Universität
Augsburg
University

INSTITUT FÜR MATHEMATIK
Lehrstuhl für Algebra und Zahlentheorie

Stokes filtered local systems and Stokes shells

Masterarbeit

Alina Erl

betreut von
Prof. Dr. Marco Hien

Dezember 2023

Contents

1 Introduction

The classical Riemann-Hilbert correspondence asserts the existence of a functor mapping flat connections on algebraic vector bundles over an algebraic variety X with regular singularities to the category of local systems of finite-dimensional complex vector spaces on X. Generalizing this to connections with irregular singularities raises the question of how to encode the "extra data" to the local systems. This can be done by using Stokes filtrations (shown in [7]) or by Stokes matrices. The notion of Stokes shells, invented by T. Mochizuki in [5], can also be used to describe the data of the irregular singularity.

In this work we will study differential systems of pure Gaussian type, which are modules with connections on the projective line that have only one singularity: an irregular one located at infinity. Using the Riemann-Hilbert correspondence for differential systems that are of pure Gaussian type, we can study these systems from an algebraic-topological point of view. Specifically, we shift our focus towards investigating Stokes filtered local systems and - the more classical approach - Stokes data of Gaussian type.

Our primary objective is to introduce and define the concept of Stokes shells within the context of Gaussian type differential systems. Furthermore, we aim to establish an equivalence between the category of Stokes shells and the category of Stokes data. To provide some context, we will begin with a brief review of local systems. Then, in Chapter 3, we will introduce a formal definition of differential systems of pure Gaussian type. Furthermore, after studying Stokes filtrations and Stokes data within our framework, we will formulate the concept of Stokes shells within the context of Gaussian type systems in Chapter 4. Ultimately, in Chapter 5, we establish an equivalence of categories, bridging the world of Stokes data with that of Stokes shells, particularly focusing on a specific class of differential systems of Gaussian type.

The reader is supposed to be familiar with basic sheaf and category theory, as well as commutative algebra.

2 Preliminary

In the first section of this chapter we want to recall one of the main objects that appear in this work, namely local systems. In the second section, we provide a concise overview of fundamental algebraic concepts, focusing on Kähler differentials and modules with connections. This serves to formally define differential systems of pure Gaussian type in chapter 3.

We will use the following notation throughout this work:

- $\mathbf{k}$ denotes a field.
- $\mathbf{Vect_k}$ denotes the category of $\mathbf{k}$-vector spaces.
- $\mathbf{Sh}_X$ denotes the category of sheaves of $\mathbf{k}$-vector spaces on a topological space X.
- For a complex vector space V and a topological space X we denote the constant sheaf with fiber V on X by $\underline{V}_X$. Recall that the constant sheaf is obtained through the sheafification of the constant presheaf.
- $\mathbb{C}[[t]]$ denotes the ring of formal power series and $\mathbb{C}((t)) = \mathbb{C}[[t]][t^{-1}]$ denotes the ring of formal Laurent series.
- $\mathcal{D} := \mathbb{C}[t]\langle \partial_t \rangle$ denotes the module of differential operators with coefficients in $\mathbb{C}[t]$.

2.1 Some sheaf theory

We begin by recalling some definitions and theorems for local systems and sheaves.

Definition 2.1. For a topological space X, a *local system* on X is a locally constant sheaf $\mathscr{L}$ on X. This means that for any point $x \in X$, there is an open neighbourhood $U \subseteq X$, $x \in U$ such that $\mathscr{L}_{|U}$ is a constant sheaf. The category of local systems $\mathbf{Loc}_X$ on X is the full subcategory of $\mathbf{Sh}_X$ where the objects are local systems.

Proposition 2.2 (Gluing sheaves)**.** *Let X be a topological space and $\bigcup_{i\in I} U_i = X$ an open covering of X. For each $i \in I$, let $\mathcal{F}_i$ be a sheaf on U_i and for each pair (i,j) of elements in I, let $\varphi_{ji} : \mathcal{F}_{i|_{U_i\cap U_j}} \to \mathcal{F}_{j|_{U_i\cap U_j}}$ be an isomorphism of sheaves such that $\varphi_{ii} = \mathrm{id}_{\mathcal{F}_i}$ and on $U_i \cap U_j \cap U_k$ the isomorphisms $\varphi_{ik}, \varphi_{ij}$ and φ_{jk} satisfy the cocycle relation, i.e.*

$$\varphi_{ik} = \varphi_{ij} \circ \varphi_{jk}.$$

Then, up to isomorphism, there exists a unique sheaf $\mathcal{F}$ on X together with a tuple of isomorphisms $(\psi_i : \mathcal{F}_{|_{U_i}} \to \mathcal{F}_i)_{i\in I}$ such that for each pair (i,j) of elements in I

$$\varphi_{ji} = \psi_j \circ \psi_i^{-1}$$

holds on $U_i \cap U_j$.

Proof. For an open $U \subseteq X$ setting

$$\mathcal{F}(U) := \left\{ (s_i)_{i\in I} \in \prod_{i\in I} \mathcal{F}_i(U_i \cap U) \;\middle|\; \varphi_{ji}\left(s_{i|_{U_i\cap U_j}}\right) = s_{j|_{U_i\cap U_j}} \text{ for all } i,j \in I \right\}$$

defines a sheaf $\mathcal{F}$ on X. Moreover for each $i \in I$ and $U \subseteq U_i$ the morphisms

$$\begin{array}{rccc} \psi_i(U): & \mathcal{F}_{|_{U_i}}(U) & \longrightarrow & \mathcal{F}_i(U) \\ & (s_i)_{i\in I} & \longmapsto & s_i \end{array}$$

are isomorphisms of vector spaces with inverse

$$\begin{array}{rccc} \psi_i(U)^{-1}: & \mathcal{F}_i(U) & \longrightarrow & \mathcal{F}_{|_{U_i}}(U). \\ & s & \longmapsto & (\varphi_{ji}(s_{|_{U\cap U_j}}))_{j\in I} \end{array}$$

Remark that $\psi_i^{-1}(U)$ is well-defined because of the cocycle relation. Thus we get isomorphisms of sheaves $(\psi_i)_{i\in I}$. On $U_i \cap U_j$ the relation $\varphi_{ji} = \psi_j \psi_i^{-1}$ holds by definition of the isomorphisms $(\psi_i)_{i\in I}$. It is left to prove that the sheaf is unique up to isomorphism. Thus let $\mathcal{G}$ be a sheaf with isomorphisms $(\lambda_i : \mathcal{G}_{|U_i} \to \mathcal{F}_i)_{i\in I}$ that satisfy

$$\varphi_{ji} = \lambda_j \lambda_i^{-1}$$

on $U_i \cap U_j$. Then for any $U \subseteq S^1$ define

$$\begin{array}{rccc} \alpha(U): & \mathcal{G}(U) & \to & \mathcal{F}(U). \\ & s & \mapsto & (\lambda_i(s_{|_{U_i\cap U}}))_{i\in I} \end{array}$$

The morphism $\alpha(U)$ is well-defined as $\varphi_{ji} = \lambda_j \lambda_i^{-1}$ on $U_i \cap U_j$ holds. The sheaf axioms of $\mathcal{G}$ show that $\alpha(U)$ is indeed an isomorphism. Hence we get an isomorphism of sheaves $\alpha : \mathcal{G} \to \mathcal{F}$. $\square$

Stokes filtered local systems are local systems on the unit circle $S^1 \subseteq \mathbb{C}$, together with a Stokes filtration. So in this thesis we will study objects of $\mathbf{Loc}_{S^1}$. Since we often work with local systems on intervals $I \subsetneq S^1$, we will need the following observations.

Lemma 2.3. *Let $\mathcal{F}$ be a local system of finite dimensional $\mathbf{k}$-vector spaces on a non-empty, connected topological space X. Then all stalks of $\mathcal{F}$ have the same dimension.*

Proof. Let X be a non-empty, connected topological space and let $\mathcal{F}$ be a local system on X. Then for every point $x \in X$ there is an open subset $U_x \subseteq X$, $x \in U_x$ and a finite dimensional $\mathbf{k}$-vector space V_x with

$$\mathcal{F}_{|U_x} \cong \underline{V_x}_{U_x}.$$

Let x_o be an element in X. Set

- $W_1 \coloneqq \{x \in X \mid \dim V_x = \dim V_{x_o}\}$ and
- $W_2 \coloneqq \{x \in X \mid \dim V_x \neq \dim V_{x_o}\}$.

For any point $x \in X$ we have $\mathcal{F}_{|U_x} \cong \underline{V_x}_{U_x}$, so for every $x' \in U_x$

$$V_{x'} \cong \left(\mathcal{F}_{|U_{x'}}\right)_{x'} = \mathcal{F}_{x'} = \left(\mathcal{F}_{|U_x}\right)_{x'} \cong V_x$$

holds. Thus W_1 and W_2 are open sets. Since $X = W_1 \cup W_2, W_1 \cap W_2 = \varnothing$ and $x_o \in W_1$ we can conclude that $W_2 = \varnothing$, because X is connected. Hence, all stalks of $\mathcal{F}$ have the same dimension. $\square$

Lemma 2.4. *Let $\mathcal{F}$ be a local system on a closed interval $[a, b] \subseteq \mathbb{R}$. Then $\mathcal{F}$ is a constant sheaf on $[a, b]$.*

Remark. More generally, one can show that a local system on a contractible space X is a constant sheaf on X. However, as we specifically require the statement for intervals, proving (2.4) suffices.

Proof. Let $\mathcal{F}$ be an object in $\mathbf{Loc}_{[a,b]}$. Then for every $x \in [a, b]$ there is an open neighbourhood $U_x \subseteq [a, b], x \in U_x$ so that $\mathcal{F}_{|U_x}$ is constant, i.e. there exists a finite dimensional $\mathbf{k}$-vector space V_x and an isomorphism of sheaves $\varphi_x : \mathcal{F}_{|U_x} \to \underline{V_x}_{U_x}$. Since $[a, b]$ is connected, we can set $V \coloneqq V_a$ and assume $\mathcal{F}_{|U_x} \cong \underline{V}_{U_x}$ for all points $x \in [a, b]$ by using lemma (2.3). $\bigcup_{x\in[a,b]} U_x$ is an open covering of $[a, b]$ and by compactness of the closed interval we find a finite subcover $[a, b] = \bigcup_{i=1}^{n} U_{x_i}$. By refining and renumbering the cover if necessary, we can assume without loss of generality that the subcover satisfies the following properties:

1. Any open set U_{x_i} of the subcover is connected.

2. There are no triple intersections, namely for three pairwise different indices $i, j, k \in \{1, \dots, n\}$ we have $U_{x_i} \cap U_{x_j} \cap U_{x_k} = \varnothing$.

3. For $i, j \in \{1, \dots n\}$ we have $x_i < x_j$ if $i < j$.

4. For $i \in \{1, \dots, n-1\}$ we have $U_{x_i} \cap U_{x_{i+1}} \neq \varnothing$.

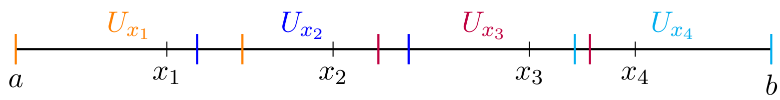

Figure 2.1: Sketch of an example subcover of the interval $[a, b]$ satisfying the previous properties.

Using the notation $U_{ij} := U_{x_i} \cap U_{x_j}$ we get the following diagram on U_{12}:

$$\begin{tikzcd}
\mathcal{F}_{|U_{x_1}} \arrow[d, "(\bullet)_{|U_{12}}"] \arrow[ddd, bend right, "\varphi_{x_1}"'] & \mathcal{F}_{|U_{x_2}} \arrow[d, "(\bullet)_{|U_{12}}"'] \arrow[ddd, bend left, "\varphi_{x_2}"] \\
\mathcal{F}_{|U_{12}} \arrow[r, "\mathrm{id}_{\mathcal{F}_{|U_{12}}}"'] \arrow[d, "\varphi_{x_1|U_{12}}"] & \mathcal{F}_{|U_{12}} \arrow[d, "\varphi_{x_2|U_{12}}"'] \\
\underline{V}_{U_{12}} \arrow[r, "\lambda"] & \underline{V}_{U_{12}} \\
\underline{V}_{U_{x_1}} \arrow[u, "(\bullet)_{|U_{12}}"] & \underline{V}_{U_{x_2}} \arrow[u, "(\bullet)_{|U_{12}}"']
\end{tikzcd}$$

Here λ is given by $\varphi_{x_2|U_{12}} \circ \varphi^{-1}_{x_1|U_{12}}$. Since $\mathcal{F}_{|U_{x_i}}$ is constant, the restriction maps $(\bullet)_{|U_{ij}}$ are isomorphisms. Thus we can change $\varphi_{x_2} : \mathcal{F}_{|U_{x_2}} \to \underline{V}_{U_{x_2}}$ by taking the composition with an automorphism $\alpha_2 : \underline{V}_{U_{x_2}} \to \underline{V}_{U_{x_2}}$ in such way that we get $(\alpha_2 \varphi_{x_2})_{|U_{12}} = \varphi_{x_1|U_{12}}$. In particular, we obtain $\lambda = \mathrm{id}_{\underline{V}_{U_{12}}}$. We set $\tilde{\varphi}_{x_2} := \alpha_2 \circ \varphi_{x_2}$. Using (2.2), $\mathcal{F}_{|U_{x_1}}$, $\mathcal{F}_{|U_{x_2}}$ glue together to $\mathcal{F}_{|U_{x_1} \cup U_{x_2}}$. Moreover, since $\underline{V}_{U_{x_1} \cup U_{x_2}}$ together with $\varphi^{-1}_{x_1} : \underline{V}_{U_{x_1}} \to \mathcal{F}_{|U_{x_1}}$ and $\tilde{\varphi}^{-1}_{x_2} : \underline{V}_{U_{x_2}} \to \mathcal{F}_{|U_{x_2}}$ fulfill

$$\mathrm{id}_{\mathcal{F}_{|U_{12}}} = \tilde{\varphi}^{-1}_{x_2|U_{12}} \varphi_{x_1|U_{12}},$$

we get an isomorphism $\mathcal{F}_{|U_{x_1}\cup U_{x_2}} \to \underline{V}_{U_{x_1}\cup U_{x_2}}$ by application of lemma (2.2). We repeat this successively for $\mathcal{F}_{|U_1\cup\cdots\cup U_{x_i}}$ and $\mathcal{F}_{|U_{x_{i+1}}}$ until we finally get $\mathcal{F} \cong \underline{V}_{[a,b]}$. □

Using lemma (2.4), it becomes apparent that a local system $\mathscr{L}$ defined on S^1 and restricted to an interval $I \subsetneq S^1$ is a constant sheaf if its closure is not equal to S^1.

Remark. Local systems on S^1 can also be studied from a different point of view using the following theorem:

Theorem 2.5. *Let X be a connected and locally simply connected topological space, and x a point in X. The category $\mathbf{Loc}_X$ is equivalent to the category of finite dimensional representations of $\pi_1(X, x)$.*

Proof. See [10], Corollary 2.6.2. □

The theorem shows that giving a local system of dimension n on S^1 is equivalent to giving a homomorphism of groups $\mathbb{Z} \cong \pi_1(S^1, 1) \to \mathrm{GL}(\mathbb{C}^n)$, i.e. an element in $\mathrm{GL}(\mathbb{C}^n)$. This element is called the *monodromy representation* of the local system. In the following we will not need this equivalence of categories, but it could be beneficial to keep it in mind.

2.2 Kähler differentials and modules with connections

In this section we recall the module of Kähler differentials for algebras over some commutative ring R with unit in order to define modules with connections afterwards. This is necessary to give the formal definition of differential system of pure Gaussian type in the next chapter. For more details see chapter 4 in [9].

Definition 2.6. Let R be a commutative ring with unit and A be an R-algebra. A *derivation* from A to a left A-module M is an R-linear mapping $D : A \to M$ that fulfills the Leibnitz rule, namely

$$D(fg) = fD(g) + gD(f)$$

holds for all $f, g \in A$. The set of derivations from A to M is denoted by $\mathrm{Der}_R(A, M)$.

One can check that $\mathrm{Der}_R(A, M)$ is an A-module.

Definition 2.7. Let R be a commutative ring with unit and let A be a commutative R-algebra. The *module of Kähler differentials of A over R*, denoted by $\Omega_{A/R}$, is an A-module together with a derivation $d_{A/R} \in \mathrm{Der}_R(A, \Omega_{A/R})$, which satisfies the following universal property: For any A-module M and any derivation $D \in \mathrm{Der}_R(A, M)$ there is a unique A-module homomorphism $\varphi : \Omega^1_{A/R} \to M$ with $D = \varphi \circ d_{A/R}$, i.e. the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "d_{A/R}"] \arrow[dr, "D"'] & \Omega_{A/R} \arrow[d, dashed, "\varphi"] \\ & M \end{tikzcd}$$

Remark. For a commutative R-algebra A one can check that there always exists the module of Kähler differentials $(\Omega_{A/R}, d_{A/R})$. For details see ([9], 4.2.1. and 4.2.2).

Example 2.8. For $A = \mathbf{k}[x_1, \ldots, x_n]$ one has $(\Omega_{A/\mathbf{k}}, d_{A/\mathbf{k}}) = (\bigoplus_{i=1}^n A\,\mathrm{d}x_i, d_{A/\mathbf{k}})$ with

$$\begin{array}{rccc} d_{A/\mathbf{k}}: & A & \to & \bigoplus_{i=1}^n A\,\mathrm{d}x_i \\ & f & \mapsto & (\frac{\partial f}{\partial x_i}\,\mathrm{d}x_i)_{i\in\{1,\ldots n\}} \end{array}$$

as $(\bigoplus_{i=1}^n A\,\mathrm{d}x_i, d_{A/\mathbf{k}})$ satisfies the universal property. Analogously for the ring of formal power series $A = \mathbf{k}[[x]]$ one has $\Omega_{A/\mathbf{k}} = \mathbf{k}[[x]]\,\mathrm{d}x$ with $d_{A/\mathbf{k}}(f(x)) = \frac{\partial f}{\partial x}(x)\,\mathrm{d}x$ where $\frac{\partial f}{\partial x}(x)$ is the formal differentiation of the formal power series f.

Lemma 2.9. *The module of Kähler differentials is compatible with localization, i.e. for an R-algebra A and a multiplicative system S in A*

$$\Omega_{S^{-1}A/R} = S^{-1}\Omega_{A/R}$$

holds.

Proof. See [2], proposition 8.2A. $\square$

Definition 2.10. Let A be a commutative $\mathbf{k}$-algebra and $(\Omega^1_{A/\mathbf{k}}, d_{A/\mathbf{k}})$ the module of Kähler differentials of A over $\mathbf{k}$. An *A-module with connection* is an A-module M together with a $\mathbf{k}$-linear connection, i.e. a $\mathbf{k}$-linear morphism

$$\nabla : M \to \Omega^1_{A/\mathbf{k}} \otimes_A M$$

satisfying the Leibniz rule, namely

$$\nabla(am) = d_{A/\mathbf{k}}(a) \otimes m + a\nabla(m)$$

holds for all $a \in A$ and $m \in M$.

Let $(M, \nabla_M), (N, \nabla_N)$ be two A-modules with connections. Then an A-linear morphism $\varphi : M \to N$ is called a *morphism of A-modules with connections* if $(\mathrm{id} \otimes \varphi) \circ \nabla_M = \nabla_N \circ \varphi$. We denote the category of A-modules with connection by $\mathbf{Conn}(A)$.

Example 2.11. With example (2.8) we get $(\Omega_{\mathbb{C}[t]/\mathbb{C}}, d_{\mathbb{C}[t]/\mathbb{C}}) = (\mathbb{C}[t]\,\mathrm{d}t, d_{\mathbb{C}[t]/\mathbb{C}} : f \mapsto \frac{\partial f}{\partial t}\,\mathrm{d}t)$. Thus a $\mathbb{C}[t]$-module with connection is a $\mathbb{C}[t]$-module M together with a $\mathbb{C}$-linear morphism

$$\nabla : M \to \mathbb{C}[t]\,\mathrm{d}t \otimes_{\mathbb{C}[t]} M$$

that satisfies the Leibniz rule

$$\nabla(fm) = d_{\mathbb{C}[t]/\mathbb{C}}(f) \otimes m + f\nabla(m) = \frac{\partial f}{\partial t}\,\mathrm{d}t \otimes m + f\nabla(m).$$

Example 2.12. A $\mathbb{C}((t))$-vector space $\mathcal{M}$ of finite dimension with a connection $\nabla : \mathcal{M} \to \Omega_{\mathbb{C}((t))/\mathbb{C}} \otimes_{\mathbb{C}((t))} \mathcal{M}$ is also called a *meromorphic connection.*

After we have clarified foundational concepts, we can apply them in the following chapter to get to the definitions of differential systems of Gaussian type and Stokes filtered local systems. Moreover we will link both concepts in the end of the next chapter using a Riemann-Hilbert correspondence.

3 Differential systems and Stokes filtered local systems of pure Gaussian type

In this chapter we begin by introducing differential systems of pure Gaussian type as invented by C. Sabbah in [6]. Nevertheless our primary objective in this section is to define Stokes filtered local systems and Stokes data. Their usefulness will become apparent at the end of the chapter where we will formulate a Riemann-Hilbert correspondence that links differential systems of pure Gaussian type with Stokes filtered local systems of pure Gaussian type.

3.1 Differential systems of pure Gaussian type

The study of partial differential systems can be approached algebraically using the theory of $\mathcal{D}$-modules. This will be our method for investigating differential systems of pure Gaussian type. Therefore, we will start by considering $\mathcal{D}$-modules to define differential systems of pure Gaussian type afterwards. We follow ([6], chapter 1).

We consider the projective line $\mathbb{P}^1$ with an affine open covering $\mathbb{P}^1 = \mathbb{A}^1_t \cup \mathbb{A}^1_{t'}$ with $t = \frac{1}{t'}$ on the intersection.

Lemma 3.1. *A left $\mathcal{D}$-module is a $\mathbb{C}[t]$-module with $\mathbb{C}$-linear connection and vice versa.*

Proof. Let M be a $\mathbb{C}[t]$-module with connection $\nabla : M \to \mathbb{C}[t]\,\mathrm{d}t \otimes_{\mathbb{C}[t]} M$ (cf. example (2.11)). Defining $\partial_t \cdot m \coloneqq \nabla(m)$ with $\nabla(m) \in M$ via the isomorphism

$$\begin{array}{rcl} \mathbb{C}[t]\,\mathrm{d}t \otimes_{\mathbb{C}[t]} M & \to & M, \\ f\,\mathrm{d}t \otimes m & \mapsto & fm \end{array}$$

we get a $\mathcal{D}$-module structure on M: For $t, \partial_t \in \mathcal{D}$ and $m \in M$ one has

$$\partial_t \cdot (t \cdot m) = \nabla(tm) = \frac{\partial t}{\partial t} m + t\nabla(m) = m + t\nabla(m) = (1 + t \cdot \partial_t)(m) = (\partial_t \cdot t)(m)$$

and

$$t \cdot (\partial_t \cdot m) = t \cdot (\nabla(m)) = t\nabla(m) = \nabla(tm) - \frac{\partial t}{\partial t} m = (\partial_t \cdot t - 1) \cdot m = (t \cdot \partial_t)(m)$$

as in $\mathcal{D}$ the relation $[\partial_t, t] = 1$ holds and ∇ is a $\mathbb{C}$-linear morphism satisfying the Leibnitz rule.
Conversely for a left $\mathcal{D}$-module M we define

$$\begin{array}{rrcl} \nabla : & M & \to & \mathbb{C}[t]\,\mathrm{d}t \otimes_{\mathbb{C}[t]} M. \\ & m & \mapsto & 1\,\mathrm{d}t \otimes \partial_t \cdot m \end{array}$$

∇ is $\mathbb{C}$-linear, as ∂_t is $\mathbb{C}$-linear and ∇ fulfills the Leibnitz rule since for any element $f \in \mathbb{C}[t]$ and any $m \in M$ we have

$$\begin{aligned} \nabla(fm) &= 1\,\mathrm{d}t \otimes \partial_t \cdot fm = 1\,\mathrm{d}t \otimes (\partial_t \cdot f)m + f(\partial_t \cdot m) \\ &= 1\,\mathrm{d}t \otimes \frac{\partial f}{\partial t} m + 1\,\mathrm{d}t \otimes f(\partial_t \cdot m) = \frac{\partial f}{\partial t}\,\mathrm{d}t \otimes m + f\nabla(m). \end{aligned}$$

Thus (M, ∇) is a $\mathbb{C}[t]$-module with a $\mathbb{C}$-linear connection. □

Remark. The same holds for $\mathbb{C}[t, t^{-1}]$-modules (resp. $\mathbb{C}((t))$-modules) with $\mathbb{C}$-linear connections and $\mathbb{C}[t, t^{-1}]\langle\partial_t\rangle$-modules (resp. $\mathbb{C}((t))\langle\partial_t\rangle$-modules), as

- $\Omega_{\mathbb{C}[t,t^{-1}]/\mathbb{C}} = \Omega_{\mathbb{C}[t]_t/\mathbb{C}} \overset{2.9}{=} (\Omega_{\mathbb{C}[t]/\mathbb{C}})_t = (\mathbb{C}[t]\,\mathrm{d}t)_t = \mathbb{C}[t]_t\,\mathrm{d}t$ and
- $\Omega_{\mathbb{C}((t))/\mathbb{C}} = \Omega_{\mathbb{C}[[t]]_t/\mathbb{C}} \overset{2.9}{=} (\Omega_{\mathbb{C}[[t]]/\mathbb{C}})_t = (\mathbb{C}[[t]]\,\mathrm{d}t)_t = \mathbb{C}((t))\,\mathrm{d}t.$

Remark. Let (M, ∇) be a free $\mathbb{C}[t]$-module of finite rank r with two connections $\nabla_1, \nabla_2 : M \to \Omega^1_{\mathbb{C}[t]/\mathbb{C}} \otimes_{\mathbb{C}[t]} M$. Then $\nabla_1 - \nabla_2$ is $\mathbb{C}[t]$-linear as they both satisfy the Leibnitz rule. Therefore every connection is of the form $\nabla = \mathrm{d} + A(t)\,\mathrm{d}t$ with $A(t) \in \mathrm{Mat}_{r\times r}(\mathbb{C}[t])$ and

$$\begin{array}{rcccl} \mathrm{d} : M \cong & \mathbb{C}[t]^r & \to & \mathbb{C}[t]^r & \cong \Omega_{\mathbb{C}[t]/\mathbb{C}} \otimes_{\mathbb{C}[t]} M \\ & \begin{pmatrix} u_1(t) \\ \vdots \\ u_r(t) \end{pmatrix} & \mapsto & \begin{pmatrix} \frac{\partial u_1}{\partial t}(t) \\ \vdots \\ \frac{\partial u_r}{\partial t}(t) \end{pmatrix} & \end{array}$$

being the canonical connection after choosing a basis of $\mathbb{C}[t]^r$. The analogous statement for $\mathbb{C}((t))$-modules with connections can be proven by using the same arguments.

Definition 3.2. Let $(\mathcal{M}, \nabla)$ be a meromorphic connection with $\dim(\mathcal{M}) = r$. We say $\mathcal{M}$ has a *regular singular* connection if there exists an isomorphism $\psi : \mathcal{M} \to \mathbb{C}((t'))^r$, such that $\nabla = \mathrm{d} + A(t')\,\mathrm{d}t'$ and all entries of $A(t')$ have pole order at most 1 at $t' = 0$.

Example 3.3. For an arbitrary $f \in \mathbb{C}[t]$ consider

$$\begin{array}{rccl} \mathrm{d} + \mathrm{d}(f): & \mathbb{C}[t] & \to & \Omega_{\mathbb{C}[t]/\mathbb{C}} \otimes_{\mathbb{C}[t]} \mathbb{C}[t]. \\ & g & \mapsto & 1\,\mathrm{d}t \otimes \frac{\partial g}{\partial t} + \frac{\partial f}{\partial t} \cdot g \end{array}$$

Then $E^{f(t)} \coloneqq (\mathbb{C}[t], \mathrm{d} + \mathrm{d}(f))$ is a $\mathcal{D}$-module as $\mathrm{d} + \mathrm{d}(f)$ is $\mathbb{C}$-linear and fulfills the Leibnitz rule:

$$\begin{aligned} (\mathrm{d} + \mathrm{d}(f))(gh) &= 1\,\mathrm{d}t \otimes \frac{\partial(gh)}{\partial_t} + \frac{\partial f}{\partial_t}(gh) = 1\,\mathrm{d}t \otimes \left(\frac{\partial g}{\partial t} h + g\frac{\partial h}{\partial t}\right) + \frac{\partial f}{\partial t}(gh) \\ &= 1\,\mathrm{d}t \otimes \frac{\partial g}{\partial t} h + 1\,\mathrm{d}t \otimes g\frac{\partial h}{\partial t} + \frac{\partial f}{\partial t}(gh) \\ &= \frac{\partial g}{\partial t}\,\mathrm{d}t \otimes h + g\left(1\,\mathrm{d}t \otimes \frac{\partial h}{\partial t} + \frac{\partial f}{\partial t} h\right) = \frac{\partial g}{\partial t}\,\mathrm{d}t \otimes h + g(\mathrm{d} + \mathrm{d}(f))(h). \end{aligned}$$

Extending scalars of a $\mathcal{D}$-module M, viewed as a $\mathbb{C}[t]$-module with connection (cf. 3.1), to a $\mathbb{C}[t, t^{-1}]$-module $M' \coloneqq M \otimes_{\mathbb{C}[t]} \mathbb{C}[t, t^{-1}]$ we obtain a $\mathbb{C}[t', t'^{-1}]\langle \partial_{t'} \rangle$-module via

$$(t', m) \mapsto t^{-1}m; \quad (t'^{-1}, m) \mapsto tm \quad \text{and} \quad (\partial_{t'}, m) \mapsto (-t^2\partial_t) \cdot m.$$

Example 3.4. The extension to $\mathbb{C}[t', t'^{-1}]$ of the $\mathcal{D}$-module we considered in example (3.3) gives us the $\mathbb{C}[t', t'^{-1}]\langle \partial_{t'} \rangle$-module

$$E^{f(t'^{-1})} = (\mathbb{C}[t', t'^{-1}], \mathrm{d} + \mathrm{d}(f(t'^{-1}))).$$

Further extending $\mathbb{C}[t', t'^{-1}]$ to $\mathbb{C}((t'))$ we get a $\mathbb{C}((t'))\langle \partial_{t'} \rangle$-module

$$\mathcal{E}^{f(t'^{-1})} \coloneqq E^{f(t'^{-1})} \otimes_{\mathbb{C}[t', t'^{-1}]} \mathbb{C}((t')) = (\mathbb{C}((t')), \mathrm{d} + \mathrm{d}(f(t'^{-1}))).$$

Now we can give the definition for differential systems of pure Gaussian type:

Definition 3.5. Let $C \subseteq \mathbb{C}^\times$ be a non-empty, finite subset. A differential system of *pure Gaussian type* C is a free $\mathbb{C}[t]$-module M of finite rank r together with a connection $\nabla = \mathrm{d} + A(t)\,\mathrm{d}t$ with $A(t) \in \mathrm{Mat}_{r \times r}(\mathbb{C}[t])$ so that for the $\mathbb{C}[t, t^{-1}]$-module $M' \coloneqq M \otimes_{\mathbb{C}[t]} \mathbb{C}[t, t^{-1}]$ one has

$$(M' \otimes_{\mathbb{C}[t', t'^{-1}]} \mathbb{C}((t')), \nabla) \cong \bigoplus_{c \in C} (\mathcal{E}^{-\frac{c}{2t'^2}} \otimes_{\mathbb{C}((t'))} R_c)$$

with $\mathcal{E}^{-\frac{c}{2t'^2}} = (\mathbb{C}((t')), \mathrm{d} + \frac{c}{t'^3})$ and R_c is a finite dimensional $\mathbb{C}((t'))$-vector space with regular singular connection.

Remark. In definition (3.5), $\mathcal{E}^{-\frac{c}{2t'^2}}$ is given by $E^{f(t'^{-1})} \otimes_{\mathbb{C}[t',t'^{-1}]} \mathbb{C}((t'))$ for the polynomial $f(t) = -\frac{1}{2}ct^2 \in \mathbb{C}[t]$ as in example (3.4). Furthermore, a differential system of pure Gaussian type C is by definition purely irregular at infinity of irregularity $2r$ (see [6], chapter 1).

Definition 3.6. Let $C \subseteq \mathbb{C}^\times$ be a non-empty, finite subset. The category $\mathbf{Mod}_{\text{Gauß}}(C)$ of differential systems of pure Gaussian type C is the full subcategory of $\mathbf{Conn}(\mathbb{C}[t])$ where the objects are differential systems of pure Gaussian type.

3.2 Stokes data of Gaussian type

Having given a formal definition of differential systems of pure Gaussian type, we now focus on defining Stokes filtered local systems and the category of Stokes data in this section. In order to do so, we mostly follow ([6], chapter 2) and ([7], chapter 2).

3.2.1 Stokes directions

To get to the definition of a Stokes filtration for a local system, we first have to establish Stokes directions.

Let $\mathbf{k}$ be a field and $\mathscr{L}$ be a local system of finite dimensional $\mathbf{k}$-vector spaces on the unit circle $S^1 \subseteq \mathbb{C}$ with coordinate $e^{i\vartheta}$. With $\vartheta \in S^1$ we abbreviate $e^{i\vartheta} \in S^1$. On $\mathbb{C}$ we define a partial order for each direction $\vartheta \in S^1$ as follows: For any $c \in \mathbb{C}$ we set

$$c \leq_\vartheta 0 \;:\Longleftrightarrow\; c = 0 \;\text{ or }\; \arg(c) - 2\vartheta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \mod 2\pi.$$

Then for $c_1, c_2 \in \mathbb{C}$ we set $c_1 \leq_\vartheta c_2$ if and only if $c_1 - c_2 \leq_\vartheta 0$.

Remark. For each $\vartheta \in S^1$, the relation $\leq_\vartheta$ is reflexive and antisymmetric by definition. To see that $\leq_\vartheta$ is also transitive, we have to check that for two complex numbers $z_1, z_2 \in \mathbb{C}$ with $z_1 \leq_\vartheta 0$ and $0 \leq_\vartheta z_2$, one has $z_1 \leq_\vartheta z_2$. In order to prove that, it is sufficient to show that for $z_1, z_2 \in \mathbb{C}$ with $\arg(z_1), \arg(z_2) \in (\varphi, \varphi + \pi)$ for some $\varphi \in [0, 2\pi)$, then $\arg(z_1 + z_2) \mod 2\pi \in (\varphi, \varphi + \pi)$. One can prove this by using the rotation matrices

$$R_{2\pi-\varphi} = \begin{pmatrix} \cos(2\pi - \varphi) & -\sin(2\pi - \varphi) \\ \sin(2\pi - \varphi) & \cos(2\pi - \varphi) \end{pmatrix} \text{ and } R_\varphi = \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi), \end{pmatrix}$$

that rotate a complex number (considered as a vector in $\mathbb{R}^2$) counterclockwise by the angle $2\pi - \varphi$ (resp. φ) and the fact that if $\mathrm{Im}(z_1) > 0$ and $\mathrm{Im}(z_2) > 0$, then $\mathrm{Im}(z_1 + z_2) > 0$. Thus, $\leq_\vartheta$ is indeed a partial order.

Furthermore we define a strict partial order on $\mathbb{C}$ for each $\vartheta \in S^1$ by setting $c_1 <_\vartheta c_2$ if and only if $c_1 \leq_\vartheta c_2$ and $c_1 - c_2 \neq 0$ for two complex numbers $c_1, c_2 \in \mathbb{C}$.

Lemma 3.7. *Let $\vartheta \in S^1$ be an arbitrary direction and $c \in \mathbb{C}$ be a complex number with $c \leq_\vartheta 0$. Then there exists an open neighbourhood $U \subseteq S^1$ of ϑ, so that $c \leq_{\vartheta'} 0$ holds for any $\vartheta' \in U$.*

Proof. For $c \in \mathbb{C}$ with $c \leq_\vartheta 0$, we have either $c = 0$ or $c <_\vartheta 0$. In the case $c = 0$ we can set $U = S_1$. Thus let c be a complex number with $c <_\vartheta 0$. Then by definition of $<_\vartheta$ we have

$$\begin{aligned}\arg(c) - 2\vartheta &\in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \mod 2\pi \\ \iff \vartheta &\in \left(\frac{2\arg(c) - 3\pi}{4}, \frac{2\arg(c) - \pi}{4}\right) \mod 2\pi.\end{aligned}$$

Therefore there is a $k \in \mathbb{Z}$, so that $\vartheta \in \left(\frac{2\arg(c)-3\pi}{4} + k2\pi, \frac{2\arg(c)-\pi}{4} + k2\pi\right) \subseteq S^1$. Setting $U := \left(\frac{2\arg(c)-3\pi}{4} + k2\pi, \frac{2\arg(c)-\pi}{4} + k2\pi\right)$ proves the statement, since for any $\vartheta' \in U$ the computation above implies that $c \leq_{\vartheta'} 0$. $\square$

Due to lemma (3.7), we also say that the relation $\leq_\vartheta$ is open with respect to ϑ.

Remark. One can check that $c \leq_\vartheta 0$ holds if and only if $\exp(\frac{c}{2t'^2})$ has moderate growth on some open neighbourhood of $t' = re^{i\vartheta}$ for $r \to 0$. For details see ([7], example 1.4).

Notation 3.8. For $c_1, c_2 \in \mathbb{C}$ we define

$$\begin{aligned}S^1_{c_1 \leq c_2} &:= \{\vartheta \in S^1 \mid c_1 \leq_\vartheta c_2\} \subseteq S^1 \text{ and} \\ S^1_{c_1 < c_2} &:= \{\vartheta \in S^1 \mid c_1 <_\vartheta c_2\} \subseteq S^1.\end{aligned}$$

Lemma (3.7) shows that $S^1_{c_1 \leq c_2}$ and $S^1_{c_1 < c_2}$ are open in S^1, therefore the inclusions $j_{c_1 \leq c_2} : S^1_{c_1 \leq c_2} \hookrightarrow S^1$ and $j_{c_1 < c_2} : S^1_{c_1 < c_2} \hookrightarrow S^1$ are open. With

$$\beta_{c_1 \leq c_2} := (j_{c_1 \leq c_2})_! j^{-1}_{c_1 \leq c_2} : \mathbf{Sh}_{S^1} \to \mathbf{Sh}_{S^1}$$

we denote the functor that restricts a sheaf on S^1 to the open set $S^1_{c_1 \leq c_2}$ and extends it by 0. Analogously we define $\beta_{c_1 < c_2} := (j_{c_1 < c_2})_! j^{-1}_{c_1 < c_2}$.

Recall that for a topological space X and an open inclusion $j : U \hookrightarrow X$ the functor $j_! : \mathbf{Sh}_U \to \mathbf{Sh}_X$ maps a sheaf $\mathcal{F}$ on U to the sheafification of the presheaf $\mathcal{F}'$ on X that is given as follows. For an open set $V \subseteq X$ one has

$$\mathcal{F}' : V \mapsto \mathcal{F}'(V) = \begin{cases} \mathcal{F}(V) & \text{if } V \subseteq U, \\ 0 & \text{if } V \nsubseteq U. \end{cases}$$

Remark. Let X be a topological space and let $j : U \hookrightarrow X$ be the inclusion of an open subset. Then the functor $j_!$ is left adjoint to j^{-1}. Thus for each $\mathcal{G} \in \mathbf{Sh}_{S^1}$ one has a morphism of sheaves $\varepsilon_{\mathcal{G}} : \beta_{c_1 \leq c_2}(\mathcal{G}) \to \mathcal{G}$ given by the counit of the adjunction. Moreover, this is a monomorphism of sheaves as for each $\vartheta \in S^1$ the stalk $\beta_{c_1 \leq c_2}(\mathcal{G})_\vartheta$ is given by

$$\beta_{c_1 \leq c_2}(\mathcal{G})_\vartheta = \begin{cases} \mathcal{G}_\vartheta & \text{if } \vartheta \in S_{c_1 \leq c_2}, \\ 0 & \text{if } \vartheta \notin S_{c_1 \leq c_2}. \end{cases}$$

For details see ([2], chapter 2).

We can now define Stokes directions.

Definition 3.9. Let $c_1, c_2 \in \mathbb{C}$ be two complex numbers. A direction $\vartheta \in S^1$ with $\vartheta \notin S^1_{c_1 \leq c_2}$ and $\vartheta \notin S^1_{c_2 \leq c_1}$ is called a *Stokes direction* of (c_1, c_2). The set of Stokes directions of (c_1, c_2) is denoted by $\mathrm{St}(c_1, c_2)$.

To calculate the set of Stokes directions of a given pair of two complex numbers, we use the following lemma.

Lemma 3.10. *Let $c_1, c_2 \in \mathbb{C}$ with $c_1 \neq c_2$. Then there are exactly four Stokes directions of (c_1, c_2). Moreover all Stokes directions of (c_1, c_2) differ by multiples of $\frac{\pi}{2}$, in particular*

$$\mathrm{St}(c_1, c_2) = \left\{ \frac{\pi}{4} + \frac{\arg(c_1 - c_2)}{2} + \mathbb{Z} \cdot \frac{\pi}{2} \mod 2\pi \right\}$$

holds.

Proof. Let $c_1, c_2 \in \mathbb{C}$ with $c_1 \neq c_2$. Define $\varphi := \arg(c_1 - c_2) \in [0, 2\pi)$. Then $\arg(c_2 - c_1) = \varphi + \pi \mod 2\pi$ and therefore $c_1 \nleq_\vartheta c_2$ and $c_2 \nleq_\vartheta c_1$ only holds if

$$\varphi - 2\vartheta \in \left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right) \mod 2\pi \text{ and } \varphi + \pi - 2\vartheta \in \left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right) \mod 2\pi.$$

Hence, ϑ is a Stokes direction if $\varphi - 2\vartheta \mod 2\pi \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$, which is equivalent to $\vartheta \mod \pi \in \{\frac{\pi}{4} + \frac{\varphi}{2}, \frac{3\pi}{4} + \frac{\varphi}{2}\}$. Since there are exactly four values for $\vartheta \in [0, 2\pi)$ that satisfy the previous, there are exactly four Stokes directions of (c_1, c_2). It also shows that if ϑ_0 is a Stokes direction of (c_1, c_2), then all elements of $\mathrm{St}(c_1, c_2)$ are of the form $\vartheta_\nu \coloneqq \vartheta_0 + \nu\frac{\pi}{2} \mod 2\pi$ for some $\nu \in \mathbb{Z}/4\mathbb{Z}$, thus

$$\mathrm{St}(c_1, c_2) = \left\{\frac{\pi}{4} + \frac{\varphi}{2} + \mathbb{Z} \cdot \frac{\pi}{2} \mod 2\pi\right\}.$$

□

Notation 3.11. As shown in lemma (3.10), for a pair of two different complex numbers $c_1, c_2 \in \mathbb{C}$ there are exactly four directions in $\mathrm{St}(c_1, c_2)$. Hence $S^1 \smallsetminus \mathrm{St}(c_1, c_2)$ has four connected components. Let $T(c_1, c_2)$ be the set of these components, i.e. for $\vartheta_0 \in \mathrm{St}(c_1, c_2)$ and $\vartheta_\nu \coloneqq \vartheta_0 + \nu\frac{\pi}{2}$ one has

$$T(c_1, c_2) \coloneqq \{(\vartheta_\nu, \vartheta_{\nu+1}) \subseteq S^1 \mid \nu \in \mathbb{Z}/4\mathbb{Z}\}.$$

Since the Stokes directions differ by a multiple of $\frac{\pi}{2}$, the elements of $T(c_1, c_2)$ are open intervals of length $\frac{\pi}{2}$. Besides, using lemma (3.7) we can observe that for every element $I \in T(c_1, c_2)$, either $c_1 \leq_\vartheta c_2$ for all $\vartheta \in I$ or $c_2 \leq_\vartheta c_1$ for all $\vartheta \in I$.

Example 3.12. Let $C = \{0, 1, c\} \subseteq \mathbb{C}$ with $\arg(c) = \varphi$. Then

- $\mathrm{St}(0, 1) = \{\frac{\pi}{4} + \nu\frac{\pi}{2} \mid \nu \in \mathbb{Z}/4\mathbb{Z}\}$,
- $\mathrm{St}(0, c) = \{\frac{\pi}{4} + \frac{\varphi}{2} + \nu\frac{\pi}{2} \mid \nu \in \mathbb{Z}/4\mathbb{Z}\}$,
- $T(0, 1) = \left\{(-\frac{\pi}{4}, \frac{\pi}{4}), \left(\frac{\pi}{4}, \frac{3\pi}{4}\right), \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right), \left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)\right\}$ and
- $T(0, c) = \left\{\left(\frac{-\pi+2\varphi}{4}, \frac{\pi+2\varphi}{4}\right), \left(\frac{\pi+2\varphi}{4}, \frac{3\pi+2\varphi}{4}\right), \left(\frac{3\pi+2\varphi}{4}, \frac{5\pi+2\varphi}{4}\right), \left(\frac{5\pi+2\varphi}{4}, \frac{8\pi+2\varphi}{4}\right)\right\}$.

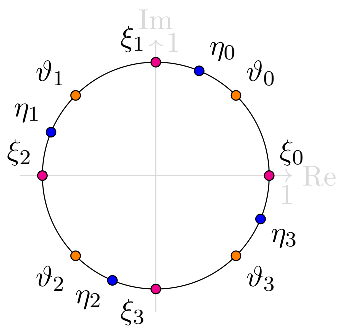

Figure 3.1: $\mathrm{St}(0, 1)$ in orange, $\mathrm{St}(0, c)$ in blue and $\mathrm{St}(1, c)$ in pink for $c = 1 + i$.

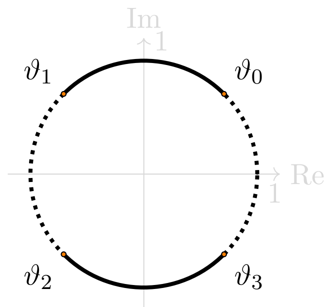

Figure 3.2: The sets $S^1_{0\leq 1}$, outlined by the dotted line and $S^1_{1\leq 0}$ outlined by the full line.

3.2.2 Stokes filtered local systems

Now, having seen Stokes directions, we can give the definitions for (pre-)Stokes filtered local systems. To do this, we specify the contents of chapter 2 of [7] to fit our framework, namely we define Stokes filtrations of local systems that are of Gaussian type.

Definition 3.13. A *pre-Stokes filtered local system of Gaussian type* is a pair $(\mathscr{L}, (\mathscr{L}_{\leq c})_{c\in\mathbb{C}})$, where

1. $\mathscr{L}$ is a local system of finite dimensional $\mathbf{k}$-vector spaces on S^1 and
2. $(\mathscr{L}_{\leq c})_{c\in\mathbb{C}}$ is a family of subsheaves of $\mathscr{L}$ in such way that for any $c_1, c_2 \in \mathbb{C}$ and $\vartheta \in S^1$ the implication
$$c_1 \leq_\vartheta c_2 \Longrightarrow \mathscr{L}_{\leq c_1,\vartheta} \subseteq \mathscr{L}_{\leq c_2,\vartheta}$$
holds and forms an exhaustive increasing filtration of $\mathscr{L}_\vartheta$.

We say that $(\mathscr{L}_{\leq c})_{c\in\mathbb{C}}$ is a *pre-Stokes filtration of Gaussian type* of $\mathscr{L}$.

Example 3.14. Consider the constant sheaf $\mathscr{L} \coloneqq \underline{\mathbf{k}}_{S^1}$ and for each $c \in \mathbb{C}$ the subsheaf $\mathscr{L}_{\leq c} \coloneqq \beta_{0\leq c}(\underline{\mathbf{k}}_{S^1})$. Then $(\mathscr{L}, (\mathscr{L}_{\leq c})_{c\in\mathbb{C}})$ is a pre-Stokes filtered local system of Gaussian type since $\mathscr{L}$ is a local system and for any $\vartheta \in S^1, c \in \mathbb{C}$ we have
$$\mathscr{L}_{\leq c,\vartheta} = \beta_{0\leq c}(\underline{\mathbf{k}}_{S^1})_\vartheta = \begin{cases} \mathbf{k} & \text{if } 0 \leq_\vartheta c, \\ 0 & \text{if } 0 \not\leq_\vartheta c. \end{cases}$$
Transitivity of $\leq_\vartheta$ implies the property $c_1 \leq_\vartheta c_2 \Longrightarrow \mathscr{L}_{\leq c_1,\vartheta} \subseteq \mathscr{L}_{\leq c_2,\vartheta}$. Since $\mathscr{L}_{\leq 0} = \mathscr{L}$ the filtration of $\mathscr{L}_\vartheta$ is exhaustive.

Let $(\mathscr{L}, (\mathscr{L}_{\leq c})_{c\in\mathbb{C}})$ be a pre-Stokes filtered local system of Gaussian type. For any $\vartheta \in S^1$ and $c \in \mathbb{C}$ the $\mathbf{k}$-vector spaces
$$\mathscr{L}_{<c,\vartheta} \coloneqq \sum_{c'<_\vartheta c} \mathscr{L}_{\leq c',\vartheta}$$
glue together to a subsheaf $\mathscr{L}_{<c}$ of $\mathscr{L}_{\leq c}$, given by
$$\mathscr{L}_{<c}(U) \coloneqq \left\{ \sigma \in \mathscr{L}(U) \;\middle|\; [\sigma]_\vartheta \in \sum_{c'<_\vartheta c} \mathscr{L}_{\leq c',\vartheta} \text{ for all } \vartheta \in U \right\}$$
for an open subset $U \subseteq S^1$, because $\leq_\vartheta$ is open with respect to ϑ. By $\operatorname{gr}_c \mathscr{L}$ we denote the quotient sheaf $\mathscr{L}_{\leq c}/\mathscr{L}_{<c}$. Then by
$$\operatorname{gr} \mathscr{L} \coloneqq \bigoplus_{c\in\mathbb{C}} \operatorname{gr}_c \mathscr{L} = \bigoplus_{c\in\mathbb{C}} \mathscr{L}_{\leq c}/\mathscr{L}_{<c}$$

we get a graded sheaf. Furthermore for any $c \in \mathbb{C}$ and $\vartheta \in S^1$ the $\mathbf{k}$-vector spaces

$$\operatorname{gr} \mathscr{L}_{\leq c,\vartheta} := \bigoplus_{c' \leq_\vartheta c} \operatorname{gr}_{c'} \mathscr{L}_\vartheta$$

glue together to a subsheaf $\operatorname{gr} \mathscr{L}_{\leq c}$ of $\operatorname{gr} \mathscr{L}$ that is given by

$$\operatorname{gr} \mathscr{L}_{\leq c} = \bigoplus_{c' \in \mathbb{C}} \beta_{c' \leq c}(\operatorname{gr}_{c'} \mathscr{L}).$$

Now we can give the definition of a Stokes filtration of Gaussian type.

Definition 3.15. Let $\mathscr{L}$ be a local system of finite dimensional $\mathbf{k}$-vector spaces on S^1. A pre-Stokes filtration of Gaussian type $(\mathscr{L}_{\leq c})_{c\in\mathbb{C}}$ of $\mathscr{L}$ is a *Stokes filtration of Gaussian type* if it satisfies the following properties:

1. $\operatorname{gr}_c \mathscr{L}$ is an object in $\mathbf{Loc}_{S^1}$ for any $c \in \mathbb{C}$.
2. $(\mathscr{L}, (\mathscr{L}_{\leq c})_{c\in\mathbb{C}})$ is locally isomorphic to $(\operatorname{gr} \mathscr{L}, (\operatorname{gr} \mathscr{L}_{\leq c})_{c\in\mathbb{C}})$, i.e. for any direction $\vartheta \in S^1$ there is a neighbourhood $U \subseteq S^1$ of ϑ and an isomorphism of local systems $\mathscr{L}_{|U} \to \operatorname{gr} \mathscr{L}_{|U}$. Moreover for any $\vartheta \in S^1$ and $c \in \mathbb{C}$ there is an open neighbourhood $U_c \subseteq S^1$ of ϑ and an isomorphism $\mathscr{L}_{\leq c|_{U_c}} \to \operatorname{gr} \mathscr{L}_{\leq c|_{U_c}}$.

We also say that $(\mathscr{L}, (\mathscr{L}_{\leq c})_{c\in\mathbb{C}})$ is a Stokes filtered local system of Gaussian type.

We can classify Stokes filtered local systems using the following insight:

Lemma 3.16. *Let $(\mathscr{L}, (\mathscr{L}_{\leq c})_{c\in\mathbb{C}})$ be a Stokes filtered local system of Gaussian type. Then the set $\{c \in \mathbb{C} \mid \operatorname{gr}_c \mathscr{L} \neq 0\}$, called* the set of exponential factors *of the Stokes filtration, is finite.*

Proof. Since $(\mathscr{L}, (\mathscr{L}_{\leq c})_{c\in\mathbb{C}})$ is a Stokes filtered local system of Gaussian type, $\mathscr{L}_\vartheta$ has finite dimension for every $\vartheta \in S^1$. Therefore there are only finitely many $c \in \mathbb{C}$ with $\mathscr{L}_{\leq \tilde{c},\vartheta} \subsetneq \mathscr{L}_{\leq c,\vartheta}$ for all $\tilde{c} \in \mathbb{C}$ with $\tilde{c} <_\vartheta c$, as this forms a filtration of $\mathscr{L}_\vartheta$. Hence for all except a finite number of $c \in \mathbb{C}$, there is a $\tilde{c} \in \mathbb{C}$ with $\tilde{c} <_\vartheta c$ and $\mathscr{L}_{\leq \tilde{c},\vartheta} = \mathscr{L}_{\leq c,\vartheta}$. For these we have

$$\mathscr{L}_{<c,\vartheta} = \sum_{\tilde{c} <_\vartheta c} \mathscr{L}_{\leq \tilde{c},\vartheta} = \mathscr{L}_{\leq c,\vartheta},$$

and consequently $\operatorname{gr}_c \mathscr{L}_\vartheta = \mathscr{L}_{\leq c,\vartheta}/\mathscr{L}_{<c,\vartheta} = 0$. As $\operatorname{gr}_c \mathscr{L}$ is a local system on S^1 and S^1 is connected we can conclude that $\operatorname{gr}_c \mathscr{L} = 0$ using lemma (2.3). Thus the set of exponential factors $\{c \in \mathbb{C} \mid \operatorname{gr}_c \mathscr{L} \neq 0\}$ is finite. $\square$

Notation 3.17. Let $(\mathscr{L}, (\mathscr{L}_{\leq c})_{c\in\mathbb{C}})$ be a Stokes filtered local system of Gaussian type and $C \subseteq \mathbb{C}$ be a finite subset. We say $(\mathscr{L}, (\mathscr{L}_{\leq c})_{c\in\mathbb{C}})$ is *of type* C if its set of exponential factors is a subset of C. In the following we abbreviate the notation $(\mathscr{L}, (\mathscr{L}_{\leq c})_{c\in\mathbb{C}})$ to $(\mathscr{L}, \mathscr{L}_{\leq\bullet})$.

Example 3.18. Let $(\mathscr{L}, \mathscr{L}_{\leq\bullet})$ be a Stokes filtered local system of Gaussian type. Given that its set of exponential factors is finite, $\operatorname{gr}\mathscr{L}$ is a local system. The implication

$$c_1 \leq_\vartheta c_2 \Longrightarrow \operatorname{gr}\mathscr{L}_{\leq c_1,\vartheta} \subseteq \operatorname{gr}\mathscr{L}_{\leq c_2,\vartheta}$$

follows directly form the definition of $\operatorname{gr}\mathscr{L}_{\leq\bullet}$. Furthermore for an arbitrary direction $\vartheta \in S^1$, property 2. of (3.15) implies $\mathscr{L}_\vartheta \cong \operatorname{gr}\mathscr{L}_\vartheta$ and $\mathscr{L}_{\leq c,\vartheta} \cong \operatorname{gr}\mathscr{L}_{\leq c,\vartheta}$. As the implication $c_1 \leq_\vartheta c_2 \Longrightarrow \mathscr{L}_{\leq c_1,\vartheta} \subseteq \mathscr{L}_{\leq c_2,\vartheta}$ forms an exhaustive filtration of $\mathscr{L}_\vartheta$, the same holds for $\operatorname{gr}\mathscr{L}_\vartheta$, so $(\operatorname{gr}\mathscr{L}, \operatorname{gr}\mathscr{L}_{\leq\bullet})$ is a pre-Stokes filtered local system of Gaussian type. Since $\operatorname{gr}_c(\operatorname{gr}\mathscr{L}) \cong \operatorname{gr}_c\mathscr{L}$ it follows that $(\operatorname{gr}\mathscr{L}_{\leq\bullet})$ fulfills the two properties of definition (3.15), hence $(\operatorname{gr}\mathscr{L}_{\leq\bullet})$ is a Stokes filtration. We also say that $(\operatorname{gr}\mathscr{L}, \operatorname{gr}\mathscr{L}_{\leq\bullet})$ is a *graded Stokes filtered local system of Gaussian type.*

More general we have the following:

Definition 3.19. For a finite set $C \subseteq \mathbb{C}$ a *graded Stokes filtered local system of Gaussian type* C is a pair $(\operatorname{gr}\mathscr{L}, (\operatorname{gr}\mathscr{L}_{\leq c})_{c\in\mathbb{C}})$ where for each $c \in C$ there is a local system $\operatorname{gr}_c\mathscr{L}$ on S^1 such that

$$\operatorname{gr}\mathscr{L} = \bigoplus_{c\in C} \operatorname{gr}_c\mathscr{L}$$

and for each $c' \in \mathbb{C}$

$$\operatorname{gr}\mathscr{L}_{\leq c'} = \bigoplus_{c\in C} \beta_{c\leq c'}(\operatorname{gr}_c\mathscr{L}).$$

In chapter (3.1) we gave the definition of differential systems that are of *pure* Gaussian type. Thus, the question arises which property Stokes filtered local systems need to fulfill to be of pure Gaussian type as well, so we can formulate the Riemann-Hilbert correspondence for differential systems that are of pure Gaussian type later on.

Definition 3.20. Let $C \subseteq \mathbb{C}$ be a non-empty, finite subset and $(\mathscr{L}, \mathscr{L}_{\leq\bullet})$ be a Stokes filtered local system of Gaussian type C. We say that $(\mathscr{L}, \mathscr{L}_{\leq\bullet})$ is *of pure Gaussian type* C if $\mathscr{L}$ is a constant sheaf and $0 \notin C$.

To finally define the category of Stokes filtered local systems that are of Gaussian type, we have to determine morphisms within this category.

Definition 3.21. Let $(\mathscr{L}, \mathscr{L}_{\leq \bullet}), (\mathscr{L}', \mathscr{L}'_{\leq \bullet})$ be two Stokes filtered local systems of Gaussian type C. A *morphism* $\lambda : (\mathscr{L}, \mathscr{L}_{\leq \bullet}) \to (\mathscr{L}', \mathscr{L}'_{\leq \bullet})$ *of (pre-)Stokes filtered local systems of Gaussian type* C is a morphism $\lambda \in \mathrm{Hom}_{\mathbf{Loc}_{S^1}}(\mathscr{L}, \mathscr{L}')$ satisfying $\lambda(\mathscr{L}_{\leq c}) \subseteq \mathscr{L}'_{\leq c}$ for each $c \in \mathbb{C}$. We denote the category of Stokes filtered local systems of Gaussian type C by $\mathbf{Loc}_{\mathrm{St}}(C)$.
Furthermore the category of Stokes filtered local systems of pure Gaussian type C, indicated by $\mathbf{Loc}^{\star}_{\mathrm{St}}(C)$, is the full subcategory of $\mathbf{Loc}_{\mathrm{St}}(C)$ where the objects are of pure Gaussian type C.

Remark. If C' is a non-empty subset of the finite set $C \subseteq \mathbb{C}$, then $\mathbf{Loc}_{\mathrm{St}}(C')$ forms a full subcategory of $\mathbf{Loc}_{\mathrm{St}}(C)$. Furthermore if $0 \notin C$, then $\mathbf{Loc}^{\star}_{\mathrm{St}}(C')$ is a full subcategory of $\mathbf{Loc}^{\star}_{\mathrm{St}}(C)$.

As we will also consider the category of graded Stokes filtered local systems later on, we need to define morphisms for this category as well.

Definition 3.22. Let $(\mathrm{gr}\,\mathscr{L}, \mathrm{gr}\,\mathscr{L}_{\leq \bullet}), (\mathrm{gr}\,\mathscr{L}', \mathrm{gr}\,\mathscr{L}'_{\leq \bullet})$ be two graded Stokes filtered local systems of Gaussian type C. A *morphism* $\lambda : (\mathrm{gr}\,\mathscr{L}, \mathrm{gr}\,\mathscr{L}_{\leq \bullet}) \to (\mathrm{gr}\,\mathscr{L}', \mathrm{gr}\,\mathscr{L}'_{\leq \bullet})$ *of graded Stokes filtered local systems of Gaussian type* C is a graded morphism of sheaves

$$\lambda : \mathrm{gr}\,\mathscr{L} = \bigoplus_{c \in C} \mathrm{gr}_c\,\mathscr{L} \to \bigoplus_{c \in C} \mathrm{gr}_c\,\mathscr{L}' = \mathrm{gr}\,\mathscr{L}',$$

satisfying $\lambda(\mathrm{gr}\,\mathscr{L}_{\leq c}) \subseteq \mathrm{gr}\,\mathscr{L}'_{\leq c}$. We denote the category of graded Stokes filtered local systems by $\mathbf{Loc}^{\mathrm{gr}}_{\mathrm{St}}(C)$.

Remark. More precisely, after decomposing $\lambda : \bigoplus_{c \in C} \mathrm{gr}_c\,\mathscr{L} \to \bigoplus_{c \in C} \mathrm{gr}_c\,\mathscr{L}'$ into blocks of morphisms of local systems

$$\lambda = (\lambda_{ji} : \mathrm{gr}_{c_i}\,\mathscr{L} \to \mathrm{gr}_{c_j}\,\mathscr{L}')_{c_i, c_j \in C},$$

we say λ is graded if $\lambda_{ji} = 0$ for $c_i \neq c_j$.

By definition (3.15) each Stokes filtered local system $(\mathscr{L}, \mathscr{L}_{\leq \bullet})$ of type C is locally isomorphic to the graded Stokes filtered local system $(\mathrm{gr}\,\mathscr{L}, \mathrm{gr}\,\mathscr{L}_{\leq \bullet})$. In fact for an interval I of "correct length", there is a unique splitting $\mathscr{L}_{|I} \cong \bigoplus_{c \in C} \mathrm{gr}_c\,\mathscr{L}_{|I}$, that is compatible with the Stokes filtration $(\mathscr{L}_{\leq \bullet})$. In order to give the precise statement, we first need to establish what "correct length" means.

Definition 3.23. Let $C \subseteq \mathbb{C}$ be a non-empty, finite subset. A direction $\vartheta_0 \in S^1$ is said to be *generic* with respect to C if ϑ_0 is not a Stokes direction for any pair (c, c') with $c \neq c'$ in C. Besides, a *C-good closed interval* $I \subseteq \mathbb{R}/2\pi\mathbb{Z}$ is a closed interval such that its interior contains exactly one Stokes direction for each pair of distinct elements (c, c') in C.

Remark. Given that we exclusively deal with finite sets $C \subseteq \mathbb{C}$, and every pair of two different complex numbers offers precisely four Stokes directions, there always exists a generic direction for C. Moreover if $\vartheta_0 \in S^1$ is generic with respect to C, then $\vartheta_\nu := \vartheta_0 + \nu\frac{\pi}{2}$ is generic and $[\vartheta_\nu, \vartheta_\nu + \frac{\pi}{2}]$ is a C-good closed interval for all $\nu \in \mathbb{Z}/4\mathbb{Z}$. Therefore we can always consider C-good closed intervals of the form $I = [\vartheta_\nu, \vartheta_\nu + \frac{\pi}{2}]$.

Example 3.24. Let $C := \{0, 1, 1+i\} \subseteq \mathbb{C}$. Then $\vartheta_0 = \frac{\pi}{16} \in S^1$ is generic with respect to C and the closed interval $[\vartheta_0, \vartheta_0 + \frac{\pi}{2}]$ is C-good.

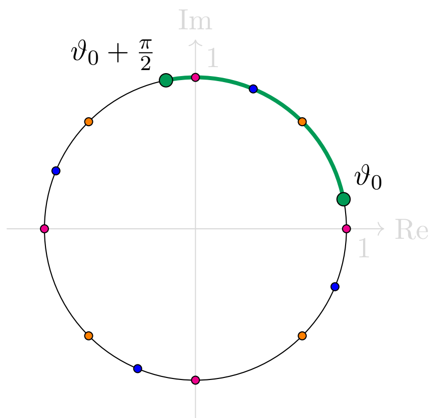

Figure 3.3: A generic direction $\vartheta_0 = \frac{\pi}{16}$ with respect to $C = \{0, 1, 1+i\}$ in green, together with the C-good closed interval $I = [\vartheta_0, \vartheta_0 + \frac{\pi}{2}]$. The other dots are Stokes directions of C (compare to figure 3.1).

Using C-good intervals we obtain a unique splitting $\mathscr{L}_{|I} \cong \bigoplus_{c \in C} \operatorname{gr}_c \mathscr{L}_{|I}$. More precisely, we get the splitting lemma:

Proposition 3.25. *Let $(\mathscr{L}, \mathscr{L}_{\leq \bullet})$ be a Stokes filtered local system of Gaussian type C, $\vartheta_0 \in S^1$ be a generic direction with respect to C and consider the C-good closed interval $I := [\vartheta_0, \vartheta_0 + \frac{\pi}{2}] \subseteq \mathbb{R}/2\pi\mathbb{Z}$. Then the following properties hold:*

1. *There exists a unique splitting $\mathscr{L}_{|I} \cong \bigoplus_{c \in C} \operatorname{gr}_c \mathscr{L}_{|I}$ that is compatible with the*

Stokes filtration. With respect to this splitting, we have

$$\mathscr{L}_{\leq c|_I} = \bigoplus_{c' \in C} \beta_{c' \leq c} (\mathrm{gr}_{c'} \mathscr{L})_{|_I}.$$

2. *Let $\lambda : (\mathscr{L}, \mathscr{L}_{\leq \bullet}) \to (\mathscr{L}', \mathscr{L}'_{\leq \bullet})$ be a morphism of Stokes filtered local systems of Gaussian Type C. Then the morphism $\lambda_{|_I}$ is graded with respect to the splittings in 1.*

Proof. See [3], proposition 2.2. □

Remark. We want to make 2. of proposition (3.25) precise.
Let $\lambda : (\mathscr{L}, \mathscr{L}_{\leq \bullet}) \to (\mathscr{L}', \mathscr{L}'_{\leq \bullet})$ be a morphism of Stokes filtered local systems. Using the unique splitting for a C-good closed interval I we get in 1. of proposition (3.25), we can decompose

$$\lambda_{|_I} : \mathscr{L}_{|_I} \cong \bigoplus_{c \in C} \mathrm{gr}_c \mathscr{L}_{|_I} \to \bigoplus_{c \in C} \mathrm{gr}_c \mathscr{L}'_{|_I} \cong \mathscr{L}'_{|_I}$$

into blocks of morphisms of local systems

$$((\lambda_{|_I})_{ji} : \mathrm{gr}_{c_i} \mathscr{L} \to \mathrm{gr}_{c_j} \mathscr{L}')_{c_i, c_j \in C}.$$

Then, similar to definition (3.22), $\lambda_{|_I}$ is graded with respect to the splittings in 1. if $(\lambda_{|_I})_{ji} = 0$ for $i \neq j$.

Stokes filtrations can be used to encode the Stokes structure near an irregular singular point of a differential equation in the local system (details can be found in [7]). In the following, we will get to know another approach to understand the Stokes structure.

3.2.3 Stokes data

In this section, our main objective is to define the category of Stokes data. This category describes Stokes structures using linear data, specifically vector spaces and linear morphisms. Moreover, Stokes filtered local systems of pure Gaussian type can be understood using the category of Stokes data. We follow the approach outlined in ([6], chapter 2b) and ([4], chapter 2.5).

Let $C \subseteq \mathbb{C}$ be a non-empty, finite subset and $\vartheta_0 \in S^1$ a generic direction with respect to C. Then for any pair of two different elements $c, c' \in C$ we can compare c, c' using $<_{\vartheta_0}$. In this way we get a unique numbering of C, satisfying $c_1 <_{\vartheta_0} c_2 <_{\vartheta_0} \cdots <_{\vartheta_0} c_r$ with $r \in \mathbb{N}$ being the cardinality of C.

Definition 3.26. Let C be a non-empty finite subset of $\mathbb{C}$, ϑ_0 a generic direction with respect to C and let $\{c_1, \dots, c_r\}$ be the unique numbering of C given by ϑ_0. The *category* $\mathfrak{SD}(C, \vartheta_0)$ *of Stokes data of Gaussian type* (C, ϑ_0) is defined by the following:

- An object $\sigma = ((G_c^{(\nu)})_{c \in C}, S^{(\nu+1,\nu)})_{\nu \in \mathbb{Z}/4\mathbb{Z}}$ consists of four families of finite-dimensional $\mathbf{k}$-vector spaces $(G_c^{(\nu)})_{c \in C}$, together with four linear morphisms $S^{(\nu+1,\nu)}$, shown in the following diagram

$$\begin{tikzcd}
& \bigoplus_{c \in C} G_c^{(0)} \arrow[dl, "S^{(1,0)}"'] & \\
\bigoplus_{c \in C} G_c^{(1)} \arrow[dr, "S^{(2,1)}"'] & & \bigoplus_{c \in C} G_c^{(3)} \arrow[ul, "S^{(0,3)}"'] \\
& \bigoplus_{c \in C} G_c^{(2)} \arrow[ur, "S^{(3,2)}"'] &
\end{tikzcd}$$

  such that writing $S^{(\nu+1,\nu)} = (S_{ij}^{(\nu+1,\nu)} : G_{c_j}^{(\nu)} \to G_{c_i}^{(\nu+1)})_{i,j \in \{0,\dots,r\}}$ the following properties hold:

  – $S^{(0,3)}$ and $S^{(2,1)}$ are block-upper triangular, i.e. $S_{ij}^{(\nu+1,\nu)}$ is zero if $i > j$ for $\nu \in \{1, 3\}$.

  – $S^{(1,0)}$ and $S^{(3,2)}$ are block-lower triangular, i.e. $S_{ij}^{(\nu+1,\nu)}$ is zero if $i < j$ for $\nu \in \{0, 2\}$.

  – $S_{ii}^{(\nu+1,\nu)}$ is invertible for all $i \in \{1, \dots, r\}$ and $\nu \in \mathbb{Z}/4\mathbb{Z}$.

- A morphism $\lambda \in \mathrm{Hom}_{\mathfrak{SD}}(\sigma, \tilde{\sigma})$ between $\sigma = ((G_c^{(\nu)})_{c \in C}, S^{(\nu+1,\nu)})_{\nu \in \mathbb{Z}/4\mathbb{Z}}$ and $\tilde{\sigma} = ((\tilde{G}_c^{(\nu)})_{c \in C}, \tilde{S}^{(\nu+1,\nu)})_{\nu \in \mathbb{Z}/4\mathbb{Z}}$ is a family of four linear morphisms

$$\left(\lambda^{(\nu)} : \bigoplus_{c \in C} G_c^{(\nu)} \to \bigoplus_{c \in C} \tilde{G}_c^{(\nu)} \right)_{\nu \in \mathbb{Z}/4\mathbb{Z}},$$

  such that writing $\lambda^{(\nu)} = (\lambda_{ij}^{(\nu)} : G_{c_j}^{(\nu)} \to \tilde{G}_{c_i}^{(\nu)})_{i,j \in \{0,\dots,r\}}$ the following properties hold:

  – $\lambda_{ij}^{(\nu)}$ is zero if $i \neq j$, i.e. $\lambda^{(\nu)}$ is block-diagonal for every $\nu \in \mathbb{Z}/4\mathbb{Z}$.

– For any $\nu \in \mathbb{Z}/4\mathbb{Z}$, one has $\tilde{S}^{(\nu+1,\nu)}\lambda^{(\nu)} = \lambda^{(\nu+1)}S^{(\nu+1,\nu)}$.

The composition of two morphisms of Stokes data $\lambda \in \mathrm{Hom}_{\mathfrak{SD}(C,\vartheta_0)}(\sigma_1,\sigma_2)$ and $\mu \in \mathrm{Hom}_{\mathfrak{SD}(C,\vartheta_0)}(\sigma_2,\sigma_3)$ is given by $\mu\circ\lambda = (\mu^{(\nu)}\circ\lambda^{(\nu)})_{\nu\in\mathbb{Z}/4\mathbb{Z}}$ where $\mu^{(\nu)}\circ\lambda^{(\nu)}$ is the composition in $\mathbf{Vect_k}$.

Remark. As $S_{ii}^{(\nu+1,\nu)}$ is invertible for every $c_i \in C$ and $\nu \in \mathbb{Z}/4\mathbb{Z}$, we get $\dim G_{c_i}^{(\nu)} = \dim G_{c_i}^{(\nu+1)}$. Furthermore, by definition (3.26), $S^{(\nu+1,\nu)}$ is a block-upper (resp. block-lower) matrix with invertible matrices on its diagonal. Therefore $S^{(\nu+1,\nu)}$ is invertible.

Again, we also want the notion of Stokes data that are of *pure* Gaussian type.

Definition 3.27. Let $C \subseteq \mathbb{C}^\times$ be a non-empty, finite subset and let ϑ_0 be a generic direction with respect to C. Then the *category* $\mathfrak{SD}^\star(C,\vartheta_0)$ *of Stokes data of pure Gaussian type* (C,ϑ_0) is the full subcategory of $\mathfrak{SD}(C,\vartheta_0)$ where the objects $\sigma = ((G_c^{(\nu)})_{c\in C}, S^{(\nu+1,\nu)})_{\nu\in\mathbb{Z}/4\mathbb{Z}}$ satisfy

$$S^{(0,3)}S^{(3,2)}S^{(2,1)}S^{(1,0)} = \mathrm{id}_{\bigoplus_{c\in C} G_c^{(0)}} \in \mathrm{Hom}_{\mathbf{Vect_k}}\left(\bigoplus_{c\in C} G_c^{(0)}, \bigoplus_{c\in C} G_c^{(0)}\right).$$

Proposition 3.28. *Let $C \subseteq \mathbb{C}$ be a non-empty finite subset and $\vartheta_0 \in S^1$ a generic direction with respect to C. The category $\mathfrak{SD}(C,\vartheta_0)$ of Stokes data of Gaussian type (C,ϑ_0) is abelian.*

Proof. Since $\mathbf{Vect_k}$ is an additive category, $\mathfrak{SD}(C,\vartheta_0)$ is also additive. Now let $\{c_1,\dots,c_r\}$ be the unique numbering of C given by ϑ_0 and let $\lambda \in \mathrm{Hom}_{\mathfrak{SD}}(\sigma,\tilde{\sigma})$ be a morphism from $\sigma = ((G_c^{(\nu)})_{c\in C}, S^{(\nu+1,\nu)})_{\nu\in\mathbb{Z}/4\mathbb{Z}}$ to $\tilde{\sigma} = ((\tilde{G}_c^{(\nu)})_{c\in C}, \tilde{S}^{(\nu+1,\nu)})_{\nu\in\mathbb{Z}/4\mathbb{Z}}$. Then in $\mathbf{Vect_k}$ the kernel of $\lambda^{(\nu)}$ is given by $\bigoplus_{i=1}^r \ker(\lambda_{ii}^{(\nu)})$. This implies that the kernel of λ in $\mathfrak{SD}(C,\vartheta_0)$ is given by

$$\ker(\lambda) = \left(\left(\ker(\lambda_{ii}^{(\nu)})\right)_{i\in\{1,\dots,r\}}, S^{(\nu+1,\nu)}_{|_{\ker(\lambda^{(\nu)})}}\right)_{\nu\in\mathbb{Z}/4\mathbb{Z}},$$

since the composition of morphisms in $\mathfrak{SD}(C,\vartheta_0)$ is defined by the composition of morphisms in $\mathbf{Vect_k}$. The kernel of λ can be illustrated in the following diagram:

$$\begin{tikzcd}
 & \bigoplus_{i=1}^{r} \ker(\lambda_{ii}^{(0)}) \arrow[dl, "S^{(1,0)}_{|\ker(\lambda^{(0)})}"'] & \\
\bigoplus_{i=1}^{r} \ker(\lambda_{ii}^{(1)}) \arrow[dr, "S^{(2,1)}_{|\ker(\lambda^{(1)})}"'] & & \bigoplus_{i=1}^{r} \ker(\lambda_{ii}^{(3)}) \arrow[ul, "S^{(0,3)}_{|\ker(\lambda^{(3)})}"'] \\
 & \bigoplus_{i=1}^{r} \ker(\lambda_{ii}^{(2)}) \arrow[ur, "S^{(3,2)}_{|\ker(\lambda^{(2)})}"'] &
\end{tikzcd}$$

$\ker(\lambda)$ is well defined, because $\lambda^{(\nu+1)}S^{(\nu+1,\nu)} = \tilde{S}^{(\nu+1,\nu)}\lambda^{(\nu)}$ holds. Analogous arguments show that $\operatorname{coker}(\lambda) = ((\operatorname{coker}(\lambda_{ii}^{(\nu)}))_{i\in\{1,\ldots,r\}}, \tilde{S}^{(\nu+1,\nu)}_{|\operatorname{coker}(\lambda^{\nu})})_{\nu\in\mathbb{Z}/4\mathbb{Z}}$ is the cokernel of λ in $\mathfrak{SD}(C,\vartheta_0)$. Since the canonical morphism $\hat{\lambda}^{(\nu)} : \operatorname{coim}(\lambda^{(\nu)}) \to \operatorname{im}(\lambda^{(\nu)})$ is given by the $\hat{\lambda}_{ii}^{(\nu)} : \operatorname{coim}(\lambda_{ii}^{(\nu)}) \to \operatorname{im}(\lambda_{ii}^{(\nu)})$ and these are isomorphisms, $\hat{\lambda}^{(\nu)}$ is an isomorphism, hence $\hat{\lambda} : \operatorname{coim}(\lambda) \to \operatorname{im}(\lambda)$ is an isomorphism. □

Remark. Using identical reasoning, it becomes apparent that for a non-empty, finite set $C \subseteq \mathbb{C}^{\times}$ and a generic direction $\vartheta_0 \in S^1$, the category $\mathfrak{SD}^{\star}(C,\vartheta_0)$ is likewise abelian.

Let $\sigma = ((G_c^{(\nu)})_{c\in C}, S^{(\nu+1,\nu)})_{\nu\in\mathbb{Z}/4\mathbb{Z}} \in \mathfrak{SD}(C,\vartheta_0)$. If we fix bases for each $\mathbf{k}$-vector space $G_c^{(\nu)}$ we can represent the linear transformations $(S^{(\nu+1,\nu)})_{\nu\in\mathbb{Z}/4\mathbb{Z}}$ as matrices $(\Sigma^{(\nu,\nu+1)})_{\nu\in\mathbb{Z}/4\mathbb{Z}}$. The same properties that hold for $S^{(\nu+1,\nu)}$ also hold for the block matrix $\Sigma^{(\nu+1,\nu)}$, i.e. we have

$$\Sigma^{(\nu+1,\nu)} = \begin{pmatrix} \Sigma_{11}^{(\nu+1,\nu)} & \Sigma_{12}^{(\nu+1,\nu)} & \cdots & \Sigma_{1r}^{(\nu+1,\nu)} \\ 0 & \Sigma_{22}^{(\nu+1,\nu)} & \cdots & \Sigma_{2r}^{(\nu+1,\nu)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Sigma_{rr}^{(\nu+1,\nu)} \end{pmatrix}$$

for $\nu \in \{1,3\}$ and

$$\Sigma^{(\nu+1,\nu)} = \begin{pmatrix} \Sigma_{11}^{(\nu+1,\nu)} & 0 & \cdots & 0 \\ \Sigma_{21}^{(\nu+1,\nu)} & \Sigma_{22}^{(\nu+1,\nu)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{r1}^{(\nu+1,\nu)} & \Sigma_{2r}^{(\nu+1,\nu)} & \cdots & \Sigma_{rr}^{(\nu+1,\nu)} \end{pmatrix}$$

for $\nu \in \{0,2\}$ with $\Sigma_{ii}^{(\nu)}$ is invertible for each $\nu \in \mathbb{Z}/4\mathbb{Z}$ and $i \in \{1,\ldots,r\}$. The family $(\Sigma^{(\nu+1,\nu)})_{\nu\in\mathbb{Z}/4\mathbb{Z}}$ is also called family of *Stokes matrices* (cf. [4], chapter 2.5).

The following proposition links the category of Stokes data with the category of Stokes filtered local systems. We will not prove the statement in this thesis, since we focus on defining Stokes shells.

Proposition 3.29. *Let $C \subseteq \mathbb{C}^\times$ be a non-empty finite set and $\vartheta_0 \in S^1$ a generic direction with respect to C. Then there exists a category equivalence*

$$F : \mathbf{Loc}^\star_{St}(C) \to \mathfrak{S}\mathfrak{D}^\star(C, \vartheta_0).$$

Proof. See [6], proposition 2.13. □

Corollary 3.30. *Let $C \subseteq \mathbb{C}^\times$ be a non-empty finite set and $\vartheta_0 \in S^1$ a generic direction with respect to C. Then $\mathbf{Loc}^\star_{St}(C)$ is abelian.*

Proof. Using the fact, that a category equivalence $F : \mathbf{C} \to \mathbf{D}$ with $\mathbf{C}$ abelian implies that $\mathbf{D}$ is abelian (cf. [8], Satz 16.2.4), the assertion follows directly from proposition (3.28) and (3.29). □

3.2.4 A Riemann-Hilbert correspondence

Equipped with knowledge about Stokes filtered local systems and differential systems of Gaussian type, we can formulate a Riemann-Hilbert correspondence that connects both categories. For a comprehensive understanding, refer to [1], [6] proposition 2.16, and the references cited within ([6], chapter 2.e).

Theorem 3.31. *For every finite, non-empty $C \subseteq \mathbb{C}^\times$ there is an equivalence of categories between the category of differential systems of pure Gaussian type C and the category of Stokes filtered local system of pure Gaussian type C, i.e.*

$$\mathbf{Mod}_{Gauß}(C) \to \mathbf{Loc}^\star_{St}(C).$$

Remark. Using proposition (3.29) we can also use the category of Stokes data of pure Gaussian type C to study differential systems that are of pure Gaussian type C.

In this chapter, we have presented two techniques to investigate differential systems of Gaussian type using the Riemann-Hilbert correspondence (3.31). In the following chapter we will explore an alternative approach: The perspective of Stokes shells. Upon defining Stokes shells of Gaussian type in detail, we will find an equivalence between Stokes shells and Stokes data of a certain type C.

4 Stokes shells of Gaussian type

In this chapter we introduce the notion of Stokes shells of Gaussian type, which provides another approach to describe Stokes structures. For this purpose, we specify the content of chapter 4 in [5] to fit our context.

4.1 Stokes tuples of vector spaces

Our initial move towards defining the category of Stokes Shells involves introducing Stokes tuples of vector spaces and establishing a connection with chapter three. We will discover that every graded Stokes filtered local system corresponds to a Stokes tuple of vector spaces.

Notation 4.1. Let $C \subseteq \mathbb{C}$ be a non-empty, finite subset and $C^\times := C \setminus \{0\}$. We set

$$S_0(C) := \bigcup_{c \in C^\times} \mathrm{St}(0, c) \quad \text{and} \quad T(C) := \bigcup_{c \in C^\times} T(0, c),$$

where $T(0, c)$ is the set of connected components of $S^1 \setminus \mathrm{St}(0, c)$ as explained in (3.11).

On $\mathbb{C}$ we define an equivalence relation setting $c_1 \sim c_2$ for $c_1, c_2 \in \mathbb{C}$ if there exists a $\lambda \in \mathbb{R}_{>0}$ with $c_1 = \lambda c_2$. For a subset $C \subseteq \mathbb{C}$ we define $[C]$ to be the quotient of C with respect to $\sim$ and $-C := \{-c \mid c \in C\} \subseteq \mathbb{C}$.

Assumption 4.2. In this study, our focus is solely on the non-aligned scenario, meaning we postulate that $C \cong [C]$. Moreover we suppose $C = C^\times$. To simplify matters, we can assume without loss of generality that $C = -C$ by including absent elements if needed.

Notation 4.3. Let $C \subseteq \mathbb{C}$ be a non-empty finite subset as in (4.2). We apply the notation of [5] to our context and get:

- $\mathcal{A}(C) := \coprod_{c \in [C^\times]} \{(c, I) \mid I \in T(0, c)\}$,
- for any open interval $I = (\vartheta_0, \vartheta_1) \subseteq \mathbb{R}/2\pi\mathbb{Z}$ we set $\overline{I} := [\vartheta_0, \vartheta_1] \subseteq \mathbb{R}/2\pi\mathbb{Z}$ and
- for any open interval $I = (\vartheta_0, \vartheta_1) \subseteq \mathbb{R}/2\pi\mathbb{Z}$ and any $r \in \mathbb{R}$ we set $I + r = (\vartheta_0 + r, \vartheta_1 + r) \subseteq \mathbb{R}/2\pi\mathbb{Z}$.

Now we can come to the definition of Stokes tuples of **k**-vector spaces.

Definition 4.4. Let $C \subseteq \mathbb{C}$ be a non-empty, finite subset as in (4.2). A *Stokes tuple of* $\mathbf{k}$*-vector spaces of Gaussian type* over C is a tuple $(\boldsymbol{K}, \boldsymbol{\Phi}, \boldsymbol{\Psi})$ where

- $\boldsymbol{K} = (K_{c,I})_{(c,I)\in\mathcal{A}(C)}$ is a tuple of $\mathbf{k}$-vector spaces and
- $\boldsymbol{\Phi} = (\Phi_c^{I+\frac{\pi}{2},I} : K_{c,I} \to K_{c,I+\frac{\pi}{2}})_{(c,I)\in\mathcal{A}(C)}$, $\boldsymbol{\Psi} = (\Psi_{c,I} : K_{c,I} \to K_{c,I})_{(c,I)\in\mathcal{A}(C)}$ denote tuples of isomorphisms so that the diagram

$$\begin{tikzcd} K_{c,I} \arrow[r, "\Psi_{c,I}"] \arrow[d, "\Phi_c^{I+\frac{\pi}{2},I}"'] & K_{c,I} \arrow[d, "\Phi_c^{I+\frac{\pi}{2},I}"] \\ K_{c,I+\frac{\pi}{2}} \arrow[r, "\Psi_{c,I+\frac{\pi}{2}}"'] & K_{c,I+\frac{\pi}{2}} \end{tikzcd}$$

  commutes for any $(c, I) \in \mathcal{A}(C)$.

Remark. From lemma (3.10) we get that if (c, I) is an element in $\mathcal{A}(C)$, then also $(c, I + \frac{\pi}{2}) \in \mathcal{A}(C)$ since $I \in T(0, c)$ takes the form $(\vartheta, \vartheta + \frac{\pi}{2})$ for some $\vartheta \in \mathrm{St}(0, c)$ and if $\vartheta \in \mathrm{St}(0, c)$, then also $\vartheta + \nu\frac{\pi}{2} \mod 2\pi \in \mathrm{St}(0, c)$ for $\nu \in \mathbb{Z}/4\mathbb{Z}$. Thus, $\boldsymbol{\Phi}$ in (4.4) is well-defined.

Next, we will observe that every graded Stokes filtered local system can be viewed as a Stokes tuple of $\mathbf{k}$-vector spaces.

Lemma 4.5. *Let $C \subseteq \mathbb{C}$ be a non-empty, finite subset as in (4.2). Then every graded Stokes filtered local system of Gaussian type* $(\mathrm{gr}\,\mathscr{L}, \mathrm{gr}\,\mathscr{L}_{\leq\bullet})$ *induces a Stokes tuple of* $\mathbf{k}$*-vector spaces. We denote the associated Stokes tuple by* $\mathfrak{D}((\mathrm{gr}\,\mathscr{L}, \mathrm{gr}\,\mathscr{L}_{\leq\bullet}))$.

Proof. Let $C \subseteq \mathbb{C}$ be a non-empty, finite subset as in (4.2) and $(\mathrm{gr}\,\mathscr{L}, \mathrm{gr}\,\mathscr{L}_{\leq\bullet})$ be a graded Stokes filtered local System of Gaussian type C. Thus for each $c \in C$ there is a local system $\mathrm{gr}_c\,\mathscr{L}$ on S^1 so that

$$\mathrm{gr}\,\mathscr{L} = \bigoplus_{c\in C} \mathrm{gr}_c\,\mathscr{L}$$

and for each $c' \in \mathbb{C}$ the subsheaf $\mathrm{gr}\,\mathscr{L}_{\leq c'}$ of $\mathrm{gr}\,\mathscr{L}$ is given by

$$\mathrm{gr}\,\mathscr{L}_{\leq c'} = \bigoplus_{c\in C} \beta_{c\leq c'}(\mathrm{gr}_c\,\mathscr{L}).$$

For each $c \in C$, an interval $I \in T(0, c)$ is of the form $(\vartheta_\nu, \vartheta_{\nu+1})$ with $\vartheta_\nu \in \mathrm{St}(0, c)$ and $\vartheta_{\nu+1} = \vartheta_\nu + \frac{\pi}{2}$. We set $I^{(\nu)} := (\vartheta_\nu, \vartheta_{\nu+1})$.

For each $(c, I^{(\nu)}) \in \mathcal{A}(C)$ we get a $\mathbf{k}$-vector space

$$K_{c,I^{(\nu)}} := H^0\left(\overline{I^{(\nu)}}, \iota^{-1}(\mathrm{gr}_c \mathscr{L})\right) \cong \iota^{-1}(\mathrm{gr}_c \mathscr{L})(\overline{I^{(\nu)}}),$$

where ι^{-1} is the inverse image functor of the inclusion $\iota : \overline{I^{(\nu)}} \hookrightarrow S^1$. We abbreviate $\iota^{-1}(\mathrm{gr}_c \mathscr{L})(\overline{I^{(\nu)}})$ with $\mathrm{gr}_c \mathscr{L}(\overline{I^{(\nu)}})$.
Since $\mathrm{gr}_c \mathscr{L}$ is a local system on S^1 and $\overline{I^{(\nu)}}$ is simply connected, we get $\mathrm{gr}_c \mathscr{L}(\overline{I^{(\nu)}}) \cong \mathrm{gr}_c \mathscr{L}_{\vartheta^{(\nu+1)}} \cong \mathrm{gr}_c \mathscr{L}(\overline{I^{(\nu+1)}})$, hence for each $(c, I^{(\nu)}), (c, I^{(\nu+1)}) \in \mathcal{A}(C)$ we obtain an isomorphism

$$\Phi_c^{(\nu+1,\nu)} := \Phi_c^{I^{(\nu)}+\frac{\pi}{2}, I^{(\nu)}} : K_{c,I^{(\nu)}} \to K_{c,I^{(\nu+1)}}.$$

Moreover for each $(c, I^{(\nu)}) \in \mathcal{A}(C)$ we define $\Psi_{c,I^{(\nu)}}$ to be the identity on $K_{c,I^{(\nu)}}$:

$$\Psi_{c,I^{(\nu)}} := \mathrm{id}_{K_{c,I^{(\nu)}}} \in \mathrm{Hom}_{\mathbf{Vect_k}}(K_{c,I^{(\nu)}}, K_{c,I^{(\nu)}}).$$

Then $(\boldsymbol{K}, \boldsymbol{\Phi}, \boldsymbol{\Psi})$ given by the tuples $\boldsymbol{K} := (K_{c,I})_{(c,I)\in\mathcal{A}(C)}$, $\boldsymbol{\Phi} := (\Phi_c^{\nu+1,\nu})_{(c,I^{(\nu)})\in\mathcal{A}(C)}$ and $\boldsymbol{\Psi} := (\Psi_{c,I})_{(c,I)\in\mathcal{A}(C)}$ is a Stokes tuple of $\mathbf{k}$-vector spaces of Gaussian type.
We set $\mathfrak{D}((\mathrm{gr}\,\mathscr{L}, \mathrm{gr}\,\mathscr{L}_{\leq\bullet})) := (\boldsymbol{K}, \boldsymbol{\Phi}, \boldsymbol{\Psi})$. □

4.2 Stokes shells

Having learned about Stokes tuples of $\mathbf{k}$-vector spaces we can take the next step and give the definition of a deformation datum of a graded Stokes filtered local system that is of Gaussian type C. This is essential to finally get to the formal definition of a Stokes shell. In order to do so, we have to give some more notation.

Let $C \subseteq \mathbb{C}$ be a non-empty, finite subset. For an element $I \in T(C)$ we define

- $C_{I,<0} := \{c \in C \mid c <_\vartheta 0 \text{ for all } \vartheta \in I\}$ and
- $C_{I,>0} := \{c \in C \mid 0 <_\vartheta c \text{ for all } \vartheta \in I\}$.

Lemma 4.6. *Let $C \subseteq \mathbb{C}$ be a non-empty, finite subset with $[C] = [-C]$. Then for each $I \in T(C)$ the sets $[C_{I,<0}]$ and $[C_{I,>0}]$ consist of exactly one element.*

Proof. Let $I \in T(C)$ be any interval. First we prove that both $[C_{I,<0}]$ and $[C_{I,>0}]$ are non-empty. Since I belongs to $T(C)$, it takes the form $(\vartheta, \vartheta + \frac{\pi}{2})$, where ϑ is in $\mathrm{St}(0, c)$ for some element $c \in C^\times$. From (3.11) we get that either $c \in C_{I,<0}$ or $c \in C_{I,>0}$. Given that $[C] = [-C]$, there exists a $d \in C$ and $\lambda \in \mathbb{R}_{>0}$ with $-c = \lambda d$. Thus $[-c] = [d]$. Now if $c \in C_{I,<0}$, then $0 <_\vartheta -c$ for all $\vartheta \in I$ and consequently $[d] \in [C_{I,>0}]$ and $[c] \in [C_{I,<0}]$. Analogously if $c \in C_{I,>0}$, then $[d] \in [C_{I,<0}]$ and $[c] \in [C_{I,>0}]$. Hence both sets $[C_{I,>0}]$ and $[C_{I,<0}]$ contain at least one element.

Now consider $c_1, c_2 \in C_{I,>0}$ with $\arg(c_1), \arg(c_2) \in [0, 2\pi)$. Then the Stokes directions of the pairs $(0, c_1)$ and $(0, c_2)$ have to be the same as I has length $\frac{\pi}{2}$. Thus $\mathrm{St}(0, c_1) = \mathrm{St}(0, c_2)$, which can be expressed as

$$\left\{\frac{\pi + 2\arg(c_1)}{4} + \mathbb{Z} \cdot \frac{\pi}{2} \mod 2\pi\right\} = \left\{\frac{\pi + 2\arg(c_2)}{4} + \mathbb{Z} \cdot \frac{\pi}{2} \mod 2\pi\right\}$$

using lemma (3.10). As a consequence, we get either $\arg(c_1) = \arg(c_2)$ or we get $\arg(c_1) = \arg(c_2) + \pi \mod 2\pi$. Since $c_1 \in C_{I,>0}$, for any direction $\vartheta \in I$ we have $0 <_\vartheta c_1$. By definition, this implies $\arg(c_1) - 2\vartheta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \mod 2\pi$.
If $\arg(c_1) = \arg(c_2) + \pi \mod 2\pi$, then $\arg(c_2) + \pi - 2\vartheta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \mod 2\pi$. Consequently $\arg(c_2) - 2\vartheta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \mod 2\pi$, which implies that $c_2 <_\vartheta 0$. Thus we have $c_2 \notin C_{I,>0}$ contradicting the assumption that $c_2 \in C_{I,>0}$. Therefore $\arg(c_1) = \arg(c_2)$ and as a result $[c_1] = [c_2]$. The proof that $[C_{I,<0}]$ contains at most one element follows a similar logic. $\square$

Notation 4.7. Again we apply the notation of [5] to the Gaussian case and a subset $C \subseteq \mathbb{C}$ as in (4.2). We therefore set

- $T_2(C) \coloneqq \{(I_1, I_2) \mid I_1, I_2 \in T(C), I_1 \cap I_2 \neq \varnothing, I_1 \neq I_2\}$,
- for any $I \in T(C)$, c_+^I is the element in C such that $\{[c_+^I]\} = [C_{I,>0}]$ and c_-^I is the element in C such that $\{[c_-^I]\} = [C_{I,<0}]$ and
- $\mathcal{B}_2(C) \coloneqq \{(c_+^I, c_-^I; I) \mid I \in T(C)\}$.

Remark. Since we only consider subsets $C \subseteq \mathbb{C}$ that hold the properties of (4.2), in particular $C \cong [C]$, the elements c_+^I, c_-^I in (4.7) are well-defined.

Now we have the required notation to define the deformation datum.

Definition 4.8. Let $(\mathrm{gr}\,\mathscr{L}, \mathrm{gr}\,\mathscr{L}_{\leq\bullet})$ be a graded Stokes filtered local system of Gaussian type C and $\mathfrak{D}((\mathrm{gr}\,\mathscr{L}, \mathrm{gr}\,\mathscr{L}_{\leq\bullet})) = (\boldsymbol{K}, \boldsymbol{\Phi}, \boldsymbol{\Psi})$ the corresponding Stokes tuple of $\mathbf{k}$-vector spaces of Gaussian type. A *deformation datum* of $(\mathrm{gr}\,\mathscr{L}, \mathrm{gr}\,\mathscr{L}_{\leq\bullet})$ is a tuple of morphisms $\boldsymbol{R}$ containing

- a morphism $\mathcal{R}_{I_2}^{I_1} : K_{c_+^{I_1}, I_1} \longrightarrow K_{c_-^{I_2}, I_2}$ for each pair $(I_1, I_2) \in T_2(C)$ and
- a morphism $\mathcal{R}_{c_2, I_+}^{c_1, I_-} : K_{c_1, I} \longrightarrow K_{c_2, I}$ for each tuple $(c_1, c_2; I) \in \mathcal{B}_2(C)$.

And finally we can define the category of Stokes shells.

Definition 4.9. Let $C \subseteq \mathbb{C}^\times$ be a non-empty, finite subset with $C = -C$ and that only includes non-aligned elements, i.e. $C \cong [C]$. A *Stokes shell of Gaussian type* C is a graded Stokes filtered local system of Gaussian type C $(\operatorname{gr}\mathscr{L}, \operatorname{gr}\mathscr{L}_{\leq\bullet})$ together with a deformation datum $\boldsymbol{R}$ of $(\operatorname{gr}\mathscr{L}, \operatorname{gr}\mathscr{L}_{\leq\bullet})$. We denote it by $\boldsymbol{Sh} = ((\operatorname{gr}\mathscr{L}, \operatorname{gr}\mathscr{L}_{\leq\bullet}), \boldsymbol{R})$.
Let $(\boldsymbol{Sh}_i)_{i\in\{1,2\}} = (((\operatorname{gr}\mathscr{L}_i, \operatorname{gr}\mathscr{L}_{i,\leq\bullet}), \boldsymbol{R}_i))_{i\in\{1,2\}}$ be two Stokes shells of Gaussian type C. A morphism between two Stokes shells $\lambda : \boldsymbol{Sh}_1 \to \boldsymbol{Sh}_2$ is a morphism $\lambda \in \operatorname{Hom}_{\mathbf{Loc}^{\mathrm{gr}}_{\mathrm{St}}(C)}((\operatorname{gr}\mathscr{L}_1, \operatorname{gr}\mathscr{L}_{1,\leq\bullet}), (\operatorname{gr}\mathscr{L}_2, \operatorname{gr}\mathscr{L}_{2,\leq\bullet}))$ that is compatible with the deformation datum, meaning

$$\lambda \circ (\mathcal{R}_1)^{I_2}_{I_1} = (\mathcal{R}_2)^{I_2}_{I_1} \circ \lambda \qquad \text{for all } (I_1, I_2) \in T_2(C),$$
$$\lambda \circ (\mathcal{R}_1)^{c_1,I_-}_{c_2,I_+} = (\mathcal{R}_2)^{c_1,I_-}_{c_2,I_+} \circ \lambda \qquad \text{for all } (c_1, c_2; I) \in \mathcal{B}_2(C).$$

The category of Stokes shells of Gaussian type C is denoted by $\mathfrak{S}\mathfrak{H}(C)$.

Remark. We shortly want to explain how

$$\lambda \circ (\mathcal{R}_1)^{I_2}_{I_1} = (\mathcal{R}_2)^{I_2}_{I_1} \circ \lambda \qquad \text{for all } (I_1, I_2) \in T_2(C),$$
$$\lambda \circ (\mathcal{R}_1)^{c_1,I_-}_{c_2,I_+} = (\mathcal{R}_2)^{c_1,I_-}_{c_2,I_+} \circ \lambda \qquad \text{for all } (c_1, c_2; I) \in \mathcal{B}_2(C)$$

in definition (4.9) can be understood. Since λ is a graded morphism of local systems on S^1, for any interval $I \subseteq S^1$ we get $\iota^{-1}(\lambda)(\overline{I}) : \iota^{-1}(\operatorname{gr}\mathscr{L})(\overline{I}) \to \iota^{-1}(\operatorname{gr}\tilde{\mathscr{L}})(\overline{I})$ where ι^{-1} is the inverse image functor of the embedding $\iota : \overline{I} \hookrightarrow S^1$. Again we abbreviate $\iota^{-1}(\lambda)(\overline{I})$ with $\lambda(\overline{I})$. Then for $(I_1, I_2) \in T_2(C)$ we have to check that the diagram

$$\begin{tikzcd}
\operatorname{gr}_{c_+^{I_1}} \mathscr{L}(\overline{I_1}) \arrow[r, "\lambda(\overline{I_1})"] \arrow[d, "(\mathcal{R}_1)^{I_1}_{I_2}"'] & \operatorname{gr}_{c_+^{I_1}} \tilde{\mathscr{L}}(\overline{I_1}) \arrow[d, "(\mathcal{R}_2)^{I_1}_{I_2}"] \\
\operatorname{gr}_{c_-^{I_2}} \mathscr{L}(\overline{I_2}) \arrow[r, "\lambda(\overline{I_2})"'] & \operatorname{gr}_{c_-^{I_2}} \tilde{\mathscr{L}}(\overline{I_2})
\end{tikzcd}$$

commutes. Remark that $\lambda(\overline{I_1})$ and $\lambda(\overline{I_2})$ in the diagram are well-defined as λ is graded. Furthermore for $(c_1, c_2; I) \in \mathcal{B}_2(C)$ we have to prove that

$$\begin{tikzcd}
\operatorname{gr}_{c_1} \mathscr{L}(\overline{I}) \arrow[r, "\lambda(\overline{I})"] \arrow[d, "(\mathcal{R}_1)^{c_1,I_-}_{c_2,I_+}"'] & \operatorname{gr}_{c_1} \tilde{\mathscr{L}}(\overline{I}) \arrow[d, "(\mathcal{R}_2)^{c_1,I_-}_{c_2,I_+}"] \\
\operatorname{gr}_{c_2} \mathscr{L}(\overline{I}) \arrow[r, "\lambda(\overline{I})"'] & \operatorname{gr}_{c_2} \tilde{\mathscr{L}}(\overline{I})
\end{tikzcd}$$

is commutative. Again, since λ is graded, $\lambda(\overline{I})$ as given in the diagram is well-defined.

Remark. If C is not equal to $-C$, set $\tilde{C} := C \cup -C$. Then, the category $\mathfrak{S}\mathfrak{H}(C)$ can be identified as the full subcategory of $\mathfrak{S}\mathfrak{H}(\tilde{C})$, where the objects are tuples of the form $((\operatorname{gr}\mathscr{L}, \operatorname{gr}\mathscr{L}_{\leq\bullet}), \boldsymbol{R})$ and $(\operatorname{gr}\mathscr{L}, \operatorname{gr}\mathscr{L}_{\leq\bullet})$ is a graded Stokes filtered local system of Gaussian type C.

After having defined the category of Stokes shells $\mathfrak{S}\mathfrak{H}(C)$, we want to establish an equivalence to the category of Stokes data in the following chapter.

5 The correspondence between Stokes data and Stokes shells of Gaussian type

In this chapter, our aim is to establish a correspondence between the category of Stokes shells and the category of Stokes data for differential systems of Gaussian type $\{r, z\} \subseteq \mathbb{C}$, where $r \in \mathbb{R}_{>0}$ and $z \in \mathbb{C} \smallsetminus \mathbb{R}$. Particularly, considering a direction $\vartheta_0 \in S^1$ that is generic with respect to $C \coloneqq \{r, z\}$, we will prove that there exists an equivalence of categories between $\mathfrak{SD}(C, \vartheta_0)$ and $\mathfrak{SH}(C)$. Moreover, we will give a precise description of the involved functors.

5.1 From shells to data

Let $\boldsymbol{Sh} = ((\operatorname{gr}\mathscr{L}, \operatorname{gr}\mathscr{L}_{\leq\bullet}), \boldsymbol{R})$ be a Stokes shell of Gaussian type $C \coloneqq \{r, z\}$ with elements $r \in \mathbb{R}_{>0}$ and $z \in \mathbb{C} \smallsetminus \mathbb{R}$. Since $C \neq -C$ we add $-r, -z$ to C and consider $(\operatorname{gr}\mathscr{L}, \operatorname{gr}\mathscr{L}_{\leq\bullet})$ of type $\tilde{C} \coloneqq \{r, -r, z, -z\}$. Remark that $\operatorname{gr}_{-r}\mathscr{L} = 0$ and $\operatorname{gr}_{-z}\mathscr{L} = 0$ because the set of exponential factors of $(\operatorname{gr}\mathscr{L}, \operatorname{gr}\mathscr{L}_{\leq\bullet})$ is a subset of C.
We start by specifying the notations from the previous chapter to apply to this particular case. For improved readability we use the following notation:

Notation 5.1. We set

- $\vartheta_r^{(\nu)} \coloneqq \frac{\pi}{4} + \nu\frac{\pi}{2}$ for $\nu \in \mathbb{Z}/4\mathbb{Z}$,
- $\vartheta_z^{(\nu)} \coloneqq \frac{\pi + 2\arg(z)}{4} + \nu\frac{\pi}{2}$ for $\nu \in \mathbb{Z}/4\mathbb{Z}$ and
- $I_c^{(\nu)} \coloneqq (\vartheta_c^{(\nu)}, \vartheta_c^{(\nu+1)}) \subseteq \mathbb{R}/2\pi\mathbb{Z}$ for $\nu \in \mathbb{Z}/4\mathbb{Z}$ and $c \in \{r, z\}$.

Given that $\operatorname{St}(0, c) = \operatorname{St}(0, -c)$ for all $c \in \mathbb{C}$, we obtain

- $S_0(\tilde{C}) = \operatorname{St}(0, r) \cup \operatorname{St}(0, z) = \left\{\vartheta_r^{(\nu)} \mid \nu \in \mathbb{Z}/4\mathbb{Z}\right\} \cup \left\{\vartheta_z^{(\nu)} \mid \nu \in \mathbb{Z}/4\mathbb{Z}\right\}$.
- $T(\tilde{C}) = \left\{I_r^{(\nu)} \mid \nu \in \mathbb{Z}/4\mathbb{Z}\right\} \cup \left\{I_z^{(\nu)} \mid \nu \in \mathbb{Z}/4\mathbb{Z}\right\}$ and
- $\mathcal{A}(\tilde{C}) = \bigcup_{\nu \in \mathbb{Z}/4\mathbb{Z}} \left\{(r, I_r^{(\nu)}), (-r, I_r^{(\nu)}), (z, I_z^{(\nu)}), (-z, I_z^{(\nu)})\right\}$.

To describe the deformation datum of a Stokes Shell of Gaussian type, we have to describe $\mathcal{B}_2(\tilde{C})$ and $T_2(\tilde{C})$. Thus we first need to specify c_+^I and c_-^I for any interval $I \in T(\tilde{C})$.

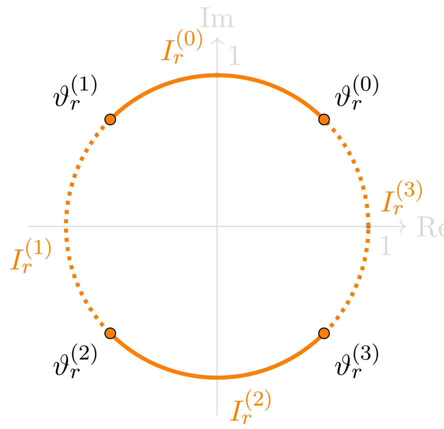

Figure 5.1: The set $\mathrm{St}(0, r)$ with open intervals $I_r^{(\nu)}$.

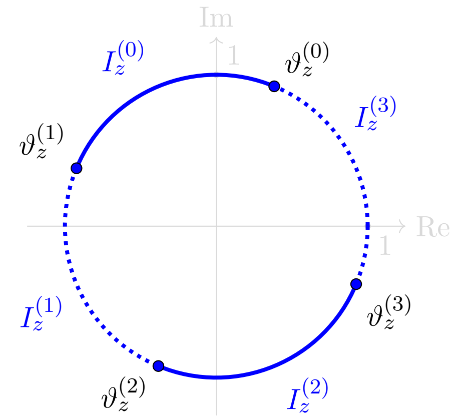

Figure 5.2: The set $\mathrm{St}(0, z)$ for $\arg(z) = \frac{\pi}{4}$ with open intervals $I_z^{(\nu)}$.

- For $c \in \{r, z\}$ and for $I \in \{I_c^{(0)}, I_c^{(2)}\}$ we get $c_+^I = -c, c_-^I = c$.
- For $c \in \{r, z\}$ and for $I \in \{I_c^{(1)}, I_c^{(3)}\}$ we get $c_+^I = c, c_-^I = -c$.

This yields to

- $\mathcal{B}_2(\tilde{C}) = \bigcup_{c \in \{r,z\}} \Big\{(-c, c; I_c^{(0)}), (-c, c; I_c^{(2)}), (c, -c; I_c^{(1)}), (c, -c; I_c^{(3)})\Big\}$.

Describing $T_2(\tilde{C})$ is slightly more complicated, since the intervals $I_z^{(\nu)}$ depend on $\arg(z)$. That is why we first have to make the following observation:

Lemma 5.2. *Let $C = \{r, z\}$ be a set with $r \in \mathbb{R}_{>0}$ and $z \in \mathbb{C} \smallsetminus \mathbb{R}$. Then there is exactly one Stokes direction $\vartheta_z^{(\nu_o)} \in \mathrm{St}(0, z)$ with $\vartheta_z^{(\nu_o)} \mod 2\pi \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$. More precisely, if $\mathrm{Im}(z) > 0$, then $\nu_o = 0$, if $\mathrm{Im}(z) < 0$, then $\nu_o = 3$.*

Proof. Let $z \in \mathbb{C} \smallsetminus \mathbb{R}$ be a complex number with $\mathrm{Im}(z) \neq 0$. If $\mathrm{Im}(z) > 0$, we have $\arg(z) \in (0, \pi)$. Then

$$\frac{\arg(z)}{2} \in \left(0, \frac{\pi}{2}\right) \Leftrightarrow \vartheta_z^{(0)} = \frac{\pi}{4} + \frac{\arg(z)}{2} \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right) = (\vartheta_r^{(0)}, \vartheta_r^{(1)}).$$

If $\mathrm{Im}(z) < 0$, we have $\arg(z) \in (\pi, 2\pi)$. Then

$$\begin{aligned}\frac{\arg(z)}{2} \in \left(\frac{\pi}{2}, \pi\right) &\Leftrightarrow \frac{\pi}{4} + \frac{\arg(z)}{2} \in \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right) \\ &\Leftrightarrow \vartheta_z^{(3)} = \frac{\pi}{4} + \frac{\arg(z)}{2} + \frac{3\pi}{2} \in \left(\frac{9\pi}{4}, \frac{11\pi}{4}\right) \\ &\Rightarrow \vartheta_z^{(3)} \mod 2\pi \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right) = (\vartheta_r^{(0)}, \vartheta_r^{(1)}).\end{aligned}$$

In both cases there is only one Stokes direction $\vartheta_z^{(\nu_o)}$ with $\vartheta_z^{(\nu_o)} \mod 2\pi \in (\vartheta_r^{(0)}, \vartheta_r^{(1)})$, since the elements in $\mathrm{St}(0,z)$ differ by a multiple of $\frac{\pi}{2}$ and $(\vartheta_r^{(0)}, \vartheta_r^{(1)})$ is an open interval of length $\frac{\pi}{2}$. □

Using (5.2) we can describe $T_2(\tilde{C})$.

1. If $\mathrm{Im}(z) > 0$, then $I_r^{(\nu)}$ has non-empty intersection with $I_z^{(\nu)}$ and with $I_z^{(\nu+3)}$, because $\vartheta_z^{(\nu)} \in I_r^{(\nu)}$. Moreover $I_r^{(\nu)} \cap I_z^{(\nu+1)} = \varnothing$ and $I_r^{(\nu)} \cap I_z^{(\nu+2)} = \varnothing$. This results in
$$T_2(\tilde{C}) = \bigcup_{\nu \in \mathbb{Z}/4\mathbb{Z}} \left\{ (I_r^{(\nu)}, I_z^{(\nu)}), (I_z^{(\nu)}, I_r^{(\nu)}), (I_r^{(\nu)}, I_z^{(\nu+3)}), (I_z^{(\nu+3)}, I_r^{(\nu)}) \right\}.$$

2. Analogously if $\mathrm{Im}(z) < 0$ we get $I_r^{(\nu)} \cap I_z^{(\nu+2)} \neq \varnothing, I_r^{(\nu)} \cap I_z^{(\nu+3)} \neq \varnothing$ and $I_r^{(\nu)} \cap I_z^{(\nu)} = \varnothing, I_r^{(\nu)} \cap I_z^{(\nu+1)} = \varnothing$, because $\vartheta_z^{(\nu+3)} \in I_r^{(\nu)}$. This results in
$$T_2(\tilde{C}) = \bigcup_{\nu \in \mathbb{Z}/4\mathbb{Z}} \left\{ (I_r^{(\nu)}, I_z^{(\nu+2)}), (I_z^{(\nu+2)}, I_r^{(\nu)}), (I_r^{(\nu)}, I_z^{(\nu+3)}), (I_z^{(\nu+3)}, I_r^{(\nu)}) \right\}.$$

In order to describe the deformation datum $\boldsymbol{R}$, it is essential to outline the Stokes tuple of $\mathbf{k}$-vector spaces $\mathfrak{D}((\mathrm{gr}\,\mathscr{L}, \mathrm{gr}\,\mathscr{L}_{\leq \bullet}))$ that is induced by $(\mathrm{gr}\,\mathscr{L}, \mathrm{gr}\,\mathscr{L}_{\leq \bullet})$.
As we saw in the previous chapter (lemma 4.5), $\mathfrak{D}((\mathrm{gr}\,\mathscr{L}, \mathrm{gr}\,\mathscr{L}_{\leq \bullet})) = (\boldsymbol{K}, \boldsymbol{\Phi}, \boldsymbol{\Psi})$ with $\boldsymbol{K} := (K_{c, I_c^{(\nu)}})_{(c, I_c^{(\nu)}) \in \mathcal{A}(\tilde{C})}$, $\boldsymbol{\Phi} := (\Phi_c^{(\nu+1,\nu)})_{(c, I_c^{(\nu)}) \in \mathcal{A}(\tilde{C})}$ and $\boldsymbol{\Psi} := (\Psi_{c, I_c^{(\nu)}})_{(c, I_c^{(\nu)}) \in \mathcal{A}(\tilde{C})}$ where

- $K_{c, I_c^{(\nu)}} := \mathrm{gr}_c\,\mathscr{L}(\overline{I_c^{(\nu)}})$,
- $\Phi_c^{(\nu+1,\nu)} : K_{c, I_c^{(\nu)}} \to K_{c, I_c^{(\nu+1)}}$ is given by $\mathrm{gr}_c\,\mathscr{L}(\overline{I_c^{(\nu)}}) \cong \mathrm{gr}_c\,\mathscr{L}_{\vartheta_c^{(\nu+1)}} \cong \mathrm{gr}_c\,\mathscr{L}(\overline{I_c^{(\nu+1)}})$
- and $\Psi_{c, I_c^{(\nu)}} = \mathrm{id}_{K_{c, I_c^{(\nu)}}} : K_{c, I_c^{(\nu)}} \to K_{c, I_c^{(\nu)}}$.

Given that $(\mathrm{gr}\,\mathscr{L}, \mathrm{gr}\,\mathscr{L}_{\leq \bullet})$ is of type $C = \{r, z\}$ for $c \in \{-r, -z\}$ and $\nu \in \mathbb{Z}/4\mathbb{Z}$, we obtain $K_{c, I_c^{(\nu)}} = \mathrm{gr}_c\,\mathscr{L}(\overline{I_c^{(\nu)}}) = 0$.
By definition (4.8), $\boldsymbol{R}$ consists of

- a morphism $\mathcal{R}_{I_1}^{I_2} : K_{c_+^{I_1}, I_1} \to K_{c_-^{I_2}, I_2}$ for each pair $(I_1, I_2) \in T_2(\tilde{C})$ and
- a morphism $\mathcal{R}_{c_2, I_+}^{c_1, I_-} : K_{c_1, I} \to K_{c_2, I}$ for each tuple $(c_1, c_2; I) \in \mathcal{B}_2(\tilde{C})$.

First we notice, that for any given tuple $(c_1, c_2; I) \in \mathcal{B}_2(\tilde{C})$ either $c_1 \in \{-r, -z\}$ or $c_2 \in \{-r, -z\}$ and therefore $K_{c_1,I} = 0$ or $K_{c_2,I} = 0$, so $\mathcal{R}^{c_1,I_-}_{c_2,I_+}$ is the zero morphism. In order to specify $\mathcal{R}^{I_2}_{I_1}$ for a given pair $(I_1, I_2) \in T_2(\tilde{C})$ notice that if $I_1 \in \{I_c^{(0)}, I_c^{(2)}\}$ or $I_2 \in \{I_c^{(1)}, I_c^{(3)}\}$ for some $c \in C$ the morphism $\mathcal{R}^{I_2}_{I_1}$ is the zero morphism as $c_-^{I_c^{(1)}} = c_-^{I_c^{(3)}} = -c$, and $c_+^{I_c^{(0)}} = c_+^{I_c^{(2)}} = -c$.
Consequently, the deformation datum $\boldsymbol{R}$ is given by four linear morphisms:

1. $\mathcal{R}_0 \coloneqq \mathcal{R}^{I_r^{(0)}}_{I_z^{(3)}} : K_{z,I_z^{(3)}} \to K_{r,I_r^{(0)}}$,
2. $\mathcal{R}_1 \coloneqq \mathcal{R}^{I_z^{(0)}}_{I_r^{(1)}} : K_{r,I_r^{(1)}} \to K_{z,I_z^{(0)}}$,
3. $\mathcal{R}_2 \coloneqq \mathcal{R}^{I_r^{(2)}}_{I_z^{(1)}} : K_{z,I_z^{(1)}} \to K_{r,I_r^{(2)}}$,
4. $\mathcal{R}_3 \coloneqq \mathcal{R}^{I_z^{(2)}}_{I_r^{(3)}} : K_{r,I_r^{(3)}} \to K_{z,I_z^{(2)}}$.

Having seen that in our context $\boldsymbol{R}$ can be described by four morphisms, we can take the next step and show that each Stokes shell of Gaussian type C can be viewed as an object in $\mathfrak{SD}(C, \vartheta_0)$.

Lemma 5.3. *Consider $C = \{r, z\}$ with $r \in \mathbb{R}_{>0}$, $z \in \mathbb{C} \smallsetminus \mathbb{R}$ and let $\vartheta_0 \in S^1$ be a generic direction with respect to C. A Stokes shell of Gaussian type C induces an object in $\mathfrak{SD}(C, \vartheta_0)$.*

Proof. Let $\boldsymbol{Sh} = ((\operatorname{gr}\mathscr{L}, \operatorname{gr}\mathscr{L}_{\leq\bullet}), \boldsymbol{R})$ be a Stokes shell of Gaussian type C, and $\mathfrak{D}((\operatorname{gr}\mathscr{L}, \operatorname{gr}\mathscr{L}_{\leq\bullet})) = (\boldsymbol{K}, \boldsymbol{\Phi}, \boldsymbol{\Psi})$. Then $\boldsymbol{R}$ is given by

1. $\mathcal{R}_0 \coloneqq \mathcal{R}^{I_r^{(0)}}_{I_z^{(3)}} : K_{z,I_z^{(3)}} \to K_{r,I_r^{(0)}}$,
2. $\mathcal{R}_1 \coloneqq \mathcal{R}^{I_z^{(0)}}_{I_r^{(1)}} : K_{r,I_r^{(1)}} \to K_{z,I_z^{(0)}}$,
3. $\mathcal{R}_2 \coloneqq \mathcal{R}^{I_r^{(2)}}_{I_z^{(1)}} : K_{z,I_z^{(1)}} \to K_{r,I_r^{(2)}}$,
4. $\mathcal{R}_3 \coloneqq \mathcal{R}^{I_z^{(2)}}_{I_r^{(3)}} : K_{r,I_r^{(3)}} \to K_{z,I_z^{(2)}}$.

Set $G_r^{(\nu)} \coloneqq K_{r,I_r^{(\nu)}} = \operatorname{gr}_r \mathscr{L}(\overline{I_r^{(\nu)}})$ and $G_z^{(\nu)} \coloneqq K_{z,I_z^{(\nu+3)}} = \operatorname{gr}_z \mathscr{L}(\overline{I_z^{(\nu+3)}})$ for each $\nu \in \mathbb{Z}/4\mathbb{Z}$.
Since ϑ_0 is generic with respect to C, we either have $r \leq_{\vartheta_0} z$ or $z \leq_{\vartheta_0} r$. So we distinguish between the two cases:

1. Consider the case $r \leq_{\vartheta_0} z$, so the unique numbering of C given by ϑ_0 is $c_1 = r$, $c_2 = z$. Set

$$S^{(1,0)} \coloneqq \begin{pmatrix} \Phi_r^{(1,0)} & 0 \\ \mathcal{R}_1 \circ \Phi_r^{(1,0)} & \Phi_z^{(0,3)} \end{pmatrix}, \quad S^{(2,1)} \coloneqq \begin{pmatrix} \Phi_r^{(2,1)} & \mathcal{R}_2 \circ \Phi_z^{(1,0)} \\ 0 & \Phi_z^{(1,0)} \end{pmatrix},$$
$$S^{(3,2)} \coloneqq \begin{pmatrix} \Phi_r^{(3,2)} & 0 \\ \mathcal{R}_3 \circ \Phi_r^{(3,2)} & \Phi_z^{(2,1)} \end{pmatrix}, \quad S^{(0,3)} \coloneqq \begin{pmatrix} \Phi_r^{(0,3)} & \mathcal{R}_0 \circ \Phi_z^{(3,2)} \\ 0 & \Phi_z^{(3,2)} \end{pmatrix}.$$

Then $((G_c^{(\nu)})_{c\in C}, S^{(\nu+1,\nu)})_{\nu\in\mathbb{Z}/4\mathbb{Z}}$ is an object in $\mathfrak{SD}(C,\vartheta_0)$, since $\mathbf{\Phi}$ is a tuple of isomorphisms, $S^{(1,0)}, S^{(3,2)}$ are block-lower and $S^{(2,1)}, S^{(0,3)}$ are block-upper triangular. We can illustrate $((G_c^{(\nu)})_{c\in C}, S^{(\nu+1,\nu)})_{\nu\in\mathbb{Z}/4\mathbb{Z}}$ in the following diagram:

$$\begin{tikzcd}
& K_{r,I_r^{(0)}} \oplus K_{z,I_z^{(3)}} \arrow[dl, "S^{(1,0)}"'] & \\
K_{r,I_r^{(1)}} \oplus K_{z,I_z^{(0)}} \arrow[dr, "S^{(2,1)}"'] & & K_{r,I_r^{(3)}} \oplus K_{z,I_z^{(2)}} \arrow[ul, "S^{(0,3)}"'] \\
& K_{r,I_r^{(2)}} \oplus K_{z,I_z^{(1)}} \arrow[ur, "S^{(3,2)}"'] &
\end{tikzcd}$$

2. Consider the case $z \leq_{\vartheta_0} r$, then the unique numbering of C given by ϑ_0 is $c_1 = z$, $c_2 = r$. Setting

$$S^{(1,0)} := \begin{pmatrix} \Phi_z^{(0,3)} & 0 \\ \Phi_r^{(1,0)} \circ \mathcal{R}_0 & \Phi_r^{(1,0)} \end{pmatrix}, \; S^{(2,1)} := \begin{pmatrix} \Phi_z^{(1,0)} & \Phi_z^{(1,0)} \circ \mathcal{R}_1 \\ 0 & \Phi_r^{(2,1)} \end{pmatrix},$$
$$S^{(3,2)} := \begin{pmatrix} \Phi_z^{(2,1)} & 0 \\ \Phi_r^{(3,2)} \circ \mathcal{R}_2 & \Phi_r^{(3,2)} \end{pmatrix}, \; S^{(0,3)} := \begin{pmatrix} \Phi_z^{(3,2)} & \Phi_z^{(3,2)} \circ \mathcal{R}_3 \\ 0 & \Phi_r^{(0,3)} \end{pmatrix},$$

defines an object $((G_c^{(\nu)})_{c\in C}, S^{(\nu+1,\nu)})_{\nu\in\mathbb{Z}/4\mathbb{Z}}$ in $\mathfrak{SD}(C,\vartheta_0)$, since $\mathbf{\Phi}$ is a tuple of isomorphisms, $S^{(1,0)}, S^{(3,2)}$ are block-lower and $S^{(2,1)}, S^{(0,3)}$ are block-upper triangular. We can illustrate $((G_c^{(\nu)})_{c\in C}, S^{(\nu+1,\nu)})_{\nu\in\mathbb{Z}/4\mathbb{Z}}$ in the following diagram:

$$\begin{tikzcd}
& K_{z,I_z^{(3)}} \oplus K_{r,I_r^{(0)}} \arrow[dl, "S^{(1,0)}"'] & \\
K_{z,I_z^{(0)}} \oplus K_{r,I_r^{(1)}} \arrow[dr, "S^{(2,1)}"'] & & K_{z,I_z^{(2)}} \oplus K_{r,I_r^{(3)}} \arrow[ul, "S^{(0,3)}"'] \\
& K_{z,I_z^{(1)}} \oplus K_{r,I_r^{(2)}} \arrow[ur, "S^{(3,2)}"'] &
\end{tikzcd}$$

Thus in both cases we get an object in $\mathfrak{SD}(C,\vartheta_0)$. $\square$

Next we will show that this extends to a functor $\mathfrak{SH}(C) \to \mathfrak{SD}(C,\vartheta_0)$.

Proposition 5.4. *Consider $C = \{r, z\}$ with $r \in \mathbb{R}_{>0}$ and $z \in \mathbb{C} \setminus \mathbb{R}$ and let $\vartheta_0 \in S^1$ be a generic direction with respect to C. Define*

$$\begin{array}{rccc} F_{\vartheta_0}: & \mathfrak{S}\mathfrak{H}(C) & \longrightarrow & \mathfrak{S}\mathfrak{D}(C, \vartheta_0), \\ & \boldsymbol{Sh} & \longmapsto & F_{\vartheta_0}(\boldsymbol{Sh}) \end{array}$$

where $F_{\vartheta_0}(\boldsymbol{Sh}) \coloneqq ((G_c^{(\nu)})_{c \in C}, S^{(\nu+1,\nu)})_{\nu \in \mathbb{Z}/4\mathbb{Z}}$ is the Stokes data induced by $\boldsymbol{Sh}$ as defined in lemma (5.3). Then F_{ϑ_0} is a functor between $\mathfrak{S}\mathfrak{H}(C)$ and $\mathfrak{S}\mathfrak{D}(C, \vartheta_0)$.

Proof. First we have to define F_{ϑ_0} on $\mathrm{Hom}_{\mathfrak{S}\mathfrak{H}(C)}(\boldsymbol{Sh}, \tilde{\boldsymbol{Sh}})$. Let $\lambda \in \mathrm{Hom}_{\mathfrak{S}\mathfrak{H}(C)}(\boldsymbol{Sh}, \tilde{\boldsymbol{Sh}})$, $F_{\vartheta_0}(\boldsymbol{Sh}) \coloneqq ((G_c^{(\nu)})_{c \in C}, S^{(\nu+1,\nu)})_{\nu \in \mathbb{Z}/4\mathbb{Z}}$ and $F_{\vartheta_0}(\tilde{\boldsymbol{Sh}}) \coloneqq ((\tilde{G}_c^{(\nu)})_{c \in C}, \tilde{S}^{(\nu+1,\nu)})_{\nu \in \mathbb{Z}/4\mathbb{Z}}$. Since

$$\lambda : \mathrm{gr}\,\mathscr{L} = \mathrm{gr}_r\,\mathscr{L} \oplus \mathrm{gr}_z\,\mathscr{L} \to \mathrm{gr}_r\,\tilde{\mathscr{L}} \oplus \mathrm{gr}_z\,\tilde{\mathscr{L}} = \mathrm{gr}\,\tilde{\mathscr{L}}$$

is a graded morphism, in particular one has $\lambda(\overline{I_c^{(\nu)}})(\mathrm{gr}_c\,\mathscr{L}(\overline{I_c^{(\nu)}})) \subseteq \mathrm{gr}_c\,\tilde{\mathscr{L}}(\overline{I_c^{(\nu)}})$. Thus for each $(c, I_c^{(\nu)}) \in \mathcal{A}(C)$ we get a morphism of vector spaces

$$\lambda_c^{(\nu)} \coloneqq \lambda(\overline{I_c^{(\nu)}})\big|_{\mathrm{gr}_c\,\mathscr{L}(\overline{I_c^{(\nu)}})} : \mathrm{gr}_c\,\mathscr{L}(\overline{I_c^{(\nu)}}) \to \mathrm{gr}_c\,\tilde{\mathscr{L}}(\overline{I_c^{(\nu)}}).$$

Define

$$\begin{array}{rccc} \lambda^{(\nu)}: & \mathrm{gr}_r\,\mathscr{L}(\overline{I_r^{(\nu)}}) \oplus \mathrm{gr}_z\,\mathscr{L}(\overline{I_z^{(\nu+3)}}) & \longrightarrow & \mathrm{gr}_r\,\tilde{\mathscr{L}}(\overline{I_r^{(\nu)}}) \oplus \mathrm{gr}_z\,\tilde{\mathscr{L}}(\overline{I_z^{(\nu+3)}}) \\ & (x, y) & \longmapsto & (\lambda_r^{(\nu)}(x), \lambda_z^{(\nu+3)}(y)) \end{array}.$$

Then $(\lambda^{(\nu)})_{\nu \in \mathbb{Z}/4\mathbb{Z}} \in \mathrm{Hom}_{\mathfrak{S}\mathfrak{D}(C,\vartheta_0)}(F_{\vartheta_0}(\boldsymbol{Sh}), F_{\vartheta_0}(\tilde{\boldsymbol{Sh}}))$, because if writing

$$\lambda^{(\nu)} = (\lambda_{ij}^{(\nu)} : G_{c_j}^{(\nu)} \to \tilde{G}_{c_i}^{(\nu)})_{c_i, c_j \in \{r, z\}},$$

then $\lambda_{ij}^{(\nu)} = 0$ if $i \neq j$ and if $r \leq_{\vartheta_0} z$ and $\nu \in \{1, 3\}$:

$$\begin{aligned} \tilde{S}^{(\nu+1,\nu)}\lambda^{(\nu)} &= \begin{pmatrix} \tilde{\Phi}_r^{(\nu+1,\nu)} & \tilde{\mathcal{R}}_{\nu+1}\tilde{\Phi}_z^{(\nu,\nu+3)} \\ 0 & \tilde{\Phi}_z^{(\nu,\nu+3)} \end{pmatrix} \begin{pmatrix} \lambda_r^{(\nu)} & 0 \\ 0 & \lambda_z^{(\nu+3)} \end{pmatrix} \\ &= \begin{pmatrix} \tilde{\Phi}_r^{(\nu+1,\nu)}\lambda_r^{(\nu)} & \tilde{\mathcal{R}}_{\nu+1}\tilde{\Phi}_z^{(\nu,\nu+3)}\lambda_z^{(\nu+3)} \\ 0 & \tilde{\Phi}_z^{(\nu,\nu+3)}\lambda_z^{(\nu+3)} \end{pmatrix} \\ &\overset{(1)}{=} \begin{pmatrix} \lambda_r^{(\nu+1)}\Phi_r^{(\nu+1,\nu)} & \tilde{\mathcal{R}}_{\nu+1}\lambda_z^{(\nu)}\Phi_z^{(\nu,\nu+3)} \\ 0 & \lambda_z^{(\nu)}\Phi_z^{(\nu,\nu+3)} \end{pmatrix} \\ &\overset{(2)}{=} \begin{pmatrix} \lambda_r^{(\nu+1)}\Phi_r^{(\nu+1,\nu)} & \lambda_z^{(\nu)}\mathcal{R}_{\nu+1}\Phi_z^{(\nu,\nu+3)} \\ 0 & \lambda_z^{(\nu)}\Phi_z^{(\nu,\nu+3)} \end{pmatrix} \\ &= \begin{pmatrix} \lambda_r^{(\nu+1)} & 0 \\ 0 & \lambda_z^{(\nu)} \end{pmatrix} \begin{pmatrix} \Phi_r^{(\nu+1,\nu)} & \mathcal{R}_{\nu+1}\Phi_z^{(\nu,\nu+3)} \\ 0 & \Phi_z^{(\nu,\nu+3)} \end{pmatrix} = \lambda^{(\nu+1)}S^{(\nu+1,\nu)} \end{aligned}$$

where (1) holds, as λ is an morphism of sheaves and $\Phi, \tilde{\Phi}$ are given by the restriction morphisms of the sheaves $\operatorname{gr}\mathscr{L}$, $\operatorname{gr}\tilde{\mathscr{L}}$ and (2) holds, as λ is compatible with the deformation datum. The same arguments show that $\tilde{S}^{(\nu+1,\nu)}\lambda^{(\nu)} = \lambda^{(\nu+1)}S^{(\nu+1,\nu)}$ for $\nu \in \{0,2\}$ and also in the in the setting $z \leq_{\vartheta_0} r$.
We set $F_{\vartheta_0}(\lambda) \coloneqq ((\lambda^{(\nu)})_{\nu\in\mathbb{Z}/4\mathbb{Z}})$. Then by definition of F_{ϑ_0} on morphisms, $F_{\vartheta_0}(\mathrm{id}_{\boldsymbol{Sh}}) = \mathrm{id}_{F_{\vartheta_0}(\boldsymbol{Sh})}$ and for morphisms $\lambda \in \operatorname{Hom}_{\mathfrak{S}\mathfrak{H}(C)}(\boldsymbol{Sh}_1, \boldsymbol{Sh}_2)$ and $\mu \in \operatorname{Hom}_{\mathfrak{S}\mathfrak{H}(C)}(\boldsymbol{Sh}_2, \boldsymbol{Sh}_3)$ we have

$$\begin{aligned} F_{\vartheta_0}(\mu \circ \lambda) &= \left(\begin{pmatrix} \mu_r^{(\nu)}\lambda_r^{(\nu)} & 0 \\ 0 & \mu_z^{(\nu+3)}\lambda_z^{(\nu+3)} \end{pmatrix}^{(\nu)} \right)_{\nu\in\mathbb{Z}/4\mathbb{Z}} \\ &= \left(\begin{pmatrix} \mu_r^{(\nu)} & 0 \\ 0 & \mu_z^{(\nu+3)} \end{pmatrix} \begin{pmatrix} \lambda_r^{(\nu)} & 0 \\ 0 & \lambda_z^{(\nu+3)} \end{pmatrix}^{(\nu)} \right)_{\nu\in\mathbb{Z}/4\mathbb{Z}} \\ &= F_{\vartheta_0}(\mu) \circ F_{\vartheta_0}(\lambda). \end{aligned}$$

Thus, F_{ϑ_0} is indeed a functor between $\mathfrak{S}\mathfrak{H}(C)$ and $\mathfrak{S}\mathfrak{D}(C, \vartheta_0)$. □

Having established a functor from Stokes shells to Stokes data, we will continue with determining a functor from Stokes data to Stokes shells in the next section.

5.2 From data to shells

To receive a Stokes shell of Gaussian type C for each object in $\mathfrak{S}\mathfrak{D}(C, \vartheta_0)$, we first have to get a graded Stokes filtered local system as defined in (3.19).

Lemma 5.5. *Let $\vartheta_0 \in S^1$ be generic direction with respect to $C = \{r, z\}$ with $r \in \mathbb{R}_{>0}$ and $z \in \mathbb{C} \setminus \mathbb{R}$. Each object $\sigma = ((G_c^{(\nu)})_{c\in C}, S^{(\nu+1,\nu)})_{\nu\in\mathbb{Z}/4\mathbb{Z}}$ in $\mathfrak{S}\mathfrak{D}(C, \vartheta_0)$ induces a graded Stokes filtered local system that we note with $\mathfrak{L}(\sigma)$.*

Proof. Let $\sigma = ((G_c^{(\nu)})_{c\in C}, S^{(\nu+1,\nu)})_{\nu\in\mathbb{Z}/4\mathbb{Z}}$ be an arbitrary object in $\mathfrak{S}\mathfrak{D}(C, \vartheta_0)$. We will write $S^{(\nu+1,\nu)} = (S_{c_2c_1}^{(\nu+1,\nu)} : G_{c_1}^{(\nu)} \to G_{c_2}^{(\nu+1)})_{c_1,c_2\in C}$. By definition of $\mathfrak{S}\mathfrak{D}(C, \vartheta_0)$, $S_{cc}^{(\nu+1,\nu)}$ is an isomorphism for each $c \in C, \nu \in \mathbb{Z}/4\mathbb{Z}$. We set $\vartheta_\nu \coloneqq \vartheta_0 + \nu\frac{\pi}{2}$ and $I^{(\nu)} \coloneqq (\vartheta_\nu, \vartheta_{\nu+1})$.
Since ϑ_0 is a generic direction with respect to C, for each $\nu \in \mathbb{Z}/4\mathbb{Z}$ the closed interval $\overline{I^{(\nu)}}$ is C-good, precisely there are no Stokes directions on the boundary of the interval. Thus there exists an $\varepsilon > 0$ with $\varepsilon < \frac{\pi}{4}$, such that for any $\nu \in \mathbb{Z}/4\mathbb{Z}$ the open interval $I_\varepsilon^{(\nu)} \coloneqq (\vartheta_\nu - \varepsilon, \vartheta_{\nu+1} + \varepsilon)$ contains exactly one Stokes direction of $\mathrm{St}(r, z)$. Besides, $(I_\varepsilon^{(\nu)})_{\nu\in\mathbb{Z}/4\mathbb{Z}}$ is an open cover of S^1.
For each $c \in C$ and $\nu \in \mathbb{Z}/4\mathbb{Z}$ we define

$$\operatorname{gr}_c \mathscr{L}_{I_\varepsilon^{(\nu)}} \coloneqq \underline{G_c^{(\nu)}}_{I_\varepsilon^{(\nu)}}$$

to be the constant sheaf with fiber $G_c^{(\nu)}$ on $I_\varepsilon^{(\nu)}$. On the intersection $I_\varepsilon^{(\nu)} \cap I_\varepsilon^{(\nu+1)}$ we get an isomorphism of sheaves

$$\varphi_c^{(\nu+1,\nu)} : \operatorname{gr}_c \mathscr{L}_{I_\varepsilon^{(\nu)}}|_{I_\varepsilon^{(\nu)} \cap I_\varepsilon^{(\nu+1)}} \to \operatorname{gr}_c \mathscr{L}_{I_\varepsilon^{(\nu+1)}}|_{I_\varepsilon^{(\nu)} \cap I_\varepsilon^{(\nu+1)}}$$

that is given by

$$\varphi_c^{(\nu+1,\nu)}(U) : \operatorname{gr}_c \mathscr{L}_{I_\varepsilon^{(\nu)}}(U) \cong G_c^{(\nu)} \overset{S_{cc}^{(\nu+1,\nu)}}{\to} G_c^{(\nu+1)} \cong \operatorname{gr}_c \mathscr{L}_{I_\varepsilon^{(\nu+1)}}(U)$$

for each open, connected subset $U \subseteq I_\varepsilon^{(\nu)} \cap I_\varepsilon^{(\nu+1)} = (\vartheta_{\nu+1} - \varepsilon, \vartheta_{\nu+1} + \varepsilon)$.
Via the isomorphisms $(\varphi_c^{(\nu+1,\nu)})_{\nu \in \mathbb{Z}/4\mathbb{Z}}$ the sheaves $(\operatorname{gr}_c \mathscr{L}_{I_\varepsilon^{(\nu)}})_{\nu \in \mathbb{Z}/4\mathbb{Z}}$ glue together to a sheaf $\operatorname{gr}_c \mathscr{L}$ on S^1 that can be described by

$$\operatorname{gr}_c \mathscr{L}(U) = \left\{ (s_\nu) \in \prod_{\nu \in \mathbb{Z}/4\mathbb{Z}} \operatorname{gr}_c \mathscr{L}_{I_\varepsilon^{(\nu)}}(U \cap I_\varepsilon^{(\nu)}) \;\middle|\; \varphi_c^{(\nu+1,\nu)}(W)(s_{\nu|_W}) = s_{\nu+1|_W} \right\}$$

with $W := U \cap I_\varepsilon^{(\nu)} \cap I_\varepsilon^{(\nu+1)}$ (see 2.2). Remark that we do not need to check the cocycle relation, because there are no triple intersections in the cover $(I_\varepsilon^{(\nu)})_{\nu \in \mathbb{Z}/4\mathbb{Z}}$ of S^1 as $\varepsilon < \frac{\pi}{4}$. The sheaf $\operatorname{gr}_c \mathscr{L}$ is locally constant, i.e. a local system on S^1. We get a graded Stokes filtered local system $(\operatorname{gr} \mathscr{L}, \operatorname{gr} \mathscr{L}_{\leq \bullet})$ of Gaussian type C by setting

$$\operatorname{gr} \mathscr{L} := \operatorname{gr}_r \mathscr{L} \oplus \operatorname{gr}_c \mathscr{L}$$

and for any $c \in \mathbb{C}$

$$\operatorname{gr} \mathscr{L}_{\leq c} := \beta_{r \leq c}(\operatorname{gr}_r \mathscr{L}) \oplus \beta_{z \leq c}(\operatorname{gr}_z \mathscr{L}).$$

We set $\mathfrak{L}(\sigma) := (\operatorname{gr} \mathscr{L}, \operatorname{gr} \mathscr{L}_{\leq \bullet})$ to be the induced graded Stokes filtered local system. $\square$

Next, we have to define a deformation datum $\boldsymbol{R}$ for $\mathfrak{L}(\sigma)$. We will use the same notation as in section (5.1), that is

- $\vartheta_r^{(\nu)} = \frac{\pi}{4} + \nu \frac{\pi}{2}$ for $\nu \in \mathbb{Z}/4\mathbb{Z}$,
- $\vartheta_z^{(\nu)} = \frac{\pi + 2\arg(z)}{4} + \nu \frac{\pi}{2} \mod 2\pi$ for $\nu \in \mathbb{Z}/4\mathbb{Z}$ and
- $I_c^{(\nu)} = (\vartheta_c^{(\nu)}, \vartheta_c^{(\nu+1)})$ for $\nu \in \mathbb{Z}/4\mathbb{Z}$ and $c \in \{r, z\}$.

For $\mathfrak{L}(\sigma)$ we get the induced Stokes tuple of $\mathbf{k}$-vector spaces of Gaussian type $\mathfrak{D}(\mathfrak{L}(\sigma)) = (\boldsymbol{K}, \boldsymbol{\Phi}, \boldsymbol{\Psi})$ that is given by

- $K_{c, I_c^{(\nu)}} := \operatorname{gr}_c \mathscr{L}(\overline{I_c^{(\nu)}})$ for each pair $(c, I_c^{(\nu)}) \in \mathcal{A}(\tilde{C})$,

- $\Phi_c^{(\nu+1,\nu)} : K_{c,I_c^{(\nu)}} \to K_{c,I_c^{(\nu+1)}}$ from $\mathrm{gr}_c \mathscr{L}(\overline{I_c^{(\nu)}}) \cong \mathrm{gr}_c \mathscr{L}_{\vartheta_c^{(\nu+1)}} \cong \mathrm{gr}_c \mathscr{L}(\overline{I_c^{(\nu+1)}})$ for each pair $(c, I_c^{(\nu)}) \in \mathcal{A}(\tilde{C})$ and
- $\Psi_{c,I_c^{(\nu)}} = \mathrm{id}_{K_{c,I_c^{(\nu)}}} : K_{c,I_c^{(\nu)}} \to K_{c,I_c^{(\nu)}}$.

To define a deformation datum $\boldsymbol{R}$ for $\mathfrak{L}(\sigma)$ we give a precise description of the vector spaces $(\mathrm{gr}_c \mathscr{L}(\overline{I_c^{(\nu)}}))_{c\in C,\nu\in\mathbb{Z}/4\mathbb{Z}}$ first.
Let $(I_\varepsilon^{(\nu)})_{\nu\in\mathbb{Z}/4\mathbb{Z}}$ be the open covering of S^1 as in the proof of lemma (5.5). The interval $\overline{I_r^{(0)}} = [\frac{\pi}{4}, \frac{3\pi}{4}]$ has length $\frac{\pi}{2}$. Since all intervals of the open covering $(I_\varepsilon^{(\nu)})_{\nu\in\mathbb{Z}/4\mathbb{Z}}$ have length $\frac{\pi}{2} + 2\varepsilon$ with $0 < \varepsilon < \frac{\pi}{4}$, $\overline{I_r^{(0)}} \cap I_\varepsilon^{(\nu)}$ is a connected set or it is empty. Let $\nu_o \in \mathbb{Z}/4\mathbb{Z}$ be such that $\vartheta_{\nu_o} \in [0, \frac{\pi}{2}) \mod 2\pi$, then

- $\overline{I_r^{(0)}} \cap I_\varepsilon^{(\nu_o)} \neq \varnothing$, so $\mathrm{gr}_r \mathscr{L}_{I_\varepsilon^{(\nu_o)}}(\overline{I_r^{(0)}} \cap I_\varepsilon^{(\nu_o)}) = G_r^{(\nu_o)}$ and
- $\overline{I_r^{(0)}} \cap I_\varepsilon^{(\nu_o+2)} = \varnothing$, so $\mathrm{gr}_r \mathscr{L}_{I_\varepsilon^{(\nu_o)}}(\overline{I_r^{(0)}} \cap I_\varepsilon^{(\nu_o+2)}) = 0$.

The intersections of the intervals $I_\varepsilon^{(\nu_o+1)}, I_\varepsilon^{(\nu_o+3)}$ with $\overline{I_r^{(0)}}$ depend on ε and ϑ_{ν_o}. Nevertheless, we can give a precise description of $\mathrm{gr}_r \mathscr{L}(\overline{I_r^{(0)}})$ distinguishing the different cases that may occur:

1. If $I_\varepsilon^{(\nu_o+1)} \cap \overline{I_r^{(0)}} \neq \varnothing$ and $I_\varepsilon^{(\nu_o+3)} \cap \overline{I_r^{(0)}} = \varnothing$, then we get
$$\mathrm{gr}_r \mathscr{L}(\overline{I_r^{(0)}}) = \left\{(s_{\nu_o}, s_{\nu_o+1}) \in G_r^{(\nu_o)} \times G_r^{(\nu_o+1)} \mid S_{rr}^{(\nu_o+1,\nu_o)}(s_{\nu_o}) = s_{\nu_o+1}\right\}.$$
2. If $I_\varepsilon^{(\nu_o+1)} \cap \overline{I_r^{(0)}} = \varnothing$ and $I_\varepsilon^{(\nu_o+3)} \cap \overline{I_r^{(0)}} \neq \varnothing$, then we get
$$\mathrm{gr}_r \mathscr{L}(\overline{I_r^{(0)}}) = \left\{(s_{\nu_o+3}, s_{\nu_o}) \in G_r^{(\nu_o+3)} \times G_r^{(\nu_o)} \mid S_{rr}^{(\nu_o,\nu_o+3)}(s_{\nu_o+3}) = s_{\nu_o}\right\}.$$
3. If $I_\varepsilon^{(\nu_o+1)} \cap \overline{I_r^{(0)}} \neq \varnothing$ and $I_\varepsilon^{(\nu_o+3)} \cap \overline{I_r^{(0)}} \neq \varnothing$, then $\mathrm{gr}_r \mathscr{L}(\overline{I_r^{(0)}})$ is equal to the set
$$\left\{(s_{\nu_o+i}) \in \prod_{i\in\{0,1,3\}} G_r^{(\nu_o+i)} \;\middle|\; S_{rr}^{(\nu_o+i+1,\nu_o+i)}(s_{\nu_o+i}) = s_{\nu_o+i+1} \ \forall i \in \{0,3\}\right\}.$$

In every case we clearly have $\mathrm{gr}_r \mathscr{L}(\overline{I_r^{(0)}}) \cong G_r^{(\nu_o)}$. Similar arguments show that $\mathrm{gr}_r \mathscr{L}(\overline{I_r^{(\nu)}}) \cong G_r^{(\nu_o+\nu)}$ for $\nu \in \mathbb{Z}/4\mathbb{Z}$.
Using this insight we can clarify $\Phi_r^{(\nu+1,\nu)} : \mathrm{gr}_r \mathscr{L}(\overline{I_r^{(\nu)}}) \to \mathrm{gr}_r \mathscr{L}(\overline{I_r^{(\nu+1)}})$:

$$\Phi_r^{(\nu+1,\nu)} : \mathrm{gr}_r\,\mathscr{L}(\overline{I_r^{(\nu)}}) \cong G_r^{(\nu_o+\nu)} \xrightarrow{S_{rr}^{(\nu_o+\nu+1,\nu_o+\nu)}} G_r^{(\nu_o+\nu+1)} \cong \mathrm{gr}_r\,\mathscr{L}(\overline{I_r^{(\nu+1)}}).$$

Moving on to $z \in C$, we first describe $\overline{I_z^{(0)}} \cap I_\varepsilon^{(\nu)}$ for all $\nu \in \mathbb{Z}/4\mathbb{Z}$. Since $I_z^{(0)}$ depends on $\arg(z)$, the intersections are also dependent on the argument of z. Thus a few more cases than by the consideration of r can occur. Again, let $\nu_o \in \mathbb{Z}/4\mathbb{Z}$ be so that $\vartheta_{\nu_o} \in [0, \frac{\pi}{2}) \mod 2\pi$.

1. Consider the case $\mathrm{Im}(z) > 0$. Then with lemma (5.2) we get $\vartheta_z^{(0)} \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$.
    a) If $\vartheta_{\nu_o} \in [\frac{\pi}{4}, \frac{\pi}{2}) \mod 2\pi$, then $\overline{I_z^{(0)}} \cap I_\varepsilon^{(\nu_o)} \neq \varnothing$. Similar arguments as when r was considered lead to $\mathrm{gr}_z\,\mathscr{L}(\overline{I_z^{(\nu)}}) \cong G_z^{(\nu_o+\nu)}$ for $\nu \in \mathbb{Z}/4\mathbb{Z}$.
    b) If $\vartheta_{\nu_o} \in [0, \frac{\pi}{4}) \mod 2\pi$, then $\vartheta_{\nu_o+1} \in [\frac{\pi}{2}, \frac{3\pi}{4}) \mod 2\pi$, so $\overline{I_z^{(0)}}$ and $I_\varepsilon^{(\nu_o+1)}$ have non-empty intersection. This results in $\mathrm{gr}_z\,\mathscr{L}(\overline{I_z^{(\nu)}}) \cong G_z^{(\nu_o+\nu+1)}$ for $\nu \in \mathbb{Z}/4\mathbb{Z}$.

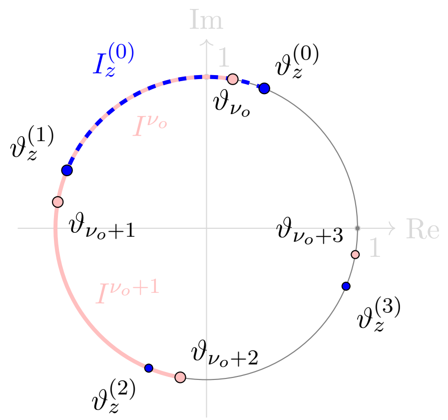

Figure 5.3: Case 1a) with $\arg(z) = \frac{\pi}{4}$ and $\vartheta_{\nu_o} = \frac{4\pi}{9} \in [\frac{\pi}{4}, \frac{\pi}{2})$.

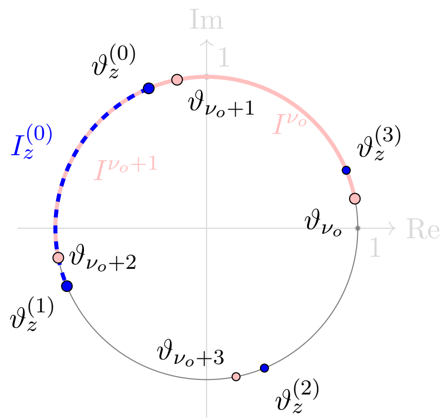

Figure 5.4: Case 1b) with $\arg(z) = \frac{3\pi}{4}$ and $\vartheta_{\nu_o} = \frac{\pi}{16} \in [0, \frac{\pi}{4})$.

2. Consider the case $\mathrm{Im}(z) < 0$. Then with lemma (5.2) we get $\vartheta_z^{(3)} \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right) \mod 2\pi$.
    a) If $\vartheta_{\nu_o} \in [\frac{\pi}{4}, \frac{\pi}{2}) \mod 2\pi$, then $\overline{I_z^{(3)}} \cap I_\varepsilon^{(\nu_o)} \neq \varnothing$. Similar arguments as when r was considered lead to $\mathrm{gr}_z\,\mathscr{L}(\overline{I_z^{(\nu+3)}}) \cong G_z^{(\nu_o+\nu)}$ for $\nu \in \mathbb{Z}/4\mathbb{Z}$. Remark that this is equivalent to $\mathrm{gr}_z\,\mathscr{L}(\overline{I_z^{(\nu)}}) \cong G_z^{(\nu_o+1+\nu)}$ for $\nu \in \mathbb{Z}/4\mathbb{Z}$, so we get the same result as in case 1b).

b) If $\vartheta_{\nu_o} \in [0, \frac{\pi}{2})\mod 2\pi$, then $\overline{I_z^{(3)}} \cap I_\varepsilon^{(\nu_o+1)} \neq \varnothing$. As before, this results in $\operatorname{gr}_z \mathscr{L}(\overline{I_z^{(\nu+3)}}) \cong G_z^{(\nu_o+\nu+1)}$ for $\nu \in \mathbb{Z}/4\mathbb{Z}$. Remark that this is equivalent to $\operatorname{gr}_z \mathscr{L}(\overline{I_z^{(\nu)}}) \cong G_z^{(\nu_o+\nu)}$ for $\nu \in \mathbb{Z}/4\mathbb{Z}$, so we get the same result as in case 1a).

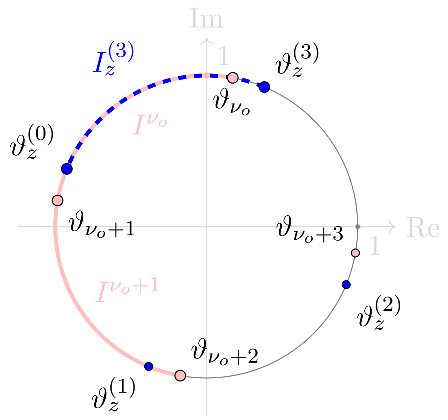

Figure 5.5: Case 2a) with $\arg(z) = \frac{5\pi}{4}$ and $\vartheta_{\nu_o} = \frac{4\pi}{9} \in [\frac{\pi}{4}, \frac{\pi}{2})$.

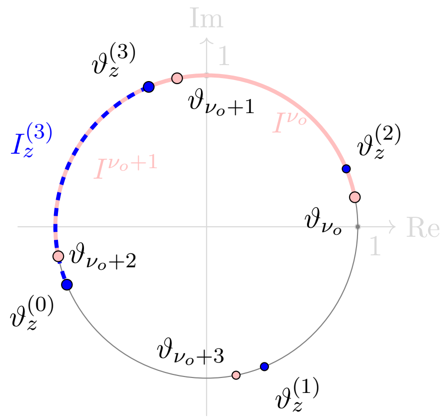

Figure 5.6: Case 2b) with $\arg(z) = \frac{7\pi}{4}$ and $\vartheta_{\nu_o} = \frac{\pi}{8} \in [0, \frac{\pi}{4})$.

This allows us to clarify $\Phi_z^{(\nu+1,\nu)} : \operatorname{gr}_z \mathscr{L}(\overline{I_z^{(\nu)}}) \to \operatorname{gr}_z \mathscr{L}(\overline{I_z^{(\nu+1)}})$:
If $\operatorname{gr}_z \mathscr{L}(\overline{I_z^{(\nu)}}) \cong G_z^{(\nu_o+\nu)}$, then

$$\Phi_z^{(\nu+1,\nu)} : \operatorname{gr}_z \mathscr{L}(\overline{I_z^{(\nu)}}) \cong G_z^{(\nu_o+\nu)} \xrightarrow{S_{zz}^{(\nu_o+\nu+1,\nu_o+\nu)}} G_z^{(\nu_o+\nu+1)} \cong \operatorname{gr}_z \mathscr{L}(\overline{I_z^{(\nu+1)}})$$

and similar if $\operatorname{gr}_z \mathscr{L}(\overline{I_z^{(\nu)}}) \cong G_z^{(\nu_o+\nu+1)}$, then

$$\Phi_z^{(\nu+1,\nu)} : \operatorname{gr}_z \mathscr{L}(\overline{I_z^{(\nu)}}) \cong G_z^{(\nu_o+\nu+1)} \xrightarrow{S_{zz}^{(\nu_o+\nu+2,\nu_o+\nu+1)}} G_z^{(\nu_o+\nu+2)} \cong \operatorname{gr}_z \mathscr{L}(\overline{I_z^{(\nu+1)}}).$$

After having a precise description of $\mathfrak{D}(\mathfrak{L}(\sigma))$, we want to show that σ induces a deformation datum of $\mathfrak{L}(\sigma)$ in the following.

Lemma 5.6. *Let $\vartheta_0 \in S^1$ be generic direction with respect to $C = \{r, z\}$ with $r \in \mathbb{R}_{>0}$ and $z \in \mathbb{C} \setminus \mathbb{R}$ and let $\sigma = ((G_c^{(\nu)})_{c\in C}, S^{(\nu+1,\nu)})_{\nu\in\mathbb{Z}/4\mathbb{Z}}$ be an object in $\mathfrak{SD}(C, \vartheta_0)$. Then σ induces a deformation datum of $\mathfrak{L}(\sigma)$, that we note with $\mathfrak{R}(\sigma)$.*

Proof. Let $\sigma = ((G_c^{(\nu)})_{c\in C}, S^{(\nu+1,\nu)})_{\nu\in\mathbb{Z}/4\mathbb{Z}}$ be an object in $\mathfrak{SD}(C, \vartheta_0)$ and $\mathfrak{L}(\sigma)$ be the induced object in $\mathbf{Loc}^{\mathrm{gr}}_{\mathrm{St}}(C)$, defined in the proof of lemma (5.5). As we saw in section (5.1), a deformation datum for a graded Stokes filtered local system $(\operatorname{gr}\mathscr{L}, \operatorname{gr}\mathscr{L}_{\leq\bullet})$ of Gaussian type $C = \{r, z\}$ with $r \in \mathbb{R}_{>0}$ and $z \in \mathbb{C} \setminus \mathbb{R}$ can be described by 4 linear morphisms

1. $\mathcal{R}_0 \coloneqq \mathcal{R}^{I_r^{(0)}}_{I_z^{(3)}} : K_{z,I_z^{(3)}} \to K_{r,I_r^{(0)}}$,
2. $\mathcal{R}_1 \coloneqq \mathcal{R}^{I_z^{(0)}}_{I_r^{(1)}} : K_{r,I_r^{(1)}} \to K_{z,I_z^{(0)}}$,
3. $\mathcal{R}_2 \coloneqq \mathcal{R}^{I_r^{(2)}}_{I_z^{(1)}} : K_{z,I_z^{(1)}} \to K_{r,I_r^{(2)}}$,
4. $\mathcal{R}_3 \coloneqq \mathcal{R}^{I_z^{(2)}}_{I_r^{(3)}} : K_{r,I_r^{(3)}} \to K_{z,I_z^{(2)}}$.

All other morphisms that appear in (4.8) are given by the zero morphism. Since $\mathfrak{D}(\mathfrak{L}(\sigma))$ is dependant on $\mathrm{Im}(z)$ and ϑ_0 we have to consider different cases in order to define $(\mathcal{R}_\nu)_{\nu\in\mathbb{Z}/4\mathbb{Z}}$. Let $\nu_o \in \mathbb{Z}/4\mathbb{Z}$ be so that $\vartheta_{\nu_o} \in [0, \frac{\pi}{2}) \mod 2\pi$.

1. First consider the case $\mathrm{gr}_z \mathscr{L}(\overline{I_z^{(\nu)}}) \cong G_z^{(\nu_o+\nu)}$ for $\nu \in \mathbb{Z}/4\mathbb{Z}$, so $\mathrm{Im}(z) > 0$ and $\vartheta_{\nu_o} \in [\frac{\pi}{4}, \frac{\pi}{2}) \mod 2\pi$ or $\mathrm{Im}(z) < 0$ and $\vartheta_{\nu_o} \in [0, \frac{\pi}{4}) \mod 2\pi$.
   a) In the case $z \leq_{\vartheta_{\nu_o}} r$ the deformation datum is given by
      i. $\mathcal{R}_0 \coloneqq \mathcal{R}^{I_r^{(0)}}_{I_z^{(3)}} : \mathrm{gr}_z \mathscr{L}(\overline{I_z^{(3)}}) \cong G_z^{(\nu_o+3)} \xrightarrow{S_0} G_r^{(\nu_o)} \cong \mathrm{gr}_r \mathscr{L}(\overline{I_r^{(0)}})$, with $S_0 \coloneqq (S_{rr}^{(\nu_o+1,\nu_o)})^{-1} \circ S_{rz}^{(\nu_o+1,\nu_o)} \circ S_{zz}^{(\nu_o,\nu_o+3)}$,
      ii. $\mathcal{R}_1 \coloneqq \mathcal{R}^{I_z^{(0)}}_{I_r^{(1)}} : \mathrm{gr}_r \mathscr{L}(\overline{I_r^{(1)}}) \cong G_r^{(\nu_o+1)} \xrightarrow{S_1} G_z^{(\nu_o)} \cong \mathrm{gr}_z \mathscr{L}(\overline{I_z^{(0)}})$, with $S_1 \coloneqq (S_{zz}^{(\nu_o+1,\nu_o)})^{-1} \circ (S_{zz}^{(\nu_o+2,\nu_o+1)})^{-1} \circ S_{zr}^{(\nu_o+2,\nu_o+1)}$,
      iii. $\mathcal{R}_2 \coloneqq \mathcal{R}^{I_r^{(2)}}_{I_z^{(1)}} : \mathrm{gr}_z \mathscr{L}(\overline{I_z^{(1)}}) \cong G_z^{(\nu_o+1)} \xrightarrow{S_2} G_r^{(\nu_o+2)} \cong \mathrm{gr}_r \mathscr{L}(\overline{I_r^{(2)}})$, with $S_2 \coloneqq (S_{rr}^{(\nu_o+3,\nu_o+2)})^{-1} \circ S_{rz}^{(\nu_o+3,\nu_o+2)} \circ S_{zz}^{(\nu_o+2,\nu_o+1)}$ and
      iv. $\mathcal{R}_3 \coloneqq \mathcal{R}^{I_z^{(2)}}_{I_r^{(3)}} : \mathrm{gr}_r \mathscr{L}(\overline{I_r^{(3)}}) \cong G_r^{(\nu_o+3)} \xrightarrow{S_3} G_z^{(\nu_o+2)} \cong \mathrm{gr}_z \mathscr{L}(\overline{I_z^{(2)}})$, with $S_3 \coloneqq (S_{zz}^{(\nu_o+3,\nu_o+2)})^{-1} \circ (S_{zz}^{(\nu_o,\nu_o+3)})^{-1} \circ S_{zr}^{(\nu_o,\nu_o+3)}$.
   b) Considering the case $r \leq_{\vartheta_{\nu_o}} z$ we get the following deformation datum:
      i. $\mathcal{R}_0 \coloneqq \mathcal{R}^{I_r^{(0)}}_{I_z^{(3)}} : \mathrm{gr}_z \mathscr{L}(\overline{I_z^{(3)}}) \cong G_z^{(\nu_o+3)} \xrightarrow{S_0} G_r^{(\nu_o)} \cong \mathrm{gr}_r \mathscr{L}(\overline{I_r^{(0)}})$, with $S_0 \coloneqq S_{rz}^{(\nu_o,\nu_o+3)}$,
      ii. $\mathcal{R}_1 \coloneqq \mathcal{R}^{I_z^{(0)}}_{I_r^{(1)}} : \mathrm{gr}_r \mathscr{L}(\overline{I_r^{(1)}}) \cong G_r^{(\nu_o+1)} \xrightarrow{S_1} G_z^{(\nu_o)} \cong \mathrm{gr}_z \mathscr{L}(\overline{I_z^{(0)}})$, with $S_1 \coloneqq (S_{zz}^{(\nu_o+1,\nu_o)})^{-1} \circ S_{zr}^{(\nu_o+1,\nu_o)} \circ (S_{rr}^{(\nu_o+1,\nu_o)})^{-1}$,
      iii. $\mathcal{R}_2 \coloneqq \mathcal{R}^{I_r^{(2)}}_{I_z^{(1)}} : \mathrm{gr}_z \mathscr{L}(\overline{I_z^{(1)}}) \cong G_z^{(\nu_o+1)} \xrightarrow{S_2} G_r^{(\nu_o+2)} \cong \mathrm{gr}_r \mathscr{L}(\overline{I_r^{(2)}})$, with $S_2 \coloneqq S_{rz}^{(\nu_o+2,\nu_o+1)}$ and
      iv. $\mathcal{R}_3 \coloneqq \mathcal{R}^{I_z^{(2)}}_{I_r^{(3)}} : \mathrm{gr}_r \mathscr{L}(\overline{I_r^{(3)}}) \cong G_r^{(\nu_o+3)} \xrightarrow{S_3} G_z^{(\nu_o+2)} \cong \mathrm{gr}_z \mathscr{L}(\overline{I_z^{(2)}})$, with $S_3 \coloneqq (S_{zz}^{(\nu_o+3,\nu_o+2)})^{-1} \circ S_{zr}^{(\nu_o+3,\nu_o+2)} \circ (S_{rr}^{(\nu_o+3,\nu_o+2)})^{-1}$.
2. Now consider the case $\mathrm{gr}_z \mathscr{L}(\overline{I_z^{(\nu)}}) \cong G_z^{(\nu_o+1+\nu)}$ for $\nu \in \mathbb{Z}/4\mathbb{Z}$, so $\mathrm{Im}(z) > 0$ and $\vartheta_{\nu_o} \in [0, \frac{\pi}{4}) \mod 2\pi$ or $\mathrm{Im}(z) < 0$ and $\vartheta_{\nu_o} \in [\frac{\pi}{4}, \frac{\pi}{2}) \mod 2\pi$.

a) In the case $z \leq_{\vartheta_{\nu_o}} r$ we get

i. $\mathcal{R}_0 \coloneqq \mathcal{R}^{I_r^{(0)}}_{I_z^{(3)}} : \operatorname{gr}_z \mathscr{L}(\overline{I_z^{(3)}}) \cong G_z^{(\nu_o)} \xrightarrow{S_0} G_r^{(\nu_o)} \cong \operatorname{gr}_r \mathscr{L}(\overline{I_r^{(0)}})$ with $S_0 \coloneqq (S_{rr}^{(\nu_o+1,\nu_o)})^{-1} \circ S_{rz}^{(\nu_o+1,\nu_o)}$,

ii. $\mathcal{R}_1 \coloneqq \mathcal{R}^{I_z^{(0)}}_{I_r^{(1)}} : \operatorname{gr}_r \mathscr{L}(\overline{I_r^{(1)}}) \cong G_r^{(\nu_o+1)} \xrightarrow{S_1} G_z^{(\nu_o+1)} \cong \operatorname{gr}_z \mathscr{L}(\overline{I_z^{(0)}})$, with $S_1 \coloneqq (S_{zz}^{(\nu_o+2,\nu_o+1)})^{-1} \circ S_{zr}^{(\nu_o+2,\nu_o+1)}$

iii. $\mathcal{R}_2 \coloneqq \mathcal{R}^{I_r^{(2)}}_{I_z^{(1)}} : \operatorname{gr}_z \mathscr{L}(\overline{I_z^{(1)}}) \cong G_z^{(\nu_o+2)} \xrightarrow{S_2} G_r^{(\nu_o+2)} \cong \operatorname{gr}_r \mathscr{L}(\overline{I_r^{(2)}})$ with $S_2 \coloneqq (S_{rr}^{(\nu_o+3,\nu_o+2)})^{-1} \circ S_{rz}^{(\nu_o+3,\nu_o+2)}$ and

iv. $\mathcal{R}_3 \coloneqq \mathcal{R}^{I_z^{(2)}}_{I_r^{(3)}} : \operatorname{gr}_r \mathscr{L}(\overline{I_r^{(3)}}) \cong G_r^{(\nu_o+3)} \xrightarrow{S_3} G_z^{(\nu_o+3)} \cong \operatorname{gr}_z \mathscr{L}(\overline{I_z^{(2)}})$ with $S_3 \coloneqq (S_{zz}^{(\nu_o,\nu_o+3)})^{-1} \circ S_{zr}^{(\nu_o,\nu_o+3)}$.

b) Considering the case $r \leq_{\vartheta_{\nu_o}} z$ we get

i. $\mathcal{R}_0 \coloneqq \mathcal{R}^{I_r^{(0)}}_{I_z^{(3)}} : \operatorname{gr}_z \mathscr{L}(\overline{I_z^{(3)}}) \cong G_z^{(\nu_o)} \xrightarrow{S_0} G_r^{(\nu_o)} \cong \operatorname{gr}_r \mathscr{L}(\overline{I_r^{(0)}})$ with $S_0 \coloneqq S_{rz}^{(\nu_o,\nu_o+3)} \circ (S_{zz}^{(\nu_o,\nu_o+3)})^{-1}$,

ii. $\mathcal{R}_1 \coloneqq \mathcal{R}^{I_z^{(0)}}_{I_r^{(1)}} : \operatorname{gr}_r \mathscr{L}(\overline{I_r^{(1)}}) \cong G_r^{(\nu_o+1)} \xrightarrow{S_1} G_z^{(\nu_o+1)} \cong \operatorname{gr}_z \mathscr{L}(\overline{I_z^{(0)}})$, with $S_1 \coloneqq S_{zr}^{(\nu_o+1,\nu_o)} \circ (S_{rr}^{(\nu_o+1,\nu_o)})^{-1}$

iii. $\mathcal{R}_2 \coloneqq \mathcal{R}^{I_r^{(2)}}_{I_z^{(1)}} : \operatorname{gr}_z \mathscr{L}(\overline{I_z^{(1)}}) \cong G_z^{(\nu_o+2)} \xrightarrow{S_2} G_r^{(\nu_o+2)} \cong \operatorname{gr}_r \mathscr{L}(\overline{I_r^{(2)}})$ with $S_2 \coloneqq S_{rz}^{(\nu_o+2,\nu_o+1)} \circ (S_{zz}^{(\nu_o+2,\nu_o+1)})^{-1}$ and

iv. $\mathcal{R}_3 \coloneqq \mathcal{R}^{I_z^{(2)}}_{I_r^{(3)}} : \operatorname{gr}_r \mathscr{L}(\overline{I_r^{(3)}}) \cong G_r^{(\nu_o+3)} \xrightarrow{S_3} G_z^{(\nu_o+3)} \cong \operatorname{gr}_z \mathscr{L}(\overline{I_z^{(2)}})$ with $S_3 \coloneqq S_{zr}^{(\nu_o+3,\nu_o+2)} \circ (S_{rr}^{(\nu_o+3,\nu_o+2)})^{-1}$.

We define $\mathfrak{R}(\sigma) \coloneqq (\mathcal{R}_\nu)_{\nu \in \mathbb{Z}/4\mathbb{Z}}$ to be the induced deformation datum of $\mathfrak{L}(\sigma)$. $\square$

Next we show that $\sigma \mapsto (\mathfrak{L}(\sigma), \mathfrak{R}(\sigma))$ extends to a functor.

Proposition 5.7. *Consider $C = \{r, z\}$ with $r \in \mathbb{R}_{>0}$ and $z \in \mathbb{C} \setminus \mathbb{R}$ and let $\vartheta_0 \in S^1$ be a generic direction with respect to C. Define*

$$\begin{array}{rccc} G_{\vartheta_0} : & \mathfrak{SD}(C, \vartheta_0) & \longrightarrow & \mathfrak{SH}(C) \\ & \sigma = ((G_c^{(\nu)})_{c \in C}, S^{(\nu+1,\nu)})_{\nu \in \mathbb{Z}/4\mathbb{Z}} & \longmapsto & G_{\vartheta_0}(\sigma) \end{array}$$

where $G_{\vartheta_0}(\sigma) \coloneqq (\mathfrak{L}(\sigma), \mathfrak{R}(\sigma))$. Then G_{ϑ_0} is a functor between $\mathfrak{SD}(C, \vartheta_0)$ and $\mathfrak{SH}(C)$.

Proof. We already saw, that $G_{\vartheta_0}(\sigma)$ is an object in $\mathfrak{SH}(C)$, as $\mathfrak{L}(\sigma)$ is a graded Stokes filtered local system and $\mathfrak{R}(\sigma)$ is a deformation datum of $\mathfrak{L}(\sigma)$.
Now we define G_{ϑ_0} for morphisms in $\mathfrak{SD}(C,\vartheta_0)$. Let $\lambda \in \mathrm{Hom}_{\mathfrak{SD}(C,\vartheta_0)}(\sigma,\tilde{\sigma})$ with the Stokes data $\sigma = ((G_c^{(\nu)})_{c\in C}, S^{(\nu+1,\nu)})_{\nu\in\mathbb{Z}/4\mathbb{Z}}$, $\tilde{\sigma} = ((\tilde{G}_c^{(\nu)})_{c\in C}, \tilde{S}^{(\nu+1,\nu)})_{\nu\in\mathbb{Z}/4\mathbb{Z}}$ and the induced graded Stokes filtered local systems $\mathfrak{L}(\sigma) = (\mathrm{gr}\,\mathscr{L}, \mathrm{gr}\,\mathscr{L}_{\leq\bullet})$ and $\mathfrak{L}(\tilde{\sigma}) = (\mathrm{gr}\,\tilde{\mathscr{L}}, \mathrm{gr}\,\tilde{\mathscr{L}}_{\leq\bullet})$. Then $\lambda = (\lambda^{(\nu)} : G_r^{(\nu)} \oplus G_z^{(\nu)} \to \tilde{G}_r^{(\nu)} \oplus \tilde{G}_z^{(\nu)})_{\nu\in\mathbb{Z}/4\mathbb{Z}}$ is given by

$$\lambda^{(\nu)} = \begin{pmatrix} \lambda_{r,r}^{(\nu)} & 0 \\ 0 & \lambda_{z,z}^{(\nu)} \end{pmatrix} : G_r^{(\nu)} \oplus G_z^{(\nu)} \longrightarrow \tilde{G}_r^{(\nu)} \oplus \tilde{G}_z^{(\nu)}.$$

We will define $\varphi_c : \mathrm{gr}_c\,\mathscr{L} \to \mathrm{gr}_c\,\tilde{\mathscr{L}}$ for $c \in C$ which gives us a graded morphism of local systems $G_{\vartheta_0}(\lambda) : \mathrm{gr}\,\mathscr{L} \to \mathrm{gr}\,\tilde{\mathscr{L}}$ with $G_{\vartheta_0}(\lambda)(\mathrm{gr}\,\mathscr{L}_{\leq c}) \subseteq \mathrm{gr}\,\tilde{\mathscr{L}}_{\leq c}$, i.e. a morphism in $\mathrm{Hom}_{\mathbf{Loc}^{\mathrm{gr}}_{\mathrm{St}}(C)}(\mathfrak{L}(\sigma), \mathfrak{L}(\tilde{\sigma}))$. Then it is left to check that $G_{\vartheta_0}(\lambda)$ is compatible with the deformation datum $\mathfrak{R}(\sigma)$ and $\mathfrak{R}(\tilde{\sigma})$.
Since $\mathcal{B} \coloneqq \{U = (\eta_0,\eta_1) \subseteq \mathbb{R}/2\pi\mathbb{Z} \mid 0 \leq \eta_0 - \eta_1 \mod 2\pi < \frac{\pi}{2}\}$ is a basis for the topology of S^1, it is enough to define φ_c for open sets $U \in \mathcal{B}$. Recall that in the proof of (5.5) we gave a construction for $\mathrm{gr}_c\,\mathscr{L}$ that is

$$\mathrm{gr}_c\,\mathscr{L}(U) = \left\{ (s_\nu) \in \prod_{\nu\in 4\mathbb{Z}} \underline{G_c^{(\nu)}}_{I_\varepsilon^{(\nu)}}(U \cap I_\varepsilon^{(\nu)}) \;\middle|\; S_{cc}^{(\nu+1,\nu)}(s_{\nu|W}) = s_{\nu+1|W}\ \forall \nu \in \mathbb{Z}/4\mathbb{Z} \right\}.$$

Here, $(I_\varepsilon^{(\nu)})_{\nu\in\mathbb{Z}/4\mathbb{Z}}$ is a covering of open intervals with length $\frac{\pi}{2} + 2\varepsilon$ of S^1 with $0 < \varepsilon < \frac{\pi}{4}$ as in the proof of lemma (5.5) and $W \coloneqq U \cap I_\varepsilon^{(\nu)} \cap I_\varepsilon^{(\nu+1)}$. Remark that $U \cap I_\varepsilon^{(\nu)}$ is connected for each $U \in \mathcal{B}$ and $\nu \in \mathbb{Z}/4\mathbb{Z}$, so $\underline{G_c^{(\nu)}}_{I_\varepsilon^{(\nu)}}(U \cap I_\varepsilon^{(\nu)})$ is equal to $G_c^{(\nu)}$ if $U \cap I_\varepsilon^{(\nu)} \neq \varnothing$ and equal to 0 if not. We set

$$\begin{array}{rccc} \varphi_c(U): & \mathrm{gr}_c\,\mathscr{L}(U) & \longrightarrow & \mathrm{gr}_c\,\tilde{\mathscr{L}}(U). \\ & (s_\nu)_{\nu\in\mathbb{Z}/4\mathbb{Z}} & \longmapsto & (\lambda_{c,c}^{(\nu)}(s_\nu))_{\nu\in\mathbb{Z}/4\mathbb{Z}} \end{array}$$

As $\lambda_{c,c}^{(\nu)}(s_\nu) \in \tilde{G}_c^{(\nu)}$ and $\tilde{S}_{cc}^{(\nu+1,\nu)}(\lambda_{c,c}^{(\nu)}(s_\nu)) = \lambda_{c,c}^{(\nu+1)}(S_{cc}^{(\nu+1,\nu)}(s_\nu)) = \lambda_{c,c}^{(\nu+1)}(s_{\nu+1})$, the morphism $\varphi_c(U)$ is well-defined. Since φ_c commutes with the restriction morphisms of the sheaves $\mathrm{gr}\,\mathscr{L}$ and $\mathrm{gr}\,\tilde{\mathscr{L}}$, φ_c is a morphism of local systems. Given that $\mathrm{gr}\,\mathscr{L}$ is the direct sum of $\mathrm{gr}_r\,\mathscr{L}$ and $\mathrm{gr}_z\,\mathscr{L}$ and similarly $\mathrm{gr}\,\tilde{\mathscr{L}} = \mathrm{gr}_r\,\tilde{\mathscr{L}} \oplus \mathrm{gr}_z\,\tilde{\mathscr{L}}$, it is straightforward to define $G_{\vartheta_0}(\lambda) \in \mathrm{Hom}_{\mathbf{Loc}_{S^1}}(\mathrm{gr}\,\mathscr{L}, \mathrm{gr}\,\tilde{\mathscr{L}})$: When we decompose $G_{\vartheta_0}(\lambda)$ into blocks of morphisms of local systems

$$(\varphi_{ji} : \mathrm{gr}_{c_i}\,\mathscr{L} \to \mathrm{gr}_{c_j}\,\tilde{\mathscr{L}})_{c_i,c_j\in\{r,z\}}$$

we have $\varphi_{ji} = 0$ if $c_i \neq c_j$ and $\varphi_{ji} = \varphi_{c_i}$ if $c_i = c_j$.

We have to show that $G_{\vartheta_0}(\lambda)(\operatorname{gr}\mathscr{L}_{\leq c}) \subseteq \operatorname{gr}\tilde{\mathscr{L}}_{\leq c}$ for all $c \in \mathbb{C}$ which can be checked on the stalks, namely $G_{\vartheta_0}(\lambda)(\operatorname{gr}\mathscr{L}_{\leq c})_\vartheta \subseteq \operatorname{gr}\tilde{\mathscr{L}}_{\leq c,\vartheta}$ for all $c \in \mathbb{C}$ and $\vartheta \in S^1$. For any $c \in \mathbb{C}$ and $\vartheta \in S^1$ we have

$$(\operatorname{gr}\mathscr{L}_{\leq c})_\vartheta = \bigoplus_{c' \in C} \beta_{c' \leq c}(\operatorname{gr}_{c'}\mathscr{L})_\vartheta = \bigoplus_{c' \leq_\vartheta c} \operatorname{gr}_{c'}\mathscr{L}_\vartheta$$

and similarly $\operatorname{gr}\tilde{\mathscr{L}}_{\leq c,\vartheta} = \bigoplus_{c' \leq_\vartheta c} \operatorname{gr}_{c'}\tilde{\mathscr{L}}_\vartheta$. Given that $\varphi_c(\operatorname{gr}_c \mathscr{L}_\vartheta) \subseteq \operatorname{gr}_c \tilde{\mathscr{L}}_\vartheta$ for $c \in C$, it follows that $G_{\vartheta_0}(\lambda)(\operatorname{gr}\mathscr{L}_{\leq c})_\vartheta \subseteq \operatorname{gr}\tilde{\mathscr{L}}_{\leq c,\vartheta}$. Thus $G_{\vartheta_0}(\lambda) \in \operatorname{Hom}_{\mathbf{Loc}^{\mathrm{gr}}_{\mathrm{St}}(C)}(\mathfrak{L}(\sigma), \mathfrak{L}(\tilde{\sigma}))$.

To see that $G_{\vartheta_0}(\lambda)$ is a morphism of Stokes shells, it is left to check that it is compatible with the deformation datum $\mathfrak{R}(\sigma) = (\mathcal{R}_\nu)_{\nu \in \mathbb{Z}/4\mathbb{Z}}$ and $\mathfrak{R}(\tilde{\sigma}) = (\tilde{\mathcal{R}}_\nu)_{\nu \in \mathbb{Z}/4\mathbb{Z}}$, i.e. that $G_{\vartheta_0}(\lambda) \circ \mathcal{R}_\nu = \tilde{\mathcal{R}}_\nu \circ G_{\vartheta_0}(\lambda)$ for each $\nu \in \mathbb{Z}/4\mathbb{Z}$. As we have seen before, the deformation datum depends on z and ϑ_0, so we have to investigate different cases. Again, let $\nu_o \in \mathbb{Z}/4\mathbb{Z}$ be such that $\vartheta_{\nu_o} = \vartheta_0 + \nu_o \frac{\pi}{2} \in [0, \frac{\pi}{2}) \mod 2\pi$.
We only consider the case $\operatorname{Im}(z) > 0$ and $\vartheta_{\nu_o} \in [\frac{\pi}{4}, \frac{\pi}{2}) \mod 2\pi$ (resp. $\operatorname{Im}(z) < 0$ and $\vartheta_{\nu_o} \in [0, \frac{\pi}{4}) \mod 2\pi$), because all other cases can be proven analogously. Thus, in our scenario, we have $\operatorname{gr}_c \mathscr{L}(\overline{I_c^{(\nu)}}) \cong G_c^{(\nu_o+\nu)}$ and $\operatorname{gr}_c \tilde{\mathscr{L}}(\overline{I_c^{(\nu)}}) \cong \tilde{G}_c^{(\nu_o+\nu)}$ for $\nu \in \mathbb{Z}/4\mathbb{Z}$, $c \in C$. We denote these isomorphisms by

$$\psi_c^{(\nu)} : \operatorname{gr}_c \mathscr{L}(\overline{I_c^{(\nu)}}) \to G_c^{(\nu_o+\nu)}.$$

In particular, $\psi_c^{(\nu)}$ maps a tuple $(s_\nu)_{\nu \in \mathbb{Z}/4\mathbb{Z}} \in \operatorname{gr}_c \mathscr{L}(\overline{I_c^{(\nu)}})$ to $s_{\nu_o+\nu} \in G_c^{(\nu_o+\nu)}$. Analogously we define $\tilde{\psi}_c^{(\nu)} : \operatorname{gr}_c \tilde{\mathscr{L}}(\overline{I_c^{(\nu)}}) \to \tilde{G}_c^{(\nu_o+\nu)}$.
In the case $z \leq_{\vartheta_{\nu_o}} r$ the deformation datum $\mathfrak{R}(\sigma)$ is given by

$$\begin{aligned}
\mathcal{R}_k &: \operatorname{gr}_z \mathscr{L}(\overline{I_z^{(k+3)}}) \cong G_z^{(\nu_o+k+3)} \xrightarrow{S_k} G_r^{(\nu_o+k)} \cong \operatorname{gr}_r \mathscr{L}(\overline{I_r^{(k)}}) \text{ with} \\
&\quad S_k := (S_{rr}^{(\nu_o+k+1,\nu_o+k)})^{-1} \circ S_{rz}^{(\nu_o+k+1,\nu_o+k)} \circ S_{zz}^{(\nu_o+k,\nu_o+k+3)} \text{ for } k \in \{0,2\} \\
\mathcal{R}_k &: \operatorname{gr}_r \mathscr{L}(\overline{I_r^{(k)}}) \cong G_r^{(\nu_o+k)} \xrightarrow{S_k} G_z^{(\nu_o+k+3)} \cong \operatorname{gr}_z \mathscr{L}(\overline{I_z^{(k+3)}}) \text{ with} \\
&\quad S_k := (S_{zz}^{(\nu_o+k,\nu_o+k+3)})^{-1} \circ (S_{zz}^{(\nu_o+k+1,\nu_o+k)})^{-1} \circ S_{zr}^{(\nu_o+k+1,\nu_o+k)} \text{ for } k \in \{1,3\}
\end{aligned}$$

and the same holds for $\tilde{\mathcal{R}}_k$.
As for each interval $I \in \{I_c^{(\nu)} \mid c \in C, \nu \in \mathbb{Z}/4\mathbb{Z}\}$ we have

$$\begin{aligned}
G_{\vartheta_0}(\lambda)(\overline{I_c^{(\nu)}})_{|\operatorname{gr}_c \mathscr{L}(\overline{I_c^{(\nu)}})} = \varphi_c(\overline{I_c^{(\nu)}}) : \quad \operatorname{gr}_c \mathscr{L}(\overline{I_c^{(\nu)}}) &\to \operatorname{gr}_c \tilde{\mathscr{L}}(\overline{I_c^{(\nu)}}), \\
(s_\nu)_{\nu \in \mathbb{Z}/4\mathbb{Z}} &\mapsto (\lambda_{c,c}^{(\nu)}(s_\nu))_{\nu \in \mathbb{Z}/4\mathbb{Z}}
\end{aligned}$$

we have to prove that for $k \in \{0,2\}$ the diagram

$$
\begin{tikzcd}
\mathrm{gr}_z \mathscr{L}(\overline{I_z^{(k+3)}}) \arrow[r, "\varphi_z(\overline{I_z^{(k+3)}})"] \arrow[d, "\mathcal{R}_k"'] & \mathrm{gr}_z \tilde{\mathscr{L}}(\overline{I_z^{(k+3)}}) \arrow[d, "\tilde{\mathcal{R}}_k"] \\
\mathrm{gr}_r \mathscr{L}(\overline{I_r^{(k)}}) \arrow[r, "\varphi_r(\overline{I_r^{(k)}})"'] & \mathrm{gr}_r \tilde{\mathscr{L}}(\overline{I_r^{(k)}})
\end{tikzcd}
$$

is commutative. This holds as $\lambda^{(k+1)}S^{(k+1,k)} = \tilde{S}^{(k+1,k)}\lambda^{(k)}$. More precisely, by definition of φ_c for $c \in C, \nu \in \mathbb{Z}/4\mathbb{Z}$

$$\tilde{\psi}_c^{(\nu)}\varphi_c(\overline{I_c^{(\nu)}}) = \lambda_{c,c}^{(\nu_o+\nu)}\psi_c^{(\nu)}$$

holds. Thus for $k \in \{0, 2\}$

$$
\begin{aligned}
\tilde{\mathcal{R}}_k \circ \varphi_z(\overline{I_z^{(k+3)}}) &= (\tilde{\psi}_r^{(k)})^{-1}\tilde{S}_k\tilde{\psi}_z^{(k+3)}\varphi_z(\overline{I_z^{(k+3)}}) = (\tilde{\psi}_r^{(k)})^{-1}\tilde{S}_k\lambda_{z,z}^{(\nu_o+k+3)}\psi_z^{(k+3)} \\
&= (\tilde{\psi}_r^{(k)})^{-1}\lambda_{r,r}^{(\nu_o+k)}S_k\psi_z^{(k+3)} = \varphi_r(\overline{I_r^{(k)}})(\psi_r^{(k)})^{-1}S_k\psi_z^{(k+3)} \\
&= \varphi_r(\overline{I_r^{(k)}}) \circ \mathcal{R}_k.
\end{aligned}
$$

Similarly for $k \in \{1, 3\}$ one can show that the following diagram commutes:

$$
\begin{tikzcd}
\mathrm{gr}_r \mathscr{L}(\overline{I_r^{(k)}}) \arrow[r, "\varphi_r(\overline{I_r^{(k)}})"] \arrow[d, "\mathcal{R}_k"'] & \mathrm{gr}_r \tilde{\mathscr{L}}(\overline{I_r^{(k)}}) \arrow[d, "\tilde{\mathcal{R}}_k"] \\
\mathrm{gr}_z \mathscr{L}(\overline{I_z^{(k+3)}}) \arrow[r, "\varphi_z(\overline{I_z^{(k+3)}})"'] & \mathrm{gr}_z \tilde{\mathscr{L}}(\overline{I_z^{(k+3)}})
\end{tikzcd}
$$

Thus $G_{\vartheta_0}(\lambda)$ is compatible with the deformation datum $\mathfrak{R}(\sigma)$ and $\mathfrak{R}(\tilde{\sigma})$ and finally $G_{\vartheta_0}(\lambda) \in \mathrm{Hom}_{\mathfrak{S}\mathfrak{H}(C)}(\boldsymbol{Sh}, \tilde{\boldsymbol{Sh}})$. The case $r \leq_{\vartheta_{\nu_o}} z$ can be proven analogously.
By the definition of G_{ϑ_0} on morphisms of $\mathfrak{S}\mathfrak{D}(C, \vartheta_0)$, the functoriality of G_{ϑ_0} follows immediately. $\square$

5.3 Equivalence of categories: $\mathfrak{S}\mathfrak{D}(C, \vartheta_0)$ and $\mathfrak{S}\mathfrak{H}(C)$

After constructing the functors F_{ϑ_0} and G_{ϑ_0} we finally want to show that these are equivalences of categories. Without loss of generality we assume $\vartheta_0 \in [0, \frac{\pi}{2})$.

Theorem 5.8. *Let $\vartheta_0 \in [0, \frac{\pi}{2})$ be a generic direction with respect to $C = \{r, z\}$ with $r \in \mathbb{R}_{>0}$ and $z \in \mathbb{C} \smallsetminus \mathbb{R}$. Then G_{ϑ_0} (defined in 5.7) is a quasi-inverse functor of F_{ϑ_0} (defined in 5.4).*

Proof. We first check $F_{\vartheta_0} \circ G_{\vartheta_0} \cong \mathrm{id}_{\mathfrak{SD}(C,\vartheta_0)}$. Let $\sigma = ((G_c^{(\nu)})_{c\in C}, S^{(\nu+1,\nu)})_{\nu\in\mathbb{Z}/4\mathbb{Z}}$ be an object in $\mathfrak{SD}(C, \vartheta_0)$ and $G_{\vartheta_0}(\sigma) = (\mathfrak{L}(\sigma), \mathfrak{R}(\sigma))$ the associated Stokes shell. We solely consider $\mathrm{Im}(z) > 0$, $\vartheta_0 \in [\frac{\pi}{4}, \frac{\pi}{2})$ (resp. $\mathrm{Im}(z) < 0$ and $\vartheta_0 \in [0, \frac{\pi}{4})$) and $z \leq_{\vartheta_0} r$ as all other cases can be proven similar. In this scenario we already saw that $\mathrm{gr}_c \mathscr{L}(\overline{I_c^{(\nu)}}) \cong G_c^{(\nu)}$ for each $\nu \in \mathbb{Z}/4\mathbb{Z}$ and $c \in C$. As in the proof of (5.7), we label this isomorphism by $\psi_c^{(\nu)} : \mathrm{gr}_c \mathscr{L}(\overline{I_c^{(\nu)}}) \longrightarrow G_c^{(\nu)}$.
We get $F_{\vartheta_0}(G_{\vartheta_0}(\sigma)) = ((\tilde{G}_c^{(\nu)})_{c\in C}, \tilde{S}^{(\nu+1,\nu)})_{\nu\in\mathbb{Z}/4\mathbb{Z}}$ where

- $\tilde{G}_r^{(\nu)} = \mathrm{gr}_r \mathscr{L}(\overline{I_r^{(\nu)}}) \overset{\psi_r^{(\nu)}}{\cong} G_r^{(\nu)}$,
- $\tilde{G}_z^{(\nu)} = \mathrm{gr}_z \mathscr{L}(\overline{I_z^{(\nu+3)}}) \overset{\psi_z^{(\nu+3)}}{\cong} G_z^{(\nu+3)}$,
- $\tilde{S}^{(\nu+1,\nu)} = \begin{pmatrix} \Phi_z^{(\nu,\nu+3)} & 0 \\ \Phi_r^{(\nu+1,\nu)}\mathcal{R}_\nu & \Phi_r^{(\nu+1,\nu)} \end{pmatrix}$ for $\nu \in \{0, 2\}$ and
- $\tilde{S}^{(\nu+1,\nu)} = \begin{pmatrix} \Phi_z^{(\nu,\nu+3)} & \Phi_z^{(\nu,\nu+3)}\mathcal{R}_\nu \\ 0 & \Phi_r^{(\nu+1,\nu)} \end{pmatrix}$ for $\nu \in \{1, 3\}$,

where $\Phi_c^{(\nu+1,\nu)} : \mathrm{gr}_c \mathscr{L}(\overline{I_c^{(\nu)}}) \to \mathrm{gr}_c \mathscr{L}(\overline{I_c^{(\nu+1)}})$ is given by $(\psi_c^{(\nu+1)})^{-1} S_{cc}^{(\nu+1,\nu)} \psi_c^{(\nu)}$ for $c \in C$ and

- for $\nu \in \{0, 2\}$, $\mathcal{R}_\nu : \mathrm{gr}_z \mathscr{L}(\overline{I_z^{(\nu+3)}}) \longrightarrow \mathrm{gr}_r \mathscr{L}(\overline{I_r^{(\nu)}})$ is given by
$$\mathcal{R}_\nu = (\psi_r^{(\nu)})^{-1} (S_{rr}^{(\nu+1,\nu)})^{-1} S_{rz}^{(\nu+1,\nu)} S_{zz}^{(\nu,\nu+3)} \psi_z^{(\nu+3)}$$
and
- for $\nu \in \{1, 3\}$, $\mathcal{R}_\nu : \mathrm{gr}_r \mathscr{L}(\overline{I_r^{(\nu)}}) \longrightarrow \mathrm{gr}_z \mathscr{L}(\overline{I_z^{(\nu+3)}})$ is given by
$$\mathcal{R}_\nu = (\psi_z^{(\nu+3)})^{-1} (S_{zz}^{(\nu,\nu+3)})^{-1} (S_{zz}^{(\nu+1,\nu)})^{-1} S_{zr}^{(\nu+1,\nu)} \psi_r^{(\nu)}.$$

Setting
$$\alpha_\sigma^{(\nu)} := \begin{pmatrix} S_{zz}^{(\nu,\nu+3)} \psi_z^{(\nu+3)} & 0 \\ 0 & \psi_r^{(\nu)} \end{pmatrix} : \tilde{G}_z^{(\nu)} \oplus \tilde{G}_r^{(\nu)} \to G_z^{(\nu)} \oplus G_r^{(\nu)}$$

for $\nu \in \mathbb{Z}/4\mathbb{Z}$ gives a morphism $\alpha_\sigma := (\alpha_\sigma^{(\nu)})_{\nu\in\mathbb{Z}/4\mathbb{Z}} \in \mathrm{Hom}_{\mathfrak{SD}(C,\vartheta_0)}(F_{\vartheta_0}(G_{\vartheta_0}(\sigma)),\sigma)$ as for $\nu \in \{0,2\}$

$$\begin{aligned}
\alpha_\sigma^{(\nu+1)}\tilde{S}^{(\nu+1,\nu)} &= \begin{pmatrix} S_{zz}^{(\nu+1,\nu)}\psi_z^{(\nu)} & 0 \\ 0 & \psi_r^{(\nu+1)} \end{pmatrix}\begin{pmatrix} \Phi_z^{(\nu,\nu+3)} & 0 \\ \Phi_r^{(\nu+1,\nu)}\mathcal{R}_\nu & \Phi_r^{(\nu+1,\nu)} \end{pmatrix} = \\
&= \begin{pmatrix} S_{zz}^{(\nu+1,\nu)}S_{zz}^{(\nu,\nu+3)}\psi_z^{(\nu+3)} & 0 \\ S_{rz}^{(\nu+1,\nu)}S_{zz}^{(\nu,\nu+3)}\psi_z^{(\nu+3)} & S_{rr}^{(\nu+1,\nu)}\psi_r^{(\nu)} \end{pmatrix} = \\
&= \begin{pmatrix} S_{zz}^{(\nu+1,\nu)} & 0 \\ S_{rz}^{(\nu+1,\nu)} & S_{rr}^{(\nu+1,\nu)} \end{pmatrix}\begin{pmatrix} S_{zz}^{(\nu,\nu+3)}\psi_z^{(\nu+3)} & 0 \\ 0 & \psi_r^{(\nu)} \end{pmatrix} = S^{(\nu+1,\nu)}\alpha_\sigma^{(\nu)}
\end{aligned}$$

and analogue for $\nu \in \{1,3\}$

$$\begin{aligned}
\alpha_\sigma^{(\nu+1)}\tilde{S}^{(\nu+1,\nu)} &= \begin{pmatrix} S_{zz}^{(\nu+1,\nu)}\psi_z^{(\nu)} & 0 \\ 0 & \psi_r^{(\nu+1)} \end{pmatrix}\begin{pmatrix} \Phi_z^{(\nu,\nu+3)} & \Phi_z^{(\nu,\nu+3)}\mathcal{R}_\nu \\ 0 & \Phi_r^{(\nu+1,\nu)} \end{pmatrix} = \\
&= \begin{pmatrix} S_{zz}^{(\nu+1,\nu)}S_{zz}^{(\nu,\nu+3)}\psi_z^{(\nu+3)} & S_{zr}^{(\nu+1,\nu)}\psi_r^{(\nu)} \\ 0 & S_{rr}^{(\nu+1,\nu)}\psi_r^{(\nu)} \end{pmatrix} = \\
&= \begin{pmatrix} S_{zz}^{(\nu+1,\nu)} & S_{rz}^{(\nu+1,\nu)} \\ 0 & S_{rr}^{(\nu+1,\nu)} \end{pmatrix}\begin{pmatrix} S_{zz}^{(\nu,\nu+3)}\psi_z^{(\nu+3)} & 0 \\ 0 & \psi_r^{(\nu)} \end{pmatrix} = S^{(\nu+1,\nu)}\alpha_\sigma^{(\nu)}.
\end{aligned}$$

Since $\alpha_\sigma^{(\nu)}$ is an isomorphism of vector spaces for each ν, α_σ is an isomorphism in $\mathfrak{SD}(C,\vartheta_0)$. Thus we have indeed $F_{\vartheta_0}(G_{\vartheta_0}(\sigma)) \cong \sigma$. It is left to show that $\alpha := (\alpha_\sigma)_{\sigma\in\mathfrak{SD}(C,\vartheta_0)}$ defines a natural transformation $F_{\vartheta_0}G_{\vartheta_0} \Rightarrow \mathrm{id}_{\mathfrak{SD}(C,\vartheta_0)}$.
Let $\lambda \in \mathrm{Hom}_{\mathfrak{SD}(C,\vartheta_0)}(\sigma,\tilde{\sigma})$ be a morphism between $\sigma = ((G_c^{(\nu)})_{c\in C}, S^{(\nu+1,\nu)})_{\nu\in\mathbb{Z}/4\mathbb{Z}}$ to $\tilde{\sigma} = ((\tilde{G}_c^{(\nu)})_{c\in C}, \tilde{S}^{(\nu+1,\nu)})_{\nu\in\mathbb{Z}/4\mathbb{Z}}$. Again we write $\lambda = (\lambda^{(\nu)})_{\nu\in\mathbb{Z}/4\mathbb{Z}}$ with

$$\lambda^{(\nu)} = \begin{pmatrix} \lambda_{z,z}^{(\nu)} & 0 \\ 0 & \lambda_{r,r}^{(\nu)} \end{pmatrix} : G_z^{(\nu)} \oplus G_r^{(\nu)} \to \tilde{G}_z^{(\nu)} \oplus \tilde{G}_r^{(\nu)}.$$

Then

$$F_{\vartheta_0}(G_{\vartheta_0}(\lambda))^{(\nu)} : \mathrm{gr}_z\,\mathscr{L}(\overline{I_z^{(\nu+3)}}) \oplus \mathrm{gr}_r\,\mathscr{L}(\overline{I_r^{(\nu)}}) \to \mathrm{gr}_z\,\tilde{\mathscr{L}}(\overline{I_z^{(\nu+3)}}) \oplus \mathrm{gr}_r\,\tilde{\mathscr{L}}(\overline{I_r^{(\nu)}})$$

is given by

$$\begin{aligned}
F_{\vartheta_0}(G_{\vartheta_0}(\lambda))^{(\nu)} &= \begin{pmatrix} (G_{\vartheta_0}(\lambda))(\overline{I_z^{(\nu+3)}})|_{\mathrm{gr}_z\,\mathscr{L}(\overline{I_z^{(\nu+3)}})} & 0 \\ 0 & (G_{\vartheta_0}(\lambda))(\overline{I_r^{(\nu)}})|_{\mathrm{gr}_r\,\mathscr{L}(\overline{I_r^{(\nu)}})} \end{pmatrix} \\
&= \begin{pmatrix} (\tilde{\psi}_z^{(\nu+3)})^{-1}\lambda_{z,z}^{(\nu+3)}\psi_z^{(\nu+3)} & 0 \\ 0 & (\tilde{\psi}_r^{(\nu)})^{-1}\lambda_{r,r}^{(\nu)}\psi_r^{(\nu)} \end{pmatrix}.
\end{aligned}$$

Then

$$\begin{aligned}\alpha_{\tilde{\sigma}}^{(\nu)} F_{\vartheta_0}(G_{\vartheta_0}(\lambda))^{(\nu)} &= \begin{pmatrix} \tilde{S}_{zz}^{(\nu,\nu+3)} \tilde{\psi}_z^{(\nu+3)} (\tilde{\psi}_z^{(\nu+3)})^{-1} \lambda_{z,z}^{(\nu+3)} \psi_z^{(\nu+3)} & 0 \\ 0 & \tilde{\psi}_r^{(\nu)} (\tilde{\psi}_r^{(\nu)})^{-1} \lambda_{r,r}^{(\nu)} \psi_r^{(\nu)} \end{pmatrix} \\ &= \begin{pmatrix} \tilde{S}_{zz}^{(\nu,\nu+3)} \lambda_{z,z}^{(\nu+3)} \psi_z^{(\nu+3)} & 0 \\ 0 & \lambda_{r,r}^{(\nu)} \psi_r^{(\nu)} \end{pmatrix} \\ &= \begin{pmatrix} \lambda_{z,z}^{(\nu)} \tilde{S}_{zz}^{(\nu,\nu+3)} \psi_z^{(\nu+3)} & 0 \\ 0 & \lambda_{r,r}^{(\nu)} \psi_r^{(\nu)} \end{pmatrix} \\ &= \begin{pmatrix} \lambda_{z,z}^{(\nu)} & 0 \\ 0 & \lambda_{r,r}^{(\nu)} \end{pmatrix} \begin{pmatrix} S_{zz}^{(\nu,\nu+3)} \psi_z^{(\nu+3)} & 0 \\ 0 & \psi_r^{(\nu)} \end{pmatrix} = \lambda^{(\nu)} \alpha_\sigma^{(\nu)},\end{aligned}$$

which shows that $\alpha : F_{\vartheta_0} \circ G_{\vartheta_0} \Rightarrow \mathrm{id}_{\mathfrak{SD}(C,\vartheta_0)}$ is a natural Isomorphism. Thus $F_{\vartheta_0} \circ G_{\vartheta_0} \cong \mathrm{id}_{\mathfrak{SD}(C,\vartheta_0)}$.

Now we check $G_{\vartheta_0} \circ F_{\vartheta_0} \cong \mathrm{id}_{\mathfrak{SH}(C)}$. Let $\boldsymbol{Sh} = ((\operatorname{gr} \mathscr{L}, \operatorname{gr} \mathscr{L}_{\leq \bullet}), \boldsymbol{R})$ be a Stokes shell of Gaussian type C and $F_{\vartheta_0}(\boldsymbol{Sh}) = ((G_c^{(\nu)})_{c \in C}, S^{(\nu+1,\nu)})_{\nu \in \mathbb{Z}/4\mathbb{Z}}$ the associated object in $\mathfrak{SD}(C, \vartheta_0)$. Since all cases follow a similar logic, we solely proof $G_{\vartheta_0} \circ F_{\vartheta_0} \cong \mathrm{id}_{\mathfrak{SH}(C)}$ for $\operatorname{Im}(z) > 0$, $\vartheta_0 \in [\frac{\pi}{4}, \frac{\pi}{2})$ (resp. $\operatorname{Im}(z) < 0$, $\vartheta_0 \in [0, \frac{\pi}{4})$) and $z \leq_{\vartheta_0} r$. First notice that by construction $\mathfrak{L}(F_{\vartheta_o}(\boldsymbol{Sh})) = (\operatorname{gr} \tilde{\mathscr{L}}, \operatorname{gr} \tilde{\mathscr{L}}_{\leq \bullet}) \cong (\operatorname{gr} \mathscr{L}, \operatorname{gr} \mathscr{L}_{\leq \bullet})$:
It is enough to check $\operatorname{gr}_c \mathscr{L} \cong \operatorname{gr}_c \tilde{\mathscr{L}}$ for all $c \in C$. We begin with $r \in C$. Recall that $G_r^{(\nu)} = \operatorname{gr}_r \mathscr{L}(\overline{I_r^{(\nu)}})$ and $\operatorname{gr}_r \tilde{\mathscr{L}}$ is the sheaf that one receives gluing $(\underline{G_r^{(\nu)}}_{I_\varepsilon^{(\nu)}})_{\nu \in \mathbb{Z}/4\mathbb{Z}}$ via the isomorphisms $(S_{rr}^{(\nu+1,\nu)})_{\nu \in \mathbb{Z}/4\mathbb{Z}}$ of $F_{\vartheta_0}(\boldsymbol{Sh})$, that are $(\Phi_r^{(\nu+1,\nu)})_{\nu \in \mathbb{Z}/4\mathbb{Z}}$ of $\mathfrak{D}(\boldsymbol{Sh})$. Since $\operatorname{gr}_r \mathscr{L}$ is a local system, in particular constant on the interval $I_\varepsilon^{(\nu)}$ (given as in the proof of lemma (5.5)) and since $\overline{I_r^{(\nu)}}$ is connected we get isomorphisms

$$\varphi_r^{(\nu)} : \operatorname{gr}_r \mathscr{L}_{|_{I_\varepsilon^{(\nu)}}} \longrightarrow \underline{\operatorname{gr}_r \mathscr{L}(\overline{I_r^{(\nu)}})}_{I_\varepsilon^{(\nu)}} = \underline{G_r^{(\nu)}}_{I_\varepsilon^{(\nu)}}.$$

As $\Phi_r^{(\nu+1,\nu)}$ is defined by $\operatorname{gr}_r \mathscr{L}(\overline{I_r^{(\nu)}}) \cong \operatorname{gr}_r \mathscr{L}_{\vartheta_r^{(\nu+1)}} \cong \operatorname{gr}_r \mathscr{L}(\overline{I_r^{(\nu+1)}})$ and since $\vartheta_r^{(\nu+1)} \in I_\varepsilon^{(\nu)} \cup I_\varepsilon^{(\nu+1)}$ the morphisms $(\varphi_r^{(\nu)})_{\nu \in \mathbb{Z}/4\mathbb{Z}}$ satisfy the condition

$$\Phi_r^{(\nu+1,\nu)} = \varphi_r^{(\nu+1)} \circ (\varphi_r^{(\nu)})^{-1}$$

on $I_\varepsilon^{(\nu)} \cap I_\varepsilon^{(\nu+1)}$. Since the glued sheaf $\operatorname{gr}_r \tilde{\mathscr{L}}$ is unique up to isomorphim (cf. lemma (2.2)), there exists an isomorphism $\lambda_r : \operatorname{gr}_r \mathscr{L} \to \operatorname{gr}_r \tilde{\mathscr{L}}$, mapping $s \in \operatorname{gr} \mathscr{L}(U)$ to $(\varphi_r^{(\nu)}(U \cap I_\varepsilon^{(\nu)})(s_{|_{U \cap I_\varepsilon^{(\nu)}}}))_{\nu \in \mathbb{Z}/4\mathbb{Z}}$, that lets the diagram

$$\begin{tikzcd}
\operatorname{gr}_r \mathscr{L} \arrow[rr, "\lambda_r"] \arrow[d, "(\bullet)|_{I_\varepsilon^{(\nu)}}"'] & & \operatorname{gr}_r \tilde{\mathscr{L}} \arrow[d, "(\bullet)|_{I_\varepsilon^{(\nu)}}"] \\
\operatorname{gr}_r \mathscr{L}|_{I_\varepsilon^{(\nu)}} \arrow[rr, "(\lambda_r)|_{I_\varepsilon^{(\nu)}}"'] \arrow[dr, "\varphi_r^{(\nu)}"] & & \operatorname{gr}_r \tilde{\mathscr{L}}|_{I_\varepsilon^{(\nu)}} \arrow[dl, "\psi_r^{(\nu)}"] \\
& \underline{\operatorname{gr}_r \mathscr{L}(\overline{I_r^{(\nu)}})}_{I_\varepsilon^{(\nu)}} &
\end{tikzcd}$$

commute. Here, $\psi_r^{(\nu)}$ is the isomorphism of the glued sheaf $\operatorname{gr}_r \tilde{\mathscr{L}}$, precisely for an open set $U \subseteq I_\varepsilon^{(\nu)}$, $\psi_r^{(\nu)}(U)$ sends a tuple $(s_\nu)_{\nu\in\mathbb{Z}/4\mathbb{Z}}$ to $s_\nu \in \underline{\operatorname{gr}_r \mathscr{L}(\overline{I_r^{(\nu)}})}_{I_\varepsilon^{(\nu)}}$. Analogously for z, recall that $G_z^{(\nu)} = \operatorname{gr}_z \mathscr{L}(\overline{I_z^{(\nu+3)}})$ and $\operatorname{gr}_r \tilde{\mathscr{L}}$ is the sheaf that one receives gluing $(\underline{G_z^{(\nu)}}_{I_\varepsilon^{(\nu)}})_{\nu\in\mathbb{Z}/4\mathbb{Z}}$ via the isomorphisms $(S_{zz}^{(\nu+1,\nu)})_{\nu\in\mathbb{Z}/4\mathbb{Z}}$ of $F_{\vartheta_0}(\boldsymbol{Sh})$, that are $(\Phi_z^{(\nu,\nu+3)})_{\nu\in\mathbb{Z}/4\mathbb{Z}}$ of $\mathfrak{D}(\boldsymbol{Sh})$. Since $\operatorname{gr}_z \mathscr{L}$ is a local system, we get isomorphisms

$$\varphi_z^{(\nu)} : \operatorname{gr}_z \mathscr{L}_{I_\varepsilon^{(\nu)}} \to \underline{\operatorname{gr}_z \mathscr{L}(\overline{I_z^{(\nu+3)}})}_{I_\varepsilon^{(\nu)}} = \underline{G_z^{(\nu)}}_{I_\varepsilon^{(\nu)}}$$

for each $\nu \in \mathbb{Z}/4\mathbb{Z}$. As $\Phi_z^{(\nu,\nu+3)}$ is given by $\operatorname{gr}_z \mathscr{L}(\overline{I_z^{(\nu+3)}}) \cong \operatorname{gr}_z \mathscr{L}_{\vartheta_z^{(\nu)}} \cong \operatorname{gr}_z \mathscr{L}(\overline{I_z^{(\nu)}})$ and since in our case $\vartheta_z^{(\nu)} \in I_\varepsilon^{(\nu)} \cup I_\varepsilon^{(\nu+3)}$ the morphisms $(\varphi_z^{(\nu)})_{\nu\in\mathbb{Z}/4\mathbb{Z}}$ satisfy the equation $\Phi_z^{(\nu,\nu+3)} = \varphi_z^{(\nu+1)}\varphi_z^{(\nu)-1}$ on $I_\varepsilon^{(\nu)} \cap I_\varepsilon^{(\nu+1)}$. Again we get a commutative diagram

$$\begin{tikzcd}
\operatorname{gr}_z \mathscr{L} \arrow[rr, "\lambda_z"] \arrow[d, "(\bullet)|_{I_\varepsilon^{(\nu)}}"'] & & \operatorname{gr}_z \tilde{\mathscr{L}} \arrow[d, "(\bullet)|_{I_\varepsilon^{(\nu)}}"] \\
\operatorname{gr}_z \mathscr{L}|_{I_\varepsilon^{(\nu)}} \arrow[rr, "(\lambda_z)|_{I_\varepsilon^{(\nu)}}"'] \arrow[dr, "\varphi_z^{(\nu)}"] & & \operatorname{gr}_z \tilde{\mathscr{L}}|_{I_\varepsilon^{(\nu)}} \arrow[dl, "\psi_z^{(\nu)}"] \\
& \underline{\operatorname{gr}_z \mathscr{L}(\overline{I_z^{(\nu+3)}})}_{I_\varepsilon^{(\nu)}} &
\end{tikzcd}$$

The morphisms λ_r, λ_z induce an isomorphism of graded Stokes filtered local systems $\beta_{\boldsymbol{Sh}} : \operatorname{gr}\mathscr{L} \to \operatorname{gr}\tilde{\mathscr{L}}$ (compare to proof of proposition (5.7)). It is left to show that $\beta_{\boldsymbol{Sh}}$ is compatible with the deformation datum. To do so, we first have to describe the deformation datum of $G_{\vartheta_0}(F_{\vartheta_0}(\boldsymbol{Sh}))$. In the case we are considering ($\operatorname{Im}(z) > 0, \vartheta_0 \in [\frac{\pi}{4}, \frac{\pi}{2})$ and $z \leq_{\vartheta_0} r$), the deformation datum $\mathfrak{R}(F_{\vartheta_0}(\boldsymbol{Sh})) = (\tilde{\mathcal{R}}_\nu)_{\nu\in\mathbb{Z}/4\mathbb{Z}}$ is given by

$$\begin{aligned}\tilde{\mathcal{R}}_\nu &: \operatorname{gr}_z\tilde{\mathscr{L}}(\overline{I_z^{(\nu+3)}}) \cong G_z^{(\nu+3)} \xrightarrow{S_\nu} G_r^{(\nu)} \cong \operatorname{gr}_r\tilde{\mathscr{L}}(\overline{I_r^{(\nu)}}) \text{ with}\\ &\quad S_\nu := (S_{rr}^{(\nu+1,\nu)})^{-1} \circ S_{rz}^{(\nu+1,\nu)} \circ S_{zz}^{(\nu,\nu+3)} \text{ for } \nu \in \{0,2\},\\ \tilde{\mathcal{R}}_\nu &: \operatorname{gr}_r\tilde{\mathscr{L}}(\overline{I_r^{(\nu)}}) \cong G_r^{(\nu)} \xrightarrow{S_\nu} G_z^{(\nu+3)} \cong \operatorname{gr}_z\tilde{\mathscr{L}}(\overline{I_z^{(\nu+3)}}) \text{ with}\\ &\quad S_\nu := (S_{zz}^{(\nu,\nu+3)})^{-1} \circ (S_{zz}^{(\nu+1,\nu)})^{-1} \circ S_{zr}^{(\nu+1,\nu)} \text{ for } \nu \in \{1,3\},\end{aligned}$$

where

$$S^{(\nu+1,\nu)} = \begin{pmatrix} \Phi_z^{(\nu,\nu+3)} & 0 \\ \Phi_r^{(\nu+1,\nu)} \circ \mathcal{R}_\nu & \Phi_r^{(\nu+1,\nu)} \end{pmatrix} \qquad \text{for } \nu \in \{0,2\} \text{ and}$$

$$S^{(\nu+1,\nu)} = \begin{pmatrix} \Phi_z^{(\nu,\nu+3)} & \Phi_z^{(\nu,\nu+3)} \circ \mathcal{R}_\nu \\ 0 & \Phi_r^{(\nu+1,\nu)} \end{pmatrix} \qquad \text{for } \nu \in \{1,3\}.$$

Thus the deformation datum $(\tilde{\mathcal{R}}_\nu)_{\nu\in\mathbb{Z}/4\mathbb{Z}}$ simplifies to

$$\begin{aligned}\tilde{\mathcal{R}}_\nu &: \operatorname{gr}_z\tilde{\mathscr{L}}(\overline{I_z^{(\nu+3)}}) \overset{\psi_z^{(\nu+3)}}{\cong} G_z^{(\nu+3)} \xrightarrow{S_\nu} G_r^{(\nu)} \overset{\psi_r^{(\nu)}}{\cong} \operatorname{gr}_r\tilde{\mathscr{L}}(\overline{I_r^{(\nu)}}) \text{ with}\\ &\quad S_\nu := \mathcal{R}_\nu \circ \Phi_z^{(\nu+3,\nu+2)} \text{ for } \nu \in \{0,2\},\\ \tilde{\mathcal{R}}_\nu &: \operatorname{gr}_r\tilde{\mathscr{L}}(\overline{I_r^{(\nu)}}) \overset{\psi_r^{(\nu)}}{\cong} G_r^{(\nu)} \xrightarrow{S_\nu} G_z^{(\nu+3)} \overset{\psi_z^{(\nu+3)}}{\cong} \operatorname{gr}_z\tilde{\mathscr{L}}(\overline{I_z^{(\nu+3)}}) \text{ with}\\ &\quad S_\nu := (\Phi_z^{(\nu+3,\nu+2)})^{-1} \circ \mathcal{R}_\nu \text{ for } \nu \in \{1,3\}.\end{aligned}$$

The proof that $\beta_{\boldsymbol{Sh}}$ is compatible with the deformation datum can be done analogously to the proof of (5.7). Therefore we consider $\beta_{\boldsymbol{Sh}|\operatorname{gr}_c\mathscr{L}} = \lambda_c : \operatorname{gr}_c\mathscr{L} \to \operatorname{gr}_c\tilde{\mathscr{L}}$ for $c \in \{r,z\}$ and for $\nu \in \{0,2\}$ one has to check that

$$\lambda_r(\overline{I_r^{(\nu)}}) \circ \mathcal{R}_\nu = \tilde{\mathcal{R}}_\nu \circ \lambda_z(\overline{I_z^{(\nu+3)}}),$$

which follows directly by the definition of $\tilde{\mathcal{R}}_\nu$ and the gluing data $\varphi_r^{(\nu)}, \varphi_z^{(\nu+3)}$. Since λ_r, λ_z are isomorphisms of local systems, $\beta_{\boldsymbol{Sh}} : \boldsymbol{Sh} \to G_{\vartheta_0}(F_{\vartheta_0}(\boldsymbol{Sh}))$ is an isomorphism of Stokes shells, that gives the data of a natural transformation from $\mathrm{id}_{\mathfrak{S}\mathfrak{D}(C,\vartheta_0)}$ to $G_{\vartheta_0}F_{\vartheta_0}$.

Therefore, F_{ϑ_0} and G_{ϑ_0} are quasi-inverse functors. $\square$

Considering the Stokes structure on a differential systems of pure Gaussian type C by using Stokes shells instead of Stokes data has a big advantage: One does not have to choose a generic direction with respect to C. The deformation datum only depends on the set of exponential factors of the graded Stokes filtered local system, while the Stokes data may vary when choosing different generic directions.

6 Outlook

In this thesis we have excusively considered the category equivalence

$$\mathfrak{S}\mathfrak{H}(\{r, z\}) \to \mathfrak{S}\mathfrak{D}(\{r, z\}, \vartheta_0)$$

for differential systems of pure Gaussian type $\{r, z\}$. However, there are several possibilities for further research. For instance, it is possible to extend our analysis by introducing non-aligned elements to $\{r, z\}$ and investigate the deformation datum of a graded Stokes filtered local system of Gaussian type $C = \{r, z_1, z_2\}$. Moreover one could try to describe the effect on the deformation datum when a non-aligend element is added to an arbitrary, non-empty finite subset $C \subseteq \mathbb{C}$ where $C \cong [C]$. Besides one could also add aligned elements to $\{r, z\}$. To define Stokes shells of type C that contains aligned elements $z, \lambda z \in C$ with $\lambda > 0$, one needs to generalize the definitions that occur in chapter 4, as in T. Mochizuki's work [5]. One could also define Stokes shells for differential systems that are not necessarily of Gaussian type. To do that one has to generalize the definition of Stokes structures, as C. Sabbah does in [7].

Bibliography

[1] P. Deligne. “Lettre à B. Malgrange du 19/4/1978”. In: *Singularités irrégulières, Correspondance et documents.* Documents mathématiques, vol. 5 Société Mathématique de France, Paris, 2007.

[2] Robin Hartshorne. *Algebraic Geometry.* Springer New York, NY, 1977. URL: https://doi.org/10.1007/978-1-4757-3849-0.

[3] C. Hertling and C. Sabbah. “Examples of non-commutative Hodge structures”. In: *Journal of the Institute of Mathematics of Jussieu* 10.3 (Mar. 2011), pp. 635–674. DOI: 10.1017/s147474801100003x. URL: https://doi.org/10.1017%2Fs147474801100003x.

[4] A. Hohl. “D-modules of pure Gaussian type and enhanced ind-sheaves”. In: *manuscripta mathematica* 167.3-4 (Apr. 2021), pp. 435–467. DOI: 10.1007/s00229-021-01281-y. URL: https://doi.org/10.1007%2Fs00229-021-01281-y.

[5] T. Mochizuki. *Stokes shells and Fourier transforms.* 2021. arXiv: 1808.01037 [math.AG].

[6] C. Sabbah. “Differential systems of pure Gaussian type”. In: *Izvestiya: Mathematics* 80.1 (Feb. 2016), pp. 189–220. DOI: 10.1070/im8308. URL: https://doi.org/10.1070%2Fim8308.

[7] C. Sabbah. *Introduction to Stokes Structures.* Springer Berlin Heidelberg, 2013. DOI: 10.1007/978-3-642-31695-1. URL: https://doi.org/10.1007%2F978-3-642-31695-1.

[8] H. Schubert. *Kategorien II.* Springer Berlin Heidelberg, 1970. URL: https://doi.org/10.1007/978-3-642-95156-5.

[9] T. A. Springer. *Linear Algebraic Groups.* Birkhäuser Boston, MA, 2008. URL: https://doi.org/10.1007/978-0-8176-4840-4.

[10] T. Szamuely. *Galois Groups and Fundamental Groups.* Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2009. DOI: 10.1017/CBO9780511627064.

Eidesstattliche Erklärung

Hiermit erkläre ich, dass ich die vorliegende Masterarbeit mit dem Titel *Stokes filtered local systems and Stokes shells* selbständig verfasst habe, dass ich sie zuvor an keiner anderen Hochschule und in keinem anderen Studiengang als Prüfungsleistung eingereicht habe und dass ich keine anderen als die angegebenen Quellen benutzt habe.

Augsburg, Dezember 2023	
Ort, Datum	Unterschrift