Topic :

The change of bases and the change of coordinates.

Discussion :

Let $A_{old} = (v_1, v_2, \cdots, v_n)$,

$A_{new} = (v_1', v_2', \cdots, v_n')$,

where $\{v_1, v_2, \cdots, v_n\}$ and $\{v_1', v_2', \cdots, v_n'\}$ are two bases of $\mathbb{R}^n$, and the relation between the two sets of basis vectors is

$$\begin{cases} v_1' = a_{11} \cdot v_1 + a_{21} \cdot v_2 + \cdots + a_{n1} \cdot v_n \\ v_2' = a_{12} \cdot v_1 + a_{22} v_2 + \cdots + a_{n2} v_n \\ \cdots\cdots \\ v_n' = a_{1n} \cdot v_1 + a_{2n} \cdot v_2 + \cdots + a_{nn} \cdot v_n \end{cases}$$

$$\Rightarrow \underbrace{(v_1' \ v_2' \ \cdots \ v_n')}_{A_{new}} = \underbrace{(v_1 \ v_2 \ \cdots \ v_n)}_{A_{old}} \cdot \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}}_{P_{n\times n}}$$

$\Rightarrow$ $A_{new} = A_{old} P$, where P is called the transition matrix.

(Since $v_1, \cdots v_n$ are linearly indepent vectors in $\mathbb{R}^n$, then A_{old} is invertible

Similarly, we can show that A_{new} is invertible.

$\Rightarrow P = A_{old}^{-1} \cdot A_{new} \Rightarrow \det(A) = \det(A_{old}^{-1} \cdot A_{new}) = \frac{1}{\det(A_{old})} \cdot \det(A_{new}) \neq 0$

$\Rightarrow$ P is invertible $\Rightarrow$ A_{old}, A_{new}, P are all invertible.

Let the coordinate of a vector u, $u \in \mathbb{R}^n$ relative to the two bases are

$X_{old} = (x_1, x_2, \cdots, x_n)^T$, and $X_{new} = (x_1', x_2', \cdots, x_n')^T$.

Then $u = x_1' \cdot v_1' + x_2' \cdot v_2' + \cdots + x_n' v_n' = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n$

$\Rightarrow (v_1' \ v_2' \ \cdots \ v_n') \cdot (x_1', x_2', \cdots x_n')^T = (v_1 \ v_2 \ \cdots \ v_n) \cdot (x_1, x_2, \cdots x_n)^T$

$$\Rightarrow \quad A_{new} \cdot x_{new} = A_{old} \cdot x_{old}$$

$$\Rightarrow \quad x_{new} = A_{new}^{-1} \cdot A_{old} \cdot x_{old} \quad (*)$$

Since $A_{new} = A_{old} \cdot P$, then $A_{old} = A_{new} \cdot P^{-1}$.

$$\Rightarrow \quad x_{new} = A_{new}^{-1} \cdot A_{new} \cdot P^{-1} \cdot x_{old}$$

$$= P^{-1} \cdot x_{old}. \quad \blacksquare$$

Conclusion:

If the change of bases is $A_{new} = A_{old} \cdot P$,

the change of coordinates is $x_{new} = P^{-1} \cdot x_{old}$.