# AMS 572 Data Analysis I
# Inference on two population means and two population variances

Pei-Fen Kuan

Applied Math and Stats, Stony Brook University

# Two population setting

one time pt

- Single cross-sectional sample, comparing two sub-samples
- Comparing samples from two different populations (two cross-sectional samples, case control study)
- Single sample; subjects randomly allocated to different interventions (experiment, clinical trials)
- Paired data; examples:
  - Study of how a characteristic changes from before to after a treatment
  - Study of twins
  - Individually-matched case-control study

# Overview of tests for two population means

- The samples are paired ⇒ paired samples t-test
- The samples are independent ⇒ independent samples t-test
  - $\sigma_1^2 = \sigma_2^2 \Rightarrow$ pooled-variance t-test (also known as equal variance t-test)
  - $\sigma_1^2 \neq \sigma_2^2 \Rightarrow$ unpooled-variance t-test (also known as unequal variance t-test)

# Inference on two population means: Paired samples

Example: Suppose that we are interested in comparing the IQ between fraternal twins consisting of a male and female

| Pair | IQ Male | IQ Female | Difference $d_i$ |
|---|---|---|---|
| 1 | 200 | 199 | 1 |
| 2 | 150 | 120 | 30 |
| ... | ... | ... | ... |
| n | 109 | 112 | -3 |

- Because of the inherent pairing or matching structure, a natural way is to consider the difference $d_i$ between each pair.
- By considering the paired differences, the two population/sample problem reduces to a one population/one sample problem.
- One can apply the tests from Chapter 7 by treating $d_i$'s as the observations.

# Inference on two population means: Paired samples

The corresponding one or two-tailed hypothesis tests for paired samples reduce to: $\mu_d = \mu_m - \mu_f$

$$\begin{cases} H_0 : \mu_m = \mu_f \\ H_a : \mu_m > \mu_f \end{cases} \Leftrightarrow \begin{cases} H_0 : \mu_d = 0 \\ H_a : \mu_d > 0 \end{cases}$$

$$\begin{cases} H_0 : \mu_m = \mu_f \\ H_a : \mu_m < \mu_f \end{cases} \Leftrightarrow \begin{cases} H_0 : \mu_d = 0 \\ H_a : \mu_d < 0 \end{cases}$$

$$\begin{cases} H_0 : \mu_m = \mu_f \\ H_a : \mu_m \neq \mu_f \end{cases} \Leftrightarrow \begin{cases} H_0 : \mu_d = 0 \\ H_a : \mu_d \neq 0 \end{cases}$$

where $\mu_f$ = population mean IQ of female twin, $\mu_m$ = population mean IQ of male twin, $\mu_d$ = population mean difference in IQ

# SAS Code

```
data twins;
input IQmale IQfemale;
diff = IQmale-IQfemale;
datalines;
200 199
150 120
...
109 112
;
run;

proc univariate data=twins normal;
var diff;
run;
```

Example: To study the effectiveness of wall insulation in saving energy for home heating, the energy consumption (in MWh) for 5 houses in Bristol, England, was recorded for two winters; the first winter was before insulation and the second winter was after insulation:

| House | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Before | 12.1 | 10.6 | 13.4 | 13.8 | 15.5 |
| After | 12.0 | 11.0 | 14.1 | 11.2 | 15.3 |

(a) Provide a 95% confidence interval for the difference between the mean energy consumption before and after the wall insulation is installed. What assumptions are necessary for your inference?

(b) Can you conclude that there is a difference in mean energy consumption before and after the wall insulation is installed at the significance level 0.05? Perform the hypothesis test and compute the p-value of your test. What assumptions are necessary for your inference?

(c) Write a SAS and R program to perform the test and examine the necessary assumptions given in (b).

# SAS Code

```
Data energy;
Input before after @@;
Diff=before - after;
Datalines;
12.1 12.0 10.6 11.0 13.4 14.1 13.8 11.2 15.5 15.3
;
run;
Proc univariate data = energy normal;
Var Diff;
run;
```

# Main SAS Output

```
 The UNIVARIATE Procedure
          Variable:  Diff
Tests for Location: Mu0=0

Test           -Statistic-    -----p Value------

Student's t    t  0.616851    Pr > |t|    0.5707
Sign           M       0.5    Pr >= |M|   1.0000
Signed Rank    S       0.5    Pr >= |S|   1.0000

Tests for Normality

Test                  --Statistic---    -----p Value------

Shapiro-Wilk          W       0.80253    Pr < W       0.0850
Kolmogorov-Smirnov    D      0.348791    Pr > D       0.0442
Cramer-von Mises      W-Sq    0.10017    Pr > W-Sq    0.0859
Anderson-Darling      A-Sq   0.551915    Pr > A-Sq    0.0779
```

# R Code and Output

```
> before <- c(12.1,10.6,13.4,13.8,15.5)
> after <- c(12,11,14.1,11.2,15.3)
> t.test(before,after,paired=TRUE)

Paired t-test

data:  before and after
t = 0.6169, df = 4, p-value = 0.5707
alternative hypothesis: true difference in means is not equal to 0
95 percent confidence interval:
 -1.26036  1.98036
sample estimates:
mean of the differences
                   0.36

> shapiro.test(before-after)

Shapiro-Wilk normality test

data:  before - after
W = 0.8025, p-value = 0.08496
```

## Inference on two population means: pooled variance t-test

# Setting

- Notation:

$$X_1, \ldots, X_{n_1} \overset{iid}{\sim} N(\mu_1, \sigma_1^2)$$

$$Y_1, \ldots, Y_{n_2} \overset{iid}{\sim} N(\mu_2, \sigma_2^2)$$

- Two independent samples of sample sizes $n_1$ and $n_2$, the population variances are unknown but equal ($\sigma_1^2 = \sigma_2^2 = \sigma^2$)
- Both samples are normally distributed.
- Goal: Compare $\mu_1$ and $\mu_2$

# Review

- Two random variables $X$ and $Y$ are *independent* if for all $x$ and $y$

- Result A. If $X$ and $Y$ are independent random variables, then for any two constants $a$ and $b$ the random variable

$$W = aX + bY$$

has mean and variance

$$E(W) = aE(X) + bE(Y)$$

$$\mathrm{Var}(W) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y)$$

# Review

- Result B. If $X$ and $Y$ are independent random variables that are **normally** distributed, then

$$W = aX + bY$$

is normally distributed with mean and variance given by Result A

- Corollary: If $\bar{X}$ and $\bar{Y}$ are based on two independent random samples of size $n_1$ and $n_2$ from two normal distributions with means $\mu_1$ and $\mu_2$ and variances $\sigma_1^2$ and $\sigma_2^2$, then

# Point estimator for $\mu_1 - \mu_2$

A point estimator for $\mu_1 - \mu_2$

$$\hat{\mu}_1 - \hat{\mu}_2 = \bar{X} - \bar{Y} \sim N\left(\mu_1 - \mu_2, \left(\frac{1}{n_1} + \frac{1}{n_2}\right)\sigma^2\right)$$

under equal variance assumption $\sigma_1^2 = \sigma_2^2 = \sigma^2$

# Review

- $W = Z_1^2 + Z_2^2 + \cdots + Z_k^2$, where $Z_i \overset{i.i.d.}{\sim} N(0,1)$, then

- $Z \sim N(0,1)$, $W \sim \chi_k^2$, and $Z$ and $W$ are independent, then

# Pivotal quantity for inference on $\mu_1 - \mu_2$

- Result C. If $\bar{X}$ and $\bar{Y}$ are based on two independent random samples of size $n_1$ and $n_2$ from two normal distributions with means $\mu_1$ and $\mu_2$ and the same variances $\sigma_1^2 = \sigma_2^2 = \sigma^2$, then

$$T = \frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{s_p\sqrt{1/n_1 + 1/n_2}} \sim t_{n_1+n_2-2}$$

where

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

- Note: If $n_1 = n_2$ then

$$s_p^2 = (s_1^2 + s_2^2)/2$$

# Confidence interval for $\mu_1 - \mu_2$

$$
\begin{aligned}
&P(-t_{n_1+n_2-2,\frac{\alpha}{2}} \le T \le t_{n_1+n_2-2,\frac{\alpha}{2}}) = 1-\alpha \\
&P(-t_{n_1+n_2-2,\frac{\alpha}{2}} \le \frac{(\bar{X}-\bar{Y})-(\mu_1-\mu_2)}{s_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} \le t_{n_1+n_2-2,\frac{\alpha}{2}}) = 1-\alpha \\
&P(\bar{X}-\bar{Y}-t_{n_1+n_2-2,\frac{\alpha}{2}} \cdot s_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}} \le \mu_1-\mu_2 \\
&\le \bar{X}-\bar{Y}+t_{n_1+n_2-2,\frac{\alpha}{2}} \cdot s_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}) = 1-\alpha
\end{aligned}
$$

A $100(1-\alpha)\%$ C.I for $(\mu_1-\mu_2)$

$$
\bar{X}-\bar{Y} \pm t_{n_1+n_2-2,\frac{\alpha}{2}} \cdot s_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}
$$

# Hypothesis test for $\mu_1 - \mu_2$

Suppose $H_0 : \mu_1 - \mu_2 = c_0$,
Test statistic:

$$T_0 = \frac{(\bar{X} - \bar{Y}) - c_0}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_{n_1+n_2-2} \text{ under } H_0$$

# Hypothesis test for $\mu_1 - \mu_2$

1. One tailed: $H_0 : \mu_1 - \mu_2 = c_0$ vs $H_a : \mu_1 - \mu_2 > c_0$
   - If $c_0 = 0$, this reduces to $H_0 : \mu_1 = \mu_2$
   - At the significance level $\alpha$, reject $H_0$ in favor of $H_a$ if $T_0 \geq t_{n_1+n_2-2,\alpha}$
   - Alternatively, if p-value$= P(T_0 \geq t_0 | H_0) \leq \alpha$, reject $H_0$
2. One tailed: $H_0 : \mu_1 - \mu_2 = c_0$ vs $H_a : \mu_1 - \mu_2 < c_0$
   - At the significance level $\alpha$, reject $H_0$ in favor of $H_a$ if $T_0 \leq -t_{n_1+n_2-2,\alpha}$
   - Alternatively, if p-value$= P(T_0 \leq t_0 | H_0) \leq \alpha$, reject $H_0$
3. Two tailed: $H_0 : \mu_1 - \mu_2 = c_0$ vs $H_a : \mu_1 - \mu_2 \neq c_0$
   - At the significance level $\alpha$, reject $H_0$ in favor of $H_a$ if $|T_0| \geq t_{n_1+n_2-2,\alpha/2}$
   - Alternatively, if p-value$= 2 \cdot P(T_0 \geq |t_0| | H_0) \leq \alpha$, reject $H_0$

Example: An experiment was conducted to compare the mean number of tapeworms in the stomachs of sheep that had been treated for worms against the mean number in those that were untreated. A sample of 14 worm-infected lambs was randomly divided into 2 groups. Seven were injected with the drug and the remainders were left untreated. After a 6-month period, the lambs were slaughtered and the following worm counts were recorded:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Drug-treated sheep | 18 | 43 | 28 | 50 | 16 | 32 | 13 |
| Untreated sheep | 40 | 54 | 26 | 63 | 21 | 37 | 39 |

(a) Test at $\alpha = 0.05$ whether the treatment is effective or not.

(b) What assumptions do you need for the inference in part (a)?

(c) Write a SAS and R program necessary to answer questions raised in (a) and (b).

# SAS Code

```
data sheep;
input group worms @@;
datalines;
1 18  1 43 1 28 1 50 1 16 1 32 1 13
2 40 2 54 2 26 2 63 2 21 2 37 2 39
;
run;
proc univariate data=sheep normal;
class group;
var worms;
title 'Check for normality';
run;
proc ttest data=sheep sides=L;
class group;
var worms;
title 'Independent samples t-test';
run;
```

# SAS Output

```
proc npar1way data=sheep wilcoxon;
class group;
var worms;
title 'Nonparametric test for two-mean comparisons';
run;
```

# SAS Output

```
                          Check for normality

                      The UNIVARIATE Procedure
                         Variable:  worms
                             group = 1

                              Moments

N                           7    Sum Weights                  7
Mean               28.5714286    Sum Observations           200
Std Deviation       14.093227    Variance            198.619048
Skewness             0.495124    Kurtosis            -1.2527576
Uncorrected SS           6906    Corrected SS        1191.71429
Coeff Variation    49.3262945    Std Error Mean      5.32673912


                     Basic Statistical Measures

        Location                    Variability

    Mean     28.57143     Std Deviation            14.09323
    Median   28.00000     Variance                198.61905
    Mode      .           Range                    37.00000
                          Interquartile Range      27.00000


                     Tests for Location: Mu0=0

       Test           -Statistic-    -----p Value------

       Student's t    t  5.363775    Pr > |t|    0.0017
       Sign           M       3.5    Pr >= |M|   0.0156
       Signed Rank    S        14    Pr >= |S|   0.0156
```

# SAS Output

```
                    Tests for Normality

Test                  --Statistic---    -----p Value------

Shapiro-Wilk          W      0.952397    Pr < W      0.7515
Kolmogorov-Smirnov    D      0.214286    Pr > D     >0.1500
Cramer-von Mises      W-Sq   0.041652    Pr > W-Sq  >0.2500
Anderson-Darling      A-Sq   0.240103    Pr > A-Sq  >0.2500
```

# SAS Output

```
                    Check for normality

                 The UNIVARIATE Procedure
                    Variable:  worms
                       group = 2

                        Moments

N                           7    Sum Weights                 7
Mean                       40    Sum Observations          280
Std Deviation      14.6742405    Variance           215.333333
Skewness           0.38989154    Kurtosis           -0.4979133
Uncorrected SS          12492    Corrected SS             1292
Coeff Variation    36.6856012    Std Error Mean     5.54634157


               Basic Statistical Measures

        Location                    Variability

    Mean     40.00000     Std Deviation            14.67424
    Median   39.00000     Variance                215.33333
    Mode       .          Range                    42.00000
                          Interquartile Range      28.00000


               Tests for Location: Mu0=0

       Test           -Statistic-    -----p Value------

       Student's t    t  7.211961    Pr > |t|    0.0004
       Sign           M       3.5    Pr >= |M|   0.0156
       Signed Rank    S        14    Pr >= |S|   0.0156
```

# SAS Output

```
                   Tests for Normality

Test                  --Statistic---    -----p Value------

Shapiro-Wilk          W     0.952397    Pr < W      0.7515
Kolmogorov-Smirnov    D     0.214286    Pr > D     >0.1500
Cramer-von Mises      W-Sq  0.041652    Pr > W-Sq  >0.2500
Anderson-Darling      A-Sq  0.240103    Pr > A-Sq  >0.2500
```

# SAS Output

Independent samples t-test 5

The TTEST Procedure

Variable: worms

| group | N | Mean | Std Dev | Std Err | Minimum | Maximum |
|---|---|---|---|---|---|---|
| 1 | 7 | 28.5714 | 14.0932 | 5.3267 | 13.0000 | 50.0000 |
| 2 | 7 | 40.0000 | 14.6742 | 5.5463 | 21.0000 | 63.0000 |
| Diff (1-2) | | -11.4286 | 14.3867 | 7.6900 | | |

| group | Method | Mean | 95% CL Mean | | Std Dev | 95% CL Std Dev | |
|---|---|---|---|---|---|---|---|
| 1 | | 28.5714 | 15.5374 | 41.6055 | 14.0932 | 9.0816 | 31.0342 |
| 2 | | 40.0000 | 26.4286 | 53.5714 | 14.6742 | 9.4560 | 32.3136 |
| Diff (1-2) | Pooled | -11.4286 | -Infty | 2.2772 | 14.3867 | 10.3165 | 23.7486 |
| Diff (1-2) | Satterthwaite | -11.4286 | -Infty | 2.2791 | | | |

| Method | Variances | DF | t Value | Pr < t |
|---|---|---|---|---|
| Pooled | Equal | 12 | -1.49 | 0.0815 |
| Satterthwaite | Unequal | 11.98 | -1.49 | 0.0815 |

Equality of Variances

| Method | Num DF | Den DF | F Value | Pr > F |
|---|---|---|---|---|
| Folded F | 6 | 6 | 1.08 | 0.9244 |

# R Code and Output

shapiro.test(drug)
shapiro.test(untrt)

} test for normality

```
> drug <- c(18,43,28,50,16,32,13)
> untrt <- c(40,54,26,63,21,37,39)
> var.test(drug,untrt)

F test to compare two variances

data:  drug and untrt
F = 0.9224, num df = 6, denom df = 6, p-value = 0.9244
alternative hypothesis: true ratio of variances is not equal to 1
95 percent confidence interval:
 0.1584911 5.3680240
sample estimates:
ratio of variances
         0.9223795
```

CH 14.
nonparametric test
usually less powerful

# R Code and Output

```
> t.test(drug,untrt,alternative='less',var.equal=TRUE)

Two Sample t-test

data:  drug and untrt
t = -1.4862, df = 12, p-value = 0.08152
alternative hypothesis: true difference in means is less than 0
95 percent confidence interval:
     -Inf 2.277215
sample estimates:
mean of x mean of y
 28.57143  40.00000
```

# R Code and Output

```
> wilcox.test(drug,untrt,alternative='less')

Wilcoxon rank sum test

data:  drug and untrt
W = 14, p-value = 0.1043
alternative hypothesis: true location shift is less than 0
```

# Inference on two population means: unequal variance $\sigma_1^2 \neq \sigma_2^2$

# Effect on testing $\mu_1 = \mu_2$

What if $\sigma_1^2 \neq \sigma_2^2$ and unknown?

1. Large sample approximation
2. Normality: Welch-Satterthwaite approximation (Behrens-Fisher problem)

# Large sample approximation

- If $n_1$ and $n_2$ are large, equal variance assumption is not important
- Recall CLT plus Slutsky implies

$$\bar{X} \sim N(\mu, \frac{s^2}{n})$$

- Thus

$$Z = \bar{X} - \bar{Y} \sim N(\mu_1 - \mu_2, \frac{s_1^2}{n_1} + \frac{s_2^2}{n_2})$$

- $Z$ is the pivotal quantity for inference on $\mu_1 - \mu_2$ for large sample
- Generally, require $n_j \geq 30$ for $j = 1, 2$
- Note assumption that $X_i$'s and $Y_i$'s normally distributed no longer needed either (CLT)

# Confidence interval for $\mu_1 - \mu_2$ for large sample

A $100(1-\alpha)\%$ C.I for $(\mu_1 - \mu_2)$

$$\bar{X} - \bar{Y} \pm z_{\alpha/2}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

# Hypothesis test for $\mu_1 - \mu_2$ for large sample

Suppose $H_0 : \mu_1 - \mu_2 = c_0$,
Test statistic:

$$Z_0 = \frac{(\bar{X} - \bar{Y}) - c_0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} \sim N(0, 1) \text{ under } H_0$$

# Hypothesis test for $\mu_1 - \mu_2$ for large sample

1. One tailed: $H_0 : \mu_1 - \mu_2 = c_0$ vs $H_a : \mu_1 - \mu_2 > c_0$
   - If $c_0 = 0$, this reduces to $H_0 : \mu_1 = \mu_2$
   - At the significance level $\alpha$, reject $H_0$ in favor of $H_a$ if $Z_0 \geq z_\alpha$
   - Alternatively, if p-value$= P(Z_0 \geq z_0 | H_0) \leq \alpha$, reject $H_0$

     $P_R$
2. One tailed: $H_0 : \mu_1 - \mu_2 = c_0$ vs $H_a : \mu_1 - \mu_2 < c_0$
   - At the significance level $\alpha$, reject $H_0$ in favor of $H_a$ if $Z_0 \leq -z_\alpha$
   - Alternatively, if p-value$= P(Z_0 \leq z_0 | H_0) \leq \alpha$, reject $H_0$

     $P_L$
3. Two tailed: $H_0 : \mu_1 - \mu_2 = c_0$ vs $H_a : \mu_1 - \mu_2 \neq c_0$
   - At the significance level $\alpha$, reject $H_0$ in favor of $H_a$ if $|Z_0| \geq z_{\alpha/2}$
   - Alternatively, if p-value$= 2 \cdot P(Z_0 \geq |z_0| | H_0) \leq \alpha$, reject $H_0$

     $= 2 \min(P_R . P_L)$

# Welch-Satterthwaite approximation for normal distribution and small samples

- Assume normality; $n_1, n_2$ small
- Pivotal quantity for inference on $\mu_1 - \mu_2$

$$T = \bar{X} - \bar{Y} \sim N(\mu_1 - \mu_2, \frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}) \sim t_{df}$$

- Welch (Biometrika 1938),

$$df = \frac{(s_1^2/n_1 + s_2^2/n_2)^2}{\frac{(s_1^2/n_1)^2}{n_1 - 1} + \frac{(s_2^2/n_2)^2}{n_2 - 1}}$$

# Confidence interval for $\mu_1 - \mu_2$ for normal distribution and small samples

A $100(1-\alpha)\%$ C.I for $(\mu_1 - \mu_2)$

$$\bar{X} - \bar{Y} \pm t_{df, \frac{\alpha}{2}} \cdot \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

# Hypothesis test for $\mu_1 - \mu_2$ for normal distribution and small samples

Suppose $H_0 : \mu_1 - \mu_2 = c_0$,
Test statistic:

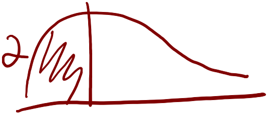
$$T_0 = \frac{(\bar{X} - \bar{Y}) - c_0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} \sim t_{df} \text{ under } H_0$$

# Hypothesis test for $\mu_1 - \mu_2$ for normal distribution and small samples

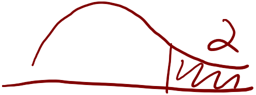
1. One tailed: $H_0 : \mu_1 - \mu_2 = c_0$ vs $H_a : \mu_1 - \mu_2 > c_0$
   - If $c_0 = 0$, this reduces to $H_0 : \mu_1 = \mu_2$
   - At the significance level $\alpha$, reject $H_0$ in favor of $H_a$ if $T_0 \geq t_{df,\alpha}$
   - Alternatively, if p-value$= P(T_0 \geq t_0|H_0) \leq \alpha$, reject $H_0$
2. One tailed: $H_0 : \mu_1 - \mu_2 = c_0$ vs $H_a : \mu_1 - \mu_2 < c_0$
   - At the significance level $\alpha$, reject $H_0$ in favor of $H_a$ if $T_0 \leq -t_{df,\alpha}$
   - Alternatively, if p-value$= P(T_0 \leq t_0|H_0) \leq \alpha$, reject $H_0$
3. Two tailed: $H_0 : \mu_1 - \mu_2 = c_0$ vs $H_a : \mu_1 - \mu_2 \neq c_0$
   - At the significance level $\alpha$, reject $H_0$ in favor of $H_a$ if $|T_0| \geq t_{df,\alpha/2}$
   - Alternatively, if p-value$= 2 \cdot P(T_0 \geq |t_0||H_0) \leq \alpha$, reject $H_0$

# Inference on two population variances

# F distribution




$F_{v_1,v_2,\alpha,L}$




$F_{v_1,v_2,\alpha,U}$

If $X_1$ and $X_2$ are independent rvs with $X_1 \sim \chi^2_{v_1}$ and $X_2 \sim \chi^2_{v_2}$, then

$$\frac{X_1/v_1}{X_2/v_2} \sim F_{v_1,v_2}$$

Note:

- If $F \sim F_{v_1,v_2}$, then $\frac{1}{F} \sim F_{v_2,v_1}$
- Thus, if the F-table only gives the upper bound $F_{v_1,v_2,\alpha,U}$, i.e., $P(F \geq F_{v_1,v_2,\alpha,U}) = \alpha$, the lower bound can be obtained using the relationship above.

$$\alpha = P(F \geq F_{v_1,v_2,\alpha,U}) = P\left(\frac{1}{F} \leq \frac{1}{F_{v_1,v_2,\alpha,U}}\right)$$

$$\Rightarrow \frac{1}{F_{v_1,v_2,\alpha,U}} = F_{v_2,v_1,\alpha,L}$$

# Inference on $\sigma_1^2/\sigma_2^2$

- Want to test

$$H_0 : \sigma_1^2 = \sigma_2^2 \text{ versus } H_A : \sigma_1^2 \neq \sigma_2^2$$

- We know that (assuming normality)

$$\frac{(n_k - 1)s_k^2}{\sigma_k^2} \sim \chi^2_{n_k - 1} \text{ for } k = 1, 2$$

# Inference on $\sigma_1^2/\sigma_2^2$

- Let
$$X = \frac{(n_1-1)s_1^2}{\sigma_1^2} \text{ and } Y = \frac{(n_2-1)s_2^2}{\sigma_2^2}$$
- It follows that
$$F = \frac{X/(n_1-1)}{Y/(n_2-1)} \sim F_{n_1-1,\, n_2-1}$$
- Thus
$$F = \frac{s_1^2/\sigma_1^2}{s_2^2/\sigma_2^2} \sim F_{n_1-1,\, n_2-1}$$
is a pivotal quantity for inference on $\sigma_1^2/\sigma_2^2$

# Point estimator and confidence interval for $\sigma_1^2/\sigma_2^2$

$$\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2}$$

- Point estimator

$$\frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2} = \frac{s_1^2}{s_2^2}$$

- $$1-\alpha = P(F_{n_1-1,n_2-1,L} \le F \le F_{n_1-1,n_2-1,U})$$

$$= P\left(\frac{\frac{S_1^2}{S_2^2}}{F_U} \le \frac{\sigma_1^2}{\sigma_2^2} \le \frac{\frac{S_1^2}{S_2^2}}{F_L}\right)$$

A $100(1-\alpha)\%$ C.I for $\sigma_1^2/\sigma_2^2$

$$\left[\frac{1}{F_{n_1-1,n_2-1,\alpha/2,U}}\frac{s_1^2}{s_2^2}, \frac{1}{F_{n_1-1,n_2-1,\alpha/2,L}}\frac{s_1^2}{s_2^2}\right]$$

# Hypothesis test for $\sigma_1^2/\sigma_2^2$

Suppose $H_0 : \sigma_1^2 = \sigma_2^2 \Leftrightarrow H_0 : \sigma_1^2/\sigma_2^2 = 1$
Test statistic:

$$F_0 = \frac{s_1^2/\sigma_1^2}{s_2^2/\sigma_2^2} \sim F_{n_1-1,n_2-1} \text{ under } H_0$$

$$= \frac{s_1^2}{s_2^2}$$

# Hypothesis test for $\sigma_1^2/\sigma_2^2$

$F_0 = \frac{S_1^2}{S_2^2}$

1. One tailed: $H_0 : \sigma_1^2/\sigma_2^2 = 1$ vs $H_a : \sigma_1^2/\sigma_2^2 > 1$
   - At the significance level $\alpha$, reject $H_0$ in favor of $H_a$ if $F_0 \geq F_{n_1-1,n_2-1,\alpha,U}$
2. One tailed: $H_0 : \sigma_1^2/\sigma_2^2 = 1$ vs $H_a : \sigma_1^2/\sigma_2^2 < 1$
   - At the significance level $\alpha$, reject $H_0$ in favor of $H_a$ if $F_0 \leq F_{n_1-1,n_2-1,\alpha,L}$
3. Two tailed: $H_0 : \sigma_1^2/\sigma_2^2 = 1$ vs $H_a : \sigma_1^2/\sigma_2^2 \neq 1$
   - At the significance level $\alpha$, reject $H_0$ in favor of $H_a$ if $F_0 \geq F_{n_1-1,n_2-1,\alpha/2,U}$ or $F_0 \leq F_{n_1-1,n_2-1,\alpha/2,L}$

old blood pressure — new blood pressure

Example: A new drug for reducing blood pressure (BP) is compared to an old drug. 20 patients with comparable high BP were recruited and randomized evenly to the 2 drugs. Reductions in BP after 1 month of taking the drugs are as follows:

New drug: 0, 10, -3, 15, 2, 27, 19, 21, 18, 10 X grp 1

Old drug: 8, -4, 7, 5, 10, 11, 9, 12, 7, 8 Y grp 2

(a) Assume both populations are normal, test at $\alpha = 0.05$ whether the new drug is better than the old one

(b) Write a SAS and R program to do the above tests (including the normality test)

# SAS Code

```
PROC UNIVARIATE DATA=BP NORMAL;
CLASS DRUG;
VAR BPR;
RUN;

PROC TTEST DATA=BP SIDES=U;
CLASS DRUG;
VAR BPR;
RUN;
```

# Selected SAS Output

```
Tests for Normality

Test (class=1)         --Statistic---     -----p Value------

Shapiro-Wilk           W      0.955526    Pr < W      0.7339
Kolmogorov-Smirnov     D      0.142059    Pr > D     >0.1500
Cramer-von Mises       W-Sq   0.037679    Pr > W-Sq  >0.2500
Anderson-Darling       A-Sq   0.238555    Pr > A-Sq  >0.2500

Test (class =2)        --Statistic---     -----p Value------

Shapiro-Wilk           W      0.805147    Pr < W      0.0167
Kolmogorov-Smirnov     D      0.273266    Pr > D      0.0334
Cramer-von Mises       W-Sq   0.124461    Pr > W-Sq   0.0447
Anderson-Darling       A-Sq   0.776159    Pr > A-Sq   0.0294
```

# Selected SAS Output

| Method | Variances | DF | t Value | Pr > t |
|---|---|---|---|---|
| Pooled | Equal | 18 | 1.34 | 0.0981 |
| Satterthwaite | Unequal | 12.547 | 1.34 | 0.1016 |

Equality of Variances

| Method | Num DF | Den DF | F Value | Pr > F |
|---|---|---|---|---|
| Folded F | 9 | 9 | 4.87 | 0.0274 |

# SAS Code and Output for Wilcoxon Rank Sum test

```
PROC NPAR1WAY DATA=BP;
CLASS DRUG;
VAR BPR;
RUN;

Wilcoxon Two-Sample Test

Statistic                 123.0000

Normal Approximation
Z                          1.3259
One-Sided Pr > Z           0.0924
Two-Sided Pr > |Z|         0.1849

t Approximation
One-Sided Pr > Z           0.1003
Two-Sided Pr > |Z|         0.2006
```

# R Code and Output

```
> new <- c(0,10,-3,15,2,27,19,21,18,10)
> old <- c(8,-4,7,5,10,11,9,12,7,8)
> var.test(new,old)

F test to compare two variances

data:  new and old
F = 4.869, num df = 9, denom df = 9, p-value = 0.02736
alternative hypothesis: true ratio of variances is not equal to
95 percent confidence interval:
  1.209381 19.602411
sample estimates:
ratio of variances
          4.868962
```

# R Code and Output

```
> t.test(new,old,alternative='greater',var.equal=FALSE)

Welch Two Sample t-test

data:  new and old
t = 1.3423, df = 12.547, p-value = 0.1016
alternative hypothesis: true difference in means is greater than
95 percent confidence interval:
 -1.485809       Inf
sample estimates:
mean of x mean of y
     11.9       7.3
```

# R Code and Output

```
> wilcox.test(new,old,alternative='greater',exact=FALSE)

Wilcoxon rank sum test with continuity correction

data:  new and old
W = 68, p-value = 0.09244
alternative hypothesis: true location shift is greater than 0
```