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The Puppe Sequence and Two Variations

$$\cdots \longrightarrow \vec{\pi}_0.\mathrm{obj}\ (\vec{\Omega}.\mathrm{obj}\ \mathrm{C}) \longrightarrow \vec{\pi}_0.\mathrm{obj}\ (\vec{\Omega}.\mathrm{obj}\ \mathrm{D}) \circlearrowright \vec{\pi}_0.\mathrm{obj}\ (\vec{\omega}.\mathrm{obj}\ ???) \longrightarrow (\vec{\pi}_0.\mathrm{obj}\ \mathrm{C}) \longrightarrow (\vec{\pi}_0.\mathrm{obj}\ \mathrm{D})$$

$$\cdots \longrightarrow \overleftrightarrow{\pi}_0.\mathrm{obj}\ (\overleftrightarrow{\Omega}.\mathrm{obj}\ \mathrm{E}) \longrightarrow \overleftrightarrow{\pi}_0.\mathrm{obj}\ (\overleftrightarrow{\Omega}.\mathrm{obj}\ \mathrm{B}) \circlearrowright \overleftrightarrow{\pi}_0.\mathrm{obj}\ (\overleftrightarrow{\omega}.\mathrm{obj}\ ???) \longrightarrow (\overleftrightarrow{\pi}_0.\mathrm{obj}\ \mathrm{E}) \longrightarrow (\overleftrightarrow{\pi}_0.\mathrm{obj}\ \mathrm{B})$$

$$\cdots \longrightarrow \pi_0.\mathrm{obj}\ (\Omega.\mathrm{obj}\ \mathrm{E}_0) \longrightarrow \pi_0.\mathrm{obj}\ (\Omega.\mathrm{obj}\ \mathrm{B}_0) \longrightarrow \pi_0.\mathrm{obj}\ (\omega.\mathrm{obj}\ ???) \longrightarrow \pi_0.\mathrm{obj}\ (\mathrm{E}_0) \longrightarrow \pi_0.\mathrm{obj}(\mathrm{B}_0)$$

Plans to define three variations of the
Puppe sequence of homotopy groups in and
Lean 4, and to prove their exactness,
with extensive use of Mathlib 4

We wish to acknowledge the collaborative efforts of E. Dean Young and Jiazhen Xia. Dean Young initially formulated the introduction with twelve goals, posting them on the Lean Zulip in August of 2023. Together the authors are pursuing these plans as a long term project.

1. Contents

The table of contents below reflects the tentative long-term goals of the authors, with the main goal the pursuit of the Whitehead theorem for a point-set model involving Mathlib's predefined homotopy groups.

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$\wedge$_(Pair ∞-Grpd$_0$), [,]_(Pair ∞-Grpd$_0$)	The monoidal closed structure on Pair ∞-Grpd$_0$
D(Pair ∞-Grpd$_0$)	The derived category of pairs
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D(Pair ∞-Cat$_0$)	The derived category of pairs
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Pair Grpd	The category of pairs of ∞-groupoids

∧_(Pair ∞-Grpd), [,]_(Pair ∞-Grpd)	The cartesian closed structure on Pair ∞-Grpd
D(Pair ∞-Grpd)	The derived category of pairs of ∞-groupoids
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2. Introduction

The main goal of this repository is to define the Puppe sequence and to prove its exactness. I would also like to pursue two variations of the Puppe sequence, featuring internal groupoids and internal groupoid actions, and internal categories and internal presheaves, respectively, instead of internal groups and internal group actions. Note that we may construct the Puppe sequence so as to feature groups, group actions, abelian groups, and abelian group actions, which will allow us to use Mathlib's pre-existing structures.

The document is very incomplete, but expresses intentions and plans that I have. I hope that the highly tabulated approach can make for something maintainable and usable. The Puppe sequence formalization will take place after the completion of the Whitehead theorem, which Jiazhen Xia and I have already make a lot of progress on.

In the thread I started originally, I had arranged for the most difficult Puppe sequence, involving internal categories and internal presheaves, to be defined and proven first. Unfortunately this approach requires a difficult analogue of “‘jar filling”’. Later I assimilated the suggestion of Joël Riou that we start with the Whitehead theorem and Puppe sequence for CW-complexes, in which a jar shape and its higher dimensional analogues can be filled. Jiazhen and I have completed this and the code can be found in the repository of “‘TheWhiteheadTheoremandTwoVariations“‘.

We will use two models of each of the following categories in the theorems above:

(i) We model $\infty\text{-Grpd}_0 : \mathrm{Cat}$ firstly as the based connected objects of a convenient category of topological spaces, and secondly as the category of based connected Kan complexes.

(ii) We model $\infty\text{-Grpd} : \mathrm{Cat}$ firstly as a convenient category of topological spaces, and secondly as the category of Kan complexes.

(iii) We model $\infty\text{-Cat} : \mathrm{Cat}$ both combinatorially with quasicategories and using a point-set model ().

This choice accords with the standard approach to the third theorem, in which one typically chooses both a combinatorial and point-set model, with the former featuring a geometric realization functor into the latter (`Mathlib` already has this).

We will use `Mathlib 4`'s category theory, particularly their categories, functors, and natural transformations:

1. Categories (see Mathlib's `Category X` here; these can be bundled into `category`)
2. Functors (see Mathlib's `Functor C D` here; these can be bundled into `functor`)
3. Natural transformations (see Mathlib's `NatTrans F G` here; these can be bundled into `natural_transform`)
4. Equations between natural transformations (see Mathlib's `NatExt` here; these are related to our `equation`)

I also am maintaining a category theory repository I have with Shanghe Chen, which follows the book "'Galois Theories"' by George Janelidze and Francis Borceux. The main addition it makes is that of a seven-entry strict twocategory (as opposed to a particular bicategory). Perhaps the most interesting feature is that both the seven entry category structure and the seven entry strict twocategory structure are instances of the thirteen entry internal category structure used in this repository and which will be defined in the repository concerning the Whitehead theorem.

While the functors π_n occuring in the main theorems above are already defined in `Mathlib` for the desired point-set model, the functors $\vec{\pi}_n$ and $\overleftrightarrow{\pi}_n$ are not, and their definition will require great care. Here are their types:

(i) π_n : Functor ∞-Grpd$_0$ Set

(ii) $\overleftrightarrow{\pi}_n$: Functor ∞-Grpd Set

(iii) $\vec{\pi}_n$: Functor ∞-Cat Set

The π_n's in the above are not quite the same as the ones defined in Mathlib already, but several simple lemmas concerning categorical equivalences will solve this small issue.

We may wish to modify these types out of convenience and to accord with the pre-existing functors π_n in `Mathlib 4`.

The existence of a base point makes π_n relatively straightforward to define, while $\vec{\pi}_n$ and $\overleftrightarrow{\pi}_n$ 'grow' as n does. We also form their derived functors:

(i) $\mathrm{D}(\vec{\pi}_n) : \mathrm{D}(\infty\text{-Cat}) \longrightarrow \mathrm{D}(\mathrm{Set}) \simeq \mathrm{Set}$

(ii) $\mathrm{D}(\overleftrightarrow{\pi}_n) : \mathrm{D}(\infty\text{-Grpd}) \longrightarrow \mathrm{D}(\mathrm{Set}) \simeq \mathrm{Set}$

(iii) $\mathrm{D}(\pi_n) : \mathrm{D}(\infty\text{-Grpd}_0) \longrightarrow \mathrm{D}(\mathrm{Set}) \simeq \mathrm{Set}$

In the course of the repository we will need the directed path space, path space, and loop space functors as well, which fit with the analogy formed by the Whitehead theorem and its two variations:

1. $\vec{\Omega} : \infty\text{-Cat} \longrightarrow \infty\text{-Cat}$ is the internal hom functor $[\Delta^1,\text{-}]$ (directed path space)
2. $\overleftrightarrow{\Omega} : \infty\text{-Grpd} \longrightarrow \infty\text{-Grpd}$ is the internal hom functor [I,-] (path space)
3. Ω is the loop space functor

The third theorem (c), is the one from Whitehead's original papers (these are included in the repository concerning the Whitehead theorem).

With the choice of quasicategories as a combinatorial model, we hope to give good integration with Mathlib's existing features (though technically only the inner horns and simplices are defined, not even the category of quasicategories itself).

In the directed context, a homotopy between two maps in ∞-Cat/C consists of a sequence of compatible directed homotopies with the odd morphisms in the sequence formed from reversed copies of Δ^1. Really we have two such categories, one of which consists of formal words, and another which involves ∞-categories and ∞-functors in the image of `repl`).

Besides the exactness of the Puppe sequence and its two variations, my hope is that.

We will define three different kinds of derived category:

1. $D(\infty\text{-Cat}) : \text{Cat}$ (the directed derived category of ∞-categories)
2. $D(\infty\text{-Grpd}) : \text{Cat}$ (the derived category of ∞-groupoids)
3. $D(\infty\text{-Grpd}_0) : \text{Cat}$ (the derived category of based ∞-groupoids)

We then create the second kind of derived category, one for each of the objects in the respective categories above:

1. For $C : D(\infty\text{-Cat})$, a category $D(\infty\text{-Cat}/C) : \text{Cat}$
2. For $G : D(\infty\text{-Grpd})$, a category $D(\infty\text{-Grpd}/G) : \text{Cat}$
3. For $G_0 : D(\infty\text{-Grpd}_0)$, a category $D(\infty\text{-Grpd}_0/G_0) : \text{Cat}$

For the model built on simplicial sets, $\vec{\Omega}$ will be representable by Δ^1 with respect to an internal hom, and $\overleftrightarrow{\Omega}$ will be representable by a model of the unit interval $I := [0,1]$.

I would like to use the "internal" structures defined in "'TheWhiteheadTheoremandTwoVariations"', whose types are as follows:

1. $\text{InternalCategory} : \text{Cat} \to \text{Cat}$

2. InternalPresheaf : (X : Cat) → (C : (IntCat X)) → Cat
3. InternalGroupoid : Cat → Cat
4. InternalGroupoidAction : (X : Cat) → (G : (IntGrpd X)) → Cat
5. InternalGroup : Cat → Cat
6. InternalGroupAction : (X : Cat) → (G_0 : (IntGrp X)) → Cat

The book “Galois theories” by Borceux and Janelidze features the middle two internal structures, and the nlab article here defines the first. The internal presheaf structure is fascinating and subtle, as well as similar to the internal groupoid action structure.

The six internal structures above arise here in relation to six functors:

(I) $\vec{\Omega} : \infty\text{-Cat} \longrightarrow \infty\text{-Cat}$ (notation for the directed path space functor, related to $[\Delta^1,\text{-}]$). D($\vec{\Omega}$) factors through internal categories in D(∞-Cat) by a categorical equivalence D(∞-Cat) $\cong$ IntCat D(∞-Cat) (internal categories in D(∞-Cat))

(II) $\vec{\omega}\,(\mathbb{1}\ C) : \infty\text{-Cat}/C \longrightarrow \infty\text{-Cat}/C$, the derived directed homotopy pullback with $\mathbb{1}$ C. D($\vec{\omega}\,(\mathbb{1}\ C)$) factors through a categorical equivalence between D(∞-Cat/C) and internal $\vec{P}$C-presheaves in D(∞-Cat/C).

(III) $\vec{\Omega} : \infty\text{-Grpd} \longrightarrow \infty\text{-Grpd}$ (notation for the path space functor [I,-]), the derived homotopy pullback of an ∞-groupoid with itself. D($\vec{\Omega}$) factors through a categorical equivalence between D(∞-Grpd) and internal groupoids in D(∞-Grpd)

(IV) $\vec{\omega}\,(\mathbb{1}\ X) : \infty\text{-Grpd}/X \longrightarrow \infty\text{-Grpd}/X$, the derived homotopy pullback with $\mathbb{1}$ X. D($\vec{\omega}\,(\mathbb{1}\ X)$) factors through internal $\vec{P}$X

(V) $\Omega : \infty\text{-Grpd}_0 \longrightarrow \infty\text{-Grpd}_0$, the loop space functor. D(Ω) factors through a categorical equivalence between D(∞-Grpd_0) and internal groups in D(∞-Grpd_0) (the loop space functor on connected based ∞-groupoids)

(VI) $\omega\,(\mathbb{1}\ X) : \infty\text{-Grpd}_0/X_0 \longrightarrow \infty\text{-Grpd}_0/X_0$, the homotopy pullback with the base of X_0. D($\omega\,(\mathbb{1}\ X)$) factors through internal PX_0-actions in based connected spaces over X_0.

(v) in the above is shown here and (vi) in the above is shown in a typical exposition of G-principal bundles.

The functors $\vec{\omega}\,(\mathbb{1}\ C)$, $\vec{\omega}\,(\mathbb{1}\ X)$, and $\omega\,(\mathbb{1}\ C)$ in the above ensue from a more general construction:

1. For C, D : D(∞-Cat), and f : C $\longrightarrow$ D, $\vec{\omega}$ f : D(∞-Cat/D) $\longrightarrow$ D(∞-Cat/C) (derived directed homotopy pullback)
2. For B, E : D(∞-Grpd), and f : E $\longrightarrow$ B, $\overset{\leftrightarrow}{\omega}$ f : D(∞-Grpd/B) $\longrightarrow$ D(∞-Grpd/E) (derived homotopy pullback)
3. For B_0, E_0 : D(∞-$Grpd_0$), and f : $E_0 \longrightarrow B_0$, ω f : D(∞-$Grpd_0/B_0$) $\longrightarrow$ D(∞-$Grpd_0/E_0$) (homotopy pullback with the base)

We obtain six categorical equivalences witnessed by these twelve functors (along with twelve natural isomorphisms). Here are the types of $\vec{P}$, $\overset{\leftrightarrow}{P}$, P : D(∞-$Grpd_0$), $\vec{p}$ ($\mathbb{1}$ C), $\overset{\leftrightarrow}{p}$ ($\mathbb{1}$ X), p:

1. The directed path space, the path space, and loop space form components of the functors $\vec{P}$, $\overset{\leftrightarrow}{P}$, and P, which are valued in internal categories, internal groupoids, and internal groups respectively.
   (a) $\vec{P}$: D(∞-Cat) $\longrightarrow$ Cat D(∞-Cat)
   (b) $\overset{\leftrightarrow}{P}$: D(∞-Grpd) $\longrightarrow$ Grpd D(∞-Grpd)
   (c) P : D(∞-$Grpd_0$) $\longrightarrow$ Grp D(∞-Grpd) (see here)
2. The directed homotopy pullback, the homotopy pullback, and the homotopy pullback with the base form components of the functors Alg(Mon($\vec{\omega}$)), Alg(Mon($\overset{\leftrightarrow}{\omega}$)), and Alg(Mon(p)), respectively.
   (a) $\vec{p}$ ($\mathbb{1}$ C) : D(∞-Cat⁄C) $\longrightarrow$ InfPreShf D(∞-Cat⁄C) $\vec{P}$.obj C
   (b) $\overset{\leftrightarrow}{p}$ ($\mathbb{1}$ X) : D(∞-Grpd⁄X) $\longrightarrow$ IntAct D(∞-Grpd⁄X) $\overset{\leftrightarrow}{P}$.obj X
   (c) p ($\mathbb{1}$ X_0) : D(∞-$Grpd_0$⁄X_0) $\longrightarrow$ $IntAct_0$ D(∞-$Grpd_0$⁄X_0) P.obj X_0

Above, the functors $\vec{P}$, $\overset{\leftrightarrow}{P}$, P, $\vec{p}$, $\overset{\leftrightarrow}{p}$, and p feature $\vec{\Omega}$, $\overset{\leftrightarrow}{\Omega}$, Ω, $\vec{\omega}$, $\overset{\leftrightarrow}{\omega}$, and ω in their components, and can be related to them using constructions from Eilenberg-Moore theory.

These six new functors combine with the functors below to form categorical equivalences:

1. The directed homotopy colimit of a point with an internal category in D(∞-Cat) as a diagram, the homotopy colimit of a constant functor with an internal internal group as a diagram
   (a) $\vec{B}$: essential_image $\vec{P}$ $\longrightarrow$ D(∞-Cat)
   (b) $\overset{\leftrightarrow}{B}$: essential_image $\overset{\leftrightarrow}{P}$ $\longrightarrow$ D(∞-Grpd)
   (c) B : essential_image P $\longrightarrow$ D(∞-$Grpd_0$) (see here)
2. The clutching functors are inverse to the above functors up to natural isomorphism:

(a) $\vec{\mathrm{b}} : \mathrm{essential\_image}\ \vec{\mathrm{p}} \longrightarrow \mathrm{D}(\infty\text{-Cat}/\mathrm{C})$

(b) $\overleftrightarrow{\mathrm{b}} : \mathrm{essential\_image}\ \overleftrightarrow{\mathrm{p}} \longrightarrow \mathrm{D}(\infty\text{-Cat}/\mathrm{C})$

(c) $\mathrm{b} : \mathrm{essential\_image}\ \mathrm{p} \longrightarrow \mathrm{D}(\infty\text{-Grpd}_0/\mathrm{X}_0)$

Take special note that each of these six involves a condition ensuring that the functor $\vec{\mathrm{B}}$ be well defined. Consider the functors:

1. $\mathrm{D}(\mathrm{IntCat}\ \infty\text{-Cat}) \longrightarrow \mathrm{IntCat}\ \mathrm{D}(\infty\text{-Cat})$
2. $\mathrm{D}(\mathrm{IntGrpd}\ \infty\text{-Grpd}) \longrightarrow \mathrm{IntGrpd}\ \mathrm{D}(\infty\text{-Grpd})$
3. $\mathrm{D}(\mathrm{IntGrp}\ \infty\text{-Grpd}_0) \longrightarrow \mathrm{IntGrp}\ \mathrm{D}(\infty\text{-Grpd}_0)$
4. $\mathrm{D}(\mathrm{InfPreShf}\ \infty\text{-Cat}/\mathrm{C})\ \vec{\mathrm{P}}\mathrm{C} \longrightarrow \mathrm{InfPreShf}\ \mathrm{D}(\infty\text{-Cat}/\mathrm{C})\ \vec{\mathrm{P}}\mathrm{C}$
5. $\mathrm{D}(\mathrm{IntAct}\ \infty\text{-Cat}/\mathrm{C})\ \overleftrightarrow{\mathrm{P}}\mathrm{X} \longrightarrow \mathrm{IntAct}\ \mathrm{D}(\infty\text{-Cat}/\mathrm{C})\ \overleftrightarrow{\mathrm{P}}\mathrm{X}$
6. $\mathrm{D}(\mathrm{IntAct}_0\ \infty\text{-Grpd}_0/\mathrm{X}_0)\ \mathrm{PX}_0 \longrightarrow \mathrm{IntAct}_0\ \mathrm{D}(\infty\text{-Grpd}_0/\mathrm{X}_0)\ \mathrm{PX}_0$

It may happen that a given object in the codomain of these six functors lies in their essential image. In this case, any of the six of $\vec{\mathrm{B}}$, $\overleftrightarrow{\mathrm{B}}$, B, $\vec{\mathrm{b}}$, $\overleftrightarrow{\mathrm{b}}$, b can sometimes but not always be obtained as a quotient of six functors $\vec{\mathrm{E}}$, $\overleftrightarrow{\mathrm{E}}$, E, $\vec{\mathrm{e}}$, $\overleftrightarrow{\mathrm{e}}$, e, respectively:

1. $\vec{\mathrm{E}} : \mathrm{IntCat}\ \infty\text{-Cat} \longrightarrow \infty\text{-Cat}$
2. $\overleftrightarrow{\mathrm{E}} : \mathrm{IntGrpd}\ \infty\text{-Grpd} \longrightarrow \infty\text{-Grpd}$
3. $\mathrm{E} : \mathrm{IntGrp}\ \infty\text{-Grpd}_0 \longrightarrow \infty\text{-Grpd}_0$
4. $\vec{\mathrm{e}} : \mathrm{PreShf}\ \infty\text{-Cat}/\mathrm{C}\ \vec{\mathrm{P}}.\mathrm{obj}\ \mathrm{C} \longrightarrow \infty\text{-Cat}/\mathrm{C}\ \vec{\mathrm{P}}.\mathrm{obj}\ \mathrm{C}$
5. $\overleftrightarrow{\mathrm{e}} : \mathrm{IntAct}\ \infty\text{-Cat}/\mathrm{C}\ \overleftrightarrow{\mathrm{P}}.\mathrm{obj}\ \mathrm{X} \longrightarrow \infty\text{-Cat}/\mathrm{C}\ \overleftrightarrow{\mathrm{P}}.\mathrm{obj}\ \mathrm{X}$
6. $\mathrm{e} : \mathrm{IntAct}_0\ \infty\text{-Grpd}_0/\mathrm{X}_0\ \mathrm{P}.\mathrm{obj}\ \mathrm{X}_0 \longrightarrow \infty\text{-Grpd}_0/\mathrm{X}_0\ \mathrm{P}.\mathrm{obj}\ \mathrm{X}_0$

The functors above will be defined as certain homotopy colimits, themselves certain coequilizers. On the condition that an internal category is internally filtered and internally cofiltered, we can further construct the $\vec{\mathrm{B}}$.

We will make extensive use of Mathlib's bicategory of categories and material on simplicial sets. We further use Mathlib's pullbacks and categorical products, as well as their Eilenberg-Moore theory constructions. I'd like to extend my appreciation to Scott Morison, Eric Wieser, Floris Van Doorn, and all the contributors who have put their efforts into creating these robust features for Mathlib 4.

Altogether, the project gets the following "periodic table" of 30 functors featured on the front cover:

D(∞-Cat)	$\vec{\Sigma}$	$\vec{\Omega}$	$\vec{\mathrm{P}}$	$\vec{\mathrm{B}}$	$\vec{\mathrm{E}}$	D(∞-Cat/C)	$\vec{\sigma}$	$\vec{\omega}$	$\vec{\mathrm{b}}$	$\vec{\mathrm{p}}$	$\vec{\mathrm{e}}$
D(∞-Grpd)	$\overleftrightarrow{\Sigma}$	$\overleftrightarrow{\Omega}$	$\overleftrightarrow{\mathrm{P}}$	$\overleftrightarrow{\mathrm{B}}$	$\overleftrightarrow{\mathrm{E}}$	D(∞-Grpd/G)	$\overleftrightarrow{\sigma}$	$\overleftrightarrow{\omega}$	$\overleftrightarrow{\mathrm{b}}$	$\overleftrightarrow{\mathrm{p}}$	$\overleftrightarrow{\mathrm{e}}$
D(∞-Grpd$_0$)	Σ	Ω	P	B	E	D(∞-Grpd$_0$/G$_0$)	σ	ω	b	p	e

Here are the names of the symbols featured above:

Suspensional	Deductive	Remembrant	Delooping	Free
$\vec{\Sigma}$ (Directed suspension)	$\vec{\Omega}$ (Directed path space)	$\vec{\mathrm{P}}$ (Remembrant derived directed path space)	$\vec{\mathrm{B}}$ (Classifying space for internal categories)	$\vec{\mathrm{E}}$
$\overleftrightarrow{\Sigma}$ (Suspensionoid)	$\overleftrightarrow{\Omega}$ (Path space)	$\overleftrightarrow{\mathrm{P}}$ (Remembrant derived path space)	$\overleftrightarrow{\mathrm{B}}$ (Classifying space for internal groupoids)	$\overleftrightarrow{\mathrm{E}}$
Σ (Suspension)	Ω (Loop space)	P (Remembrant derived loop space)	B (Classifying space for internal groups)	E
$\vec{\sigma}$	$\vec{\omega}$ (Directed homotopy pushout with a point)	$\vec{\mathrm{p}}$ (Remembrant derived directed homotopy pullback)	$\vec{\mathrm{b}}$ (Classifying space for internal presheaves)	$\vec{\mathrm{e}}$
$\overleftrightarrow{\sigma}$	$\overleftrightarrow{\omega}$ (Homotopy pushout with a point)	$\overleftrightarrow{\mathrm{p}}$ (Remembrant derived homotopy pullback)	$\overleftrightarrow{\mathrm{b}}$ (Classifying space for internal groupoid actions)	$\overleftrightarrow{\mathrm{e}}$
σ	ω (Homotopy fiber)	p (Remembrant derived homotopy fiber)	b (Classifying space for internal group actions)	e

The term “remembrant” in the above is not common terminology. It is intended to mean that the second collumn features functors which are valued in categories of internal objects wheras the left collumn forms particular components of those structures.

The notation here is both an attempt to make the three-fold division of the project (three Whitehead theorems, three Puppe sequences, etc.) manifest while sticking to the standard notation for the established theorems (Σ, Ω, B, E). In the above, P could be said to stand for “(remembrant) path space” and p for “(remembrant) pullback”, while at the same time this matches the theme that our capital letters reflect various internal structures and that their lower-case forms reflect the corresponding actions.

The mentioned “delooping principals”, which identify inverses to the remembrant functors *on their essential image*, form important consequences of the three Puppe sequences. All in all, there are nine important theorems we want to show:

Twelve Goals

(I) Define the Puppe sequence for ∞-categories and prove its exactness.

(II) Define and prove the `internal_category_delooping_principal : Type`.

(III) Define and prove the `internal_sheaf_delooping_principal : Type`.

(IV) Define the Puppe sequence for ∞-groupoids and prove its exactness

(V) Define and prove the `internal_groupoid_delooping_principal : Type`.

(VI) Define and prove the `internal_groupoid_action_delooping_principal : Type`.

(VII) Define the Puppe sequence for based connected ∞-groupoids and prove its exactness

(VIII) Define and prove the `internal_group_delooping_principal : Type`.

(IX) Define and prove the `internal_group_action_delooping_principal : Type`.

None of these theorems are currently contained in Mathlib. The last three are well-known.

In the work that ensues, we plan to take an approach which establishes the known results before the original ones, taking advantage of the predefined π_n functors in `Mathlib 4` in the process. This decision will also help to start with smaller pull requests.

3. Unicode

Implementation Progress

Writing Progress

Here is a list of the unicode characters we will use:

Symbol	Unicode	VSCode shortcut	Use
Lean's Kernel			
×	2A2F	`\times`	Product of types
→	2192	`\rightarrow`	Hom of types
⟨,⟩	27E8,27E9	`\langle,\rangle`	Product term introduction
↦	21A6	`\mapsto`	Hom term introduction
∧	2227	`\wedge`	Conjunction
∨	2228	`\vee`	Disjunction
∀	2200	`\forall`	Universal quantification
∃	2203	`\exists`	Existential quantification
¬	00AC	`\neg`	Negation
Variables and Constants			
a,b,c,...,z	1D52,1D56		Variables and constants
0,1,2,3,4,5,6,7,8,9	1D52,1D56		Variables and constants
-	207B		Variables and constants
0,1,2,3,4,5,6,7,8,9	2080 - 2089	`\0-\9`	Variables and constants
𝔸,...,ℤ	1D538		
𝕒,...,𝕫	1D552		
A,...,Z	1D41A		
a,...,z	1D41A		
α-ω,A-Ω	03B1-03C9		Variables and constants
Categories			
𝟙	1D7D9	`\b1`	The identity morphism
∘	2218	`\circ`	Composition
Bicategories			
•	2022	`\smul`	Horizontal composition of objects
Adjunctions			
⇄	21C4	`\rightleftarrows`	Adjunctions
⇆	21C6	`\leftrightarrows`	Adjunctions
.	1BC94		Right adjoints
·	0971		Left adjoints
⊣	22A3	`\dashv`	The condition that two functors are adjoint
Monads and Comonads			
?,¿	003F, 00BF	?,\?	The corresponding (co)monad of an adjunction
!,¡	0021, 00A1	!, \!	The (co)-Eilenberg-(co)-Moore adjunction
ꜝ,ꜞ	A71D, A71E		The (co)exponential maps
Miscellaneous			
∼	223C	`\sim`	Homotopies
≃	2243	`\equiv`	Equivalences
≅	2245	`\cong`	Isomorphisms
⊥	22A5	`\bot`	The overobject classifier
∞	221E	`\infty`	Infinity categories and infinity groupoids
↔	20D7		Homotopical operations on ∞-categories
→	20E1		Homotopical operations on ∞-groupoids

PART 1: BASED CONNECTED ∞-GROUPOIDS

Chapter 1: The Puppe Sequence

This chapter establishes the well know Puppe sequence for the based homotopy groups π_n. This is the well known Puppe sequence of homotopy groups.

Chapter 2: The Smash Product for $\infty\text{-}\mathtt{Grpd}_0$

Chapter 3: The Triangle Theorems

4. The Cycle Theorem

5. The Derived Exact Sequence

6. The Octahedral Axiom

7. Colimits of Inclusions

8. The Mayer-Vietoris Sequence

9. Fibrations and the exactness of $- \wedge X_0$

10. Cofibrations and the exactness of $[X_0,-]$

PART 2: BASED CONNECTED ∞-CATEGORIES

Chapter 4: The Puppe Sequence

This chapter establishes the well know Puppe sequence for the based homotopy groups π_n. This is the well known Puppe sequence of homotopy groups.

Chapter 5: The Smash Product for $\infty\text{-}\mathrm{Cat}_0$

Chapter 6: The Triangle Theorems

11. The Cycle Theorem

12. The Derived Exact Sequence

13. The Octahedral Axiom

14. Colimits of Inclusions

15. The Mayer-Vietoris Sequence

16. Fibrations and the exactness of $- \wedge X_0$

17. Cofibrations and the exactness of $[X_0,-]$

PART 3: ∞-GROUPOIDS

Chapter 7: The Puppe Sequence for ∞-Grpd

Chapter 8: The Smash Product for ∞-Grpd

Chapter 9: The Triangle Theorems for ∞-Grpd

18. The Cycle Theorem

19. The Derived Exact Sequence

20. The Octahedral Axiom

21. Colimits of Inclusions

22. The Mayer-Vietoris Sequence

23. Fibrations and the exactness of $- \wedge X$

24. Cofibrations and the exactness of [X,-]

PART 4: ∞-CATEGORIES

Chapter 10: The Puppe Sequence for ∞-Cat

In this chapter we construct the Puppe sequence for $\vec{\pi}_n$. Note: one joint in this exact sequence consists not of a map but an action.} This will be used in the next chapter two establish two of the six categorical equivalences.

Chapter 11: The Smash Product for ∞-Cat

Chapter 12: The Triangle Theorems for ∞-Cat

25. The Cycle Theorem

26. The Derived Exact Sequence

27. The Octahedral Axiom

28. Colimits of Inclusions

29. The Mayer-Vietoris Sequence

30. Fibrations and the exactness of $- \wedge X$

31. Cofibrations and the exactness of [X,-]

Bibliography

Further reading:

1. J. Beck, “Distributive laws,” in Seminar on Triples and Categorical Homology Theory, Springer-Verlag, 1969, pp. 119-140.
2. Saunders Mac Lane, ”Categories for the Working Mathematician,” Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1971.
3. Samuel Eilenberg and Saunders Mac Lane, “General Theory of Natural Equivalences,” Transactions of the American Mathematical Society, vol. 58, no. 2, pp. 231-294, 1945.
4. Daniel M. Kan, “Adjoint Functors,” Transactions of the American Mathematical Society, vol. 87, no. 2, pp. 294-329, 1958.
5. Chris Heunen, Jamie Vicary, and Stefan Wolf, “Categories for Quantum Theory: An Introduction,” Oxford Graduate Texts, Oxford University Press, Oxford, 2018.
6. S. Eilenberg and J. C. Moore, “Adjoint Functors and Triples,” Proceedings of the Conference on Categorical Algebra, La Jolla, California, 1965, pp. 89-106.
7. Daniel M. Kan, “On Adjoints to Functors” (1958): In this paper, Kan further explored the theory of adjoint functors, focusing on the existence and uniqueness of adjoints. His work provided important insights into the fundamental aspects of adjoint functors and their role in category theory.

Lectures, Videos, and Stackexchange questions:

1. `https://www.youtube.com/watch?v=Ob9tOgWumPI`
2. `https://www.youtube.com/watch?v=xYenPIeX6MY`
3. `https://mathoverflow.net/questions/5901/do-the-signs-in-puppe-sequences-matter`

Relevant discussions on the `Lean 4` Zulip:

1.

Ideas for future applications:

1. `https://arxiv.org/pdf/2206.13563.pdf`