Lab00-Proof

CS214-Algorithm and Complexity, Xiaofeng Gao, Spring 2020.

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1. Prove that for any integer $n > 2$, there is a prime p satisfying $n < p < n!$. (Hint: consider a prime factor p of $n! - 1$ and prove by contradiction)

   Proof. Assume $n > 2$ and there doesn't exist a prime p satisfying $n < p < n!$

   Then there must exist a prime factor of $n! - 1$, p, and $p \leq n! - 1 < n!$

   Assume $p \leq n$,so $p \mid n!$ and $p \mid n! - 1$, then $p \mid 1$

   It's impossible. So $p > n$.

   Therefore there exists such a prime p satisfying $n < p < n!$, which contradicts the assumption that there doesn't exist such a prime p.

   □

2. Use the minimal counterexample principle to prove that for any integer $n > 17$, there exist integers $i_n \geq 0$ and $j_n \geq 0$, such that $n = i_n \times 4 + j_n \times 7$.

   Proof. Define $P(n)$ be the statement that "there exist integers $i_n \geq 0$ and $j_n \geq 0$, such that $n = i_n \times 4 + j_n \times 7$".

   If $P(n)$ is not true for every $n > 17$, then there are values of n for which $P(n)$ is false, and there must be a smallest such value, say $n = k$.

   Since we have $k > 17$, and $k - 1 > 16$.

   Since k is the smallest value for which $P(k)$ is false, $P(k-1)$ is true.

   Thus $k - 1 = i_n \times 4 + j_n \times 7$.

   If $j_n = 0$, so $i_n > 4$,

   However we have

   $$k = k - 1 - 4 \times 5 + 7 \times 3 = (i_n - 5) \times 4 + (j_n + 3) \times 7$$

   If $j_n \neq 0$

   However we have

   $$k = k - 1 + 2 \times 4 - 7 = (i_n + 2) \times 4 + (j_n - 1) \times 7$$

   We have derived a contradiction, which allows us to conclude that our original assumption is false.

   □

3. Let $P = \{p_1, p_2, \cdots\}$ the set of all primes. Suppose that $\{p_i\}$ is monotonically increasing, i.e., $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $\cdots$. Please prove: $p_n < 2^{2^n}$. (Hint: $p_i \nmid (1 + \prod_{j=1}^{n} p_j), i = 1, 2, \cdots, n.$)

Proof. Define $P(n)$ be the statement that "$p_n < 2^{2^n}$".

Basis step. $P(1)$ is true

Induction hypothesis. For $k \geq 1$ and $1 \leq n \leq k$,$P(n)$ is true.

Proof of induction step. Let's prove $P(k+1)$.

If $p_{k+1} \geq 2^{2^{k+1}}$, there were only k prime in $2^{2^{k+1}}$.

So any composite numbers own the factor in $p_1, p_2, \cdots, p_k$

However $p_i \nmid (1 + \prod_{j=1}^{n} p_j), i = 1, 2, \cdots, n$

since $\prod_{j=1}^{k} p_j + 1 = 2^{2^{k+1}-2} + 1 < 2^{2^{k+1}}$, We have derived a contradiction.

So $p_{k+1} < 2^{2^{k+1}}$

Thus by induction hypothesis,$P(k+1)$ is true.

Finally we can conclude that $p_n < 2^{2^n}$.

□

4. Prove that a plane divided by n lines can be colored with only 2 colors, and the adjacent regions have different colors.

Proof. Define $P(n)$ be the statement that "a plane divided by n lines can be colored with only 2 colors, and the adjacentregions have different colors".

Basis step. $P(1)$ is true.

Induction hypothesis. For $k > 1$, $P(k)$ is true.

Proof of induction step. Let's prove $P(k+1)$.

After drawing the n line, we divide the plane into two sides. Now, let's think about two adjacent regions R_1 and R_2. Change all of the colors into the different color in one side which includes R_1.

If they are on the same side, their color is different since $P(k)$ is true.

If they are on the different side, the color of the region including R_1 has been changed. So their color are different. $P(k+1)$ is true.

Finally we can conclude that $P(n)$ is true.

□

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