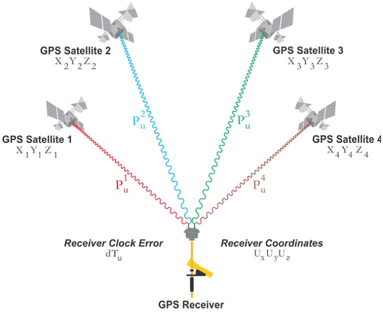
Lecture 2: Example applications

COMP5930M Scientific Computation

Today

1. Navigation by GPS

- Receiver calculates its position (x, y, z) based on signals sent by four GPS satellites and received at times t_i^R.
- Each satellite $i = 1, 2, 3, 4$ transmits messages that include the transmission time t_i^S and the satellite position (x_i, y_i, z_i) at the time of transmission.



- The clock error, $\Delta t = t_{\text{satellite}} - t_{\text{receiver}}$, is also unknown.

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- True range from satellite i:

$$d_i = \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}$$

Pseudorange from satellite i:

$$p_i = c\left(t_i^R - t_i^S\right)$$

where c is the speed of light (assuming no clock error)

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Navigation equations (four satellites)

By collecting all four equations into one system, we can in theory solve for the four unknowns $(x, y, z, \Delta t)$:

$$\begin{cases} \sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2} - c\left(t_1^R - t_1^S + \Delta t\right) = 0 \\ \sqrt{(x-x_2)^2+(y-y_2)^2+(z-z_2)^2} - c\left(t_2^R - t_2^S + \Delta t\right) = 0 \\ \sqrt{(x-x_3)^2+(y-y_3)^2+(z-z_3)^2} - c\left(t_3^R - t_3^S + \Delta t\right) = 0 \\ \sqrt{(x-x_4)^2+(y-y_4)^2+(z-z_4)^2} - c\left(t_4^R - t_4^S + \Delta t\right) = 0 \end{cases}.$$

$\Rightarrow$ nonlinear equation system $F(\vec{x}) = 0$

Possible issues in solving the navigation equations

- The equations may have multiple solutions or no solutions[1]

  ⇒ Are the solutions **unique** (up to some conditions)?

- Receiver motion introduces uncertainty to the times t_i^R

  ⇒ How **sensitive** are the solutions to the coefficients?

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2. Simple differential equation

Predator-prey -model in computational ecology:

"Assume we have an ecological system with two species: prey and predator. The prey reproduce without limits due to large quantities of food. The predator needs to eat the prey to reproduce."

Objective: describe the dynamics of the two populations

- t is our time variable
- $x(t)$ is the number of animals in the prey species
- $y(t)$ is the number of animals in the predator species
- The initial populations are $x(0) = x_0$ and $y(0) = y_0$
- The change in time of the populations are $\frac{dx}{dt}$ and $\frac{dy}{dt}$

2. Simple differential equation

Lotka-Volterra model:

$$\begin{aligned} \frac{dx}{dt} &= \alpha x(t) - \beta x(t)y(t) \\ \frac{dy}{dt} &= \delta x(t)y(t) - \gamma y(t) \end{aligned} \tag{1}$$

where $\alpha, \beta, \gamma, \delta$ are positive constants

2. Simple differential equation

Interpretation of terms in the equation:

- $\alpha x(t)$ is the reproduction rate of new prey species

  Since $\frac{dx}{dt} = \alpha x(t)$ has solution $x(t) = C \exp(\alpha t)$, the prey population grows exponentially when there are no predators

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  Predators will die off exponentially if there is no prey

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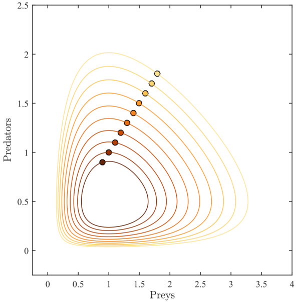
In many cases, the solution can be shown to approach a periodic orbit where the number of predators/prey behave cyclically:



Phase-space plot in the (x, y) space tracks the solution from a given initial solution (x_0, y_0)

Numerical approximation of differential equations

In practice, differential equations are often **discretised** in time on a numerical grid with discrete time step $\Delta t = t_j - t_{j-1}$:

$$X_j = x(t_j) \text{ for } t_0 < t_1 < \ldots < t_N$$

by applying difference approximations for the derivatives, e.g.

$$\frac{\partial x}{\partial t}(t_j) \approx \frac{X_j - X_{j-1}}{\Delta t}.$$

This leads to a system $F(\vec{X}) = 0$ of algebraic nonlinear equations for the X_j to be solved at each time step $t_{j-1} \to t_j$.

Return to these techniques later in the module...

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Nonlinear equations for continuous Lotka-Volterra model:

$$\begin{aligned} \frac{dx}{dt} &= \alpha x(t) - \beta x(t) y(t) \\ \frac{dy}{dt} &= \delta x(t) y(t) - \gamma y(t) \end{aligned} \tag{2}$$

2. Simple differential equation

Nonlinear equations for discrete Lotka-Volterra model:

$$\frac{X_j - X_{j-1}}{\Delta t} = \alpha X_j - \beta X_j Y_j$$
$$\frac{Y_j - Y_{j-1}}{\Delta t} = \delta X_j Y_j - \gamma Y_j \qquad (3)$$

for $j = 1, 2, \ldots, N$

Replace derivatives with discrete difference approximations

2. Simple differential equation

Nonlinear equations for discrete Lotka-Volterra model:

$$
\begin{aligned}
X_j - X_{j-1} &= \Delta t \left[\alpha X_j - \beta X_j Y_j\right] \\
Y_j - Y_{j-1} &= \Delta t \left[\delta X_j Y_j - \gamma Y_j\right]
\end{aligned}
\tag{4}
$$

for $j = 1, 2, \ldots, N$

Multiply both sides by Δt

2. Simple differential equation

Nonlinear equations for discrete Lotka-Volterra model:

$$\begin{aligned} X_j - X_{j-1} - \Delta t \left[\alpha X_j - \beta X_j Y_j\right] &= 0 \\ Y_j - Y_{j-1} - \Delta t \left[\delta X_j Y_j - \gamma Y_j\right] &= 0 \end{aligned} \tag{5}$$

for $j = 1, 2, \ldots, N$

Move all terms to the left-hand side

2. Simple differential equation

Nonlinear equations for discrete Lotka-Volterra model:

$$
\begin{aligned}
F_1(X_j, Y_j) &= X_j - X_{j-1} - \Delta t \left[\alpha X_j - \beta X_j Y_j\right] = 0 \\
F_2(X_j, Y_j) &= Y_j - Y_{j-1} - \Delta t \left[\delta X_j Y_j - \gamma Y_j\right] = 0
\end{aligned}
\tag{6}
$$

for $j = 1, 2, \ldots, N$

Identify the non-linear equations

Numerical approximation of differential equations

- Depending on application, number of equations can be large

  $\Rightarrow$ Do our algorithms **scale** for large problems?

- Does the approximate solution U approach u as $N \to \infty$?

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