H10.1 a) $E_{p,\sigma} = E_p + E_\sigma = \frac{p}{2m} + g\mu_B \sigma B$

$$Z_G = \sum_{\substack{n_{\alpha_i}=0,1 \\ i=1,2,\dots}} \exp\left(-\beta\left(E(\{n_{\alpha_i}\}) - \mu N\right)\right) \quad , \alpha = p,\sigma$$

$$E(\{n_{\alpha_i}\}) = \sum_i n_{\alpha_i} E_{\alpha_i} \ , \ N = \sum_i n_{\alpha_i}$$

$$= \prod_i^\infty \left(\sum_{n_{\alpha_i}=0,1} \exp\left(-\beta n_{\alpha_i} (E_{\alpha_i} - \mu)\right) \right)$$

$$= \prod_i^\infty 1 + \exp\left(-\beta(E_{\alpha_i} - \mu)\right)$$

b) $$\Omega = -k_B T \ln Z_G$$

$$= -k_B T \sum_i^\infty \ln\left(1 + \exp(-\beta(E_{\alpha_i} - \mu))\right)$$

Skript Formel S. 61

$$= -k_B T \sum_\sigma \int_{-\infty}^{\infty} dE \, D_\sigma(E) \ln\left(1 + \exp(-\beta(E - \mu))\right)$$

part. Int. (gegen Pfeil weg) $\int_{-\infty}^{E} d\varepsilon \, D_\sigma(\varepsilon) = a_\sigma(E)$ → Anzahl belegter Zustände bis Energie E

$$= +k_B T \sum_\sigma \int_{-\infty}^{\infty} dE \, a_\sigma(E) \frac{-\beta \exp(-\beta(E-\mu))}{\exp(-\beta(E-\mu)) + 1}$$

$$= -\sum_\sigma \int_{-\infty}^{\infty} dE \, a_\sigma(E) \frac{1}{\exp(\beta(E-\mu)) + 1}$$

$$= -\sum_\sigma \int_{-\infty}^{\infty} dE \, a_\sigma(E) \, f(E)$$

erneute partielle Integration mit $b_\sigma(E) = \int_{-\infty}^{E} d\varepsilon \, a_\sigma(\varepsilon) = \int_{-\infty}^{E} d\varepsilon \int_{-\infty}^{\varepsilon} d\varepsilon' \, D_\sigma(\varepsilon')$

$$I = \int_{-\infty}^{\infty} dE \, a_\sigma(E) f(E) = \underbrace{\left[b_\sigma(E) f(E) \right]_{-\infty}^{\infty}}_{0} - \int_{-\infty}^{\infty} dE \, b_\sigma(E) f'(E)$$

$f'(E)$ nur um μ von 0 verschieden, daher Taylor von $b_\sigma(E)$ um $E = \mu$:

$$b_\sigma(E) = b_\sigma(\mu) + \sum_{n=1}^{\infty} \frac{(E-\mu)^n}{n!} \left. \frac{d^n b_\sigma(E)}{dE^n} \right|_{E=\mu}$$

Erster Term im Integral:

$$I_0 \quad - b_\sigma(\mu) \underbrace{\int_{-\infty}^{\infty} dE \, f'(E)}_{-1} = b_\sigma(E_F) = \int_{-\infty}^{\mu} dE \, a_\sigma(E)$$

mit $$-\frac{f'(E)}{\beta} = \frac{\exp(\beta(E-\mu))}{(\exp(\beta(E-\mu))+1)^2}$$

$$= \frac{1}{\exp(\beta(E-\mu)) + \exp(-\beta(E-\mu)) + 2}$$

$$= \frac{1}{\exp(\beta(E-\mu)) + 2\frac{\exp(\beta(E-\mu))}{\exp(\beta(E-\mu))} + \exp(-\beta(E-\mu))}$$

$$= \frac{1}{(\exp(\beta(E-\mu)) + \exp(-\beta(E-\mu)))^2}$$

$$= \frac{1}{4\cosh^2(\beta(E-\mu))}$$

folgt für die weiteren Glieder des Integrals

$$I_1 = \beta a_\sigma(\mu)\int_{-\infty}^{\infty} dE \frac{(E-\mu)}{4\cosh^2(\beta(E-\mu))} = \beta a_\sigma(\mu)\left(\frac{(E^2-E\mu)}{4\cosh^2(\beta(E-\mu))}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dE \frac{1}{4\cosh^2(\beta(E-\mu))}\right)$$

$$= \beta a_\sigma(\mu)\left[\frac{E^2-E\mu}{4\cosh^2(\beta(E-\mu))} + \frac{\tanh(\beta(E-\mu))}{4\beta}\right]_{-\infty}^{\infty}$$

c) $$N = \sum_\sigma \int_0^\infty dE\ D_\sigma(E)\ f(E)$$

$$N_\uparrow = \int_{-\infty}^{\infty} dE\ D_\uparrow(E)\ f(E) \overset{\text{Nolting S.525}}{=} \frac{1}{2}\int_{2\sigma\mu_B B}^{\infty} dE\ D(E-\mu_B B)\ f(E)$$

$y = E-\mu_B B$
$dy = dE$

$$\rightarrow = \frac{1}{2}\int_0^\infty dy\ D(y)\ f(y+\mu_B B)$$

Taylor da $\mu_B B \ll 1$

$$\approx \frac{1}{2}\int_0^\infty dy\ D(y)\left(f(y) + \mu_B B f'\right)$$

$$= \frac{1}{2}\int_0^\infty dy\ D(y)\ f(y) + \frac{\mu_B B}{2}\int_0^\infty dy\ D(y)\ f'(y)$$

Analog

$$N_\downarrow = \frac{1}{2}\int dy\ D(y)\left(f(y) - \mu_B B f'\right)$$

$$= \frac{1}{2}\int_0 dy\ D(y)\ f(y) - \frac{\mu_B B}{2}\int_0^\infty dy\ D(y)\ f'(y)$$

d)

$$M = \frac{\mu_B}{V}\left(N_\downarrow - N_\uparrow\right) = -\frac{\mu_B^2 B}{V}\int_0^\infty dy \; D(y)\, f'(y)$$

Sommerfeld:

$$\approx \frac{\mu_B^2 B}{V}\left(D(\mu) + \frac{\pi^2}{6}(k_B T)^2 \; D''(\mu)\right)$$

e) $$\chi = \mu_0 \left.\frac{\partial M}{\partial B}\right|_{T,\mu} = \frac{\mu_0 \mu_B^2}{V}\left(D(\mu) + \frac{\pi^2}{6}(k_B T)^2 \; D''(\mu)\right)$$

Mit Nolting 3.50 und 3.74

$$D(\mu) = \frac{3N}{2 E_F^{3/2}}\sqrt{\mu} \quad \Rightarrow \quad D''(\mu) = -\frac{3}{8}\frac{N}{E_F^{3/2}}\,\mu^{-3/2} \qquad , c_3 = \frac{m^{3/2}}{\sqrt{2}\,\pi^2 \hbar^3}$$

$$\Rightarrow \chi = \frac{3N}{2\mu^{3/2}}\frac{\mu_0 \mu_B^2}{V}\left(\sqrt{\mu} - \frac{1}{4}\frac{1}{\mu^{3/2}}(k_B T)^2\right)$$

$$= \frac{3N}{2}\frac{\mu_0 \mu_B^2}{V\mu}\left(1 - \frac{1}{4}\left(\frac{k_B T}{\mu}\right)^2\right)$$

Für $T \to 0$ gilt $\chi \to \dfrac{3N}{2}\dfrac{\mu_0 \mu_B^2}{V \mu}$