input* will be the elevator deflection angle δ and the *output* will be the pitch angle θ of the aircraft.

To simplify the modeling equations, we can plug in some numerical values:

$$\begin{aligned} \dot{\alpha} &= -0.313\alpha + 56.7q + 0.232\delta \\ \dot{q} &= -0.0139\alpha - 0.426q + 0.0203\delta \\ \dot{\theta} &= 56.7q \end{aligned}$$

These values are taken from the data from one of Boeing's commercial aircraft.

Problem 2-1: Submit the following: (20 points)

- **[P1-2.a] (2)** Write the state space representation of the system.
- **[P1-2.b] (2)** How many states are needed ?
- **[P1-2.c] (4)** Write the dimensions of the A,B,C,D matrices of the state-space model.
- **[P1-2.d] (10)** Write a script `aircraft.m` to create a state-space model object for the vehicle in `MATLAB`. Submit your code.
- **[P1-2.e] (2)** Using the `MATLAB` command `step`, plot and submit the step response of this dynamical system. What is the physical interpretation of the step-response plot ?

Acknowledgments

A few examples in the worksheet have been adapted from the control tutorials by Prof. Rick Hill.