1. (i) Use both the standard approach by setting up the vector equations, and some "shortcut" methods to determine whether the following sets are linearly independent.

(ii) Which of these sets are bases for $\mathbb{R}^3$?

(a) $S_1 = \{(1, 0, -1), (-1, 2, 3)\}$.

(b) $S_2 = \{(1, 0, -1), (-1, 2, 3), (0, 3, 0)\}$.

(c) $S_3 = \{(1, 0, -1), (-1, 2, 3), (0, 3, 3)\}$.

(d) $S_4 = \{(1, 0, -1), (-1, 2, 3), (0, 3, 0), (1, -1, 1)\}$.

Solution :

(i) Standard Approach

(a) ① set up the vector equation,

$$c_1(1, 0, -1) + c_2(-1, 2, 3) = (0, 0, 0)$$

② Convert them into linear system,

$$\begin{cases} 1 \cdot c_1 + (-1) c_2 = 0 \\ 0 \cdot c_1 + 2 c_2 = 0 \\ -1 \cdot c_1 + 3 c_2 = 0 \end{cases}$$

③ write down the augmented matrix of this linear system,

$$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 2 & 0 \\ -1 & 3 & 0 \end{array}\right) \xrightarrow{G.E.} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

As the last column is non-pivot, and the rest columns are pivot, this homogeneous system has exactly one solution — trivial solution, i.e. $c_1 = c_2 = 0$.

Hence, the two vectors are linearly independent.

(b). ① Set up the vector equation,

$$c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} = 0$$

② Write down the augmented matrix of this linear system directly,

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ -1 & 3 & 0 & 0 \end{array}\right) \xrightarrow{G.E.} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right).$$

As the last column is non-pivot, and the rest columns are pivot, this homogeneous system has exactly one solution — trivial solution, i.e. $C_1 = C_2 = C_3 = 0$.

Hence, the three vectors are linearly independent.

(c) ① Set up the vector equation,

$$C_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + C_2 \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + C_3 \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} = 0$$

② Write down the augmented matrix of this linear system directly,

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ -1 & 3 & 3 & 0 \end{array}\right) \xrightarrow{G.E.} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

As the last column is non-pivot, and the third column is also non-pivot, this homogeneous system has non-trivial solutions, i.e. C_1, C_2, C_3 are not all zero.

Hence, the three vectors are linearly dependent.

(d) ① Set up the vector equation,

$$C_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + C_2 \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + C_3 \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} + C_4 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0$$

② Write down the augmented matrix of this linear system directly,

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 0 & 2 & 3 & -1 & 0 \\ -1 & 3 & 0 & 1 & 0 \end{array}\right) \xrightarrow{G.E.} \left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 0 & 2 & 3 & -1 & 0 \\ 0 & 0 & -3 & 3 & 0 \end{array}\right).$$

As the last column is non-pivot, and the fourth column is also non-pivot, this homogeneous system has non-trivial solutions, i.e. c_1, c_2, c_3, c_4 are not all zero.

Hence, the four vectors are linearly dependent.

"Shortcut" methods

(a) $S_1 = \{(1,0,-1), (-1,2,3)\}$.

(b) $S_2 = \{(1,0,-1), (-1,2,3), (0,3,0)\}$.

(c) $S_3 = \{(\underbrace{1,0,-1}_{u_1}), (\underbrace{-1,2,3}_{u_2}), (\underbrace{0,3,3}_{u_3})\}$.

(d) $S_4 = \{(1,0,-1), (-1,2,3), (0,3,0), (1,-1,1)\}$.

(a). The two vectors $(1,0,-1)$, $(-1,2,3)$ are not scalar multiples of each other,

So the 2 vectors are linearly independent.

(b). We can form the 3×3 matrix A, $A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 3 \\ -1 & 3 & 0 \end{pmatrix}$

$$\det(A) = 1 \times \begin{vmatrix} 2 & 3 \\ 3 & 0 \end{vmatrix} - (-1) \times \begin{vmatrix} 0 & 3 \\ -1 & 0 \end{vmatrix} + 0$$

$$= 1\times(-9) + 1\times 3 = -6 \neq 0$$

So the three column vectors are linearly independent (c).

Method 1. By inspection,

$$\frac{3}{2}u_1 + \frac{3}{2}u_2 = \frac{3}{2}(u_1 + u_2) = \frac{3}{2}\cdot\begin{pmatrix}0\\2\\2\end{pmatrix} = \begin{pmatrix}0\\3\\3\end{pmatrix} = u_3.$$

So the three column vectors are linearly dependent.

Method 2. We can form the 3×3 matrix B, $B = \begin{pmatrix}1 & -1 & 0\\0 & 2 & 3\\-1 & 3 & 3\end{pmatrix}$

$$\det(B) = 1\times\begin{vmatrix}2 & 3\\3 & 3\end{vmatrix} - (-1)\times\begin{vmatrix}0 & 3\\-1 & 3\end{vmatrix} + 0$$

$$= 1\times(-3) + 1\times3 + 0 = 0$$

So the three column vectors are linearly dependent.

(d). For the given four vectors, all vectors are from R^3.

So these 4 vectors are linearly dependent.

(ii) Which of these sets are bases for $\mathbb{R}^3$?

(a) $S_1 = \{(1, 0, -1), (-1, 2, 3)\}$.

(b) $S_2 = \{(1, 0, -1), (-1, 2, 3), (0, 3, 0)\}$.

(c) $S_3 = \{(1, 0, -1), (-1, 2, 3), (0, 3, 3)\}$.

(d) $S_4 = \{(1, 0, -1), (-1, 2, 3), (0, 3, 0), (1, -1, 1)\}$.

(ii) Since $\dim R^3 = 3$,

any basis for R^3 must contain 3 vectors.

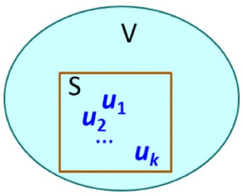
And in fact, any three vectors in $\mathbb{R}^3$ that are linearly independent will form a basis for R^3.

As you see, (b) is a set of 3 linearly independent vectors.

So (b) is indeed a basis for R^3.

5. **Basis and Dimension (slide 3)**

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be a subset of a vector space V. (Recall that a vector space can either be $\mathbf{R}^n$ or any subspace of the $\mathbf{R}^n$.) Then



S is called a basis for V if

1. $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\} = V$; and
2. $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ are linearly independent.

In other words, S is a smallest possible set of vectors in V that generate every vector in V.

The number of vectors in a basis for V is called the dimension of V and is denoted by dim V. In other words, if $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is a basis for V, then dim V = *k*. It is the smallest possible number of vectors that can generate V.

★ Remark

1. **Linear Independence and Dependence (slide 4)**

Let $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ be a set of vectors in $\mathbf{R}^n$.

Set up the vector equation

$$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k = \mathbf{0}.$$

This is equivalent to a homogeneous linear system with variables $c_1, c_2, \ldots, c_k$. (See examples below on vector equation form of linear system.)

Recall that a homogeneous system has either only the trivial solution (case I) or infinitely many solutions with non-trivial solutions (case II).

Case I: The system has only the trivial solution,
i.e. the only possible scalars are: $c_1 = 0$, $c_2 = 0$, ..., $c_k = 0$

In this case, we say the set of vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ are linearly independent.

Case II: The system has non-trivial solutions,
i.e. there are scalars c_1, c_2, ..., c_k that are not all zero.

In this case, we say the set $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ are linearly dependent.

2. In general, when there are n vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ in $\mathbf{R}^n$, we can form an n x n matrix $\boldsymbol{A}$ using $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ (written in column form) as the n columns of the matrix.

- If $\det(\boldsymbol{A}) = 0$, then $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ are linearly dependent
- If $\det(\boldsymbol{A}) \neq 0$, then $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ are linearly independent.

3. **Linear Independence of Two Vectors (slide 9)**

Let $\boldsymbol{u}, \boldsymbol{v}$ be a set with two vectors in $\mathbf{R}^n$.

This gives us a very simple way to decide whether two vectors are linearly independent by inspecting their components:

- If $\boldsymbol{u}$ and $\boldsymbol{v}$ are scalar multiples of each other, they are linearly dependent;
- If $\boldsymbol{u}$ and $\boldsymbol{v}$ are not scalar multiples of each other, they are linearly independent.

4. **A Condition for Linear Dependence (slide 10)**

Let $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ be a set of k vectors in $\mathbf{R}^n$.

If $k > n$, then the set of vectors are linearly dependent.

2. Find a basis for and the dimension of the solution space of each of the following homogeneous systems.

(a) $\begin{cases} x_1 + \ \ 3x_2 - x_3 + 2x_4 = 0 \\ \quad\ \ -3x_2 + x_3 \qquad\quad = 0. \end{cases}$ (b) $\begin{cases} x_1 + \ \ 3x_2 - x_3 + 2x_4 = 0 \\ \quad\ \ -3x_2 + x_3 \qquad\quad = 0 \\ x_1 \qquad\qquad\qquad - \ x_4 = 0. \end{cases}$

Solution:

(a). the augmented matrix of the homogeneous system is

$$\left(\begin{array}{cccc|c} 1 & 3 & -1 & 2 & 0 \\ 0 & -3 & 1 & 0 & 0 \end{array}\right),$$ which is already in REF.

As the last column, and the 3rd, 4th column of REF are non-pivot columns, there are infinitely many solutions for the system.

We set $x_3 = s$, $x_4 = t$, where s and t are arbitrary parameters.

Then we perform the back substitution to get the general solution,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2t \\ \frac{s}{3} \\ s \\ t \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \frac{s}{3} \\ s \\ 0 \end{pmatrix} + \begin{pmatrix} -2t \\ 0 \\ 0 \\ t \end{pmatrix} = s \cdot \begin{pmatrix} 0 \\ \frac{1}{3} \\ 1 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$ where $s \in \mathbb{R}$, $t \in \mathbb{R}$.

So $\{(0, \frac{1}{3}, 1, 0), (-2, 0, 0, 1)\}$ is a basis for the solution space, and the dimension of the solution space is 2.

(b). The augmented matrix of this homogeneous system is

$$\left(\begin{array}{cccc|c} 1 & 3 & -1 & 2 & 0 \\ 0 & -3 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \end{array}\right) \xrightarrow{G.E.} \left(\begin{array}{cccc|c} 1 & 3 & -1 & 2 & 0 \\ 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \end{array}\right)$$

As the last column, and the 3rd column of REF are non-pivot columns, there are infinitely many solutions for the system.

We set $x_3 = t$, where t is an arbitrary parameter.

We perfor the back substitution to attain the general solution,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{t}{3} \\ t \\ 0 \end{pmatrix}$$

$$= t \cdot \begin{pmatrix} 0 \\ \frac{1}{3} \\ 1 \\ 0 \end{pmatrix} \quad \text{where } t \in R.$$

Hence, $\{(0, \frac{1}{3}, 1, 0)\}$ is a basis for the solution space, and the dimension of the solution space is 1.

3. The regular coordinate axes for the $xyz-$ space is rotated about the z-axis through $45°$ counterclockwise, viewing from the positive z-axis.

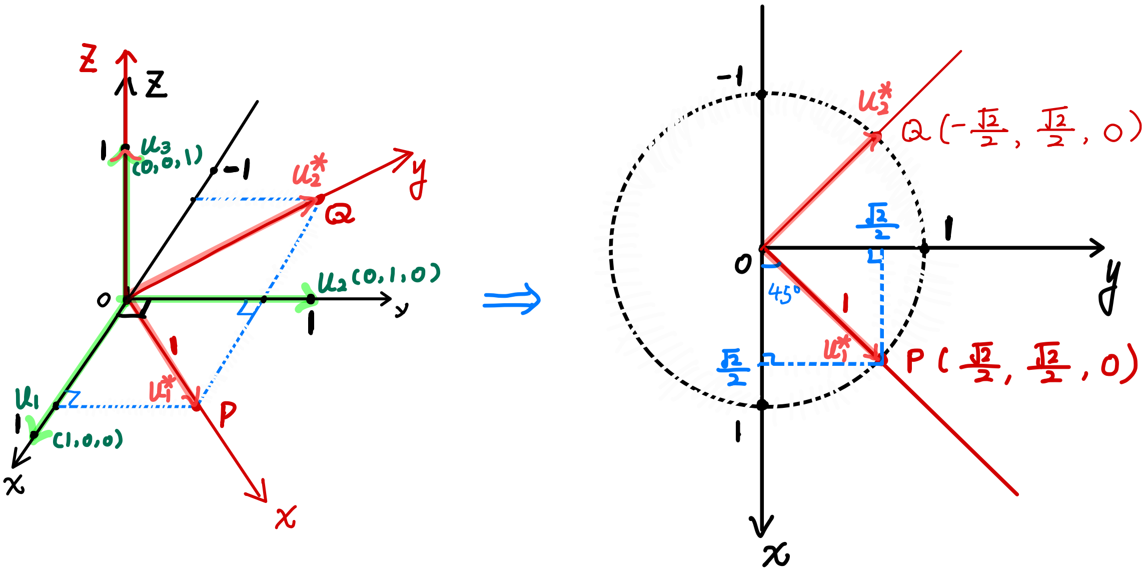
(a) Find a basis S consisting of unit vectors that determine the new coordinate axes.

(b) What are the coordinates of the vector $\boldsymbol{v} = (1, 1, 1)$ relative to the new axes?

(c) Find a matrix $\boldsymbol{M}$ such that $\boldsymbol{M}\boldsymbol{v} = (\boldsymbol{v})_S$.

Solution:

(a).



The unit vector $\underbrace{(1, 0, 0)}_{u_1}$ will be rotated to $\underbrace{(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)}_{u_1^*}$

The unit vector $\underbrace{(0, 1, 0)}_{u_2}$ will be rotated to $\underbrace{(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)}_{u_2^*}$

The unit vector $\underbrace{(0, 0, 1)}_{u_3}$ remains unchanged, still $\underbrace{(0, 0, 1)}_{u_3}$.

Hence, the basis S determing the new coordinates axes is

$$S = \{(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0), (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0), (0, 0, 1)\}.$$

(b). We need to find a, b, c such that

$$a \cdot (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0) + b \cdot (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0) + c \cdot (0, 0, 1) = (1, 1, 1) \quad (**)$$

That is
$$\begin{cases} \frac{\sqrt{2}}{2}a - \frac{\sqrt{2}}{2}b = 1 \\ \frac{\sqrt{2}}{2}a + \frac{\sqrt{2}}{2}b = 1 \\ 1 \cdot c = 1 \end{cases} \xrightarrow[\text{Substitution}]{\text{Back}} \begin{cases} a = \sqrt{2} \\ b = 0 \quad (*) \\ c = 1 \end{cases}$$

Hence, the coordinates of the vector (1, 1, 1) relative to the new axes are $(\sqrt{2}, 0, 1)$, i.e. $(v)_S = (\sqrt{2}, 0, 1)$

(c). From part (b),

the coordinate vector relative to the basis S is $(v)_S = (\sqrt{2}, 0, 1)$,

Substitute (*) into (**), and write the vectors in the column form,

$$\begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix} \cdot \sqrt{2} + \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix} \cdot 0 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot 1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Which can be further expressed in the matrix form,

$$\underbrace{\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} \sqrt{2} \\ 0 \\ 1 \end{pmatrix}}_{(v)_S} = \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{v} \implies A \cdot (v)_S = v$$

$$\underbrace{\left(\begin{array}{ccc|ccc} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 1 & 0 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)}_{A} \xrightarrow{R_1 \times (-1) + R_2} \left(\begin{array}{ccc|ccc} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_1 \times \sqrt{2}]{R_2 \times \frac{1}{\sqrt{2}}} \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2 + R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 1 & 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) A^{-1}$$

Pre-multiply A^{-1} to $(v)_S$ to move it to the right hand side, and

we get $\quad (v)_S = A^{-1} v$.

we require $\quad (v)_S = M v$.

So we have the matrix M,

$$M = A^{-1} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4. Let $\boldsymbol{A} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, $\boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

(a) Show that the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ has no solution.

(b) Find the least squares solution to $\boldsymbol{Ax} = \boldsymbol{b}$.

(c) Find the projection of $\boldsymbol{b}$ onto span$\{\underbrace{(1,0,-1,1)}_{u_1}, \underbrace{(-1,1,0,1)}_{u_2}, \underbrace{(0,-1,1,1)}_{u_3}\}$.

(d) Use (c) to find a vector that is orthogonal to all the three vectors: $\underbrace{(1,0,-1,1)}_{u_1}, \underbrace{(-1,1,0,1)}_{u_2}, \underbrace{(0,-1,1,1)}_{u_3}$.

Solution:

(a).

$$(\boldsymbol{A} \mid \boldsymbol{b}) = \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right) \xrightarrow{G.E.} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 3 \end{array}\right)$$

Since the last column in REF is pivot, this linear system has no solution.

(b). Step 1. We start with the system $Ax=b$.

$$\underbrace{\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}_{b} \Rightarrow Ax = b$$

Step 2. Form the new linear system $A^TAx = A^Tb$,

$$A^T = \begin{pmatrix} 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 \end{pmatrix}, \quad A^TA = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad A^Tb = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

which is given by
$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Step 3. Solve the above system to get its solution u,

$$u = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

Hence, this vector u is the least squares solution of $Ax=b$.

Remark: Matlab

Command Window

```
>> A=[1 -1 0; 0 1 -1; -1 0 1; 1 1 1];
b=[1;1;1;1];
rref([A'*A A'*b])

ans =

    1.0000         0         0    0.3333
         0    1.0000         0    0.3333
         0         0    1.0000    0.3333
```

(c). $\text{span}\{u_1, u_2, u_3\} = \text{span}\{u_1^T, u_2^T, u_3^T\} =$ the column space of A,

So the required projection p is given by Au, where u is the least squares solution in part (b), which is

$$p = Au = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

(d) Since p is the projection of b onto the supspace $= \text{span}\{u_1, u_2, u_3\}$, then $b-p$ is orthogonal to this subspace, which is,

$$b-p = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

As $u_1, u_2, u_3 \in \text{span}\{u_1, u_2, u_3\}$,

this vector is orthogonal to the three vectors u_1, u_2, u_3

Remark

1. **Finding Least Squares Solutions (slide 11)**

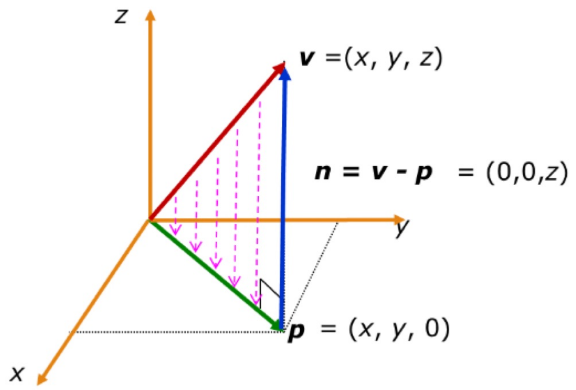
- Starting with the system $\boldsymbol{Ax} = \boldsymbol{b}$ (1)
- Form the new linear system $\boldsymbol{A}^T\boldsymbol{Ax} = \boldsymbol{A}^T\boldsymbol{b}$ (2)
- Solve the system (2)
- A solution of (2) gives a least squares solution of (1)

System (2) $\boldsymbol{A}^T\boldsymbol{Ax} = \boldsymbol{A}^T\boldsymbol{b}$ is always consistent. It can have exactly one solution, or infinitely many solutions.

2. $\boldsymbol{Au} = \boldsymbol{p}$ where $\boldsymbol{p}$ is the projection of $\boldsymbol{b}$ onto the column space* of $\boldsymbol{A}$.

3. **Projection of a Vector onto a Plane in R^3 (slide 2)**

For a 3-vector (x y z) in $\mathbf{R}^3$, its projection onto the xy-plane is given geometrically as follow.



From this observation, we have:

p is a projection of ***v*** onto the plane $\rightleftarrows$ ***v*** - ***p*** is orthogonal to the plane

5. (**MATLAB**) To measure the takeoff performance of an airplane, the horizontal position of the plane was measured every second, from $t = 0$ to $t = 12$. The positions (in meter) were: 0, 2.9, 9.8, 20.1, 34.3, 52.8, 74.0, 98.5, 126.7, 157.2, 190.3, 228.6, and 269.0.

(a) Find the least squares cubic curve: $y = a + bt + ct^2 + dt^3$ for these data.

(b) Use the result of part (a) to estimate the velocity of the plane when $t = 4.5$ second.

Solution:

(a). Substituting the 13 data pairs of (t, y) into the equation $y = a + bt + ct^2 + dt^3$,

$$\text{we get } \begin{cases} a & & & & 0 \\ a+ & b+ & c+ & d= & 2.9 \\ a+ & 2b+ & 4c+ & 8d= & 9.8 \\ a+ & 3b+ & 9c+ & 27d= & 20.1 \\ a+ & 4b+ & 16c+ & 64d= & 34.3 \\ a+ & 5b+ & 25c+ & 125d= & 52.8 \\ a+ & 6b+ & 36c+ & 216d= & 74.0 \\ a+ & 7b+ & 49c+ & 343d= & 98.5 \\ a+ & 8b+ & 64c+ & 512d= & 126.7 \\ a+ & 9b+ & 81c+ & 729d= & 157.2 \\ a+ & 10b+ & 100c+ & 1000d= & 190.3 \\ a+ & 11b+ & 121c+ & 1331d= & 228.6 \\ a+ & 12b+ & 144c+ & 1728d= & 269.0 \end{cases} \Rightarrow AX = b$$

where A denotes the coefficient matrix,
b denotes the constant matrix,
and $X = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$

Command Window

```
>> A=[1 0 0 0;1 1 1 1;1 2 4 8;1 3 9 27;1 4 16 64;1 5 25 125;1 6 36 216;1 7 49 343;1 8 64 512;
1 9 81 729;1 10 100 1000;1 11 121 1331;1 12 144 1728;];
>> A

A =

     1     0     0     0
     1     1     1     1
     1     2     4     8
     1     3     9    27
     1     4    16    64
     1     5    25   125
     1     6    36   216
     1     7    49   343
     1     8    64   512
     1     9    81   729
     1    10   100  1000
     1    11   121  1331
     1    12   144  1728
```

```
>> rref([A'*A A'*b])

ans =

    1.0000         0         0         0   -0.1758
         0    1.0000         0         0    1.1716
         0         0    1.0000         0    1.9453
         0         0         0    1.0000   -0.0147
```

This will give us the least squares solution $a = -0.1758$, $b = 1.1716$

$c = 1.9453$, $d = -0.0147$.

Hence, the least squares cubic curve is

$$y = -0.1758 + 1.1716t + 1.9453t^2 - 0.0147t^3.$$

(b). $\frac{dy}{dt} = 1.1716 + 1.9453 \times 2t - 0.0147 \times 3t^2$

when $t = 4.5$, $\frac{dy}{dt} = 17.9453 \approx 17.79$ (m/s)

So the velocity at 4.5 second is given by 17.79 m/s.

6.

$$\boldsymbol{A}=\begin{pmatrix}1&2&0\\0&1&1\\-1&3&6\\2&1&0\\3&1&-1\end{pmatrix},\quad \boldsymbol{B}=\begin{pmatrix}2&1&4&1&2\\4&2&2&3&2\\2&1&-2&2&0\\6&3&6&4&4\end{pmatrix},\quad \boldsymbol{C}=\begin{pmatrix}1&4&5&8\\-1&4&3&0\\2&0&2&1\end{pmatrix}.$$

For each matrix, find

(i) a basis for the row space and a basis for the column space;

(ii) a basis for the nullspace;

(iii) the rank and nullity of the matrix, and verify the Dimension Theorem for Matrices.

Solution :

In general,

> the non-zero rows in the r.e.f. $\boldsymbol{R}$ of a matrix $\boldsymbol{A}$ → a basis for the row space of $\boldsymbol{A}$.

> the columns of $\boldsymbol{A}$ corresponding to pivot columns of $\boldsymbol{R}$ → a basis for the column space of $\boldsymbol{A}$.

For A, perform Gaussian Elimination,

$$\boldsymbol{A}=\begin{pmatrix}1&2&0\\0&1&1\\-1&3&6\\2&1&0\\3&1&-1\end{pmatrix}\xrightarrow{G.E}\underbrace{\begin{pmatrix}1&2&0\\0&1&1\\0&0&1\\0&0&0\\0&0&0\end{pmatrix}}_{\text{REF of A}}$$

(i) $\{(1,2,0),(0,1,1),(0,0,1)\}$ is a basis for the row space.

Since the pivot columns in REF of A are 1st, 2nd and 3rd column. We will look at the corresponding three columns of the original matrix A. So $\{(1,0,-1,2,3)^T,(2,1,3,1,1)^T,(0,1,6,0,-1)^T\}$ is a basis for the column space.

(Note: here $(a,b,c,d,e)^T$ refer to the column form of the vector.)

(ii) Recall that $\boxed{\text{the nullspace of } \mathbf{A} = \text{the solution space of } \mathbf{Ax} = \mathbf{0}}$,

Since every column is pivot in REF of A,

then this system $Ax = 0$ has only trivial solution $x = 0$.

As the vectors in a basis for a vector space must be non-zero,

then there's no basis for this solution space.

So the basis for the nullspace is the empty set.

(iii). Recall nullity (A) = dim(nullspace of A)

= the number of vectors in a basis for "nullspace of A"

So nullity $(A) = 0$

Recall "the Dimension Theorem for Matrices"

If A is a matrix with n columns, then $\text{rank}(A) + \text{nullity}(A) = n$.

Since $LHS = 3 + 0 = 3$,

$RHS = 3 = LHS$.

Hence, verified.

For $\boldsymbol{B} = \begin{pmatrix} 2 & 1 & 4 & 1 & 2 \\ 4 & 2 & 2 & 3 & 2 \\ 2 & 1 & -2 & 2 & 0 \\ 6 & 3 & 6 & 4 & 4 \end{pmatrix}$, the row echelon form is $\begin{pmatrix} 2 & 1 & 4 & 1 & 2 \\ 0 & 0 & -6 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

(i) $\{(2,1,4,1,2), (0,0,-6,1,-2)\}$ is a basis for the row space.
$\{(2,4,2,6)^{\mathrm{T}}, (4,2,-2,6)^{\mathrm{T}}\}$ is a basis for the column space.
(Using reduced row-echelon form: $\begin{pmatrix} 1 & \frac{1}{2} & 0 & \frac{5}{6} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{6} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, basis for the row space is given by $\{(1, \frac{1}{2}, 0, \frac{5}{6}, \frac{1}{3}),\ (0, 0, 1, -\frac{1}{6}, \frac{1}{3})\}$.

(ii) This is the same as finding basis for the solution space of the homogeneous system $\boldsymbol{Bx} = \boldsymbol{0}$.
$\{(-\frac{1}{2}, 1, 0, 0, 0)^{\mathrm{T}},\ (-\frac{5}{6},\ 0,\ \frac{1}{6},\ 1, 0)^{\mathrm{T}},\ (-\frac{1}{3}, 0, -\frac{1}{3}, 0, 1)^{\mathrm{T}}\}$ is the basis for the nullspace.

(iii) $\mathrm{rank}(\boldsymbol{B}) = 2$, $\mathrm{nullity}(\boldsymbol{B}) = 3$.
$\mathrm{rank}(\boldsymbol{B}) + \mathrm{nullity}(\boldsymbol{B}) = 5 =$ number of columns of $\boldsymbol{B}$.

For $\boldsymbol{C} = \begin{pmatrix} 1 & 4 & 5 & 8 \\ -1 & 4 & 3 & 0 \\ 2 & 0 & 2 & 1 \end{pmatrix}$, the row echelon form is $\begin{pmatrix} 1 & 4 & 5 & 8 \\ 0 & 8 & 8 & 8 \\ 0 & 0 & 0 & -7 \end{pmatrix}$.

(i) $\{(1,4,5,8),\ (0,8,8,8),\ (0,0,0,-7)\}$ is a basis for the row space.
$\{(1,-1,2)^{\mathrm{T}},\ (4,4,0)^{\mathrm{T}},\ (8,0,1)^{\mathrm{T}}\}$ is a basis for the column space.
Using reduced row-echelon form: $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.
the basis for row space is given by $\{(1,0,1,0),\ (0,1,1,0),\ (0,0,0,1)\}$.

(ii) This is the same as finding basis for the solution space of the homogeneous system $\boldsymbol{Cx} = \boldsymbol{0}$.
$\{(-1,-1,1,0)^{\mathrm{T}}\}$ is a basis for the nullspace.

(iii) $\mathrm{rank}(\boldsymbol{C}) = 3$, $\mathrm{nullity}(\boldsymbol{C}) = 1$.
$\mathrm{rank}(\boldsymbol{C}) + \mathrm{nullity}(\boldsymbol{C}) = 4 =$ number of columns of $\boldsymbol{C}$.

7. Given two 3×4 matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ with respective row echelon forms

$$\boldsymbol{R_A} = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{R_B} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Do we have enough information to find the following? Justify your answers.

(i) Matrices $\boldsymbol{A}$ and $\boldsymbol{B}$.

(ii) The row spaces of $\boldsymbol{A}$ and $\boldsymbol{B}$.

(iii) The column spaces of $\boldsymbol{A}$ and $\boldsymbol{B}$.

Write down a basis for each of the row spaces and column spaces of $\boldsymbol{A}$ and $\boldsymbol{B}$ if possible.

Solution:

(i) No. As we do not know what elementary row operations performed on A and B to get the row echelon forms R_A and R_B, then we can not find the original A and B from R_A and R_B respectively.

(ii). **Recall:**

In general,

the non-zero rows in the r.e.f. ***R*** of a matrix ***A*** → a basis for the row space of ***A***.

Hence,

a basis for the row space of $A = \{(1\ 0\ 1\ 2), (0\ 0\ 1\ 1)\}$

row space of $A = \text{span}\{(1\ 0\ 1\ 2), (0\ 0\ 1\ 1)\}$

a basis for the row space of $B = \{(0\ 1\ 0\ 1), (0\ 0\ 1\ -1), (0\ 0\ 0\ 1)\}$

row space of $B = \text{span}\{(0\ 1\ 0\ 1), (0\ 0\ 1\ -1), (0\ 0\ 0\ 1)\}$

(iii). **Recall:**

the columns of ***A*** corresponding to pivot columns of ***R*** → a basis for the column space of ***A***.

In R_A, the 1st and 3rd columns are pivot, so

a basis of the column space of A = { 1st column of A, 3rd column of A }.

Since we do not know A, we can not find a basis of the column space of A, and hence we can not get the column space of A either.

In R_B, the 2nd, 3rd and 4th columns are pivot, so

a basis of the column space of B

= { 2nd column of B, 3rd column of B, 4th column of B }.

= { 3 linearly independent vectors in $\mathbb{R}^3$ }

We know, the span of any 3 linearly independent vectors in $\mathbb{R}^3$ $= \mathbb{R}^3$.

Hence, the column space of $B = \mathbb{R}^3$.

Therefore, we can use the standard basis of $\mathbb{R}^3$ as a basis of the column space of B, which is

a basis of the column space of $B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

7. An engineer has found two solutions to a homogeneous system $\boldsymbol{Ax} = \mathbf{0}$ of 40 equations and 42 variables. These two solutions are not scalar multiple of each other, and every solution of the system can be constructed by a linear combination of these two solutions.

Suppose now the engineer needs to replace the homogeneous system with an associated non-homogeneous system $\boldsymbol{Ax} = \boldsymbol{b}$ for some non-zero vector $\boldsymbol{b}$. Can the engineer be certain that he will be able to find a solution?

Solution:

We assume that u_1, u_2 are two solutions that the engineer found for homogeneous system $Ax = 0$.

Since $u_1 \neq c \cdot u_2$ for all $c \in \mathbb{R}$,

then u_1 and u_2 are linearly independent.

Since every solution of $Ax=0$ can be expressed as a linear combination of u_1, u_2,

then the solution space $= \operatorname{span}\{u_1, u_2\}$

Hence, $\{u_1, u_2\}$ is a basis for the solution space, which is also a basis for the nullspace of A.

Therefore, $\operatorname{nullity}(A) = 2$.

Since $Ax=0$ has 40 equations and 42 variables,

then A is a 40×42 matrix.

By the Dimension theorem,

$$\begin{aligned} \operatorname{rank}(A) &= \text{total number of columns of } A - \operatorname{nullity}(A) \\ &= 42 - 2 = 40. \end{aligned}$$

This implies, in the RREF of A, all the 40 rows are non-zero.

(That is to say, in the RREF of A, the leading entry in the last row is 1. Which implies, the system $Ax=b$ is always consistent, no matter what the value of the vector b is). e.g. One possible RREF of the corresponding augmented matrix of $Ax=b$ can be:

$$\begin{array}{c} \\ \\ \\ \\ \text{40th row} \end{array} \begin{array}{c} \begin{array}{ccccccc} & & & \text{40th column} & & & \end{array} \\ \left(\begin{array}{ccccccc|c} 1 & 0 & \cdots & 0 & a_{1,41} & a_{1,42} & b_1 \\ 0 & 1 & \cdots & 0 & a_{2,41} & a_{2,42} & b_2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_{40,41} & a_{40,42} & b_{40} \end{array}\right) \end{array} \; 40 \times 43$$

So for the associated non-homogeneous system $Ax=b$,

it will have a solution regardless of the vector b.