.py file
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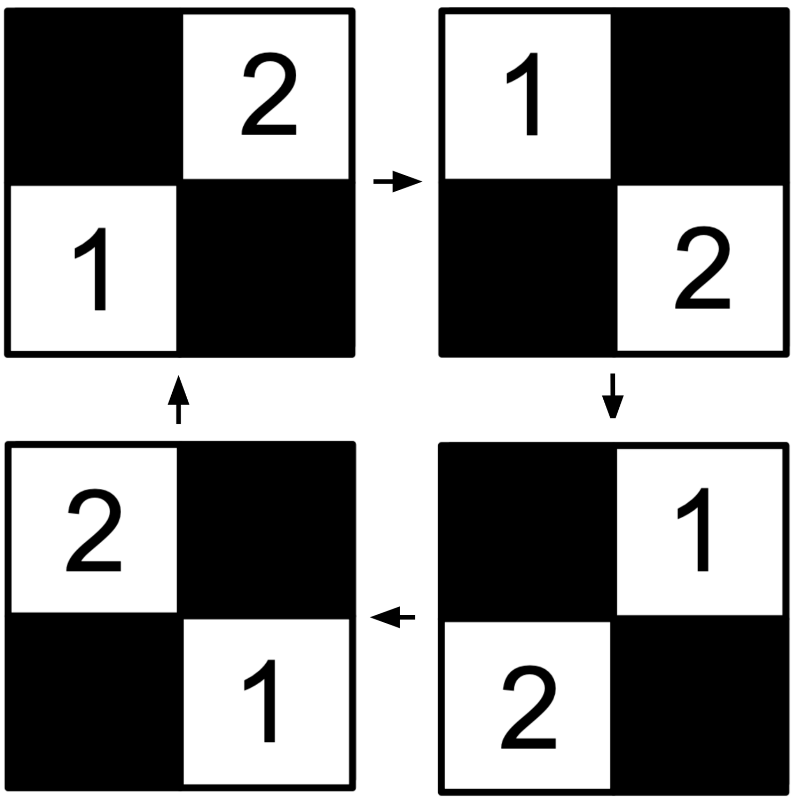
.lean file

∞-Spaces



1. Unicode

Here is a list of the unicode characters we will use:

Symbol	Unicode	VSCode shortcut	Use
		Lean's Kernel	
×	2A2F	\times	Product of types
→	2192	\rightarrow	Hom of types
⟨,⟩	27E8,27E9	\langle,\rangle	Product term introduction
↦	21A6	\mapsto	Hom term introduction
∧	2227	\wedge	Conjunction
∨	2228	\vee	Disjunction
∀	2200	\forall	Universal quantification
∃	2203	\exists	Existential quantification
¬	00AC	\neg	Negation
		Variables and Constants	
$^{a},^{b},^{c},...,^{z}$	1D52,1D56		Variables and constants
$^{0},^{1},^{2},^{3},^{4},^{5},^{6},^{7},^{8},^{9}$	1D52,1D56		Variables and constants
$^{-}$	207B		Variables and constants
$_{0},_{1},_{2},_{3},_{4},_{5},_{6},_{7},_{8},_{9}$	2080 - 2089	\0-\9	Variables and constants
$\mathbb{A}$,...,$\mathbb{Z}$	1D538		
𝕒,...,𝕫	1D552		
A,...,Z	1D41A		
a,...,z	1D41A		
α-ω,Α-Ω	03B1-03C9		Variables and constants
		Categories	
𝟙	1D7D9	\b1	The identity morphism
∘	2218	\circ	Composition
		Bicategories	
•	2022	\smul	Horizontal composition of objects
		Adjunctions	
⇄	21C4	\rightleftarrows	Adjunctions
⇆	21C6	\leftrightarrows	Adjunctions
.	1BC94		Right adjoints
˙	0971		Left adjoints
⊣	22A3	\dashv	The condition that two functors are adjoint
		Monads and Comonads	
?,¿	003F, 00BF	?,\?	The corresponding (co)monad of an adjunction
!,¡	0021, 00A1	!, \!	The (co)-Eilenberg-(co)-Moore adjunction
ꜝ,ꜞ	A71D, A71E		The (co)exponential maps
		Miscellaneous	
∼	223C	\sim	Homotopies
≃	2243	\equiv	Equivalences
≅	2245	\cong	Isomorphisms
⊥	22A5	\bot	The overobject classifier
∞	221E	\infty	Infinity categories and infinity groupoids
↔	20D7		Homotopical operations on ∞-categories
→	20E1		Homotopical operations on ∞-groupoids

2. Introduction

Implementation Progress

Writing Progress

How d is to D as unitial is to actional, and yet d^2 is 0 for chain complexes (different than globular sets), and not so for D. This suggests that we consider complexes in general rather than those particular complexes arising from globular abelian groups or globular ∞-spaces, in which $d^2 = 0$. What remains is an understanding of tensor product and a "sign" which accompanies it. Even though tensor product is determined up to isomorphism by being adjoint to a graded hom, the addition of the sign allows for the free DGA and free differential graded ∞-space constructions.

Hence in this document we take the approach of thinking about presheaves and ∞-presheaves over the diagram $\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$, without a square-zero condition. Those presheaves (in abelian groups) and ∞-presheaves (in ∞-spaces) which arise from ∞-groupoids have this condition, but reside within a larger situation in which the main constructions extend.

All of this produces less confusion in regards to

$E\mathbb{N}$...

1. A functorial construction of the classifying space in homotopy which can be applied indefinitely.
2. This construction will be an endofunctor of operadic groups in operadic groups in ∞-Grpd.
3. I would like to notate it as B^1.

The topics here feature the Eckman-Hilton argument and the abelian nature of π_2 at their core. We can understand these using continuous functions out of the square of the unit interval $\vec{\gamma}^2$ or the square of the directed unit interval $\vec{\gamma}^2$. In the case of a based ∞-groupoid X, π_2 is first defined using continuous functions f : $I^2 \longrightarrow$ X such that f((x,y)) is sent to the base of X when either x or y is 0.

This notation distinguishes
my attempt at an abelian-classifying-space construction which is an endofunctor of grouplike E^∞-spaces, here refered to as ∞-spaces, from the classifying-space construction involving ∞-groupoids and grouplike A^∞-spaces. Note that our ∞-categories always have r = 1, i.e. ∞ is short for (∞,1).

I will write B^n for the n-fold composition of B^1, Pow B n. Note that B is not an endofunctor so that it cannot be iterated. It is more typical to divide up the construction B, for instance constructing the Grassmanians $Gr^\infty(\mathbb{C},n) \cong$ B.obj $GL_n(\mathbb{C})$, from which it follows that [-,$Gr^\infty(\mathbb{C},n)$] is equivalent to the category of $GL_n(\mathbb{C})$-principal bundles and therefore to n-dimensional vector bundles bundles.

In "'TheWhiteheadTheoremandTwoVariations"', we developed two models of ∞-Grpd_(A) and ∞-Grpd_(B). These will produce two models on which B^1 can be defined:

$$B^1 : \text{OperadicGroup} \bullet \text{OperadicGroup } \infty\text{-Grpd} \longrightarrow \text{OperadicGroup} \bullet \text{OperadicGroup } \infty\text{-Grpd}$$

after this we develop chain complexes of these.

The table of contents below reflects the tentative long-term goals of the authors, with the main goal the pursuit of the Whitehead theorem for a point-set model involving Mathlib's predefined homotopy groups.

1. cup and cap product

Ideas for future applications:

1. `https://arxiv.org/pdf/2206.13563.pdf`

1. One of the basic things I wanted out of this was homotopy colimit preserving maps $(E^{\mathtt{inf}}\text{-Alg } A)^{\mathtt{op}} \longrightarrow \infty$-Grpd

Note that this repository does not implement `Q X := colimit`$_n$ Ω^n Σ^n `X` or the stable homotopy groups. It concerns the relationship between O and B and not Ω and Σ.

3. Contents

ring	The type of rings
Ring	The category of rings
commutative_ring	The type of commutative rings
CommutativeRing	The category of commutative rings
`Chapter 8:` A^∞-Rings and E^∞-Rings	
A^∞-ring	The type of A^∞-rings
A^∞-Ring	The category of A^∞-Rings
E^∞-ring	The type of E^∞-rings
E^∞-Ring	The category of E^∞-Rings
`Chapter 9:` Modules and Modules over Commutative Rings	
InternalMonoidAction (InternalMonoid C) $\cong$ InternalMonoid (InteralMonoidAction C)	The ??? theorem
CommutativeAlgebra : CommutativeRing $\to$ Cat	The category of commutative algebras
Maps (Algebra A) : Cat	The category of maps of commutative A-algebras
`Chapter 10:` A^∞-Modules and E^∞-Modules	
A^∞-RingAction (A^∞-Ring C) $\cong$ A^∞-Ring (A^∞-RingAction C)	The ??? theorem
Maps A^∞-Algebras	
`PART III:` DERIVATIONS AND CONNECTIONS	
`Chapter 11:` Lie Algebras	
lie_algebra	The type of Lie-algebras
LieAlgebra	The category of Lie-algebras
`Chapter 12:` Derivations	
InternalAbelianGroup (Maps (Algebra A)) $\cong$ MonoidActionObject A	???
??? : (Maps (Algebra A)) $\rightleftarrows$ InternalAbelianGroup (Maps (Algebra A)) : ???	The free abelian group functor for (Maps (Algebra A))
Λ : ??? $\rightleftarrows$??? : FstDeg	
??? : (Algebra A) $\rightleftarrows$ Chn (Algebra A) : ???	The free DGA functor
derivation	Definition of a derivation
Der : () $\rightleftarrows$ (InternalMonoidAction A) : ???	A derivation is a primitive element
`Chapter 13:` L^∞ Algebras	
L`inf`_algebra	The type of L^∞-algebras
L^∞Algebra	
`Chapter 14:` ∞-Derivations	
OperadicAbelianGroup (Maps (∞-Algebra A)) $\cong$ E^∞-MonoidAction A	???
??? : A^∞-Algebras $\rightleftarrows$???	The free abelian group
Λ : ??? $\rightleftarrows$??? : FstDeg	
??? : (???) $\rightleftarrows$ (???) : ???	The free ???
∞-derivation	Definition of an ∞-derivation
∞-Der : () $\rightleftarrows$ () : ???	A derivation is an ∞-primitive element
`Chapter 15:` Tensor Product of Lie Algebras	
- $\otimes$_() -	
LieAlgebra : ???	The monoidal category of Lie algebras
`Chapter 16:` Tensor Product of L^∞-Algebras	
- $\otimes$_() -	
: ???	The symmetric monoidal category of L^∞-algebras
`Chapter 17:` Lie Algebra Representations	
`Chapter 18:` Connections	
???	The ??? equivalence
???	The free internal abelian group action functor
???	The first degree of the free E^∞-DGM on an algebra is s_()

???	???
connection	Definition of a connection
???	A connection is a d-action
`Chapter 19:` L^∞-Representations	
`Chapter 20:` ∞-Connections	
???	The ??? equivalence
???	The free operadic abelian group action functor
???	The first degree of the free E^∞-DGM on an algebra is s`inf`_
???	???
connection	Definition of a connection
???	A connection is a d`inf`-action
`Chapter 21:` Tensor Product of Lie-Algebra Representations	
- ⊗_() -	
[-,-]_()	
???	The symmetric monoidal closed category of Lie-algebra repr
`Chapter 22:` Tensor Product of L^∞-Algebra Representations	
- ⊗_() -	
[-,-]_() -	
???	The symmetric monoidal closed category of L^∞-algebra rep

PART 1: ABELIAN GROUPS AND ∞-SPACES

In this first section I will construct eight structures for monoidal categories. These structures will be constructed so as to be endofunctions of a particular kind of monoidal category (as opposed to a cartesian category).

Eight Structures			
Strict		Lax	
Unitial	Actional	Unitial	Actional
InternalCommutativeMonoid	InternalCommutativeMonoidAction	OperadicCommutativeMonoid	OperadicCommutativeMonoidActio
InternalAbelianGroup	InternalAbelianGroupAction	OperadicAbelianGroup	OperadicAbelianGroupAction

One particular kind of monoidal category is a *cartesian category*, i.e. a monoidal category in which the monoidal operation is cartesian product. Fox's theorem says that a symmetric monoidal category is cartesian if and only if it is isomorphic to co-commutative comonoids in itself.

We will also be interested to form the tensor product of abelian groups and the tensor product of ∞-spaces.

Abelian Groups

4. The Category of Commutative Monoids and the Category

5. AbelianGroup $\longrightarrow$ Group

In this chapter I cover the type of abelian groups as well as the category of abelian groups.

1. Mathlib.Algebra.Group.Defs

6. Negation

7. The Eckman-Hilton Argument

The Eckman-Hilton Argument demonstrates that internal groups in the monoidal category of groups with product as monoidal operation is equivalent to the category of abelian groups.

8. Eilenberg-Maclane Spaces

Definition 1. Let `A` be an abelian group, and let $n \in \mathbb{N}$ be a non-negative integer. An Eilenberg-Maclane space .

1. The Postnikov Tower
2. The Whitehead Tower

9. Chain Complexes

Definition 2. ...

10. Realization of Chain Complexes

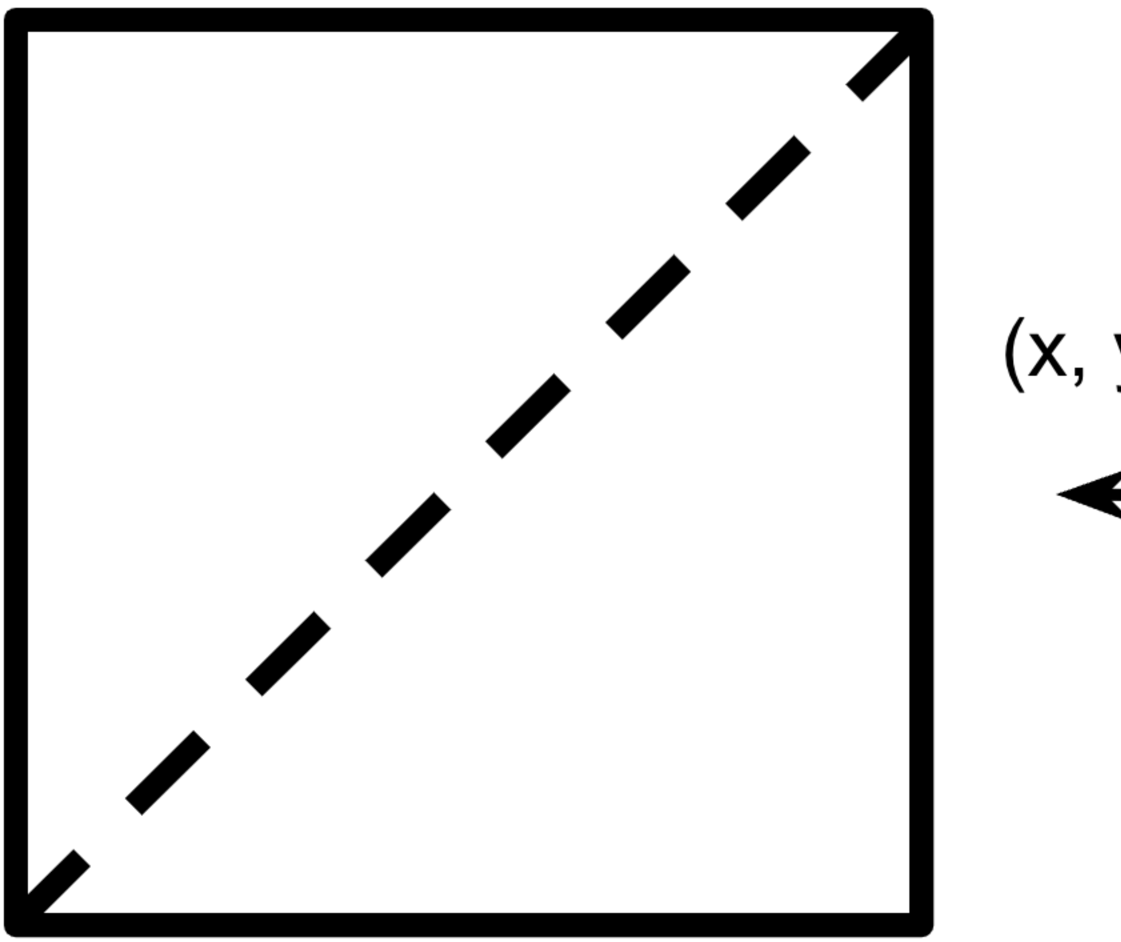
Definition 3. ...

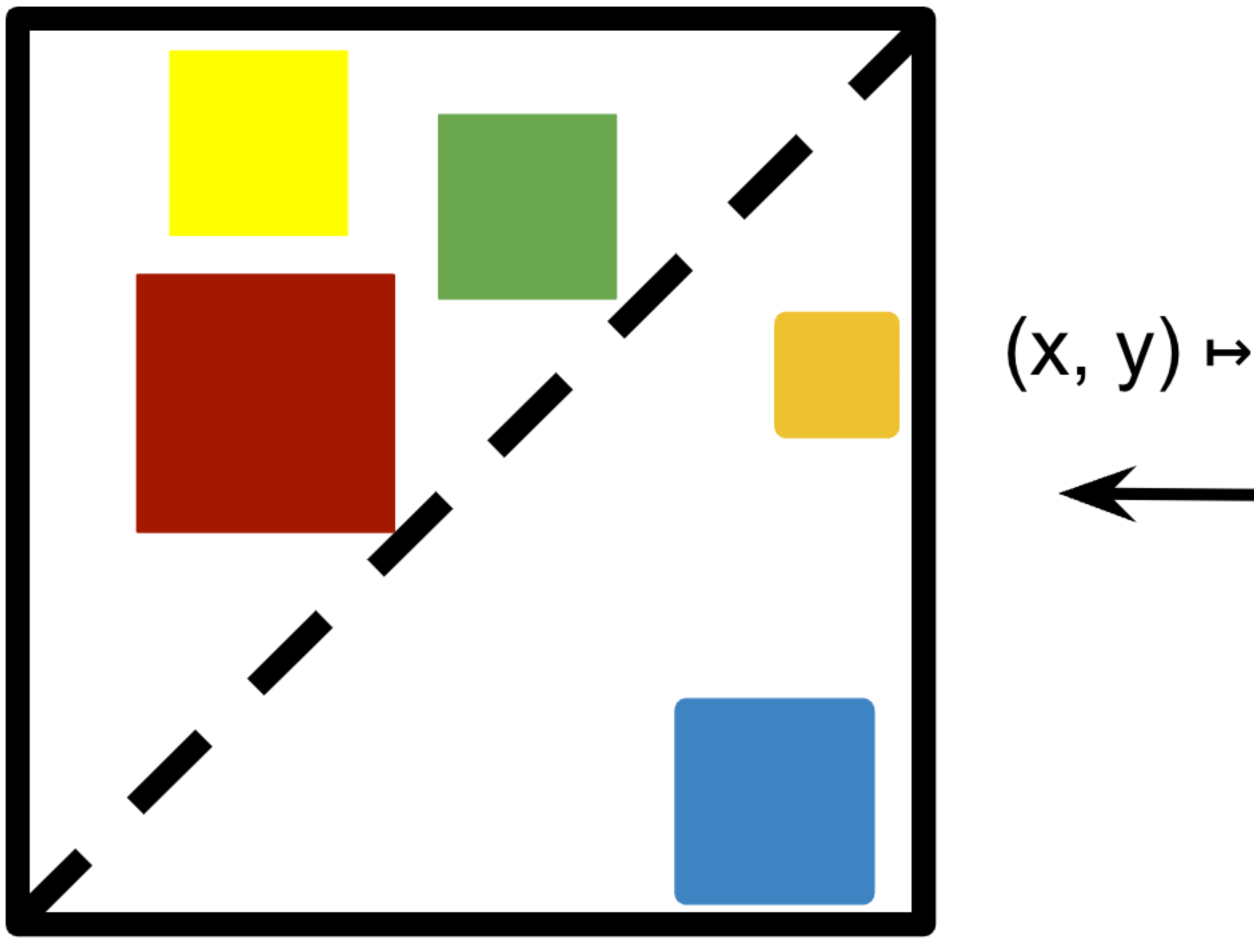
11. Tensor Product of Chain Complexes

Definition 4. ...

∞-Spaces

12. The category of ∞-Spaces and the category of E^{∞}-Spa


(x,


(x, y) ↦

13. ∞-Spaces $\longrightarrow$ OperadicGroup • OperadicGroup ∞-Grpd_

An ∞-space is an algebra for the little-squares operad.

The little-squares operad is OperadicGroup 2 ∞-Grpd.

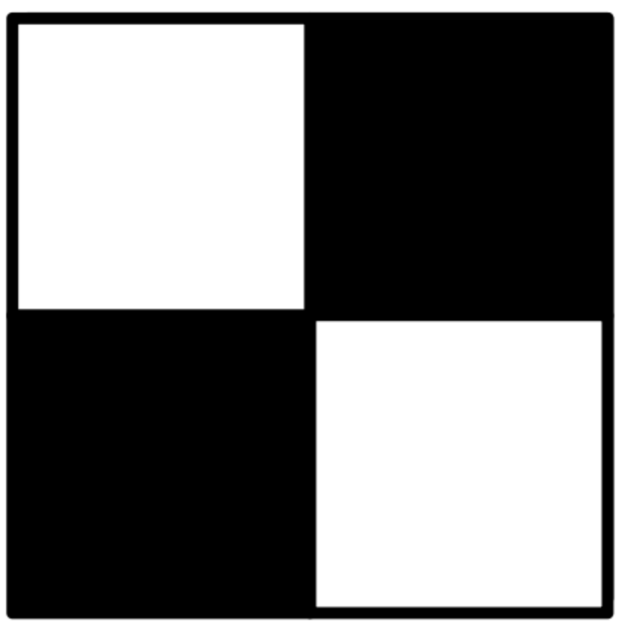
, which we have here taken to mean an operadic group in operadic groups in based connected ∞-groupoids, is a particular operadic group. In this approach, we have considered OperadicGroup to have type $\mathbb{N} \longrightarrow$ Cat rather than ∞-Cat $\longrightarrow$ ∞-Cat or ∞_(∞-Cat) $\longrightarrow$ ∞_(∞-Cat). Specifically, we can supply a non-negative integer to obtain a particular operad resembling little n-cubes but which features no ”empty space”.

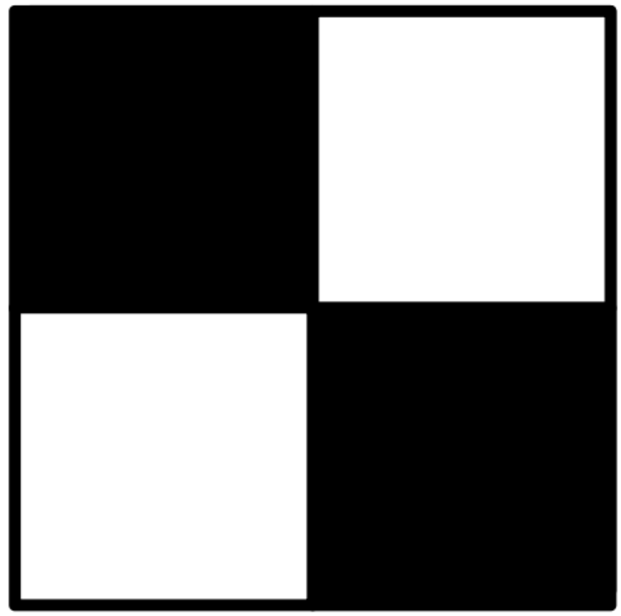
14. Negation

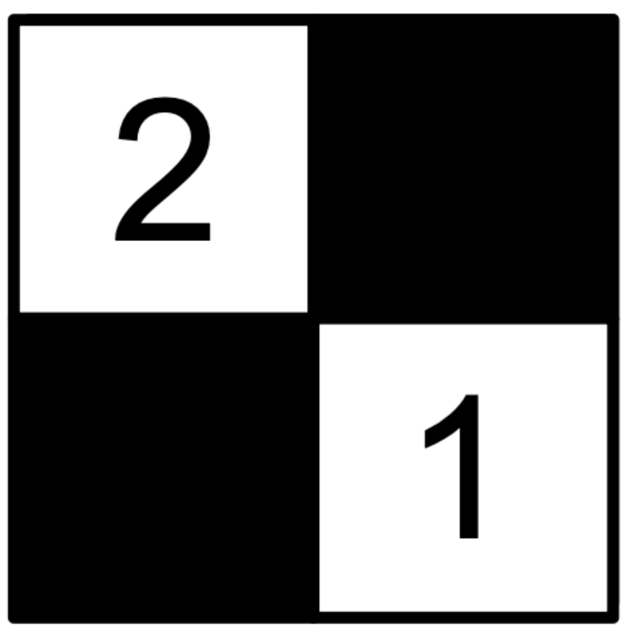
For an ∞-space A, negation $\neg : A \longrightarrow A$ ...

∞-spaces

15. The Eckman-Hilton Argument



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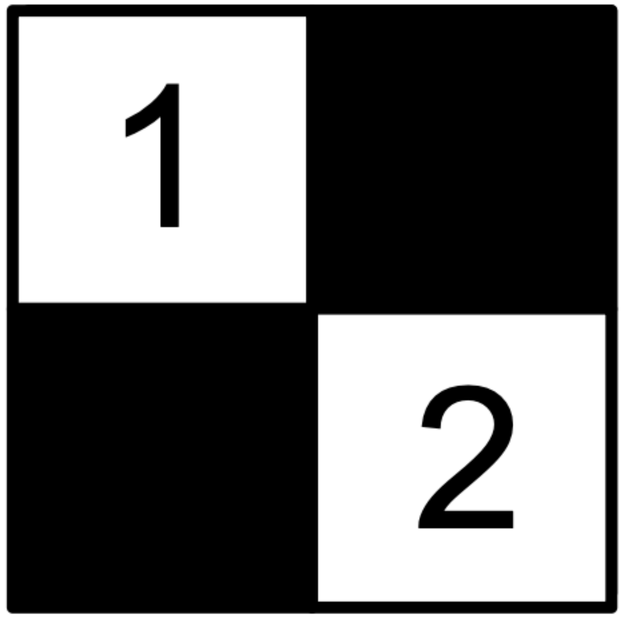
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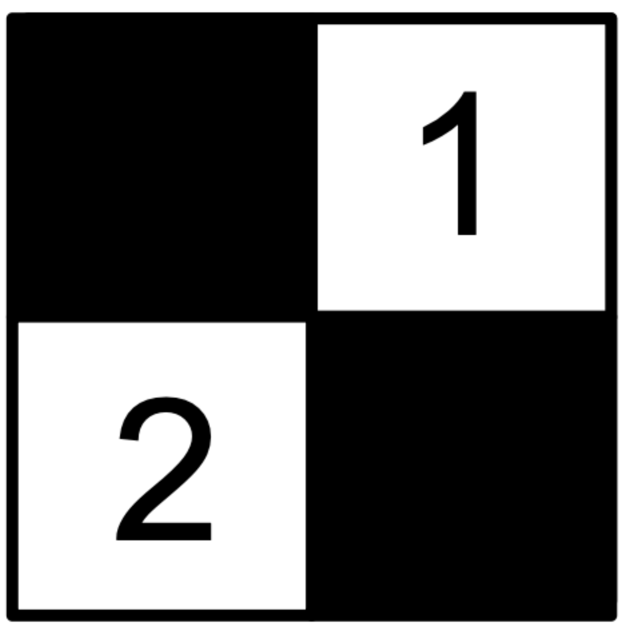
2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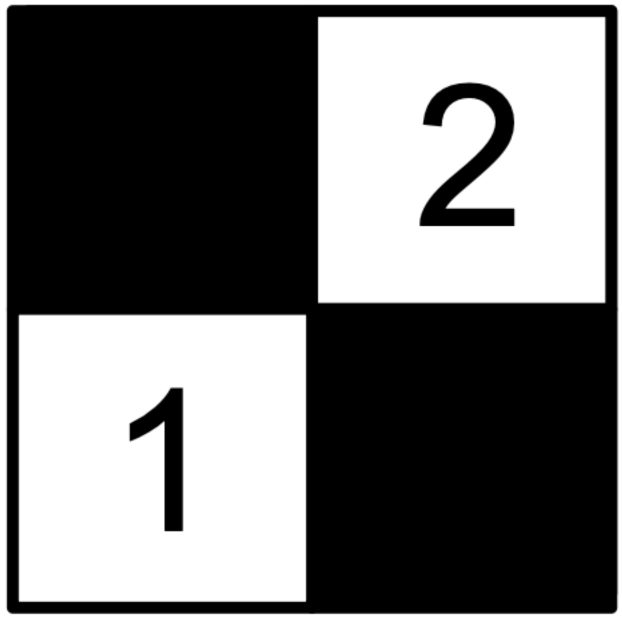

OperadicGroup 2 ∞-Grpd$_{-1}$

16. B^1 and B^n

Definition 5. B^1

Definition 6. B^n is $B^1 \bullet B^{n-1}$.

1. The Postnikov Tower

Here I would like to construct the Postnikov tower from a different perspective, using B^1.

2. The Whitehead Tower

Here I would like to construct the Whitehead tower from a different perspective, using B^1.

17. Chain Complexes

18. Realization of Chain Complexes

19. Tensor Product of Chain Complexes

Tensor Product

1.

Tensor Product of ∞-Spaces

∞-Grpd$_{-1}$ with smash product forms an operadic commutative monoid, but not a monoidal category.

$\mathtt{Set}_{-1} \rightleftarrows \mathtt{AbelianGroup}$

Construction	Description
??? : InternalAbelianGroup $\mathrm{Set}_{-1} \cong$ AbelianGroup : ???	The ??? theorem
??? : $\mathrm{Set}_{-1} \rightleftarrows$ AbelianGroup : ???	The ??? adjunction

Abelian groups are internal groups in internal groups in sets.

In forming the free group on a set, based sets intermediate the construction.

$\infty\texttt{-Grpd}_{-1} \rightleftarrows \infty\texttt{-Space}$

In this section, we construct a

Construction	Description
??? : InternalAbelianGroup $\mathrm{Set}_{-1} \cong$ AbelianGroup : ???	The ??? theorem
??? : ∞-Grpd$_{-1} \rightleftarrows \infty$-Space : ???	The ??? adjunction

1. π_n of an ∞-space arising from an ∞-groupoid
2. H_n of an
3. Dold-Thom theorem
4.
5. B is iterable on this
6. Chn ...
7. μ : Chn $\times$ Chn $\longrightarrow$ Chn

S_(???) in general...

PART 2: RINGS, COMMUTATIVE RINGS, A$^{\infty}$-RINGS, AND E$^{\infty}$-RINGS

In this second section, I consider four constructions for monoidal categories which have occured less generally elsewhere for cartesian categories: internal monoids, internal commutative monoids, algebras for A^{∞}-operads, and algebras for E^{∞}-operads. The main difference with the structures featured previously is that these structures concern an operation which is more general than product, but less general than pullback. Four of the sixteen structures formed in the repository concerning the Whitehead theorem can re-create the rest, are not instances of the structures here. Meanwhile, the four structures defined here will be constructed using Mathlib 4's monoidal categories and symmetric monoidal categories. Tensor product of abelian groups and smash product of ∞-spaces, coincide with the previous structures in the case where the monoidal operation is product (see Fox's theorem).

To reflect the use of monoidal categories as opposed to categories (in which the cartesian monoidal structure can be recovered from the structure as a seven entry with the addition of a single Lean universe), I use different names for the constructions:

Categories of Internal Objects			
Strict		Lax	
Unitial	Actional	Unitial	Actional
`MonoidObjects : ??? → ???`	`MonoidActionObjects : ??? → ???`	A$^{\infty}$-Monoid ∞-Space	A$^{\infty}$-Monoid
`CommutativeMonoidObjects : ??? → ???`	`CommutativeMonoidActionObjects : ??? → ???`	E$^{\infty}$-Monoid ∞-Space	E$^{\infty}$-Monoid

Rings and Commutative Rings

Rings and DGAs
Commutative rings and CDGAs

1. A thread on creating the six-entry category of commutative algebras.
2. In this section we use slightly different internal structures than the internal monoid in the last section; these internal monoids are defined in a monoidal category and the others are defined for product only. As such we may like to re-examine those structures, or alternatively to keep separate definitions.
3. What's more clear is that the ∞-analogous are more difficult to reconcile with the choices made for the first sixteen structures and the four doubled structures (see the section on the Eckman-Hilton argument)

$\mathtt{A}^{\infty}$-Rings and $\mathtt{E}^{\infty}$-Rings

make sure to include Alg
make sure to include ∞-Alg...

1. What's more clear is that the ∞-analogous are more difficult to reconcile with the choices made for the first sixteen structures and the four doubled structures (see the section on the Eckman-Hilton argument).

Modules over Rings and Commutative Rings

A^{∞}-Modules and E^{∞}-Modules

PART 3: DERIVATIONS AND CONNECTIONS

In this section, I define four more structures:

	Four Definitions	
	Strict	Lax
Unitial	Derivation	∞-Derivation
Actional	Connection	∞-Connection

I also construct four adjunctions which feature the internal abelian group structure:

	Four Free Constructions in Algebra	
	Strict	Lax
Unitial	??? : Map ??? $\rightleftarrows$ InternalAbelianGroup (Map ???) : ???	??? : Map ??? $\rightleftarrows$ (OperadicAbelianGroup (Map ???)) : ???
Actional	??? : ??? $\rightleftarrows$ InternalAbelianGroupAction ??? : ???	??? : ??? $\rightleftarrows$ OperadicAbelianGroupAction ??? : ???

In this second part, I define eight adjunctions associated to the algebraic structures defined in the last section.

Lie Algebras

Definition 7 (Lie Algebra). Let A be a (commutative unitial) ring. A Lie-algebra is an A-module M such that...

Definition 8 (Lie Algebra Map). Let A be a (commutative unitial) ring.

1. A thread on the classification of Lie algebras.

   The Jacobi identity says that the lie-bracket is a self-derivation.

Derivations

In a blog post here,
Let `A` be a ring and suppose that `B : Alg A`. `B.dom`.

$$S\_(\mathrm{Mon}\ (\mathrm{Act}\ R))_{\cdot} : (\mathrm{ComMon}\ R) \rightleftarrows ??? : S\_(\mathrm{Mon}\ (\mathrm{Act}\ R))^{\cdot}$$

Theorem 1. the category of internal abelian groups in Mon (Act R) is equivalent to Act R

$$S\_(\mathrm{Mon}\ (\mathrm{Ch}\ (\mathrm{Act}\ R)))_{\cdot} : \mathrm{Mon}\ (\mathrm{Ch}\ (\mathrm{Act}\ R)) \rightleftarrows \mathrm{Mod}\ R : S\_(\mathrm{Mon}\ (\mathrm{Ch}\ (\mathrm{Act}\ R)))^{\cdot}$$

$$(\mathrm{Alg}\ A)/A \rightleftarrows (\mathrm{Alg}\ A)$$

1. I would like to first construct the lie-algebra of derivations using the spectrum `Ωinf.obj X`. It seems related to coalgebra endomorphisms from `Ωinf.obj X` to itself.
2. Lie algebras and Der�(A,A)

L^∞ Algebras

∞-Derivations

$$\Omega\_()_{\cdot} : (\mathrm{E}^{\mathtt{inf}}\text{-Alg A})/\mathrm{A} \rightleftarrows \mathrm{E}^{\mathtt{inf}}\text{-Mod A} : \Omega\_()^{\cdot}$$

$$\Lambda\_()_{\cdot} : \mathrm{Ch}\ (\mathrm{E}^{\mathtt{inf}}\text{-Alg A}) \rightleftarrows \mathrm{E}^{\mathtt{inf}}\text{-Mod A} : \Lambda\_()^{\cdot}$$

In this section, I construct:

$$\Omega^{\mathtt{inf}}\_(\mathrm{A}) : (\mathrm{E}^{\infty}\text{-Alg A})/\mathrm{A} \rightleftarrows \mathrm{E}^{\infty}\text{-Mod A} : \Lambda^{\mathtt{inf}\cdot}\_(\mathrm{A})$$

And I also define the concept of an ∞-derivation.

This adjunction factors like so:

$$\Omega^{\mathtt{inf}}\_(\mathrm{A}) : (\mathrm{E}^{\infty}\text{-Alg A})/\mathrm{A} \rightleftarrows ??? \rightleftarrows \mathrm{E}^{\infty}\text{-Mod A} : \Lambda^{\mathtt{inf}\cdot}\_(\mathrm{A})$$

The more typical concept of a free abelian group .

Given an E^{∞}-algebra...., we can form an

Tensor Product of Lie-Algebras

Tensor Product of L^∞-Algebras

Connections

s instead of ω and λ
The set of connections forms an affine space. One can then define:

1. Free abelian group action.

A connection on a vector bundle can be understood as an element of:

1. the free graded module given a connection
2. $\Omega^0(X) \otimes V \longrightarrow \Omega^1(X) \otimes V$ such that $(\square X)\, f \otimes x = (df) \otimes x + f((\square X)\,(1 \otimes x))$
3. $\Omega^1([V,V])$

This can be understood in terms of the theorem that $\mathrm{End}_A(V \otimes A)$ is isomorphic to $\mathrm{End}_k(V) \otimes A$ for a free A-module V and certain condition on the k-algebra A, where we here take A to be $\Omega^{\cdot}(X)$.

We can understand the first in terms of the $\Omega^0(X)$-module $\Omega^n(X) \otimes [V,V] \longrightarrow \Omega^{n+1}(X) \otimes [V,V]$ in which we extend $\square$ to feature a the rule of $(df) \otimes x + (-1)^{ij} * f((\square X)\,(1 \otimes x))$, and write $d_\square$ for this, but it doesn't form a chain complex. Instead, we obtain an element of $\Omega^2([V,V])$ from $d_\square d_\square x$ for any section x of V. Defining $F_\square$ to be $dA - A \wedge A$, wedge with $F_\square$ is the same as $d_\square d_\square$.

∞-Connections

Connections are elements of the free abelian group action on an A-algebra over A, and ∞-connections are ...

1. I would like to first construct the lie-algebra representation of flat connections using the spectrum $\omega^{\mathtt{inf}}$`.obj X).obj V`.
2.
3. Lie algebra representations and ...
4.
5.
6.
7. Somehow connections are the dual of the free abelian group action

Some goals:

1.

1. smooth (etale locally ???)
2. analytic (etale locally ???)

$\mathbb{C}\mathrm{P}^1$,$\mathbb{C}\mathrm{P}^1$ $\cong \mathbb{C}$(x)

3. Our $\mathbb{R}$ is ∂.obj $\mathbb{Z}$ regarded as an object in [$\vec{\gamma}$,∞_(∞-Grpd)]
4. B.obj det : B.obj U(n) $\longrightarrow$ B.obj U(1)
5. Which E_∞ Space have a Chern class?

Cohomology with coefficients in `[-,B`$\mathbb{C}^{\mathtt{X}}$`]` plays a .

1.
2. d2 is wedge with (representable...)

Given an E^∞-algebra...., we can form an

Tensor Product of Lie-Algebra Representations

L^{∞}-Algebra Representations

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1. Samuel Eilenberg and Saunders Mac Lane, ”On the Groups $H(\pi, n)$. I”, Annals of Mathematics, Second Series, Vol. 58, No. 1 (Jul., 1953), pp. 55-106.
2. Samuel Eilenberg and Saunders Mac Lane, ”On the Groups $H(\pi, n)$. II”, Annals of Mathematics, Second Series, Vol. 60, No. 1 (Jul., 1954), pp. 49-139.
3. Saunders Mac Lane, ”On the Homology Theory of Eilenberg-Mac Lane”, Proceedings of the National Academy of Sciences of the United States of America, Vol. 35, No. 11 (Nov. 15, 1949), pp. 657-663.
4. Eilenberg, S., & MacLane, S. (1945). Relations Between Homology and Homotopy Groups of Spaces. Proceedings of the National Academy of Sciences of the United States of America, 31(2), 83–87.

Further reading:

1. The nlab article on ∞-spaces
2. A blog post of Akhil Matthew explaining how $B^n X \cong \Omega B^{n+1} X$ for an ∞-space X and n ⍰ 2
3. The n-lab article on the Eckman-Hilton argument
4. Operads, Algebras, and Modules, an exposition of J. P. May.