

# Tutorial 2: Bisection method and rate of convergence

COMP5930M Scientific Computation

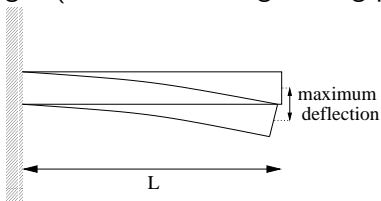
# Today

Further examples

Bisection method

## Example: Cantilever beam

Consider a cantilever beam of length  $L$  which bends under its own weight (a model civil engineering problem).



The beam's maximum deflection  $\delta_{\max}$  is at the unsupported end, and the deflection  $\delta$  at a point  $\alpha L$  ( $0 \leq \alpha \leq 1$ ) along the beam is given by the nonlinear equation  $f(\alpha) = 0$  where

$$f(\alpha) = \alpha^4 - 4\alpha^3 + 6\alpha^2 - 3\frac{\delta}{\delta_{\max}}$$

Find the point  $\alpha L$  where the beam deflection  $\delta$  is  $0.6 \delta_{\max}$

## Cantilever beam example

cantilever.m

```
function y = cantilever(x,def)
y = x^4 - 4*x^3 + 6*x^2 - 3*def;
```

- ▶ We know  $\alpha \in [0, 1]$  by definition which sets a range for the initial guess
- ▶ For `def=0.6, fzero()` converges to a root at 0.6979 from any point in  $[0, 1]$  or with bracket  $[0, 1]$
- ▶ Note that basic Newton will fail if  $x_0 = 0$  due to zero derivative at that point.

## Example: Airfoil modelling

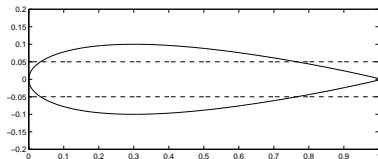
Consider the NACA0012 prototype wing section, which is often used for testing computational methods for simulating flows in aerodynamics: see, for example,

[http://www2.icfd.co.jp/examples/naca0012\\_2d/na3.htm](http://www2.icfd.co.jp/examples/naca0012_2d/na3.htm)

Its profile is given by

$$y(x) = \pm(0.2969\sqrt{x} - 0.126x - 0.3516x^2 + 0.2843x^3 - 0.1015x^4)$$

in which  $+$  gives the upper surface and  $-$  the lower surface.



Find the point  $x$  at which the thickness  $t$  of the aerofoil is 0.1

## Airfoil example

naca0012.m

```
function f = naca0012(x,th)
y = -0.1015*x^4 + 0.2843*x^3 - 0.3516*x^2
    - 0.126*x + 0.2969*sqrt(x);
f = 2*y - th;
```

- ▶ We know  $x \in [0, 1]$  by definition which sets a range for the initial guess
- ▶ But, there are 2 roots and also regions of small derivative which will cause problems for the basic algorithms
- ▶ **fzero()** converges to 0.0339 from  $x_0 = 0$  and to 0.7652 from  $x_0 = 1$
- ▶ Easier to use a single start point in this case

## Bisection method

bisection.m (basic algorithm):

```
% Bisection method for zero finding
%
% function [ x,f ] = bisection( fnon, xL, xR, tol, maxit )
%
% Input:
%     fnon - function handle for nonlinear equation
%     xL - left endpoint of initial bracket
%     xR - left endpoint of initial bracket
%     tol - convergence tolerance
%     maxIt - maximum allowed number of iterations
%
% Output:
%     x - final point
%     f - final function value
```

## Bisection method

```
fL = feval(fnon,xL);  
fR = feval(fnon,xR);  
  
% Test whether the sign of f changes within the bracket  
if(fL * fR > 0)  
    error('Sign change within interval not guaranteed!')  
end  
  
k=0;  
fprintf('    i    x_i            |F(x_i)|\n')
```



## Bisection method

```
while ((xR - xL) > tol && k < maxit)
    xC = (xL + xR) / 2;
    fC = feval(fnon,xC);
    fprintf(' %3d    %1.8f    %1.2e\n',k,xC,abs(fC));

    if (fL * fC < 0)
        xR = xC;
    else
        xL = xC;
    end
    k=k+1;
    x(k) = xC;
    f(k) = feval(fnon,xC);
end
xC = (xL + xR) / 2;
x(k) = xC;
f(k) = feval(fnon,xC);
```

## Example

Test the bisection method on the same function as before:

$$F(x) = x^2 - 2.$$

We can plot the convergence of the error  $|x^* - x_i|$ :

```
% Plot the convergence history
semilogy(abs(sqrt(2) - x), 'o-', 'LineWidth', 2)
xlabel('Iteration i')
ylabel('Error |x^* - x_i|')
```

## Comparison with Newton's method


- ▶ To estimate the convergence order of a method, we compute

$$\alpha_i := \frac{|x^* - x_i|}{|x^* - x_{i-1}|^q}$$

numerically for different  $q = 1, 2, \dots$

- ▶ If  $\lim_{i \rightarrow \infty} \alpha_i = \alpha > 0$ , the method is convergent of order  $q$ .

## Observations

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- ▶ Convergence of bisection method is not monotone
  - ▶ Linear convergence at roughly rate  $\alpha = 1/2$
  - ▶ Newton converges quadratically to  $tol = 10^{-12}$  in 8 iterations, while bisection takes 41 iterations

## Issues with Newton's method

Recall before we noted that Newton's method runs into trouble if  $F'(x_i) = 0$ . It can be shown that Newton's method iterates satisfy:

$$|x^* - x_i| \leq M|x^* - x_{i-1}|^2$$

for  $M > \frac{|F''(x^*)|}{|F'(x^*)|}$ . Therefore the method converges quadratically if

$$F'(x^*) \neq 0.$$

If  $F'(x^*) = 0$  Newton's method may converge but does so slower than quadratically. Example function:  $F(x) = x^3$ .

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## Further work...

### Self study...

- ▶ Solve the beam and airfoil examples (Tutorial 1) using the bisection method
- ▶ Compare with Newton's method from different initial guesses

### Next...

### Lecture

Improvements on Newton and bisection methods