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School of Science

S&P 500 Volatility Analysis: Bayesian Inference with ARCH, GARCH and Stochastic Volatility Models

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S&P 500 Volatility Analysis

- Understanding market volatility is a fundamental aspect of financial modelling.
- Data: S&P 500, a benchmark for U.S. financial markets. It serves as a key indicator of economic health and investor sentiment (2014 and 2024).
- **Problem**: whether relatively simple models, such as ARCH and GARCH, can adequately capture periods of high volatility and their underlying patterns in the S&P 500 data.
- Applications in financial risk management: volatility forecasting, risk measurement and stress testing, option pricing, and portfolio management.

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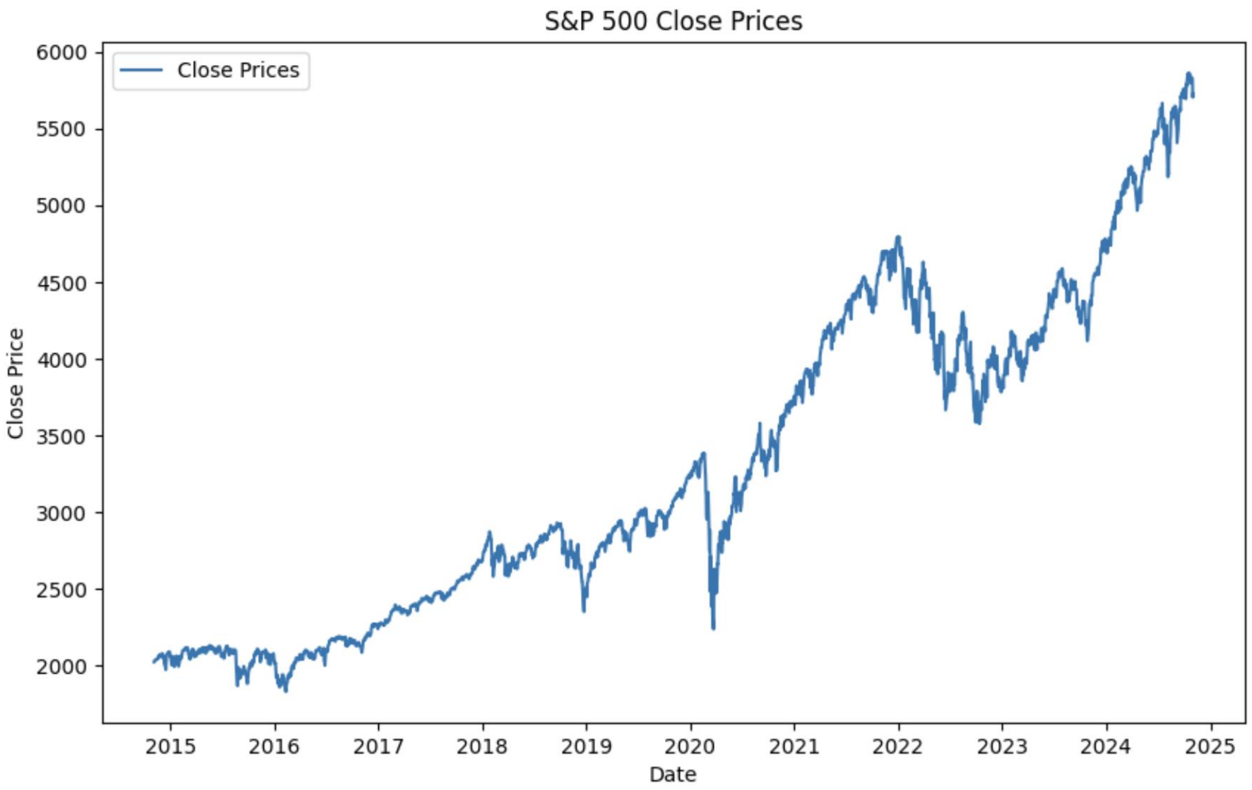

Figure 1: Historical closing prices of the S&P 500 index, showing the general upward trend and fluctuations.

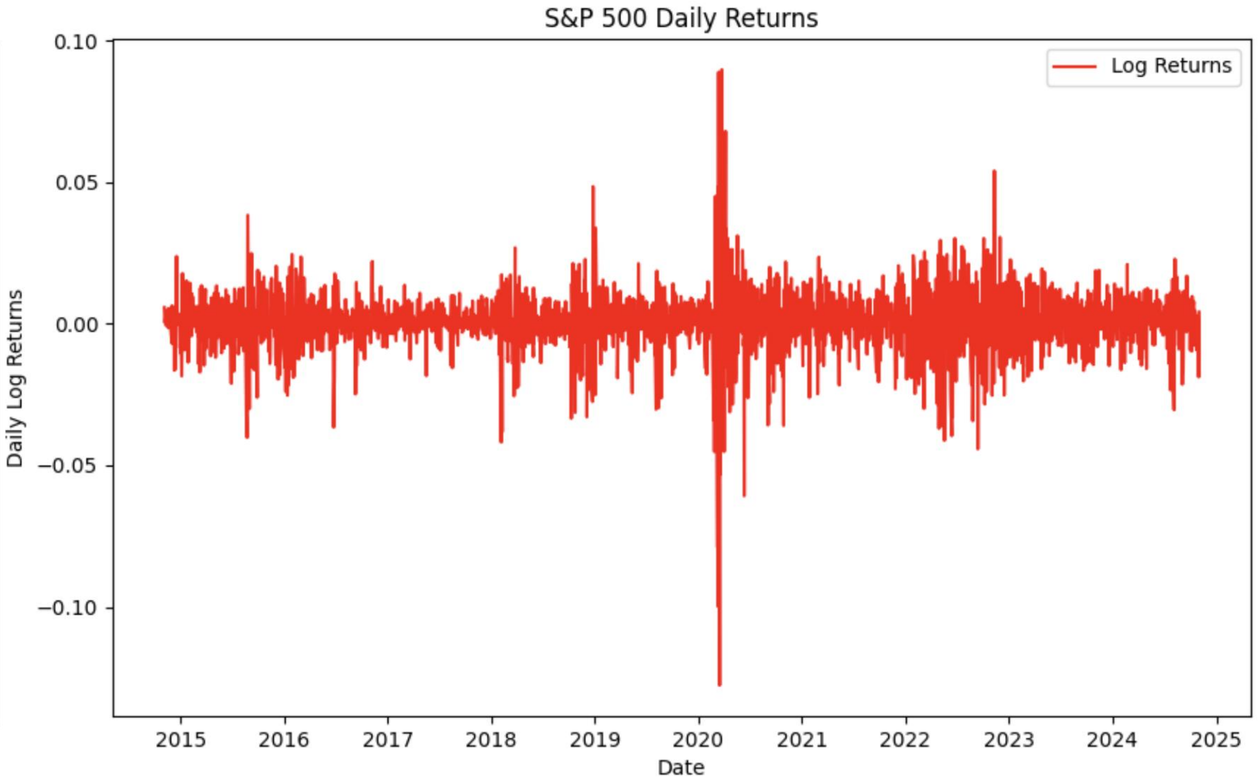

Figure 2: Daily log returns of the S&P 500 index, illustrating periods of high and low volatility over time.

Volatility: how much of the price of a stock move up and down overtime.
Return: the profit or loss one's make on an investment.

ARCH and GARCH Models

- Statistical models used to estimate the volatility of stock returns in finance.
- As we want to compare the volatility prediction performance, we use time series non-linear models and compare.

- ARCH(1) models:

$$\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2.$$

σ^2_t - conditional variance at time t
α_0 - constant term (baseline variance)
α_1 - ARCH parameter measures impact of previous shocks
a^2_{t-1} - squared return from previous period

- GARCH(1,1) models:

$$\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2.$$

β_1 - GARCH parameter measures persistence of volatility
Σ^2_{t-1} - previous period's conditional variance

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ARCH and GARCH - Prior Selections

- Mostly informative priors estimated from previous S&P500 study for the years 2012-2022. (with subperiods to include 2008 crisis)

```
// INFORMATIVE PRIORS - Correct - Original Priors:
mu ~ normal(prior_mean_mu, 1.5 * prior_sd_mu); // Informative prior for mean
alpha0 ~ normal(0.1, 0.05);              // Base volatility prior/weakly informative prior
alpha1 ~ beta(2, 5);                     // ARCH parameter prior/ informative prior
// https://www.shs-conferences.org/articles/shsconf/pdf/2023/18/shsconf_fems2023_01077.pdf
// Scale parameter error can be ignored!
```

```
prior_mean_mu = log_return_mean,
prior_sd_mu = log_return_sd
```

```
# Calculate Mean and SD for Priors
log_return_mean <- mean(data$Log_Returns, na.rm = TRUE)
log_return_sd <- sd(data$Log_Returns, na.rm = TRUE)
```


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ARCH and GARCH - Prior Selections

Difference (2002-2007 vs 2002-2010)					Percentage				
	α_0	α_1	β	γ		α_0	α_1	β	γ
ARCH	0.0000591	0.2938558			ARCH	96.25%	210.13%		
GARCH	-3.45E-07	0.0113187	0.0013508		GARCH	-25.94%	20.76%	0.15%	
EGARCH	0.0780327	-0.0062659	0.0076176	-0.0155777	EGARCH	45.20%	-6.57%	0.78%	-17.06%
GJR-GARCH	-2.20E-07	-0.0020502	0.0147492	-0.0131719	GJR-GARCH	-20.95%	-2.01%	1.58%	-12.81%
Difference (2012-2020.01 vs 2012-2020.12)					Percentage				
	α_0	α_1	β	γ		α_0	α_1	β	γ
ARCH	0.000005	0.2492201			ARCH	10.68%	87.99%		
GARCH	-2.20E-07	0.0268766	-0.0041007		GARCH	-4.61%	14.03%	-0.55%	
EGARCH	0.0643097	0.1512663	0.0044431	0.0680251	EGARCH	9.18%	100.50%	0.48%	27.72%
GJR-GARCH	4.90E-07	0.074407	-0.067822	0.0314114	GJR-GARCH	12.34%	25.46%	-8.40%	9.70%

alpha_1 ~ 0.3, modelled by beta distribution, bounded from 0,1. Represents skewness well.
(mean of a Beta(2,5) distribution approximately 0.2857)

alpha_0 selected from domain knowledge, since smaller value caused scale errors while running the code.

Chen, X. (2023). Comparing various GARCH-type models in the estimation and forecasts of volatility of S&P 500 returns during Global Finance Crisis of 2008 and COVID-19 financial crisis. SHS Web of Conferences, 169, 01077.

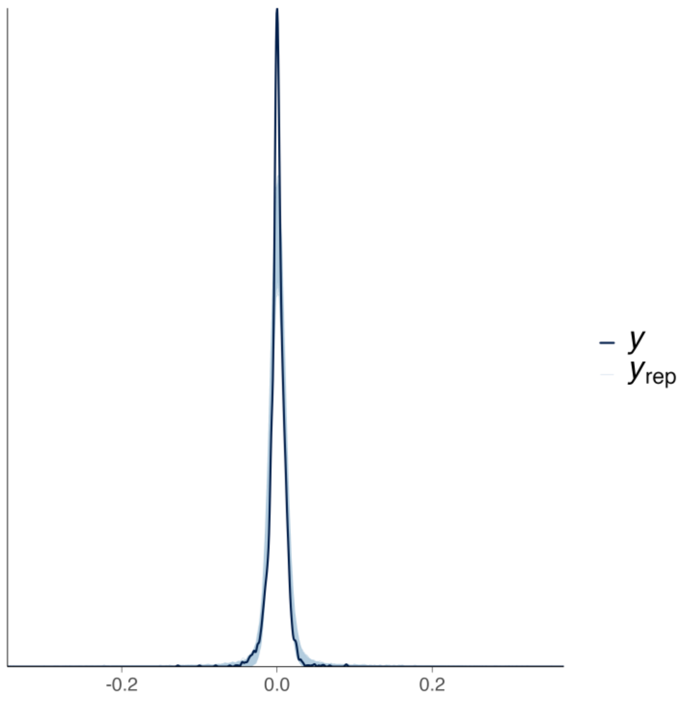
ARCH and GARCH - Posterior Predictive Check

- Plot of density of draws of daily log returns from the posterior distribution against the data.

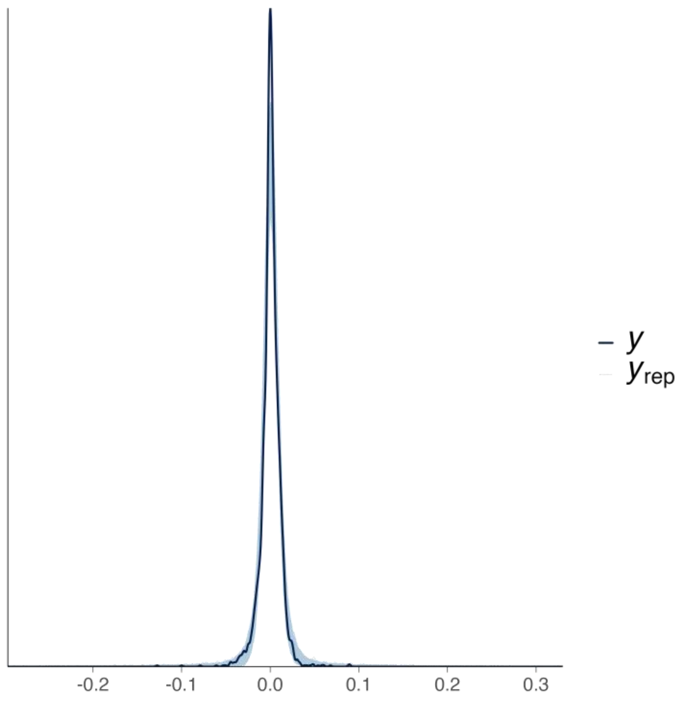
- Dark blue line represents the data while light blue field represents the samples from the posterior distribution.

- Distribution seems to explore the data very closely.

- GARCH has slightly thicker bulk (next slide)



ARCH Model – Posterior Predictive Check



GARCH Model – Posterior Predictive Check

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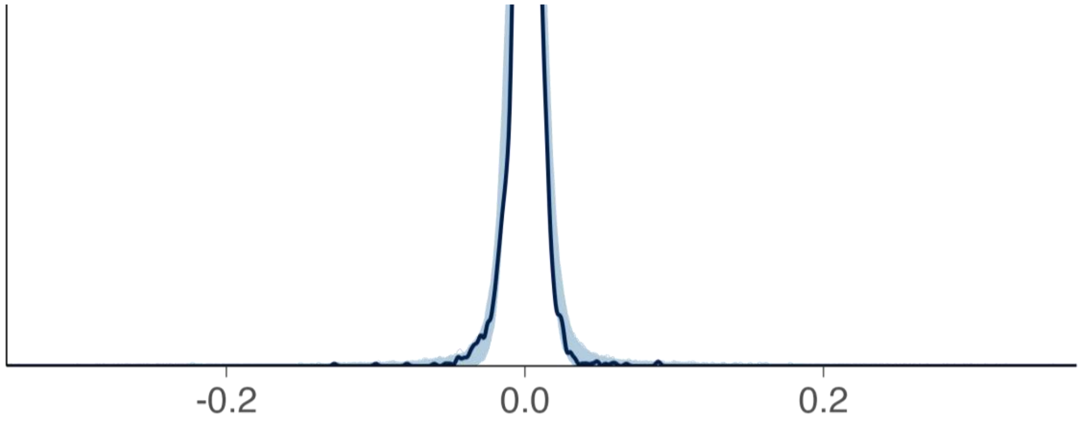
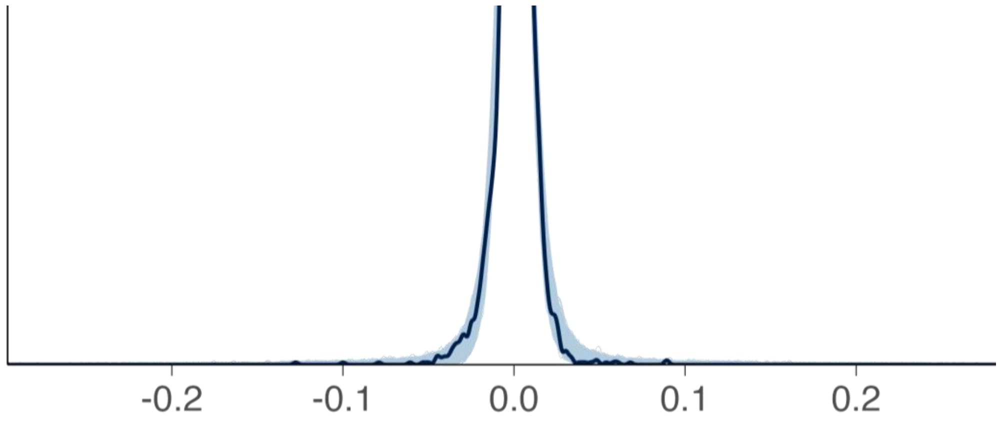
ARCH and GARCH - Posterior Predictive Check



ARCH Model – Posterior Predictive Check



GARCH Model – Posterior Predictive Check

- Table 4 (on slide: LOO-CV Model) shows that there isn't a risk of overfitting and both ARCH and GARCH perform quite well with high expected log predictive density and low number of parameters. To distinguish between the both we see GARCH does perform better (ref. LOO-CV) as GARCH seems to explore the peaks better than ARCH and also has a more informative bulk.

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Stochastic Volatility Model

$$y_t = \epsilon_t \cdot \exp\left(\frac{h_t}{2}\right)$$

$$h_{t+1} = \mu + \phi(h_t - \mu) + \delta_t \sigma$$

$$h_1 \sim \mathcal{N}\left(\mu, \frac{\sigma}{\sqrt{1-\phi^2}}\right)$$

$$\epsilon_t \sim \mathcal{N}(0,1)$$

$$\delta_t \sim \mathcal{N}(0,1)$$

$$y_t \sim normal(0, exp(h_t/2))$$

$$h_t \sim normal(\mu + \phi(h_{t-1} - \mu), \sigma)$$

y_t - log-returns at time t
h_t - volatility at time t

μ - mean log-volatility
ϕ_t - persistence of volatility term at time t
δ_t - shock on volatility at time t
σ_t - white noise shock scale at time t
e_t - white noise shock (multiplicative error) at time t

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Stochastic Volatility Model - Prior Selections

- Prior distribution for mu is picked informatively based on the data.
- Prior distributions for phi and sigma_vol are picked weakly informatively based on the domain knowledge.

```
// Priors
// INFORMATIVE PRIORS - CORRECT ONES DON'T DELETE:
mu ~ normal(prior_sd_log_return, 0.025);// Informative prior for mean
phi ~ normal(0.5, 0.1);;                // Base volatility prior/weakly informative prior
sigma_vol ~ cauchy(0, 0.1);             // Volatiltiy of volatility parameter prior/ informative prior
```


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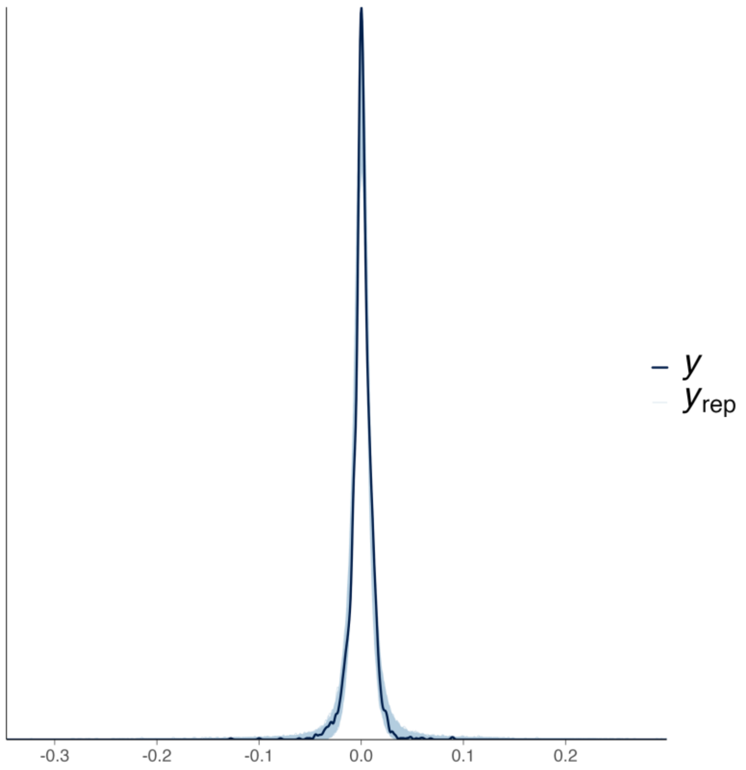
Stochastic Volatility Model - Posterior Predictive Check

- Posterior distribution follows the data very closely.
- This is either due to very successful model or due to overfitting.
- Therefore, further analysis is needed to make a decision.



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Model Validation – Prior Sensitivity Analysis

Three different approaches to test prior sensitivity:

- Priors based on financial theory (ARCH; GARCH)
- Priors informed by empirical studies (ARCH; GARCH)
- Dummy priors: potentially lead to unreliable parameter estimates, theoretical violations, nonsensical for financial time series, and convergence issues (ARCH; stochastic volatility model)

Not all implementations are included in the report due to practical constraints.

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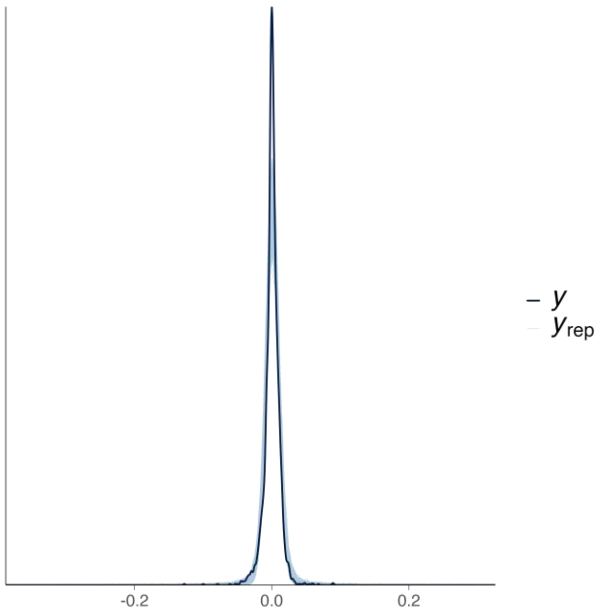
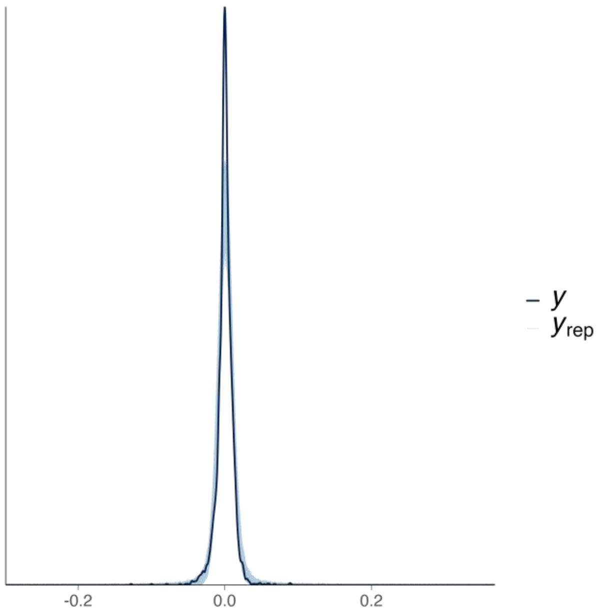
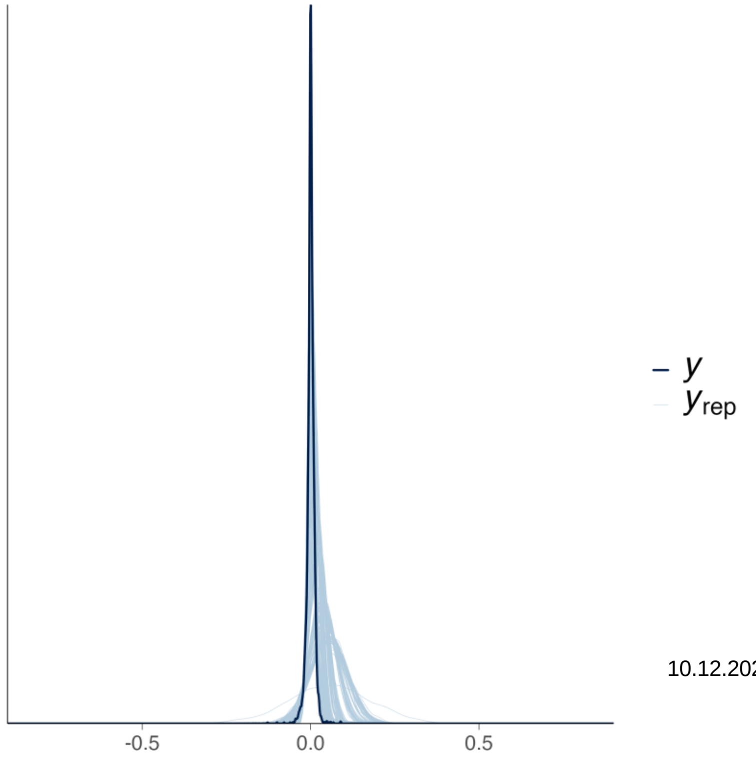
ARCH Model

```
// INFORMATIVE PRIORS - Correct - Original Priors:
mu ~ normal(prior_mean_mu, 1.5 * prior_sd_mu); // Informative prior for mean
alpha0 ~ normal(0.1, 0.05);             // Base volatility prior/weakly informative prior
alpha1 ~ beta(2, 5);                    // ARCH parameter prior/ informative prior
// https://www.shs-conferences.org/articles/shsconf/pdf/2023/18/shsconf_fems2023_01077.pdf
// Scale parameter error can be ignored!

// Priors based on financial theory - Alternative priors 1:
// mu ~ normal(0, 0.2);           // Small mean, reflecting the assumption that returns hover near zero
// alpha0 ~ lognormal(-2, 0.5);   // Lognormal ensures positivity, with a small mean and moderate uncertainty
// alpha1 ~ beta(2, 8);           // Slightly stronger belief that alpha1 is close to zero

// Priors informed by empirical studies - Alternative priors 2:
// mu ~ student_t(3, 0, 0.5);      // Heavy-tailed prior centered at 0, allowing for more uncertainty
// alpha0 ~ normal(0.2, 0.1);      // Base volatility with a mean around 0.2 and wider uncertainty
// alpha1 ~ uniform(0, 1);         // Weakly informative prior, allowing exploration of the entire valid range

// Dummy priors
// mu ~ normal(100, 1000);      // Implausibly large prior for mean return
// alpha0 ~ normal(0, 10);      // Allows negative values, invalid for volatility
// alpha1 ~ uniform(-1, 2);     // Invalid range for ARCH parameter
```



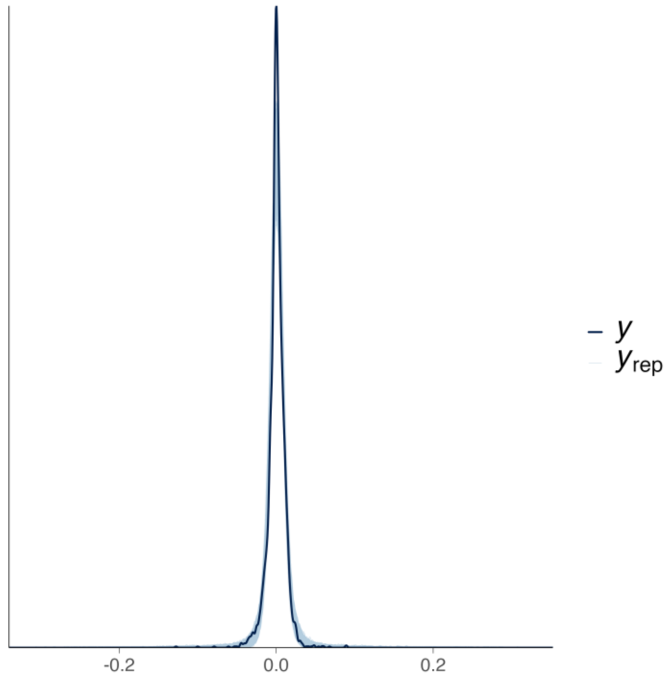


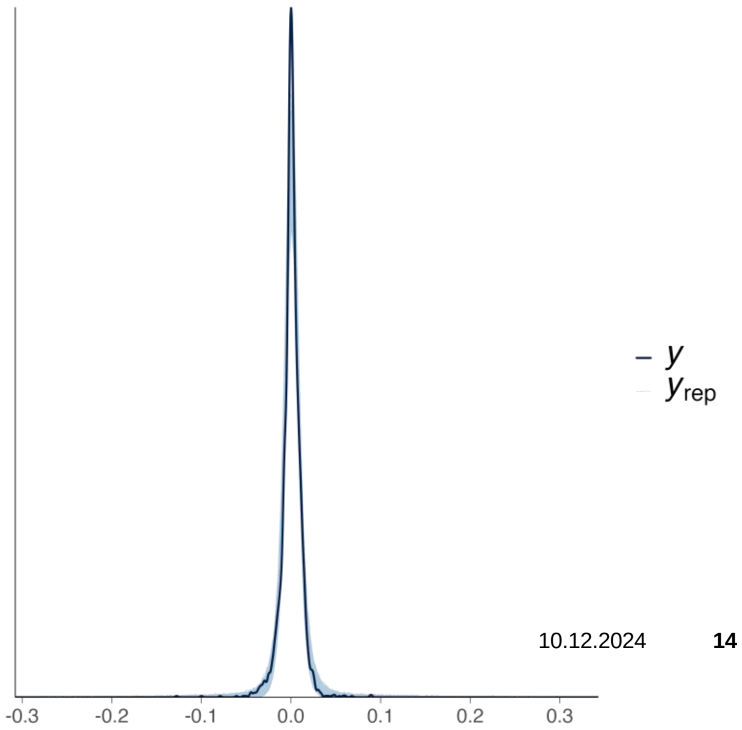


The model remains robust when using reasonable alternative priors due to minimal changes in estimates of key quantities, but becomes fragile under unrealistic (dummy) priors.

GARCH Model

```
// Priors
// INFORMATIVE PRIORS - CORRECT ONES DON'T DELETE:
mu ~ normal(prior_mean_mu, 1.5 * prior_sd_mu); // Informative prior for mean
alpha0 ~ normal(0.1, 0.05);             // Base volatility prior/weakly informative prior
alpha1 ~ beta(2, 5);                    // ARCH parameter prior/ informative prior
beta1 ~ beta(5, 2);               // Informative prior for GARCH term
// https://www.shs-conferences.org/articles/shsconf/pdf/2023/18/shsconf_fems2023_01077.pdf
// Scale parameter error can be ignored!

// Priors based on financial theory - Alternative Priors 1:
// mu ~ normal(0, 0.2);            // Small mean, reflecting the assumption that returns hover near zero
// alpha0 ~ lognormal(-2, 0.5);    // Lognormal ensures positivity, with a small mean and moderate uncertainty
// alpha1 ~ beta(2, 8);            // Slightly stronger belief that alpha1 is close to zero
// beta1 ~ beta(10, 2);            // Stronger belief in high persistence

// Priors informed by empirical studies - Alternative Priors 2:
// mu ~ student_t(3, 0, 0.5);        // Heavy-tailed prior centered at 0 for robustness
// alpha0 ~ normal(0.2, 0.1);        // Base volatility with a mean around 0.2 and moderate uncertainty
// alpha1 ~ uniform(0, 1);           // Weakly informative prior for ARCH term
// beta1 ~ uniform(0, 1);            // Weakly informative prior for GARCH term
```





Comparable results were obtained using the GARCH model, consistent with those observed from the ARCH model.

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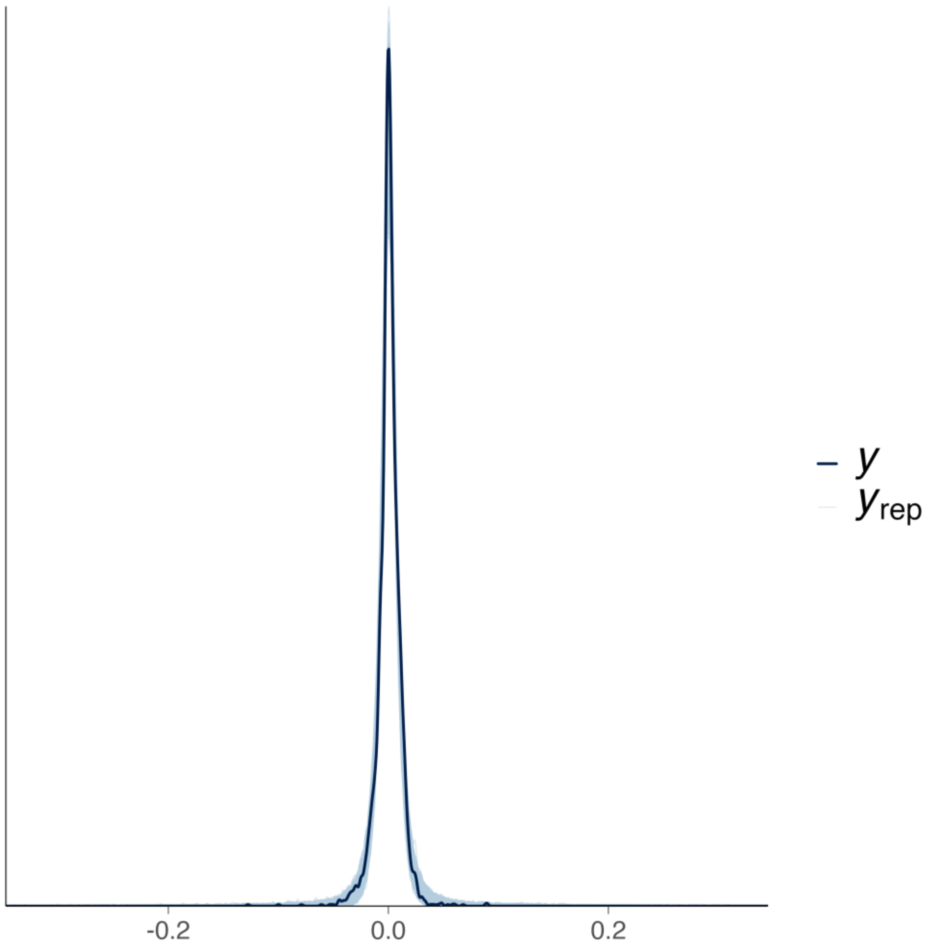
Stochastic Volatility Model

```
// Priors
// INFORMATIVE PRIORS - CORRECT ONES DON'T DELETE:
mu ~ normal(prior_sd_log_return, 0.025);// Informative prior for mean
phi ~ normal(0.5, 0.1);;                // Base volatility prior/weakly informative prior
sigma_vol ~ cauchy(0, 0.1);             // Volatiltiy of volatility parameter prior/ informative prior

// Updated Dummy Priors
// mu ~ uniform(-1000, 1000);    // Extremely vague and nonsensical prior
// phi ~ uniform(-10, 10);       // Exaggerated range, allowing completely unrealistic values
// sigma_vol ~ gamma(0.01, 0.01);// Extremely vague prior, leading to highly dispersed values
```



The unexpected findings from the sensitivity analysis, which align with the earlier posterior predictive check results with informative priors, indicate potential issues with this model.

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Model Validation - Convergence Diagnostics

- Models are run with 3000 iterations: 1000 warm-up, 2000 sampling. Max Tree Depth is default, 10.
- There isn't any divergent chain for any model.
- All models had acceptable Rhat values: almost always 1.00.
- All models had very high ESS Bulk and ESS tail values: almost always higher four digits with very few exceptions.
- MCMC sampling performance is well based on the usual convergence diagnostics.

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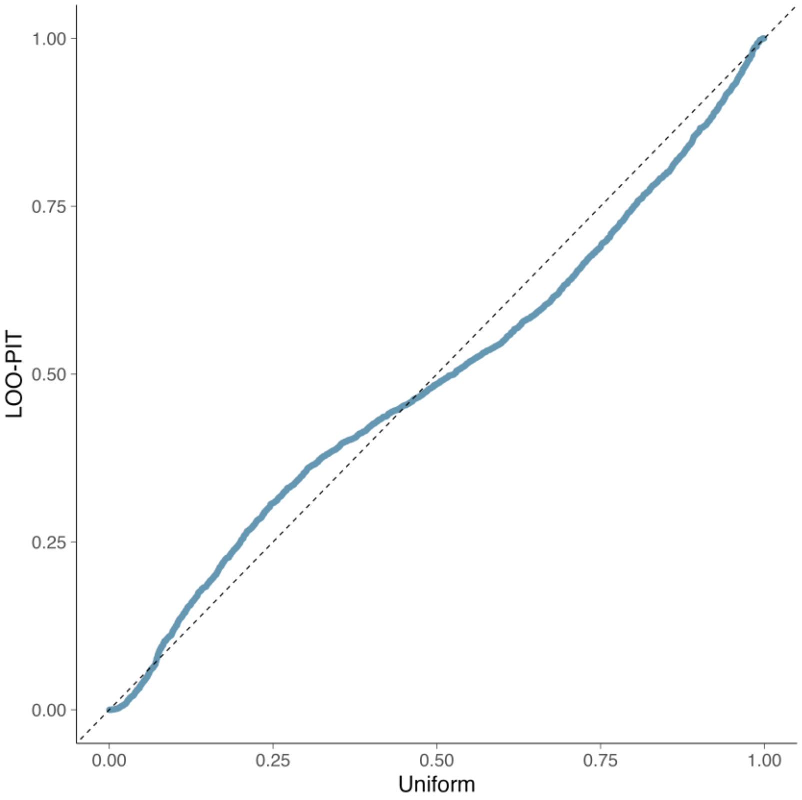
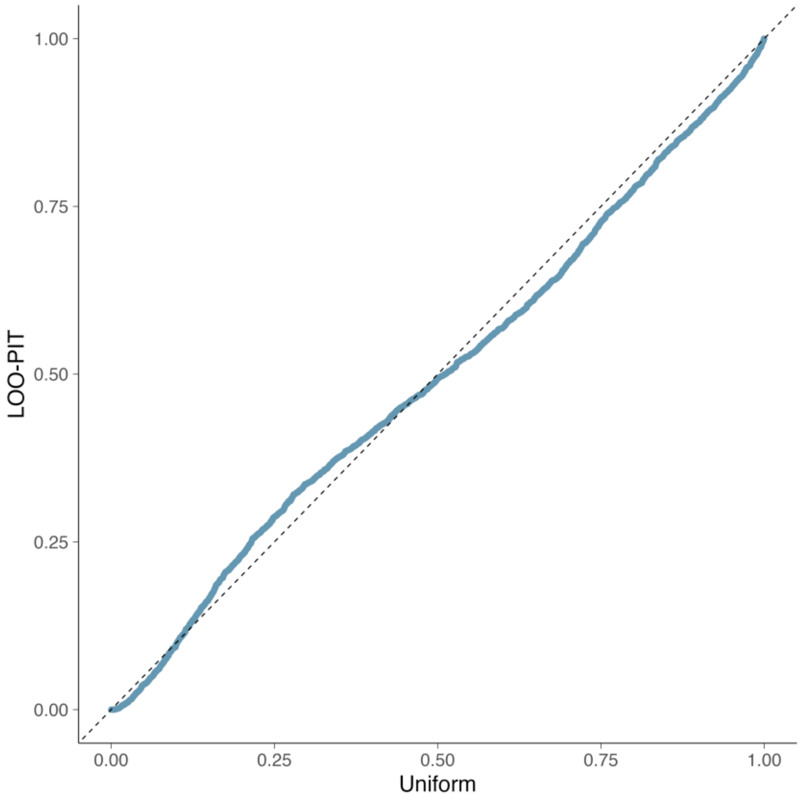
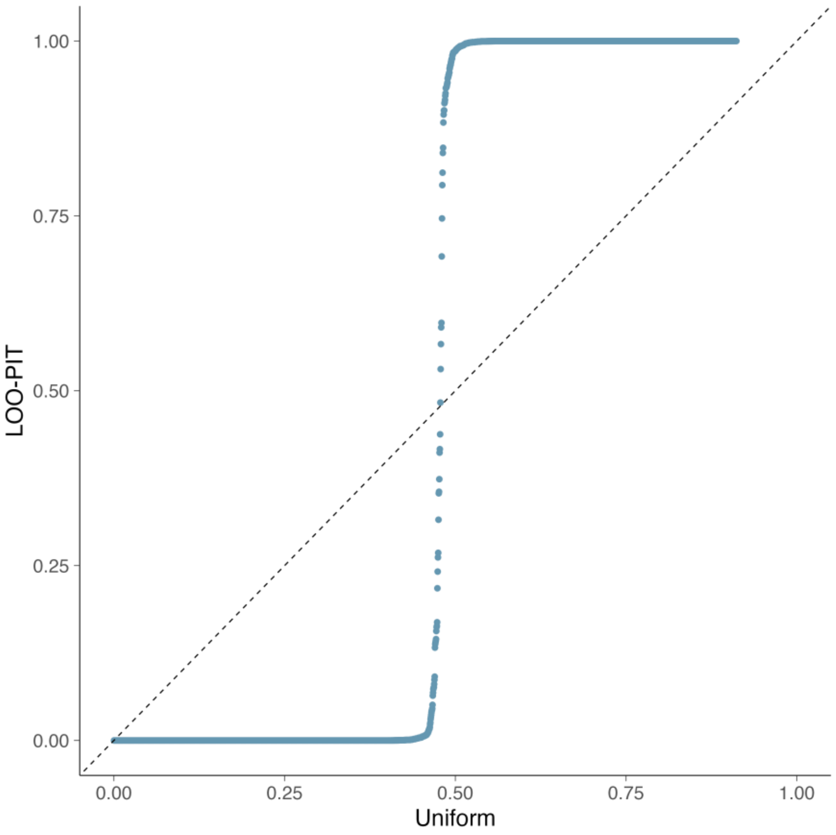
LOO-PIT Checking



ARCH Model



GARCH Model



Stochastic Volatility Model

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LOO-CV Model Comparison

Parameter	Estimate	Standard Error
Expected Log Predictive Density (ARCH)	8056.4	61.6
Effective Number of Parameters (ARCH)	6.9	1.1
Leave-One-Out Information Criterion (ARCH)	-16112.7	123.2
Expected Log Predictive Density (GARCH)	8359.6	56.6
Effective Number of Parameters (GARCH)	3.7	0.7
Leave-One-Out Information Criterion (GARCH)	-16719.1	113.2
Expected Log Predictive Density (SV)	-419549.5	11467.2
Effective Number of Parameters (SV)	425998.9	11469.8
Leave-One-Out Information Criterion (SV)	839098.9	22934.3

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Conclusion

- Based on the model checking diagnostics, both ARCH and GARCH models are reasonable while GARCH has the slightly better performance.
- Stochastic volatility's complexity was unnecessary for this stable index and has the worst performance due to overfitting.
- The model choice should align with data complexity and the problem's nature. Simpler models often suffice for stable indices such as the S&P 500.
- More results at: https://github.com/linm5/sp500-volatility-analysis

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References

- Stan Development Team, “Time series: Modeling temporal heteroscedasticity,” Stan User’s Guide, https://mc-stan.org/docs/stan-users-guide/time-series.html#modeling-temporal-heteroscedasticity.
- X. Chen, “Comparing various GARCH-type models in the estimation and forecasts of volatility of S&P 500 returns during Global Finance Crisis of 2008 and COVID-19 financial crisis,” SHS Web of Conferences, vol. 169, p. 01077, 2023, doi: 10.1051/shsconf/202316901077.
- H. Sun and B. Zhang, “Volatility Clustering in the S&P 500 Market Index: An ARCH/GARCH Approach,” California State Polytechnic University, Pomona, Pomona, 2017.
- J. Shu and J. E. Zhang, “Pricing S&P 500 Index Options under Stochastic Volatility with the Indirect Inference Method,” Journal of Derivatives Accounting, vol. 1, no. 2, pp. 171–186, Sep. 2004, doi: 10.1142/s021986810400021x.

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Additional Information/Appendix

arch.stan

```
data {
  int<lower=0> T;             // Number of observations
  vector[T] y;                // Returns data
  real prior_mean_mu;         // Informative prior mean for mu
  real<lower=0> prior_sd_mu; // Informative prior SD for mu
}
parameters {
  real mu;                    // Mean return
  real<lower=0> alpha0;       // Base volatility
  real<lower=0,upper=1> alpha1; // ARCH parameter
}
model {
  vector[T] h;  // Conditional variances - log volatilities
  // Priors
  // INFORMATIVE PRIORS - Correct - Original Priors:
  mu ~ normal(prior_mean_mu, 1.5 * prior_sd_mu); // Informative prior for mean
  alpha0 ~ normal(0.1, 0.05);             // Base volatility prior/weakly informative prior
  alpha1 ~ beta(2, 5);                    // ARCH parameter prior/ informative prior
  // ARCH(1) model
  h[1] = alpha0 / (1 - alpha1); // Stationary variance initialization
  for (t in 2:T) {
    h[t] = fmax(alpha0 + alpha1 * square(y[t-1] - mu), 1e-8); // Ensure positivity
  }
  // Likelihood
  for (t in 1:T) {
    y[t] ~ normal(mu, sqrt(h[t]));
  }
}
```

garch.stan

```
data {
  int<lower=1> N;           // Number of observations
  vector[N] y;              // Observed returns or log-returns
  real prior_mean_mu;         // Informative prior mean for mu
  real<lower=0> prior_sd_mu; // Informative prior SD for mu
}
parameters {
  real mu;                  // Mean of the returns
  real<lower=0> alpha0;     // Constant term in the GARCH model
  real<lower=0, upper=1> alpha1;    // Coefficient for lagged squared residuals
  real<lower=0, upper=1> beta1;     // Coefficient for lagged conditional variance
}
transformed parameters {
  vector[N] volatility;         // Conditional variances
  // Initialize the first conditional variance
  volatility[1] = alpha0 / (1.0 - alpha1 - beta1);  // Stationary assumption
  volatility[1] = sqrt(fmax(volatility[1], 1e-8));  // Ensure positivity
  // Calculate the conditional variances
  for (n in 2:N) {
    volatility[n] = alpha0 + alpha1 * square(y[n-1] - mu) + beta1 * square(volatility[n-1]);
    volatility[n] = sqrt(fmax(volatility[n], 1e-8));          // Ensure positivity
  }
}
model {
  // Priors
  // INFORMATIVE PRIORS
  mu ~ normal(prior_mean_mu, 1.5 * prior_sd_mu); // Informative prior for mean
  alpha0 ~ normal(0.1, 0.05);             // Base volatility prior/weakly informative prior
  alpha1 ~ beta(2, 5);                    // ARCH parameter prior/ informative prior
  beta1 ~ beta(5, 2);              // Informative prior for GARCH term
  // Likelihood
  for (n in 1:N) {
    y[n] ~ normal(mu, volatility[n]);
    target += normal_lpdf(y[n] | mu, volatility[n]);
  }
}
```

stochastic_volatility.stan

```stan
data {
    int<lower=1> N;           // Number of observations
    vector[N] y;              // Observed log-returns
    real prior_mean_log_return; // Mean of our daily returns,
    // we use this for meaningful sampling for daily log-returns
    real<lower=0> prior_sd_log_return; // Standard deviation of our daily returns (volatility),
    //we use this for building informative prior
}
parameters {
    real mu;                  // Mean of the returns
    real<lower=-1,upper=1> phi;   // Persistence of volatility (AR1 coefficient)
    real<lower=0> sigma_vol;      // Standard deviation of volatility process - white noise shock scale
    vector[N] h;              // Log-volatilities
}
transformed parameters {
    vector[N] volatility;         // Volatilities (exponential of log-volatilities)

    // Convert log-volatilities to volatilities
    volatility = exp(h / 2);
}
model {
    // Priors
    // INFORMATIVE PRIORS - CORRECT ONES DON'T DELETE:
    mu ~ normal(prior_sd_log_return, 0.025);// Informative prior for mean
    phi ~ normal(0.5, 0.1);;                // Base volatility prior/weakly informative prior
    sigma_vol ~ cauchy(0, 0.1);             // Volatiltiy of volatility parameter prior/ informative prior
    // AR(1) process for log-volatilities
    h[1] ~ normal(mu, sigma_vol / sqrt(1 - phi * phi));
    for (t in 2:N) {
        h[t] ~ normal(mu + phi * (h[t-1] - mu), sigma_vol);
    }
    // Estimated returns on time t
    for (t in 1:N){
        y[t] ~ normal(prior_mean_log_return, exp(h[t] / 2));
    }
}
```

```r
stan_data <- list(
  T = nrow(data),
  y = data$Log_Returns,
  prior_mean_mu = log_return_mean,
  prior_sd_mu = log_return_sd
)

# Compile Stan Model
arch_model <- cmdstan_model("models/arch_model.stan", force_recompile = TRUE, quiet = FALSE)

# Explanation of MCMC options
cat("MCMC Inference:\n")
cat("- The model was run with 4 chains, each with 1000 warmup iterations and 2000 post-warmup iterations.\n")
cat("- A seed value (4911) was used for reproducibility.\n")

# Fit the model
fit <- arch_model$sample(
  data = stan_data,
  seed = 4911,
  chains = 4,
  iter_warmup = 1000,
  iter_sampling = 2000,
  refresh = 100
)
```

Example R code to run a model

Table 1: MCMC Analysis for ARCH Results and Convergence Diagnostics

Param	Mean	Med	SD	MAD	Q5	Q95	$\hat{R}$	ESS_bulk	ESS_tail
lp__	1.04e4	1.04e4	1.26	1.04	1.04e4	1.04e4	1.00	2627	3673
μ	9.62e-4	9.61e-4	1.71e-4	1.69e-4	6.83e-4	1.24e-3	1.00	8432	6049
α_0	6.44e-5	6.44e-5	2.60e-6	2.62e-6	6.03e-5	6.88e-5	1.00	2534	2461
α_1	0.506	0.506	0.0443	0.0445	0.433	0.579	1.00	2463	2474
vol[1]	1.15e-2	1.14e-2	4.61e-4	4.51e-4	1.07e-2	1.23e-2	1.00	2883	2850
vol[2]	8.70e-3	8.70e-3	1.43e-4	1.42e-4	8.47e-3	8.94e-3	1.00	3105	4035
vol[3]	8.27e-3	8.27e-3	1.54e-4	1.53e-4	8.02e-3	8.53e-3	1.00	2702	2840
vol[4]	8.04e-3	8.04e-3	1.61e-4	1.61e-4	7.78e-3	8.31e-3	1.00	2543	2470
vol[5]	8.17e-3	8.17e-3	1.57e-4	1.58e-4	7.92e-3	8.43e-3	1.00	2626	2674
vol[6]	8.03e-3	8.03e-3	1.62e-4	1.62e-4	7.77e-3	8.30e-3	1.00	2536	2462

Table 2: Convergence Diagnostics for GARCH

Param	Mean	Med	SD	MAD	Q5	Q95	$\hat{R}$	ESS Bulk	ESS Tail
lp__	1.90e+4	1.90e+4	1.42	1.19	1.90e+4	1.90e+4	1.00	2378.0	3412.0
μ	7.91e-4	7.92e-4	0.000104	0.000102	6.20e-4	9.58e-4	1.00	7738.0	5725.0
alpha0	4.04e-6	4.02e-6	0.000000436	0.000000434	3.34e-6	4.77e-6	1.00	1707.0	2249.0
alpha1	1.84e-1	1.83e-1	0.0138	0.0134	1.61e-1	2.07e-1	1.00	1614.0	1940.0
beta1	7.80e-1	7.81e-1	0.0144	0.0142	7.56e-1	8.03e-1	1.00	1454.0	1697.0
volatility[1]	1.07e-2	1.06e-2	0.000751	0.000697	9.58e-3	1.20e-2	1.00	4465.0	4288.0
volatility[2]	9.86e-3	9.78e-3	0.000628	0.000589	8.95e-3	1.10e-2	1.00	4937.0	4408.0
volatility[3]	9.03e-3	8.96e-3	0.000531	0.000496	8.27e-3	9.98e-3	1.00	5492.0	4892.0
volatility[4]	8.23e-3	8.17e-3	0.000454	0.000425	7.58e-3	9.03e-3	1.00	5942.0	4984.0
volatility[5]	7.61e-3	7.56e-3	0.000388	0.000368	7.05e-3	8.30e-3	1.00	6114.0	4715.0

Table 3: Convergence Diagnostics for Stochastic Volatility

Param	Mean	Med	SD	MAD	Q5	Q95	$\hat{R}$	ESS Bulk	ESS Tail
lp__	13214.00	13209.25	192.89	196.59	12906.68	13540.51	1.06	65.61	240.87
μ	0.0113	0.0115	0.02	0.0247	-0.03	0.0512	1.00	9489.48	4971.08
ϕ	1	1	0.0003	0.0003	1	1	1.00	3190.45	3116.19
σ	0.2422	0.24	0.02	0.02	0.21	0.27	1.06	66.99	253.00
h[1]	-11.43	-11.46	0.55	0.55	-12.28	-10.49	1	8122.97	6604.24
h[2]	-11.49	-11.52	0.54	0.54	-12.34	-10.57	1.00	7952.55	6277.15
h[3]	-11.57	-11.6	0.54	0.54	-12.43	-10.65	1.00	7228.50	6668.71
h[4]	-11.61	-11.63	0.53	0.53	-12.46	-10.71	1.00	7044.94	6213.35
h[5]	-11.65	-11.68	0.53	0.53	-12.50	-10.74	1.00	6750.55	6476.77
h[6]	-11.66	-11.6	0.54	0.54	-12.53	-10.77	1.00	6449.24	6405.98