Tutorial 2: Bisection method and rate of convergence

COMP5930M Scientific Computation

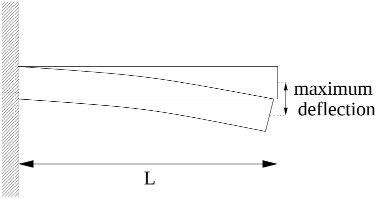
Today

Further examples

Bisection method

Example: Cantilever beam

Consider a cantilever beam of length L which bends under its own weight (a model civil engineering problem).



The beam's maximum deflection δ_{max} is at the unsupported end, and the deflection δ at a point αL $(0 \leq \alpha \leq 1)$ along the beam is given by the nonlinear equation $f(\alpha) = 0$ where

$$f(\alpha) = \alpha^4 - 4\alpha^3 + 6\alpha^2 - 3\frac{\delta}{\delta_{\text{max}}}$$

Find the point αL where the beam deflection δ is $0.6\,\delta_{\text{max}}$

Cantilever beam example

cantilever.m

```
function y = cantilever(x,def)
y = x^4 - 4*x^3 + 6*x^2 - 3*def;
```

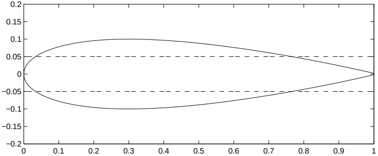
- We know $\alpha \in [0, 1]$ by definition
  which sets a range for the initial guess
- For `def=0.6`, `fzero()` converges to a root at 0.6979
  from any point in $[0, 1]$ or with bracket $[0, 1]$
- Note that basic Newton will fail if $x_0 = 0$
  due to zero derivative at that point.

Example: Airfoil modelling

Consider the NACA0012 prototype wing section, which is often used for testing computational methods for simulating flows in aerodynamics: see, for example,
`http://www2.icfd.co.jp/examples/naca0012_2d/na3.htm`
Its profile is given by

$$y(x) = \pm(0.2969\sqrt{x} - 0.126x - 0.3516x^2 + 0.2843x^3 - 0.1015x^4)$$

in which $+$ gives the upper surface and $-$ the lower surface.



Find the point x at which the thickness t of the aerofoil is 0.1

Airfoil example

naca0012.m

```
function f = naca0012(x,th)
y = -0.1015*x^4 + 0.2843*x^3 - 0.3516*x^2
         - 0.126*x + 0.2969*sqrt(x);
f = 2*y - th;
```

- We know $x \in [0, 1]$ by definition
  which sets a range for the initial guess
- But, there are 2 roots and also regions of small derivative
  which will cause problems for the basic algorithms
- `fzero()` converges to 0.0339 from $x_0 = 0$
  and to 0.7652 from $x_0 = 1$
- Easier to use a single start point in this case

Bisection method

bisection.m (basic algorithm):

```
% Bisection method for zero finding
%
% function [ x,f ] = bisection( fnon, xL, xR, tol, maxit )
%
% Input:
%     fnon - function handle for nonlinear equation
%     xL - left endpoint of initial bracket
%     xR - left endpoint of initial bracket
%     tol - convergence tolerance
%     maxIt - maximum allowed number of iterations
%
% Output:
%     x - final point
%     f - final function value
```

Bisection method

```
fL = feval(fnon,xL);
fR = feval(fnon,xR);

% Test whether the sign of f changes within the bracket
if(fL * fR > 0)
    error('Sign change within interval not guaranteed!')
end

k=0;
fprintf('   i   x_i          |F(x_i)|\n')
```

Bisection method

```
while ((xR - xL) > tol && k < maxit)
    xC = (xL + xR) / 2;
    fC = feval(fnon,xC);
    fprintf(' %3d   %1.8f   %1.2e\n',k,xC,abs(fC));

    if (fL * fC < 0)
        xR = xC;
    else
        xL = xC;
    end
    k=k+1;
    x(k) = xC;
    f(k) = feval(fnon,xC);
end
xC = (xL + xR) / 2;
x(k) = xC;
f(k) = feval(fnon,xC);
```

Example

Test the bisection method on the same function as before:

$$F(x) = x^2 - 2.$$

We can plot the convergence of the error $|x^* - x_i|$:

```
% Plot the convergence history
semilogy(abs(sqrt(2) - x),'o-','LineWidth',2)
xlabel('Iteration i')
ylabel('Error |x^* - x_i|')
```

Comparison with Newton's method

- To estimate the convergence order of a method, we compute

$$\alpha_i := \frac{|x^* - x_i|}{|x^* - x_{i-1}|^q}$$

numerically for different $q = 1, 2, \ldots$

- If $\lim_{i\to\infty} \alpha_i = \alpha > 0$, the method is convergent of order q.

Observations

- Convergence of bisection method is not monotone
- Linear convergence at roughly rate $\alpha = 1/2$
- Newton converges quadratically to $tol = 10^{-12}$ in 8 iterations, while bisection takes 41 iterations

Issues with Newton's method

Recall before we noted that Newton's method runs into trouble if $F'(x_i) = 0$. It can be shown that Newton's method iterates satisfy:

$$|x^* - x_i| \leq M|x^* - x_{i-1}|^2$$

for $M > \frac{|F''(x^*)|}{|F'(x^*)|}$. Therefore the method converges quadratically if

$$F'(x^*) \neq 0.$$

If $F'(x^*) = 0$ Newton's method may converge but does so slower than quadratically. Example function: $F(x) = x^3$.

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Further work...

Self study...

- Solve the beam and airfoil examples (Tutorial 1) using the bisection method
- Compare with Newton's method from different initial guesses

Next...

Lecture
Improvements on Newton and bisection methods