# Adaptive Control - Homework 1

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## 1 MRAC of first-order

### 1.1 Design and simulate controller

Assuming that the plant looks like the following

$$\dot{x}_p = a_p x_p + k_p u + \alpha^T f(z) \tag{1}$$

and a reference model

$$\dot{x}_m = a_m x_m + k_m r \tag{2}$$

The point of the controller is so that the plant model 1 to behave and act like the reference model 2. Using the input $u$ that we can control to try to change the plant model to follow the reference model the following equation is derived

$$u^\star = \frac{1}{k_p}(a_m - a_p)x_p + \frac{k_m}{k_p} r + \frac{-\alpha}{k_p} f(z) \tag{3}$$

Now this is the optimal controller, in reality $a_p, k_p$ and $\alpha$ is unknown, this means that to be able to find this optimal $u^\star$ we have to estimate these parameters.

To make things easier we rewrite the control input as

$$u = \theta^T \phi \tag{4}$$

Where $\theta$ is the estimations of the parameters. The difference between the estimations and the actual parameters is denoted as $\tilde{\theta} = \theta - \theta^\star$.

From the error of the estimations of the system and the real system a adaptive control law can be derived. This adaptive control law is given as

$$\dot{\theta} = -\Gamma \mathrm{sign}(k_p) \phi e \tag{5}$$

## 1.2 Linear Plant

Comparing different reference signals:

### Unit step

The model error goes to zero, so it results in no steady state error, but does not estimate the correct parameters. But since the system is steady state, it only needs to learn the DC gain and not all of its dynamics. The unit step contains theoretically infinite number of frequencies, but we do not have enough time to learn the parameters.

### Sinus wave

As soon as we have a sinus wave with a frequency higher than the pole frequency (cutoff frequency) of the linear, first order plant, the adaptive controller is able to learn the real parameters. Since one point in the Bode plot (At a frequency higher than the pole) is enough to characterize a first order system.

### Square wave

Then the controller learns the parameters perfectly, and the state error as well as the parameter error goes to zero. Since the square wave contains enough frequencies, and since it continuously changing so that the controller gets enough information and enough time to learn the parameters correctly.

So to be able to correctly estimate the system parameters, for a linear plant, then the system reference has to be continuously changing and contain enough frequencies.

## 1.3 Prove Stability and convergence of error

The reference model for the plant is of course stable, so what we want to do is get the input $u$ to change the actual plant into a copy of the reference model. The problem with doing this is that the system or plant has parameters that is unknown, the adaptive controller will then estimate these parameter to try to control the system correctly to the reference. So since the reference model is stable and we can correctly estimate the parameters then the plant will also be stable. So to check stability we propose a candidate *Lyapunov*-like function, based on the error between the estimations and the true parameters. Here it is important to notice that this is not a proper Lyapunov function because we are not measuring all the states. I.e. we don't know $\tilde{\theta}$, only our estimate of $\theta$.

$$V(e, \tilde{\theta}) = \frac{1}{2}e^2 + \frac{1}{2}|k_p|\ (\tilde{\theta}\,\Gamma^{-1}\,\tilde{\theta}) \geq 0 \tag{6}$$

Clearly, $V(e,\tilde{\theta})$ is globally positive semidefinite ($V(e,\tilde{\theta}) \geq 0, \forall\, e, \tilde{\theta}$), if $\Gamma^{-1} \succ 0$, which implies that:

a) $V(e,\tilde{\theta}) = 0$ iff $e = 0$, $\tilde{\theta} = 0$

b) $V(e,\tilde{\theta}) > 0$ iff $e > 0$ or $\tilde{\theta} > 0$

c) $\dot{V}(e,\tilde{\theta}) < 0$

All three of these should be fulfilled for a proper Lyapunov function to be stable. The third condition is the one that we still need to prove, so the next step to prove asymptotic stability is to show that $\dot{V}(e,\tilde{\theta}) < 0, \forall\, e, \tilde{\theta} \neq 0$. Taking the derivative of $V(e,\tilde{\theta})$ with respect to time gives us

$$\dot{V} = e\dot{e} + \frac{1}{2}\,|k_p|\left(\dot{\tilde{\theta}}^T\Gamma^{-1}\tilde{\theta} + \tilde{\theta}^T\Gamma^{-1}\dot{\tilde{\theta}}\right) \tag{7}$$

where one can show that $\dot{e} = a_m e + k_p\theta^T\phi$. After some algebraic manipulations, it then follows that:

$$\dot{V} = a_m e^2 + |k|\,\tilde{\theta}^T\left(\mathrm{sign}(k_p)\,\phi\,e + \Gamma^{-1}\dot{\tilde{\theta}}\right) \tag{8}$$

Plugging in our adaptive controller update law (eq. 5) in (8), we observe that the second term on the right side vanishes, which leads us to the following result:

$$\dot{V}(e,\tilde{\theta}) = a_m e^2 \tag{9}$$

For obvious reasons of stability of the reference model, we choose $a_m < 0$, which implies that:

$$\dot{V} = a_m e^2 \leq 0 \tag{10}$$

With statements *a)*, *b)* and *c)* only, we can not show stability of the closed-loop system. For asymptotic stability, it would be required that $\dot{V}(e,\tilde{\theta}) < 0$ for $e, \tilde{\theta} \neq 0$, which is not the case.

Indeed, having $V$ lower bounded ($V(e,\tilde{\theta}) \geq 0$) and decreasing ($\dot{V}(e,\tilde{\theta}) \leq 0$) implies it converges to a limit, but it does not say whether or not $\dot{V}(e,\tilde{\theta}) \to 0$ as $t \to \infty$. Since $\dot{V} = a_m e^2$, this implies that we can also not guarantee that $e \to 0$ as $t \to \infty$ by having only access to the statements *a)*, *b)* and *c)*. To this end, lets introduce the Barbalat's Lemma. which is an alternative to prove stability for non-autonomous systems whose $\dot{V}(e,\tilde{\theta}) \leq 0$:

**Lemma 1.1 (Barbalat's Lemma:)** *If $f(x)$ has a finite limit as $t \to \infty$ and if $\dot{f}(t)$ is uniformly continuous (or $\dot{f}(t)$ is bounded), then $\dot{f}(t) \to 0$ as $t \to \infty$*

This means that, if we can prove that $\dot{V}(e,\tilde{\theta})$ has a finite limit as $t \to \infty$ and $\ddot{V}(e,\tilde{\theta})$ is bounded, then $\dot{V}(e,\tilde{\theta}) \to 0$ and consequently $e \to 0$ as $t \to \infty$. We have already shown that $V$ is lower bounded and decreasing, and therefore it has a limit. What lasts now is to prove that $\ddot{V}(e, \tilde{\theta})$ is bounded.

$$\ddot{V}(e, \tilde{\theta}) = 2a_m e \dot{e} \tag{11}$$

As we have shown that $0 \leq V$ and $\dot{V} \leq 0$, this means that $\tilde{\theta}$ and $e$ are bounded.

Solving for $x_p$ from $e = x_p - x_m$ yields $x_p = e + x_m$ and since $e$ and $x_m$ is bounded, we know that $x_p$ also is bounded.

We can say that $u$ is bounded, since $u = \theta^\top [r \ x_p \ f(z)]$, where $[r \ x_p \ f(z)]$ all are bounded.

Then we can see from $\dot{x}_p = a_p + k_p u + \alpha f(z)$ that $\dot{x}_p$ is bounded, and thereby we also know that $\dot{e} = \dot{x}_p - \dot{x}_m$ is bounded.

Thus we know that $e$ and $\dot{e}$ is bounded, and thereby also $\ddot{V}$

Finally, using Barbalat's Lemma we get.

$$\begin{aligned} \lim_{t \to \infty} \dot{V} = 0 \\ \lim_{t \to \infty} e = 0 \end{aligned} \tag{12}$$

## 1.4 Lyapunov-like function in simulation

As can be seen in figures 1 and 2, the Lyapunov-like function are lower bounded by zero and monotonically decreasing ($V \geq 0, \dot{V} \leq 0$).

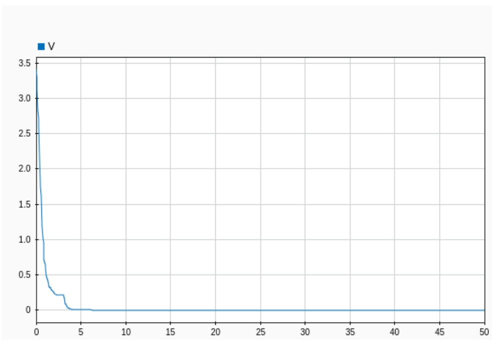


*Figure 1: Lyapunov-like function for the linear plant and square input*

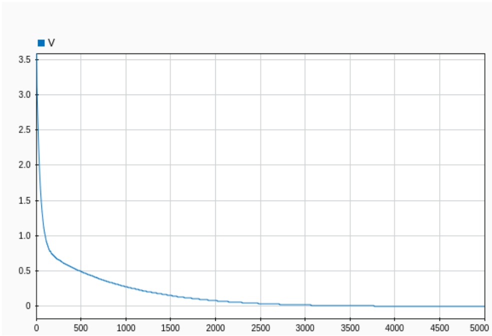


*Figure 2: Lyapunov-like function for the nonlinear plant and combination of sinusoid inputs*

# 2 Non-linear Plant

## 2.1 Parameter Estimation

Using the reference signals given to us, we restrict the values that the state $x_p$ can take. This restriction will affect our parameter estimation of the non-linear parameters in the controller. To get the parameter estimation error to zero we would have to excite the system to every possible $x_p$ value so that the adaptive law can estimated the full dynamic of the non-linear parts of the system.

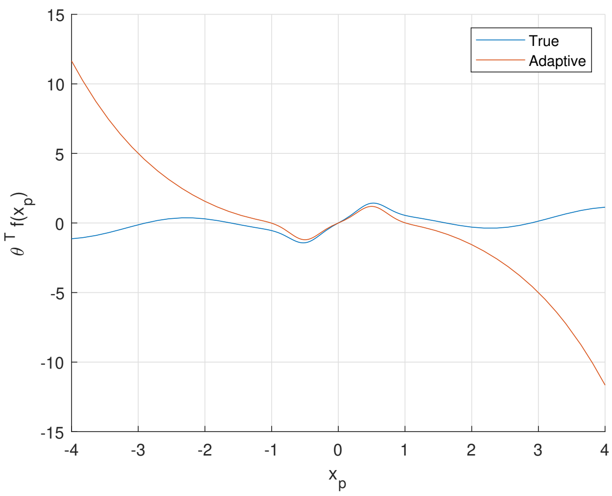


*Figure 3: Comparison of the between the real non-linear function $f(x_p)$ and the estimated version of the function $\tilde{f}(x_p)$.*

As can be see from the figure 3 the controller estimates the non-linear function very well in the interval $[-2, 2]$. If we investigates the $x_p$ values of the system they go between $[-2, 2]$, this means that only for this input interval can the system estimate the non-linear function $f(x_p)$. Outside of this the controller has no way in determining the behaviour of this function.

### 2.2 Change in non-linear parameters

Removing the first and the last $f(z)$ functions, makes our controller estimate the coefficients $\alpha$ perfectly. This happens, because as one can see in Figure 4, the two base functions left are bell-shaped functions, which in this case are not redundant. This means that there is only one combination of $\alpha$'s that minimizes the error between the ideal non-linear function and the adaptive one (for the interval $x \in [-2, 2]$), which are the ideal $\alpha$'s. This also means that the Lyapunov function will converge to zero.

The problem of the 4 non-linear functions $f(z)$ used previously is that they are redundant in the interval $z = x_p = [-2, 2]$, which is the interval of values taken by our states during simulation. This means that, given the input signal, there were combinations of parameters $\alpha$'s that make the Lyapunov function converge to a value that is not equal zero.

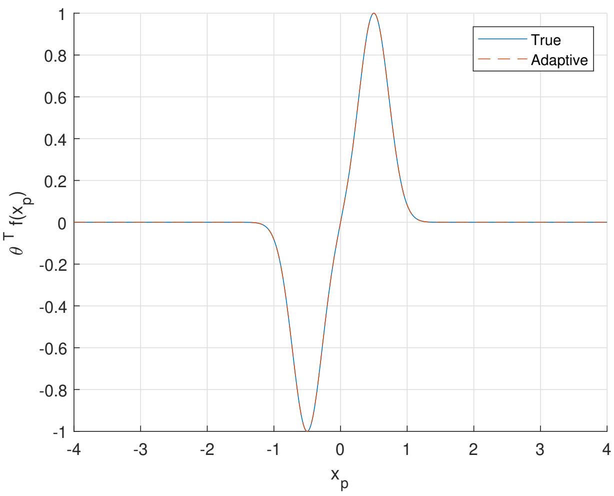


*Figure 4: Estimated $f(x_p)$ when having known zeros in $\theta_f$.*

## 3 Parameter convergence

From the homework we are given.

$$e(t) = M(s)[\tilde{\theta}^\top \phi] \tag{13}$$

We can set the error to zero.

$$0 = M(s)[\tilde{\theta}^\top \phi] \tag{14}$$

Since $M(s)$ is a stable system, with a nonzero DC-gain, the output of the system can only be zero when the input is zero or if the system has a zero at the $j\omega$ axis and the input is that frequency.

The input to the system is zero, i.e. $[\tilde{\theta}^\top \phi]$ is zero, either because $\tilde{\theta}$ is zero, i.e. that the controller has learned the correct values, or the product $(\tilde{\theta}^\top \phi)$ is zero because the controller has solved the linear equation (if $\phi$ is stationary).

Thus we can specify conditions such that $e(t) = 0 \Rightarrow \tilde{\theta}(t) \to 0$:

- $\phi(t)$ should not be always equal to zero. $\exists t,\ \phi(t) \neq 0$.
- $\phi(t)$ should not be any of the zero-frequencies of $M(s)$ (if there is any). In a more formal way: $\exists w, |M(jw)|\ |\Phi(w)| > 0$, where $\Phi(w)$ is the Fourrier transform of $\phi(t)$.
- $\tilde{\theta}(t)$ should not be orthogonal to $\phi(t)\ \ \forall t$. This implies that $\exists t,\ \phi(t) \not\perp \tilde{\theta}(t)$.

For these reason we have found that the best reference input to the system would be a reference $r$ that is changing over time and contains several frequencies. As we could also see from the different inputs that we tested the square wave was the best for parameter estimation.

# 4 Investigate the high-frequency gain

First, we rewrite the system such that it matches the one given in the homework.

$$G(s) = \frac{b_1 \cdot s + b_0}{a_3 \cdot s^3 + a_2 \cdot s^2 + a_1 \cdot s + a_0} = \frac{b_1}{a_3} \frac{s + b_0/b_1}{s^3 + a_2/a_3 \cdot s^2 + a_1/a_3 \cdot s + a_0/a_3} \tag{15}$$

Thus we see that the high frequency gain is $\frac{b_1}{a_3}$.

The relative degree of a transfer function tells the minimum number of integration's between the input and the output of the system, for a transfer function this means subtract the order of the denominator with the order of the numerator. This also reflects how fast the system responds to the input signal, basically a higher relative degree yields a slower response to the input signal.

To get the DC-gain of a transfer function, we set $s = 0$ and see how the system behave. With the given transfer function $G(s)$ we get

$$G(s = 0) = \frac{b_0}{a_0} \tag{16}$$

For our system the relative degree $r = 2$, in the frequency domain we know that a derivation is just to multiply the system by $s$.

Deriving the system twice yields

$$W(s) = s^2 G(s) = \frac{b_1}{a_3} \frac{s^3 + b_0/b_1 \cdot s^2}{s^3 + a_2/a_3 \cdot s^2 + a_1/a_3 \cdot s + a_0/a_3} \tag{17}$$

Applying a step as input and applying the initial value theorem

$$f(0^+) = \lim_{s \to \infty} sF(s)$$

yields

$$\lim_{s \to \infty} \frac{1}{s} sW(s) = \lim_{s \to \infty} \frac{b_1}{a_3} \frac{s^3 + b_0/b_1 \cdot s^2}{s^3 + a_2/a_3 \cdot s^2 + a_1/a_3 \cdot s + a_0/a_3} = \frac{b_1}{a_3} \cdot 1 = k \tag{18}$$

Which is our high frequency gain $k$.

## 4.1 Sign of the high-frequency gain

Given the first system

$$G_a(s) = \frac{\lambda}{s + \lambda} \tag{19}$$

and $\lambda \in [-1, 1]$, the DC-gain of the system is $G(0) = 1$ and the High Frequency Gain is $\lambda$. Using a P-controller for this system, i.e. $u = ky$ then the closed-loop (CL) transfer function will be given by.

$$CL = \frac{kG(s)}{1 + kG(s)} \tag{20}$$

From this the poles of the $CL_a$ will be given by

$$\begin{aligned} 1 + k\frac{\lambda}{s + \lambda} &= 0 \\ \Rightarrow -\lambda(k + 1) &< 0 \end{aligned} \tag{21}$$

To choose a stable $k$, we must know the sign of $\lambda$, the high frequency gain, since as can be seen, we get two different cases depending on the sign of $\lambda$. We assume $\lambda \neq 0$, since otherwise the system will just be $G(s) = 0$.

For $\lambda < 0$ (unstable plant), $k$ must be less than $-1$ to have a stable closed loop system.

When $\lambda > 0$ (stable plant), $k$ must be greater than $-1$ to have a stable closed loop system.

For the second system

$$G_b(s) = \frac{1}{s+\lambda} \tag{22}$$

the DC-gain will be $1/\lambda$ and the high frequency gain is 1. Using the same approach to determine the poles for the closed loop system as for $G_a(s)$, we get the following expression for the pole polynominal of the closed loop.

$$1 + \frac{k}{s+\lambda} = 0 \Rightarrow s = -\lambda - k < 0 \Rightarrow -\lambda < k$$

Here we can see that we are not dependent of the sign of $\lambda$, assuming $\lambda \in [-1, 1]$, then we should always get a stable system when $k > 1$.

Hence, knowing the High Frequency Gain is essential to the design of a stable control system.