## Lecture 4: Extending Newton's method

COMP5930M Scientific Computation

# Today

Recap

Secant method

A more robust algorithm

Next

## Recap

We have seen two basic methods for scalar nonlinear equations

- Bisection
  Robust, slow, requires valid starting bracket
- Newton
   Fast, unreliable, requires derivative,
   requires good initial guess

We would like the best features of both

### The Secant Method

Replace the derivative required by Newton's Method with a numerical approximation

$$\frac{dF}{dx}(x_n) \approx \frac{F(x_n + \delta) - F(x_n)}{\delta}$$

Secant method:

$$x_{n+1} = x_n - \frac{\delta F(x_n)}{F(x_n + \delta) - F(x_n)}$$

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### Numerical derivatives

- ▶ Allowing for finite-precision arithmetic we can show that the optimal value of  $\delta \propto \sqrt{eps}$
- ▶ eps is the machine precision, typically  $10^{-16}$ , which gives roughly  $10^{-8}$  accuracy for the derivative
- In practice  $\delta = 10\sqrt{eps}$  is often a default value

## In practice

- The previous method requires an additional function evaluation at each step inefficient
- ▶ Instead use  $F(x_{n-1})$ , the previous iterate

$$\frac{dF}{dx}(x_n) \approx \frac{F(x_n) - F(x_{n-1})}{x_n - x_{n-1}}$$

- Less accurate ( $\delta$  larger) but more efficient
- Requires two data points to start the algorithm

### Secant method

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{F(x_n) - F(x_{n-1})}F(x_n)$$

- lacktriangle Order of convergence can be shown to be  $q=rac{\sqrt{5}+1}{2}pprox 1.618$
- ▶ The performance in practice is similar to Newton's method
- Still lacks robustness

Combine the advantages of bisection and Newton/secant

- 1. Define an initial bracket  $[x_0, x_1]$  s.t.  $F(x_0)F(x_1) < 0$ .
- 2. If  $|F(x_0)| < |F(x_1)|$  we swap their order,  $x_0 \leftrightarrow x_1$ . Take a step with the secant method:

$$x_2 = x_1 - \frac{x_1 - x_0}{F(x_1) - F(x_0)}F(x_1).$$

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### Algorithm outline:

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3a. If  $x_2 \in [x_0, x_1]$  define a new bracket  $[x_0, x_2]$  or  $[x_2, x_1]$  depending on if  $F(x_0)F(x_2) \le 0$  or  $F(x_2)F(x_1) < 0$ .

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  - 4. Test if converged. If not, increment *n* and iterate from 2.

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### Comments

- Requires a search for an initial bracket
- Will use (fast) secant whenever possible
- Will use robust (slow) bisection whenever secant appears unreliable
- The basis for Matlab's fzero() function

## Summary

- Scalar nonlinear equations can be solved in a robust way
- ▶ The initial guess (or bracket) is often a bigger challenge

#### Next week

Systems of nonlinear equations