AMS 572 Data Analysis I
Power and sample size for two population means

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Based on Z test

Sample size determination for a given margin of error E

Based on exact or the large sample approximate z-test ;

$\boxed{n_1 = n_2 = n}$

Suppose the margin of error is E with probability $(1-\alpha)$

$$P(|(\bar{X}-\bar{Y})-(\mu_1-\mu_2)| \le E) = 1-\alpha$$

$$\frac{\bar{X}-\bar{Y}-(\mu_1-\mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}} \dot{\sim} N(0,1)$$

$$P(-E \le \bar{X}-\bar{Y}-(\mu_1-\mu_2) \le E) = 1-\alpha$$

$$P\left(\frac{-E}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}} \le \frac{\bar{X}-\bar{Y}-(\mu_1-\mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}} \le \frac{E}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}\right) = 1-\alpha$$

$$\frac{E}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}} = Z_{\alpha/2} \Longrightarrow n = \frac{(Z_{\alpha/2})^2(\sigma_1^2+\sigma_2^2)}{E^2}$$

Sample size determination for a given CI length

$100(1-\alpha)\%$ CI for $(\mu_1-\mu_2)$ $\quad \bar{X}_1-\bar{X}_2 \pm Z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}$

$$L = 2 \times E$$

$$n = \frac{4(Z_{\alpha/2})^2(\sigma_1^2+\sigma_2^2)}{L^2}$$

Sample size determination for a given power

$H_a: \mu_1 - \mu_2 > \Delta_1$

type II error

$$H_0 : \mu_1 - \mu_2 = \Delta_1$$
$$H_a : \mu_1 - \mu_2 = \Delta_2 > (or <, \neq)\Delta_1$$

$$1 - \beta = \text{power} = P(\text{reject } H_0 | H_a)$$

$$= P(Z_0 \geq Z_\alpha | \mu_1 - \mu_2 = \Delta_2)$$
$$= P\left(\frac{\bar{X} - \bar{Y} - \Delta_1}{\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{n}}} \geq Z_\alpha | \mu_1 - \mu_2 = \Delta_2\right)$$
$$= P\Bigg(\underbrace{\frac{\bar{X} - \bar{Y} - \Delta_2}{\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{n}}}}_{H_a \sim N(0,1)} \geq Z_\alpha - \frac{\Delta_2 - \Delta_1}{\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{n}}} | \mu_1 - \mu_2 = \Delta_2\Bigg)$$

$$Z_\alpha - \frac{\Delta_2 - \Delta_1}{\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{n}}} = -Z_\beta$$

$$Z_0 \overset{H_a}{\sim} N\left(\frac{\Delta_2 - \Delta_1}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}, 1\right)$$

Sample size determination for a given power

One-tailed test, exact Z

$$n = \frac{(Z_\alpha + Z_\beta)^2 (\sigma_1^2 + \sigma_2^2)}{(\Delta_2 - \Delta_1)^2}$$

One-tailed test, approximate Z

$$n \doteq \frac{(Z_\alpha + Z_\beta)^2 (\sigma_1^2 + \sigma_2^2)}{(\Delta_2 - \Delta_1)^2}$$

Two-tailed test, exact or approximate Z

$$n \doteq \frac{(Z_{\alpha/2} + Z_\beta)^2 (\sigma_1^2 + \sigma_2^2)}{(\Delta_2 - \Delta_1)^2}$$

Example: A new method of making concrete blocks has been proposed. To test whether or not the new method increases the compressive strength, 5 sample blocks are made by each method.

New Method	14	15	13	15	16	μ_1
Old Method	13	15	13	12	14	μ_2

(a) Get a 95% CI for the mean difference of the 2 methods.

(b) At $\alpha = 0.05$, can you conclude the new method is better?

Write a SAS and R program for part (b).

Pooled-variance statistic (PQ)

$$t = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_{n_1+n_2-2}$$

95% CI for $(\mu_1 - \mu_2)$ is $(\bar{X}_1 - \bar{X}_2) \pm t_{n_1+n_2-2,0.025} \cdot s_p \cdot \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$

$$s_p = \sqrt{\frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2}}$$

SAS Code

```
data block ;
input method $ strength ;
datalines ;
new 14
new 15
new 13
new 15
new 16
old 13
old 15
old 13
old 12
old 14
;
run ;
```

SAS Code

```
proc univariate data=block normal plot ;
class method ;
var strength ;
run ;

proc ttest data=block sides=U ;
class method ;
var strength ;
run ;

proc npar1way data=block ;
class method ;
var strength ;
run ;
```

proc t test

The SAS System

Method	Variances	DF	t Value	Pr > t
Pooled	Equal	8	1.66	0.0673
Satterthwaite	Unequal	8	1.66	0.0673

Equality of Variances

Method	Num DF	Den DF	F Value	Pr > F
Folded F	4	4	1.00	1.0000

R Code and Output

```
> new <- c(14,15,13,15,16)
> old <- c(13,15,13,12,14)
> var.test(new,old)

F test to compare two variances

data:  new and old
F = 1, num df = 4, denom df = 4, p-value = 1
alternative hypothesis: true ratio of variances is not equal to
95 percent confidence interval:
 0.1041175 9.6045299
sample estimates:
ratio of variances
                 1
```

R Code and Output

```
> t.test(new,old,var.equal=TRUE,alternative='greater')

Two Sample t-test

data:  new and old
t = 1.6641, df = 8, p-value = 0.06733
alternative hypothesis: true difference in means is greater than
95 percent confidence interval:
 -0.1409392        Inf
sample estimates:
mean of x mean of y
     14.6      13.4
```

R Code and Output

```
> wilcox.test(new,old,alternative='greater',exact=FALSE)

Wilcoxon rank sum test with continuity correction

data:  new and old
W = 19.5, p-value = 0.08129
```

Assume $\sigma_1^2 = \sigma_2^2$

Example: An experiment was done to determine the effect on dairy cattle of a diet supplement with liquid whey. While no differences were noted in milk production between the group with a standard diet (hay + grain + water) and the experimental group with whey supplement (hay + grain + whey), a considerable difference was noted in the amount of hay ingested. For a 2-tailed test with α=0.05, determine the approximate number of cattle that should be included in each group if we want $\beta \leq 0.1$ for $|\mu_1 - \mu_2| \geq 0.5$. Previous study has shown $\sigma \approx 0.8$.

$$n = \frac{(\sigma_1^2 + \sigma_2^2)(z_{\alpha/2} + z_\beta)^2}{(\mu_1 - \mu_2)^2} = \frac{2(0.8)^2(1.96+1.28)^2}{0.5^2}$$

$$= 53.75$$

$n = 54.$

Example: Do fraternities help or hurt your academic progress at college? To investigate this question, 5 students who joined fraternities in 1998 were randomly selected. It was shown that their GPA before and after they joined the fraternities are as follows.

Student	1	2	3	4	5
Before	3	4	3	3	2
After	2	3	3	2	1
Diff	1	1	0	1	1

Test the hypothesis at α=0.05

Chapter 7 method.

$$H_0 : \mu_d = 0$$
$$H_a : \mu_d \neq 0$$

Assumption : the difference follows a normal distribution.

$$\bar{X}_d = 0.8, s_d = 0.447, n = 5, \alpha = 0.05$$

Test statistic : $T_0 = \frac{\bar{X}_d - 0}{s_d / \sqrt{n}} \overset{H_0}{\sim} t_{n-1}$

$$|T_0| = 4.02 > t_{4,0.025} = 2.776$$

We reject H_0 at α=0.05 and conclude fraternities does hurt

SAS Code

```
data frat ;
input before after ;
diff = before  after ;
datalines ;
3 2
4 3
3 3
3 2
2 1
;
run ;

proc univariate data=frat normal ;
var diff ;
run ;
```

R Code and Output

```
> before <- c(3,4,3,3,2)
> after <- c(2,3,3,2,1)
> t.test(before,after,paired=TRUE)

Paired t-test

data:  before and after
t = 4, df = 4, p-value = 0.01613
alternative hypothesis: true difference in means is not equal to
95 percent confidence interval:
 0.244711 1.355289
sample estimates:
mean of the differences
                    0.8
```

R Code and Output

```
> d <- c(1,1,0,1,1)
> t.test(d)

One Sample t-test

data:  d
t = 4, df = 4, p-value = 0.01613
alternative hypothesis: true mean is not equal to 0
95 percent confidence interval:
 0.244711 1.355289
sample estimates:
mean of x
      0.8
```