.py file
.tex file
.pdf file
.lean file

Two Theorems in Classifying Space Theory and Two Variations Each

$\vec{E}$: Functor (OperadicCategory ∞-Cat) ∞-Cat
$\vec{B}$: Functor (OperadicCategory ∞-Cat) ∞-Cat
$\vec{\partial}$: (C : OperadicCategory ∞-Cat) $\to$ ∞-Cat.hom ($\vec{E}$.obj C) ($\vec{B}$.obj C)
$\overleftrightarrow{E}$: Functor (OperadicGroupoid ∞-Grpd) ∞-Grpd
$\overleftrightarrow{B}$: Functor (OperadicGroupoid ∞-Grpd) ∞-Grpd
$\overleftrightarrow{\partial}$: (G : OperadicCategory ∞-Grpd) $\to$ ∞-Grpd.hom ($\overleftrightarrow{E}$.obj G) ($\overleftrightarrow{B}$.obj G)
E : OperadicGroup ∞-Grpd$_{-1}$ $\longrightarrow$ ∞-Grpd$_{-1}$
B : OperadicGroup ∞-Grpd$_{-1}$ $\longrightarrow$ ∞-Grpd$_{-1}$
∂ : (G$_{-1}$: OperadicGroup ∞-Grpd$_{-1}$) $\to$ ∞-Grpd$_{-1}$.hom (E.obj G$_{-1}$) (B.obj G$_{-1}$)
$\vec{e}$: {C : ∞-Cat } $\to$ { D : ∞-Cat } $\to$ (F : ∞-Cat.hom C D) $\to$ Functor (OperadicPresheaf ($\vec{O}$.obj D)) (∞-Cat/D)
$\vec{b}$: { C : ∞-Cat } $\to$ { D : ∞-Cat } $\to$ (F : ∞-Cat.hom C D) $\to$ Functor (OperadicPresheaf ($\vec{O}$.obj D)) (∞-Cat/D)
$\boxed{?}$: { C : ∞-Cat } $\to$ { D : ∞-Cat } $\to$ (F : ∞-Cat.hom C D) $\to$ (∞-Cat/D).hom ($\vec{e}$.obj F) ($\vec{b}$.obj F)
$\overleftrightarrow{e}$: { X : ∞-Cat } $\to$ { Y : ∞-Cat } $\to$ (F : ∞-Cat.hom X Y) $\to$ Functor (OperadicGroupoidAction ($\overleftrightarrow{O}$.obj Y)) (∞-Grpd/Y)
$\overleftrightarrow{b}$: { X : ∞-Grpd } $\to$ { Y : ∞-Grpd } $\to$ (F : ∞-Cat.hom X Y) $\to$ Functor (OperadicGroupoidAction ($\overleftrightarrow{O}$.obj Y)) (∞-Grpd/Y)
$\boxed{?}$: { X : ∞-Grpd } $\to$ { Y : ∞-Grpd } $\to$ (F : ∞-Grpd.hom X Y) $\to$ (∞-Cat/D).hom ($\overleftrightarrow{e}$.obj F) ($\overleftrightarrow{b}$.obj F)
e : { X$_{-1}$: ∞-Grpd$_{-1}$ } $\to$ { Y$_{-1}$: ∞-Grpd$_{-1}$ } $\to$ (F : ∞-Grpd$_{-1}$.hom X$_{-1}$ Y$_{-1}$) $\to$ Functor (∞-Grpd$_{-1}$/Y$_{-1}$) (OperadicGroupAction (O.obj Y$_{-1}$))
b : { X$_{-1}$: ∞-Grpd$_{-1}$ } $\to$ { Y$_{-1}$: ∞-Grpd$_{-1}$ } $\to$ (F : ∞-Grpd$_{-1}$.hom X$_{-1}$ Y$_{-1}$) $\to$ Functor (∞-Grpd$_{-1}$/Y$_{-1}$) (OperadicGroupAction (O.obj Y$_{-1}$))
$\boxed{?}$: { G$_{-1}$: ∞-Grpd$_0$ } $\to$ { Y$_0$: ∞-Grpd$_0$ } $\to$ (F : ∞-Grpd$_{-1}$.hom X$_{-1}$ Y$_{-1}$) $\to$ (∞-Cat/D).hom (e.obj F) (b.obj F)

E. Dean Young

1. Contents

$\vec{\mathrm{E}}$	
$\vec{\mathrm{e}}$	
$\vec{\mathrm{B}}$	
$\vec{\mathrm{b}}$	
∂	
$\boxed{?}$	
`Chapter 11:` The Recognition Theorem for ∞-Categories	
`Chapter 12:` The Classifying Space Theorem for ∞-Categories	

Introduction

In "'TheWhiteheadTheoremandTwoVariations"', we will define six "internal" structures based on the ones found in "'Galois Theories"' by Janelidze and Borceux, as well as six "'operadic"' structures.

$\vec{E}$: Functor (OperadicCategory ∞-Cat) ∞-Cat
$\vec{B}$: Functor (OperadicCategory ∞-Cat) ∞-Cat
$\vec{\partial}$: (C : OperadicCategory ∞-Cat) $\to$ ∞-Cat.hom ($\vec{E}$.obj C) ($\vec{B}$.obj C)
$\overleftrightarrow{E}$: Functor (OperadicGroupoid ∞-Grpd) ∞-Grpd
$\overleftrightarrow{B}$: Functor (OperadicGroupoid ∞-Grpd) ∞-Grpd
$\overleftrightarrow{\partial}$: (G : OperadicCategory ∞-Grpd) $\to$ ∞-Grpd.hom ($\overleftrightarrow{E}$.obj G) ($\overleftrightarrow{B}$.obj G)
E : OperadicGroup ∞-Grpd$_{-1}$ $\longrightarrow$ ∞-Grpd$_{-1}$
B : OperadicGroup ∞-Grpd$_{-1}$ $\longrightarrow$ ∞-Grpd$_{-1}$
∂ : (G$_{-1}$: OperadicGroup ∞-Grpd$_{-1}$) $\to$ ∞-Grpd$_{-1}$.hom (E.obj G$_{-1}$) (B.obj G$_{-1}$)
$\vec{e}$: {C : ∞-Cat } $\to$ { D : ∞-Cat } $\to$ (F : ∞-Cat.hom C D) $\to$ Functor (OperadicPresheaf ($\vec{O}$.obj D)) (∞-Cat/D)
$\vec{b}$: { C : ∞-Cat } $\to$ { D : ∞-Cat } $\to$ (F : ∞-Cat.hom C D) $\to$ Functor (OperadicPresheaf ($\vec{O}$.obj D)) (∞-Cat/D)
⍰ : { C : ∞-Cat } $\to$ { D : ∞-Cat } $\to$ (F : ∞-Cat.hom C D) $\to$ (∞-Cat/D).hom ($\vec{e}$.obj F) ($\vec{b}$.obj F)
$\overleftrightarrow{e}$: { X : ∞-Cat } $\to$ { Y : ∞-Cat } $\to$ (F : ∞-Cat.hom X Y) $\to$ Functor (OperadicGroupoidAction ($\overleftrightarrow{O}$.obj Y)) (∞-Grpd/Y)
$\overleftrightarrow{b}$: { X : ∞-Grpd } $\to$ { Y : ∞-Grpd } $\to$ (F : ∞-Cat.hom X Y) $\to$ Functor (OperadicGroupoidAction ($\overleftrightarrow{O}$.obj Y)) (∞-Grpd/Y)
⍰ : { X : ∞-Grpd } $\to$ { Y : ∞-Grpd } $\to$ (F : ∞-Grpd.hom X Y) $\to$ (∞-Cat/D).hom ($\overleftrightarrow{e}$.obj F) ($\overleftrightarrow{b}$.obj F)
e : { X$_{-1}$: ∞-Grpd$_{-1}$ } $\to$ { Y$_{-1}$: ∞-Grpd$_{-1}$ } $\to$ (F : ∞-Grpd$_{-1}$.hom X$_{-1}$ Y$_{-1}$) $\to$ Functor (∞-Grpd$_{-1}$/Y$_{-1}$) (OperadicGroupAction (O.obj Y$_{-1}$))
b : { X$_{-1}$: ∞-Grpd$_{-1}$ } $\to$ { Y$_{-1}$: ∞-Grpd$_{-1}$ } $\to$ (F : ∞-Grpd$_{-1}$.hom X$_{-1}$ Y$_{-1}$) $\to$ Functor (∞-Grpd$_{-1}$/Y$_{-1}$) (OperadicGroupAction (O.obj Y$_{-1}$))
⍰ : { G$_{-1}$: ∞-Grpd$_0$ } $\to$ { Y$_0$: ∞-Grpd$_0$ } $\to$ (F : ∞-Grpd$_{-1}$.hom X$_{-1}$ Y$_{-1}$) $\to$ (∞-Cat/D).hom (e.obj F) (b.obj F)

and "'ThePuppeSequenceandTwoVariations"'. In "'InternalUniverses"', I considered straightening and unstraightening and three variations of it, which were each considered before and after the application of D(-). This made for the six diagrams depicted on page ???. In this repository, we consider the classifying space B.

Let F : C $\longrightarrow$ D : ∞-Cat.hom C D be an ∞-functor. Given either the C-infinity presheaf in ∞-Cat/C arising from F : ∞-Cat/D or the C-infinity presheaf in ∞-Cat/D arising from Id$_C$: ∞-Cat/C, we obtain in both cases an internal presheaf in the corresponding derived category. However, not all internal categories D : InternalCategory D(∞-Cat/C) arise from and not all internal presheaves S : InternalPresheaf D D(∞-Cat/C) arise from C-infinity presheaves over some C-infinity category in ∞-Cat/C.

In "'InternalUniverses"', we showed the straightening/unstraightening categorical equivalence and three variations using the six Ω-functors and six E-functors, treating the situations before and after the application of D(-) seperately for a total of six goals.

In this section, we consider classifying spaces as well as a perspective about remembering information concerning a right or left adjoint applied to a particular functor or object in the following way: E and Ω and their respective five variations give "'remembrant"' functors E-infinity and Ω-infinity, which each produce internal presheaves in respective derived categories.

`Plans to prove three variations of the`

```
Whitehead theorem of homotopy groups in
Lean 4, with extensive use of Mathlib 4
```

$\vec{\vec{E}} : \infty\text{-Cat} \longrightarrow \infty\text{-Cat}$	$\vec{\vec{B}} : \infty\text{-Cat} \longrightarrow \infty\text{-Cat}$	$\vec{\vec{\partial}} : \infty\text{-Cat} \longrightarrow \infty\text{-Cat}$	$\vec{e} : (C : \infty\text{-Cat}) \to (D : \infty\text{-Cat}) \to \text{Adjunc}$
$\vec{E} : \infty\text{-Grpd} \longrightarrow \infty\text{-Grpd}$	$\vec{B} : \infty\text{-Grpd} \longrightarrow \infty\text{-Grpd}$	$\vec{\partial} : \infty\text{-Grpd} \longrightarrow \infty\text{-Grpd}$	$\vec{e} : (X : \infty\text{-Grpd}) \to (Y : \infty\text{-Grpd}) \to \text{Adju}$
$E : \infty\text{-Grpd}_0 \longrightarrow \infty\text{-Grpd}_0$	$B : \infty\text{-Grpd}_0 \longrightarrow \infty\text{-Grpd}_0$	$\partial : \infty\text{-Grpd}_0 \longrightarrow \infty\text{-Grpd}_0$	$e : (X : \infty\text{-Grpd}_0) \to (Y: \infty\text{-Grpd}_0) \to \text{Ad}$

By the time this repository is seen to in 2025, I will have filled out a certain six operadic structures to do with ∞-categories and ∞-groupoids. Each of these structures will be made to work together with Pow X : Type : $\lambda(\mathrm{n} : \mathbb{N}), \mathrm{X} \to \mathrm{X}$. The six operadic structures are endofunctions of one of six mathematical objects, here with an option for 12 based on models A (simplicial set model) and B (point-set model).

$\mathtt{B}^{1}$ `: Functor (Pow OperadicGroup 2) ??? (Pow OperadicGroup 2) ???`

$\mathtt{B}^{n}$ `: Functor (Pow OperadicGroup 2) ??? (Pow OperadicGroup 2) ???`

In this repository I construct six categorical equivalences:

1. Between certain operadic categories and
2. Between certain internal categories and
3. Between certain operadic presheaves and
4. Between certain internal presheaves and
5. Between certain operadic groupoids and
6. Between certain internal groupoids and
7. Between certain operadic groups in ??? and ???
8. Between certain internal groups and
9. Between certain operadic group actions
10. Between certain internal group actions

PART 1: BASED CONNECTED ∞-GROUPOIDS

Operadic Groups and Operadic Group Actions

The Classifying Space and the Total Space

PART 2: ∞-GROUPOIDS

Operadic Groupoids and Operadic Groupoid Actions

The Recognition Theorem for ∞-Groupoids

The Classifying Space Theorem for ∞-Groupoids

PART 3: ∞-CATEGORIES

Operadic Categories and Operadic Presheaves

The Recognition Theorem for ∞-Categories

The Classifying Space Theorem for ∞-Categories

Implementation	2024 ∞-category project identity		Lax		Strict	
	ND	Filling Number	Unitial	Actional	Unitial	Actional
First	ND00	Filling number (n : Nat)	C-infinity categories	C-infinity presheaves	Internal categories	Internal presheaves
	ND01	Filling number (n : Nat)	∞-Cat	∞-Cat ⁄ C	D(∞-Cat)	D(∞-Cat/C)
Second	ND10	Filling number (n : Nat)	The lax lifted Whitehead theorem	The lax lifted Puppe sequence	The strict Whitehead theorem	The strict Puppe sequence
	ND11	Filling number (n : Nat)	The lax lifted B-Ω equivalence	The lax lifted b-ω equivalence	The strict strict B-Ω equivalence	The strict b-ω equivalence

In "'Internal Universes"' I thought about the six variations of straightening and unstraightening featured in the diagrams below:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
D([X^{op},\infty_(\infty\text{-}Grpd_0)]) \arrow[r, shift left, "D((e.hom\ F)^{\cdot})"] \arrow[d, shift left, "D((\chi_{\cdot}).obj\ C)"] & D([Y^{op},\infty_(\infty\text{-}Grpd_0)]) \arrow[l, shift left, "D((e.hom\ F)_{\cdot})"] \arrow[d, shift left, "D((\chi_{\cdot}).obj\ Y)"] \\
D(\infty_(\infty\text{-}Grpd_0)/X) \arrow[u, shift left, "D((\chi^{\cdot}).obj\ X)"] \arrow[r, shift left, "D((\omega.hom\ F)^{\cdot})"] & \infty_(\infty\text{-}Grpd_0)/Y \arrow[l, shift left, "D((\omega.hom\ F)_{\cdot})"] \arrow[u, shift left, "D((\chi^{\cdot}).obj\ Y)"]
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge, row sep=huge]
D([X^{op},\infty_(\infty\text{-}Cat)]) \arrow[r, shift left, "D((\vec{e}.hom\ F)^{\cdot})"] \arrow[d, shift left, "D((\vec{\chi}_{\cdot}).obj\ X)"] & D([Y^{op},\infty_(\infty\text{-}Cat)]) \arrow[l, shift left, "D((\vec{e}.hom\ F)_{\cdot})"] \arrow[d, shift left, "D((\vec{\chi}_{\cdot}).obj\ Y)"] \\
D(\infty_(\infty\text{-}Cat)/X) \arrow[u, shift left, "D((\vec{\chi}^{\cdot}).obj\ X)"] \arrow[r, shift left, "D((\vec{\omega}.hom\ F)^{\cdot})"] & \infty_(\infty\text{-}Cat)/Y \arrow[l, shift left, "D((\vec{\omega}.hom\ F)_{\cdot})"] \arrow[u, shift left, "D((\vec{\chi}^{\cdot}).obj\ Y)"]
\end{tikzcd}$$

$$
\begin{tikzcd}[column sep=huge, row sep=large]
D([X^{op},\infty\_(\infty\text{-Grpd})]) \arrow[r, shift left, "D((\overleftrightarrow{e}.\mathrm{hom}\ F)^{\cdot})"] \arrow[d, shift left, "D((\overleftrightarrow{\chi}_{\cdot}).\mathrm{obj}\ X)"] & D([Y^{op},\infty\_(\infty\text{-Grpd})]) \arrow[l, shift left, "D((\overleftrightarrow{e}.\mathrm{hom}\ F)_{\cdot})"] \arrow[d, shift left, "D((\overleftrightarrow{\chi}_{\cdot}).\mathrm{obj}\ Y)"] \\
D(\infty\_(\infty\text{-Grpd})/X) \arrow[u, shift left, "D((\overleftrightarrow{\chi}^{\cdot}).\mathrm{obj}\ X)"] \arrow[r, shift left, "D((\overleftrightarrow{\omega}.\mathrm{hom}\ F)^{\cdot})"] & \infty\_(\infty\text{-Grpd})/Y \arrow[l, shift left, "D((\overleftrightarrow{\omega}.\mathrm{hom}\ F)_{\cdot})"] \arrow[u, shift left, "D((\overleftrightarrow{\chi}^{\cdot}).\mathrm{obj}\ Y)"]
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
{[X^{op},\infty\_(\infty\text{-Grpd}_0)]} \arrow[r, shift left=2, "(e.\mathrm{hom}\ F)^{\cdot}"] \arrow[d, shift left=2, "(\chi_{\cdot}).\mathrm{obj}\ X"] & {[Y^{op},\infty\_(\infty\text{-Grpd}_0)]} \arrow[l, shift left=2, "(e.\mathrm{hom}\ F)_{\cdot}"] \arrow[d, shift left=2, "(\chi_{\cdot}).\mathrm{obj}\ Y"] \\
\infty\_(\infty\text{-Grpd}_0)/X \arrow[u, shift left=2, "(\chi^{\cdot}).\mathrm{obj}\ X"] \arrow[r, shift left=2, "(\omega.\mathrm{hom}\ F)^{\cdot}"] & \infty\_(\infty\text{-Grpd}_0)/Y \arrow[u, shift left=2, "(\chi^{\cdot}).\mathrm{obj}\ Y"] \arrow[l, shift left=2, "(\omega.\mathrm{hom}\ F)_{\cdot}"]
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
{[C^{op},\infty\_(\infty\text{-Cat})]} \arrow[r, shift left=2, "(\vec{e}.\mathrm{hom}\ F)^{\cdot}"] \arrow[d, shift left=2, "(\vec{\chi}_{\cdot}).\mathrm{obj}\ C"] & {[D^{op},\infty\_(\infty\text{-Cat})]} \arrow[l, shift left=2, "(\vec{e}.\mathrm{hom}\ F)_{\cdot}"] \arrow[d, shift left=2, "(\vec{\chi}_{\cdot}).\mathrm{obj}\ D"] \\
\infty\_(\infty\text{-Cat})/C \arrow[u, shift left=2, "(\vec{\chi}^{\cdot}).\mathrm{obj}\ C"] \arrow[r, shift left=2, "(\vec{\omega}.\mathrm{hom}\ F)^{\cdot}"] & \infty\_(\infty\text{-Cat})/D \arrow[u, shift left=2, "(\vec{\chi}^{\cdot}).\mathrm{obj}\ D"] \arrow[l, shift left=2, "(\vec{\omega}.\mathrm{hom}\ F)_{\cdot}"]
\end{tikzcd}
$$

6 goals 6 structures
With these goals I want to create several “‘remembrant”’ adjunctions:

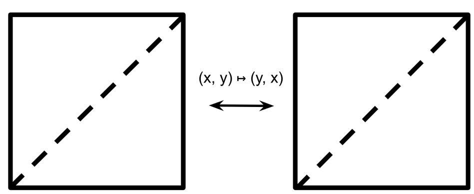
1. $\vec{\gamma}\ \overleftrightarrow{\gamma}\ \gamma$
2. $\vec{\Sigma}\ \overleftrightarrow{\Sigma}\ \Sigma$
3. $\vec{\sigma}\ \overleftrightarrow{\sigma}\ \sigma$
4. Pullback of two homs and a single hom vs. a pushout of two products and a single product

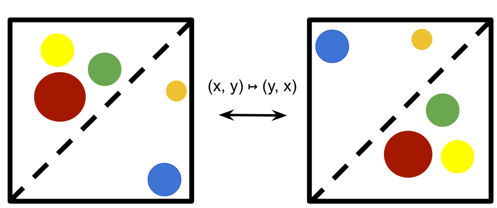
$$\vec{\mathrm{o}} : (\mathrm{C} : \infty\text{-Cat}) \to \infty\text{-Cat}/\mathrm{C} \longrightarrow \mathrm{OperadicPresheaf}\,(\vec{\mathrm{O}}.\mathrm{obj}\ \mathrm{C})$$

defining B

1. It is possible that the B lifts under slightly different conditions than those under which it is an endomorphism.
2. After use of the ∞-box, whose product is difficult, we can invert certain maps to obtain complexes. For this to work we need both biproducts and minus.
3. Not only must these spaces be based; B necessitates that they be A∞ or E∞ (plus some other thing about grouplike, for me).
4. After this we can consider the ”free ???”, but the product is a bit difficult.

$[\mathbb{N}, \vec{\gamma}\ \ , \mathrm{X}]$





2. Bibliography

1. Serre, Jean-Pierre. ”Homologie singulière des espaces fibrés. Applications.” Annals of Mathematics 54, no. 3 (1951): 425-505.

2.

About the Author

Dean Young is a master's student at New York University, where he studies mathematics.