complex X and a subcomplex A, then $X/A \simeq X \cup CA$, the mapping cone of the inclusion $A \hookrightarrow X$. For we have $X/A = (X \cup CA)/CA \simeq X \cup CA$ since CA is a contractible subcomplex of $X \cup CA$.

Example 0.14. If (X, A) is a CW pair and A is contractible in X, that is, the inclusion $A \hookrightarrow X$ is homotopic to a constant map, then $X/A \simeq X \vee SA$. Namely, by the previous example we have $X/A \simeq X \cup CA$, and then since A is contractible in X, the mapping cone $X \cup CA$ of the inclusion $A \hookrightarrow X$ is homotopy equivalent to the mapping cone of a constant map, which is $X \vee SA$. For example, $S^n/S^i \simeq S^n \vee S^{i+1}$ for $i < n$, since S^i is contractible in S^n if $i < n$. In particular this gives $S^2/S^0 \simeq S^2 \vee S^1$, which is Example 0.8 again.

The Homotopy Extension Property

In this final section of the chapter we will actually prove a few things. In particular we prove the two criteria for homotopy equivalence described above, along with the fact that any two homotopy equivalent spaces can be embedded as deformation retracts of the same space.

The proofs depend upon a technical property that arises in many other contexts as well. Consider the following problem. Suppose one is given a map $f_0 : X \to Y$, and on a subspace $A \subset X$ one is also given a homotopy $f_t : A \to Y$ of $f_0 | A$ that one would like to extend to a homotopy $f_t : X \to Y$ of the given f_0. If the pair (X, A) is such that this extension problem can always be solved, one says that (X, A) has the **homotopy extension property**. Thus (X, A) has the homotopy extension property if every map $X \times \{0\} \cup A \times I \to Y$ can be extended to a map $X \times I \to Y$.

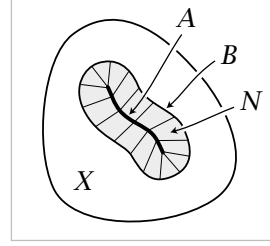In particular, the homotopy extension property for (X, A) implies that the identity map $X \times \{0\} \cup A \times I \to X \times \{0\} \cup A \times I$ extends to a map $X \times I \to X \times \{0\} \cup A \times I$, so $X \times \{0\} \cup A \times I$ is a retract of $X \times I$. The converse is also true: If there is a retraction $X \times I \to X \times \{0\} \cup A \times I$, then by composing with this retraction we can extend every map $X \times \{0\} \cup A \times I \to Y$ to a map $X \times I \to Y$. Thus the homotopy extension property for (X, A) is equivalent to $X \times \{0\} \cup A \times I$ being a retract of $X \times I$. This implies for example that if (X, A) has the homotopy extension property, then so does $(X \times Z, A \times Z)$ for any space Z, a fact that would not be so easy to prove directly from the definition.

If (X, A) has the homotopy extension property, then A must be a closed subspace of X, at least when X is Hausdorff. For if $r : X \times I \to X \times I$ is a retraction onto the subspace $X \times \{0\} \cup A \times I$, then the image of r is the set of points $z \in X \times I$ with $r(z) = z$, a closed set if X is Hausdorff, so $X \times \{0\} \cup A \times I$ is closed in $X \times I$ and hence A is closed in X.

A simple example of a pair (X, A) with A closed for which the homotopy extension property fails is the pair (I, A) where $A = \{0, 1, 1/2, 1/3, 1/4, \cdots\}$. It is not hard to show that there is no continuous retraction $I \times I \to I \times \{0\} \cup A \times I$. The breakdown of homotopy extension here can be attributed to the bad structure of (X, A) near 0.

With nicer local structure the homotopy extension property does hold, as the next example shows.

Example 0.15. A pair (X, A) has the homotopy extension property if A has a mapping cylinder neighborhood in X, by which we mean a closed neighborhood N containing a subspace B, thought of as the boundary of N, with $N - B$ an open neighborhood of A, such that there exists a map $f : B \rightarrow A$ and a homeomorphism $h : M_f \rightarrow N$ with $h | A \cup B = \mathbb{1}$. Mapping cylinder neighborhoods like this occur fairly often. For example, the thick letters discussed at the beginning of the chapter provide such neighborhoods of the thin letters, regarded as subspaces of the plane. To verify the homotopy extension property, notice first that $I \times I$ retracts onto $I \times \{0\} \cup \partial I \times I$, hence $B \times I \times I$ retracts onto $B \times I \times \{0\} \cup B \times \partial I \times I$, and this retraction induces a retraction of $M_f \times I$ onto $M_f \times \{0\} \cup (A \cup B) \times I$. Thus $(M_f, A \cup B)$ has the homotopy extension property. Hence so does the homeomorphic pair $(N, A \cup B)$. Now given a map $X \rightarrow Y$ and a homotopy of its restriction to A, we can take the constant homotopy on $X - (N - B)$ and then extend over N by applying the homotopy extension property for $(N, A \cup B)$ to the given homotopy on A and the constant homotopy on B.

Proposition 0.16. *If (X, A) is a CW pair, then $X \times \{0\} \cup A \times I$ is a deformation retract of $X \times I$, hence (X, A) has the homotopy extension property.*

Proof: There is a retraction $r : D^n \times I \rightarrow D^n \times \{0\} \cup \partial D^n \times I$, for example the radial projection from the point $(0, 2) \in D^n \times \mathbb{R}$. Then setting $r_t = tr + (1 - t)\mathbb{1}$ gives a deformation retraction of $D^n \times I$ onto $D^n \times \{0\} \cup \partial D^n \times I$. This deformation retraction gives rise to a deformation retraction of $X^n \times I$ onto $X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times I$ since $X^n \times I$ is obtained from $X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times I$ by attaching copies of $D^n \times I$ along $D^n \times \{0\} \cup \partial D^n \times I$. If we perform the deformation retraction of $X^n \times I$ onto $X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times I$ during the t-interval $[1/2^{n+1}, 1/2^n]$, this infinite concatenation of homotopies is a deformation retraction of $X \times I$ onto $X \times \{0\} \cup A \times I$. There is no problem with continuity of this deformation retraction at $t = 0$ since it is continuous on $X^n \times I$, being stationary there during the t-interval $[0, 1/2^{n+1}]$, and CW complexes have the weak topology with respect to their skeleta so a map is continuous iff its restriction to each skeleton is continuous. $\square$
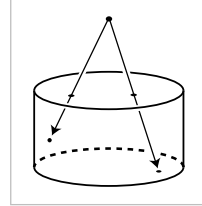
Now we can prove a generalization of the earlier assertion that collapsing a contractible subcomplex is a homotopy equivalence.

Proposition 0.17. *If the pair (X, A) satisfies the homotopy extension property and A is contractible, then the quotient map $q : X \rightarrow X/A$ is a homotopy equivalence.*

Proof: Let $f_t : X \rightarrow X$ be a homotopy extending a contraction of A, with $f_0 = \mathbb{1}$. Since $f_t(A) \subset A$ for all t, the composition $qf_t : X \rightarrow X/A$ sends A to a point and hence factors as a composition $X \xrightarrow{q} X/A \rightarrow X/A$. Denoting the latter map by $\overline{f}_t : X/A \rightarrow X/A$, we have $qf_t = \overline{f}_t q$ in the first of the two diagrams at the right. When $t = 1$ we have $f_1(A)$ equal to a point, the point to which A contracts, so f_1 induces a map $g : X/A \rightarrow X$ with $gq = f_1$, as in the second diagram. It follows that $qg = \overline{f}_1$ since $qg(\overline{x}) = qgq(x) = qf_1(x) = \overline{f}_1 q(x) = \overline{f}_1(\overline{x})$. The maps g and q are inverse homotopy equivalences since $gq = f_1 \simeq f_0 = \mathbb{1}$ via f_t and $qg = \overline{f}_1 \simeq \overline{f}_0 = \mathbb{1}$ via $\overline{f}_t$. $\qquad\square$

Another application of the homotopy extension property, giving a slightly more refined version of one of our earlier criteria for homotopy equivalence, is the following:

Proposition 0.18. *If (X_1, A) is a CW pair and we have attaching maps $f, g : A \rightarrow X_0$ that are homotopic, then $X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1$ rel X_0.*

Here the definition of $W \simeq Z$ rel Y for pairs (W, Y) and (Z, Y) is that there are maps $\varphi : W \rightarrow Z$ and $\psi : Z \rightarrow W$ restricting to the identity on Y, such that $\psi\varphi \simeq \mathbb{1}$ and $\varphi\psi \simeq \mathbb{1}$ via homotopies that restrict to the identity on Y at all times.

Proof: If $F : A \times I \rightarrow X_0$ is a homotopy from f to g, consider the space $X_0 \sqcup_F (X_1 \times I)$. This contains both $X_0 \sqcup_f X_1$ and $X_0 \sqcup_g X_1$ as subspaces. A deformation retraction of $X_1 \times I$ onto $X_1 \times \{0\} \cup A \times I$ as in Proposition 0.16 induces a deformation retraction of $X_0 \sqcup_F (X_1 \times I)$ onto $X_0 \sqcup_f X_1$. Similarly $X_0 \sqcup_F (X_1 \times I)$ deformation retracts onto $X_0 \sqcup_g X_1$. Both these deformation retractions restrict to the identity on X_0, so together they give a homotopy equivalence $X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1$ rel X_0. $\qquad\square$

We finish this chapter with a technical result whose proof will involve several applications of the homotopy extension property:

Proposition 0.19. *Suppose (X, A) and (Y, A) satisfy the homotopy extension property, and $f : X \rightarrow Y$ is a homotopy equivalence with $f|A = \mathbb{1}$. Then f is a homotopy equivalence rel A.*

Corollary 0.20. *If (X, A) satisfies the homotopy extension property and the inclusion $A \hookrightarrow X$ is a homotopy equivalence, then A is a deformation retract of X.*

Proof: Apply the proposition to the inclusion $A \hookrightarrow X$. $\qquad\square$

Corollary 0.21. *A map $f : X \rightarrow Y$ is a homotopy equivalence iff X is a deformation retract of the mapping cylinder M_f. Hence, two spaces X and Y are homotopy equivalent iff there is a third space containing both X and Y as deformation retracts.*

Proof: In the diagram at the right the maps i and j are the inclu-
sions and r is the canonical retraction, so $f = ri$ and $i \simeq jf$. Since
j and r are homotopy equivalences, it follows that f is a homotopy
equivalence iff i is a homotopy equivalence, since the composition
of two homotopy equivalences is a homotopy equivalence and a map homotopic to a
homotopy equivalence is a homotopy equivalence. Now apply the preceding corollary
to the pair (M_f, X), which satisfies the homotopy extension property by Example 0.15
using the neighborhood $X \times [0, {}^1/_2]$ of X in M_f. □
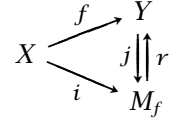
**Proof of 0.19**: Let $g : Y \to X$ be a homotopy inverse for f. There will be three steps
to the proof:

(1) Construct a homotopy from g to a map g_1 with $g_1 | A = \mathbb{1}$.

(2) Show $g_1 f \simeq \mathbb{1}$ rel A.

(3) Show $f g_1 \simeq \mathbb{1}$ rel A.

(1) Let $h_t : X \to X$ be a homotopy from $gf = h_0$ to $\mathbb{1} = h_1$. Since $f | A = \mathbb{1}$, we
can view $h_t | A$ as a homotopy from $g | A$ to $\mathbb{1}$. Then since we assume (Y, A) has the
homotopy extension property, we can extend this homotopy to a homotopy $g_t : Y \to X$
from $g = g_0$ to a map g_1 with $g_1 | A = \mathbb{1}$.

(2) A homotopy from $g_1 f$ to $\mathbb{1}$ is given by the formulas

$$k_t = \begin{cases} g_{1-2t} f, & 0 \le t \le {}^1/_2 \\ h_{2t-1}, & {}^1/_2 \le t \le 1 \end{cases}$$

Note that the two definitions agree when $t = {}^1/_2$. Since $f | A = \mathbb{1}$ and $g_t = h_t$ on A,
the homotopy $k_t | A$ starts and ends with the identity, and its second half simply re-
traces its first half, that is, $k_t = k_{1-t}$ on A. We will define a 'homotopy of homotopies'
$k_{tu} : A \to X$ by means of the figure at the right showing the param-
eter domain $I \times I$ for the pairs (t, u), with the t-axis horizontal
and the u-axis vertical. On the bottom edge of the square we de-
fine $k_{t0} = k_t | A$. Below the 'V' we define k_{tu} to be independent
of u, and above the 'V' we define k_{tu} to be independent of t.
This is unambiguous since $k_t = k_{1-t}$ on A. Since $k_0 = \mathbb{1}$ on A,
we have $k_{tu} = \mathbb{1}$ for (t, u) in the left, right, and top edges of the square. Next we
extend k_{tu} over X, as follows. Since (X, A) has the homotopy extension property,
so does $(X \times I, A \times I)$ by a remark in the paragraph following the definition of the ho-
motopy extension property. Viewing k_{tu} as a homotopy of $k_t | A$, we can therefore
extend $k_{tu} : A \to X$ to $k_{tu} : X \to X$ with $k_{t0} = k_t$. If we restrict this k_{tu} to the left, top,
and right edges of the (t, u)-square, we get a homotopy $g_1 f \simeq \mathbb{1}$ rel A.
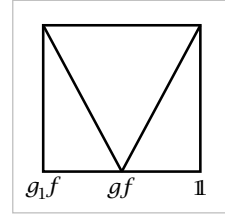
(3) Since $g_1 \simeq g$, we have $f g_1 \simeq fg \simeq \mathbb{1}$, so $f g_1 \simeq \mathbb{1}$ and steps (1) and (2) can be
repeated with the pair f, g replaced by g_1, f. The result is a map $f_1 : X \to Y$ with
$f_1 | A = \mathbb{1}$ and $f_1 g_1 \simeq \mathbb{1}$ rel A. Hence $f_1 \simeq f_1 (g_1 f) = (f_1 g_1) f \simeq f$ rel A. From this
we deduce that $f g_1 \simeq f_1 g_1 \simeq \mathbb{1}$ rel A. □