An illustrated primer

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One of the best things about modern cryptography is the beautiful terminology. You could start any number of punk bands (or [Tumblrs](#)) named after cryptography terms like ‘hard-core predicate’, ‘trapdoor function’, ‘ or ‘impossible differential cryptanalysis’. And of course, I haven’t even mentioned the one term that surpasses all of these. That term is ‘*zero knowledge*‘.

In fact, the term ‘zero knowledge’ is so appealing that it leads to problems. People misuse it, assuming that zero knowledge must be synonymous with '*really, really secure*'. Hence it gets tacked onto all kinds of stuff — like encryption systems and anonymity networks — that really have nothing to do with true zero knowledge protocols.

This all serves to underscore a point: zero-knowledge proofs are one of the most powerful tools cryptographers have ever devised. But unfortunately they're also relatively poorly understood. In this series of posts I'm going try to give a (mostly) *non–mathematical* description of what ZK proofs are, and what makes them so special. In this post and the next I'll talk about some of the ZK protocols we actually use.

Origins of Zero Knowledge

The notion of 'zero knowledge' was first proposed in the 1980s by MIT researchers Shafi Goldwasser, Silvio Micali and Charles Rackoff. These researchers were working on problems related to interactive proof systems, theoretical systems where a first party (called a 'Prover') exchanges messages with a second party ('Verifier') to convince the Verifier that some mathematical statement is true.*

Prior to Goldwasser *et al.*, most work in this area focused the soundness of the proof system. That is, it considered the case where a malicious Prover attempts to 'trick' a Verifier into believing a false statement. What

Goldwasser, Micali and Rackoff did was to turn this problem on its head. Instead of worrying only about the Prover, they asked: what happens if you don't trust the *Verifier?*

The specific concern they raised was *information leakage.* Concretely, they asked, how much extra information is the Verifier going to learn during the course of this proof, beyond the mere fact that the statement is true?

It's important to note that this is not simply of theoretical interest. There are real, practical applications where this kind of thing matters.

Here's one: imagine that a real-world client wishes to log into a web server using a password. The standard 'real world' approach to this problem involves storing a [hashed version of the password] on the server. The login can thus be viewed as a sort of 'proof' that a given password hash is the output of a hash function on some password — and more to the point, that the client actually *knows* the password.

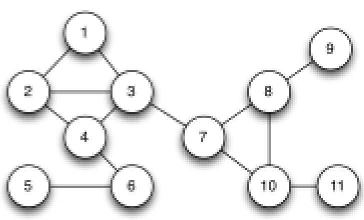
Most real systems implement this 'proof' in the absolute worst possible way. The client simply transmits the original password to the server, which re-computes the password hash and compares it to the stored value. The problem here is obvious: at the conclusion of the protocol, *the server has learned my cleartext password.* Modern password hygiene therefore involves a good deal of praying that servers aren't compromised.

What Goldwasser, Micali and Rackoff proposed was a new hope for conducting such proofs. If fully realized, zero knowledge proofs would allow us to prove statements like the one above, while provably revealing *no information* beyond the single bit of information corresponding to 'this statement is true'.

A 'real world' example

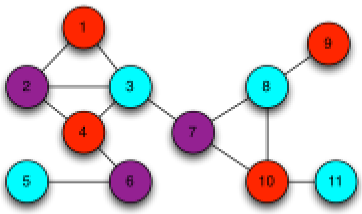
So far this discussion has been pretty abstract. To make things a bit more concrete, let's go ahead and give a 'real' example of a (slightly insane) zero knowledge protocol.

For the purposes of this example, I'd like you to imagine that I'm a telecom magnate in the process of deploying a new cellular communications network. My network structure is represented by the graph below. Each vertex in this graph represents a cellular radio tower, and the connecting lines (edges) indicate locations where two cells *overlap*, meaning that their transmissions are likely to interfere with each other.



This overlap is problematic, since it means that signals from adjacent towers are likely to scramble reception. Fortunately my network design allows me to configure each tower to one of three different frequency bands to avoid such interference.

Thus the challenge in deploying my network is to assign frequency bands to the towers such that no two overlapping cells share the same frequencies. If we use colors to represent the frequency bands, we can quickly work out one solution to the problem:



Of course, many of you will notice that what I'm describing here is simply an instance of the famous theory problem called the graph three-coloring problem. You might also know that what makes this problem interesting is that, for some graphs, it can be quite hard to find a solution, or even to determine *if* a solution exists. In fact, graph three-coloring — specifically, the decision problem of whether a given graph supports a solution with three colors — is known to be in the complexity class NP-complete.

It goes without saying that the toy example above is easy to solve by hand. But what if it wasn't? For example, imagine that my cellular network was very large and complex, so much so that the computing power at my disposal was not sufficient to find a solution. In this instance, it would be desirable to *outsource* the problem to someone else who has plenty of computing power. For example, I might hire my friends at Google to solve it for me on spec.

But this leads to a problem.

Suppose that Google devotes a large percentage of their computing infrastructure to searching for a valid coloring for my graph. I'm certainly not going to pay them until I know that they really have such a coloring. At the same time, Google isn't going to give me a copy of their solution until I've paid up. We'll wind up at an impasse.

In real life there's probably a common-sense answer to this dilemma, one that involves lawyers and escrow accounts. But this is not a blog about real life, it's a blog about cryptography. And if you've ever read a crypto paper, you'll understand that the right way to solve this problem is *to dream up an absolutely crazy technical solution*.

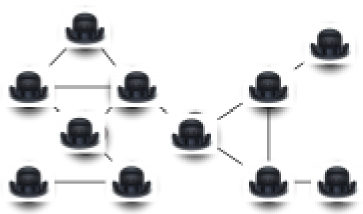
A crazy technical solution (with hats!)

The engineers at Google consult with Silvio Micali at MIT, who in consultation with his colleagues Oded Goldreich and Avi Wigderson, comes up with the following clever protocol — one so elegant that it doesn't even require any computers. All it requires is a large warehouse, lots of crayons, and plenty of paper. Oh yes, and a whole bunch of hats.**

Here's how it works.

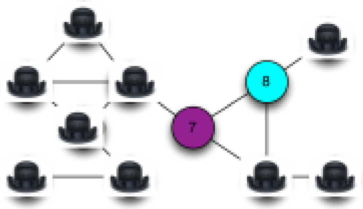
First I will enter the warehouse, cover the floor with paper, and draw a blank representation of my cell network graph. Then I'll exit the warehouse. Google can now enter enter, shuffle a collection of three crayons to *pick a random assignment of the three agreed-upon crayon colors* (red/blue/purple, as in the example above), and color in the graph in with their solution. Note that it doesn't matter which specific crayons they use, only that the coloring is valid.

Before leaving the warehouse, Google covers up each of the vertices with a hat. When I come back in, this is what I'll see:



Obviously this approach protects Google's secret coloring perfectly. But it doesn't help me at all. For all I know, Google might have filled in the graph with a random, invalid solution. They might not even have colored the graph at all.

To address my valid concerns, Google now gives me an opportunity to 'challenge' their solution to the graph coloring. I'm allowed to pick — at random — a single 'edge' of this graph (that is, one line between two adjacent hats). Google will then remove the two corresponding hats, revealing a small portion of their solution:



Notice that there are two outcomes to my experiment:

1. If the two revealed vertices are the same color (or aren't colored in at all!) then I definitely know that Google is lying to me. Clearly I'm not going to pay Google a cent.
2. If the two revealed vertices are different colors, then Google *might not* be lying to me.

Hopefully the first proposition is obvious. The second one requires a bit more consideration. The problem is that *even after our experiment*, Google could still be lying to me — after all, I only looked under two of the hats. If there are *E* different edges in the graph, then Google could fill in an invalid solution and still get away with it most of the time. Specifically, after one test they could succeed in cheating me with probability up to (E-1)/E (which for a 1,000 edge graph works out to 99.9% of the time).

Fortunately Google has an answer to this. We'll just run the protocol *again!*

We put down fresh paper with a new, blank copy of the graph. *Google now picks a new (random) shuffle of the three crayons*. Next they fill in the graph with a valid solution, but using the new random ordering of the three colors.

The hats go back on. I come back in and repeat the challenge process, picking a new random edge. Once again the logic above applies. Only this time if all goes well, I should now be slightly more confident that Google is telling me the truth. That's because in order to cheat me, Google would have had to get lucky twice in a row. That can happen — but it happens with relatively lower probability. The chance that Google fools me twice in a row is now $(E\text{-}1)/E * (E\text{-}1)/E$ (or about 99.8% probability for our 1,000 edge example above).

Fortunately we don't have to stop at two challenges. In fact, we can keep trying this over and over again until I'm confident that Google is probably telling me the truth.

But don't take my word for it. Thanks to some neat Javascript, you can [go try it yourself].

Note that I'll never be perfectly certain that Google is being honest — there's always going to be a tiny probability that they're cheating me. But after a large number of iterations (*E^2,* as it happens) I can eventually raise my confidence to the point where Google can only cheat me with negligible probability — low enough that *for all practical purposes* it's not worth worrying about. And then I'll be able to safely hand Google my money.

What you need to believe is that Google is also protected. Even if I try to learn something about their solution by keeping notes between protocol runs, it shouldn't matter. I'm foiled by Google's decision to *randomize* their color choices between each iteration. The limited information I obtain does me no good, and there's no way for me to *link* the data I learn between interactions.

What makes it 'zero knowledge'?

I've claimed to you that this protocol leaks no information about Google's solution. But don't let me get away with this! The first rule of modern cryptography is *never to trust people* who claim such things without proof.

Goldwasser, Micali and Rackoff proposed three following properties that every zero-knowledge protocol must satisfy. Stated informally, they are:

1. *Completeness.* If Google is telling the truth, then they will eventually convince me (at least with high probability).
2. *Soundness.* Google can *only* convince me *if* they're actually telling the truth.

3. *Zero-knowledgeness.* (Yes it's really called this.) I don't learn anything else about Google's solution.

We've already discussed the argument for completeness. The protocol will eventually convince me (with a negligible error probability), provided we run it enough times. Soundness is also pretty easy to show here. If Google ever tries to cheat me, I will detect their treachery with overwhelming probability.

The hard part here is the 'zero knowledgeness' property. To do this, we need to conduct a very strange thought experiment.

A thought experiment (with time machines)

First, let's start with a crazy hypothetical. Imagine that Google's engineers aren't quite as capable as people make them out to be. They work on this problem for weeks and weeks, *but they never manage to come up with a solution*. With twelve hours to go until showtime, the Googlers get desperate. They decide to *trick* me into thinking they have a coloring for the graph, even though they don't.

Their idea is to sneak into the GoogleX workshop and borrow Google's prototype time machine. Initially the plan is to travel backwards a few years and use the extra working time to take another crack at solving the problem. Unfortunately it turns out that, like most Google prototypes, the time machine has some limitations. Most critically: it's only capable of going backwards in time *four and a half minutes.*

So using the time machine to manufacture more working time is out. But still, it turns out that even this very limited technology can still be used to trick me.

I don't really know what's going on here but it seemed apropos.

The plan is diabolically simple. Since Google *doesn't actually know* a valid coloring for the graph, they'll simply color the paper with a bunch of random colors, then put the hats on. If by sheer luck, I challenge them on a pair of vertices that happen to be different colors, everyone will heave a sigh of relief and we'll continue with the protocol. So far so good.

Inevitably, though, I'm going to pull off a pair of hats and discover two vertices of the *same* color. In the normal protocol, Google would now be totally busted. And this is where the time machine comes in. Whenever Google finds themselves in this awkward situation, they simply fix it. That is, a designated Googler pulls a switch, 'rewinds' time about four minutes, and the Google team recolors the graph with a completely new random solution. Now they let time roll forward and try again.

In effect, the time machine allows Google to 'repair' any accidents that happen during their bogus protocol execution, which makes the experience look totally legitimate to me. Since bad challenge results will occur only 1/3 of the time, the expected runtime of the protocol (from Google's perspective) is only moderately greater than the time it takes to run the honest protocol. From my perspective I don't even know that the extra time machine trips are happening.

This last point is the most important. In fact, from my perspective, being unaware that the time machine is in the picture, the resulting interaction *is exactly the same as the real thing.* It's statistically identical. And yet it's worth pointing out again that in the time machine version, *Google has absolutely no information about how to color the graph.*

What the hell is the point of this?

What we've just shown is an example of a *simulation*. Note that in a world where time runs only forward and nobody can trick me with a time machine, the hat-based protocol is correct and *sound*, meaning that after *E^2* rounds I should be convinced (with all but negligible probability) that the graph really is colorable and that Google is putting valid inputs into the protocol.

What we've just shown is that if time doesn't run only forward — specifically, if Google can 'rewind' my view of time — then they can fake a valid protocol run *even if they have no information at all about the actual graph coloring.*

From my perspective, what's the difference between the two protocol transcripts? When we consider the statistical distribution of the two, there's no difference at all. Both convey exactly the same amount of useful information.

Believe it or not, this proves something very important.

Specifically, assume that I (the Verifier) have some strategy that 'extracts' useful information about Google's coloring after observing an execution of the honest protocol. Then my strategy should work equally well in the case where I'm being fooled with a time machine. The protocol runs are, from my perspective, statistically identical. I physically cannot tell the difference.

Thus if the amount of information I can extract is identical in the 'real experiment' and the 'time machine experiment', yet the amount of information Google puts into the 'time machine' experiment is exactly zero — then this implies that even in the real world the protocol must not leak any useful information.

Thus it remains only to show that computer scientists have time machines. We do! (It's a well-kept secret.)

Getting rid of the hats (and time machines)

Of course we don't actually want to run a protocol with hats. And even Google (probably?) doesn't have a literal time machine.

To tie things together, we first need to bring our protocol into the digital world. This requires that we construct the digital equivalent of a 'hat': something that both hides a digital value, while simultaneously 'binding' (or 'committing') the maker to it, so she can't change her mind after the fact.

Fortunately we have a perfect tool for this application. It's called a digital commitment scheme. A commitment scheme allows one party to 'commit' to a given message while keeping it secret, and then later 'open' the resulting commitment to reveal what's inside. They can be built out of various ingredients, including (strong) cryptographic hash functions.******

Given a commitment scheme, we now have all the ingredients we need to run the zero knowledge protocol electronically. The Prover first encodes its vertex colorings as a set of digital messages (for example, the numbers 0, 1, 2), then generates digital commitments to each one. These commitments get sent over to the Verifier. When the Verifier challenges on an edge, the Prover simply reveals the opening values for the commitments corresponding to the two vertices.

So we've managed to eliminate the hats. But how do we prove that this protocol is zero knowledge?

Fortunately now that we're in the digital world, we no longer need a real time machine to prove things about this protocol. A key trick is to specify in our setting that the protocol is not going to be run between two *people*, but rather between two different *computer programs* (or, to be more formal, probabilistic Turing machines.)

What we can now prove is the following theorem: if you could ever come up with a computer program (for the Verifier) that extracts useful information after participating in a run of the protocol, then it would be possible to use a 'time machine' on that program in order to make it extract the same amount of useful information from a 'fake' run of the protocol where the Prover doesn't put in any information to begin with.

And since we're now talking about *computer programs*, it should be obvious that rewinding time isn't such an extraordinary feat at all. In fact, we rewind computer programs all the time. For example, consider using virtual machine software with a snapshot capability.

Example of rewinding through VM snapshots. An initial VM is played forward, rewound to an
initial snapshot, then execution is forked to a new path.

Even if you don't have fancy virtual machine software, any computer program can be 'rewound' to an earlier state, simply by starting the program over again from the beginning and feeding it exactly the same inputs. Provided that the inputs — including all random numbers — are fixed, the program will always follow the same execution path. Thus you can rewind a program just by running it from the start and 'forking' its execution when it reaches some desired point.

Ultimately what we get is the following theorem. If there exists any Verifier computer program that successfully extracts information by interactively running this protocol with some Prover, then we can simply use the rewinding trick on that program to commit to a random solution, then 'trick' the Verifier by rewinding its execution whenever we can't answer its challenge correctly. The same logic holds as we gave above: if such a Verifier succeeds in extracting information after running the real protocol, then it should be able to extract the *same amount of information* from the simulated, rewinding-based protocol. But since there's no information going into the simulated protocol, there's no information to extract. Thus the information the Verifier can extract must always be zero.

Ok, so what does this all mean?

So let's recap. We know that the protocol is complete and sound, based on our analysis above. The soundness argument holds in any situation where we know that nobody is fiddling with time — that is, the Verifier is running normally and nobody is rewinding its execution.

At the same time, the protocol is also zero knowledge. To prove this, we showed that any Verifier program that succeeds in extracting information must also be able to extract information from a protocol run where rewinding is used and *no information is available in the first place.* Which leads to an obvious contradiction, and tells us that the protocol can't leak information in either situation.

There's an important benefit to all this. Since it's trivial for anyone to 'fake' a protocol transcript, even after Google proves to me that they have a solution, I can't re-play a recording of the protocol transcript to prove anything to anyone else (say, a judge). That's because the judge would have no guarantee that the video was recorded honestly, and that I didn't simply *edit* in the same way Google might have done using the time machine. This means that protocol transcripts themselves contain no information. The protocol is only meaningful if I myself participated, and I can be sure that it happened in real time.

Proofs for all of NP!

If you've made it this far, I'm pretty sure you're ready for the big news. Which is that 3-coloring cellphone networks isn't all that interesting of a problem — at least, not in and of itself.

The really interesting thing about the 3-coloring problem is that it's in the class NP-complete. To put this informally, the wonderful thing about such problems is that *any other problem in the class NP* can be translated into an instance of that problem.In a single stroke, this result — due to Goldreich, Micali and Wigderson — proves that 'efficient' ZK proofs exists for a vast class of useful statements, many of which are *way more* interesting than assigning frequencies to cellular networks. You simply find a statement (in NP) that you wish to prove, such as our hash function example from above, then translate it into an instance of the 3-coloring problem. At that point you simply run the digital version of the hat protocol.

In summary, and next time

Of course, actually running this protocol for interesting statements would be an insanely silly thing for anyone to do, since the cost of doing so would include the total size of the original statement and witness, plus the reduction cost to convert it into a graph, plus the *E^2* protocol rounds you'd have to conduct in order to convince someone that the proof is valid. Theoretically this is 'efficient', since the total cost of the proof would be polynomial in the input size, but in practice it would be anything but.

So what we've shown so far is that such proofs are *possible*. It remains for us to actually find proofs that are practical enough for real-world use.

In the next post I'll talk about some of those — specifically, the *efficient* proofs that we use for various useful statements. I'll give some examples (from real applications) where these things have been used. Also at reader request: I'll also talk about why I dislike SRP so much.

See here for Part 2.

Notes:

* Formally, the goal of an interactive proof is to convince the Verifier that a particular string belongs to some language. Typically the Prover is very powerful (unbounded), but the Verifier is limited in computation.

** This example is based on the original solution of Goldwasser, Micali and Rackoff, and the teaching example using hats is based on an explanation by Silvio Micali. I take credit only for the silly mistakes.

****** A simple example of a commitment can be built using a hash function. To commit to the value “x” simply generate some (suitably long) string of random numbers, which we’ll call ‘salt’, and output the commitment *C = Hash(salt || x)*. To open the commitment, you simply reveal ‘x’ and ‘salt’. Anyone can check that the original commitment is valid by recomputing the hash. This is secure under some (moderately strong) assumptions about the function itself.