Lecture 14: Direct solvers for linear systems

COMP5930M Scientific Computation

Today

The Newton algorithm

- A (large) linear system of equations $Ax = b$ must be solved at each Newton iteration
- For n equations the classical algorithms (Gaussian elimination etc.) have $\mathcal{O}(n^3)$ expense

Direct solvers

- We define a direct solution algorithm as one which produces a solution in a fixed number of operations at fixed expense for a given problem size
- Typical direct solvers operate on the matrix A and the right-hand side b and apply algebraic operations to reduce the problem (matrix decompositions)
- Our goal is introduce direct methods to sparse systems that achieve a computational cost that is less than $\mathcal{O}(n^3)$

Standard algorithms

- Gaussian elimination
  - Often the first method covered in linear algebra
  - Forward elimination of A into upper triangular form, followed by back substitution
  - Special case: Thomas algorithm for tridiagonal systems
- LU factorisation
  - Considered a more practical approach
  - Factorisation into upper/lower triangular blocks, followed by upper and lower triangular solves
- They are fundamentally the same algorithm with a different sequence of operations

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LU factorisation

For any square matrix $A \in \mathbb{R}^{n \times n}$, we look for a decomposition:

$$A = LU,$$

where L is lower triangular and U is upper triangular $n \times n$ matrix.

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \begin{pmatrix} L_{11} & & \\ L_{21} & L_{22} & \\ L_{31} & L_{32} & L_{33} \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ & U_{22} & U_{23} \\ & & U_{33} \end{pmatrix}$$

Implementation: Doolittle algorithm

LU factorisation

If such a decomposition can be found, we proceed in two steps:

1. Solve $Lz = b$ to find z;
2. Solve $Ux = z$ to find x.

These sub-problems are solved efficiently with $\mathcal{O}(n^2)$ cost using forward/backward subsitution ($\mathcal{O}(n)$ in Thomas algorithm)

However: Cost of LU factorisation is $\mathcal{O}(n^3)$ for dense matrices.

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Sparse matrix problems

- Assumption: The matrix A is sparse
- Both Gauss elimination and LU factorisation algorithms have been extended to sparse matrices
- Large-scale, widely-used implementations for both
  - Gauss elimination: UMFPACK
  - LU factorisation: SuperLU
  - Multifrontal parallel LU factorisation: MUMPS
  - Many other variants

Improving the basic algorithms

- To improve from $\mathcal{O}(n^3)$, need to exploit sparsity of A
- Column-based sparse algorithm: storage
- Reordering
  - Fill-reduction: efficiency
  - Pivoting: numerical accuracy

Why reorder the unknowns of the problem?

- Efficiency
  - The sparse structure of a factorised matrix is determined by the sparse structure of the matrix itself
  - We can reorder the matrix to minimise the size of the factorised matrix

- Accuracy
  - The diagonal entries (or pivots) in the factorisation algorithms are critical to the accuracy
  - Round-off error is minimised if the pivot magnitude is controlled through reordering

An example with round-off problems

Given some small ϵ, solve $\mathbf{Ax} = \mathbf{b}$ to find $\mathbf{x}$

$$\begin{pmatrix} \epsilon & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 + \epsilon \\ 2 \end{pmatrix}$$

Standard Gaussian elimination

$$\begin{pmatrix} \epsilon & 1 \\ 0 & 1 - \frac{1}{\epsilon} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 + \epsilon \\ 2 - \frac{1+\epsilon}{\epsilon} \end{pmatrix}$$

Back-substitution

$$x_2 = \frac{2 - \frac{1+\epsilon}{\epsilon}}{1 - \frac{1}{\epsilon}}, \quad x_1 = \frac{1 + \epsilon - x_2}{\epsilon}$$

Round-off errors without reordering

As $\epsilon \to 0$, we observe catastrophic round-off errors:

ϵ	x_2	x_1
10^{-7}	1	1.000000000583867
10^{-8}	1	0.999999993922529
10^{-9}	1	0.999999860695766
$\vdots$	$\vdots$	$\vdots$
10^{-14}	1	0.999200722162641
10^{-15}	1	0.888178419700125
10^{-16}	1	2.220446049250313

Reorder the equations

ie. reorder the rows before elimination (row pivoting)

$$\begin{pmatrix} 1 & 1 \\ \epsilon & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1+\epsilon \end{pmatrix}$$

Standard elimination

$$\begin{pmatrix} 1 & 1 \\ 0 & 1-\epsilon \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1-\epsilon \end{pmatrix}$$

Back-substitution

$$x_2 = \frac{1-\epsilon}{1-\epsilon} = 1, \quad x_1 = 1$$

An example with fill-in problems

Solve $\mathbf{Ax} = \mathbf{b}$ to find $\mathbf{x}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & & \\ 1 & & 4 & \\ 1 & & & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

This kind of matrix is called an arrow matrix (very sparse).

Standard algorithm

LU factorisation:

$$\mathbf{A} = \mathbf{LU} = \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & -1 & 1 & \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ & 1 & -1 & -1 \\ & & 2 & -2 \\ & & & -1 \end{pmatrix}$$

Solution: $\mathbf{x} = (25, -11.5, -5.5, -7)^T$

LU factors of an arrow matrix are dense if no reordering is applied

(1) Reorder the equations

ie. reorder the rows

Equations 1,2,3,4 $\rightarrow$ Equations 4,3,2,1

$$\begin{pmatrix} 1 & & & 3 \\ 1 & & 4 & \\ 1 & 2 & & \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}$$

(2) Reorder the variables

ie. reorder the columns

Variables $x_1, x_2, x_3, x_4 \rightarrow$ variables x_4, x_3, x_2, x_1

$$\begin{pmatrix} 3 & & & 1 \\ & 4 & & 1 \\ & & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_4 \\ x_3 \\ x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}$$

Solve the reordered system

Standard LU factorisation:

$$\mathbf{LU} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 3 & & & 1 \\ & 4 & & 1 \\ & & 2 & 1 \\ & & & -\frac{1}{12} \end{pmatrix}$$

Solution: $(\mathbf{x}_4, \mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1)^T = (-7, -5.5, -11.5, 25)^T$

LU factors of reordered arrow matrix have sparsity pattern of A