

Moving Least-Squares Are Backus–Gilbert Optimal

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1. INTRODUCTION

Moving least-squares methods for the interpolation of scattered data in the plane are well known [6]. The simplest of them is known more commonly as Shepard's method [5, 7]. The value of the interpolant at any point is obtained from a weighted least-squares polynomial approximation to the data, the weighting of a data point being inversely related to its distance from the point at which the interpolant is being evaluated. In Shepard's method, the polynomial is a constant.

The Backus–Gilbert theory [2] has been developed in a geophysical context, but it is a theory of interpolation in that the values of a number of functionals on an unknown function f are given, and an approximation to the value of f at some point is required. The basic principle is to optimize the approximation of the Dirac delta by a linear combination of the representers of the given functionals.

Recently, Abramovici [1] showed that Shepard's method could be obtained from the Backus–Gilbert theory. Here we will demonstrate that all moving least-squares approximants can be generated from a slightly modified Backus–Gilbert theory.

2. MOVING LEAST-SQUARES

Let $M := \binom{n+m}{n}$, i.e., the dimension of the space of polynomials of degree at most m in n variables, and $\{\varphi_1(\mathbf{x}), \varphi_2(\mathbf{x}), \dots, \varphi_M(\mathbf{x})\}$ be the monomials of degree at most m in n real variables. Suppose that we are given $N \geq M$ distinct points $\mathbf{x}_i \in \mathbb{R}^n$, not all of which lie on the zero set of a polynomial of degree at most m . We approximate $f: \mathbb{R}^n \rightarrow \mathbb{R}$, at $\mathbf{x}_0 \in \mathbb{R}^n$, by the weighted least-squares polynomial of degree m at the points \mathbf{x}_i , $1 \leq i \leq N$, with weights $w_i := w(\mathbf{x}_i - \mathbf{x}_0)$. Here $w: \mathbb{R}^n \rightarrow \mathbb{R}_+$. Typically, $w(\mathbf{x}) = |\mathbf{x}|^{-k}$

for some even k although many other choices are possible. This particular choice ensures that the approximation is actually an interpolant and also has a certain smoothness (see Lancaster and Salkauskas [6]).

In order to obtain an explicit expression for this approximation, let $B^T \in \mathbb{R}^{N \times M}$ be the Vandermonde matrix of the monomials of degree at most m evaluated at the points \mathbf{x}_i . Specifically,

$$B_{ij}^T = \varphi_j(\mathbf{x}_i). \quad (2.1)$$

Further, let $W = \text{diag}(w_1, w_2, \dots, w_N) \in \mathbb{R}^{N \times N}$, $\mathbf{f} = (f(\mathbf{x}_1), \dots, f(\mathbf{x}_N))^T \in \mathbb{R}^N$, and $\boldsymbol{\varphi} = (\varphi_1(\mathbf{x}_0), \varphi_2(\mathbf{x}_0), \dots, \varphi_M(\mathbf{x}_0))^T \in \mathbb{R}^M$. Thus the moving least-squares approximation to $f(\mathbf{x}_0)$ is $\sum_{i=1}^M c_i \varphi_i(\mathbf{x}_0)$, where $\mathbf{c} \in \mathbb{R}^M$ minimizes

$$(\mathbf{B}^T \mathbf{c} - \mathbf{f})^T W (\mathbf{B}^T \mathbf{c} - \mathbf{f}).$$

Under our assumptions, W is positive definite and B^T is of full rank. Thus \mathbf{c} is the solution of the normal equations

$$(BWB^T) \mathbf{c} = BW\mathbf{f}, \quad (2.2)$$

and the approximation is therefore given by

$$\sum_{i=1}^M c_i \varphi_i(\mathbf{x}_0) = \mathbf{c}^T \boldsymbol{\varphi} = \mathbf{f}^T W B^T (BWB^T)^{-1} \boldsymbol{\varphi}. \quad (2.3)$$

For further properties of moving least-squares approximations see, for instance, [4, 6].

3. BACKUS-GILBERT OPTIMALITY

We give a brief, univariate account of those features of the Backus-Gilbert theory that apply to the problem at hand. It will be clear how this can be extended to \mathbb{R}^n .

Let $\{\lambda_i\}_{i=1}^N$ be a linearly independent set of bounded linear functionals on $L_2[a, b]$, and for some (unknown) $f \in L_2[a, b]$ let the values of $\lambda_i(f)$, $1 \leq i \leq N$, be given. It is required to obtain an approximation $\tilde{f}(x_0)$, $x_0 \in [a, b]$, in terms of this information. To this end we seek coefficients $a_i(x_0)$ such that $\sum_{i=1}^N a_i(x_0) \lambda_i(f)$ will in some sense be a good approximation to $f(x_0)$. Let $\{L_i\}_{i=1}^N$ be the representers of the functionals λ_i . Then $W_0(x) := \sum_{i=1}^N a_i(x_0) L_i(x)$ is to be a good approximation to the Dirac distribution, $\delta(x - x_0)$. In order to measure the "deltaness" of W_0 , Backus and Gilbert propose that a symmetric, non-negative "sink function,"

$J(x, x_0)$, vanishing only at $x = x_0$, be selected, and that the a_i 's be chosen to minimize the "spread" of W_0 :

$$S(W_0; x_0) := \int_a^b J(x, x_0) W_0^2(x) dx. \quad (3.1)$$

If $J(x, x_0)$ is chosen so as to increase as $|x - x_0|$ increases, then a W_0 minimizing the spread will tend to have its relatively large values concentrated near x_0 . The choice $J(x, x_0) = (x - x_0)^2$ is typical. Clearly, the spread is just the square of a weighted L_2 norm of W_0 .

As well, since the approximation

$$\hat{f}(x_0) = \int_a^b f(x) W_0(x) dx \quad (3.2)$$

can be seen as an average of f over $[a, b]$, it is natural to impose the condition

$$\int_a^b W_0(x) dx = 1, \quad (3.3)$$

which certainly is also satisfied by $\delta(x - x_0)$.

Abramovici [1] has applied this to the case where the functionals λ_i are $\delta(x - x_i)$, and hence do not have representers. However, by working with a certain δ -sequence of functions, he has shown that the approximation $\hat{f}(x_0)$ is identical to the Shepard interpolant. In the sequel, we apply this technique with a broad class of δ -sequences. We do not require $J(x, x_0)$ to vanish when $x = x_0$. In addition, since W_0 is to approximate $\delta(x - x_0)$, we impose the conditions

$$\int_a^b x^i W_0(x) dx = x_0^i, \quad 0 \leq i \leq m, \quad m \leq N - 1,$$

which forces the approximation to be exact for polynomials of degree up to m . The resulting schemes are shown to be moving least-squares methods of approximation, and are interpolants if $J(x, x_0)$ is chosen appropriately.

4. MOVING LEAST-SQUARES ARE BACKUS-GILBERT OPTIMAL

Suppose that $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N \in \mathbb{R}^n$ form a given set of points not all of which lie on the zero set of any polynomial of degree m . Let $D \subset \mathbb{R}^n$ be a compact, connected set which contains all of the \mathbf{x}_i in its interior. Select a continuous $J: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that $J(\mathbf{x}, \mathbf{y}) = 0 \Rightarrow \mathbf{x} = \mathbf{y}$. Let $L_2(D)$ be, as

usual $\{f: D \rightarrow \mathbb{R} \mid \int_D f^2(\mathbf{x}) d\mathbf{x} < \infty\}$, with the usual inner product $\langle f, g \rangle := \int_D f(\mathbf{x}) g(\mathbf{x}) d\mathbf{x}$. Further, for fixed $\mathbf{x}_0 \in D$, let $L_2(D; J; \mathbf{x}_0) = \{f: D \rightarrow \mathbb{R} \mid \int_D J(\mathbf{x}, \mathbf{x}_0) f^2(\mathbf{x}) d\mathbf{x} < \infty\}$, with inner product $\langle f, g \rangle_J := \int_D J(\mathbf{x}, \mathbf{x}_0) f(\mathbf{x}) g(\mathbf{x}) d\mathbf{x}$, and associated norm $\|\cdot\|_J$. Now, as J is continuous, there is an $M > 0$ such that $\|f\|_J \leq M\|f\|_2$, and hence $L_2(D) \subset L_2(D; J; \mathbf{x}_0)$. Keeping in mind the form of the approximation $\tilde{f}(\mathbf{x}_0)$ of Eq. (3.2), we now make

DEFINITION 4.1. For $\mathbf{x}_0 \in D$, $W_0 \in L_2(D; J; \mathbf{x}_0)$, and $f \in C(D)$, the W_0 -approximation to $f(\mathbf{x}_0)$ is given by

$$\tilde{f}(\mathbf{x}_0) := \int_D f(\mathbf{x}) W_0(\mathbf{x}) d\mathbf{x} = \langle f, W_0 \rangle.$$

Now we wish our approximation to be exact for polynomials of a given degree.

DEFINITION 4.2. $W_0 \in L_2(D; J; \mathbf{x}_0)$ has degree of precision m if, for all polynomials p , of degree at most m ,

$$\langle p, W_0 \rangle = p(\mathbf{x}_0).$$

We set $\mathcal{A}_m := \{W_0 \in L_2(D; J; \mathbf{x}_0) \mid W_0 \text{ has degree of precision } m\}$.

As \mathcal{A}_m is the intersection of a finite number of hyperplanes, it is closed and convex in $L_2(D; J; \mathbf{x}_0)$.

The fragment of the Backus–Gilbert (abbreviated B-G in the sequel) theory described in Section 3, shows that W_0 is drawn from $\text{span}\{L_i\}_{i=1}^N$, a finite-dimensional subspace of $L_2[a, b]$. We insist on the same property of W_0 in our next definition.

DEFINITION 4.3. Let $V \subset L_2(D)$ be a finite-dimensional subspace, and fix $m \in \mathbb{Z}_+$. The approximation $\tilde{f}(\mathbf{x}_0) = \langle f, W_0 \rangle$ is B-G optimal of degree m with respect to V if $W_0 \in V \cap \mathcal{A}_m$ and $\|W_0\|_J$ is a minimum. (In the terminology of B-G, W_0 has minimal spread.)

Note that the existence and uniqueness of such a W_0 is guaranteed by the fact that $V \cap \mathcal{A}_m$ is closed, convex, and finite-dimensional.

If this approximation to $f(\mathbf{x}_0)$ is to be constructed in terms of information about f consisting of the values of a finite (say N) number of bounded, linear functionals on $L_2(D)$, with Riesz representers $L_1, \dots, L_N \in L_2(D)$, then $W_0 = \sum_{i=1}^N a_i L_i$, where the a_i 's satisfy linear constraints inherent in Definition 4.2, and the minimization of $\|W_0\|_J$ involves nothing more than the minimization of a quadratic form with linear constraints—a standard problem in linear algebra. Our aim is to examine the existence and nature

of the approximation when the given information about f consists of function values at distinct points of D . As Dirac delta functionals are unbounded, this does not fit directly into our earlier B-G formulation. However, as delta distributions are limits of ordinary functions we are able to extend Definition 4.3 to include the notion of generalized B-G optimality of degree m . For this we make use of certain delta sequences defined below.

DEFINITION 4.4. $\{\Delta_\lambda(\mathbf{x}) | \lambda > 0\}$ is said to be a delta sequence if $\lim_{\lambda \rightarrow \infty} \int_{\mathbb{R}^n} f(\mathbf{x}) \Delta_\lambda(\mathbf{x}) d\mathbf{x} = f(\mathbf{0})$ for any bounded f continuous at $\mathbf{0}$. We will say that the delta sequence is *regular* if, in addition, $\Delta_\lambda \in L_2(\mathbb{R}^n)$ and

$$\lim_{\lambda \rightarrow \infty} \int_{\mathbb{R}^n} f(\mathbf{x}) \frac{\Delta_\lambda(\mathbf{x} - \mathbf{a}) \Delta_\lambda(\mathbf{x} - \mathbf{b})}{\int_{\mathbb{R}^n} \Delta_\lambda^2(\mathbf{x}) d\mathbf{x}} d\mathbf{x} = \begin{cases} 0 & \text{if } \mathbf{a} \neq \mathbf{b}, \\ f(\mathbf{a}) & \text{if } \mathbf{a} = \mathbf{b}, \end{cases}$$

for any bounded f continuous at \mathbf{a} .

DEFINITION 4.5. Suppose that $\{\mathbf{x}_1, \dots, \mathbf{x}_N\} \subset \text{Int}(D)$ are distinct points. Let $\{\Delta_\lambda(\mathbf{x}) | \lambda > 0\}$ be a regular delta sequence, and set $V^{(\lambda)} = \text{span}\{\Delta_\lambda(\mathbf{x} - \mathbf{x}_1), \Delta_\lambda(\mathbf{x} - \mathbf{x}_2), \dots, \Delta_\lambda(\mathbf{x} - \mathbf{x}_N)\}$. Further, let $W_0^{(\lambda)} \in V^{(\lambda)} \cap \mathcal{A}_m$ be B-G optimal of degree m . If $\tilde{f}(\mathbf{x}_0) := \lim_{\lambda \rightarrow \infty} \int_D f(\mathbf{x}) W_0^{(\lambda)}(\mathbf{x}) d\mathbf{x}$ exists for all $f \in C(D)$ we will say that $\tilde{f}(\mathbf{x}_0)$ is the generalized B-G approximation of degree m to $f(\mathbf{x}_0)$.

The use of regular delta sequences is not restrictive. In fact, the common constructions of delta sequences are regular.

PROPOSITION 4.6. Suppose that $K \in L_1(\mathbb{R}^n)$ is bounded and is such that $\int_{\mathbb{R}^n} K(\mathbf{x}) d\mathbf{x} = 1$. Then $\{\Delta_\lambda(\mathbf{x}) := \lambda^n K(\lambda\mathbf{x}) | \lambda > 0\}$ is a regular delta sequence.

Proof. The fact that $\{\Delta_\lambda\}$ is a delta sequence is standard (see for instance [3, Sect. 3.2]). We must show that it is regular. Now as $K \in L_1(\mathbb{R}^n)$ is bounded, $K \in L_2(\mathbb{R}^n)$ and hence, an easy calculation shows that $\Delta_\lambda \in L_2(\mathbb{R}^n)$. Further, $\Delta_\lambda^2(\mathbf{x}) / \int_{\mathbb{R}^n} \Delta_\lambda^2(\mathbf{x}) d\mathbf{x} = \lambda^n K^2(\lambda\mathbf{x}) / \int_{\mathbb{R}^n} K^2(\mathbf{x}) d\mathbf{x}$ and so the same calculations that show that $\{\Delta_\lambda\}$ is a delta sequence show that $\{\Delta_\lambda^2 / \int_{\mathbb{R}^n} \Delta_\lambda^2 d\mathbf{x}\}$ is a delta sequence. Thus

$$\lim_{\lambda \rightarrow \infty} \int_{\mathbb{R}^n} f(\mathbf{x}) \frac{\Delta_\lambda(\mathbf{x} - \mathbf{a}) \Delta_\lambda(\mathbf{x} - \mathbf{b})}{\int_{\mathbb{R}^n} \Delta_\lambda^2(\mathbf{x}) d\mathbf{x}} d\mathbf{x} = f(\mathbf{a}) \quad \text{if } \mathbf{b} = \mathbf{a}.$$

Now if $\mathbf{b} \neq \mathbf{a}$, as we assume f to be bounded, it suffices to show that

$$\lim_{\lambda \rightarrow \infty} \int_{\mathbb{R}^n} \frac{|\Delta_\lambda(\mathbf{x} - \mathbf{a}) \Delta_\lambda(\mathbf{x} - \mathbf{b})|}{\int_{\mathbb{R}^n} \Delta_\lambda^2(\mathbf{x}) d\mathbf{x}} d\mathbf{x} = 0$$

which in turn is implied by

$$\lim_{\lambda \rightarrow \infty} \int_{\mathbb{R}^n} \lambda^n |K(\lambda(\mathbf{x} - \mathbf{a})) K(\lambda(\mathbf{x} - \mathbf{b}))| d\mathbf{x} = 0.$$

To see this let $\varepsilon > 0$ be given. As K is bounded, there is a constant $M > 0$ such that $|K(\mathbf{x})| \leq M$. Choose R sufficiently large so that $\int_{|\mathbf{u}| \geq R} |K(\mathbf{u})| d\mathbf{u} \leq \varepsilon/(2M)$. Then, upon letting $\mathbf{u} = \lambda(\mathbf{x} - \mathbf{a})$,

$$\begin{aligned} & \int_{\mathbb{R}^n} \lambda^n |K(\lambda(\mathbf{x} - \mathbf{a})) K(\lambda(\mathbf{x} - \mathbf{b}))| d\mathbf{x} \\ &= \int_{\mathbb{R}^n} |K(\mathbf{u})| \cdot |K(\mathbf{u} + \lambda(\mathbf{a} - \mathbf{b}))| d\mathbf{u} \\ &= \left(\int_{|\mathbf{u}| \leq R} + \int_{|\mathbf{u}| \geq R} \right) |K(\mathbf{u})| \cdot |K(\mathbf{u} + \lambda(\mathbf{a} - \mathbf{b}))| d\mathbf{u} \\ &\leq \{\varepsilon/(2M)\} M + \int_{|\mathbf{u}| \leq R} |K(\mathbf{u})| \cdot |K(\mathbf{u} + \lambda(\mathbf{a} - \mathbf{b}))| d\mathbf{u} \\ &\leq \varepsilon/2 + \left\{ \int_{|\mathbf{u}| \leq R} K^2(\mathbf{u}) d\mathbf{u} \right\}^{1/2} \left\{ \int_{|\mathbf{u}| \leq R} K^2(\mathbf{u} + \lambda(\mathbf{a} - \mathbf{b})) d\mathbf{u} \right\}^{1/2}. \end{aligned}$$

Now select L so large that for $|\mathbf{u}| \geq L$, $\int_{|\mathbf{u}| \geq L} K^2(\mathbf{u}) d\mathbf{u} < \varepsilon^2/4 \int_{\mathbb{R}^n} K^2(\mathbf{x}) d\mathbf{x}$. Then for λ sufficiently large, $|\mathbf{u} + \lambda(\mathbf{a} - \mathbf{b})| \geq L$ for all $|\mathbf{u}| \leq R$ and so $\left\{ \int_{|\mathbf{u}| \leq R} K^2(\mathbf{u} + \lambda(\mathbf{a} - \mathbf{b})) d\mathbf{u} \right\}^{1/2} \leq \left\{ \int_{|\mathbf{u}| \geq L} K^2(\mathbf{u}) d\mathbf{u} \right\}^{1/2} < \varepsilon/(2\|K\|_2)$. The result now follows. ■

We are now ready to state and prove our main result.

THEOREM 4.7. *Moving least-squares approximations by polynomials of degree m are generalized B-G optimal of degree m .*

Proof. We make use of the notation of Section 2. Recall that the B-G approach is to approximate $f(\mathbf{x}_0)$ by

$$\tilde{f}(\mathbf{x}_0) := \int_D f(\mathbf{x}) W_0^{(\lambda)}(\mathbf{x}) d\mathbf{x},$$

where $W_0^{(\lambda)}(\mathbf{x}) := \sum_{i=1}^N a_i^{(\lambda)} \Delta_\lambda(\mathbf{x} - \mathbf{x}_i)$. The coefficients, $a_i^{(\lambda)}$, are chosen so that the spread, $\|W_0^{(\lambda)}\|_J$, is a minimum. We have added the requirement that it reproduces all the polynomials of degree at most m , i.e.,

$$\int_D \varphi_i(\mathbf{x}) W_0^{(\lambda)}(\mathbf{x}) d\mathbf{x} = \varphi_i(\mathbf{x}_0), \quad 1 \leq i \leq M. \quad (4.1)$$

Now,

$$\int_D J(\mathbf{x}, \mathbf{x}_0) [W_0^{(\lambda)}(\mathbf{x})]^2 d\mathbf{x} = [\mathbf{a}^{(\lambda)}]^T A(\lambda) \mathbf{a}^{(\lambda)},$$

where

$$A_{ij}^{(\lambda)} = \int_D J(\mathbf{x}, \mathbf{x}_0) \Delta_\lambda(\mathbf{x} - \mathbf{x}_i) \Delta_\lambda(\mathbf{x} - \mathbf{x}_j) d\mathbf{x}, \quad 1 \leq i, j \leq N,$$

and $\mathbf{a}^{(\lambda)} \in \mathbb{R}^N$ is the coefficient vector. Note that as $A^{(\lambda)}$ is a matrix of inner products it is non-negative definite for all $\lambda > 0$. Because of our assumption that $\{\Delta_\lambda\}$ is regular, we may assume without loss of generality that $A^{(\lambda)}$ is strictly positive definite for all $\lambda > 0$. Further, the constraints, (4.1), may be expressed in matrix form as

$$B^{(\lambda)} \mathbf{a}^{(\lambda)} = \boldsymbol{\varphi},$$

where $B^{(\lambda)} \in \mathbb{R}^{M \times N}$ and $B_{ij}^{(\lambda)} = \int_D \varphi_i(\mathbf{x}) \Delta_\lambda(\mathbf{x} - \mathbf{x}_j) d\mathbf{x}$.

The approximation is then

$$\tilde{f}_\lambda(\mathbf{x}_0) := \sum_{i=1}^N a_i^{(\lambda)} f(\mathbf{x}_i) = \mathbf{f}^T \mathbf{a}^{(\lambda)},$$

where $\mathbf{a}^{(\lambda)} \in \mathbb{R}^N$ minimizes $[\mathbf{a}^{(\lambda)}]^T A^{(\lambda)} \mathbf{a}^{(\lambda)}$ subject to the constraint $B^{(\lambda)} \mathbf{a}^{(\lambda)} = \boldsymbol{\varphi}$. Note that $B_{ij}^{(\lambda)} = \int_D \varphi_i(\mathbf{x}) \Delta_\lambda(\mathbf{x} - \mathbf{x}_j) d\mathbf{x} = \int_{\mathbb{R}^n} (\chi_D(\mathbf{x}) \varphi_i(\mathbf{x})) \Delta_\lambda(\mathbf{x} - \mathbf{x}_j) d\mathbf{x} \rightarrow \varphi_i(\mathbf{x}_j)$ as $\lambda \rightarrow \infty$. Hence

$$\lim_{\lambda \rightarrow \infty} B^{(\lambda)} = B \quad (\text{of (2.1)}), \quad (4.2)$$

which by our assumptions is of full rank. Thus we may, without loss of generality, assume that $B^{(\lambda)}$ is of full rank. We now have a standard problem in linear algebra, whose solution is given by

LEMMA 4.8. *The $\mathbf{a} \in \mathbb{R}^N$ for which $B^{(\lambda)} \mathbf{a} = \boldsymbol{\varphi}$ and $\mathbf{a}^T A^{(\lambda)} \mathbf{a}$ is a minimum is given by*

$$\mathbf{a} = A^{-1} B^T (B A^{-1} B^T)^{-1} \boldsymbol{\varphi}. \quad (4.3)$$

For convenience we have suppressed the λ superscript.

Proof. As A^{-1} is also positive definite and B is of full rank, an easy argument shows that $B A^{-1} B^T$ is non-singular. Now suppose that $\mathbf{x} \in \mathbb{R}^N$ is such that $B\mathbf{x} = \boldsymbol{\varphi}$. Then $\mathbf{x}^T A \mathbf{x} = (\mathbf{x} - \mathbf{a})^T A (\mathbf{x} - \mathbf{a}) + 2\mathbf{a}^T A (\mathbf{x} - \mathbf{a}) + \mathbf{a}^T A \mathbf{a}$.

But

$$\begin{aligned}\mathbf{a}^T A(\mathbf{x} - \mathbf{a}) &= \boldsymbol{\varphi}^T (BA^{-1}B^T)^{-1} BA^{-1}A(\mathbf{x} - A^{-1}B^T(BA^{-1}B^T)^{-1}\boldsymbol{\varphi}) \\ &= \boldsymbol{\varphi}^T (BA^{-1}B^T)^{-1}(B\mathbf{x} - \boldsymbol{\varphi}) \\ &= \boldsymbol{\varphi}^T \mathbf{0} = 0.\end{aligned}$$

Hence

$$\mathbf{x}^T A\mathbf{x} = (\mathbf{x} - \mathbf{a})^T A(\mathbf{x} - \mathbf{a}) + \mathbf{a}^T A\mathbf{a} \geq \mathbf{a}^T A\mathbf{a}. \quad \blacksquare$$

We now compute $\lim_{\lambda \rightarrow \infty} \mathbf{a}^{(\lambda)}$, where $\mathbf{a}^{(\lambda)}$ is given by (4.3). We may write

$$\mathbf{a}^{(\lambda)} = [A^{(\lambda)}/\kappa]^{-1} [B^{(\lambda)}]^T \{B^{(\lambda)}[A^{(\lambda)}/\kappa]^{-1} [B^{(\lambda)}]^T\}^{-1} \boldsymbol{\varphi},$$

where $\kappa := \int_{\mathbb{R}^n} \Delta_\lambda^2(\mathbf{x}) d\mathbf{x}$. First consider $A^{(\lambda)}/\kappa$.

$$\begin{aligned}\lim_{\lambda \rightarrow \infty} (1/\kappa) A_{ii}^{(\lambda)} &= \lim_{\lambda \rightarrow \infty} (1/\kappa) \int_D J(\mathbf{x}, \mathbf{x}_0) \Delta_\lambda^2(\mathbf{x} - \mathbf{x}_i) d\mathbf{x} \\ &= \lim_{\lambda \rightarrow \infty} \int_{\mathbb{R}^n} \chi_D(\mathbf{x}) J(\mathbf{x}, \mathbf{x}_0) (1/\kappa) \Delta_\lambda^2(\mathbf{x} - \mathbf{x}_i) d\mathbf{x} \\ &= J(\mathbf{x}_i, \mathbf{x}_0) \quad \text{as } \{\Delta_\lambda\} \text{ is regular.}\end{aligned}$$

Also, for $j \neq i$,

$$\begin{aligned}\lim_{\lambda \rightarrow \infty} (1/\kappa) A_{ij}^{(\lambda)} &= \lim_{\lambda \rightarrow \infty} (1/\kappa) \int_D J(\mathbf{x}, \mathbf{x}_0) K(\lambda(\mathbf{x} - \mathbf{x}_i)) K(\lambda(\mathbf{x} - \mathbf{x}_j)) d\mathbf{x} \\ &= 0 \quad \text{again by the regularity of } \{\Delta_\lambda\}.\end{aligned}$$

Hence $\lim_{\lambda \rightarrow \infty} (1/\kappa) A^{(\lambda)} = \text{diag}(J(\mathbf{x}_1, \mathbf{x}_0), \dots, J(\mathbf{x}_N, \mathbf{x}_0))$. Set $W := \text{diag}(J^{-1}(\mathbf{x}_1, \mathbf{x}_0), \dots, J^{-1}(\mathbf{x}_N, \mathbf{x}_0)) = \lim_{\lambda \rightarrow \infty} \{(1/\kappa) A^{(\lambda)}\}^{-1}$.

We have already seen that $\lim_{\lambda \rightarrow \infty} B^{(\lambda)} = B$ (of (2.1)). Hence

$$\lim_{\lambda \rightarrow \infty} \mathbf{a}^{(\lambda)} = WB^T(BWB^T)^{-1} \boldsymbol{\varphi}$$

and

$$\lim_{\lambda \rightarrow \infty} \hat{f}_\lambda(\mathbf{x}_0) = \mathbf{f}^T \mathbf{a} = \mathbf{f}^T WB^T(BWB^T)^{-1} \boldsymbol{\varphi}. \quad (4.4)$$

Comparing (4.4) with (2.3) we see that the Backus–Gilbert approximation is exactly moving least-squares with weights $w_i = 1/J(\mathbf{x}_i, \mathbf{x}_0)$. \blacksquare

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