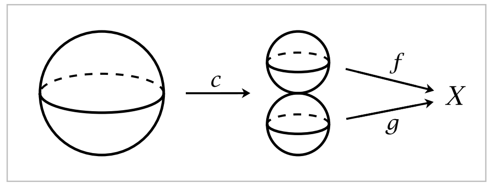
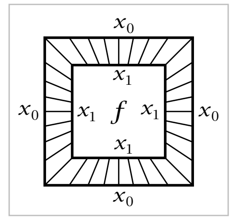
of the same form $(S^n, s_0) \to (X, x_0)$. In this interpretation of $\pi_n(X, x_0)$, the sum $f+g$ is the composition $S^n \xrightarrow{c} S^n \vee S^n \xrightarrow{f \vee g} X$ where c collapses the equator S^{n-1} in S^n to a point and we choose the basepoint s_0 to lie in this S^{n-1}.



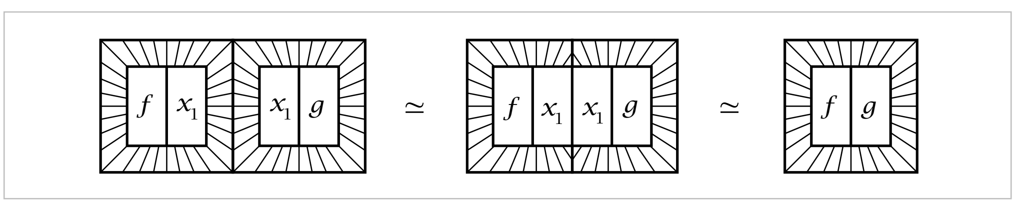
We will show next that if X is path-connected, different choices of the basepoint x_0 always produce isomorphic groups $\pi_n(X, x_0)$, just as for π_1, so one is justified in writing $\pi_n(X)$ for $\pi_n(X, x_0)$ in these cases. Given a path $\gamma : I \to X$ from $x_0 = \gamma(0)$ to another basepoint $x_1 = \gamma(1)$, we may associate to each map $f : (I^n, \partial I^n) \to (X, x_1)$ a new map $\gamma f : (I^n, \partial I^n) \to (X, x_0)$ by shrinking the domain of f to a smaller concentric cube in I^n, then inserting the path γ on each radial segment in the shell between this smaller cube and ∂I^n. When $n = 1$ the map γf is the composition of the three paths γ, f, and the inverse of γ, so the notation γf conflicts with the notation for composition of paths. Since we are mainly interested in the cases $n > 1$, we leave it to the reader to make the necessary notational adjustments when $n = 1$.



A homotopy of γ or f through maps fixing ∂I or ∂I^n, respectively, yields a homotopy of γf through maps $(I^n, \partial I^n) \to (X, x_0)$. Here are three other basic properties:

(1) $\gamma(f+g) \simeq \gamma f + \gamma g$.
(2) $(\gamma\eta) f \simeq \gamma(\eta f)$.
(3) $1f \simeq f$, where 1 denotes the constant path.

The homotopies in (2) and (3) are obvious. For (1), we first deform f and g to be constant on the right and left halves of I^n, respectively, producing maps we may call $f+0$ and $0+g$, then we excise a progressively wider symmetric middle slab of $\gamma(f+0) + \gamma(0+g)$ until it becomes $\gamma(f+g)$:



An explicit formula for this homotopy is

$$h_t(s_1, s_2, \cdots, s_n) = \begin{cases} \gamma(f+0)\big((2-t)s_1, s_2, \cdots, s_n\big), & s_1 \in [0, {}^1\!/_2] \\ \gamma(0+g)\big((2-t)s_1 + t - 1, s_2, \cdots, s_n\big), & s_1 \in [{}^1\!/_2, 1] \end{cases}$$

Thus we have $\gamma(f+g) \simeq \gamma(f+0) + \gamma(0+g) \simeq \gamma f + \gamma g$.

If we define a change-of-basepoint transformation $\beta_\gamma : \pi_n(X, x_1) \to \pi_n(X, x_0)$ by $\beta_\gamma([f]) = [\gamma f]$, then (1) shows that β_γ is a homomorphism, while (2) and (3) imply that β_γ is an isomorphism with inverse $\beta_{\overline{\gamma}}$ where $\overline{\gamma}$ is the inverse path of γ,