Lab07-Amortized Analysis

CS214-Algorithm and Complexity, Xiaofeng Gao, Spring 2020.

$*$ If there is any problem, please contact TA Shuodian Yu.

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1. For the TABLE-DELETE Operation in Dynamic Tables, suppose we construct a table by multiplying its size by $\frac{2}{3}$ when the load factor drops below $\frac{1}{3}$. Using *Potential Method* to prove that the amortized cost of a TABLE-DELETE that uses this strategy is bounded above by a constant.

   Solution.

$$\Phi(T) = size[T] - num[T]$$

   case 1: no contraction

   The amortized cost is:

$$\begin{aligned}\hat{C}_i &= C_i + \Phi_i - \Phi_{i-1} \\ &= 1 + (size_i - num_i) - (size_{i-1} - num_{i-1}) \\ &= 2\end{aligned}$$

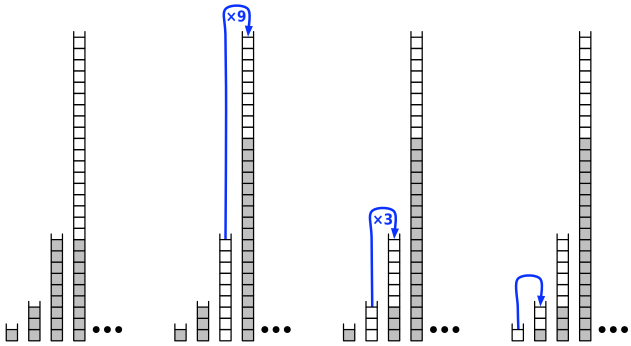
   case 2: a contraction was triggered

   The amortized cost is:

$$\begin{aligned}\hat{C}_i &= C_i + \Phi_i - \Phi_{i-1} \\ &= num_{i-1} + (size_i - num_i) - (size_{i-1} - num_{i-1}) \\ &= num_{i-1} + (\frac{2}{3}size_{i-1} - num_{i-1} + 1) - (size_{i-1} - num_{i-1}) \\ &= num_{i-1} - \frac{1}{3}size_{i-1} + 1 \leftarrow \quad num_{i-1} = \frac{1}{3}size_{i-1} \\ &= 1\end{aligned}$$

   Therefore the cost of a TABLE-DELETE is bounded above by a constant.

2. A **multistack** consists of an infinite series of stacks $S_0, S_1, S_2, \cdots$, where the i^{th} stack S_i can hold up to 3^i elements. Whenever a user attempts to push an element onto any full stack S_i, we first pop all the elements off S_i and push them onto stack S_{i+1} to make room. (Thus, if S_{i+1} is already full, we first recursively move all its members to S_{i+2} .) An illustrative example is shown in Figure 1. Moving a single element from one stack to the next takes $O(1)$ time. If we push a new element, we always intend to push it in stack S_0.



Making room for one new element in a multistack.

图 1: An example of making room for one new element in a multistack.

$$\sum_{i=0} i = k$$

(a) In the worst case, how long does it take to push a new element onto a multistack containing n elements?

(b) Prove that the amortized cost of a push operation is $O(\log n)$ by *Aggregation Analysis*.

(c) (Optional Subquestion with Bonus) Prove that the amortized cost of a push operation is $O(\log n)$ by *Potential Method*.

Solution.

(a) The worst case is when we push a new element, the n elements in the multistack are filled with the first k stacks. And all elements are needed to be moved. The time cost is $O(n)$.

(b) The worst time $T(n)$ in total for a sequence of n operations is when the n elements are stoed in the i_{th} stack.

All the elemens pushed from $stack_1$ to $stack_i$ and cost $O(i)$ time.

And it's clear that $n = \frac{3^{i-1}-1}{2}$. Thus $i = O(\log n)$.

Thus the amortized cost of a push operation is $\frac{nO(\log n)}{n} = O(\log n)$

3. Given a graph $G = (V, E)$, and let V' be a strict subset of V. Prove the following propositions.

(a) Let T be a minimum spanning tree of a G. Let T' be the subgraph of T induced by V', and let G' be the subgraph of G induced by V'. Then T' is a minimum spanning tree of G' if T' is connected.

(b) Let e be a minimum weight edge which connects V' and $V \setminus V'$. There exists a minimum weight spanning tree which contains e.

Solution.

(a) Assume T' is not a minimum spanning tree of G' if T' is connected.

And there must exist an edge $e_2 \in G'$ that own smaller weight than the edge $e_1 \in T'$.

For T, the reason why $e_2 \notin T$ is that e_2 forms a circle if we add e_2 in T. Thus in the connected tree T', the edge e_2 must form a circle in T'.

Therefore T' is a minimum spanning tree of G' if T' is connected.

(b) Assume all the minimum spanning tree in G don't contain the minimum weight edge (u, v).

Let T one of the minimum spanning tree in G. If we joing (u, v) to T, there must exist a circle containing (u, v). And for T is a spanning tree, there must exist another edge (u', v') that $u' \in V'$ and $v' \in V/V'$. Thus u is connected with u' and v is connected with v'.

Deleting (u', v') will eliminate the circle. And add (u, v) to T will form another spanning tree whose weight is smaller than that containing (u', v').

Therefore there exists a minimum weight spanning tree which contains e.

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