

Lecture 12: Nonlinear PDEs in 2d and sparsity

COMP5930M Scientific Computation

Today

Model problem

Numerical solution

Approximation in space

Nonlinear system

Sparse matrix storage

Summary

Next

2d nonlinear diffusion

Find $u(x, y, t)$ satisfying

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(c(u) \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(c(u) \frac{\partial u}{\partial y} \right) + g(u, x, y)$$

- ▶ The PDE is defined on spatial region $(x, y) \in \Omega \subset \mathbb{R}^2$
- ▶ Initial condition $u(x, y, 0)$ needs to be given
- ▶ The solution $u(x, y, t)$ is known on the boundary $\partial\Omega$
- ▶ $g(u, x, y)$ is the **source function** that can depend on u but not its derivatives

Application in image processing

Let $u_0(x, y)$ denote the intensity of a noisy grayscale image at pixel (x, y)

Perona-Malik equation: find $u(x, y, t)$ s.t.

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial}{\partial x} \left(c(u) \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(c(u) \frac{\partial u}{\partial y} \right), & t > 0 \\ u(x, y, 0) &= u_0(x, y), & t = 0.\end{aligned}$$

where $c(u) = \frac{1}{1 + \left(\frac{\|\nabla u\|}{K} \right)^2}$ is an anisotropic diffusion term

Removes noise from images without smearing boundaries:

https://www.youtube.com/watch?v=J5mZD40V_VU

Numerical approximation

We consider only spatial discretisation here

- ▶ Approximate in space
 - ▶ Fully discrete nonlinear system
- ▶ Solution algorithm
 - ▶ Numerical solution with Newton's Method

Spatial approximation

- ▶ Assume our domain Ω is rectangular
- ▶ Define an $n \times m$ uniform grid with spacing

$$\Delta x = \frac{X_2 - X_1}{n - 1}, \quad \Delta y = \frac{Y_2 - Y_1}{m - 1}$$

- ▶ A point (node) p_{ij} has coordinates (x_i, y_j) where

$$x_i = X_1 + (i - 1)\Delta x, \quad y_j = Y_1 + (j - 1)\Delta y$$

Solution data mapping

- ▶ Our discrete problem has $(n - 2)(m - 2) = N$ unknowns (assuming boundary data is known)
- ▶ ie. our nonlinear system has N equations
- ▶ Our solution algorithm stores a **vector** \mathbf{U} but our discrete data is a **matrix** u_{ij}
- ▶ We require a unique mapping between these two representations

Numbering convention

- ▶ We can define a row-based numbering system as

$$U_k \equiv u_{ij}, \quad k = (j-2)(n-2) + (i-1) \\ i = 2, \dots, n-1, \quad j = 2, \dots, m-1$$

- ▶ Could alternatively use column-based numbering

A compact FDM approximation

$$\begin{aligned}
 \frac{\partial}{\partial x} \left(c(u) \frac{\partial u}{\partial x} \right) &\approx \frac{(c(u) \frac{\partial u}{\partial x})_{i+\frac{1}{2}} - (c(u) \frac{\partial u}{\partial x})_{i-\frac{1}{2}}}{\Delta x} \\
 &\approx \frac{c(u_{i+\frac{1}{2}}) \left(\frac{u_{i+1} - u_i}{\Delta x} \right) - c(u_{i-\frac{1}{2}}) \left(\frac{u_i - u_{i-1}}{\Delta x} \right)}{\Delta x} \\
 &= \frac{c(u_{i+\frac{1}{2}}) (u_{i+1} - u_i) - c(u_{i-\frac{1}{2}}) (u_i - u_{i-1})}{\Delta x^2}
 \end{aligned}$$

- Requires only u_{i-1}, u_i, u_{i+1}
- Reduces to the second order derivative for constant $c(u)$

Computing $c(u_{i+\frac{1}{2}})$

Nonlinear $c(u)$ leads to two obvious alternatives

$$c(u_{i+\frac{1}{2}}) = \frac{c(u_i) + c(u_{i+1})}{2}$$

$$c(u_{i+\frac{1}{2}}) = c\left(\frac{u_i + u_{i+1}}{2}\right)$$

- ▶ Formally of the same accuracy
- ▶ Will produce different nonlinear equations and Jacobian matrix

Semi-discrete 2d system

$$\begin{aligned}
 \dot{u}_i = & \frac{c(u_{i+\frac{1}{2}j})(u_{i+1j} - u_{ij}) - c(u_{i-\frac{1}{2}j})(u_{ij} - u_{i-1j})}{\Delta x^2} \\
 & + \frac{c(u_{ij+\frac{1}{2}})(u_{ij+1} - u_{ij}) - c(u_{ij-\frac{1}{2}})(u_{ij} - u_{ij-1})}{\Delta y^2} \\
 & + g(u_{ij}, x_i, y_j)
 \end{aligned}$$

- Sparse
- Only 5 values required for each equation
- But not pentadiagonal

Sparse storage schemes

- ▶ We require compact storage schemes for large sparse matrices
 - ▶ Only store non-zero values
- ▶ Our algorithms will have to work with data stored in this format
- ▶ Common alternatives:
 - ▶ Coordinate format
 - ▶ Row-major or column-major format

Coordinate format

- ▶ For every non-zero item store indices i, j and value a_{ij}
- ▶ Requires 3 vectors for storage of size nz
- ▶ No ordering implied

Row-major format

- ▶ For every row i of the matrix store:
 - ▶ The column index j and value a_{ij}
 - ▶ The location l_i in that list of the start of each row
- ▶ Requires 2 vectors of size nz and 1 of size N
- ▶ Ordering is implied (at the row level)

Example of sparse row-major format

Sparse Matrix

10	0	0	0	-2
3	9	0	0	0
0	7	8	7	0
3	0	8	7	5
0	8	0	9	13

Row pointer array

0	2	4	7	11	14
---	---	---	---	----	----

Column indices array

0	4	0	1	1	2	3	0	2	3	4	1	3	4
---	---	---	---	---	---	---	---	---	---	---	---	---	---

Values array

10	-2	3	9	7	8	7	3	8	7	5	8	9	13
----	----	---	---	---	---	---	---	---	---	---	---	---	----

A sparse matrix and its corresponding CSR row pointer, column indices and values arrays

Column-major format (MATLAB)

- ▶ For every column j of the matrix store:
 - ▶ The row index i and value a_{ij}
 - ▶ The location l_j in that list of the start of each column
- ▶ Requires 2 vectors of size nz and 1 of size N
- ▶ Ordering is implied (at the column level)

Unrolling matrices into vectors in MATLAB

```
>> A=[1 2 3; 4 5 6; 7 8 9]
```

```
A =
```

1	2	3
4	5	6
7	8	9

```
>> A(:)'
```

```
ans =
```

1	4	7	2	5	8	3	6	9
---	---	---	---	---	---	---	---	---

Summary

- ▶ Discretisation of 2d problems leads to sparse Jacobian...
- ▶ ...but bandwidth in general larger than tridiagonal
- ▶ Sparse matrices can be stored efficiently using $2 \times nnz$ vectors (row- or column-major format)
- ▶ Problem is how to find out the sparsity pattern of the Jacobian (more details in tutorial)