Lecture 13: Initial guess for steady-state PDEs

COMP5930M Scientific Computation

Today

Time-dependent PDE models

Steady-state PDE models

Algorithms
- Pseudo-timestepping
- Nested iteration
- Ad-hoc continuation

Summary

Time-dependent PDE models

- Always specified with appropriate initial conditions
- On each step $t^k \to t^{k+1}$, we can use the current solution $\mathbf{U}^k$ as an initial guess to Newton
- For large time step sizes we may still face problems
- We can use forward Euler solution as an initial guess: $\mathbf{U}_0 = \mathbf{U}^k + \Delta t \mathbf{F}(\mathbf{U}^k)$
- In practice we may have to restrict Δt to ensure convergence

Steady PDE models

- No initial conditions, only boundary data
- Often difficult to *guess* an appropriate initial state
- We can employ some standard techniques to help

1. Pseudo-timestepping

Modify the nonlinear system $\mathbf{F}(\mathbf{U}) = \mathbf{0}$ to

$$\frac{\partial \mathbf{U}}{\partial \tau} + \mathbf{F}(\mathbf{U}) = \mathbf{0}$$

- τ is a *pseudo*-time variable - not physical time
- The steady-state solution satisfies the original nonlinear system
- Use standard time-stepping techniques to evolve to steady state from the initial state

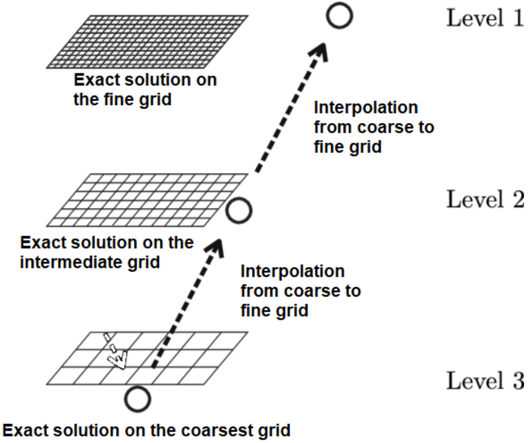
Pros/Cons

- Simple heuristic approach
  - use standard time-stepping algorithm
- Still requires some (pseudo)-initial conditions
  - should be less sensitive to a poor guess
- There will be some dependence on the step size $\Delta\tau$
  - as for real time stepping

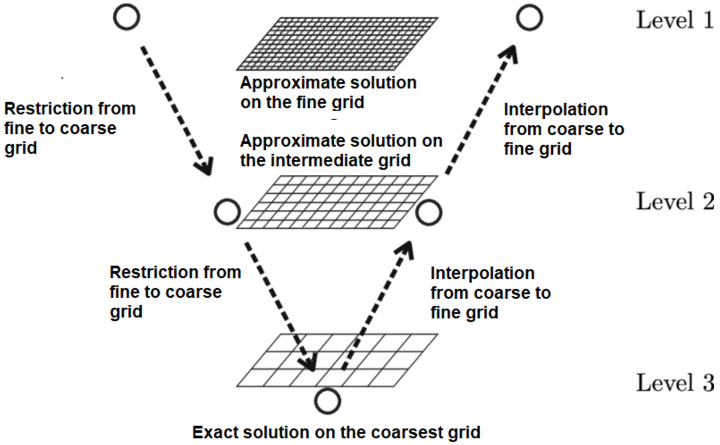
2. Nested iteration

- Convergence for PDE models is often faster (easier) for smaller problems
  - fewer grid points, ie. smaller n
- Solve a sequence of problems on successively finer grids
  - the coarse mesh solution is interpolated to the finer level as an initial guess

Example: Nested iteration method



Example: Multigrid method



Pros/Cons

- As we move to finer levels we should have an almost perfect initial state
  - Fast convergence
- Requires us to be able to solve from a poor guess at some coarse level
  - May not be possible
- In general will require experimentation

3. Continuation approaches

(less formal than Homotopy Continuation)

- Define a transformation, from a simple, easy-to-solve state to our desired nonlinear model
- Mathematically, we use a continuation parameter α to define the transformation
- Solve a sequence of nonlinear problems with the previous solution used as initial data for the next

A simple example

Find x such that

$$F(x) \; = \; -x^3 - 2x + 2$$

- For $\alpha \in [0, 1]$, define

$$G(\alpha, x) \; = \; -\alpha \, x^3 - 2x + 2$$

- For $\alpha = 0$, $G(0, x) = -2x + 2$
  we have a root at $x = 1$

- Define k steps from $\alpha = 0$ to $\alpha = 1$

A PDE example

Find $u(x)$ on $x \in [0, 1]$ that satisfies

$$u\frac{\partial u}{\partial x} - \epsilon\frac{\partial^2 u}{\partial x^2} \;=\; 0$$

with boundary conditions $u(0) = 1$ and $u(1) = 0$

- For $\alpha \in [0, 1]$, define a modified PDE
  with the same domain and boundary conditions

$$((1-\alpha) + \alpha\; u)\frac{\partial u}{\partial x} - \epsilon\frac{\partial^2 u}{\partial x^2} \;=\; 0$$

- $\alpha = 0$ is a linear PDE and hence the FDM
  leads to a linear system of equations

Pros/Cons

- We can start from an easily solvable state
  - generally linear
- Requires some mathematical *intuition*
  to define a workable sequence
  - focus on the nonlinear part of the problem
- We assume the *path* taken is well-defined
  - ie. that the sub-problems have a solution
- There may be some dependence on the step size $\Delta\alpha$
  - similar to pseudo-time stepping

Summary

- There are a variety of methods available
  if a good initial state is not known

- Knowledge of the context, or physics, of the model
  is a very useful starting point

- Formal, mathematical techniques are available
  as a last resort