Advanced Algorithms (3501)

Spring Semester, 2023

Problem Set 4

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1. In this question, we will prove some of the simpler cases of the strong duality theorem. Recall that we have the following primal-dual pair of LPs:

$$\begin{array}{lll} \text{minimize} & c^T x & \\ \text{subject to} & Ax = b, x \geq 0 & \\ \text{(P)} & & \end{array} \qquad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y \leq c \\ \text{(D)} & \end{array}$$

   (a) Suppose that (P) is infeasible and (D) is feasible. Show that (D) is unbounded.

   (b) Suppose that (D) is infeasible and (P) is feasible. Show that (P) is unbounded.

   Guidance: in (a), start from an arbitrary feasible solution y for (D). Use the version of Farkas' lemma from Problem set 3 to find a vector y' with properties as stated there, and consider the set of vectors $y - \lambda y'$ for all $\lambda > 0$. In (b), proceed similarly with the roles of (P) and (D) reversed, and the version of Farkas' lemma from class.

2. In this exercise we will prove the claim we made in class, that when we need to decide the feasibility problem for a polyhedron $P = \{x : Cx \leq d\}$ we can assume without loss of generality a lower bound on its volume.

   Recall that in class we proved the volume upper bound: that is, for every vertex x of P and for every coordinate j, $|x_j| \leq 2^L$ where L an integer whose size is at most a polynomial function of the input.

   Let $\mathbf{1}$ denote the all ones vector. In the following two parts, L' stands for some integer whose size is at most a polynomial function of the input. You don't need to compute it exactly, you just need to show that one can pick L' with that property such that the statements hold.

   (a) Prove that if $Cx \leq d$ is infeasible, then so is $Cx \leq d + 2^{-L'}\mathbf{1}$ (Hint: Farkas' Lemma)

   (b) Prove that if $Cx \leq d$ is feasible, there's a feasible point $\hat{x}$ such that the ball of radius $2^{-2L'}$ around $\hat{x}$, that is, the set $\{x : \|x - \hat{x}\|_2 \leq 2^{-2L'}\}$, is contained in $\{x : Cx \leq d + 2^{-L'}\mathbf{1}\}$.

Note that given the two items above, given the polyhedron $Cx \leq d$ we can run the ellipsoid algorithm on $Cx \leq d + 2^{-L'}\mathbf{1}$. By the first item, if the original system is infeasible, so is the new one. But if the original system is feasible, the new system has volume which is at least $2^{-\Theta(L'n)}$ (since the volume of a ball of radius r is $\Theta(r^n)$. So when we run the ellipsoid algorithm on $Cx \leq d + 2^{-L'}\mathbf{1}$, if the volume becomes smaller than $2^{-\Theta(L'n)}$, we know $Cx \leq d$ is infeasible.

3. Formulate each of the following NP-hard problems as an Integer Programming (IP) problem. That is, formulate an IP problem whose feasible solutions correspond to solutions of the original problem, and whose optimal solution (if applicable) corresponds to an optimal solution of the original problems. You don't need to write a formal detailed proof that the IP corresponds to the original problem. Please only describe what is the set of variables of the IP and provide a short explanation for the constraints.

   (a) MAX-CLIQUE: given an undirected graph $G = (V, E)$, find the clique of maximal size in G.

   (b) SUBSET-SUM: given a set of integers $U = \{w_1, \ldots, w_n\}$ and a target t, decide if there's a subset $S \subseteq U$ such that $\sum_{w \in S} w = t$.

   (c) MIN-COLORING: given an undirected graph $G = (V, E)$, find a *legal coloring* of the vertices of G that uses as few colors as possible. A coloring of the vertices of G is a function $c : V \to \{1, \ldots, n\}$ that assigns a color for each vertex. A coloring is *legal* if for every edge $(u, v) \in E$, $c(u) \neq c(v)$.