Research Interests and Objectives

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STAGE 0 (Category Theory)

1. $\mathsf{Pro} : \mathsf{Cat} \to \mathsf{Cat}$
2. $\mathsf{Ind} : \mathsf{Cat} \to \mathsf{Cat}$

STAGE I (Twenty Four Goals)

1. (Three Whitehead Theorems and Three Puppe Sequences, 2024)

   (a) See the goals listed in the pdf file under the `Three Whitehead Theorems and Three Puppe Sequences` repository.

   (b)

2. (Three Freudenthal Suspension Theorems and Three Long Exact Sequences, 2025)

   (a) See the goals listed in the pdf file under the `Three Freudenthal Suspension Theorems and Three Long Exac` repository

   (b) Term length, typeclasses, and inductive types

Stage II: Goals Concerning Geometric Maps

In *Foundations for Computable Topology*, Paul Taylor developed a synthesis of the logic of topological frames, exposing a duality between the topological concepts of local homeomorphism and proper map, and revealing a new condition (overtness and the analogous condition for maps). All topological spaces are overt, but not all locales.

These ideas have also been developed by Martin Escardo.

One of the most interesting features of this synthesis (abstract stone duality) is that it allows for compact intersections of open sets and compact unions of closed sets, a dual concept to (discrete overt) unions of open sets and (discrete) intersections of closed sets.

A frame can be endowed with a topology using the compact-open topology on the internal hom [X,2]. This topology is the same as the Scott topology.

In one ensuing line of inquiry, the preservation of cofiltered limits and filtered colimits is related to the continuity of a homomorphism of frames.

This project (Stage II) takes as its main interest the logical patterns of the work of Taylor and Escardo. Our goal is to make many of the same proofs work for higher topos theory and higher algebra, in which the open/closed classifier gets replaced by an internal universe object.

The analogues fall within the scope of topos theory and higher algebra.

1. (Topology Project I, 2026)
    (a) The construction of the following universe objects:
        i. ∞_(∞-Cat) : ∞-Cat.α
        ii. ∞_(∞-Grpd) : ∞-Cat.α
        iii. ∞_(∞-Grpd$_0$) : ∞-Cat.α
        iv. D(∞_(∞-Cat)) : ∞-Cat.α
        v. D(∞_(∞-Grpd)) : ∞-Cat.α
        vi. D(∞_(∞-Grpd$_0$)) : ∞-Cat.α
    (b) (Straightening and Unstraightening) Establishing the straightening/unstraightening categorical equivalence, featuring directed homotopy pullback of $\perp$: * $\longrightarrow$ ∞_(∞-Cat).
    (c) The construction of the following:
        i. ∞_(∞-Cat)/C : ∞-Cat.α
        ii. ∞_(∞-Grpd)/G : ∞-Cat.α
        iii. ∞_(∞-Grpd$_0$)/G$_0$: ∞-Cat.α
        iv. D(∞_(∞-Cat)/C) : ∞-Cat.α
        v. D(∞_(∞-Grpd)/G) : ∞-Cat.α
        vi. D(∞_(∞-Grpd$_0$)/G$_0$) : ∞-Cat.α
    (d) (The Projection Formulas) The equivalence between the projection formulas and the condition of being an open or closed map in frame-local theory. See here and here.
    (e) (Universally Closed/Open) ???

(f) (Unramified/Separated) The equivalence between open/closed and universally open / universally closed.

(g) (Étale/Proper)

(h)

(i)

(j)

(k) Proper intersections of closed maps are closed and etale unions of opens are open.

(l) Proper iff universally closed, Étale iff universally open.

(m) Proper iff cofiltered limit in the "category of actions", Étale iff filtered colimit in the "category of actions". In the case of frame-local theory, this is a certain structure to do with a frame (a particular join lattice), but viewed in the category of complete partial orders as opposed to frames and locales.

(n) Tychonoff's theorem:

2. (Topology Project II, 2027):

(a) The construction of the following universe objects:

i. ∞_(Directoid) : ∞-Cat.α
ii. ∞_(Spectroid) : ∞-Cat.α
iii. ∞_(Spectra) : ∞-Cat.α
iv. D(∞_(Directoid)) : ∞-Cat.α
v. D(∞_(Spectroid)) : ∞-Cat.α
vi. D(∞_(Spectra)) : ∞-Cat.α

(b) The construction of the following universe objects:

i. Mon D(∞_(Directoid)) : ∞-Cat.α
ii. Mon D(∞_(Spectroid)) : ∞-Cat.α
iii. Mon D(∞_(Spectra)) : ∞-Cat.α

(c) The construction of the following:

i. Act D : ∞-Cat.α
ii. Act S : ∞-Cat.α
iii. Act S : ∞-Cat.α

(d) Ramification

(e) The closed and open conditions and their relationship with the projection formulas.

(f) Universally closed intersections of closed maps are closed and universally open etale unions of opens are open.

(g) Showing that proper is equivalent to being universally closed with universally closed diagonal, and that Étale is equivalent to being universally open with universally open diagonal. This much can be shown first for the case of partial orders, and then for the case of an unstraightening system.

(h) Proper iff cofiltered limit in the category of actions, Étale iff filtered colimit in the category of actions.

(i) Tychonoff's theorem.

(j) What is "Conglomeration Index"?

(k) Ramified extensions, Unseparated extensions.

Stage III: Goals Concerning Number Theory

1. Constructing the E∞-ring
2. Constructing the A∞-ring $\mathbb{Z}$
3. Constructing $\mathbb{Z}[[x]] \cong [\ldots\Sigma\infty.\text{obj } B.\text{obj } B.\text{obj } \mathbb{Z},\mathbb{Z}]$
4. Constructing the E-∞ ring $\mathbb{Z}/n\mathbb{Z}$
5. Constructing Spec($\mathbb{Z}$)

$B B\mathbb{Z}, B B\mathbb{Z} \cong \mathbb{Z}$

6. How to get $\mathbb{Z}[[x]]$ as a cohomology ring.
   (a) https://math.stackexchange.com/questions/2015228/how-to-show-mathbbcp-infty-mathbbcp-infty-cong-mathbbz
7. Chern-Weil Theory and $\mathbb{Z}[[x_1,\ldots,x\ ]]$
8. How to get $\mathbb{Z}[[x_1,\ldots,x\ ]]$ as a cohomology ring.
   (a)
9. $GL_1(\ )$-representations and finite étale extensions of
10. GL ( )-representations and finite étale extensions of ((t))
11. GL (^{sep})-representations
12. $GL_1(\ )$-representations and the profinite abelian extensions of
13. GL (^{sep})-representations and
    (a)
14. GL ( )-representations.
15.
16. $((x_1,\ldots,x\ ))$
17. We begin the subsection here with the concept of a fixed subobject given an internal action for an object in one of the stable models. We then relate the fixed object to its universal property under certain conditions.
18. Unitary groups
19. One giant group which either (a) quotients onto all unitary groups... (b)
20. $\mathbb{Z}[x_1,\ldots,x\ ]$ is the derived cohomology ring.
21. I would like to form U(n) and also a group U(∞), the group of unitary operators on Hilbert space
22. BU(n) classifies rank n vector bundles
23. GL ($\mathbb{C}$)?
24. $U(1)^n \longrightarrow U(n)^1$

25. Volume of U(n)
26. U(n)
27.
28. H
29. $U(\infty) :=$
30. The classifying space construction is related to the symmetric group and to the symmetric polynoimals
31. U(n)

It will not be for many years that the ideas in this section come to fruition and take their place on the computer. The two stages before this are quite extensive.

In this section, we will define the following rings:

1.
2. $\mathbb{R}$, $\mathbb{R}^{\mathtt{sep}}$, $\mathbb{C}$
3. $\mathbb{Z}$
4. $\mathbb{Z}$
5. $\mathbb{Z} \cong \Pi\ \mathbb{Z}$
6. , $^{\mathtt{sep}}$, $\mathbb{C}$
7. $\mathbb{A}^{\mathtt{fin}}$
8. $\mathbb{A}$

Each of these will be defined first as E∞-rings using the models in the repository of `ThreeStableHomotopyCategoriesand`
We will also use the definition of unramified extensions established in the fourth repository, `GeometricMapsinStableHomotopy`
We then relate these E∞-rings to their corresponding rings via structure preserving bijections.

`0. Spectral Sequences and Exact Couples`

1.

`1. Derivations and Flat Connections`
Derivations and flat connections .

1. Derivations
   (a) Derivations as elements of the $\mathbb{R}$-linear dual of ...
   (b) H_1 ([Ω∞ X,$\mathbb{R}$])
2. Connections
   (a) Flat connections as elements of the $\mathbb{R}$-linear dual of `Ω_(Spectra)C`, where C is the cohomology content (an E∞-ring)
   (b) H_1 ([$\omega\infty$ X V, $\mathbb{R}$])

2. Trace and Determinant

1. The main properties of $\mathbb{r} := \Sigma_i\ (-1)^i * \mathrm{Tr}^{(-1)^i}(\Phi)$
2. The main properties of $\mathbb{et} := \Pi_i\ \mathrm{Det}^{(-1)^i}(\Phi)$

3. Borel-Moore Homology and Cohomology with Compact Support

Borel-Moore homology and cohomology with compact support:

1. Homology, $\mathrm{p}^{\cdot}$ p. $\mathbb{1}$
2. Cohomology $\mathrm{Cmp(p)}^{\cdot}$ p. $\mathbb{1}$
3. Borel-Moore Cohomology Cmp(p). $\mathrm{Cmp(p)}^{\cdot}$
4. Cohomology with compact support p. $\mathrm{Cmp(p)}^{\cdot}$

https://homepages.warwick.ac.uk/staff/Martin.Gallauer/docs/m6ff.pdf

I would eventually like to follow through on several analogous "étalification" based cohomology theories:

1. Homology, $\mathrm{p}^{\cdot}$ p. $\mathbb{1}$
2. ??? $\mathrm{Ét(p)}^{\cdot}$ p. $\mathbb{1}$
3. ??? Ét(p). $\mathrm{Ét(p)}^{\cdot}$
4. ??? p. $\mathrm{Ét(p)}^{\cdot}$

4. Poincare Duality

1. The Cayley-Hamilton Theorem for objects with Poincare duality
2. Pullback and cup product

5. The Kunneth Theorem

1. Tor and Ext

6. Intersection Theory

In this section I would like to establish a few core results concerning intersections. By the date of the implementation of stage III, we will have finished the following repositories:

(a) ThreeWhiteheadTheoremsandThreePuppeSequences

(b) ThreeStableHomotopyCategoriesandThreeLongExactSequences

(c) GeometricMapsinHigherToposTheory

(d) GeometricMapsinStableHomotopyTheory

We will then be in a place to consider an intersection theory using the smash product, $\otimes$. We also develop two generalizations of the smash product for spectroids and directoids.

1. Well behaved cycle maps between pullback and cohomology classes.

2. We would like to mention how these results relate to Bézout's theorem.
3.

7. The Lefschetz Fixed Point Theorem

1. The Lefschetz Fixed Point Theorem follows from results in intersection theory concerning the cycle map, the Kunneth theorem, and Poincare duality.
2.

8. The Cayley-Hamilton Theorem

1. det(Φ- λ I) = (Φ - λ I) • adj(Φ- λ I).
2. End(V $\otimes$ k[t]) $\cong$ End(V) $\otimes$ k[t] for V a free k-action.
3. The theorem is first shown for the case of a finite free module, and then closed under quotients.

9. Exp and Log

1. contains the coefficients in the power series of both exp and log
2. Cohomology with rational coefficients
3. E.obj $\mathbb{Z}$ $\times$ E.obj $\mathbb{Z}$
4. $\mathbb{Z}$ (E.obj $\mathbb{Z}$)
5.
6. (E.obj $\mathbb{Z}$) $\times$ (E.obj $\mathbb{Z}$) $\longrightarrow$ (B.obj $\mathbb{Z}$) $\times$ (E.obj $\mathbb{Z}$)
7.
8. exp : $\mathbb{C} \longrightarrow \mathbb{C}^{x}$
9. log : π_1.obj ($\mathbb{C}^{x}$) $\longrightarrow \mathbb{C}$
10. 1/(1-tz)
11. Under a wide variety of circumstances, the product over n of det($H_0(\Omega^{n}(\Phi))$) to the power $(-1)^{n}$ is the unique invariant with a multiplication rule for exact sequences.
12. adj(1 - t * Φ) * (1 - t * Φ) = det(1 - t * Φ)
13. log(1/(1 - tΦ))
14. log(1/(1 - t * Φ)) = - log(1-t * Φ)
15. log(1 - t * Φ) = - $\int$ (n : Nat) Pow n (t * Φ)
16.
17. log(1-tΦ) and both have power-series expansions around t = 0
18. exp, log
19. Tr : [V,V] $\longrightarrow \mathbb{N}$

20. Algebraically closed field
21.
22. Det : [V,V] $\longrightarrow$ k
23. $\mathbb{r}$:= Σ_i $(-1)^i$ * $\mathrm{Tr}^{(-1)^i}(\Phi)$
24. $\mathbb{e}\mathbb{t}$:= Π_i $\mathrm{Det}^{(-1)^i}(\Phi)$
25. (1 - t * Φ) * adj(1 - t * Φ) = det(1- t * Φ)
26. log(1 - t * Φ) = - $\int$ (n : $\mathbb{N}$) $(t * \Phi)^{\mathtt{n}}$
27. is_invertible det(1 - t * Φ)
28. log(1/(1-tΦ)) = $\int$ (n : $\mathbb{N}$) $(t * \Phi)^{\mathtt{n}}$
29. Functional equation given a Poincare duality
30. Sorry, I made a ton of mistakes. I guess I wanted A to be normal in the above.
31. The power series for log : Mat(n×n) $\mathbb{C}$ converges for || Φ - λ I || < 1.

`10. L-Functions`

1. Obtaining a graded ring from an A∞-ring.
2. Obtaining a graded ring from an E∞-ring.

`11. Etale Homotopy`

Consider one of the internal objects defined in the first repository of `ThreeWhiteheadTheoremsandThreePuppeSequences`

1. Internal category
2. Internal groupoid
3. Internal group

The functors satisfy a condition which ensures that smash product and its two analogues be sent to product of ∞-categories, ∞-groupoids, and based connected ∞-groupoids, respectively.

1. How to obtain an ∞-category from a map of A∞-rings
2. How to obtain an ∞-groupoid from a map of E∞-rings

`12. Topological Topoi`

`13. Topological Stable Homotopy Categories`

In the third and fourth repositories, we took inspiration from the study of topological frames in thinking about 'topological topoi', in which the topological frame arising from the frame with two points is replaced with the internal universe ∞_(∞-Cat). The fourth repository details how three different internal universes ∞_(Directoid), ∞_(Spectroid), and ∞_(Spectrum). Each of these constituents of ∞-Cat is involved in a similar logic to the situation for frames, locales, join lattices, and meet lattices. The third and four repositories, titled `GeometricMapsinHigherToposTheory` and `GeometricMapsinHigherAlgebra` detail a way to think of 'topological'. A∞-rings and E∞-rings in this way.

It's possible that some of the theorems below could feature nicely in the theory we're striving for:

1. https://ncatlab.org/nlab/show/compact+Hausdorff+rings+are+profinite
2.