The Puppe Sequence and Two Variations

$$\cdots \longrightarrow \pi_0.\text{obj }(\Omega.\text{obj C}) \longrightarrow \pi_0.\text{obj }(\Omega.\text{obj D}) \circlearrowright \pi_0.\text{obj }(\omega.\text{obj ???}) \longrightarrow (\pi_0.\text{obj C}) \longrightarrow (\pi_0.\text{obj D})$$

$$\cdots \longrightarrow \pi_0.\text{obj }(\Omega.\text{obj E}) \longrightarrow \pi_0.\text{obj }(\Omega.\text{obj B}) \circlearrowright \pi_0.\text{obj }(\omega.\text{obj ???}) \longrightarrow (\pi_0.\text{obj E}) \longrightarrow (\pi_0.\text{obj B})$$

$$\cdots \longrightarrow \pi_0.\text{obj }(\Omega.\text{obj E}_0) \longrightarrow \pi_0.\text{obj }(\Omega.\text{obj B}_0) \longrightarrow \pi_0.\text{obj }(\omega.\text{obj ???}) \longrightarrow \pi_0.\text{obj }(\text{E}_0) \longrightarrow \pi_0.\text{obj}(\text{B}_0)$$

E. Dean Young and Jiazhen Xia

Plans to define three variations of the Puppe sequence of homotopy groups in and Lean 4, and to prove their exactness, with extensive use of Mathlib 4

We wish to acknowledge the collaborative efforts of E. Dean Young and Jiazhen Xia. Dean Young initially formulated the introduction with twelve goals, posting them on the Lean Zulip in August of 2023. Together the authors are pursuing these plans as a long term project.

1. Contents

The table of contents below reflects the tentative long-term goals of the authors, with the main goal the pursuit of the Whitehead theorem for a point-set model involving Mathlib's predefined homotopy groups.

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Unfinished	
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Chapter 2: The Smash Product in ∞-Grpd_0	
Pair ∞-Grpd_0	The category of pairs
$\wedge$_(Pair ∞-Grpd_0), [,]_(Pair ∞-Grpd_0)	The monoidal closed structure on Pair ∞-Grpd_0
D(Pair ∞-Grpd_0)	The derived category of pairs
$\wedge$_(D(Pair ∞-Grpd_0)), [,]_(D(Pair ∞-Grpd_0))	The cartesian closed structure on D(Pair ∞-Grpd_0)
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Pair Grpd	The category of pairs of ∞-groupoids
$\wedge$_(Pair ∞-Grpd), [,]_(Pair ∞-Grpd)	The cartesian closed structure on Pair ∞-Grpd
D(Pair ∞-Grpd)	The derived category of pairs of ∞-groupoids
$\wedge$_(D(Pair ∞-Grpd)), [,]_(D(Pair ∞-Grpd))	The cartesian closed structure on D(Pair ∞-Grpd)
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Pair ∞-Cat	The category of pairs

∧_(Pair ∞-Cat), [,]_(Pair ∞-Cat)	The cartesian closed structure on Pair Grpd$_0$
D(Pair ∞-Cat)	The derived category of pairs
∧_(D(Pair ∞-Cat)), [,]_(D(Pair ∞-Cat))	The cartesian closed structure on D(Pair ∞-Cat)
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Colimits of Inclusions	...
The Mayer-Vietoris Sequence	...

2. Introduction

The main goals of this repository is to define the Puppe sequence and to prove its exactness, along with two variations.

(a) ...

(b) ...

(c) ...

We have stated these theorems in the above in an order reversed from the order of its implementation. This choice

We will use two models of each of the following categories in the theorems above:

(i) We model ∞-Cat : Cat firstly as the category of categories enriched over a convenient category of topological spaces, and secondly as the category of quasicategories.

(ii) We model ∞-Grpd : Cat firstly as a convenient category of topological spaces, and secondly as the category of Kan complexes.

(iii) We model $\infty\text{-Grpd}_{0}$: Cat firstly as the based connected objects of a convenient category of topological spaces, and secondly as the category of based connected Kan complexes.

This choice accords with the standard approach to the third theorem, in which one typically chooses both a combinatorial and point-set model, with the former featuring a geometric realization functor into the latter (`Mathlib` already has this).

We will make heavy use of `Mathlib 4`'s material on category theory, particularly their categories, functors, and natural transformations:

1. Categories (see Mathlib's `Category X` here; these can be bundled into `category`)
2. Functors (see Mathlib's `Functor C D` here; these can be bundled into `functor`)
3. Natural transformations (see Mathlib's `NatTrans F G` here; these can be bundled into `natural_transform`)
4. Equations between natural transformations (see Mathlib's `NatExt` here; these are related to our `equation`)

While the functors π_n occuring in the main theorems above are already defined in `Mathlib` for the desired point-set model, the functors π_n and π_n are not, and their definition will require great care. Here are their types:

(i) π_n : Functor ∞-Cat Set

(ii) π_n : Functor ∞-Grpd Set

(iii) π_n : Functor ∞-Grpd$_0$ Set

We may wish to modify these types out of convenience and to accord with the pre-existing functors π_n in `Mathlib 4`.

The existence of a base point makes π_n relatively straightforward to define, while π_n and π_n 'grow' as n does. We also form their derived functors:

(i) $\mathrm{D}(\pi_n) : \mathrm{D}(\infty\text{-Cat}) \longrightarrow \mathrm{D}(\mathrm{Set}) \simeq \mathrm{Set}$

(ii) $\mathrm{D}(\pi_n) : \mathrm{D}(\infty\text{-Grpd}) \longrightarrow \mathrm{D}(\mathrm{Set}) \simeq \mathrm{Set}$

(iii) $\mathrm{D}(\pi_n) : \mathrm{D}(\infty\text{-Grpd}_0) \longrightarrow \mathrm{D}(\mathrm{Set}) \simeq \mathrm{Set}$

In the course of the repository we will need the directed path space, path space, and loop space functors as well, which fit with the analogy formed by the Whitehead theorem and its two variations:

1. $\Omega : \infty\text{-Cat} \longrightarrow \infty\text{-Cat}$ is the internal hom functor $[\Delta^1, \text{-}]$ (directed path space)
2. $\Omega : \infty\text{-Grpd} \longrightarrow \infty\text{-Grpd}$ is the internal hom functor [I,-] (path space)
3. Ω is the loop space functor

The third theorem (c), is the one from Whitehead's original papers.

With the choice of quasicategories as a combinatorial model, we hope to give good integration with Mathlib's existing features (though technically only the inner horns and simplices are defined, not even the category of quasicategories itself).

In the directed context, a homotopy between two maps in ∞-Cat/C consists of a sequence of compatible directed homotopies with the odd morphisms in the sequence formed from reversed copies of Δ^1. Really we have two such categories, one of which consists of formal words, and another which involves ∞-categories and ∞-functors in the image of `repl`).

The main technical feature in the proofs of these theorems concerns a lifting property which successively lifts a homotopy along a single attachment of Δ^n along its boundary $\partial\Delta^n$. A homotopy $h : \partial\Delta^n \times \Delta^1 \longrightarrow Y$ between $f, g : \partial\Delta^n \longrightarrow Y$ extends to a map $H : \Delta^n \times \Delta^1 \longrightarrow Y$. The directed case requires an extra technical feature. H(-,1) and g match on $\partial\Delta^{\mathtt{n}}$, producing a map f : X $\longrightarrow$ Y, where X consists of two copies of $\Delta^{\mathtt{n}}$ glued together at the boundary.

Consider a space X' formed as a quotient of $\Delta^{\mathtt{n}} \times \Delta^{1}$ by $\partial\Delta^{\mathtt{n}} \times \Delta^{1}$. There is a map ϕ : X $\longrightarrow$ X'. An induction hypothesis on f and g involving π_n ensures that the aparent map X $\longrightarrow$ Y lifts along ϕ, producing a map from $\Delta^{\mathtt{n}} \times \Delta^{1}$ which is constant on $\partial\Delta^{\mathtt{n}} \times \Delta^{1}$. Stacking this on top of H can be done using an isomorphism between Δ^{1} and Δ^{1} glued with itself along different endpoints. Altogether this produces a homotopy between f and g.

We will define three different kinds of derived category:

1. D(∞-Cat) : Cat (the directed derived category of ∞-categories)
2. D(∞-Grpd) : Cat (the derived category of ∞-groupoids)
3. D(∞-Grpd$_0$) : Cat (the derived category of based ∞-groupoids)

We then create the second kind of derived category, one for each of the objects in the respective categories above:

1. For C : D(∞-Cat), a category D(∞-Cat/C) : Cat
2. For G : D(∞-Grpd), a category D(∞-Grpd/G) : Cat
3. For G$_0$: D(∞-Grpd$_0$), a category D(∞-Grpd$_0$/G$_0$) : Cat

For the model built on simplicial sets, Ω will be representable by Δ^{1} with respect to an internal hom, and Ω will be representable by a model of the unit interval I := [0,1].

We will use six “internal” structures in addition to the standard structures in category theory:

1. IntCat : Cat $\to$ Cat
2. InfPreShf : (X : Cat) $\to$ (C : (IntCat X)) $\to$ Cat
3. IntGrpd : Cat $\to$ Cat
4. IntAct : (X : Cat) $\to$ (G : (IntGrpd X)) $\to$ Cat
5. IntGrp : Cat $\to$ Cat
6. IntAct$_0$: (X : Cat) $\to$ (G$_0$: (IntGrp X)) $\to$ Cat

The book "Galois theories" by Borceux and Janelidze deserves special mention as an inspiration for these internal structures. That book details how to think about Galois theory using internal groupoids, internal G-presheaves, monadicity, comonadicity, and the constructions involved in Eilenberg-Moore theory.

The six internal structures above arise here in relation to six functors:

(I) $\Omega : \infty$-Cat $\longrightarrow \infty$-Cat (notation for the directed path space functor, related to $[\Delta^1$,-]). D(Ω) factors through internal categories in D(∞-Cat) by a categorical equivalence D(∞-Cat) $\cong$ IntCat D(∞-Cat) (internal categories in D(∞-Cat))

(II) ω ($\mathbb{1}$ C) : ∞-Cat/C $\longrightarrow \infty$-Cat/C, the derived directed homotopy pullback with $\mathbb{1}$ C. D(ω ($\mathbb{1}$ C)) factors through a categorical equivalence between D(∞-Cat/C) and internal PC-presheaves in D(∞-Cat/C).

(III) $\Omega : \infty$-Grpd $\longrightarrow \infty$-Grpd (notation for the path space functor [I,-]), the derived homotopy pullback of an ∞-groupoid with itself. D(Ω) factors through a categorical equivalence between D(∞-Grpd) and internal groupoids in D(∞-Grpd)

(IV) ω ($\mathbb{1}$ X) : ∞-Grpd/X $\longrightarrow \infty$-Grpd/X, the derived homotopy pullback with $\mathbb{1}$ X. D(ω ($\mathbb{1}$ X)) factors through internal PX

(V) $\Omega : \infty$-Grpd$_0 \longrightarrow \infty$-Grpd$_0$, the loop space functor. D(Ω) factors through a categorical equivalence between D(∞-Grpd$_0$) and internal groups in D(∞-Grpd$_0$) (the loop space functor on connected based ∞-groupoids)

(VI) ω ($\mathbb{1}$ X) : ∞-Grpd$_0$/X$_0 \longrightarrow \infty$-Grpd$_0$/X$_0$, the homotopy pullback with the base of X$_0$. D(ω ($\mathbb{1}$ X)) factors through internal PX$_0$-actions in based connected spaces over X$_0$.

(v) in the above is shown here and (vi) in the above is shown in a typical exposition of *G*-principal bundles.

The functors ω ($\mathbb{1}$ C), ω ($\mathbb{1}$ X), and ω ($\mathbb{1}$ C) in the above ensue from a more general construction:

1. For C, D : D(∞-Cat), and f : C $\longrightarrow$ D, ω f : D(∞-Cat/D) $\longrightarrow$ D(∞-Cat/C) (derived directed homotopy pullback)

2. For B, E : D(∞-Grpd), and f : E $\longrightarrow$ B, ω f : D(∞-Grpd/B) $\longrightarrow$ D(∞-Grpd/E) (derived homotopy pullback)

3. For B$_0$, E$_0$: D(∞-Grpd$_0$), and f : E$_0 \longrightarrow$ B$_0$, ω f : D(∞-Grpd$_0$/B$_0$) $\longrightarrow$ D(∞-Grpd$_0$/E$_0$) (homotopy pullback with the base)

These six factored functors P , P , P : D(∞-Grpd$_0$), p ($\mathbb{1}$ C), p ($\mathbb{1}$ X), p are each fully faithful and produce categorical equivalences; we later construct functors B, B, B, b, b, b defined on the essential image of these six, which are inverse to them up to natural isomorphism.

We obtain six categorical equivalences witnessed by these twelve functors (along with twelve natural isomorphisms). Here are the types of P , P , P : D(∞-Grpd$_0$), p ($\mathbb{1}$ C), p ($\mathbb{1}$ X), p:

1. The directed path space, the path space, and loop space form components of the functors P, P, and P, which are valued in internal categories, internal groupoids, and internal groups respectively.
   (a) P : D(∞-Cat) $\longrightarrow$ Cat D(∞-Cat)
   (b) P : D(∞-Grpd) $\longrightarrow$ Grpd D(∞-Grpd)
   (c) P : D(∞-Grpd$_0$) $\longrightarrow$ Grp D(∞-Grpd) (see here)
2. The directed homotopy pullback, the homotopy pullback, and the homotopy pullback with the base form components of the functors Alg(Mon(ω)), Alg(Mon(ω)), and Alg(Mon(p)), respectively.
   (a) p ($\mathbb{1}$ C) : D(∞-Cat/C) $\longrightarrow$ InfPreShf D(∞-Cat/C) P.obj C
   (b) p ($\mathbb{1}$ X) : D(∞-Grpd/X) $\longrightarrow$ IntAct D(∞-Grpd/X) P.obj X
   (c) p ($\mathbb{1}$ X$_0$) : D(∞-Grpd$_0$/X$_0$) $\longrightarrow$ IntAct$_0$ D(∞-Grpd$_0$/X$_0$) P.obj X$_0$

Above, the functors P, P, P, p, p, and p feature Ω, Ω, Ω, ω, ω, and ω in their components, and can be related to them using constructions from Eilenberg-Moore theory.

These six new functors combine with the functors below to form categorical equivalences:

1. The directed homotopy colimit of a point with an internal category in D(∞-Cat) as a diagram, the homotopy colimit of a constant functor with an internal internal group as a diagram
   (a) B : essential_image P $\longrightarrow$ D(∞-Cat)
   (b) B : essential_image P $\longrightarrow$ D(∞-Grpd)
   (c) B : essential_image P $\longrightarrow$ D(∞-Grpd$_0$) (see here)
2. The clutching functors are inverse to the above functors up to natural isomorphism:
   (a) b : essential_image p $\longrightarrow$ D(∞-Cat/C)

(b) b : essential_image p $\longrightarrow$ D(∞-Cat/C)

(c) b : essential_image p $\longrightarrow$ D(∞-Grpd$_0$/X$_0$)

We will show six categorical equivalences featuring these:

1. P $\bullet$ B $\cong \mathbb{1}$ (DlpIntCat D(∞-Cat)) and B $\bullet$ P $\cong \mathbb{1}$ D(∞-Cat)
2. P $\bullet$ B $\cong \mathbb{1}$ (DlpIntGrpd D(∞-Grpd)) and B $\bullet$ P $\cong \mathbb{1}$ D(∞-Grpd)
3. P $\bullet$ B $\cong \mathbb{1}$ (DlpIntGrp D(∞-Grpd$_0$)) and B $\bullet$ P $\cong \mathbb{1}$ D(∞-Grpd$_0$) (see here)
4. p $\bullet$ b $\cong \mathbb{1}$ (DlpInfPreShf D(∞-Cat/C) PC) and b $\bullet$ p $\cong \mathbb{1}$ D(∞-Cat/C)
5. p $\bullet$ b $\cong \mathbb{1}$ (DlpIntAct D(∞-Cat/C) PX) and b $\bullet$ p $\cong \mathbb{1}$ D(∞-Cat/C)
6. p $\bullet$ b $\cong \mathbb{1}$ (DlpIntAct$_0$ D(∞-Grpd$_0$/X$_0$) PX$_0$) and b $\bullet$ p $\cong \mathbb{1}$ D(∞-Grpd$_0$/X$_0$) (see here)

Take special note that each of these six involves a condition ensuring that the functor B be well defined. Consider the functors:

1. D(IntCat ∞-Cat) $\longrightarrow$ IntCat D(∞-Cat)
2. D(IntGrpd ∞-Grpd) $\longrightarrow$ IntGrpd D(∞-Grpd)
3. D(IntGrp ∞-Grpd$_0$) $\longrightarrow$ IntGrp D(∞-Grpd$_0$)
4. D(InfPreShf ∞-Cat/C) PC $\longrightarrow$ InfPreShf D(∞-Cat/C) PC
5. D(IntAct ∞-Cat/C) PX $\longrightarrow$ IntAct D(∞-Cat/C) PX
6. D(IntAct$_0$ ∞-Grpd$_0$/X$_0$) PX$_0$ $\longrightarrow$ IntAct$_0$ D(∞-Grpd$_0$/X$_0$) PX$_0$

It may happen that a given object in the codomain of these six functors lies in their essential image. In this case, any of the six of B, B, B, b, b, b can sometimes but not always be obtained as a quotient of six functors E, E, E, e, e, e, respectively:

1. E : IntCat ∞-Cat $\longrightarrow$ ∞-Cat
2. E : IntGrpd ∞-Grpd $\longrightarrow$ ∞-Grpd
3. E : IntGrp ∞-Grpd$_0$ $\longrightarrow$ ∞-Grpd$_0$
4. e : PreShf ∞-Cat/C P.obj C $\longrightarrow$ ∞-Cat/C P.obj C
5. e : IntAct ∞-Cat/C P.obj X $\longrightarrow$ ∞-Cat/C P.obj X
6. e : IntAct$_0$ ∞-Grpd$_0$/X$_0$ P.obj X$_0$ $\longrightarrow$ ∞-Grpd$_0$/X$_0$ P.obj X$_0$

The functors above will be defined as certain homotopy colimits, themselves certain coequilizers. On the condition that an internal category is internally filtered and internally cofiltered, we can further construct the B.

We will make extensive use of Mathlib's bicategory of categories and material on simplicial sets. We further use Mathlib's pullbacks and categorical products, as well as their Eilenberg-Moore theory constructions. I'd like to extend my appreciation to Scott Morison, Eric Wieser, Floris Van Doorn, and all the contributors who have put their efforts into creating these robust features for Mathlib 4.

Altogether, the project gets the following "periodic table" of 30 functors featured on the front cover:

$\mathtt{D}(\infty\text{-}\mathtt{Cat})$	Σ	Ω	P	B	E	$\mathtt{D}(\infty\text{-}\mathtt{Cat/C})$	σ	ω	b	p	e
$\mathtt{D}(\infty\text{-}\mathtt{Grpd})$	Σ	Ω	P	B	E	$\mathtt{D}(\infty\text{-}\mathtt{Grpd/G})$	σ	ω	b	p	e
$\mathtt{D}(\infty\text{-}\mathtt{Grpd}_0)$	Σ	Ω	P	B	E	$\mathtt{D}(\infty\text{-}\mathtt{Grpd}_0/\mathtt{G}_0)$	σ	ω	b	p	e

Here are the names of the symbols featured above:

Suspensional	Deductive	Remembrant	Delooping	Free
Σ (Directed suspension)	Ω (Directed path space)	P (Remembrant derived directed path space)	B (Classifying space for internal categories)	E
Σ (Suspensionoid)	Ω (Path space)	P (Remembrant derived path space)	B (Classifying space for internal groupoids)	E
Σ (Suspension)	Ω (Loop space)	P (Remembrant derived loop space)	B (Classifying space for internal groups)	E
σ	ω (Directed homotopy pushout with a point)	p (Remembrant derived directed homotopy pullback)	b (Classifying space for internal presheaves)	e
σ	ω (Homotopy pushout with a point)	p (Remembrant derived homotopy pullback)	b (Classifying space for internal groupoid actions)	e
σ	ω (Homotopy fiber)	p (Remembrant derived homotopy fiber)	b (Classifying space for internal group actions)	e

The term "remembrant" in the above is not common terminology. It is intended to mean that the second collumn features functors which are valued in categories of internal objects wheras the left collumn forms particular components of those structures.

The notation here is both an attempt to make the three-fold division of the project (three Whitehead theorems, three Puppe sequences, etc.) manifest while sticking to the standard notation for the established theorems (Σ, Ω, B, E). In the above, P could be said to stand for "(remembrant) path space" and p for "(remembrant) pullback", while at the same time this matches the theme that our capital letters reflect various internal structures and that their lower-case forms reflect the corresponding actions.

The mentioned "delooping principals", which identify inverses to the remembrant functors *on their essential image*, form important consequences of the three Puppe sequences. All in all, there are nine important theorems we want to show:

Twelve Goals

(I) Define the Puppe sequence for ∞-categories and prove its exactness.

(II) Define and prove the `internal_category_delooping_principal : Type`.

(III) Define and prove the `internal_sheaf_delooping_principal : Type`.

(IV) Define the Puppe sequence for ∞-groupoids and prove its exactness

(V) Define and prove the `internal_groupoid_delooping_principal : Type`.

(VI) Define and prove the `internal_groupoid_action_delooping_principal : Type`.

(VII) Define the Puppe sequence for based connected ∞-groupoids and prove its exactness

(VIII) Define and prove the `internal_group_delooping_principal : Type`.

(IX) Define and prove the `internal_group_action_delooping_principal : Type`.

None of these theorems are currently contained in Mathlib. The last three are well-known.

In the work that ensues, we plan to take an approach which establishes the known results before the original ones, taking advantage of the predefined π_n functors in `Mathlib 4` in the process. This decision will also help to start with smaller pull requests.

3. Unicode

Here is a list of the unicode characters we will use:

Symbol	Unicode	VSCode shortcut	Use
		Lean's Kernel	
×	2A2F	`\times`	Product of types
→	2192	`\rightarrow`	Hom of types
⟨,⟩	27E8,27E9	`\langle,\rangle`	Product term introduction
↦	21A6	`\mapsto`	Hom term introduction
∧	2227	`\wedge`	Conjunction
∨	2228	`\vee`	Disjunction
∀	2200	`\forall`	Universal quantification
∃	2203	`\exists`	Existential quantification
¬	00AC	`\neg`	Negation
		Variables and Constants	
$^{a},^{b},^{c},\ldots,^{z}$	1D52,1D56		Variables and constants
$^{0},^{1},^{2},^{3},^{4},^{5},^{6},^{7},^{8},^{9}$	1D52,1D56		Variables and constants
$^{-}$	207B		Variables and constants
$_{0},_{1},_{2},_{3},_{4},_{5},_{6},_{7},_{8},_{9}$	2080 - 2089	`\0-\9`	Variables and constants
𝔸,...,ℤ	1D538		
𝕒,...,𝕫	1D552		
A,...,**Z**	1D41A		
a,...,**z**	1D41A		
α-ω,Α-Ω	03B1-03C9		Variables and constants
		Categories	
𝟙	1D7D9	`\b1`	The identity morphism
∘	2218	`\circ`	Composition
		Bicategories	
•	2022	`\smul`	Horizontal composition of objects
		Adjunctions	
⇄	21C4	`\rightleftarrows`	Adjunctions
⇆	21C6	`\leftrightarrows`	Adjunctions
.	1BC94		Right adjoints
·	0971		Left adjoints
⊣	22A3	`\dashv`	The condition that two functors are adjoint
		Monads and Comonads	
?,¿	003F, 00BF	`?,\?`	The corresponding (co)monad of an adjunction
!,¡	0021, 00A1	`!, \!`	The (co)-Eilenberg-(co)-Moore adjunction
ꜝ,ꜞ	A71D, A71E		The (co)exponential maps
		Miscellaneous	
∼	223C	`\sim`	Homotopies
≃	2243	`\equiv`	Equivalences
≅	2245	`\cong`	Isomorphisms
⊥	22A5	`\bot`	The overobject classifier
∞	221E	`\infty`	Infinity categories and infinity groupoids
$^{\leftrightarrow}$	20D7		Homotopical operations on ∞-categories
$^{\rightarrow}$	20E1		Homotopical operations on ∞-groupoids

PART 1: BASED CONNECTED ∞-GROUPOIDS

Chapter 1: The Puppe Sequence

This chapter establishes the well know Puppe sequence for the based homotopy groups π_n. This is the well known Puppe sequence of homotopy groups.

Chapter 2: The Smash Product for $\infty\text{-}\mathtt{Grpd}_0$

Chapter 3: The Triangle Theorems

PART 2: ∞-GROUPOIDS

Chapter 4: The Puppe Sequence for ∞-Grpd

Chapter 5: The Smash Product for ∞-Grpd

Chapter 6: The Triangle Theorems for ∞-Grpd

PART 3: ∞-CATEGORIES

Chapter 7: The Puppe Sequence for ∞-Cat

In this chapter we construct the Puppe sequence for π_n. **Note: one joint in this exact sequence consists not of a map but an action.} This will be used in the next chapter two establish two of the six categorical equivalences.**

Chapter 8: The Smash Product for ∞-Cat

Chapter 9: The Triangle Theorems for ∞-Cat

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Further reading:

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3. Samuel Eilenberg and Saunders Mac Lane, “General Theory of Natural Equivalences,” Transactions of the American Mathematical Society, vol. 58, no. 2, pp. 231-294, 1945.
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5. Chris Heunen, Jamie Vicary, and Stefan Wolf, “Categories for Quantum Theory: An Introduction,” Oxford Graduate Texts, Oxford University Press, Oxford, 2018.
6. S. Eilenberg and J. C. Moore, “Adjoint Functors and Triples,” Proceedings of the Conference on Categorical Algebra, La Jolla, California, 1965, pp. 89-106.
7. Daniel M. Kan, “On Adjoints to Functors” (1958): In this paper, Kan further explored the theory of adjoint functors, focusing on the existence and uniqueness of adjoints. His work provided important insights into the fundamental aspects of adjoint functors and their role in category theory.

Lectures, Videos, and Stackexchange questions:

1. `https://www.youtube.com/watch?v=Ob9tOgWumPI`
2. `https://www.youtube.com/watch?v=xYenPIeX6MY`
3. `https://mathoverflow.net/questions/5901/do-the-signs-in-puppe-sequences-matter`

Relevant discussions on the `Lean 4` Zulip:

1.

Ideas for future applications:

1. `https://arxiv.org/pdf/2206.13563.pdf`

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