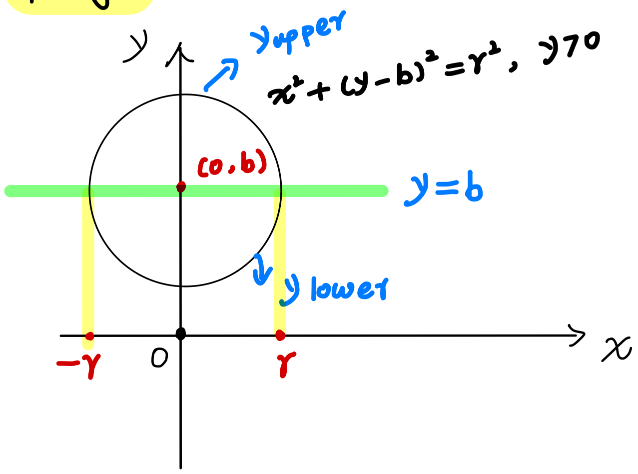
How can we use an integral to find the volume of a solid torus?

(1). If the torus is obtained by rotating the circle $x^2+(y-b)^2=r^2$, $y>0$, about the x-axis, the volume of the torus is $\boxed{V=2\pi^2 r^2 \cdot b}$.

proof:

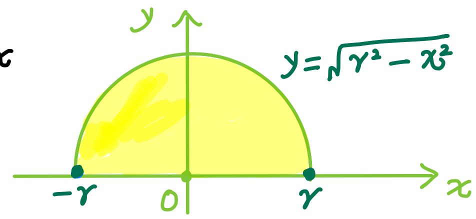


$$x^2+(y-b)^2=r^2$$
$$\Rightarrow (y-b)^2=r^2-x^2$$
$$\Rightarrow y-b=\pm\sqrt{r^2-x^2}$$
$$\Rightarrow y=b\pm\sqrt{r^2-x^2}$$
$$\Rightarrow y_{upper}(x)=b+\sqrt{r^2-x^2}$$
$$y_{lower}(x)=b-\sqrt{r^2-x^2}$$

The volume of solid of revolution can be expressed as:

$$V=\int_{-r}^{r}\pi\cdot\left(y_{upper}^2(x)-y_{lower}^2(x)\right)dx$$

$$=\pi\cdot\int_{-r}^{r}\left(\left(b+\sqrt{r^2-x^2}\right)^2-\left(b-\sqrt{r^2-x^2}\right)^2\right)dx$$



$$=\pi\cdot\int_{-r}^{r}4b\cdot\sqrt{r^2-x^2}\,dx$$

$$=4\pi b\cdot\int_{-r}^{r}\sqrt{r^2-x^2}\,dx$$

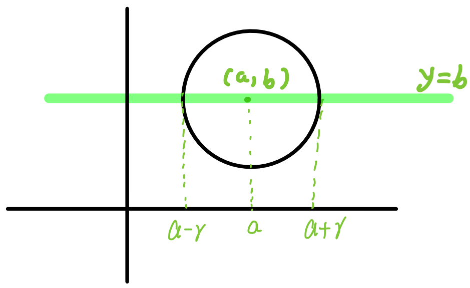
this integral is equivalent to the area of a semicircle, centered at (0,0), with the radius r.

$$=4\pi b\cdot\frac{1}{2}\pi r^2$$
$$=2\pi^2 r^2 b.$$

(2) If the torus is obtained by rotating the circle $(x-a)^2+(y-b)^2=r^2$, $y>0$, about the x-axis, the volume of the torus is $\boxed{V=2\pi^2r^2\cdot b}$.

proof: $y_{upper} = b+\sqrt{r^2-(x-a)^2}$

$y_{lower} = b-\sqrt{r^2-(x-a)^2}$



$$V = \pi\int_{a-r}^{a+r}(y_{upper}^2 - y_{lower}^2)dx$$

$$= \pi\int_{a-r}^{a+r} 4b\cdot\sqrt{r^2-(x-a)^2}\,dx$$

$$= 4\pi b\cdot\underbrace{\int_{a-r}^{a+r}\sqrt{r^2-(x-a)^2}\,dx}$$

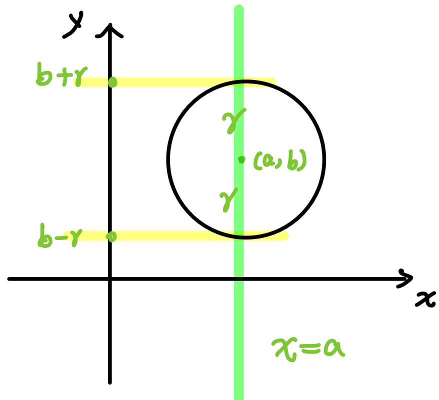
which is equivalent to the area of a semicircle, centered at $(a, 0)$, with radius r.

$$= 4\pi b\cdot\tfrac{1}{2}\pi r^2 = \underline{2\pi^2r^2b}.$$

(3) If the torus is obtained by rotating the circle $(x-a)^2+(y-b)^2=r^2$, $x>0$ about the y-axis, the volume of the torus is $\boxed{V=2\pi^2r^2\cdot a}$.

proof: $(x-a)^2+(y-b)^2=r^2$

$$\Rightarrow |x-a| = \sqrt{r^2-(y-b)^2}$$



For the right semicircle,

$$x-a = \sqrt{r^2-(y-b)^2}$$

$$\Rightarrow x_{right}(y) = a+\sqrt{r^2-(y-b)^2}$$

For the left semicircle,

$$x-a = -\sqrt{r^2-(y-b)^2}$$

$$\Rightarrow x_{left}(y) = a-\sqrt{r^2-(y-b)^2}$$

And thus, the volume of the torus is

$$V = \pi \int_{b-r}^{b+r} \left([x_{right}(y)]^2 - [x_{left}(y)]^2 \right) dy$$

$$= \pi \int_{b-r}^{b+r} 4a \sqrt{r^2 - (y-b)^2}\, dy$$

$$= 4\pi a \cdot \underbrace{\int_{b-r}^{b+r} \sqrt{r^2 - (y-b)^2}\, dy}$$

which is equivalent to the area of a semicircle (centered at (0, b)), with radius r.

$$= 4\pi a \cdot \frac{1}{2} \pi r^2$$

$$= 2\pi^2 r^2 a$$