Lecture 13: Initial guess for steady-state PDEs

COMP5930M Scientific Computation

Today

Time-dependent PDE models

Steady-state PDE models

Algorithms
Pseudo-timestepping
Nested iteration
Ad-hoc continuation

Summary

Time-dependent PDE models

- ▶ Always specified with appropriate initial conditions

- ▶ On each step $t^k \to t^{k+1}$, we can use the current solution $\mathbf{U}^k$ as an initial guess to Newton

- ▶ For large time step sizes we may still face problems

- ▶ We can use forward Euler solution as an initial guess:
  $\mathbf{U}_0 = \mathbf{U}^k + \Delta t \mathbf{F}(\mathbf{U}^k)$

- ▶ In practice we may have to restrict Δt to ensure convergence

Steady PDE models

- No initial conditions, only boundary data

- Often difficult to *guess* an appropriate initial state

- We can employ some standard techniques to help

1. Pseudo-timestepping

Modify the nonlinear system $\mathbf{F}(\mathbf{U}) = \mathbf{0}$ to

$$\frac{\partial \mathbf{U}}{\partial \tau} + \mathbf{F}(\mathbf{U}) = \mathbf{0}$$

- τ is a *pseudo*-time variable - not physical time

- The steady-state solution satisfies
  the original nonlinear system

- Use standard time-stepping techniques to
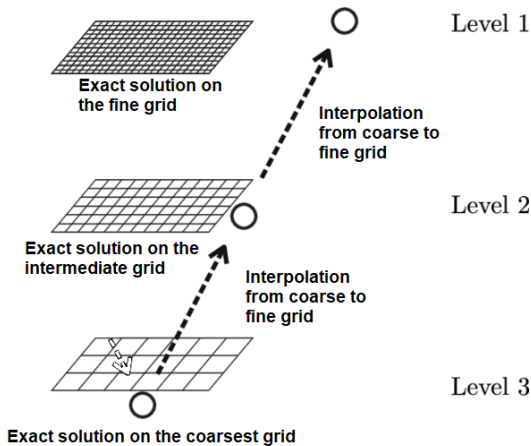 evolve to steady state from the initial state

Pros/Cons

- Simple heuristic approach
  - use standard time-stepping algorithm

- Still requires some (pseudo)-initial conditions
  - should be less sensitive to a poor guess

- There will be some dependence on the step size $\Delta\tau$
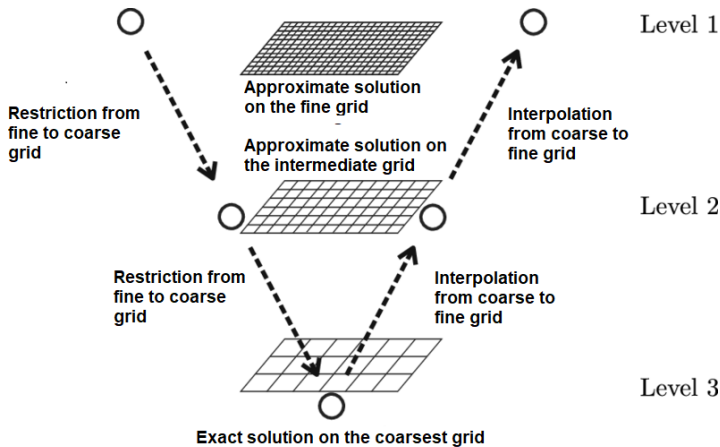 - as for real time stepping

2. Nested iteration

- Convergence for PDE models is often faster (easier)
  for smaller problems
    - fewer grid points, ie. smaller n

- Solve a sequence of problems on successively finer grids
    - the coarse mesh solution is interpolated
      to the finer level as an initial guess

Example: Nested iteration method

Exact solution on
the fine grid
Level 1

Interpolation
from coarse to
fine grid

Exact solution on the
intermediate grid
Level 2

Interpolation
from coarse to
fine grid

Exact solution on the coarsest grid
Level 3

Example: <mark>Multigrid method</mark>

Level 1

Approximate solution on the fine grid

Restriction from fine to coarse grid

Interpolation from coarse to fine grid

Approximate solution on the intermediate grid

Level 2

Restriction from fine to coarse grid

Interpolation from coarse to fine grid

Level 3

Exact solution on the coarsest grid

Pros/Cons

- As we move to finer levels we should have an almost perfect initial state
  - Fast convergence

- Requires us to be able to solve from a poor guess at some coarse level
  - May not be possible

- In general will require experimentation

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  └─ Ad-hoc continuation

3. Continuation approaches

(less formal than Homotopy Continuation)

- ▶ Define a transformation, from a simple, easy-to-solve state to our desired nonlinear model

- ▶ Mathematically, we use a continuation parameter α to define the transformation

- ▶ Solve a sequence of nonlinear problems with the previous solution used as initial data for the next

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A simple example

Find x such that

$$F(x) \;=\; -x^3 - 2x + 2$$

▶ For $\alpha \in [0, 1]$, define

$$G(\alpha, x) \;=\; -\alpha\, x^3 - 2x + 2$$

▶ For $\alpha = 0$, $G(0, x) = -2x + 2$
  we have a root at $x = 1$

▶ Define k steps from $\alpha = 0$ to $\alpha = 1$

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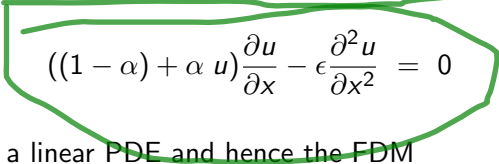└─ Algorithms
  └─ Ad-hoc continuation

A PDE example

Find $u(x)$ on $x \in [0, 1]$ that satisfies

$$u\frac{\partial u}{\partial x} - \epsilon\frac{\partial^2 u}{\partial x^2} \;=\; 0$$

with boundary conditions $u(0) = 1$ and $u(1) = 0$

▶ For $\alpha \in [0, 1]$, define a modified PDE
  with the same domain and boundary conditions

$$((1 - \alpha) + \alpha\, u)\frac{\partial u}{\partial x} - \epsilon\frac{\partial^2 u}{\partial x^2} \;=\; 0$$

▶ $\alpha = 0$ is a linear PDE and hence the FDM
  leads to a linear system of equations

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Pros/Cons

- We can start from an easily solvable state
  - generally linear

- Requires some mathematical *intuition*
  to define a workable sequence
  - focus on the nonlinear part of the problem

- We assume the *path* taken is well-defined
  - ie. that the sub-problems have a solution

- There may be some dependence on the step size $\Delta\alpha$
  - similar to pseudo-time stepping

Summary

- There are a variety of methods available
  if a good initial state is not known

- Knowledge of the context, or physics, of the model
  is a very useful starting point

- Formal, mathematical techniques are available
  as a last resort