HOMEWORK 2

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MAE 5010
Atmospheric Flight Control

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1. (a)

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 1 & -4 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Controllability matrix:

$$C_G = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

$$C_G = \begin{bmatrix} B & AB & A^2B \end{bmatrix},$$

$$AB = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 1 & -4 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 1 & -4 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 1 & -4 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 8 & -3 \\ 0 & 1 & 0 \\ 3 & -10 & 3 \end{pmatrix}, \quad A^2B = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

$$C_G = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{pmatrix}$$

Since, second column is a linear combination of first and third column, C_G is rank deficient (Rank of $C_G = 2$). Hence, the dynamics are uncontrollable with this control surface design.

(b)

$$\lambda I - A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} - \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 1 & -4 & 2 \end{pmatrix} = \begin{pmatrix} \lambda - 1 & -2 & 1 \\ 0 & \lambda - 1 & 0 \\ -1 & 4 & \lambda - 2 \end{pmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & 1 \\ 0 & \lambda - 1 & 0 \\ -1 & 4 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 1)(\lambda - 2) + (\lambda - 1)$$

$$\text{Now setting, } |\lambda I - A| = 0$$

$$\implies (\lambda - 1)(\lambda - 1)(\lambda - 2) + (\lambda - 1) = 0$$

$$\implies (\lambda - 1)(\lambda^2 - 3\lambda + 3) = 0$$

$$\implies \lambda_1 = 1 \quad \text{and } \lambda_{2,3} = 1.5 \pm 0.866i$$

Since all the eigen values are positive which implies that system is unstable as we showed in 1(a).

For $\lambda_1 = 1$

$\text{rank}(\lambda I - A \quad B) = n$

$$(\lambda I - A \quad B) = \left\{ \begin{pmatrix} \lambda_1 - 1 & -2 & 1 \\ 0 & \lambda_1 - 1 & 0 \\ -1 & 4 & \lambda_1 - 2 \end{pmatrix} \quad B \right\}$$
$$= \left\{ \begin{pmatrix} 0 & -2 & 1 \\ 0 & 0 & 0 \\ -1 & 4 & -1 \end{pmatrix} \quad B \right\}$$
$$= \begin{pmatrix} 0 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 \end{pmatrix}$$

Since the rank of $(\lambda I - A \quad B) = 2 \neq 3$, for this unstable mode, it is not stabilizable. Left eigen vector from MATLAB, $v_1 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T$,

$$v_1^T B = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$$

For $\lambda_2 = 1.5 + 0.866i$
$\text{rank}(\lambda I - A \quad B) = n$

$$(\lambda_2 I - A \quad B) = \left\{ \begin{pmatrix} \lambda_2 - 1 & -2 & 1 \\ 0 & \lambda_2 - 1 & 0 \\ -1 & 4 & \lambda_2 - 2 \end{pmatrix} \quad B \right\}$$
$$= \left\{ \begin{pmatrix} 0.5 + 0.866i & -2 & 1 \\ 0 & 0.5 + 0.866i & 0 \\ -1 & 4 & -0.5 + 0.866i \end{pmatrix} \quad B \right\}$$
$$= \begin{pmatrix} 0.5 + 0.866i & -2 & 1 & 1 \\ 0 & 0.5 + 0.866i & 0 & 0 \\ -1 & 4 & -0.5 + 0.866i & 0 \end{pmatrix}$$

Since the rank of $(\lambda I - A \quad B) = 3 = n$, for this unstable mode, it is stabilizable.
For $\lambda_3 = 1.5 - 0.866i$

rank$(\lambda_3 I - A \quad B) = n$

$$(\lambda_3 I - A \quad B) = \left\{ \begin{pmatrix} \lambda_3 - 1 & -2 & 1 \\ 0 & \lambda_3 - 1 & 0 \\ -1 & 4 & \lambda_3 - 2 \end{pmatrix} \quad B \right\}$$
$$= \left\{ \begin{pmatrix} 0.5 - 0.866i & -2 & 1 \\ 0 & 0.5 - 0.866i & 0 \\ -1 & 4 & -0.5 - 0.866i \end{pmatrix} \quad B \right\}$$
$$= \begin{pmatrix} 0.5 - 0.866i & -2 & 1 & 1 \\ 0 & 0.5 - 0.866i & 0 & 0 \\ -1 & 4 & -0.5 - 0.866i & 0 \end{pmatrix}$$

Since the rank of $(\lambda I - A \quad B) = 3 = n$, for this unstable mode, it is stabilizable.

(c) Observability matrix for $C = (1 \quad 0 \quad 0)$

$$O_G = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & -1 \\ 0 & 8 & -3 \end{pmatrix}$$

rank of $O_G = 3$, it is observable. Since the dynamics is observable, it is also detectable.

2.

$$Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & (\frac{1}{\Delta\phi_{max}})^2 & 0 \\ 0 & 0 & (\frac{1}{\Delta\delta_{v_{max}}})^2 \end{pmatrix}, \quad R = (\frac{1}{\delta_{v_{max}}})^2$$

Please look at the MATLAB code is in the appendix.

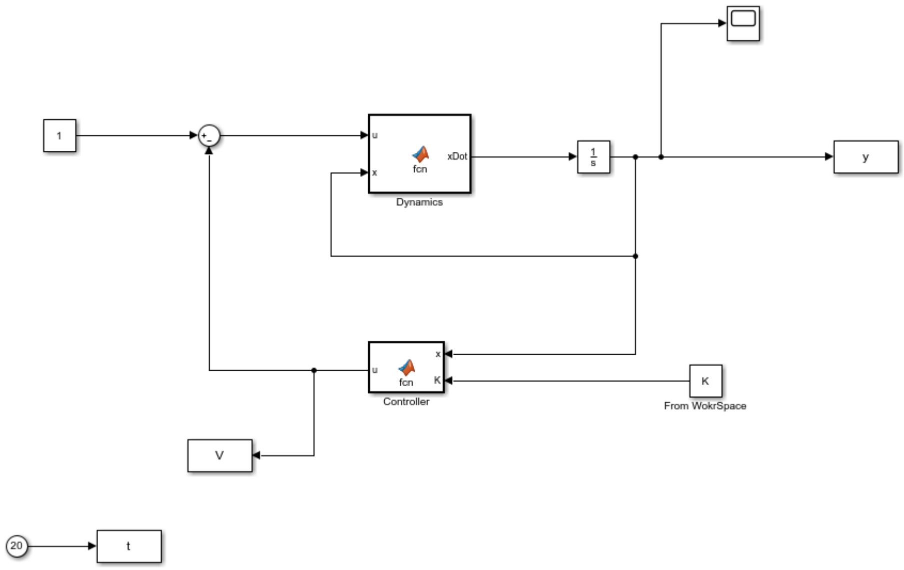

Figure 1: Block Diagram for simulation

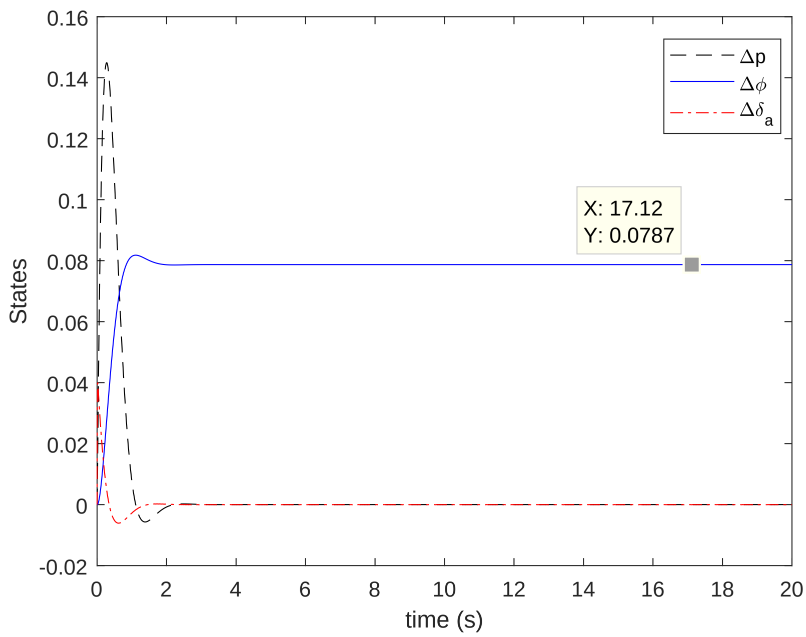

Figure 2: Step response from voltage to the states

Steady state value of $\Delta\phi = 0.0787$ rad.

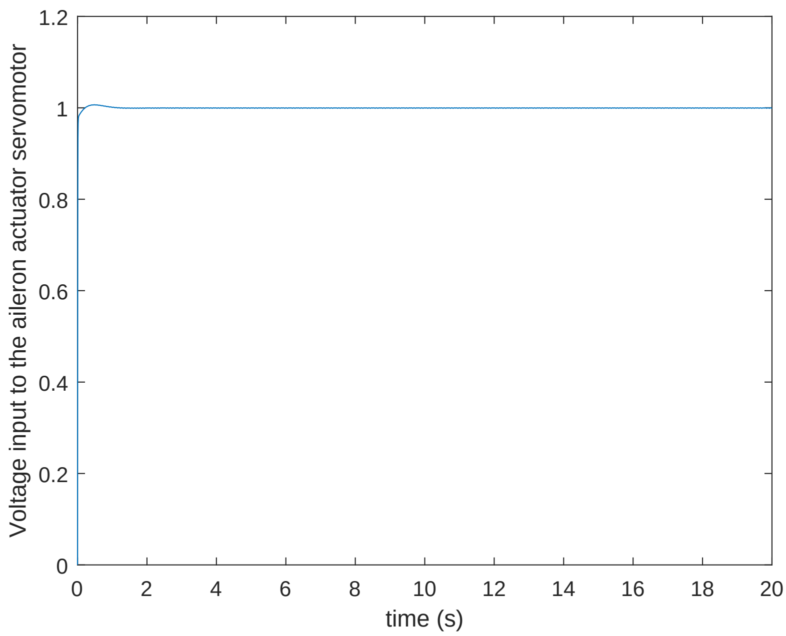

Figure 3: Actuator voltage plot

3. (a) Conditions check [1]

C_1, Q_1, R_1

i. Finding a stabilize gain K such that $A_c = A - BKC_1$ is stable which implies that the system is output stabilizable. For the sake of contradiction assume that there exists such k such that the system is stabliizable.

$$A_c = A - BKC_1$$
$$A_c = \begin{pmatrix} -1 & 0 & 30 \\ 0 & 0 & \\ -10K & 0 & -10K - 10 \end{pmatrix}$$
$$\lambda_1 = 0$$
$$\lambda_2 = -5K - (100K^2 - 1020K + 81)^(1/2)/2 - 11/2$$
$$\lambda_3 = (100K^2 - 1020K + 81)^(1/2)/2 - 5K - 11/2$$

We can observe that not all eigen values of A_c are negative since $\lambda_1 = 0$. This system is not stabilize.

C_2, Q_2, R_2

i. We were able to find a stabilize gain K (for e.g. , k = 0.1 works) such that $A_c = A - BKC$ is stable which implies that the system is output stabilizable.

ii. C matrix has full row rank in this case. (rank of C = 1).

iii. $R > 0$ and $Q \geq 0$ is true.

iv. $(A, \sqrt{Q})$ is detectable.

Rank of

$$\text{rank} \begin{pmatrix} \lambda I - A \\ -\sqrt{Q} \end{pmatrix} = n \; \forall \quad \lambda \in \mathbb{C}^+$$

$$A = \begin{pmatrix} -1 & 0 & 30 \\ 1 & 0 & 0 \\ 0 & 0 & -10 \end{pmatrix}, \tag{1}$$

$$\text{rank of} \begin{pmatrix} \sqrt{Q} \\ \sqrt{Q}A \\ \sqrt{Q}A^2 \end{pmatrix} = 3$$

So it is observable which implies the dynamics is detectable.

(b) Proceed with C_2 and Q_2, $K_0 = 0.1$, $\alpha = 0.1$

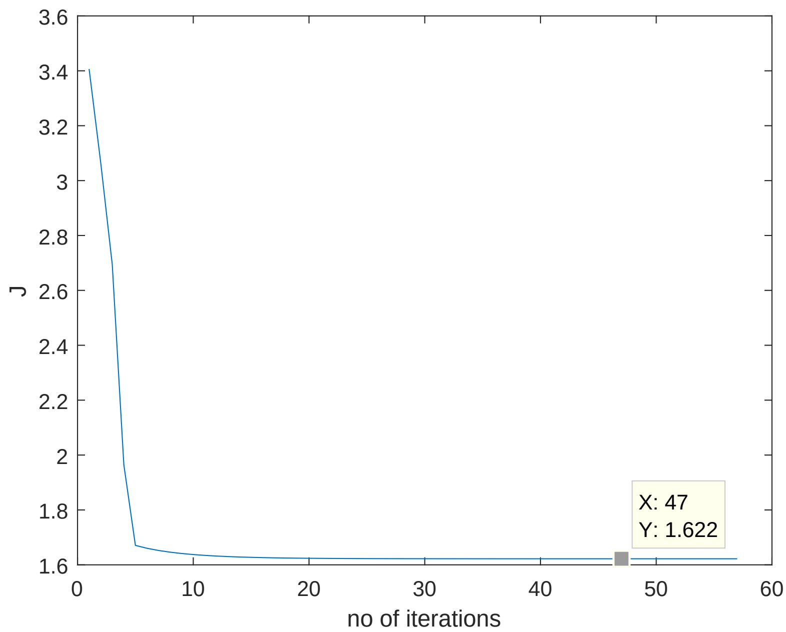

Figure 4: Initial plot for J.

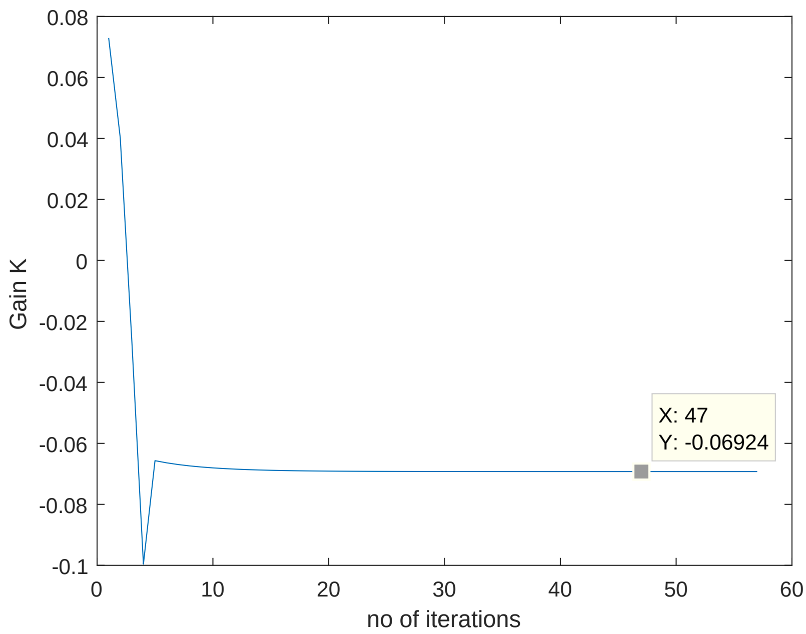

Figure 5: Initial plot for K.

(c) Used C_2 and Q_2, $K_0 = 0.1$, $\alpha = 0.01$

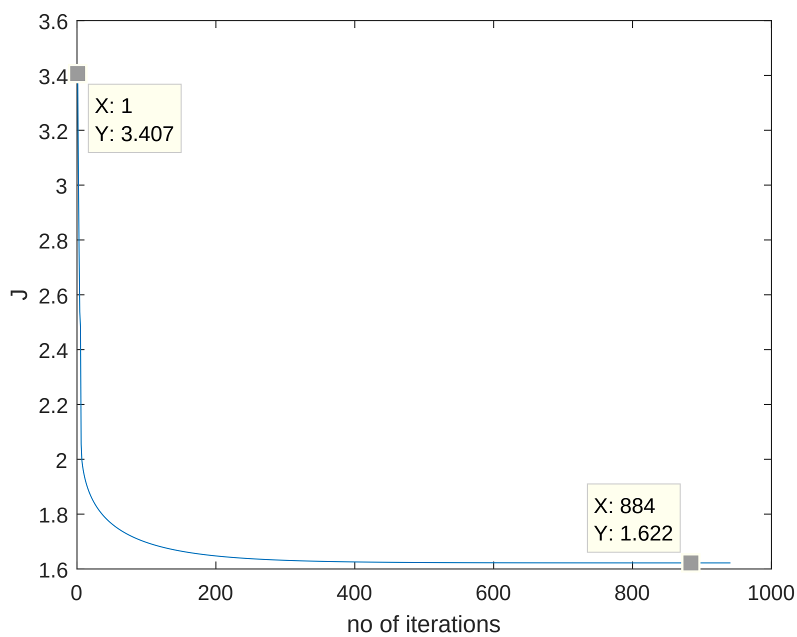

Figure 6: J plot.

We can observe that initial value of $J = 4.407$. Final value of $J = 1.622$. Error in prediction:

$$\% \quad error = \frac{3.407 - 1.622}{1.622} = 110\%.$$

Since the initial stabilizing gain only gives us a stable. system it does not provide us the local or global minimum so it is expected that final and initial prediction are different from each other.

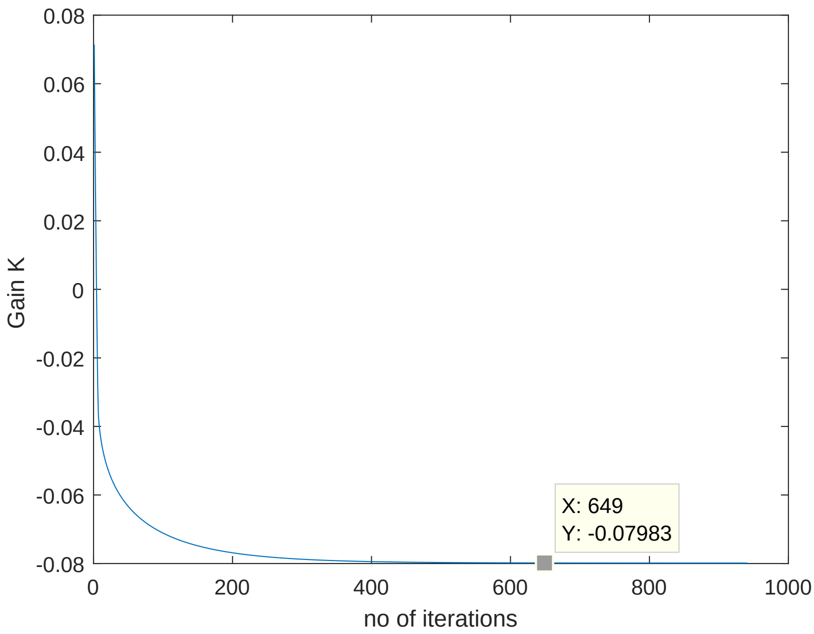

Figure 7: Gain plot.

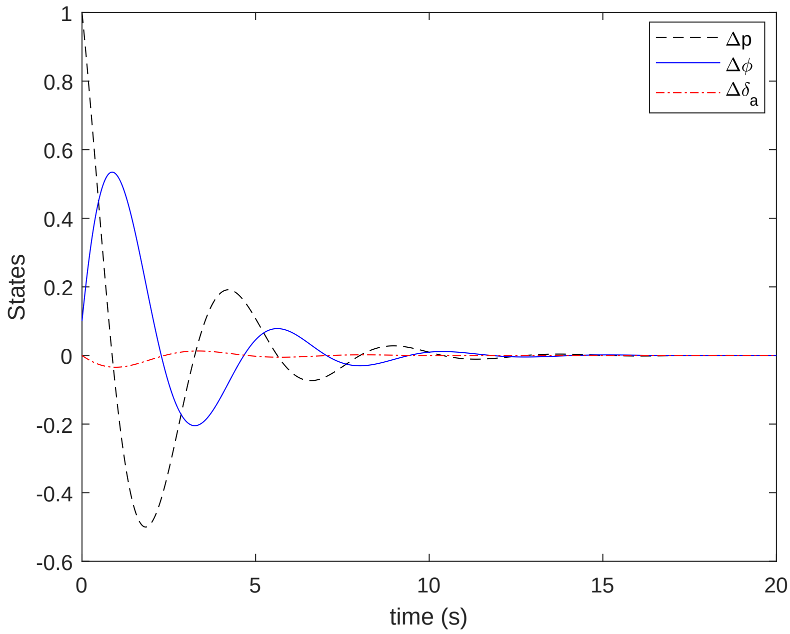

Figure 8: Step response from voltage to the states. All of the states converge to zero.

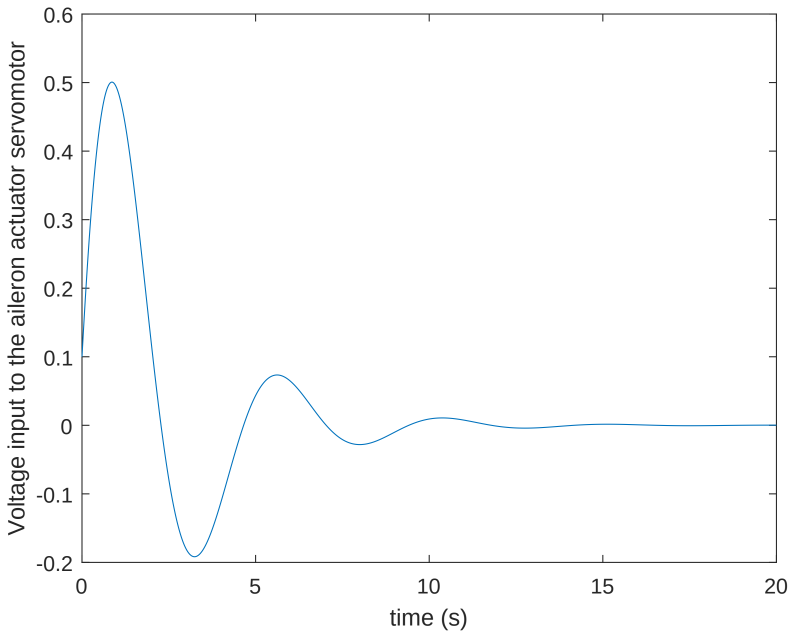

Figure 9: Actuator voltage plot. It saturates after 10 seconds.

(d) Use $H = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$, $Q = H^T H$, $\alpha = 0.01$, $\rho = 0.5$.

Conditions check:

i. We were able to find a stabilize gain K (K = 0.1 works in this case too.) such that

$A_c = A - BKC$ is stable which implies that the system is output stabilizable.

ii. H matrix has full row rank in this case. (rank of H = 1).

iii. $R > 0$ and $Q \geq 0$ is true.

iv. $(A, \sqrt{Q})$ is detectable.

Rank of

$$\text{rank} \begin{pmatrix} \lambda I - A \\ -\sqrt{Q} \end{pmatrix} = n \; \forall \quad \lambda \in \mathbb{C}^{+}$$

$$A = \begin{pmatrix} -1 & 0 & 30 \\ 1 & 0 & 0 \\ 0 & 0 & -10 \end{pmatrix}, \tag{2}$$

$$\text{rank of} \begin{pmatrix} \sqrt{Q} \\ \sqrt{Q}A \\ \sqrt{Q}A^2 \end{pmatrix} = 3$$

So it is observable which implies the dynamics is detectable.

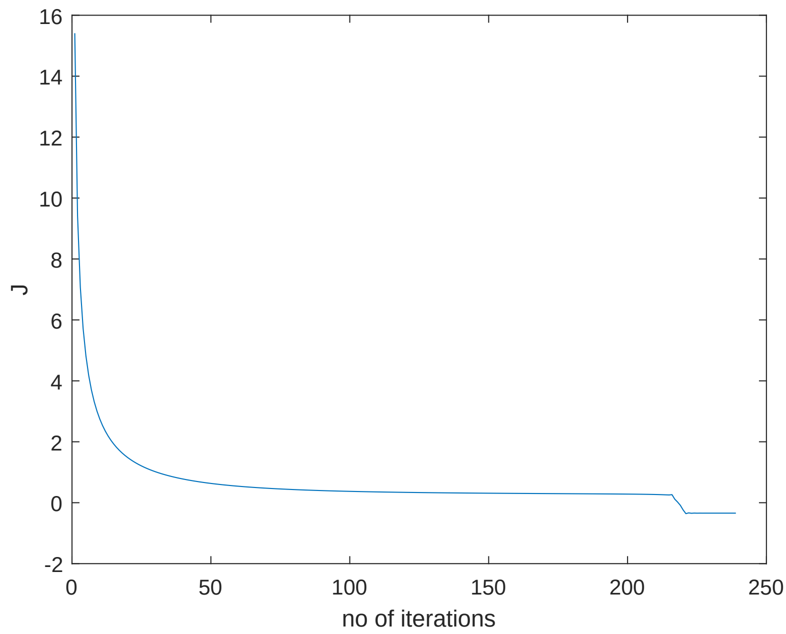

Figure 10: J plot for initial case for 3(d).

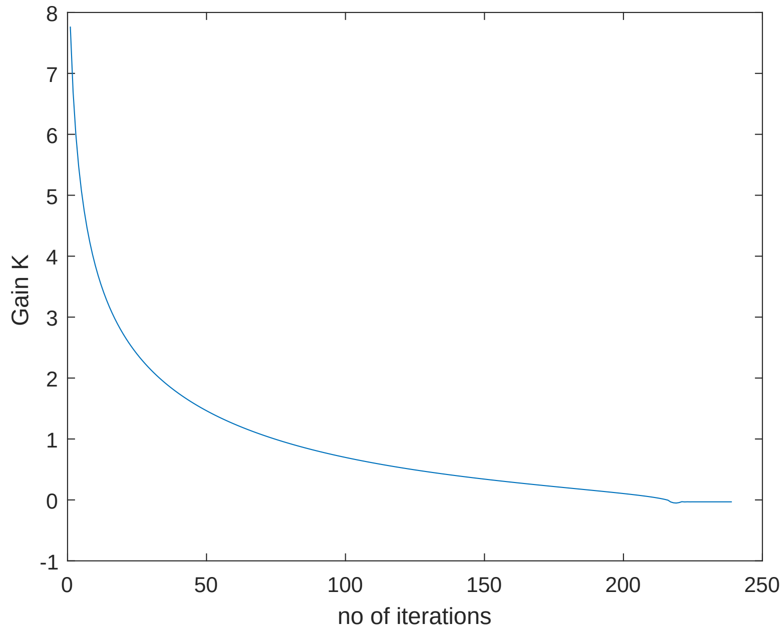

Figure 11: K plot for initial case for 3(d).

Now we use $\alpha = 0.01$, $\rho = 0.001$.

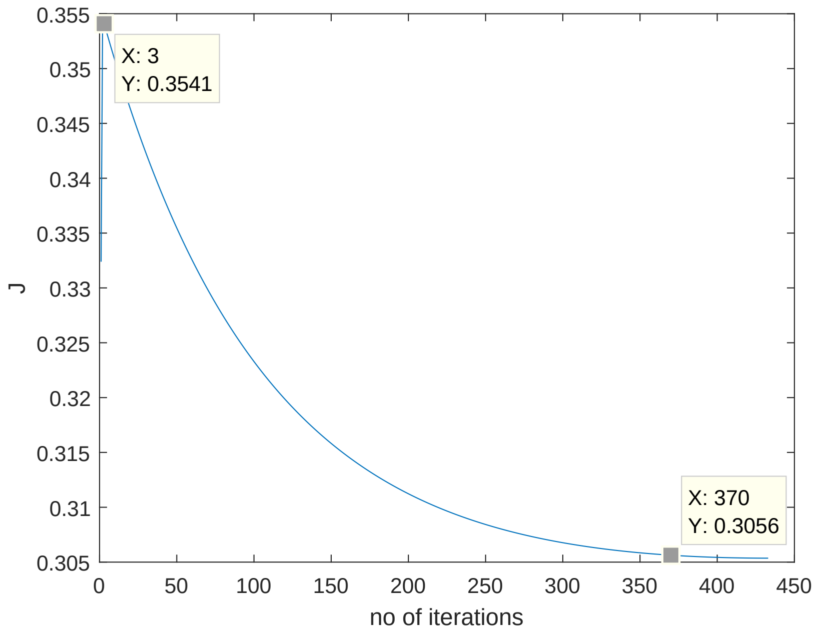

Figure 12: J plot for 3(d)

We can observe that initial value of $J = 0.3541$. Final value of $J = 0.3056$. Error in prediction:

$$\% \quad error = \frac{0.3541 - 0.3056}{0.3056} = 16\%.$$

Since the initial stabilizing gain only gives us a stable. system it does not provide us the local or global minimum so it is expected that final and initial prediction are different from each other. Also notice that $\rho = 0.001$, which causes significant difference in the error prediction.

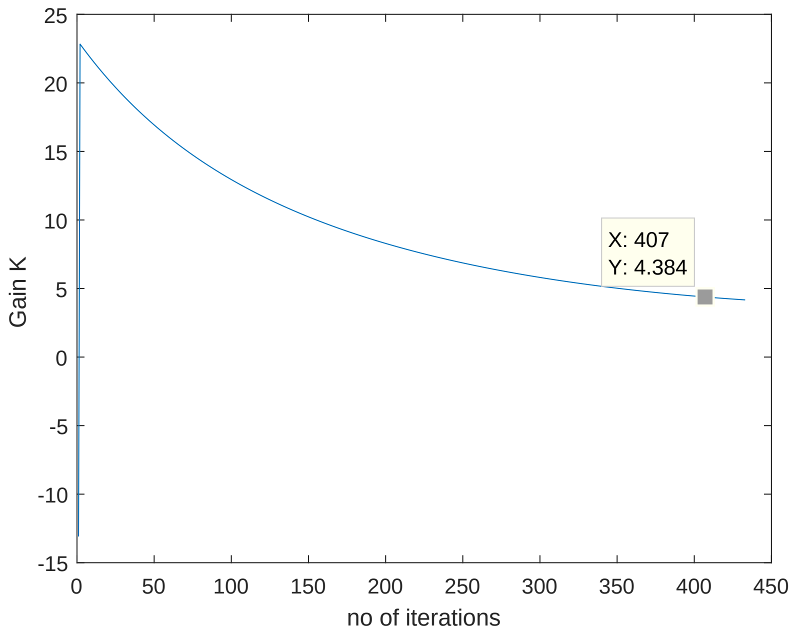

Figure 13: Gain plot for 3(d).

(e) Comparison of Output-LQR and full state feedback

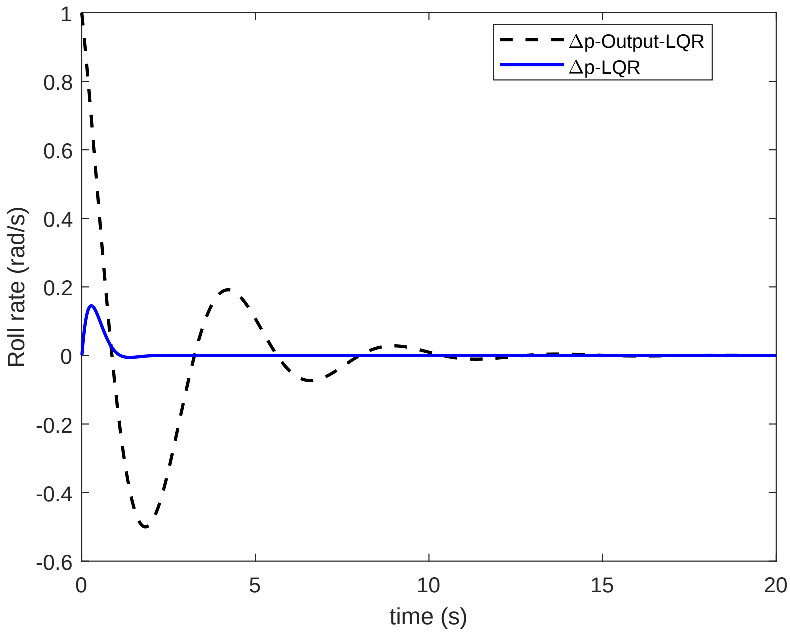

Figure 14: Roll rate Comparison.

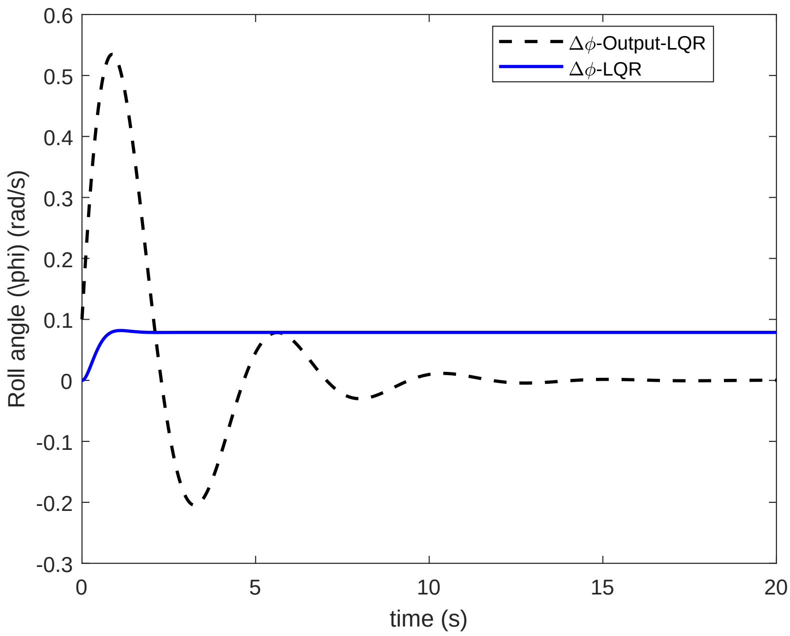

Figure 15: Roll angle Comparison.

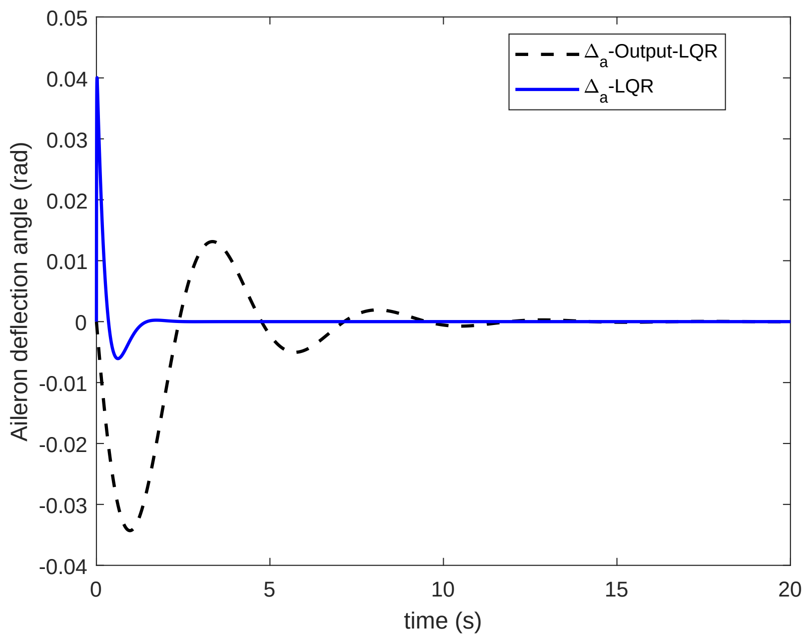

Figure 16: Aileron Comparison.

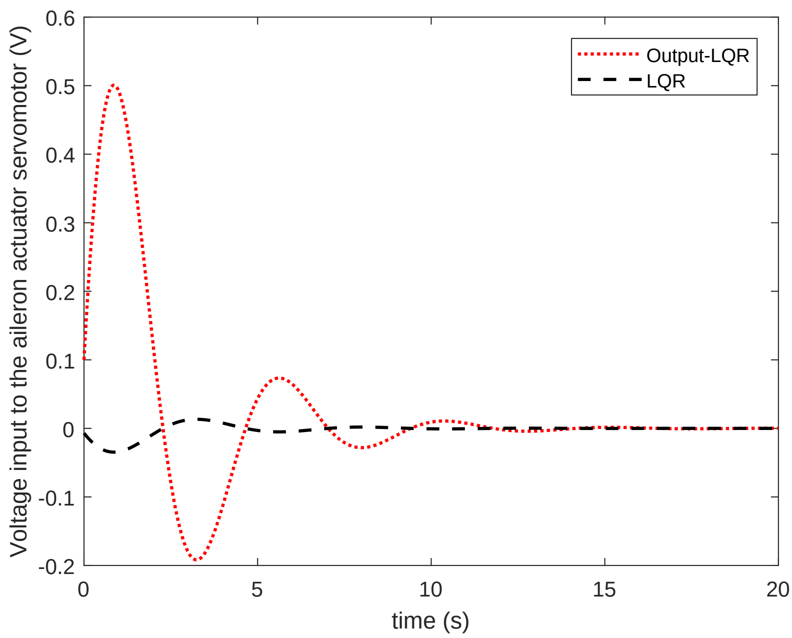

Figure 17: Voltage Comparison.

% overshoot for output LQR is higher than overshoot for full state feedback. The response from full state LQR is much faster than output LQR. Settling time is less than 3 seconds for all the states in LQR but for output LQR, it is much higher , which is around 15 seconds.

Since, Output LQR does not guarantee a global solution, it takes longer time than regular LQR.

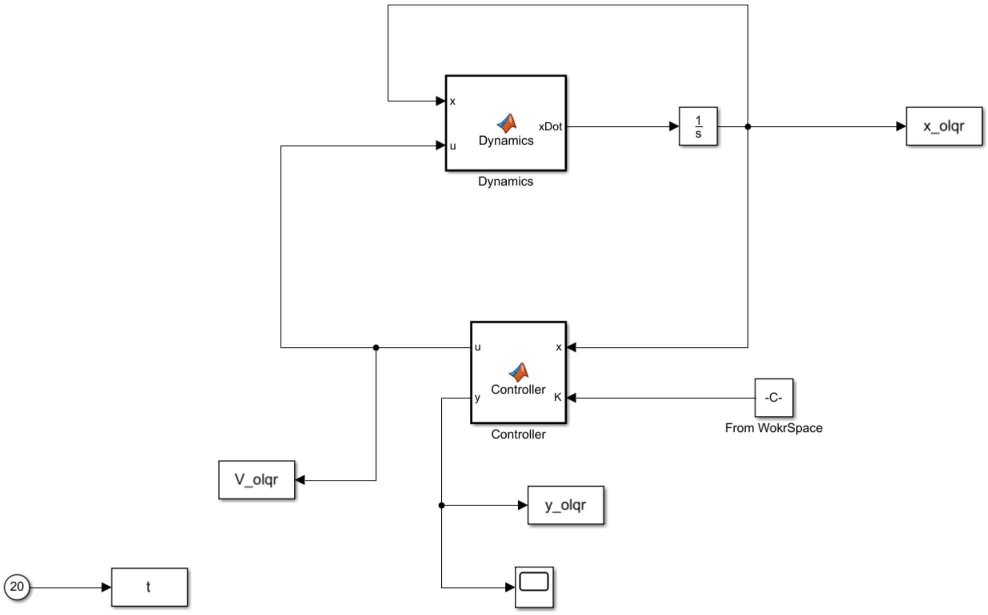

Figure 18: Simulink block diagram from implementing the output LQR.

A Matlab Code

```
clear all; close all; clc;
% Prob 2-HW2- MAE 5010- Atmospheric Flight Controls
% Sandesh Thapa

%% Prob2(a)
Lp = -1; % rad/s
Ldela = 30; % s^-2;
tau = 0.1; % sec

Deldela_max = 0.436; % rad
Delphi_max = 0.787; % rad
delv_max = 10; % volts

A = [Lp 0 Ldela;
     1  0 0;
     0  0 -1/tau];

B = [0 0 1/tau]';

```

```
Q = [0   0                    0;
      0  1/(Delphi_max)^2  0
      0  0                    1/(Deldela_max)^2];

R = 1/(delv_max)^2;

N = zeros(3,1);
[K,S,e] = lqr(A,B,Q,R,N)

%% Prob 2(b)

sim('Sim_Prob2')

figure
plot(t,y(:,1),'--k',t,y(:,2),'b',t,y(:,3),'-.r');
legend('\Deltap','\Delta\phi','\Delta\delta_a','Location','best','Interpreter','
    latex')
xlabel('time (s)')
ylabel('States')
hold on

figure
plot(t,V)
xlabel('time (s)')
ylabel('Voltage input to the aileron actuator servomotor')
```

```
%% Dynamics for Simlink block
function xDot = Dynamics(u,x)

Lp = -1; % rad/s
Ldela = 30; % s^-2;
tau = 0.1; % sec
%
% Deldela_max = 0.436; % rad
% Delphi_max = 0.787; % rad
% delv_max = 10; % volts

A = [Lp  0  Ldela;
      1   0  0;
      0   0  -1/tau];

B = [0  0  1/tau]';

xDot = A*x + B*u;
```

```
%% Controller
function u = Controller(x,K)

u = K'*x;
```

```
close all;clear all; clc
% Prob 3-HW2- MAE 5010- Atmospheric Flight Controls
% Sandesh Thapa

%%
Lp = -1; % rad/s
Ldela = 30; % s^-2;
tau = 0.1; % sec

Deldela_max = 0.436; % rad
Delphi_max = 0.787; % rad
delv_max = 10; % volts

A = [Lp 0 Ldela;
      1  0 0;
      0  0 -1/tau];

B = [0 0 1/tau]';
C = [0 1 1];

v = [1;1;1];
Q = diag(v);

R = 1;
x0 = [1 0.1 0]';
X = x0*x0';

n = 1000;

K0 = [0.1];
Jt = zeros(n,1);
Kt = zeros(n,1);
for m = 1:n

A_c = A- B*K0*C;

S = Q + C'*K0'*R*K0*C;

P = lyap(A_c,S);
```

```
Lamda = lyap(A_c,X);
x0 = [1 0.1 0]';
X = x0*x0';

J(m) = 0.5*trace(P*X);
Jt(m) = J(m);

Del_k = inv(R)*B'*P*Lamda*C'*inv(C*Lamda*C') - K0;
alpha = 0.01;

K(m) = K0 + alpha*Del_k;
Kt(m) = K(m);
eA_c = eig(A_c);
if m > 1
    del_J = (Jt(m) - Jt(m-1));
    if abs(del_J) < 0.0000002
        break
    end

    if max(real(eA_c)) < 0 && (del_J) < 0.0000002
        K0 = K(m);

    else
        m = m+1;
        A_c = A- B*K0*C;
        S = Q + C'*K0'*R*K0*C;
        x0 = [1 0.1 0]';

        P = lyap(A_c,S);

        X = x0*x0';

        Lamda = lyap(A_c,X);

        J(m) = 0.5*trace(P*X);
        Jt(m) = J(m);

        Del_k = inv(R)*B'*P*Lamda*C'*(C*Lamda*C') - K0;

        X = x0*x0';

        Lamda = lyap(A_c,X);
```

```
            alpha = alpha/4;
            K(m) = K0 + alpha*Del_k;
            K0 = K(m);
        end
end
K0 = K(m);
end

figure
plot(J)
xlabel('no of iterations')
ylabel('J')

hold on
figure
plot(K)
xlabel('no of iterations')
ylabel('Gain K ')

%%

sim('Sim_Prob3')

figure
plot(t,x_olqr(:,1),'--k',t,x_olqr(:,2),'b',t,x_olqr(:,3),'-.r');
legend('\Deltap','\Delta\phi','\Delta\delta_a','Location','best','Interpreter','
    latex')
xlabel('time (s)')
ylabel('States')
hold on

figure
plot(t,y_olqr)
xlabel('time (s)')
ylabel('Voltage input to the aileron actuator servomotor')


% % fun = @(k)A-B*K*C;
% %
% k = fminsearch(J,x0)
%%
figure
```

```
plot(t,y_olqr,':r',t,V_olqr,'--k','LineWidth',1.5)
legend('Output-LQR', 'LQR')
xlabel('time (s)')
ylabel('Voltage input to the aileron actuator servomotor (V)')
hold on


figure
plot(t,x_olqr(:,1),'--k',t,y(:,1),'b','LineWidth',1.5);
legend('\Deltap-Output-LQR','\Deltap-LQR','Location','best','Interpreter','latex')
xlabel('time (s)')
ylabel('Roll rate (rad/s)')
hold on

figure
plot(t,x_olqr(:,2),'--k',t,y(:,2),'b','LineWidth',1.5);
legend('\Delta\phi-Output-LQR','\Delta\phi-LQR','Location','best','Interpreter','latex')
xlabel('time (s)')
ylabel('Roll angle (\phi) (rad/s)','Interpreter','latex')
hold on

figure
plot(t,x_olqr(:,3),'--k',t,y(:,3),'b','LineWidth',1.5);
legend('\Delta_a-Output-LQR','\Delta_a-LQR','Location','best','Interpreter','latex')
xlabel('time (s)')
ylabel('Aileron deflection angle (rad)')
hold on
```

```
close all;clear all; clc
% Prob 3-HW2- MAE 5010- Atmospheric Flight Controls
% Sandesh Thapa

%%
Lp = -1; % rad/s
Ldela = 30; % s^-2;
tau = 0.1; % sec

Deldela_max = 0.436; % rad
Delphi_max = 0.787; % rad
delv_max = 10; % volts

```

```
A = [Lp 0 Ldela;
      1  0  0;
      0  0 -1/tau];

B = [0 0 1/tau]';
H = [1   1  0];
C = H;

Q = H'*H;

R = .001;
x0 = [1 0.1 0]';
X = x0*x0';

n = 10000;

K0 = [10];
Jt = zeros(n,1);
Kt = zeros(n,1);
for m = 1:n

A_c = A- B*K0*C;

S = Q + C'*K0'*R*K0*C;

P = lyap(A_c,S);

Lamda = lyap(A_c,X);
x0 = [1 0.1 0]';
X = x0*x0';

J(m) = 0.5*trace(P*X);
Jt(m) = J(m);

Del_k = inv(R)*B'*P*Lamda*C'*inv(C*Lamda*C') - K0;
alpha = 0.01;

K(m) = K0 + alpha*Del_k;
Kt(m) = K(m);
eA_c = eig(A_c);
if m > 1
    del_J = (Jt(m) - Jt(m-1));
    if abs(del_J) < 0.0000002
```

```
            break
        end

        if max(real(eA_c)) < 0 &&  (del_J) < 0.0000002
            K0 = K(m);

        else
            m = m+1;
            A_c = A- B*K0*C;
            S = Q + C'*K0'*R*K0*C;
            x0 = [1  0.1  0]';

            P = lyap(A_c,S);

            X = x0*x0';

            Lamda = lyap(A_c,X);

            J(m) = 0.5*trace(P*X);
            Jt(m) = J(m);

            Del_k = inv(R)*B'*P*Lamda*C'*(C*Lamda*C') - K0;

            X = x0*x0';

            Lamda = lyap(A_c,X);

            alpha = alpha/4;
            K(m) = K0 + alpha*Del_k;
            K0 = K(m);
        end
    end
    K0 = K(m);
    end

    figure
    plot(J)
    xlabel('no of iterations')
    ylabel('J')

    hold on
    figure
    plot(K)
```

```
xlabel('no of iterations')
ylabel('Gain K ')

%%

sim('Sim_Prob3')

figure
plot(t,x_olqr(:,1),'--k',t,x_olqr(:,2),'b',t,x_olqr(:,3),'-.r');
legend('\Deltap','\Delta\phi','\Delta\delta_a','Location','best','Interpreter','
    latex')
xlabel('time (s)')
ylabel('States')
hold on

figure
plot(t,y_olqr)
xlabel('time (s)')
ylabel('Voltage input to the aileron actuator servomotor')


% % fun = @(k)A-B*K*C;
% %
% k = fminsearch(J,x0)
%%
figure
plot(t,y_olqr,':r',t,V_olqr,'--k','LineWidth',1.5)
legend('Output-LQR', 'LQR')
xlabel('time (s)')
ylabel('Voltage input to the aileron actuator servomotor (V)')
hold on


figure
plot(t,x_olqr(:,1),'--k',t,y(:,1),'b','LineWidth',1.5);
legend('\Deltap-Output-LQR','\Deltap-LQR','Location','best','Interpreter','latex
    ')
xlabel('time (s)')
ylabel('Roll rate (rad/s)')
hold on

figure
plot(t,x_olqr(:,2),'--k',t,y(:,2),'b','LineWidth',1.5);
```

```
legend('\Delta\phi-Output-LQR','\Delta\phi-LQR','Location','best','Interpreter','latex')
xlabel('time (s)')
ylabel('Roll angle (\phi) (rad/s)','Interpreter','latex')
hold on

figure
plot(t,x_olqr(:,3),'--k',t,y(:,3),'b','LineWidth',1.5);
legend('\Delta_a-Output-LQR','\Delta_a-LQR','Location','best','Interpreter','latex')
xlabel('time (s)')
ylabel('Aileron deflection angle (rad)')
hold on
```

References

[1] Brian L Stevens, Frank L Lewis, and Eric N Johnson. *Aircraft control and simulation: dynamics, controls design, and autonomous systems.* John Wiley & Sons, 2015.