In the document below I state my interests and objectives concerning mathematics in the Lean 4 computer proof assistant. Specifically, I want to develop many of these ideas into PRs (pull requests) for Mathlib 4. Over the next four year period, I would like to complete what I below describe as stages I and II of a plan.

These plans are detailed in ...

1. Stage I: goals concerning homotopy and stable homotopy

Over the course of the next many years, I hope to develop the repositories at `github.com/linlib`, particularly and first of all `ThreeWhiteheadTheoremsandThreePuppeSequences`, into PRs for Mathlib 4. This is an ambitious goal, and it features objects which are of central importance in mathematics. The most significant and mentioned obstacle in this regard is the need for an API (application programming interface). Considerations in the interface associated to a theory of ∞-categories in Lean are especially important given the central role of the material. However, while the proof assistants Agda and Coq have developed extensive libraries for these concepts, the approaches to homotopy in Lean 3 and 4 remain largely personal projects. In addition, there are many desirable features that a PR for these concepts could have. Here are some other considerations:

(a) There are many models for ∞-categories, each of which satisfies the four goals expressed in the repository `ThreeWhiteheadTheoremandThreePuppeSequences` for ∞-categories (the first of the twelve mentioned goals).

(b) It is best to start small, and to get one part correct at a time. Large PRs are hard to check and more prone to error.

(c) Many mathematicans are more interested in CW-complexes and point-set approaches to ∞-groupoids than the full view presented in `ThreeWhiteheadTheoremandThreePuppeSequences`. It is desirable to have an approach which integrates modern and classical approaches. Further, the approaches which use simplicial sets (see SSet) and point-set topology benefit from being accessible and familiar.

(d) It is important to ensure that code is usable for future projects, especially those which will add important API features.

One discussion on the Lean 4 Zulip from 2020 contained a similar observation, in which the prospects of a "plain formalization" are mentioned:
The Lean community is especially suited towards such a 'plain formalization' approach.

The category of quasicategories and the category of Kan complexes are both essentially determined from the pre-existing material within Mathlib 4. Specifically, we below list those structures involved in the definition of quasicategories and Kan complexes:

1. Horns

2. Simplices

3. The horn inclusions

Given the four important considerations (a)-(d) at the outset of the chapter, we would like to consider a pull request based both on the ultimate and far-reaching goals of the first repository. After explaining the content and main features of the pull request, we proceed to explain how we address the important concerns made at the outset.

1.

API

The twelve goals in the repository `ThreeWhiteheadTheoremsandThreePuppeSequences` have allowed for an approach which is motivated by results and which is not particular as to the choice of model.

In stage one, we would like to detail twelve major theorems:

1. The Whitehead theorem
2. The Puppe sequence and its exactness
3. An analogue of the Whitehead theorem for ∞-groupoids featuring an interval object along with two different structures from internal category theory (see Janelidze and Bourceux chapter 7).
4. An analogue of the Puppe sequence for ∞-groupoids featuring an interval object along with two different structures from internal category theory (see Janelidze and Bourceux chapter 7).

STAGE 0 (Category Theory)

1. $\mathsf{Pro} : \mathsf{Cat} \to \mathsf{Cat}$
2. $\mathsf{Ind} : \mathsf{Cat} \to \mathsf{Cat}$

STAGE I (Twenty Four Goals):

1. (Three Whitehead Theorems and Three Puppe Sequences, 2024)
   (a) See the goals listed in the pdf file under the `Three Whitehead Theorems and Three Puppe Sequences` repository.
   (b)
2. (Three Freudenthal Suspension Theorems and Three Long Exact Sequences, 2025)
   (a) See the goals listed in the pdf file under the `Three Freudenthal Suspension Theorems and Three Long Exac` repository
   (b) Term length, typeclasses, and inductive types

2. Stage II: Goals concerning geometric maps in topos theory and stable homotopy

In *Foundations for Computable Topology*, Paul Taylor developed a synthesis of the logic of topological frames, exposing a duality between the topological concepts of local homeomorphism and proper map, and revealing a new condition (overtness and the analogous condition for maps). All topological spaces are overt, but not all locales.

These ideas have also been developed by Martin Escardo.

One of the most interesting features of this synthesis (abstract stone duality) is that it allows for compact intersections of open sets and compact unions of closed sets, a dual concept to (discrete overt) unions of open sets and (discrete) intersections of closed sets.

A frame can be endowed with a topology using the compact-open topology on the internal hom [X,2]. This topology is the same as the Scott topology.

In one ensuing line of inquiry, the preservation of cofiltered limits and filtered colimits is related to the continuity of a homomorphism of frames.

This project (Stage II) takes as its main interest the logical patterns of the work of Taylor and Escardo. Our goal is to make many of the same proofs work for higher topos theory and higher algebra, in which the open/closed classifier gets replaced by an internal universe object.

The analogues fall within the scope of topos theory and higher algebra.

STAGE II (TOPOLOGY):

1. (Topology Project I, 2026)
   (a) The construction of the following universe objects:
      i. ∞_(∞-Cat) : ∞-Cat.α
      ii. ∞_(∞-Grpd) : ∞-Cat.α
      iii. ∞_(∞-Grpd$_0$) : ∞-Cat.α
      iv. D(∞_(∞-Cat)) : ∞-Cat.α
      v. D(∞_(∞-Grpd)) : ∞-Cat.α
      vi. D(∞_(∞-Grpd$_0$)) : ∞-Cat.α
   (b) (Straightening and Unstraightening) Establishing the straightening/unstraightening categorical equivalence, featuring directed homotopy pullback of $\perp$: * $\longrightarrow$ ∞_(∞-Cat).
   (c) The construction of the following:
      i. ∞_(∞-Cat)/C : ∞-Cat.α
      ii. ∞_(∞-Grpd)/G : ∞-Cat.α
      iii. ∞_(∞-Grpd$_0$)/G$_0$: ∞-Cat.α
      iv. D(∞_(∞-Cat)/C) : ∞-Cat.α

v. D(∞_(∞-Grpd)/G) : ∞-Cat.α
vi. D(∞_(∞-Grpd$_0$)/G$_0$) : ∞-Cat.α

(d) (The Projection Formulas) The equivalence between the projection formulas and the condition of being an open or closed map in frame-local theory. See here and here.

(e) (Universally Closed/Open) ???

(f) (Unramified/Separated) The equivalence between open/closed and universally open / universally closed.

(g) (Étale/Proper)

(h)

(i)

(j)

(k) Proper intersections of closed maps are closed and etale unions of opens are open.

(l) Proper iff universally closed, Étale iff universally open.

(m) Proper iff cofiltered limit in the "category of actions", Étale iff filtered colimit in the "category of actions". In the case of frame-local theory, this is a certain structure to do with a frame (a particular join lattice), but viewed in the category of complete partial orders as opposed to frames and locales.

(n) Tychonoff's theorem:

2. (Topology Project II, 2027):

(a) The construction of the following universe objects:
i. ∞_(Directoid) : ∞-Cat.α
ii. ∞_(Spectroid) : ∞-Cat.α
iii. ∞_(Spectra) : ∞-Cat.α
iv. D(∞_(Directoid)) : ∞-Cat.α
v. D(∞_(Spectroid)) : ∞-Cat.α
vi. D(∞_(Spectra)) : ∞-Cat.α

(b) The construction of the following universe objects:
i. Mon D(∞_(Directoid)) : ∞-Cat.α
ii. Mon D(∞_(Spectroid)) : ∞-Cat.α
iii. Mon D(∞_(Spectra)) : ∞-Cat.α

(c) The construction of the following:
i. Act D : ∞-Cat.α
ii. Act S : ∞-Cat.α
iii. Act S : ∞-Cat.α

(d) Ramification

(e) The closed and open conditions and their relationship with the projection formulas.

(f) Universally closed intersections of closed maps are closed and universally open etale unions of opens are open.

(g) Showing that proper is equivalent to being universally closed with universally closed diagonal, and that Étale is equivalent to being universally open with universally open diagonal. This much can be shown first for the case of partial orders, and then for the case of an unstraightening system.

(h) Proper iff cofiltered limit in the category of actions, Étale iff filtered colimit in the category of actions.

(i) Tychonoff's theorem.

(j) What is "Conglomeration Index"?

(k) Ramified extensions, Unseparated extensions.

3. STAGE III:

1.
2. Constructing the E∞-ring
3. Constructing the A∞-ring $\mathbb{Z}$
4. Constructing $\mathbb{Z}[[x]] \cong$ [...Σ∞.obj B.obj B.obj $\mathbb{Z}$,$\mathbb{Z}$]
5. Constructing the E-∞ ring $\mathbb{Z}/n\mathbb{Z}$
6. Constructing Spec($\mathbb{Z}$)

BB$\mathbb{Z}$,BB$\mathbb{Z}$ $\cong \mathbb{Z}$

7. How to get $\mathbb{Z}[[x]]$ as a cohomology ring.
   (a) https://math.stackexchange.com/questions/2015228/how-to-show-mathbbcp-infty-mathbbcp-infty-cong-mathbbz
8. Chern-Weil Theory and $\mathbb{Z}[[x_1,...,x\ ]]$
9. How to get $\mathbb{Z}[[x_1,...,x\ ]]$ as a cohomology ring.
   (a)
10. $GL_1(\ )$-representations and finite étale extensions of
11. GL ( )-representations and finite étale extensions of ((t))
12. GL (^{sep})-representations
13. $GL_1(\ )$-representations and the profinite abelian extensions of
14. GL (^{sep})-representations and
    (a)
15. GL ( )-representations.
16.
17. $((x_1,...,x\ ))$
18. We begin the subsection here with the concept of a fixed subobject given an internal action for an object in one of the stable models. We then relate the fixed object to its universal property under certain conditions.
19. Unitary groups
20. One giant group which either (a) quotients onto all unitary groups... (b)

21. $\mathbb{Z}[x_1,\ldots,x\ ]$ is the derived cohomology ring.
22. I would like to form U(n) and also a group U(∞), the group of unitary operators on Hilbert space
23. BU(n) classifies rank n vector bundles
24. GL ($\mathbb{C}$)?
25. $U(1)^n \longrightarrow U(n)^1$
26. Volume of U(n)
27. U(n)
28.
29. H
30. $U(\infty) :=$
31. The classifying space construction is related to the symmetric group and to the symmetric poly-noimals
32. U(n)

It will not be for many years that the ideas in this section come to fruition and take their place on the computer. The two stages before this are quite extensive.

In this section, we will define the following rings:

1.
2. $\mathbb{R}$, $\mathbb{R}^{\mathtt{sep}}$, $\mathbb{C}$
3. $\mathbb{Z}$
4. $\mathbb{Z}$
5. $\mathbb{Z} \cong \Pi\ \mathbb{Z}$
6. , $^{\mathtt{sep}}$, $\mathbb{C}$
7. $\mathbb{A}^{\mathtt{fin}}$
8. $\mathbb{A}$

Each of these will be defined first as E∞-rings using the models in the repository of `ThreeStableHomotopyCategoriesand`
We will also use the definition of unramified extensions established in the fourth repository, `GeometricMapsinStableHomotopy`
We then relate these E∞-rings to their corresponding rings via structure preserving bijections.

`1. Spectral Sequences and Exact Couples`

1.

`2. Derivations and Flat Connections`
Derivations and flat connections .

1. Derivations

(a) Derivations as elements of the $\mathbb{R}$-linear dual of ...

(b) H_1 ([Ω∞ X,$\mathbb{R}$])

2. Connections

(a) Flat connections as elements of the $\mathbb{R}$-linear dual of `Ω_(Spectra)C`, where C is the cohomology content (an E∞-ring)

(b) H_1 ([ω∞ X V, $\mathbb{R}$])

`3. Trace and Determinant`

1. The main properties of $\mathbb{r} := \Sigma_i\ (-1)^i$ * $\mathrm{Tr}^{(-1)^i}(\Phi)$

2. The main properties of $\mathbb{et} := \Pi_i\ \mathrm{Det}^{(-1)^i}(\Phi)$

`4. Borel-Moore Homology and Cohomology with Compact Support`

Borel-Moore homology and cohomology with compact support:

1. Homology, $p^{\cdot}$ p. $\mathbb{1}$

2. Cohomology $\mathrm{Cmp(p)}^{\cdot}$ p. $\mathbb{1}$

3. Borel-Moore Cohomology $\mathrm{Cmp(p)}_{\cdot}$ $\mathrm{Cmp(p)}^{\cdot}$

4. Cohomology with compact support $p_{\cdot}$ $\mathrm{Cmp(p)}^{\cdot}$

https://homepages.warwick.ac.uk/staff/Martin.Gallauer/docs/m6ff.pdf

I would eventually like to follow through on several analogous "étalification" based cohomology theories:

1. Homology, $p^{\cdot}$ p. $\mathbb{1}$

2. ??? $\mathrm{Ét(p)}^{\cdot}$ p. $\mathbb{1}$

3. ??? $\mathrm{Ét(p)}_{\cdot}$ $\mathrm{Ét(p)}^{\cdot}$

4. ??? $p_{\cdot}$ $\mathrm{Ét(p)}^{\cdot}$

`5. Poincare Duality`

1. The Cayley-Hamilton Theorem for objects with Poincare duality

2. Pullback and cup product

`6. The Kunneth Theorem`

1. Tor and Ext

`7. Intersection Theory`

In this section I would like to establish a few core results concerning intersections. By the date of the implementation of stage III, we will have finished the following repositories:

(a) ThreeWhiteheadTheoremsandThreePuppeSequences

(b) ThreeStableHomotopyCategoriesandThreeLongExactSequences

(c) GeometricMapsinHigherToposTheory

(d) GeometricMapsinStableHomotopyTheory

We will then be in a place to consider an intersection theory using the smash product, $\otimes$. We also develop two generalizations of the smash product for spectroids and directoids.

1. Well behaved cycle maps between pullback and cohomology classes.
2. We would like to mention how these results relate to Bézout's theorem.
3.

8. The Lefschetz Fixed Point Theorem

1. The Lefschetz Fixed Point Theorem follows from results in intersection theory concerning the cycle map, the Kunneth theorem, and Poincare duality.
2.

9. The Cayley-Hamilton Theorem

1. det(Φ- λ I) = (Φ - λ I) $\bullet$ adj(Φ- λ I).
2. End(V $\otimes$ k[t]) $\cong$ End(V) $\otimes$ k[t] for V a free k-action.
3. The theorem is first shown for the case of a finite free module, and then closed under quotients.

10. Exp and Log

1. contains the coefficients in the power series of both exp and log
2. Cohomology with rational coefficients
3. E.obj $\mathbb{Z} \times$ E.obj $\mathbb{Z}$
4. $\mathbb{Z}$ (E.obj $\mathbb{Z}$)
5.
6. (E.obj $\mathbb{Z}$) $\times$ (E.obj $\mathbb{Z}$) $\longrightarrow$ (B.obj $\mathbb{Z}$) $\times$ (E.obj $\mathbb{Z}$)
7.
8. exp : $\mathbb{C} \longrightarrow \mathbb{C}^{x}$
9. log : π_1.obj ($\mathbb{C}^{x}$) $\longrightarrow \mathbb{C}$
10. 1/(1-tz)
11. Under a wide variety of circumstances, the product over n of det($H_0(\Omega^n(\Phi))$) to the power $(-1)^n$ is the unique invariant with a multiplication rule for exact sequences.
12. adj(1 - t * Φ) * (1 - t * Φ) = det(1 - t * Φ)
13. log(1/(1 - tΦ))
14. log(1/(1 - t * Φ)) = - log(1-t * Φ)

15. log(1 - t * Φ) = - $\int$ (n : Nat) Pow n (t * Φ)
16.
17. log(1-tΦ) and both have power-series expansions around t = 0
18. exp, log
19. Tr : [V,V] $\longrightarrow \mathbb{N}$
20. Algebraically closed field
21.
22. Det : [V,V] $\longrightarrow$ k
23. $\mathbb{r}$:= Σ_i $(-1)^i$ * $\mathrm{Tr}^{(-1)^i}$(Φ)
24. $\mathbb{et}$:= Π_i $\mathrm{Det}^{(-1)^i}$(Φ)
25. (1 - t * Φ) * adj(1 - t * Φ) = det(1- t * Φ)
26. log(1 - t * Φ) = - $\int$ (n : $\mathbb{N}$) $(t * \Phi)^{n}$
27. is_invertible det(1 - t * Φ)
28. log(1/(1-tΦ)) = $\int$ (n : $\mathbb{N}$) $(t * \Phi)^{n}$
29. Functional equation given a Poincare duality
30. Sorry, I made a ton of mistakes. I guess I wanted A to be normal in the above.
31. The power series for log : Mat(n×n) $\mathbb{C}$ converges for || Φ - λ I || < 1.

`11. L-Functions`

1. Obtaining a graded ring from an A∞-ring.
2. Obtaining a graded ring from an E∞-ring.

`12. Etale Homotopy`

Consider one of the internal objects defined in the first repository of `ThreeWhiteheadTheoremsandThreePuppeSequences`

1. Internal category
2. Internal groupoid
3. Internal group

The functors satisfy a condition which ensures that smash product and its two analogues be sent to product of ∞-categories, ∞-groupoids, and based connected ∞-groupoids, respectively.

1. How to obtain an ∞-category from a map of A∞-rings
2. How to obtain an ∞-groupoid from a map of E∞-rings

13. `Topological E-`∞ `and A-`∞ `Rings`

In the third and fourth repositories, we took inspiration from the study of topological frames in thinking about 'topological topoi', in which the topological frame arising from the frame with two points is replaced with the internal universe ∞_(∞-Cat). The fourth repository details how three different internal universes ∞_(Directoid), ∞_(Spectroid), and ∞_(Spectrum). Each of these constituents of ∞-Cat is involved in a similar logic to the situation for frames, locales, join lattices, and meet lattices. The third and four repositories, titled `GeometricMapsinHigherToposTheory` and `GeometricMapsinHigherAlgebra` detail a way to think of 'topological'. A∞-rings and E∞-rings in this way.

It's possible that some of the theorems below could feature nicely in the theory we're striving for:

1. https://ncatlab.org/nlab/show/compact+Hausdorff+rings+are+profinite
2.