Problem:

We assume SDE $y' = Ay$, where $A_{2\times2}$ is a real matrix, and A has only one repeated eigenvalue λ with only one linearly independent eigenvector v and a generalized eigenvector u.

Show that $y = (tv+u)\cdot e^{\lambda t}$ is a solution of $y' = Ay$.

Proof:

First, on the left,

$$y' = (1\cdot v + 0)\cdot e^{\lambda t} + (tv+u)\cdot \lambda \cdot e^{\lambda t}$$
$$= (v + \lambda v\cdot t + \lambda u)\, e^{\lambda t}.$$

Since v is an eigenvector of A, $Av = \lambda v$.

Thus, $y' = \underline{(Av\cdot t + v + \lambda u)\cdot e^{\lambda t}}$

On the right,

$$Ay = A(tv+u)\, e^{\lambda t}$$
$$= (Av\cdot t + Au)\cdot e^{\lambda t}$$

Since u is a generalized eigenvector of A,

$$(A - \lambda I)u = v$$
$$\Rightarrow Au - \lambda u = v$$
$$\Rightarrow Au = v + \lambda u$$

Thus, $Ay = \underline{(Avt + v + \lambda u)\cdot e^{\lambda t}} = y'$.

Hence,

$y = (tv+u)\cdot e^{\lambda t}$ is a solution of $y' = Ay$.