AMS 572 Data Analysis I
Nonparametric Statistical Methods

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Nonparametric inference for two independent samples

Wilcoxon (Mann-Whitney) Rank Sum Test

1. Assume $Y_{1j}, \ldots, Y_{n_j j}$ iid $F_j(y)$; $j = 1, 2$

$$H_0 : F_1(y) = F_2(y)$$

$$H_a : F_1(y) = F_2(y + \Delta)$$

   where Δ is a constant
2. Pool the two samples
3. Rank them from smallest to largest
4. Compute the sum of the ranks, W_1, in group 1

Wilcoxon Rank Sum Test

- There are $N = n_1 + n_2$ subjects in our study
- Thus there are $\binom{N}{n_1}$ possible outcomes
- Under H_0, each is equally likely
- We compute the distribution of W_1 by enumeration

Wilcoxon Rank Sum Test: Example

- A new drug is being test in humans for the first time to assess effect on CD4+ T cells in patients with HIV
- 7 individuals are randomized to 2 groups: control ($n_1 = 3$) or drug ($n_2 = 4$)
- Endpoint is percent change in CD4+ count from baseline
- Null hypothesis is the drug has no effect

$$H_0 : \Delta = 0; H_a : \Delta \neq 0$$

Wilcoxon Rank Sum Test: Example

- Data: control (65, 73, 69); drug (89, 70, 92, 88)
- There are $\binom{7}{3} = 35$ possible outcomes

10, 35 possible ranking for group 1.

Wilcoxon Rank Sum Test: $n_1 = 3, n_2 = 4$

Ranks	W_1	Ranks	W_1	Ranks	W_1
1,2,3	6	1,5,6	12	2,6,7	15
1,2,4	7	1,5,7	13	3,4,5	12
1,2,5	8	1,6,7	14	3,4,6	13
1,2,6	9	2,3,4	9	3,4,7	14
1,2,7	10	2,3,5	10	3,5,6	14
1,3,4	8	2,3,6	11	3,5,7	15
1,3,5	9	2,3,7	12	3,6,7	16
1,3,6	10	2,4,5	11	4,5,6	15
1,3,7	11	2,4,6	12	4,5,7	16
1,4,5	10	2,4,7	13	4,6,7	17
1,4,6	11	2,5,6	13	5,6,7	18
1,4,7	12	2,5,7	14		

Wilcoxon Rank Sum Test: $n_1 = 3, n_2 = 4$

$= \frac{1}{35}$

w	Freq	$F(w)$	w	Freq	$F(w)$
6	1	0.0286	13	5	0.6857
7	1	0.0571	14	4	0.8000
8	2	0.1143	15	3	0.8857
9	3	0.2000	16	2	0.9429
10	4	0.3142	17	1	0.9714
11	4	0.4286	18	1	1
12	4	0.5714			

Wilcoxon Rank Sum Test: $n_1 = 3, n_2 = 4$

- Note it is impossible to reject H_0 for a two-sided alternative when $\alpha = 0.05$.
- For a two-sided $\alpha = 0.1$ test

$$C_\alpha = \{6, 18\}$$

- Observed $W_1 = 1 + 2 + 4 = 7$; do not reject H_0

pvalue $= 2(0.0571) = 0.1142 > 0.1$

Wilcoxon Rank Sum Test

- It can be shown that

$$E(W_1) = \frac{n_1}{N}\frac{N(N+1)}{2} = \frac{n_1(N+1)}{2}$$

- Similarly

$$V(W_1) = \frac{n_1 n_2(N+1)}{12}$$

Wilcoxon Rank Sum Test: Large Sample Approx

continuity correction

- If n_1 and n_2 are large

$$Z = \frac{W_1 - E(W_1) - 0.5}{\sqrt{V(W_1)}}$$

  will be approx $N(0,1)$.
- Approximation is good for $n_1, n_2 \geq 12$
- If there are ties

$$V(W_1) = \frac{n_1 n_2 (N+1)}{12} \underbrace{- \frac{n_1 n_2}{12N(N-1)} \sum_{i=1}^{q} t_i(t_i - 1)(t_i + 1)}_{\text{Correction for ties}}$$

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Mann-Whitney Test

- Consider all $n_1 n_2$ possible pairs

$$(Y_{1i}, Y_{2j}); i = 1, 2, \ldots, n_1; j = 1, 2, \ldots, n_2$$

- Let U_1 equal the number of pairs with $Y_{1i} > Y_{2j}$ and U_2 equal the number of pairs with $Y_{1i} < Y_{2j}$
- It can be shown that

$$U_1 = W_1 - \frac{n_1(n_1+1)}{2} \qquad U_2 = W_2 - \frac{n_2(n_2+1)}{2}$$

- Mann Whitney test and Wilcoxon rank sum test are equivalent

Wilcoxon Rank Sum Test: Example

$n_1 = 15$ $\qquad$ $n_2 = 15$

Drug	Rank	Placebo	Rank
6.9	**18**	6.4	11
7.6	**25.5**	6.7	13
7.3	**23.5**	5.4	3
7.6	**25.5**	8.2	**28.5**
6.8	**15**	5.3	2
7.2	22	6.6	12
8.0	27	5.8	**8.5**
5.5	4	5.7	**6.5**
5.8	**8.5**	6.2	10
7.3	**23.5**	7.1	21
8.2	**28.5**	7.0	20
6.9	**18**	6.9	**18**
6.8	**15**	5.6	5
5.7	**6.5**	4.2	1
8.6	30	6.8	**15**

Wilcoxon Rank Sum Test: Example

$W_1 = 290.5$

$$Z = \frac{290.5 - 232.5 - 0.5}{\sqrt{579.57}} = 2.388$$

reject H_0, p-value $= 1 - \Phi(2.388) = 0.10846$

- $H_0 : \Delta = 0$; $H_A : \Delta > 0$
- $C_{.05} = \{z : z > 1.645\}$
- $E(W_1) = \frac{n_1(N+1)}{2} = \frac{15 \times (31)}{2} = 232.5$
- $Var(W_1 \mid \text{no ties}) = \frac{n_1 n_2 (N+1)}{12} = \frac{15^2\, 31}{12} = 581.25$

Ties correction: $g = 7$, $t_1 = t_2 = 2$, $t_5 = t_6 = t_7 = 2$

$t_3 = t_4 = 3$

$Var(W_1 \mid \text{with ties}) = 581.25 - \frac{78(15)^2}{12(30)(31)} = 579.57$

SAS Code

```
data drug;
input trt $ bp @@;
datalines;
drug 6.9  drug 7.6  drug 7.3  drug 7.6  drug 6.8  drug 7.2  drug 8  dru
;
run;

proc npar1way wilcoxon correct=yes;
class trt;
var bp;
run;
```

SAS Output

```
                       The NPAR1WAY Procedure

          Wilcoxon Scores (Rank Sums) for Variable bp
                   Classified by Variable trt

                          Sum of      Expected       Std Dev          Mean
trt             N         Scores      Under H0      Under H0         Score

drug           15         290.50        232.50     24.074239     19.366667
placebo        15         174.50        232.50     24.074239     11.633333

                  Average scores were used for ties.


                      Wilcoxon Two-Sample Test

                 Statistic                290.5000

                 Normal Approximation
                 Z                          2.3884
                 One-Sided Pr >  Z          0.0085
                 Two-Sided Pr > |Z|         0.0169

                 t Approximation
                 One-Sided Pr >  Z          0.0118
                 Two-Sided Pr > |Z|         0.0237

             Z includes a continuity correction of 0.5.
```

R Code and Output

mu=0 (ie δ=0)

```
> drug <- c(6.9,7.6,7.3,7.6,6.8,7.2,8.0,5.5,5.8,
7.3,8.2,6.9,6.8,5.7,8.6)
> placebo <- c(6.4,6.7,5.4,8.2,5.3,6.6,5.8,5.7,6.2,
7.1,7.0,6.9,5.6,4.2,6.8)
>
> wilcox.test(drug,placebo,exact=FALSE,correct=TRUE,
alternative='greater')

Wilcoxon rank sum test with continuity correction

data:  drug and placebo
W = 170.5, p-value = 0.00846
alternative hypothesis: true location shift is greater than 0
```

$U_1 = W_1 - \frac{n_1(n_1+1)}{2}$

Summary of nonparametric inference

9. ANOVA. Regression.

- One sample: Sign test or Wilcoxon Signed Rank test
- Two samples: Wilcoxon Rank Sum/Mann-Whitney test
- More than two samples: Kruskal Wallis test, in R: `kruskal.test()`

ANOVA

- Test for equality of variances: Levene's test, in R: `library(lawsuit), levene.test()`
- Correlation: Spearman's rank correlation or Kendall's tau, in R: `cor(method="spearman","kendall")`

cor.test ( )